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FIXED-GAIN, TWO-STAGE ESTIMATORS FOR TRACKING MANEUVERING TARGETS

BY W.D. BLAIR
WEAPONS SYSTEMS DEPARTMENT

JULY 1992

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NAVAL SURFACE WARFARE CENTER
DAHLGREN DIVISION
Dahlgren, Virginia 22448-5000

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FOREWORD

The classical problem of weapons control involves predicting the future position of a maneuvering target. Critical to successful prediction is the accurate estimation of the current target state. With the advent of guided weapons, the consequences of threat maneuver are reduced when accurate estimates of the target state can be obtained. Threat trends indicate that the conditions under which hostile targets can be engaged successfully are becoming more difficult to achieve; hence, any improvement in existing estimation algorithms is of critical importance.

This report presents the results of an investigation of an approach to the improvement of an important class of target tracking algorithms. The work was supported by the Naval Surface Warfare Center Dahlgren Division (NSWCDD) AEGIS Program Office.

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Approved by:

David S Malyevac

DAVID S. MALYEVAC, Deputy Department Head
Weapons Systems Department

ABSTRACT

The two-stage Alpha-Beta-Gamma estimator is proposed as an alternative to adaptive gain versions of the Alpha-Beta and Alpha-Beta-Gamma filters for tracking maneuvering targets. The purpose of this report is to accomplish fixed-gain, variable dimension filtering with a two-stage Alpha-Beta-Gamma estimator. The two-stage Alpha-Beta-Gamma estimator is derived from the two-stage Kalman estimator, and the noise variance reduction matrix and steady-state error covariance matrix are given as a function of the steady-state gains. A procedure for filter parameter selection is also given along with a technique for maneuver response and a gain scheduling technique for initialization. The kinematic constraint for constant speed targets is also incorporated into the two-stage estimator to form the two-stage Alpha-Beta-Gamma-Lambda estimator. Simulation results are given for a comparison of the performances of estimators with that of the Alpha-Beta-Gamma filter.

CONTENTS

CHAPTER 1
INTRODUCTION

While the Kalman filter is known to produce an optimal estimate of the target state when given the motion model and a sequence of sensor measurements corrupted with white Gaussian errors, the computational burden of maintaining the Kalman filter may prohibit its use when many targets are being tracked. A widely used approach to reducing the computational burden of the filters involves the use of an approximate filter gain instead of the optimal Kalman gain. The gain is usually based on the steady-state gains of the Kalman filter and obtained by either using a fixed gain or an easily generated gain schedule. When these steady-state or approximate gains are used, the resultant filters are called an α filter for tracking position; an α, β filter for tracking position and velocity; and an α, β, γ filter tracking position, velocity, and acceleration.

For tracking systems with a uniform data rate and stationary measurement noise, non-maneuvering targets can be accurately tracked with an α, β filter. However, when the target maneuvers, the quality of the position and velocity estimates provided by the filter can degrade significantly, and for a target undergoing a large maneuver, the target track may be lost. An α, β, γ filter can be used to track such a target, but the accelerations of a maneuvering target are seldom constant in the tracking frame during the maneuver. If the α gain of the α, β, γ filter is chosen small to achieve good noise reduction, the "optimal" γ of Kalata [1] will be very small and the acceleration estimates will respond slowly to a maneuver. Thus, in order for the α, β, γ filter to respond quickly to a maneuver, α must be maintained at a high level and the filter will provide poor noise reduction (noisy estimates) when the target is not manuevering. The noisy target state estimates may hinder the sensor pointing, data association, and other control decisions and computations based on the state estimates.

One approach to overcoming this filtering dilemma is to use a small α until a maneuver is detected. Then α is increased for tracking through the maneuver. This approach was investigated by Cantrell [2] and Blackman [3]. It has two major problems. First, the response of the filter is significantly delayed because the gains are not increased until a maneuver is detected. Furthermore, in the α, β filter, the α gain must be set artificially high to account for the absence of acceleration from the motion model. Second, after a target has returned to constant velocity from a maneuver, the decision to reduce α is often delayed because there is no bias in the higher gain α, β filter to detect.

An alternate approach, which seems to be attractive, is to continuously estimate the target acceleration and correct the position and velocity estimates of the α, β filter whenever a maneuver is detected. The purpose of this report is to accomplish fixed-gain, variable dimension filtering with a two-stage $\alpha, \beta, \bar{\gamma}$ estimator, as suggested by Alouani et. al. [4,5] for the two-stage Kalman estimator. In the two-stage $\alpha, \beta, \bar{\gamma}$ estimator, position and velocity are tracked in the first stage, which is a standard α, β filter, and the acceleration is tracked in the second stage, which is a standard single gain filter. The output of the acceleration filter can be used to adjust the estimates of the α, β filter, as shown in Figure 1-1 when a bias is detected. This approach to tracking maneuvering targets is similar to input estimation in [6-8], with the exception that the acceleration is modeled as a stochastic process instead of a deterministic process as in input estimation. The two-stage estimator has the good noise reduction associated with the α, β filter when the target is not maneuvering, and when a maneuver is detected, an acceleration estimate is available to compensate the estimates of the α, β filter. Furthermore, since the acceleration is not part of the central filter, the $\bar{\gamma}$ gain can be picked somewhat independent of α to improve maneuver response.

When targets maneuver while maintaining a constant speed, the accelerations vary with time throughout the maneuver and the estimates of both the α, β, γ filter and the two-stage $\alpha, \beta, \bar{\gamma}$ estimator will be biased. The kinematic constraint for constant speed targets can be included as a pseudomeasurement in the filtering process to reduce the filter bias as in [9]. The kinematic constraint for constant speed targets is incorporated into the two-stage estimator to form the two-stage $\alpha, \beta, \gamma, \lambda$ estimator.

This report is organized as follows. In Chapter 2, steady-state Kalman filters are discussed, and the α, β filter and α, β, γ filter are summarized. The two-stage Kalman estimator is presented in Chapter 3. In Chapter 4, the two-stage $\alpha, \beta, \bar{\gamma}$ estimator is derived, and procedures for gain selection during initialization and steady-state conditions are presented along with the concept of a soft switch for maneuver response. The two-stage $\alpha, \beta, \bar{\gamma}, \lambda$ estimator is presented in Chapter 5. Simulation results for a radar tracking system are presented in Chapter 6 for the two-stage estimators and the α, β, γ filter. Some concluding remarks are given in Chapter 7 along with a discussion of future research.

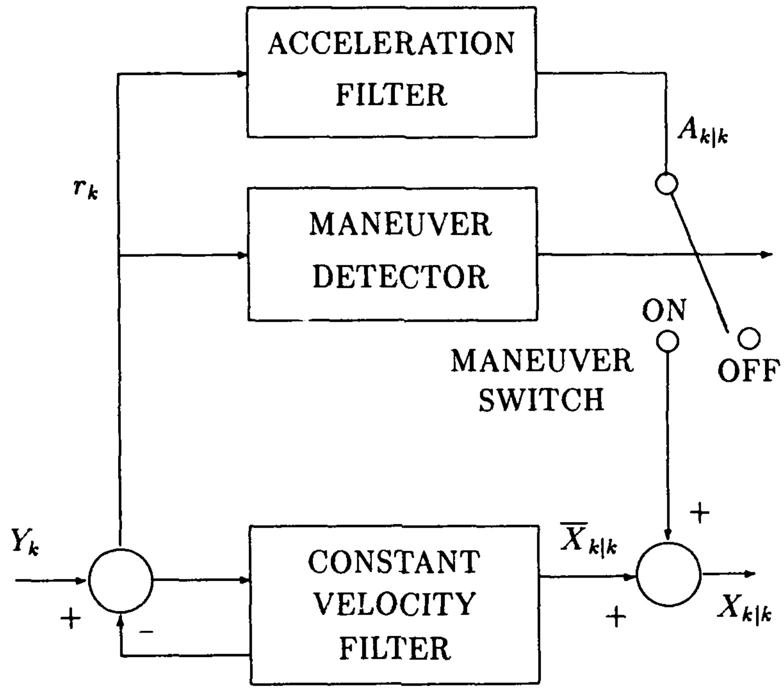

Figure 1–1. Two-Stage Estimator for Tracking Maneuvering Targets

CHAPTER 2

STEADY-STATE KALMAN FILTERS

A Kalman filter is often employed to filter the position measurements for estimating the position, velocity, and/or acceleration of a target. When the target motion and measurement models are linear and the measurement and motion modeling error processes are Gaussian, the Kalman filter provides the minimum mean-square error estimate of the target state. When the target motion and measurement models are linear, but the noise processes are not Gaussian, the Kalman filter is the best linear estimator of the target state in the mean-square error sense. The dynamics model commonly assumed for a target in track is given by

$$X_{k+1} = F_k X_k + G_k w_k \tag{2.1}$$

where $w_k \sim N(0, Q_k)$ is the process noise and F_k defines a linear constraint on the dynamics. The target state vector X_k contains the position, velocity, and acceleration of the target at time k, as well as other variables used to model the time-varying acceleration. The linear measurement model is given by

$$Y_k = H_k X_k + n_k \tag{2.2}$$

where Y_k is usually the target position measurement and $n_k \sim N(0, R_k)$. The Kalman filtering equations associated with the state model in Eq. (2.1) and the measurement model in Eq. (2.2) are given by the following equations.

Time Update:

$$X_{k|k-1} = F_{k-1} X_{k-1|k-1} \tag{2.3}$$

$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T \tag{2.4}$$

Measurement Update:

$$K_k = P_{k|k-1} H_k^T [H_k P_{k|k-1} H_k^T + R_k]^{-1} \tag{2.5}$$

$$X_{k|k} = X_{k|k-1} + K_k [Y_k - H_k X_{k|k-1}] \tag{2.6}$$

$$P_{k|k} = [I - K_k H_k] P_{k|k-1} \tag{2.7}$$

where $X_k \sim N(X_{k|k}, P_{k|k})$ with $X_{k|k}$ and $P_{k|k}$ denoting the mean and error covariance of the state estimate, respectively. The subscript notation $(k|j)$ denotes the state estimate for time k when given measurements through time j, and K_k denotes the Kalman gain. Using the matrix inversion lemma of [10] and Eqs. (2.5) and (2.7), an alternate form of the Kalman gain is given by

$$\begin{aligned} K_k &= P_{k|k-1} H_k^T [H_k P_{k|k-1} H_k^T + R_k^{-1}]^{-1} \\ &= [I - P_{k|k-1} H_k^T R_k^{-1} H_k]^{-1} P_{k|k-1} H_k^T R_k^{-1} \\ &= [I - P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k^{-1})^{-1} H_k] P_{k|k-1} H_k^T R_k^{-1} \\ &= [I - K_k H_k] P_{k|k-1} H_k^T R_k^{-1} \\ &= P_{k|k} H_k^T R_k^{-1} \end{aligned} \tag{2.8}$$

The steady-state form of the Kalman filter is often used in order to reduce the computations required to maintain each track. In steady-state, $P_{k|k} = P_{k-1|k-1}$, and $P_{k+1|k} = P_{k|k-1}$, and $K_k = K_{k-1}$. For a Kalman filter to achieve these steady-state conditions, the error processes, w_k and n_k, must have stationary statistics and the data rate must be constant. When the noise processes are not stationary or the data rate is not constant, a filter using the steady-state gains will provide suboptimal estimates. The α, β and α, β, γ filters are the steady-state Kalman filters for tracking nearly constant velocity targets and nearly constant acceleration targets, respectively. First, the α filter for tracking nearly constant position targets will be considered. Then the α, β and α, β, γ filters will be considered.

ALPHA FILTER

The α filter is a single coordinate filter that is based on the assumption that the target position is constant plus zero-mean, white Gaussian acceleration errors. Given this assumption, the filter gain α is chosen as the steady-state Kalman gain that minimize the mean-square error in the position estimates. For the α filter,

$$X_k = [\,x_k\,]^T \tag{2.9}$$

$$F_k = [\,1\,] \tag{2.10}$$

$$G_k = \left[\frac{T^2}{2}\right]^T \tag{2.11}$$

$$H_k = [\,1\,] \tag{2.12}$$

$$R_k = \sigma_v^2 \tag{2.13}$$

$$Q_k = \sigma_w^2 \tag{2.14}$$

$$K_k = [\,\alpha\,]^T \tag{2.15}$$

The steady-state error covariance of the filtered estimates for the α filter can be expressed as a function the Kalman gain using Eq. (2.8) to be

$$P_{k+1|k+1} = P_{k|k} = P = \alpha\sigma_v^2 \tag{2.16}$$

Using Eq. (2.4) in Eq. (2.7) and inserting Eq. (2.16) for P gives

$$\alpha\sigma_v^2 = [1-\alpha][\alpha\sigma_v^2 + \frac{T^4}{4}\sigma_w^2] \tag{2.17}$$

Thus, Eq. (2.17) gives

$$\Gamma^2 = T^4\frac{\sigma_w^2}{\sigma_v^2} = \frac{4\alpha^2}{1-\alpha} \tag{2.18}$$

where Γ is the Tracking Index of [11], and σ_w^2 is the variance of the acceleration modeling error. Thus, the α gain is determined as in [1] from Γ by solving the Eq. (2.18). Note the these steady-state relationships result from the assumption that the model error process is acceleration errors that are constant through each measurement sample period.

The input-output relationships between the measurements Y_k and $X_{k|k}$ can be expressed as a linear system that is given by

$$X_{k|k} = (1-\alpha)X_{k-1|k-1} + \alpha Y_k \tag{2.19}$$

Using Eq. (2.19) provides the error covariance of $X_{k|k}$ that results from the measurement errors. Let Y_k be a white noise stationary sequence with zero mean. Then $E[X_k X_k] = \overline{S}_k = \overline{S}_{k-1} = \overline{S}_\alpha$, which is given by

$$\overline{S}_\alpha = \frac{\sigma_v^2\alpha}{2-\alpha} \tag{2.20}$$

The variance reduction ratio of the filter is given by Eq. (2.20) when $\sigma_v^2 = 1$.

Since during initialization the α filter is not in a steady-state condition, Eq. (2.16) cannot be used to determine α. Using least-squares estimation in conjunction with a constant state model, a simple gain scheduling procedure for α during initialization is given by

$$\alpha_k = \max\{\frac{1}{(k+1)}, \alpha\} \tag{2.21}$$

with $X_{0|-1} = [\,0\,]^T$. The derivation of Eq. (2.21) follows closely the derivation for the gains for initializing the α, β filter in Appendix A.

ALPHA-BETA FILTER

The α, β filter is a single coordinate filter that is based on the assumption that the target is moving with constant velocity plus zero-mean, white Gaussian acceleration errors. Given this assumption, the filter gains α and β are chosen as the steady-state Kalman gains that minimize the mean-square error in the position and velocity estimates. For the α, β filter,

$$X_k = [x_k \quad \dot{x}_k]^T \tag{2.22}$$

$$F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{2.23}$$

$$G_k = \begin{bmatrix} \frac{T^2}{2} & T \end{bmatrix}^T \tag{2.24}$$

$$H_k = [1 \quad 0] \tag{2.25}$$

$$R_k = \sigma_v^2 \tag{2.26}$$

$$Q_k = \sigma_w^2 \tag{2.27}$$

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{T} \end{bmatrix}^T \tag{2.28}$$

The α, β gains are determined as in [1,3,8] by solving the simultaneous equations

$$\Gamma^2 = T^4 \frac{\sigma_w^2}{\sigma_v^2} = \frac{\beta^2}{(1-\alpha)} \tag{2.29}$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{2.30}$$

where σ_w^2 is the variance of the acceleration modeling error.

The steady-state error covariance of the filtered estimates for the α, β filter is given in [1,8] as

$$P_{k|k} = \sigma_v^2 \begin{bmatrix} \alpha & \frac{\beta}{T} \\ \frac{\beta}{T} & \frac{\beta(2\alpha - \beta)}{2(1-\alpha)T^2} \end{bmatrix} \tag{2.31}$$

The steady-state gains and error covariance in Eqs. (2.29) through (2.31) are derived in Appendix A. Note that these steady-state relationships result from the assumption that the model error process is acceleration errors that are constant through each measurement sample period.

In target tracking, a conflict arises between the goals of good noise reduction which requires small α (and thus, small β), and good tracking through maneuvers which requires a larger α. For good noise reduction, heavy filtering is used and a filter with slow system response (i.e., a large time constant) and a narrow bandwidth results. For good tracking through a maneuver, light filtering is performed and a filter with fast response (i.e., a short time constant) and a wide bandwidth results. The α and β are chosen with the "optimal" relationship in Eq. (2.29) to obtain a compromise between these two goals that meets the design requirements.

The input-output relationships between the measurements Y_k and $X_{k|k}$ can be expressed as a linear system that is given by

$$X_{k|k} = \overline{F} X_{k-1|k-1} + \overline{G}_k Y_k \tag{2.32}$$

where

$$\overline{F} = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\dfrac{\beta}{T} & (1-\beta) \end{bmatrix} \tag{2.33}$$

$$\overline{G} = \begin{bmatrix} \alpha \\ \dfrac{\beta}{T} \end{bmatrix} \tag{2.34}$$

Using Eq. (2.32) provides the error covariance of $X_{k|k}$ that results from the measurement errors. That error covariance $\overline{S}_k$ is given by

$$\overline{S}_k = \overline{F}\, \overline{S}_{k-1} \overline{F}^T + \overline{G}\, \overline{G}^T \sigma_v^2 \tag{2.35}$$

Since $\overline{S}_k = \overline{S}_{k-1} = \overline{S}_{\alpha\beta}$ in steady-state conditions, Eq. (2.35) can be used to solve for $\overline{S}_{\alpha\beta}$ in terms of the filter gains, measurement period, and the measurement error variance. As derived in Appendix A,

$$\overline{S}_{\alpha\beta} = \frac{\sigma_v^2}{\alpha d_1} \begin{bmatrix} 2\alpha^2 + \beta(2-3\alpha) & \dfrac{\beta}{T}(2\alpha-\beta) \\ \dfrac{\beta}{T}(2\alpha-\beta) & \dfrac{2\beta^2}{T^2} \end{bmatrix} \tag{2.36}$$

where $d_1 = 4 - 2\alpha - \beta$. The variance reduction ratios of the filter are given by Eq. (2.36) when $\sigma_v^2 = 1$ with the (1,1) and (2,2) elements of $\overline{S}_{\alpha\beta}$ denoting the position variance reduction ratio and velocity variance reduction ratio, respectively [3].

Since during initialization the α, β filter is not in steady-state conditions, Eqs. (2.29) and (2.30) cannot be used to determine α and β. Using least-squares estimation in conjunction with a constant velocity motion model, a simple gain scheduling procedure for α and β

during initialization is given by

$$\alpha_k = \max\{\frac{2(2k+1)}{(k+1)(k+2)}, \alpha\} \tag{2.37}$$

$$\beta_k = \max\{\frac{6}{(k+1)(k+2)}, \beta\} \tag{2.38}$$

with $X_{0|-1} = [0 \quad 0]^T$. Note that if Y_0/T can be extremely large relative to the dynamic range of the compute., the first two filter iterations should be computed analytically to use a two-point initialization procedure.

ALPHA-BETA-GAMMA FILTER

The α, β, γ filter is a single coordinate filter that is based on the assumption that the target is moving with constant acceleration plus zero-mean, white Gaussian acceleration errors. Given this assumption, the α, β, γ filter gains are chosen as the steady-state Kalman gains that minimize the mean-square error in the position, velocity, and acceleration estimates. For the α, β, γ filter,

$$X_k = [x_k \quad \dot{x}_k \quad \ddot{x}_k]^T \tag{2.39}$$

$$F_k = \begin{bmatrix} 1 & T & 0.5T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{2.40}$$

$$G_k = \begin{bmatrix} \frac{T^2}{2} & T & 1 \end{bmatrix}^T \tag{2.41}$$

$$H_k = [1 \quad 0 \quad 0] \tag{2.42}$$

$$R_k = \sigma_v^2 \tag{2.43}$$

$$Q_k = \sigma_w^2 \tag{2.44}$$

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{T} & \frac{\gamma}{T^2} \end{bmatrix}^T \tag{2.45}$$

The α, β, γ gains are determined as in [1,3,8] by solving the simultaneous equations

$$\Gamma^2 = T^4 \frac{\sigma_w^2}{\sigma_v^2} = \frac{\gamma^2}{(1-\alpha)} \tag{2.46}$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{2.47}$$

$$\gamma = \frac{\beta^2}{2\alpha} \tag{2.48}$$

The steady-state error covariance matrix of the filtered estimates for the α, β, γ filter is given in [1,8] as

$$P_{k|k} = \sigma_v^2 \begin{bmatrix} \alpha & \dfrac{\beta}{T} & \dfrac{\gamma}{T^2} \\ \dfrac{\beta}{T} & \dfrac{4\alpha\beta+\gamma(\beta-2\alpha-4)}{4(1-\alpha)T^2} & \dfrac{\beta(\beta-\gamma)}{2(1-\alpha)T^3} \\ \dfrac{\gamma}{T^2} & \dfrac{\beta(\beta-\gamma)}{2(1-\alpha)T^3} & \dfrac{\gamma(\beta-\gamma)}{(1-\alpha)T^4} \end{bmatrix} \tag{2.49}$$

The steady-state gains and error covariance in Eqs. (2.46) through (2.49) are derived in Appendix B.

The input-output relationships between the measurements Y_k and filtered state $X_{k|k}$ can be expressed as a linear system that is given by

$$X_{k|k} = \begin{bmatrix} 1-\alpha & (1-\alpha)T & (1-\alpha)\dfrac{T^2}{2} \\ -\dfrac{\beta}{T} & (1-\beta) & (1-\dfrac{\beta}{2})T \\ -\dfrac{\gamma}{T^2} & -\dfrac{\gamma}{T} & 1-\dfrac{\gamma}{2} \end{bmatrix} X_{k-1|k-1} + \begin{bmatrix} \alpha \\ \dfrac{\beta}{T} \\ \dfrac{\gamma}{T^2} \end{bmatrix} Y_k \tag{2.50}$$

Using Eq. (2.50) provides the steady-state error covariance of $X_{k|k}$ that results from the measurement errors. That error covariance $\overline{S}_{\alpha\beta\gamma}$ as given in [11] is

$$\overline{S}_{\alpha\beta\gamma} = \left(\frac{\sigma_v^2}{d_1 d_2}\right) \times$$

$$\begin{bmatrix} 2\alpha d_2 - \beta^2(6\alpha-4)+\alpha\beta\gamma & \dfrac{\beta}{T}(2\alpha-\beta)(2\beta-\gamma) & \dfrac{\gamma}{T^2}(2d_2+\beta(\gamma-2\beta)) \\ \dfrac{\beta}{T}(2\alpha-\beta)(2\beta-\gamma) & \dfrac{2}{T^2}(\gamma^2(2-\alpha)+2\beta^2(\beta-\gamma)) & \dfrac{2\beta\gamma}{T^3}(2\beta-\gamma) \\ \dfrac{\gamma}{T^2}(2d_2+\beta(\gamma-2\beta)) & \dfrac{2\beta\gamma}{T^3}(2\beta-\gamma) & \dfrac{4\beta\gamma^2}{T^4} \end{bmatrix} \tag{2.51}$$

where $d_1 = 4 - 2\alpha - \beta$ and $d_2 = 2\alpha\beta + \gamma(\alpha - 2)$. The measurement error variance reduction ratios of the filter are given by Eq. (2.51) when $\sigma_v^2 = 1$ with the (1,1), (2,2), and (3,3) elements of $\overline{S}_{\alpha\beta\gamma}$ denoting the position variance reduction ratio, velocity variance reduction ratio, and the acceleration variance reduction ratio, respectively.

Using least-squares estimation in conjunction with a constant acceleration motion model, a simple gain scheduling procedure for α, β and γ during initialization is given by

$$\alpha_k = \max\{\frac{3(3k^2+3k+2)}{(k+1)(k+2)(k+3)}, \alpha\} \tag{2.52}$$

$$\beta_k = \max\{\frac{18(2k+1)}{(k+1)(k+2)(k+3)}, \beta\} \tag{2.53}$$

$$\gamma_k = \max\{\frac{60}{(k+1)(k+2)(k+3)}, \gamma\} \tag{2.54}$$

with $X_{0|-1} = [0 \quad 0 \quad 0]^T$. Note that if $Y_{0,1}/T^2$ can be extremely large relative to the dynamic range of the computer, the first three iterations of the filter should be computed analytically to use a three-point initialization procedure.

CHAPTER 3

TWO-STAGE KALMAN ESTIMATOR

Consider the problem of estimating the state of a linear system in the presence of a random bias that influences the system dynamics and/or observations. The bias vector may represent a part of the augmented system state as suggested in [12]. The idea of using a two-stage filter to implement an augmented state filter was introduced in [12]. The idea is to decouple the central filter into two parallel filters. The first filter, the "bias-free" filter, is based on the assumption that the bias is nonexistent. The second filter, the bias filter, produces an estimate of the bias vector. The output of the first filter is then corrected with the output of the second filter. It was shown in [12] that if the bias is deterministic and constant, but unknown, the two-stage filter is equivalent to the augmented state filter. It was shown in [4,13] that under an algebraic constraint on the correlation between the state process noise and the bias process noise, the proposed two-stage Kalman estimator is equivalent to the augmented state Kalman filter.

Consider a linear system whose dynamics is modeled by

$$X_{k+1} = F_k X_k + G_k b_k + G_k^X W_k^X \tag{3.1}$$

$$b_{k+1} = b_k + G_k^b W_k^b \tag{3.2}$$

where X_k is an n dimensional system state vector, b_k is a p dimensional bias vector. This system may represent the dynamics of a maneuvering target, where the position and velocity are the system state and the bias represents the target acceleration. The W_k^X and W_k^b are white Gaussian sequences with zero means and variances given by

$$E[W_k^X (W_l^X)^T] = Q_k^X \delta_{kl} \tag{3.3}$$

$$E[W_k^b (W_l^b)^T] = Q_k^b \delta_{kl} \tag{3.4}$$

$$E[W_k^X (W_l^b)^T] = Q_k^{Xb} \delta_{kl} \tag{3.5}$$

The state measurement model at time k is given by

$$Y_k = H_k X_k + C_k b_k + v_k \tag{3.6}$$

where Y_k is the measurement and $v_k \sim N(0, R_k)$ is the measurement error. The standard approach to estimating the state and bias is to form an augmented state model which includes both the bias and the state. With this augmented state model, a Kalman filter may be used to produce the optimal state estimates.

If the bias term is ignored, ($b_k = 0$), the bias-free filter is the Kalman filter based on the model in Eqs. (3.1) and (3.6), where fictitious statistics are used for W_k^X (i.e., $\overline{Q}_k^X$ is used instead of $G_k^X Q_k^X (G_k^X)^T$). The bias-free filter is given by

$$\overline{X}_{k|k-1} = F_{k-1}\overline{X}_{k-1|k-1} \tag{3.7}$$

$$\overline{X}_{k|k} = \overline{X}_{k|k-1} + \overline{K}_k^X [Y_k - H_k \overline{X}_{k|k-1}] \tag{3.8}$$

$$\overline{P}_{k|k-1}^X = F_{k-1}\overline{P}_{k-1|k-1}^X F_{k-1}^T + \overline{Q}_{k-1}^X \tag{3.9}$$

$$\overline{P}_{k|k}^X = (I - \overline{K}_k^X H_k)\overline{P}_{k|k-1}^X \tag{3.10}$$

$$\overline{K}_k^X = \overline{P}_{k|k-1}^X H_k^T [H_k \overline{P}_{k|k-1}^X H_k^T + R_k]^{-1} \tag{3.11}$$

where $\overline{Q}_{k-1}^X$ is defined later in this chapter. The $\overline{X}_{k|k}$ represents the estimate of the state process when the bias is ignored, and $\overline{P}_{k|k}^X$ is the error covariance of $\overline{X}_{k|k}$.

As in [12], a separate filter may be used to estimate the bias vector from the residual sequence of the bias-free filter. The bias filter is given by

$$b_{k|k-1} = b_{k-1|k-1} \tag{3.12}$$

$$b_{k|k} = b_{k|k-1} + K_k^b [r_k - S_k b_{k|k-1}] \tag{3.13}$$

$$P_{k|k-1}^b = P_{k-1|k-1}^b + G_{k-1}^b Q_{k-1}^b (G_{k-1}^b)^T \tag{3.14}$$

$$K_k^b = P_{k|k-1}^b S_k^T [S_k P_{k|k-1}^b S_k^T + H_k \overline{P}_{k|k-1}^X H_k^T + R_k]^{-1} \tag{3.15}$$

$$P_{k|k}^b = (I - \overline{K}_k^b S_k) P_{k|k-1}^b \tag{3.16}$$

where

$$S_k = H_k U_k + C_k \tag{3.17}$$

$$U_k = F_{k-1} V_{k-1} + G_{k-1} \tag{3.18}$$

$$V_k = (I - \overline{K}_k^X H_k) U_k - \overline{K}_k^X C_k \tag{3.19}$$

and r_k is the residual of the bias-free filter. Initialization of this bias filter is discussed later in this chapter.

The algorithm for compensating the output of the bias-free filter with the output of the bias filter is given by

$$X_{k|k} = E[X_k] = \overline{X}_{k|k} + V_k b_{k|k} \tag{3.20}$$

$$P_{k|k} = E[(X_k - X_{k|k})(X_k - X_{k|k})^T] = \overline{P}^X_{k|k} + V_k P^b_{k|k} V_k^T \tag{3.21}$$

$$P^{Xb}_{k|k} = E[(X_k - X_{k|k})(b_k - b_{k|k})^T] = V_k P^b_{k|k} \tag{3.22}$$

$$P^b_{k|k} = E[(b_k - b_{k|k})(b_k - b_{k|k})^T] = P^b_{k|k} \tag{3.23}$$

The structure of the two-stage Kalman estimator is shown in Figure 3–1.

Theorem: If

$$G^X_k Q^{Xb}_k (G^b_k)^T = U_{k+1} G^b_k Q^b_k (G^b_k)^T \tag{3.24}$$

and the error covariance $\overline{Q}^X_k$ of the process noise of the bias-free filter model given by

$$\overline{Q}^X_k = G^X_k Q^X_k (G^X_k)^T - U_{k+1} G^b_k Q^b_k (G^b_k)^T U^T_{k+1} \tag{3.25}$$

is positive semidefinite, then the filter given by Eqs. (3.20)-(3.23) is equivalent to the augmented state filter.

Proof: See [4,13].

While the algebraic constraint of Eq. (3.24) will not be satisfied by almost all real systems, the theorem provides a basis for assessing the degree of suboptimality of a two-stage estimator when applied to specific problems. For example, let the algebraic constraint of Eq. (3.24) be expressed as

$$G^X_k Q^{Xb}_k (G^b_k)^T < (1+\epsilon) U_{k+1} G^b_k Q^b_k (G^b_k)^T \quad k \geq 0 \tag{3.26}$$

Then for a given system, the smallest positive ϵ that satisfies Eq. (3.26) indicates the degree of suboptimality of the two-stage estimate with respect to the augmented state filter.

For initialization of the two-stage estimator, $P^{Xb}_{0|0} = 0$ if the initial estimate of the bias is uncorrelated with the initial estimate of the bias-free state. If $P^{Xb}_{0|0} = 0$, Eqs. (3.22) and (3.19) imply that $V_0 = 0$ and $U_0 = 0$. Note that as shown in [4,13] the initialization does not require $P^{Xb}_{0|0} = 0$.

For the system and bias models given in Eqs. (3.1) and (3.2), the process noise covariances are given in Eqs. (3.3) through (3.5). However, for tracking targets with the two-stage estimator, the process noise covariances in Eqs. (3.3) through (3.5) are design parameters. When designing the two-stage estimator for tracking maneuvering targets, $\overline{Q}^X_k$ can be chosen to obtain the desired response of the constant velocity filter and Q^b_k can be chosen for maneuver response. In this type of design procedure, $\overline{Q}^X_k$ is chosen to be positive semi-definite and Q^X_k is allowed be a free parameter that depends on the selection of $\overline{Q}^X_k$ and Q^b_k.

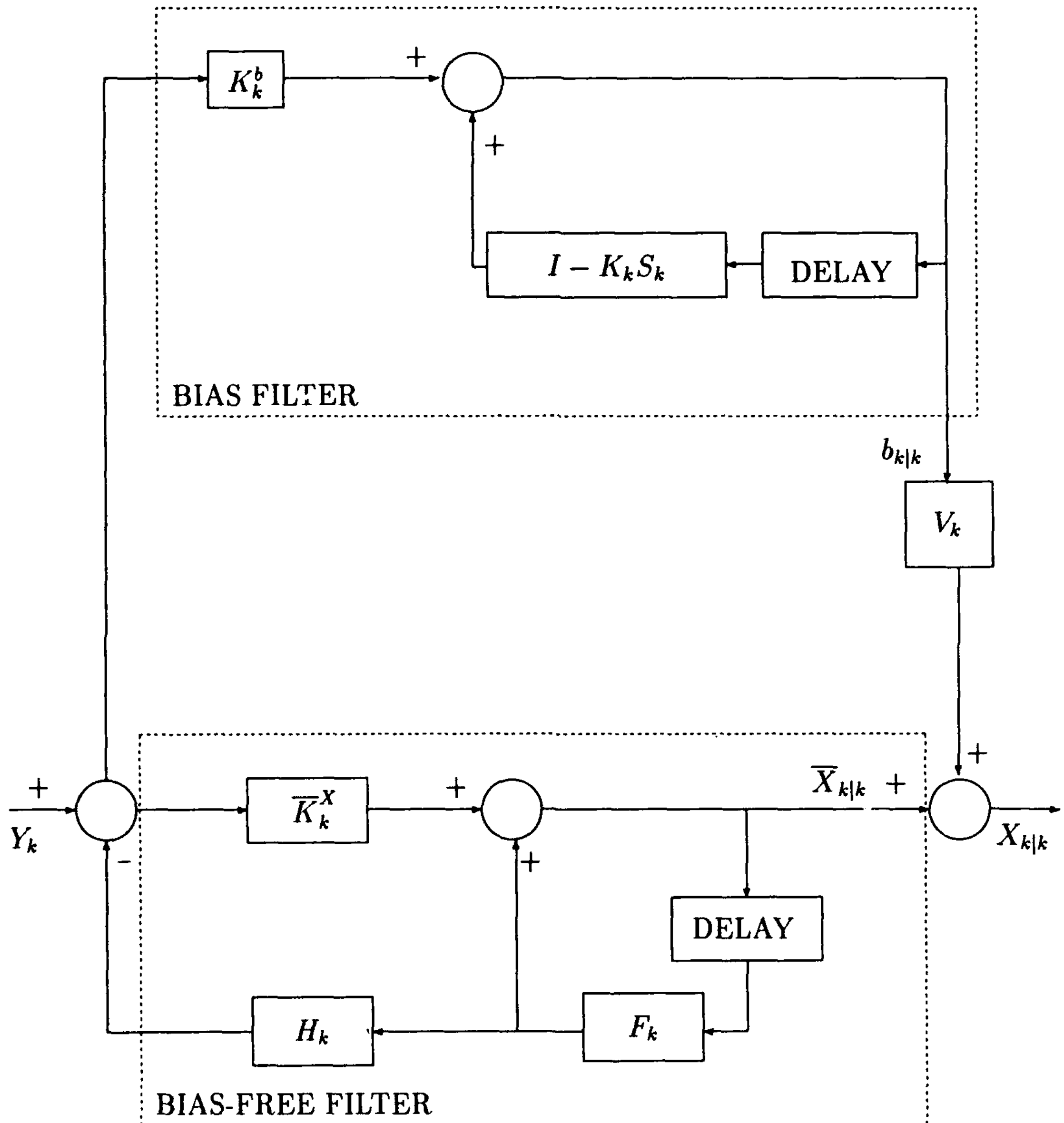

Figure 3-1. Structure of Two-Stage Kalman Estimator

CHAPTER 4

TWO-STAGE ALPHA-BETA-GAMMA ESTIMATOR

For a uniform data rate and stationary noise processes, the single coordinate version of the two-stage Kalman estimator can be approximated with a filter including three fixed gains. In this fixed-gain filter, the two-stage $\alpha, \beta, \bar{\gamma}$ estimator, the position and velocity are tracked in the first stage, which is a standard α, β filter, and the acceleration is tracked separately in the second stage, which is a standard single gain filter. The output of the acceleration filter can be used to adjust the estimates of the α, β filter, as shown in Figure 4–1, when a maneuver is detected. The parameters K_1, K_2, and K_3 will be defined later.

The bias-free filter of Eqs. (3.7)-(3.11) corresponds to the standard constant velocity filter in a single coordinate when

$$F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{4.1}$$

$$\overline{Q}_k^X = \bar{q}_k^x G_k G_k^T \tag{4.2}$$

$$G_k = G_k^X = \begin{bmatrix} \frac{T^2}{2} & T \end{bmatrix}^T \tag{4.3}$$

$$H_k = [1 \quad 0], \qquad C_k = 0 \tag{4.4}$$

$$\overline{K}_k^X = \begin{bmatrix} \alpha_k & \frac{\beta_k}{T} \end{bmatrix}^T \tag{4.5}$$

for scalar acceleration error variance $\bar{q}_k^x$. The bias filter of Eqs. (3.12)-(3.16) corresponds to the standard constant state filter with residuals of the bias-free filter, r_k, as the input and $G_k^b = 1$. Since the bias corresponds to a scalar acceleration, the bias filter gain will be a scalar. Let the bias filter gain be denoted as

$$\overline{K}_k^b = \bar{\gamma}_k S_k^{-1}, \qquad k > 0 \tag{4.6}$$

where $\bar{\gamma}_k$ are S_k are positive scalars obtained from Eqs. (3.15) and (3.17). Letting $b_k = A_k$ and inserting Eq. (4.6) into Eq. (3.13) gives

$$A_{k|k} = A_{k|k-1} + \gamma_k[S_k^{-1} r_k - A_{k|k-1}] \tag{4.7}$$

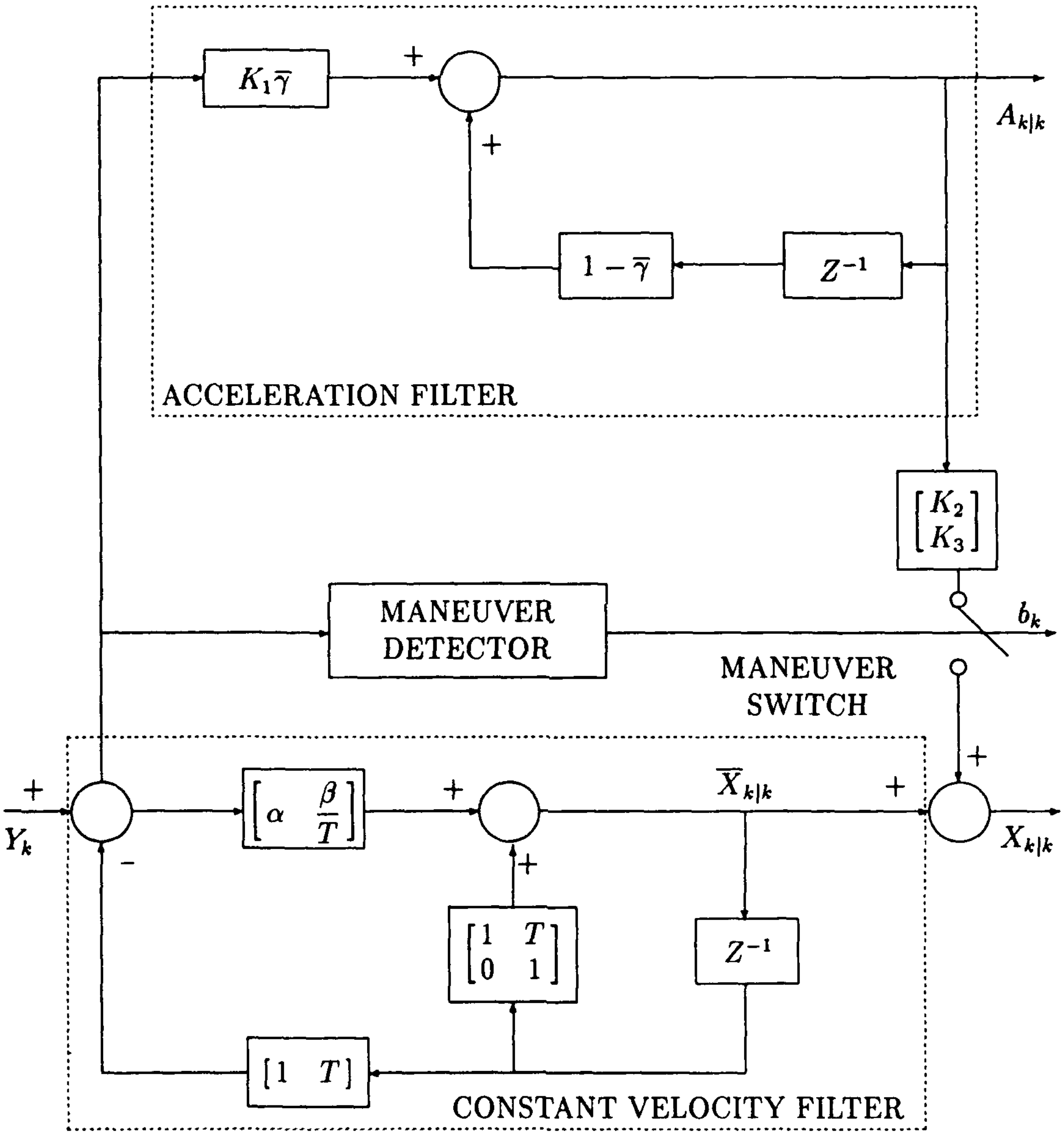

Figure 4-1. Two-Stage $\alpha, \beta, \bar{\gamma}$ Estimator

Using Eqs. (4.1) through (4.7) in the bias-free and bias filters gives the two-stage $\alpha, \beta, \bar{\gamma}$ estimator as summarized by the following equations:

Prediction

$$\overline{X}_{k|k-1} = F_{k-1}\overline{X}_{k-1|k-1} \tag{4.8}$$

$$A_{k|k-1} = A_{k-1|k-1} \tag{4.9}$$

$$U_k = \begin{bmatrix} 1-\alpha_{k-1}-\beta_{k-1} & T \\ -\dfrac{\beta_{k-1}}{T} & 1 \end{bmatrix} U_{k-1} + G_{k-1} \tag{4.10}$$

Measurement Update

$$\overline{X}_{k|k} = \overline{X}_{k|k-1} + \overline{K}_k^X (Y_k - H_k \overline{X}_{k|k-1}) \tag{4.11}$$

$$A_{k|k} = (1-\bar{\gamma}_k) A_{k|k-1} + \bar{\gamma}_k S_k^{-1} r_k \tag{4.12}$$

Output Correction

$$X_{k|k} = \overline{X}_{k|k} + V_k A_{k|k} \tag{4.13}$$

$$V_k = \begin{bmatrix} 1-\alpha_k & 0 \\ -\dfrac{\beta_k}{T} & 1 \end{bmatrix} U_k \tag{4.14}$$

where the gains are a function of time. In steady-state conditions, $\alpha_k = \alpha$, $\beta_k = \beta$, and $\bar{\gamma}_k = \bar{\gamma}$. Also, the bias filter is initialized with $A_{0|-1} = 0$ and $U_{-1} = [0 \quad 0]^T$.

STEADY-STATE RELATIONSHIPS

If the measurement rate is uniform (i.e., T is constant), the measurement error statistics are stationary (i.e., $R_k = \sigma_v^2$), and the acceleration error statistics are stationary (i.e., $\bar{q}_k^x = \bar{\sigma}_w^2$), then the bias-free filter in steady-state conditions corresponds to an α, β filter with $\alpha_k = \alpha$ and $\beta_k = \beta$. Then α and β are given by Eqs. (2.29) and (2.30) for the α, β as

$$\bar{\Gamma} = T^4 \frac{\bar{\sigma}_w^2}{\sigma_v^2} = \frac{\beta^2}{(1-\alpha)} \tag{4.15}$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{4.16}$$

where $\bar{\Gamma}$ denotes the tracking index for the bias-free filter.

Since the statistics of the residuals r_k in the α, β filter will be stationary in steady state, the bias filter will achieve steady-state conditions if the statistics of W_k^b are stationary (i.e., $Q_k^b = \sigma_w^2$). The bias filter in steady-state conditions corresponds to a single gain filter with

$$K_k^b = K^b = \bar{\gamma} S^{-1} \tag{4.17}$$

$Q_k^b = \sigma_w^2$). The bias filter in steady-state conditions corresponds to a single gain filter with

$$K_k^b = K^b = \bar{\gamma} S^{-1} \tag{4.17}$$

where S is a constant scalar in steady-state conditions. Using Eqs. (4.10) and (4.14), the steady-state parameters are

$$U_{k+1} = U_k = U = \begin{bmatrix} \frac{T^2}{\beta} & (\frac{\alpha}{\beta} + 0.5)T \end{bmatrix}^T \tag{4.18}$$

$$V_{k+1} = V_k = V = \begin{bmatrix} (1-\alpha)\frac{T^2}{\beta} & (\frac{\alpha}{\beta} - 0.5)T \end{bmatrix}^T \tag{4.19}$$

The steady state values of K_1, K_2, and K_3 in Figure 4-1 are given by

$$K_1 = S^{-1} = (H_k U)^{-1} = \frac{\beta}{T^2} \tag{4.20}$$

$$\begin{bmatrix} K_2 \\ K_3 \end{bmatrix} = V \tag{4.21}$$

Since the two-stage $\alpha, \beta, \bar{\gamma}$ estimator is an α, β filter when the switch is open, the steady-state error covariance $\overline{P}_{k|k}^X = \overline{P}^X$ is provided by Eq. (2.31) as

$$\overline{P}_{k|k}^X = \overline{P}^X = \sigma_v^2 \begin{bmatrix} \alpha & \frac{\beta}{T} \\ \frac{\beta}{T} & \frac{\beta(2\alpha - \beta)}{2(1-\alpha)T^2} \end{bmatrix} \tag{4.22}$$

The variance of the residuals of the bias-free or α, β filter is given by

$$\sigma_{RES}^2 = H_k F_k \overline{P}^X F_k^T H_k^T + \bar{q}_k^x H_k G_k G_k^T H_k^T + \sigma_v^2 \tag{4.23}$$

Inserting Eqs. (4.1) through (4.4) and (4.22) into Eq. (4.23) gives

$$\sigma_{RES}^2 = \sigma_v^2 [1 + \alpha + \frac{\beta(3-\beta)}{2(1-\alpha)}] + \frac{T^4}{4} \bar{\sigma}_w^2 \tag{4.24}$$

Using Eq. (4.15) to eliminate $\bar{\sigma}_w^2$ from Eq. (4.24) and using $\alpha^2 - 2\beta + \alpha\beta + 0.25\beta = 0$ of [8] gives

$$\sigma_{RES}^2 = \frac{\sigma_v^2}{(1-\alpha)} \tag{4.25}$$

Thus, using Eq. (4.25) in conjunction with Eq. (4.20) gives the variance of the input to the second stage or bias filter as $(K_1)^2 \sigma_{RES}^2$. Using Eqs. (2.8) and (4.15) gives the steady-state gain for the second stage as

$$\bar{\gamma} = P_{k|k}^b \left(\frac{\beta^2}{T^4} \sigma_{RES}^2\right)^{-1} = P^b \left(\frac{\beta^2}{T^4} \sigma_{RES}^2\right)^{-1} = \bar{\gamma} \left(\frac{\beta^2}{T^4 (1-\alpha)}\right)^{-1} \sigma_v^{-2} = P^b \bar{\sigma}_w^{-2} \tag{4.26}$$

Thus, the steady-state error covariance of the second stage is given by

$$P^b = \bar{\gamma}\left(\frac{\beta^2}{T^4}\sigma_{RES}^2\right) = \bar{\gamma}\left(\frac{\beta^2}{T^4(1-\alpha)}\right)\sigma_v^2 = \bar{\gamma}\bar{\sigma}_w^2 \tag{4.27}$$

Inserting Eq. (2.4) into Eq. (2.7) for the bias filter gives

$$P^b = [1-\bar{\gamma}][P^b + \sigma_w^2] \tag{4.28}$$

Thus

$$P^b = \frac{(1-\bar{\gamma})}{\bar{\gamma}}\sigma_w^2 \tag{4.29}$$

Eqs. (4.27) and (4.29) give

$$\frac{\sigma_w^2}{\bar{\sigma}_w^2} = \frac{\bar{\gamma}^2}{(1-\bar{\gamma})} \tag{4.30}$$

Using Eqs. (4.19) and (4.27) in Eq. (3.22) gives

$$P_{k|k}^{Xb} = P^{Xb} = VP^b = \bar{\gamma}\sigma_v^2 \begin{bmatrix} \dfrac{\beta}{T^2} \\ \dfrac{(\alpha-0.5\beta)\beta}{(1-\alpha)T^3} \end{bmatrix} \tag{4.31}$$

Using Eqs. (4.19) and (4.31) gives

$$VP^bV^T = \bar{\gamma}\sigma_v^2 \begin{bmatrix} 1-\alpha & \dfrac{(\alpha-0.5\beta)}{T} \\ \dfrac{(\alpha-0.5\beta)}{T} & \dfrac{(\alpha-0.5\beta)^2}{(1-\alpha)T^2} \end{bmatrix} \tag{4.32}$$

Since the output error covariance of the two-stage estimator is dependent on the maneuver switch, let the switch gain be denoted by

$$G_S = \begin{cases} 0, & \text{switch open} \\ \bar{\gamma}, & \text{switch closed} \end{cases} \tag{4.33}$$

Using Eqs. (4.22), (4.27), (4.31), and (4.32) in Eqs. (3.21) through (3.23) gives the output error covariance of the two-stage estimator as a function of the maneuver switch to be

$$P_{k|k}(1,1) = \sigma_v^2(\alpha + (1-\alpha)G_S) \tag{4.34}$$

$$P_{k|k}(1,2) = \frac{\sigma_v^2}{T}[\beta + (\alpha-0.5\beta)G_S] \tag{4.35}$$

$$P_{k|k}(1,3) = \frac{\sigma_v^2\beta}{T^2}G_S \tag{4.36}$$

$$P_{k|k}(2,2) = \sigma_v^2\frac{(\alpha-0.5\beta)}{(1-\alpha)T^2}[\beta + (\alpha-0.5\beta)G_S] \tag{4.37}$$

$$P_{k|k}(2,3) = \sigma_v^2\frac{\beta(\alpha-0.5\beta)}{(1-\alpha)T^3}G_S \tag{4.38}$$

$$P_{k|k}(3,3) = \sigma_v^2\frac{\beta^2}{(1-\alpha)T^4}G_S \tag{4.39}$$

where $P_{k|k}(i,j)$ denotes the (i,j) element of the error covariance.

SELECTING GAMMA

Since the algebraic constraint of Eq. (3.24) will not be satisfied, the two-stage estimator with the maneuver switch closed will not be identical to the augmented state filter (i.e., the α, β, γ filter). However, the position, velocity, or acceleration variances of the two-stage estimator and the α, β, γ filter can be matched through the selection of γ. Using the (1,1) elements of Eqs. (2.49) and (4.34), a match between the position variances is given by

$$\bar{\gamma} = \frac{\hat{\alpha} - \alpha}{(1 - \alpha)} \tag{4.40}$$

where $\hat{\alpha}$ denotes the alpha gain of a standard α, β, γ filter that is desired for tracking through maneuvers. Using the (2,2) elements of Eqs. (2.49) and (4.37), a match between the velocity variances is given by

$$\bar{\gamma} = \left[\left(\frac{1 - \alpha}{1 - \hat{\alpha}} \right) \left(\frac{\hat{\alpha}\hat{\beta} + 0.25\hat{\gamma}(\hat{\beta} - 2\hat{\alpha} - 4)}{(\alpha - 0.5\beta)^2} \right) - \frac{\beta}{\alpha - 0.5\beta} \right] \tag{4.41}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ denote the gains of a standard α, β, γ filter that is desired for tracking through maneuvers. Using the (3,3) element of Eq. (2.49) and Eq. (4.39) gives a match between the acceleration variances as

$$\bar{\gamma} = \frac{\hat{\gamma}(\hat{\beta} - \hat{\gamma})}{\beta^2} \left[\frac{1 - \alpha}{1 - \hat{\alpha}} \right] \tag{4.42}$$

Fig. 4-2 gives various values of $\bar{\gamma}$ versus $\hat{\alpha}$ for matching the position, velocity, or acceleration variances when $\hat{\alpha} - \alpha = 0.1$. Using Fig. 4-2, consider the two-stage estimator with $\alpha = 0.3$ and $\bar{\gamma} = 0.2$ and the α, β, γ filter with $\hat{\alpha} = 0.4$. Then the position and velocity estimates of the two-stage estimator with the maneuver switch closed will closely match those estimates of the corresponding α, β, γ filter, while the acceleration estimates of the two-stage estimator will be filtered more heavily than those of the α, β, γ filter. Fig. 4-2 also shows that the estimates of two-stage estimator for $\hat{\alpha} - \alpha = 0.1$ will more closely match those of the augmented state filter, the α, β, γ filter, when $0.4 \leq \alpha \leq 0.5$ and $0.25 \leq \bar{\gamma} \leq 0.5$.

MEASUREMENT VARIANCE REDUCTION MATRIX

The variance reductions between position, velocity, and acceleration estimates and the measurements are often considered in the design and analysis of fixed-gain filters. The

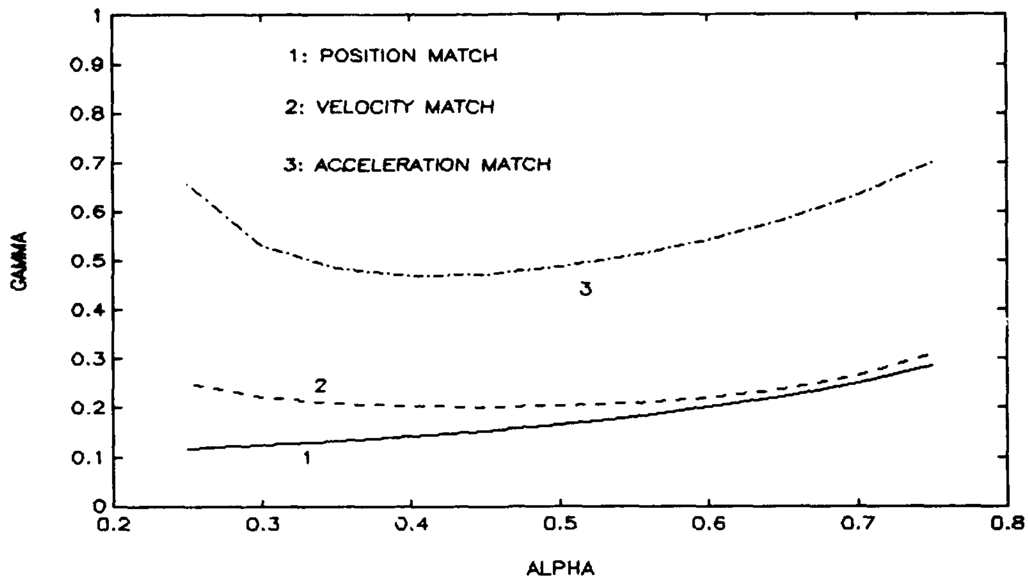

Figure 4–2. $\bar{\gamma}$ Versus $\hat{\alpha}$ for $\hat{\alpha} - \alpha = 0.1$

variance reduction matrix for the two-stage $\alpha, \beta, \bar{\gamma}$ estimator is derived by considering the estimator with the maneuver switch open as a linear system with a white noise input. The variance reduction matrix for the estimator with the maneuver switch open is used to compute the variance reduction matrix for the estimator with the maneuver switch is closed. The input-output relationships between the measurements Y_k and $\begin{bmatrix} \overline{X}_{k|k}^T & A_{k|k} \end{bmatrix}^T$ when the maneuver switch is open can be expressed as a linear system that is given by

$$\begin{bmatrix} \overline{X}_{k|k} \\ A_{k|k} \end{bmatrix} = \overline{F} \begin{bmatrix} \overline{X}_{k-1|k-1} \\ A_{k-1|k-1} \end{bmatrix} + \overline{G} Y_k \tag{4.43}$$

where

$$\overline{F} = \begin{bmatrix} 1-\alpha & (1-\alpha)T & 0 \\ -\dfrac{\beta}{T} & (1-\beta) & 0 \\ -\dfrac{\bar{\gamma}\beta}{T^2} & -\dfrac{\bar{\gamma}\beta}{T} & 1-\bar{\gamma} \end{bmatrix} \tag{4.44}$$

$$\overline{G} = \begin{bmatrix} \alpha \\ \dfrac{\beta}{T} \\ \dfrac{\bar{\gamma}\beta}{T^2} \end{bmatrix} \tag{4.45}$$

Using Eq. (4.43) provides the error covariance of $\begin{bmatrix} \overline{X}_{k|k}^T & A_{k|k} \end{bmatrix}^T$ that results from white noise measurement errors. That error covariance is denoted as

$$E\left[\begin{bmatrix}\overline{X}_k \\ A_k\end{bmatrix}\begin{bmatrix}\overline{X}_k^T & A_k\end{bmatrix}\right] = S^O_{\alpha\beta\bar{\gamma}} \tag{4.46}$$

and given by

$$S^O_{\alpha\beta\bar{\gamma}} = \overline{F}\ \overline{S}^O_{\alpha\beta\bar{\gamma}}\overline{F}^T + \overline{G}\ \overline{G}^T\sigma_v^2 \tag{4.47}$$

where σ_v^2 is the variance of the input Y_k. Eq. (4.47) can be used to solve for elements of $S^O_{\alpha\beta\bar{\gamma}}$ in terms of the filter gains, measurement period, and the measurement error variance. Let

$$S^O_{\alpha\beta\bar{\gamma}} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33}\end{bmatrix} \tag{4.48}$$

Since the position and velocity estimates of the two-stage estimator with the maneuver switch open are equivalent to those of the α, β filter, s_{11}, s_{12}, s_{22} are given by those of the α, β filter in Eqs. (A.43) through (A.45) as

$$s_{11} = \frac{2\alpha^2 + \beta(2 - 3\alpha}{\alpha[4 - 2\alpha - \beta]}\sigma_v^2 \tag{4.49}$$

$$s_{12} = \frac{(2\alpha - \beta)\beta}{\alpha[4 - 2\alpha - \beta]T}\sigma_v^2 \tag{4.50}$$

$$s_{22} = \frac{2\beta^2}{\alpha[4 - 2\alpha - \beta]T^2}\sigma_v^2 \tag{4.51}$$

Equating the (1,3), (2,3), and (3,3) elements of Eq. (4.47) gives

$$s_{13} = \frac{(\alpha - 1)\bar{\gamma}\beta}{T^2}[s_{11} + 2Ts_{12} + T^2 s_{22}] + (\bar{\gamma} - 1)(\alpha - 1)[s_{13} + Ts_{23}] + \frac{\alpha\bar{\gamma}\beta}{T^2}\sigma_v^2 \tag{4.52}$$

$$s_{23} = \frac{\beta^2}{T^3}[s_{11} + 2Ts_{12} + T^2 s_{22}] - \frac{\beta}{T^2}[s_{12} + Ts_{22}] + \frac{\beta(\bar{\gamma} - 1)}{\bar{\gamma}T}[s_{13} + Ts_{23}] + \frac{\bar{\gamma}\beta^2}{T^3}\sigma_v^2 \tag{4.53}$$

$$s_{33}(2 - \bar{\gamma})\bar{\gamma} = \frac{\bar{\gamma}^2\beta^2}{T^4}[s_{11} + 2Ts_{12} + T^2 s_{22}] + \frac{2\bar{\gamma}\beta(\bar{\gamma} - 1)}{T^2}[s_{13} + Ts_{23}] + \frac{\bar{\gamma}^2\beta^2}{T^4}\sigma_v^2 \tag{4.54}$$

Using Eqs. (4.49) through (4.51) gives

$$s_{11} + 2Ts_{12} + T^2 s_{22} = \sigma_v^2\left[\frac{2\alpha^2 + \alpha\beta + 2\beta}{\alpha[4 - 2\alpha - \beta]}\right] \tag{4.55}$$

$$s_{12} + Ts_{22} = \sigma_v^2\left[\frac{\beta(\beta + 2\alpha)}{\alpha[4 - 2\alpha - \beta]T}\right] \tag{4.56}$$

Adding Eqs. (4.51) and (4.52) and using Eqs. (4.55) and (4.56) gives

$$\begin{aligned} s_{13} + Ts_{23} &= \frac{\beta[\beta + \bar{\gamma}(\alpha - 1)]}{T^2}[s_{11} + 2Ts_{12} + T^2 s_{22}] - \frac{\beta}{T}[s_{12} + Ts_{22}] \\ &\quad + \frac{(\bar{\gamma} - 1)[\bar{\gamma}(\alpha - 1) + \beta]}{\bar{\gamma}}[s_{13} + Ts_{23}] + \frac{\beta(\beta + \alpha\bar{\gamma})}{T^2}\sigma_v^2 \end{aligned} \tag{4.57}$$

$$= \sigma_v^2\frac{\beta\bar{\gamma}}{T^2}\left[\frac{2\bar{\gamma}\alpha^2 + (\bar{\gamma} + 2)\alpha\beta + \beta(\beta - 2\bar{\gamma})}{\alpha[(1 - \alpha)\bar{\gamma}^2 + (\alpha - \beta)\bar{\gamma} + \beta][4 - 2\alpha - \beta]}\right] \tag{4.58}$$

Inserting Eqs. (4.55) and (4.58) into Eq. (4.52) gives

$$s_{13} = \sigma_v^2 \frac{\beta\bar{\gamma}}{T^2}\left[\frac{2\bar{\gamma}\alpha^2 + (2-\bar{\gamma})\alpha\beta + [(\beta-2)\bar{\gamma}-\beta]\beta}{\alpha[(1-\alpha)\bar{\gamma}^2 + (\alpha-\beta)\bar{\gamma}+\beta][4-2\alpha-\beta]}\right] \tag{4.59}$$

Inserting Eqs. (4.55) and (4.58) into Eq. (4.53) gives

$$s_{23} = \sigma_v^2 \frac{\beta^2\bar{\gamma}}{T^3}\left[\frac{\bar{\gamma}(2\alpha-\beta)+2\beta}{\alpha[(1-\alpha)\bar{\gamma}^2 + (\alpha-\beta)\bar{\gamma}+\beta][4-2\alpha-\beta]}\right] \tag{4.60}$$

Inserting Eqs. (4.55) and (4.58) into Eq. (4.54) gives

$$s_{33} = \sigma_v^2 \frac{2\beta^2\bar{\gamma}^2}{(2-\bar{\gamma})T^4}\left[\frac{\bar{\gamma}(2\alpha-\beta)+2\beta}{\alpha[(1-\alpha)\bar{\gamma}^2 + (\alpha-\beta)\bar{\gamma}+\beta][4-2\alpha-\beta]}\right] \tag{4.61}$$

Inserting Eqs. (4.49) through (4.51) and Eqs. (4.59) through (4.61) into Eq. (4.48) provides the steady-state error covariance of the filter output that results from the measurement errors when the maneuver switch is open. The elements of $S^O_{\alpha\beta\bar{\gamma}}$ are summarized as

$$S^O_{\alpha\beta\bar{\gamma}}(1,1) = \frac{\sigma_v^2}{\alpha d_1}[2\alpha^2 + \beta(2-3\alpha)] \tag{4.62}$$

$$S^O_{\alpha\beta\bar{\gamma}}(1,2) = \frac{\sigma_v^2\beta}{\alpha d_1 T}(2\alpha-\beta) \tag{4.63}$$

$$S^O_{\alpha\beta\bar{\gamma}}(1,3) = \frac{\sigma_v^2\beta^2\bar{\gamma}}{\alpha d_1 d_3 T^2}\left[\frac{2\bar{\gamma}\alpha^2}{\beta} + \alpha(2-\bar{\gamma}) + \bar{\gamma}(\beta-2) - \beta\right] \tag{4.64}$$

$$S^O_{\alpha\beta\bar{\gamma}}(2,2) = \frac{\sigma_v^2\, 2\beta^2}{\alpha d_1 T^2} \tag{4.65}$$

$$S^O_{\alpha\beta\bar{\gamma}}(2,3) = \frac{\sigma_v^2\bar{\gamma}\beta^2}{\alpha d_1 d_3 T^3}(\bar{\gamma}(2\alpha-\beta)+2\beta) \tag{4.66}$$

$$S^O_{\alpha\beta\bar{\gamma}}(3,3) = \frac{\sigma_v^2 2\bar{\gamma}^2\beta^2}{\alpha(2-\bar{\gamma})d_1 d_3 T^4}(\bar{\gamma}(2\alpha-\beta)+2\beta) \tag{4.67}$$

where $d_1 = 4-2\alpha-\beta$ and $d_3 = (1-\alpha)\bar{\gamma}^2 + (\alpha-\beta)\bar{\gamma}+\beta$.

The input-output relationships between the measurements Y_k and filter output when the maneuver switch is closed can be expressed as

$$\begin{bmatrix} X_{k|k} \\ A_{k|k} \end{bmatrix} = D \begin{bmatrix} \overline{X}_{k|k} \\ A_{k|k} \end{bmatrix} \tag{4.68}$$

where

$$D = \begin{bmatrix} 1 & 0 & (1-\alpha)\frac{T^2}{\beta} \\ 0 & 1 & (\frac{\alpha}{\beta}-0.5)T \\ 0 & 0 & 1 \end{bmatrix} \tag{4.69}$$

Using Eq. (4.68) provides the steady-state error covariance that results from the measurement errors when the maneuver switch is closed. That error covariance is given by

$$S^C_{\alpha\beta\bar{\gamma}} = D S^O_{\alpha\beta\bar{\gamma}} D^T \tag{4.70}$$

INITIALIZATION

A gain scheduling procedure is needed to facilitate the settling of the estimates of the two-stage $\alpha, \beta, \bar{\gamma}$ estimator. A gain scheduling procedure that produces estimates that are equivalent to those of a quadratic least-squares fit is developed. The estimates will be shown to be equivalent to those produced by an α, β, γ filter that is initialized with gains scheduled with Eqs. (2.52) through (2.54).

Since the bias-free filter is an α, β filter, Eqs. (2.37) and (2.38) will be used to schedule α_k and β_k as

$$\alpha_k = \max\{\frac{2(2k+1)}{(k+1)(k+2)}, \alpha\} \tag{4.71}$$

$$\beta_k = \max\{\frac{6}{(k+1)(k+2)}, \beta\} \tag{4.72}$$

with $X_{0|-1} = [0 \quad 0]^T$. Inserting Eqs. (4.71) and (4.72) into Eq. (4.10) gives

$$U_k = \begin{bmatrix} \frac{k-4}{k} & T \\ -\frac{6}{k(k+1)T} & 1 \end{bmatrix} U_{k-1} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} \tag{4.73}$$

A solution to the time-varying difference equation in Eq. (4.73) is

$$U_k = \begin{bmatrix} \frac{(k+1)(k+2)T^2}{12} \\ \frac{(k+1)T}{2} \end{bmatrix} \tag{4.74}$$

Inserting U_{k-1} from Eq. (4.74) into Eq. (4.73) confirms that Eq. (4.74) is a valid solution of the difference equation. Using Eqs. (4.71), (4.72), and (4.74) in Eq. (4.14) gives

$$V_k = \begin{bmatrix} k(k-1) & 0 \\ -\frac{6}{(k+1)(k+2)T} & 1 \end{bmatrix} U_k = \begin{bmatrix} \frac{k(k-1)T^2}{12} \\ \frac{kT}{2} \end{bmatrix} \tag{4.75}$$

Using Eq. (4.75) with Eq. (4.13) gives

$$X_{k|k} = \begin{bmatrix} x_{k|k} \\ \dot{x}_{k|k} \end{bmatrix} = \overline{X}_{k|k} + V_k A_{k|k} = \begin{bmatrix} \bar{x}_{k|k} + \dfrac{k(k-1)T^2}{12} A_{k|k} \\ \bar{v}_{k|k} + \dfrac{kT}{2} A_{k|k} \end{bmatrix} \tag{4.76}$$

Delaying the time index in Eq. (4.76) gives

$$X_{k-1|k-1} = \begin{bmatrix} \bar{x}_{k-1|k-1} + \dfrac{(k-1)(k-2)T^2}{12} A_{k-1|k-1} \\ \bar{v}_{k-1|k-1} + \dfrac{(k-1)T}{2} A_{k-1|k-1} \end{bmatrix} \tag{4.77}$$

Also,

$$x_{k-1|k-1} + T\dot{x}_{k-1|k-1} + \frac{T^2}{2} A_{k-1|k-1} = \bar{x}_{k-1|k-1} + T\bar{v}_{k-1|k-1} + \frac{(k+2)(k+3)T^2}{12} A_{k-1|k-1} \tag{4.78}$$

Using Eqs. (4.74) and (4.78) in Eq. (4.12) gives

$$\begin{aligned} A_{k|k} &= (1-\bar{\gamma}_k)A_{k-1|k-1} + \bar{\gamma}_k (H_k U_k)^{-1}[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1}] \\ &= A_{k-1|k-1} + \bar{\gamma}_k \Big(\frac{12}{(k+1)(k+2)T^2}\Big) \times \\ &\quad \Big[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1} - \frac{(k+1)(k+2)T^2}{12} A_{k-1|k-1}\Big] \\ &= A_{k-1|k-1} + \Big(\frac{12\bar{\gamma}_k}{(k+1)(k+2)T^2}\Big)\Big[Y_k - x_{k-1|k-1} - T\dot{x}_{k-1|k-1} - \frac{T^2}{2} A_{k-1|k-1}\Big] \end{aligned} \tag{4.79}$$

which is in the form of the acceleration update of the α, β, γ filter. Setting the gain equal to the acceleration gain of the α, β, γ filter in Eq. (2.54) gives

$$\frac{12\bar{\gamma}_k}{(k+1)(k+2)} = \frac{60}{(k+1)(k+2)(k+3)} \tag{4.80}$$

Thus,

$$\bar{\gamma}_k = \frac{5}{(k+3)} \tag{4.81}$$

Inserting Eq. (4.81) into Eq. (4.79) gives

$$A_{k|k} = A_{k-1|k-1} + \Big(\frac{60}{(k+1)(k+2)(k+3)T^2}\Big)\Big[Y_k - x_{k-1|k-1} - T\dot{x}_{k-1|k-1} - \frac{T^2}{2} A_{k-1|k-1}\Big] \tag{4.82}$$

which is the acceleration update equation for the α, β, γ filter when using Eq. (2.54) for initialization.

Using Eqs. (4.11), (4.12), and (4.74) in Eq. (4.77) gives

$$\begin{aligned}
x_{k|k} &= \bar{v}_{k|k} + \frac{kT}{2}A_{k|k} \\
&= \bar{v}_{k-1|k-1} + \frac{\beta_k}{T}[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1}] \\
&\quad + \frac{kT}{2}(1-\bar{\gamma}_k)A_{k-1|k-1} + \frac{kT}{2}\bar{\gamma}_k(H_kU_k)^{-1}[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1}] \\
&= \bar{v}_{k-1|k-1} + \frac{(k-1)T}{2}A_{k-1|k-1} + TA_{k-1|k-1} - \frac{3(2k+1)T}{2(k+3)}A_{k-1|k-1} \\
&\quad + \left(\frac{6}{(k+1)(k+2)T} + \frac{30k}{(k+1)(k+2)(k+3)T}\right)\left[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1}\right] \\
&= \dot{x}_{k-1|k-1} + TA_{k-1|k-1} + \left(\frac{18(2k+1)}{(k+1)(k+2)(k+3)T}\right)\times \\
&\quad \left[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1} - \frac{(k+1)(k+2)T^2}{12}A_{k-1|k-1}\right] \\
&= \dot{x}_{k-1|k-1} + TA_{k-1|k-1} + \left(\frac{18(2k+1)}{(k+1)(k+2)(k+3)T}\right)\times \\
&\quad \left[Y_k - x_{k-1|k-1} - T\dot{x}_{k-1|k-1} - \frac{T^2}{2}A_{k-1|k-1}\right]
\end{aligned} \tag{4.83}$$

which is in the form of the velocity update of the α, β, γ filter when using Eq. (2.53) for initialization.

Using Eqs. (4.11), (4.12) and (4.74) in Eq. (4.77) gives

$$\begin{aligned}
x_{k|k} &= \bar{x}_{k|k} + \frac{k(k-1)T^2}{12}A_{k|k} \\
&= \bar{x}_{k-1|k-1} + T\bar{v}_{k-1|k-1} + \frac{k(k-1)T^2}{12}(1-\bar{\gamma}_k)A_{k-1|k-1} \\
&\quad + \left(\alpha_k + \frac{k(k-1)T^2}{12}\bar{\gamma}_k(H_kU_k)^{-1}\right)\left[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1}\right] \\
&= \bar{x}_{k-1|k-1} + T\bar{v}_{k-1|k-1} + \frac{(k+1)(k+2)T^2}{12}A_{k-1|k-1} - \frac{3(3k^2+3k+2)T^2}{12(k+3)}A_{k-1|k-1} \\
&\quad + \left(\frac{3(3k^2+3k+2)}{(k+1)(k+2)(k+3)T}\right)\left[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1}\right] \\
&= x_{k|k} + \dot{x}_{k-1|k-1} + \frac{T^2}{2}A_{k-1|k-1} + \left(\frac{3(3k^2+3k+2)}{(k+1)(k+2)(k+3)T}\right)\times \\
&\quad \left[Y_k - \bar{x}_{k-1|k-1} - T\bar{v}_{k-1|k-1} - \frac{(k+1)(k+2)T^2}{12}A_{k-1|k-1}\right] \\
&= x_{k|k} + \dot{x}_{k-1|k-1} + \frac{T^2}{2}A_{k-1|k-1} + \left(\frac{3(3k^2+3k+2)}{(k+1)(k+2)(k+3)T}\right)\times \\
&\quad \left[Y_k - x_{k-1|k-1} - T\dot{x}_{k-1|k-1} - \frac{T^2}{2}A_{k-1|k-1}\right]
\end{aligned} \tag{4.84}$$

which is in the form of the position update of the α, β, γ filter when using Eq. (2.52) for initialization.

Using the definitions of the parameters K_1, K_2, and K_3 in Eqs. (4.20) and (4.21) and Eqs. (4.71), (4.72), (4.74), (4.75), and (4.81), the gain scheduling is summarized as

$$\alpha_k = \max\{\frac{2(2k+1)}{(k+1)(k+2)}, \alpha\} \tag{4.85}$$

$$\beta_k = \max\{\frac{6}{(k+1)(k+2)}, \beta\} \tag{4.86}$$

$$\bar{\gamma}_k = \max\{\frac{5}{(k+3)}, \bar{\gamma}\} \tag{4.87}$$

$$K_1 = \frac{1}{T^2} \max\{\frac{12}{(k+1)(k+2)}, \beta\} \tag{4.88}$$

$$K_2 = T^2 \min\{\frac{k(k-1)}{12}, \frac{(1-\alpha)}{\beta}\} \tag{4.89}$$

$$K_3 = T \min\{\frac{k}{2}, \frac{\alpha}{\beta} - 0.5\} \tag{4.90}$$

with $X_{0|-1} = [0 \quad 0 \quad 0]^T$.

Figure 4-3 gives a comparison of the Root-Mean Square Errors (RMSE) during intialization of the two-stage $\alpha, \beta, \bar{\gamma}$ estimator with Eqs. (4.85) through (4.90), and the α, β, γ filter with Eqs. (2.52) through (2.54). For the α, β, γ filter, the steady-state gains were chosen with $\alpha = 0.45$. For the two-stage estimator, the steady-state gains were chosen with $\alpha = 0.35$ and $\bar{\gamma} = 0.15$ to match the position variances. The errors are the average over 50 experiments. The target trajectory began at 12 km with an initial velocity of -100 m/s and the acceleration remained constant at 10 m/s/s. The measurement period was 0.25 sec and $\sigma_v = 8$ m. The RMSE of the two filters a[illegible] losely matched in the transient region from 0 to 4 sec, while the RMSE in position are clos[illegible]y matched in steady-state.

When Eqs. (4.85) through (4.90) are used to initialize the two-stage estimator, the maneuver switch can be left open until enough measurements are received so that the acceleration estimate is settled before it is used.

EXAMPLE

A simple tracking system is considered to illustrate the operation of the two-stage estimator. The tracking system measures a target's position at 4 Hz with errors that have a standard deviation of 8 m. The target is expected to perform maneuvers with up to 2 g of

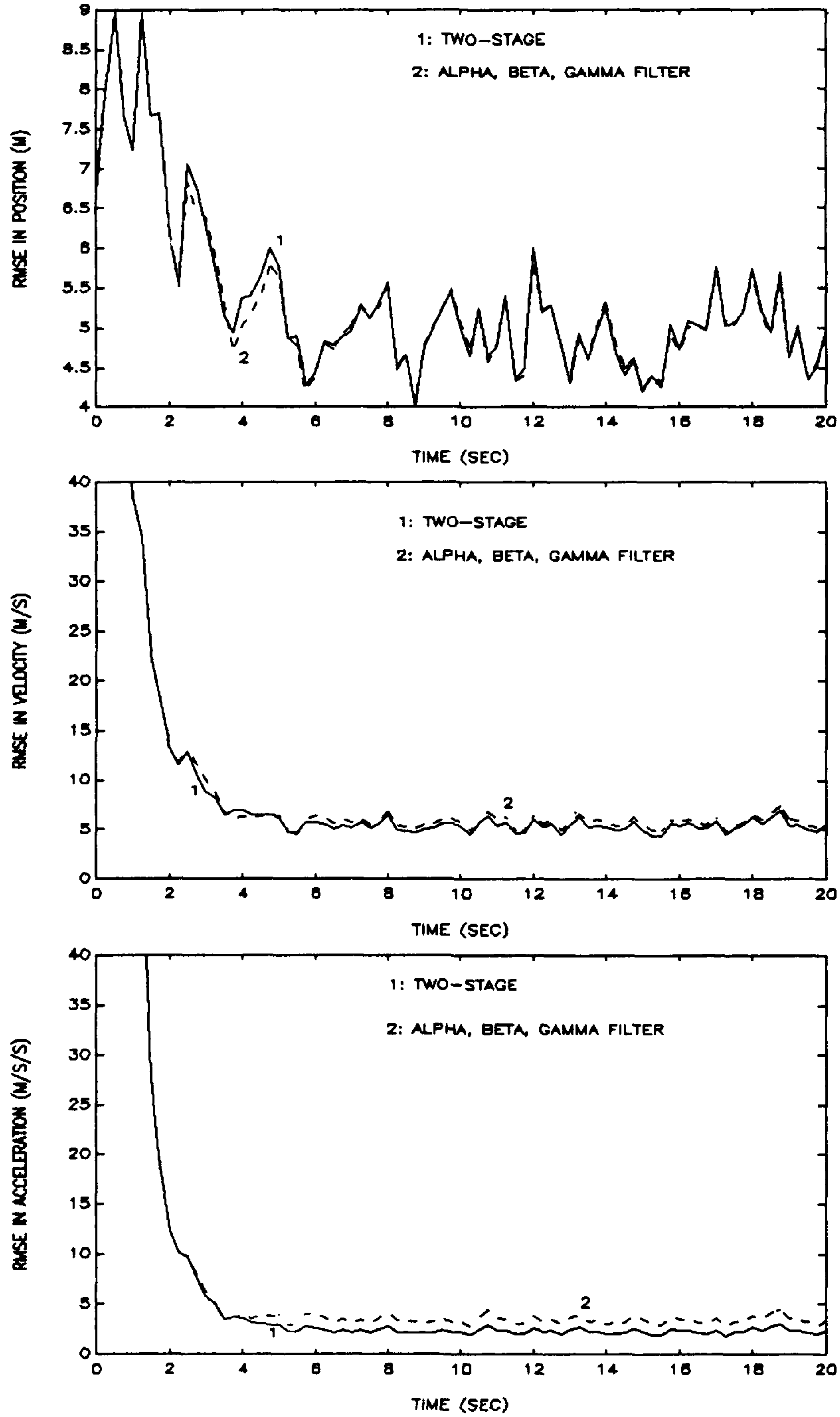

Figure 4-3. RMSE During Initialization

acceleration. For an α, β, γ filter, $\alpha = 0.45$, $\beta = 0.13$ and $\gamma = 0.02$. For the two-stage estimator, $\alpha = 0.35$, $\beta = 0.075$ and $\bar{\gamma} = 0.15$, where Eq. (4.40) was used to compute $\bar{\gamma}$. While various techniques can be used for maneuver detection, a soft switch approach was chosen. For the soft switch approach, the bias filter is duplicated in the maneuver detector with b_k as the filter output. Then the gain of the maneuver switch in Eq. (4.33) is augmented by

$$G_{SS} = \begin{cases} 0, & |b_k| < 4.0 \\ 1.0, & |b_k| > 6.0 \\ 0.5(|b_k| - 4.0), & \text{otherwise} \end{cases} \tag{4.91}$$

Also, to improve maneuver response, the magnitude of b_k is restricted so that $|b_k| \leq 6.2$. Using the soft switch approach, the correction of the output of the bias-free filter with the acceleration estimates is multiplied by G_{SS}. Simulated tracking results from a Monte Carlo simulation with 200 experiments are given in Figure 4-4 for the two filters, where the target moved with constant velocity except for a 2 g maneuver from 15 to 30 sec. A critically damped, second order system with an approximate time constant of 1 sec was used to model the dynamics of the target. As indicated in Figure 4-4, the two-stage estimator provides significantly better tracking when the target moves with constant velocity, while matching the tracking performance of the α, β, γ filter during the maneuver.

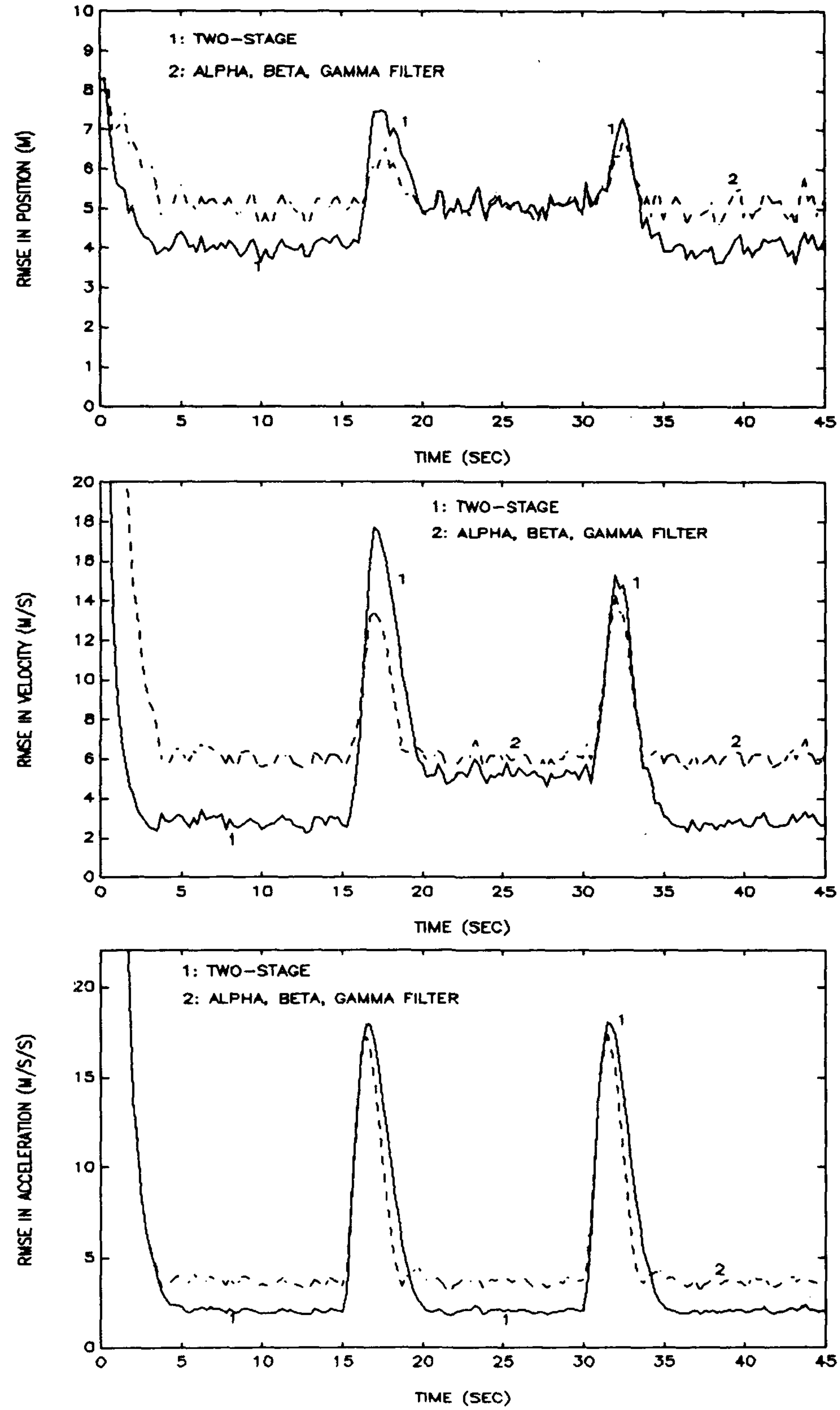

Figure 4-4. RMSE in the Estimates

CHAPTER 5

TWO-STAGE ALPHA-BETA-GAMMA-LAMBDA ESTIMATOR

In the two-stage $\alpha, \beta, \bar{\gamma}$ estimator, the position and velocity are tracked in the first stage, which is a standard α, β filter, and the acceleration is tracked separately in the second stage, which is a standard single gain filter. However, when tracking two or three dimensional target motion with two-stage $\alpha, \beta, \bar{\gamma}$ estimators operating separately in each coordinate, the state estimates will be biased for targets maneuvering with constant speed because the accelerations are time-varying. This bias in the state estimates can be reduced through the use of the kinematic constraint for constant speed targets as in [9]. The speed of a target is given by

$$S = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} \tag{5.1}$$

For a target moving at constant speed,

$$\frac{dS}{dt} = 0 \tag{5.2}$$

or

$$(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = 0 \tag{5.3}$$

This kinematic constraint can be incorporated in the system state of Eq. (2.1) or used as a pseudomeasurement in conjunction with the measurement equation. While both approaches are feasible, the second approach is more attractive because the first changes the state equation from linear to nonlinear. In the implementation of the extended Kalman filter, including nonlinearities in the measurement equation is computationally less expensive than in the state equation [9]. Thus, the kinematic constraint of Eq. (5.3) is incorporated into the filter through a pseudomeasurement as discussed in [9]. The pseudomeasurement equation is given by

$$\frac{V_{k|k}}{S_{k|k}} \cdot A_k + \mu_k = 0 \tag{5.4}$$

where

$$V_{k|k} = \begin{bmatrix} \dot{x}_{k|k} & \dot{y}_{k|k} & \dot{z}_{k|k} \end{bmatrix} \tag{5.5}$$

$$S_{k|k} = (\dot{x}^2_{k|k} + \dot{y}^2_{k|k} + \dot{z}^2_{k|k})^{\frac{1}{2}} \tag{5.6}$$

and $\mu_k \sim N(0, R^\mu_k)$. The $V_{k|k}$ and $S_{k|k}$ are the filtered velocity and speed, respectively, for time k given measurements through time k. The μ_k is a white Gaussian error process that accounts for the uncertainty in both $V_{k|k}$ and the constraint. Since the initial estimate of $V_{k|k}$ may include a significant error, R^μ_k is initialized with a large value and allowed to decrease as

$$R^\mu_k = r_1(\delta)^k + r_0 \tag{5.7}$$

where $0 < \delta < 1$, r_1 is a constant chosen large for initialization, and r_0 is a constant chosen for steady-state conditions.

The two-stage $\alpha, \beta, \bar{\gamma}$ estimator utilizing this constraint in the second stage will be called the two-stage $\alpha, \beta, \bar{\gamma}, \lambda$ estimator. Eq. (5.4) can be written as

$$\frac{-1}{S_{k|k}}(\dot{y}_{k|k}\ddot{y}_k + \dot{z}_{k|k}\ddot{z}_k) = \frac{\dot{x}_{k|k}}{S_{k|k}}\ddot{x}_k + \mu_k \tag{5.8}$$

Thus, the filtered acceleration estimates can be inserted into the left side of Eq. (5.8) to compute a measurement for updating the x-coordinate acceleration. For the two-stage approach, the measurement equation for the x-coordinate is given by Eq. (3.6) with $b_k = A_k$ as

$$\begin{bmatrix} Y_k \\ \frac{-1}{S_{k|k}}(\dot{y}_{k|k}\ddot{y}_{k|k} + \dot{z}_{k|k}\ddot{z}_{k|k}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X_k + \begin{bmatrix} 0 \\ C^x_k \end{bmatrix} A_k + \begin{bmatrix} v_k \\ \mu_k \end{bmatrix} \tag{5.9}$$

$$C^x_k = \frac{\dot{x}_{k|k}}{S_{k|k}} \tag{5.10}$$

where Y_k is the x-coordinate position measurement, $v_k \sim N(0, R_k)$ is the measurement error, $\mu_k \sim N(0, R^\mu_k)$ is the constraint error, and A_k is the x-coordinate acceleration.

Since the bias-free state is not observed through the pseudomeasurement, its gain in the bias-free filter can be shown to be zero. Thus, the bias-free filter can be defined as if the kinematic constraint is not being used. Thus, as in Chapter 4, the bias-free filter of Eqs. (3.7)–(3.11) corresponds to the standard constant velocity filter when defined as in Eqs. (4.1)–(4.5). Using Eqs. (3.17)–(3.19) in conjunction with Eq. (5.2), U_k, V_k, and the first element of S_k can be shown to be independent of the constraint and equivalent for both the two-stage $\alpha, \beta, \bar{\gamma}$ estimator and the two-stage $\alpha, \beta, \bar{\gamma}, \lambda$ estimator. The second element of S_k is given by $S^2_k = C^x_k$. Also, since the errors on the measurement and the constraint are assumed to be independent, the measurement update and the constraint update of the bias filter will be performed separately with the measurement-updated acceleration estimates being used to compensate the velocity estimates in the pseudomeasurement. Thus, the bias filter of Eqs. (3.12)–(3.16) corresponds to the standard constant state filter with the residuals of the bias-free filter r_k and the pseudomeasurement as the inputs. Since the bias corresponds to a scalar acceleration, the bias filter gain includes a scalar gain for the bias-free filter residual and a scalar gain for the constraint. Let the bias filter gain for the bias-free filter residuals be represented by

$$K_k^b = \bar{\gamma}_k (S_k^1)^{-1}, \qquad k > 0 \tag{5.11}$$

where $\bar{\gamma}_k$ and S_k^1 are positive scalars obtained from Eqs. (3.15) and (3.17) with S_k^1 being the first element of S_k. Let the gain for the constraint be represented by

$$K_k^c = \lambda_k C_k^x, \qquad k > 0 \tag{5.12}$$

where λ_k is a positive scalar. Letting $A_k = b_k$, the two-stage $\alpha, \beta, \bar{\gamma}, \lambda$ estimator is given in the following equations:

Prediction

$$\overline{X}_{k|k-1} = F_{k-1}\overline{X}_{k-1|k-1} \tag{5.13}$$

$$A_{k|k-1} = A_{k-1|k-1}^c \tag{5.14}$$

$$U_k = \begin{bmatrix} 1 - \alpha_{k-1} - \beta_{k-1} & T \\ -\dfrac{\beta_{k-1}}{T} & 1 \end{bmatrix} U_{k-1} + G_{k-1} \tag{5.15}$$

Measurement Update

$$\overline{X}_{k|k} = \overline{X}_{k|k-1} + \overline{K}_k^X (Y_k - H_k \overline{X}_{k|k-1}) \tag{5.16}$$

$$A_{k|k} = (1 - \bar{\gamma}_k) A_{k|k-1} + \bar{\gamma}_k (S_k^1)^{-1} (r_k) \tag{5.17}$$

Constraint Update

$$A_{k|k}^c = A_{k|k} - \lambda_k \frac{C_k^x}{S_{k|k}} (\dot{x}_{k|k}\ddot{x}_{k|k} + \dot{y}_{k|k}\ddot{y}_{k|k} + \dot{z}_{k|k}\ddot{z}_{k|k}) \tag{5.18}$$

Output Correction

$$X_{k|k} = \overline{X}_{k|k} + V_k A_{k|k}^c \tag{5.19}$$

$$V_k = \begin{bmatrix} 1 - \alpha_k & 0 \\ -\dfrac{\beta_k}{T} & 1 \end{bmatrix} U_k \tag{5.20}$$

The velocities of Eq. (5.18) are temporarily compensated with the acceleration estimates as in Eq. (5.11) before updating the acceleration with the kinematic constraint. The bias filter is initialized with $A_{0|-1} = 0$ and $U_{-1} = [0 \quad 0]^T$.

If the measurement rate is uniform (i.e., T is constant) and the statistics of the measurement and acceleration errors are stationary (i.e., $R_k = \sigma_v^2$ and $\bar{q}_k^x = \bar{\sigma}_w^2$), then the bias-free filter in steady-state conditions corresponds to an α, β filter with $\alpha_k = \alpha$ and $\beta_k = \beta$. The steady-state values of U_k and V_k are given by Eqs. (4.18) and (4.19), respectively. Using Eqs. (3.17) and (5.9) gives

$$S_k = \begin{bmatrix} \dfrac{T^2}{\beta} & C_k^x \end{bmatrix}^T \tag{5.21}$$

where the first element of S_k reaches a steady-state value. The bias filter is a two-gain filter with the residual of the bias-free filter multiplied by $(S_k^1)^{-1}$ and the pseudomeasurement as inputs. Using Eq. (2.8) for the Kalman gain gives

$$\begin{bmatrix} \bar{\gamma}_k \\ \lambda_k C_k^x \end{bmatrix} = \begin{bmatrix} P_{k|k}^b \bar{\sigma}_w^{-2} \\ P_{k|k}^b C_k^x \sigma_\mu^{-2} \end{bmatrix} \tag{5.22}$$

where $\bar{\sigma}_w^2$ is the variance of the residual errors entering into the bias filter as shown in Eq. (4.27), and σ_μ^2 is the variance of the constraint errors. Thus, using Eq. (5.22) gives

$$\lambda_k = \frac{\bar{\sigma}_w^2}{\sigma_\mu^2} \bar{\gamma}_k \tag{5.23}$$

and the output variance of the second stage is given by

$$P_{k|k}^b = \bar{\gamma}_k \bar{\sigma}_w^2 = \frac{\bar{\gamma}_k \beta^2 \sigma_v^2}{T^4(1-\alpha)} \tag{5.24}$$

If the statistics of r_k are stationary, the bias filter will achieve steady-state conditions when the statistics of W_k^b are stationary (i.e., $Q_k^b = \sigma_w^2$) and the kinematic constraint is not included. Thus, for simplification, the gain $\bar{\gamma}_k$ will be assumed to reach a steady-state value $\bar{\gamma}$. Thus, the covariance matrix of the output of the two-stage $\alpha, \beta, \bar{\gamma}, \lambda$ estimator will the equivalent to that of the two-stage $\alpha, \beta, \bar{\gamma}$ estimator given in Chapter 4.

When selecting λ, the scalar relationship between λ and $\bar{\gamma}$ of Eq. (5.23) can be adjusted through σ_μ^2 to tune the filter. The gain scheduling technique for initializing the two-stage $\alpha, \beta, \bar{\gamma}$ estimator given in Chapter 4 can be used for the two-stage $\alpha, \beta, \bar{\gamma}, \lambda$ estimator with

$$\frac{\bar{\sigma}_w^2}{\sigma_\mu^2} = \frac{r_0}{r_1 \delta^k + 1}, \quad \delta < 1 \tag{5.25}$$

where r_0 is chosen for steady-state conditions, r_1 is chosen large for initialization, and δ is chosen for the desired settling time.

CHAPTER 6
SIMULATION RESULTS

As an example, a tracking system that measures a target's position at 2 Hz with errors that have a standard deviation in range of 6 m and standard deviation in bearing and elevation of 2 mrad. For an α, β, γ filter, $\alpha = 0.40$, $\beta = 0.1$ and $\gamma = 0.013$. For the two-stage estimator, $\alpha = 0.3$, $\beta = 0.053$, and Eq. (4.41) was used to compute $\bar{\gamma} = 0.19$ to match the velocity variances of the filters. For Eq. (5.25), $r_0 = 0.75$, $r_1 = 400$, and $\delta = 0.75$. Thus, in steady-state conditions $\lambda = 0.14$. While various techniques can be used for maneuver detection, a soft switch approach was chosen. For the soft switch approach, the acceleration filter in each coordinate is duplicated in the maneuver detector with b_k as the filter output. As a result, the gain for compensating the α, β filter in each coordinate is given by

$$G_{SS} = \begin{cases} 0, & |b_k| < 4.0 \\ 1.0, & |b_k| > 6.0 \\ 0.5(|b_k| - 4.0), & \text{otherwise} \end{cases} \tag{6.1}$$

Also, to improve maneuver response, the magnitude of b_k is restricted so that $|b_k| \leq 6.2$. Using the soft switch approach, the correction to the output of the bias-free filter with the acceleration estimates is multiplied by G_{SS}. Simulated tracking results from a Monte Carlo simulation with 200 experiments are given in Figure 6–1 for the α, β, γ filter and the two-stage estimator with and without the kinematic constraint. For this example, the target moved with constant velocity except for a 4 g, constant speed turn from 12 to 24 s. The target began at a range of 17.5 km and moved at a constant speed of 300 m/s to a final range of 9.3 km. A critically damped, second order system with an approximate time constant of 1 s was used to model the dynamics of the target. Both two-stage estimators provide better tracking than the α, β, γ filter when the target moves with constant velocity, while the two-stage estimator with the kinematic constraint provides the best tracking through the maneuver.

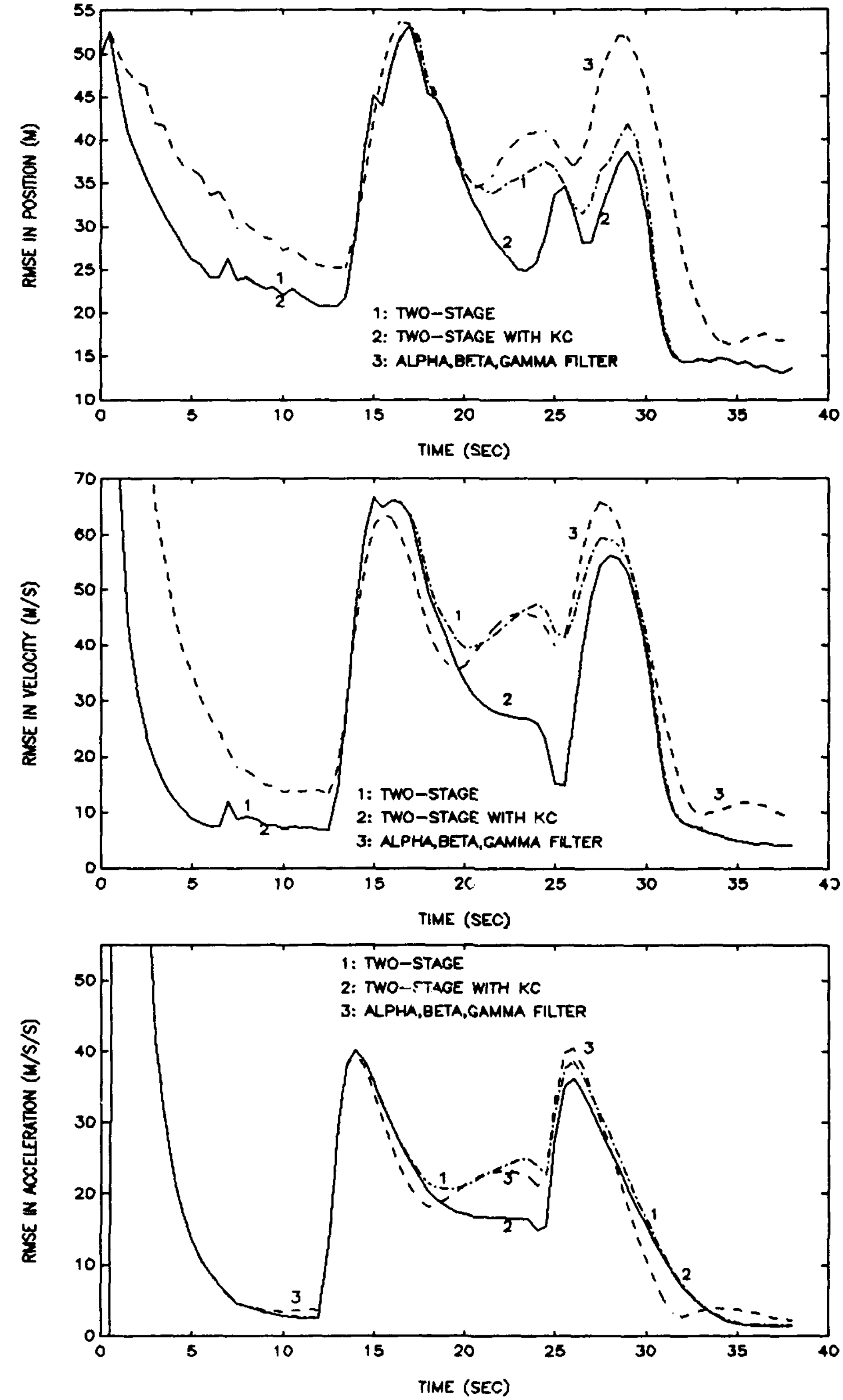

Figure 6-1. RMSE for Radar Tracking Example

CHAPTER 7

CONCLUSIONS AND FUTURE RESEARCH

The tracking performance of the two-stage estimator can be achieved by operating an α, β filter in parallel with a α, β, γ filter and selecting the output of one of the filters. However, when using the soft switch approach to maneuver detection and response as presented in Chapters 3 and 5, a second α, β, γ filter is required so that its acceleration estimate can be restricted during a maneuver to improve detecting the end of a maneuver. While this multiple filter approach is feasible, it requires approximately three times the number of computations for essentially the same tracking performance.

Since the two-stage $\alpha, \beta, \bar{\gamma}$ estimator is simple, it can be implemented in current systems with a modest increase in the computational burden and memory. The two-stage $\alpha, \beta, \bar{\gamma}$ estimator has the potential to provide significant performance gains in the tracking of maneuvering targets within systems that are responsible for tracking and engaging a large number of targets. Also, since the two-stage $\alpha, \beta, \bar{\gamma}$ estimator maintains an α, β filter track at all times, the command and control decisions, as well as maneuver detection, can be made with greater consistency with the two-stage estimator than other adaptive filtering techniques where the gains are increased during maneuvers.

Further research is needed to develop improved maneuver response procedures for the two-stage estimator. Also, further research is needed to compare the tracking performances of the fixed-gain, two-stage estimators with other simple, adaptive tracking algorithms. When using the two-stage estimator, the kinematically constrained second stage can be paralleled with the standard second stage so that constant speed and variable speed targets can be tracked by selecting the more accurate second stage. However, research is needed to develop procedures for selecting the more accurate second stage.

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APPENDIX A

DERIVATIONS FOR ALPHA-BETA FILTER

A Kalman filter is often employed to filter the position measurements for estimating the position, velocity, and/or acceleration of a target. When the target motion and measurement models are linear and the measurement and motion modeling error processes are Gaussian, the Kalman filter provides the minimum mean-square error estimate of the target state. When the target motion and measurement models are linear, but the noise processes are not Gaussian, the Kalman filter is the best linear estimator of the target state in the mean-square error sense. The dynamics model commonly assumed for a target in track is given by

$$X_{k+1} = F_k X_k + G_k w_k \tag{A.1}$$

where $w_k \sim N(0, Q_k)$ is the process noise and F_k defines a linear constraint on the dynamics. The target state vector X_k contains the position, velocity, and acceleration of the target at time k as well as other variables used to model the time-varying acceleration. The linear measurement model is given by

$$Y_k = H_k X_k + n_k \tag{A.2}$$

where Y_k is usually the target position measurement and $n_k \sim N(0, R_k)$. The Kalman filtering equations associated with the state model in Eq. (A.1) and the measurement model in Eq. (A.2) are given by the following equations.

Time Update:

$$X_{k|k-1} = F_{k-1} X_{k-1|k-1} \tag{A.3}$$

$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T \tag{A.4}$$

Measurement Update:

$$K_k = P_{k|k-1} H_k^T [H_k P_{k|k-1} H_k^T + R_k]^{-1} \tag{A.5}$$

$$X_{k|k} = X_{k|k-1} + K_k [Y_k - H_k X_{k|k-1}] \tag{A.6}$$

$$P_{k|k} = [I - K_k H_k] P_{k|k-1} \tag{A.7}$$

where $X_k \sim N(X_{k|k}, P_{k|k})$ with $X_{k|k}$ and $P_{k|k}$ denoting the mean and error covariance of the state estimate, respectively. The subscript notation $(k|j)$ denotes the state estimate for time k when given measurements through time j, and K_k denotes the Kalman gain. Using the matrix inversion lemma of [A-1] and Eqs. (A.5) and (A.7), an alternate form of the Kalman gain is given by

$$\begin{aligned} K_k &= P_{k|k-1} H_k^T [H_k P_{k|k-1} H_k^T + R_k^{-1}]^{-1} \\ &= [I - P_{k|k-1} H_k^T R_k^{-1} H_k]^{-1} P_{k|k-1} H_k^T R_k^{-1} \end{aligned}$$

$$
\begin{aligned}
&= [I - P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k^{-1})^{-1} H_k] P_{k|k-1} H_k^T R_k^{-1} \\
&= [I - K_k H_k] P_{k|k-1} H_k^T R_k^{-1} \\
&= P_{k|k} H_k^T R_k^{-1} \qquad (A.8)
\end{aligned}
$$

The steady-state form of the Kalman filter is often used in order to reduce the computations required to maintain each track. In steady-state, $P_{k|k} = P_{k-1|k-1}$, and $P_{k+1|k} = P_{k|k-1}$, and $K_k = K_{k-1}$. For a Kalman filter to achieve these steady-state conditions, the error processes, w_k and n_k, must have stationary statistics and the data rate must be constant. When the noise processes are not stationary or the data rate is not constant, a filter using the steady-state gains will provide suboptimal estimates. The α, β filter is the steady-state Kalman filter for tracking nearly constant velocity targets.

The α, β filter is a single coordinate filter that is based on the assumption that the target is moving with constant velocity plus zero-mean, white Gaussian acceleration errors. Given this assumption, the filter gains α and β are chosen as the steady-state Kalman gains that minimize the mean-square error in the position and velocity estimates. For the α, β filter,

$$X_k = [x_k \quad \dot{x}_k]^T \qquad (A.9)$$

$$F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad (A.10)$$

$$G_k = \begin{bmatrix} \frac{T^2}{2} & T \end{bmatrix}^T \qquad (A.11)$$

$$H_k = [1 \quad 0] \qquad (A.12)$$

$$R_k = \sigma_v^2 \qquad (A.13)$$

$$Q_k = \sigma_w^2 \qquad (A.14)$$

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{T} \end{bmatrix}^T \qquad (A.15)$$

The α, β gains are determined as in [A-2] by solving the simultaneous equations

$$\Gamma = T^4 \frac{\sigma_w^2}{\sigma_v^2} = \frac{\beta^2}{(1-\alpha)} \qquad (A.16)$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \qquad (A.17)$$

where Γ is the tracking index of [A-2], and σ_w^2 is the variance of the acceleration modeling error.

STEADY-STATE ERROR COVARIANCE AND GAINS

Let the steady-state error covariance matrix of the filtered estimates for the α, β filter be denoted as

$$P_{k+1|k+1} = P_{k|k} = P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{A.18}$$

Using Eqs. (A.8), (A.12), and (A.13) gives the steady-state gain

$$K = PH_k^T \sigma_v^{-2} = \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} \tag{A.19}$$

Thus using Eq. (A.15) gives

$$p_{11} = \alpha \sigma_v^2 \tag{A.20}$$

$$p_{12} = \frac{\beta}{T} \sigma_v^2 \tag{A.21}$$

Inserting Eq. (A.4) into Eq. (A.7) and setting $P_{k|k} = P_{k-1|k-1} = P$ for steady-state gives

$$[I - KH_k]^{-1} P = F_{k-1} P F_{k-1}^T + G_{k-1} G_{k-1}^T \sigma_w^2 \tag{A.22}$$

Then

$$[I - KH_k]^{-1} = \frac{1}{1-\alpha} \begin{bmatrix} 1 & 0 \\ \frac{\beta}{T} & 1-\alpha \end{bmatrix} \tag{A.23}$$

$$[I - KH_k]^{-1} P = \frac{\sigma_v^2}{1-\alpha} \begin{bmatrix} \alpha & \frac{\beta}{T} \\ \frac{\beta}{T} & \frac{\beta^2}{T^2} + (1-\alpha)\frac{p_{22}}{\sigma_v^2} \end{bmatrix} \tag{A.24}$$

$$F_k P F_k^T = \begin{bmatrix} p_{11} + 2Tp_{12} + T^2 p_{22} & p_{12} + Tp_{22} \\ p_{12} + Tp_{22} & p_{22} \end{bmatrix} \tag{A.25}$$

$$G_k G_k^T \sigma_w^2 = \sigma_w^2 \begin{bmatrix} \frac{T^4}{4} & \frac{T^3}{2} \\ \frac{T^3}{2} & T^2 \end{bmatrix} \tag{A.26}$$

Equating the (2,2) elements of Eq. (A.22) gives

$$\frac{\beta^2}{1-\alpha} = \frac{T^4 \sigma_w^2}{\sigma_v^2} = \Gamma^2 \tag{A.27}$$

Equating the (1,2) elements of Eq. (A.22) and using Eq. (A.27) to eliminate σ_w^2 gives

$$p_{22} = \frac{\beta(2\alpha - \beta)}{(1-\alpha)T^2}\sigma_v^2 \qquad (A.28)$$

Equating the (1,1) elements of Eq. (A.22) and using Eqs. (A.20) through (A.21) and Eqs. (A.27) through (A.28) gives

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \qquad (A.29)$$

Eqs. (A.27) and (A.30) give the steady-state gain relationships in Eqs. (A.16) and (A.17). The steady-state error covariance in Eq. (A.18) is given by [1,8] as

$$P_{k|k} = \sigma_v^2 \begin{bmatrix} \alpha & \dfrac{\beta}{T} \\ \dfrac{\beta}{T} & \dfrac{\beta(2\alpha-\beta)}{2(1-\alpha)T^2} \end{bmatrix} \qquad (A.30)$$

MEASUREMENT VARIANCE REDUCTION MATRIX

The variance reduction between position and velocity estimates and the measurements are often considered in the design and analysis of α, β filters. The variance reduction matrix for the α, β filter is derived by considering the filter as a linear system with a white noise input. The input-output relationships between the measurements Y_k and $X_{k|k}$ can be expressed as a linear system that is given by

$$X_{k|k} = \overline{F} X_{k-1|k-1} + \overline{G}_k Y_k \qquad (A.31)$$

where

$$\overline{F} = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\dfrac{\beta}{T} & (1-\beta) \end{bmatrix} \qquad (A.32)$$

$$\overline{G} = \begin{bmatrix} \alpha \\ \dfrac{\beta}{T} \end{bmatrix} \qquad (A.33)$$

Using Eq. (A.31) provides the error covariance of $X_{k|k}$ that results from white noise measurement errors. That error covariance $E[X_{k|k}X_{k|k}^T] = \overline{S}_k$ is given by

$$\overline{S}_k = \overline{F}\,\overline{S}_{k-1}\overline{F}^T + \overline{G}\,\overline{G}^T\sigma_v^2 \qquad (A.34)$$

where σ_v^2 is the variance of the input Y_k. Since $\overline{S}_k = \overline{S}_{k-1} = \overline{S}_{\alpha\beta}$ in steady-state conditions, Eq. (A.34) can be used to solve for elements of $\overline{S}_{\alpha\beta}$ in terms of the filter gains, measurement period, and the measurement error variance. Eq. (A.34) can be rewritten as

$$\overline{F}^{-1}\overline{S}_{\alpha\beta} = \overline{S}_{\alpha\beta}\overline{F}^{T} + \overline{F}^{-1}\overline{G}\,\overline{G}^{T}\sigma_v^2 \tag{A.35}$$

where

$$\overline{F}^{-1} = \frac{1}{1-\alpha}\begin{bmatrix} 1-\beta & -(1-\alpha)T \\ \dfrac{\beta}{T} & (1-\alpha) \end{bmatrix} \tag{A.36}$$

$$\overline{F}^{-1}\overline{G}\,\overline{G}^{T}\sigma_v^2 = \frac{\sigma_v^2}{1-\alpha}\begin{bmatrix} (\alpha-\beta)\alpha & (\alpha-\beta)\dfrac{\beta}{T} \\ \dfrac{\alpha\beta}{T} & \dfrac{\beta^2}{T^2} \end{bmatrix} \tag{A.37}$$

Let

$$S_{\alpha\beta} = \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} \tag{A.38}$$

Then, equating the (1,1), (2,1), and (2,2) elements of Eq. (A.35) gives

$$[(2-\alpha)-\beta]s_{11} - (2-\alpha)(1-\alpha)Ts_{12} = (\alpha-\beta)\alpha\sigma_v^2 \tag{A.39}$$

$$\beta s_{11} + \alpha(1-\alpha)Ts_{12} - (1-\alpha)^2T^2s_{22} = = \alpha\beta\sigma_v^2 \tag{A.40}$$

$$(2-\alpha)s_{12} + (1-\alpha)Ts_{22} = = \frac{\beta\sigma_v^2}{T} \tag{A.41}$$

Multiplying Eq. (A.41) by $(1-\alpha)T$ and adding it to Eq. (A.40) gives

$$s_{11} = -2(1-\alpha)\frac{T}{\beta}s_{12} + \sigma_v^2 \tag{A.42}$$

Inserting Eq. (A.42) into Eq. (A.39) gives

$$s_{12} = \frac{(2\alpha-\beta)\beta}{\alpha[4-2\alpha-\beta]T}\sigma_v^2 \tag{A.43}$$

Inserting Eq. (A.43) into Eq. (A.42) gives

$$s_{11} = \frac{2\alpha^2+\beta(2-3\alpha)}{\alpha[4-2\alpha-\beta]}\sigma_v^2 \tag{A.44}$$

Inserting Eq. (A.43) into Eq. (A.41) gives

$$s_{22} = \frac{2\beta^2}{\alpha[4-2\alpha-\beta]T^2}\sigma_v^2 \tag{A.45}$$

Eqs. (A.43) through (A.45) with Eq. (A.38) gives

$$\overline{S}_{\alpha\beta} = \frac{\sigma_v^2}{\alpha d_1} \begin{bmatrix} 2\alpha^2 + \beta(2-3\alpha) & \frac{\beta}{T}(2\alpha-\beta) \\ \frac{\beta}{T}(2\alpha-\beta) & \frac{2\beta^2}{T^2} \end{bmatrix} \tag{A.46}$$

where $d_1 = 4-2\alpha-\beta$. The variance reduction ratios of the filter are given by Eq. (A.46) when $\sigma_v^2 = 1$ with the (1,1) and (2,2) elements of $\overline{S}_{\alpha\beta}$ denoting the position variance reduction ratio and velocity variance reduction ratio, respectively [A-3].

INITIALIZATION GAINS

For tracking systems with a uniform data rate and stationary measurement noise, nonmaneuvering targets can be accurately tracked with a steady–state Kalman filter. However, between filter initialization and steady-state conditions, the Kalman gain and state error covariances are transient. If steady–state gains are used from initialization, the settling time of the state estimates may be extended significantly. While the initial gains can be approximated as decaying exponentials as suggested in [A-2], identifying the constants for the exponential can be quite cumbersome. The purpose of this section is to present a simple gain scheduling procedure for initializing α, β filters.

The gain scheduling procedures are developed by using the motion and measurement models to formulate a batch least-squares estimation problem for an arbitrary number of measurements. An analytical form is then obtained for the resulting error covariance that is used in conjunction with Eq. (A.8) to obtain an analytical form for the Kalman gain.

For linear least-squares estimation, an equation formulating the measurement vector Z as a linear function of the parameter vector X to be estimated is given by

$$Z = WX + V \tag{A.47}$$

where $E[V] = 0$ and $E[VV^T] = \sigma^2 I_N$ for N measurements with I_N denoting the $N \times N$ identity matrix. The least-squares estimate $\widehat{X}$ is given in [A-4] as

$$\widehat{X} = (W^T W)^{-1} W^T Z \tag{A.48}$$

with error covariance

$$P_N = \mathrm{COV}[\widehat{X}] = \sigma^2 (W^T W)^{-1} \tag{A.49}$$

A current state estimate of X denoted by $\widehat{X}_0 = [\,x_0 \quad \dot{x}_0\,]$ can be obtained from the current measurement plus the N previous measurements as

$$Z_N = W_N X_0 + V_N \tag{A.50}$$

where

$$Z_N = [\,y_{-N} \quad y_{-N+1} \quad \cdots \quad y_{-1} \quad y_0\,]^T \tag{A.51}$$

$$V_N = [\,v_{-N} \quad v_{-N+1} \quad \cdots \quad v_{-1} \quad v_0\,]^T \tag{A.52}$$

$$W_N = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -NT & (-N+1)T & \cdots & -T & 0 \end{bmatrix}^T \tag{A.53}$$

with $E[V_N] = 0$ and $E[V_N V_N^T] = \sigma_v^2 I_N$. The error covariance of the least-squares estimate $\widehat{X}_0$ is given by

$$\mathrm{COV}[\widehat{X}_0] = P_N = \sigma_v^2 (W_N^T W_N)^{-1} \tag{A.54}$$

First,

$$\begin{aligned}(W_N^T W_N) &= \begin{bmatrix} N+1 & T\displaystyle\sum_{i=0}^{N}(-N+i) \\ T\displaystyle\sum_{i=0}^{N}(-N+i) & T^2\displaystyle\sum_{i=0}^{N}(N-i)^2 \end{bmatrix} \\ &= \begin{bmatrix} N+1 & -T\displaystyle\sum_{j=0}^{N} j \\ -T\displaystyle\sum_{j=0}^{N} j & T^2\displaystyle\sum_{j=0}^{N} j^2 \end{bmatrix}\end{aligned} \tag{A.55}$$

Evaluating the finite summations of Eq. (A.55) gives

$$(W_N^T W_N) = (N+1)\begin{bmatrix} 1 & -N\dfrac{T}{2} \\ -N\dfrac{T}{2} & N(2N+1)\dfrac{T^2}{6} \end{bmatrix} \tag{A.56}$$

Then

$$\mathrm{COV}[\widehat{X}_0] = P_N =$$

$$\frac{2\sigma_v^2}{(N+1)(N+2)}\begin{bmatrix} 2N+1 & \dfrac{3}{T} \\ \dfrac{3}{T} & \dfrac{6}{(N-1)T^2} \end{bmatrix} \tag{A.57}$$

Let the Kalman gain be denoted by

$$K_k = \left[\alpha_k \quad \frac{\beta_k}{T} \right]^T \tag{A.58}$$

Letting $P_{k|k} = P_k$ in Eq. (A.8) provides a simple gain scheduling procedure for α and β during initialization that is given by

$$\alpha_k = \max\{\frac{2(2k+1)}{(k+1)(k+2)}, \alpha\} \tag{A.59}$$

$$\beta_k = \max\{\frac{6}{(k+1)(k+2)}, \beta\} \tag{A.60}$$

with $X_{0|-1} = [0 \quad 0]^T$. Note that if Y_0/T can be extremely large relative to dynamic range of the computer, the first two filter iterations should be computed analytically to use a two-point initialization procedure.

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APPENDIX B

DERIVATIONS FOR ALPHA-BETA-GAMMA FILTER

A Kalman filter is often employed to filter the position measurements for estimating the position, velocity, and/or acceleration of a target. When the target motion and measurement models are linear and the measurement and motion modeling error processes are Gaussian, the Kalman filter provides the minimum mean-square error estimate of the target state. When the target motion and measurement models are linear, but the noise processes are not Gaussian, the Kalman filter is the best linear estimator of the target state in the mean-square error sense. The dynamics model commonly assumed for a target in track is given by

$$X_{k+1} = F_k X_k + G_k w_k \tag{B.1}$$

where $w_k \sim N(0, Q_k)$ is the process noise and F_k defines a linear constraint on the dynamics. The target state vector X_k contains the position, velocity, and acceleration of the target at time k, as well as other variables used to model the time-varying acceleration. The linear measurement model is given by

$$Y_k = H_k X_k + n_k \tag{B.2}$$

where Y_k is usually the target position measurement and $n_k \sim N(0, R_k)$. The Kalman filtering equations associated with the state model in Eq. (B.1) and the measurement model in Eq. (B.2) are given by the following equations.

Time Update:

$$X_{k|k-1} = F_{k-1} X_{k-1|k-1} \tag{B.3}$$

$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T \tag{B.4}$$

Measurement Update:

$$K_k = P_{k|k-1} H_k^T [H_k P_{k|k-1} H_k^T + R_k]^{-1} \tag{B.5}$$

$$X_{k|k} = X_{k|k-1} + K_k [Y_k - H_k X_{k|k-1}] \tag{B.6}$$

$$P_{k|k} = [I - K_k H_k] P_{k|k-1} \tag{B.7}$$

where $X_k \sim N(X_{k|k}, P_{k|k})$ with $X_{k|k}$ and $P_{k|k}$ denoting the mean and error covariance of the state estimate, respectively. The subscript notation $(k|j)$ denotes the state estimate for time k when given measurements through time j, and K_k denotes the Kalman gain. Using the matrix inversion lemma of [B-1] and Eqs. (B.5) and (B.7), an alternate form of the Kalman gain is given by

$$\begin{aligned} K_k &= P_{k|k-1} H_k^T [H_k P_{k|k-1} H_k^T + R_k^{-1}]^{-1} \\ &= [I - P_{k|k-1} H_k^T R_k^{-1} H_k]^{-1} P_{k|k-1} H_k^T R_k^{-1} \end{aligned}$$

$$
\begin{aligned}
&= [I - P_{k|k-1}H_k^T(H_kP_{k|k-1}H_k^T + R_k^{-1})^{-1}H_k]P_{k|k-1}H_k^TR_k^{-1} \\
&= [I - K_kH_k]P_{k|k-1}H_k^TR_k^{-1} \\
&= P_{k|k}H_k^TR_k^{-1} \qquad (B.8)
\end{aligned}
$$

The steady-state form of the Kalman filter is often used in order to reduce the computations required to maintain each track. In steady-state, $P_{k|k} = P_{k-1|k-1}$, and $P_{k+1|k} = P_{k|k-1}$, and $K_k = K_{k-1}$. For a Kalman filter to achieve these steady-state conditions, the error processes, w_k and n_k, must have stationary statistics and the data rate must be constant. When the noise processes are not stationary or the data rate is not constant, a filter using the steady-state gains will provide suboptimal estimates. The α, β, γ filter is the steady-state Kalman filter for tracking nearly constant acceleration targets.

The α, β, γ filter is a single coordinate filter that is based on the assumption that the target is moving with constant acceleration plus zero-mean, white Gaussian acceleration errors. Given this assumption, the α, β, γ filter gains are chosen as the steady-state Kalman gains that minimize the mean-square error in the position, velocity, and acceleration estimates. For the α, β, γ filter,

$$X_k = [x_k \quad \dot{x}_k \quad \ddot{x}_k]^T \qquad (B.9)$$

$$F_k = \begin{bmatrix} 1 & T & 0.5T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \qquad (B.10)$$

$$G_k = \begin{bmatrix} \frac{T^2}{2} & T & 1 \end{bmatrix}^T \qquad (B.11)$$

$$H_k = [1 \quad 0 \quad 0] \qquad (B.12)$$

$$R_k = \sigma_v^2 \qquad (B.13)$$

$$Q_k = \sigma_w^2 \qquad (B.14)$$

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{T} & \frac{\gamma}{T^2} \end{bmatrix}^T \qquad (B.15)$$

The α, β, γ gains are determined as in [B-2] by solving the simultaneous equations

$$T^4\frac{\sigma_w^2}{\sigma_v^2} = \frac{\gamma^2}{(1-\alpha)} \qquad (B.16)$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \qquad (B.17)$$

$$\gamma = \frac{\beta^2}{2\alpha} \qquad (B.18)$$

where $\Gamma = \dfrac{T^2\sigma_w}{\sigma_v}$ is the tracking index of [B-2].

STEADY-STATE ERROR COVARIANCE AND GAINS

Let the steady-state error covariance matrix of the filtered estimates for the α, β, γ filter be denoted as

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \tag{B.19}$$

Using Eq. (B.8), (B.12), and (B.13) gives the steady-state gain

$$K = PH_k^T \sigma_v^{-2} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} \tag{B.20}$$

Thus using Eq. (B.15) gives

$$p_{11} = \alpha \sigma_v^2 \tag{B.21}$$

$$p_{12} = \frac{\beta}{T} \sigma_v^2 \tag{B.22}$$

$$p_{13} = \frac{\gamma}{T^2} \sigma_v^2 \tag{B.23}$$

Inserting Eq. (B.4) into Eq. (B.7) and setting $P_{k|k} = P_{k-1|k-1} = P$ for steady-state gives

$$[I - KH_k]^{-1} P = F_{k-1} P F_{k-1}^T + G_{k-1} G_{k-1}^T \sigma_w^2 \tag{B.24}$$

Then

$$[I - KH_k]^{-1} = \frac{1}{1-\alpha} \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\beta}{T} & 1-\alpha & 0 \\ \dfrac{\gamma}{T^2} & 0 & 1-\alpha \end{bmatrix} \tag{B.25}$$

$$[I - KH_k]^{-1} P = \frac{\sigma_v^2}{1-\alpha} \begin{bmatrix} \alpha & \dfrac{\beta}{T} & \dfrac{\gamma}{T^2} \\ \dfrac{\beta}{T} & \dfrac{\beta^2}{T^2} + (1-\alpha)\dfrac{p_{22}}{\sigma_v^2} & \dfrac{\beta\gamma}{T^3} + (1-\alpha)\dfrac{p_{23}}{\sigma_v^2} \\ \dfrac{\gamma}{T^2} & \dfrac{\beta\gamma}{T^3} + (1-\alpha)\dfrac{p_{23}}{\sigma_v^2} & \dfrac{\gamma^2}{T^4} + (1-\alpha)\dfrac{p_{33}}{\sigma_v^2} \end{bmatrix} \tag{B.26}$$

$$F_k P F_k^T = \begin{bmatrix} p_{11} + 2Tp_{12} + T^2(p_{13} + p_{22}) + T^3 p_{23} + \frac{T^4}{4} p_{33} & \cdots & \cdots \\ p_{12} + T(p_{13} + p_{22}) + \frac{3T^2}{2} p_{23} + \frac{T^3}{2} p_{33} & p_{22} + 2Tp_{23} + T^2 p_{33} & \cdots \\ p_{13} + Tp_{23} + \frac{T^2}{2} p_{33} & p_{23} + Tp_{33} & p_{33} \end{bmatrix} \tag{B.27}$$

$$G_k G_k^T \sigma_w^2 = \sigma_w^2 \begin{bmatrix} \frac{T^4}{4} & \frac{T^3}{2} & \frac{T^2}{2} \\ \frac{T^3}{2} & T^2 & T \\ \frac{T^2}{2} & T & 1 \end{bmatrix} \tag{B.28}$$

Equating the (3,3) elements of Eq. (B.24) gives

$$\frac{\gamma^2}{1-\alpha} = \frac{T^4 \sigma_w^2}{\sigma_v^2} = \Gamma^2 \tag{B.29}$$

Equating the (2,3) elements of Eq. (B.24) and using Eq. (B.29) to eliminate σ_w^2 gives

$$p_{33} = \frac{\gamma(\beta-\gamma)}{(1-\alpha)T^4} \sigma_v^2 \tag{B.30}$$

Equating the (2,2) elements of Eq. (B.24) and using Eq. (B.29) to eliminate σ_w^2 gives

$$p_{23} = \frac{\gamma(\beta-\gamma)}{2(1-\alpha)T^3} \sigma_v^2 \tag{B.31}$$

Equating the (3,1) elements of Eq. (B.24) and using Eqs. (B.20), (B.29), (B.30), and (B.31) gives

$$\gamma = \frac{\beta^2}{2\alpha} \tag{B.32}$$

Equating the (2,3) elements of Eq. (B.24) and using Eqs. (B.20), (B.29), and (B.31) gives

$$p_{22} = \frac{4\alpha\beta + \gamma(\beta - 2\alpha - 4)}{4(1-\alpha)T^2} \sigma_v^2 \tag{B.33}$$

Equating the (1,1) elements of Eq. (B.24) and using Eqs. (B.21) through (B.23) and Eqs. (B.29) through (B.33) gives

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{B.34}$$

Eqs. (B.29), (B.32), and (B.34) give the steady-state gain relationships in Eqs. (B.16) through (B.18). The steady-state error covariance in Eq. (B.19) is given by

$$P = \sigma_v^2 \begin{bmatrix} \alpha & \dfrac{\beta}{T} & \dfrac{\gamma}{T^2} \\ \dfrac{\beta}{T} & \dfrac{4\alpha\beta + \gamma(\beta - 2\alpha - 4)}{4(1-\alpha)T^2} & \dfrac{\beta(\beta-\gamma)}{2(1-\alpha)T^3} \\ \dfrac{\gamma}{T^2} & \dfrac{\beta(\beta-\gamma)}{2(1-\alpha)T^3} & \dfrac{\gamma(\beta-\gamma)}{(1-\alpha)T^4} \end{bmatrix} \tag{B.35}$$

INITIALIZATION GAINS

For tracking systems with a uniform data rate and stationary measurement noise, nonmaneuvering targets can be accurately tracked with a steady-state Kalman filter. However, between filter initialization and steady-state conditions, the Kalman gain and state error covariances are transient. If steady-state gains are used from initialization, the settling time of the state estimates can extended significantly. While the initial gains can be approximated as decaying exponentials as suggested in [B-2], identifying the constants for the exponential can be quite cumbersome. The purpose of this section is to present some simple gain scheduling procedures for initializing α, β, γ filters.

The gain scheduling procedures are developed by using the motion and measurement models to formulate a batch least-squares estimation problem for an arbitrary number of measurements. An analytical form is then obtained for the resulting error covariance that is used in conjunction with Eq. (B.8) to obtain an analytical form for the Kalman gain.

For linear least-squares estimation, an equation formulating the measurement vector Z as a linear function of the parameter vector X to be estimated is given by

$$Z = WX + V \tag{B.36}$$

where $E[V] = 0$ and $E[VV^T] = \sigma^2 I_N$ for N measurements with I_N denoting the $N \times N$ identity matrix. The least-squares estimate $\widehat{X}$ is given in [B-3] as

$$\widehat{X} = (W^T W)^{-1} W^T Z \tag{B.37}$$

with error covariance

$$P_N = \mathrm{COV}[\widehat{X}] = \sigma^2 (W^T W)^{-1} \tag{B.38}$$

A current state estimate of $\widehat{X}_0 = [x_0 \quad \dot{x}_0 \quad \ddot{x}_0]$ can be obtained from the current measurement plus the N previous measurements as

$$Z_N = W_N X_0 + V_N \tag{B.39}$$

where

$$Z_N = [y_{-N} \quad y_{-N+1} \quad \cdots \quad y_{-1} \quad y_0]^T \tag{B.40}$$

$$V_N = [v_{-N} \quad v_{-N+1} \quad \cdots \quad v_{-1} \quad v_0]^T \tag{B.41}$$

$$W_N = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -NT & (-N+1)T & \cdots & -T & 0 \\ 0.5(NT)^2 & 0.5(-N+1)^2T^2 & \cdots & 0.5T^2 & 0 \end{bmatrix}^T \tag{B.42}$$

with $E[V_N] = 0$ and $E[V_N V_N^T] = \sigma_v^2 I_N$. The covariance of the least-squares estimate $\widehat{X}_0$ is given by

$$\mathrm{COV}[\widehat{X}_0] = P_N = \sigma_v^2 (W_N^T W_N)^{-1} \tag{B.43}$$

First,

$$(W_N^T W_N) = \begin{bmatrix} N+1 & T\sum_{i=0}^{N}(-N+i) & \frac{T^2}{2}\sum_{i=0}^{N}(N-i)^2 \\ T\sum_{i=0}^{N}(-N+i) & T^2\sum_{i=0}^{N}(N-i)^2 & \frac{T^3}{2}\sum_{i=0}^{N}(-N+i)^3 \\ \frac{T^2}{2}\sum_{i=0}^{N}(N-i)^2 & \frac{T^3}{2}\sum_{i=0}^{N}(-N+i)^3 & \frac{T^4}{4}\sum_{i=0}^{N}(N-i)^4 \end{bmatrix}$$

$$= \begin{bmatrix} N+1 & -T\sum_{j=0}^{N} j & \frac{T^2}{2}\sum_{j=0}^{N} j^2 \\ -T\sum_{j=0}^{N} j & T^2\sum_{j=0}^{N} j^2 & \frac{-T^3}{2}\sum_{j=0}^{N} j^3 \\ \frac{T^2}{2}\sum_{j=0}^{N} j^2 & \frac{-T^3}{2}\sum_{j=0}^{N} j^3 & \frac{T^4}{4}\sum_{j=0}^{N} j^4 \end{bmatrix} \tag{B.44}$$

Evaluating the finite summations of Eq. (B.44) gives

$$(W_N^T W_N) = (N+1)\cdot$$
$$\begin{bmatrix} 1 & -N\frac{T}{2} & N(2N+1)\frac{T^2}{12} \\ -N\frac{T}{2} & N(2N+1)\frac{T^2}{6} & -N^2(N+1)\frac{T^3}{8} \\ N(2N+1)\frac{T^2}{12} & -N^2(N+1)\frac{T^3}{8} & N(2N+1)(3N^2+3N-1)\frac{T^4}{120} \end{bmatrix} \quad (B.45)$$

Then

$$(W_N^T W_N)P_N = \sigma_v^2 I_3 \quad (B.46)$$

Let the error covariance be denoted as

$$P_N = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (B.47)$$

Equating the elements of Eq. (B.46) gives the following six equations to solve for P_N:

$$\frac{1}{N}a_{11} - \frac{T}{2}a_{12} + (2N+1)\frac{T^2}{12}a_{13} = \frac{\sigma_v^2}{N(N+1)} \quad (B.48)$$

$$\frac{1}{N}a_{12} - \frac{T}{2}a_{22} + (2N+1)\frac{T^2}{12}a_{23} = 0 \quad (B.49)$$

$$\frac{1}{N}a_{13} - \frac{T}{2}a_{23} + (2N+1)\frac{T^2}{12}a_{33} = 0 \quad (B.50)$$

$$-\frac{T}{2}a_{12} + (2N+1)\frac{T^2}{6}a_{22} - N(N+1)\frac{T^3}{8}a_{23} = \frac{\sigma_v^2}{N(N+1)} \quad (B.51)$$

$$-\frac{T}{2}a_{13} + (2N+1)\frac{T^2}{6}a_{23} - N(N+1)\frac{T^3}{8}a_{33} = 0 \quad (B.52)$$

$$(2N+1)\frac{T^2}{12}a_{13} - N(N+1)\frac{T^3}{8}a_{23} + (2N+1)(3N^2+3N-1)\frac{T^4}{120}a_{33} = \frac{\sigma_v^2}{N(N+1)} \quad (B.53)$$

Solving Eqs. (B.48) through (B.53) gives

$$\mathrm{COV}[\widehat{X}_0] = P_N =$$
$$\frac{3\sigma_v^2}{(N+1)(N+2)(N+3)} \begin{bmatrix} 3N^2+3N+2 & \frac{6(2N+1)}{T} & \frac{20}{T^2} \\ \frac{6(2N+1)}{T} & \frac{4(2N+1)(8N-3)}{(N-1)NT^2} & \frac{120}{(N-1)T^3} \\ \frac{20}{T^2} & \frac{120}{(N-1)T^3} & \frac{240}{(N-1)NT^4} \end{bmatrix} \quad (B.54)$$

Let the Kalman gain be denoted by

$$K_k = \left[\alpha_k \quad \frac{\beta_k}{T} \quad \frac{\gamma_k}{T^2} \right]^T \tag{B.55}$$

Letting $P_{k|k} = P_k$ in Eq. (B.8) gives a simple gain scheduling procedure for α_k, β_k and γ_k during initialization that is given by

$$\alpha_k = \max\{\frac{3(3k^2+3k+2)}{(k+1)(k+2)(k+3)}, \alpha\} \tag{B.56}$$

$$\beta_k = \max\{\frac{18(2k+1)}{(k+1)(k+2)(k+3)}, \beta\} \tag{B.57}$$

$$\gamma_k = \max\{\frac{60}{(k+1)(k+2)(k+3)}, \gamma\} \tag{B.58}$$

with $X_{0|-1} = [0 \quad 0 \quad 0]^T$. Note that if $Y_{0,1}/T^2$ can be extremely large relative to the dynamic range of the computer, the first three iterations of the filter should be computed analytically to use a three-point initialization procedure.

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The two-stage Alpha-Beta-Gamma estimator is proposed as an alternative to adaptive gain versions of the Alpha-Beta and Alpha-Beta-Gamma filters for tracking maneuvering targets. This report accomplishes fixed-gain, variable dimension filtering with a two-stage Alpha-Beta-Gamma estimator. The two-stage Alpha-Beta-Gamma estimator is derived from the two-stage Kalman estimator, and the noise variance and reduction matrix and steady-state error covariance matrix are given as a function of the steady-state gains. A procedure for filter parameter selection is also given along with a technique for maneuver response and a gain scheduling technique for initialization. The kinematic constraint for constant speed targets is also incorporated into the two-stage estimator to form the two-stage Alpha-Beta-Gamma-Lambda estimator. Simulation results are given for a comparison of the performances of estimators with that of the Alpha-Beta-Gamma filter.

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