

AD-A255 254



(1)

PLASTIC SHEAR BUCKLING OF SHIP

HULL PLATING INDUCED BY GROUNDING

by

Stephen Rodgers Price (1)

B.S. Civil Eng. University of New Mexico (1977)

DTIC

ELECTE

SEP 1 1992

S

C

D

Submitted to the Department of Ocean Engineering  
in partial fulfillment of the requirements for the degrees of

NAVAL ENGINEER

and

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

at the  
Massachusetts Institute of Technology  
June, 1992

© Stephen Rodgers Price, 1992. All rights reserved

The author hereby grants to MIT permission to reproduce and to  
distribute copies of this thesis document in whole or in part.

Signature of Author .....

Department of Ocean Engineering  
May, 1992

Certified by .....

Tomasz Wierzbicki  
Professor of Applied Mechanics  
Thesis Supervisor

Certified by .....

Frank A. McClintock  
Professor of Mechanical Engineering  
Thesis Reader

Accepted by .....

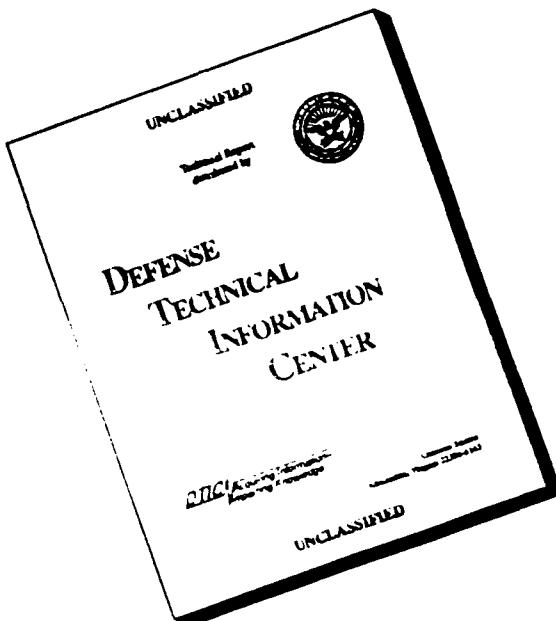
A. Douglas Carmichael  
Departmental Graduate Committee  
Department of Ocean Engineering

|   |
|---|
| Approved for public release<br>Distribution Unlimited |
|---|



148pg

# **DISCLAIMER NOTICE**



**THIS DOCUMENT IS BEST  
QUALITY AVAILABLE. THE COPY  
FURNISHED TO DTIC CONTAINED  
A SIGNIFICANT NUMBER OF  
PAGES WHICH DO NOT  
REPRODUCE LEGIBLY.**

PLASTIC SHEAR BUCKLING OF SHIP  
HULL PLATING INDUCED BY GROUNDING

by

STEPHEN RODGERS PRICE

Submitted to the Department of Ocean Engineering on May 8, 1992  
in partial fulfillment of the requirements for the degrees of  
Naval Engineer and Master of Science in Mechanical Engineering

ABSTRACT

A study was conducted to develop an analytical model for predicting the horizontal resistance due to in-plane shear of ship's hull bottom plating that is undergoing deformation induced by grounding. First, four models were developed for predicting the load carrying capacity of a flat rectangular plate element when subjected to in-plane shear loads in both the elastic and plastic buckling ranges. Seven plate buckling experiments were then conducted which were used to validate the analytical model results. Finally, the plate element model was incorporated into a global model description for the bottom plating of a very large crude carrier (VLCC) which is undergoing a deformation process induced by grounding.

The results of the study demonstrated that a suitable plate element model could be developed for accurately predicting within 10 percent both the in-plane shear load capacity as well as the plastic buckling wavelength of a flat rectangular plate. The best prediction was obtained from the buckled element which is initially rectangular.

The grounding resistance due to in-plane shear was roughly an order of magnitude greater than that from other sources not including shear. Thus it seems that fracture or some other "weaker" failure modes will pre-empt the shear deformation mode.

Thesis Supervisor: Dr. Tomasz Wierzbicki

Title: Professor of Applied Mechanics, Department of Ocean Engineering

DTIC QUALITY INSPECTED 3

CODE  
23

|                             |                                     |
|-----------------------------|-------------------------------------|
| Accession For               |                                     |
| NTIS Serial                 | <input checked="" type="checkbox"/> |
| Master File                 | <input type="checkbox"/>            |
| Supplementary               | <input type="checkbox"/>            |
| Final Evaluation            | <input type="checkbox"/>            |
| DTIC Form 50                |                                     |
| Distribution Is:            |                                     |
| Approved for Public Release |                                     |
| DTIC Distribution Codes     |                                     |
| AMERICAN LIBRARIES          |                                     |
| ALL INFORMATION CONTAINED   |                                     |
| HEREIN IS UNCLASSIFIED      |                                     |
| DATE 10-10-94 BY SP-101     |                                     |
| A-1                         |                                     |

## Table of Contents

|  | <u>Page</u> |
|--|-------------|
| <b>1. Introduction and Background</b>                            | 4           |
| <b>2. Summary of Elastic Shear Buckling Response</b>             | 7           |
| <b>3. Formulation of Analytical Models</b>                       |             |
| <b>3.1 Plastic buckling model</b>                                | 13          |
| <b>3.2 Elastic buckling models</b>                               | 35          |
| <b>3.3 Analytical model for plastic flow without buckling</b>    | 40          |
| <b>3.4 The complete analytical model</b>                         | 47          |
| <b>4. Discussion Shear Buckling Apparatus and Experiments</b>    |             |
| <b>4.1 Experimental apparatus and setup</b>                      | 51          |
| <b>4.2 Design of buckling apparatus</b>                          | 54          |
| <b>4.3 Tensile specimen test results</b>                         | 60          |
| <b>4.4 Buckling load test results</b>                            | 64          |
| <b>5. Comparison of Experimental and Analytical Results</b>      | 69          |
| <b>6. Formulation of Total Grounding Resistance due to Shear</b> |             |
| <b>6.1 The grounding model</b>                                   | 74          |
| <b>6.2 Results of grounding model calculations</b>               | 84          |
| <b>6.3 Comparison with other hull failure modes</b>              | 86          |
| <b>7. Conclusion</b>   | 93          |
| <b>8. Appendices</b>   |             |
| <b>A. Ship Total Shear Resistance Calculations</b>               | 95          |
| <b>B. Non-dimensional Shear Force Plots and Calculations</b>     | 111         |
| <b>C. Experimental Data</b>                                      | 142         |

## **1. Introduction and Background**

The serious environmental damage caused by the spillage of oil or other hazardous cargo from tankers in recent years has forced ship designers to look for hull structural designs that are more robust against the likelihood of rupture of the hull plating in the event of a grounding. In some designs this may entail addition of a second inner hull. Another idea proposed by Wierzbicki<sup>1</sup> among others is to design a uniform strength hull structure that contains a larger number of more densely positioned but smaller structural elements. An example of a uniform strength hull is the unidirectionally stiffened double hull currently being studied by the U.S. Navy<sup>2</sup>. The principle is to make the hull plating support structure flexible enough to deform with the hull, thus preventing the hull plate from rupturing at the welded joints where it is connected to the otherwise rigid structural frame. To effect such a design an understanding needs to be developed of the various structural resisting force and energy absorption mechanisms as well as the structural failure modes for a ship hull. These analyses should then lead to an optimal approach for strengthening a ship hull to better resist hull penetration and thus reduce the likelihood of cargo spillage.

During a grounding, four mechanisms of work dissipation have been identified by Wierzbicki<sup>3</sup>:

- (1) Global lifting of the ship hull against gravity
- (2) Frictional forces between hull and shallow bottom
- (3) Membrane rupture of ship's hull plate
- (4) Plastic deformation of hull plating and bottom structural elements

One of the hull plate plastic deformation mechanisms in item (4) above is due to in-plane shear deformations that develop within the hull plate during a grounding. Previous

---

<sup>1</sup> T. Wierzbicki, E. Rady, D.B. Peer J.G. Shin, "Damage Estimates in High Energy Grounding of Ships", Massachusetts Institute of Technology, Department of Ocean Engineering, *Joint Industry Program on Tanker Safety*, Report No. 1, June 1990.

<sup>2</sup> J. Beach, "Advanced Surface Ship Hull Technology, Cluster B", *Naval Engineers Journal*, vol. 103, no. 6, November 1991, pp 27-37.

<sup>3</sup> T. Wierzbicki, D.B. Peer, E. Rady, "The Anatomy of Tanker Grounding", Massachusetts Institute of Technology, Department of Ocean Engineering, *Joint Industry-MIT Program on Tanker Safety*, Report No. 2, May 1991.

work conducted by Wierzbicki <sup>1,3</sup> and Peer <sup>4</sup> which formulated an analytical model for tanker grounding recognized the existence of plastic work done by in plane shear forces but deferred the estimation of this work to future studies.

This paper presents an analytical model for plastic work done by in plane shear forces on a flat rectangular plate element. An experimental study is conducted and the results are compared with these analytical results to validate the model. The new model of a localized plastic shear buckling element is then incorporated into a global model which describes the deformation of the hull plating during a grounding assuming a simplified geometry. Finally, a comparison is made between work absorbed by shear buckling and by other mechanisms during a grounding, previously formulated by Peer<sup>4</sup>, in order to determine the importance of shear buckling mechanisms in dissipation of kinetic energy during a grounding.

Stated simply, the objectives of this study are threefold:

- (1) Develop analytical models for elastic pre and post-buckling behavior, plastic in-plane shear post-buckling behavior, and plastic in-plane deformation without buckling, of a single prismatic element
- (2) Validate this model with experimental results
- (3) Incorporate this model into the global ship grounding model and determine the importance of shear buckling as an energy dissipation mechanism

Three limits on the applicability of the theory must be considered before studying the details of this model and how it can be applied to the ship grounding problem. First, hull membrane rupture is not permitted since hull rupture changes the geometry and boundary conditions of the problem and the point at which hull plate rupture occurs is difficult to quantify. Therefore, the present model is applicable only to grounding geometries which allow the hull structure to "dish" inward while the plating rides up and over the obstacle on the ocean bottom.

Second, the present model assumes symmetric deformation about the keel of the ship. This assumption is necessary to avoid the need to describe the hydrodynamic stability characteristics of the grounded vessel, which otherwise would greatly complicate the analysis.

---

<sup>4</sup> D.B. Peer, "Coupling Global Motion and Local Deformation in Tanker Grounding", Massachusetts Institute of Technology, Department of Ocean Engineering, *Joint Mit-Industry Program on Tanker Safety*, Report No. 4, May 1991.

Third, this local model is incorporated into a ship that is assumed to be longitudinally framed at certain intervals with transverse webs at a specified spacing  $l$  and longitudinal bulkheads spaced at an interval  $l_1$ . The transition zone of deformation forward of the rock  $\eta$  is assumed to be a multiple of the transverse web spacing while the transverse extent of damage is assumed to be limited to the distance between longitudinal bulkheads. This structural geometry is chosen to be representative of today's very large crude carrier, (VLCC). It should be noted, therefore, that while the results from the local buckling element model are applicable to any ship with unsupported areas of rectangular flat plate, the results from the global ship grounding model are only applicable to the particular structural geometry chosen and would have to be redone for ships with a different hull structure. The main reason for this is the differing degree of constraints placed on hull deformation by the transverse webs and the longitudinal bulkheads in different designs.

## 2. Summary of Elastic Shear Buckling Response

To gain an insight into the plastic shear buckling problem, it is necessary to look first at shear buckling of a flat plate in the elastic range. The exact solution to the elastic shear buckling problem for plates of infinite length has been derived by Southwell and Skan<sup>5</sup>. Since that time plates of finite length have been analyzed by other investigators. A brief discussion of this previous work follows so that a foundation can be laid on which to base the problem formulation for the plastic buckling model discussed herein.

A plate subjected to uniform in-plane shear has an out of plane displacement  $w(x,y)$  that obeys the following differential equation in terms of the shear force per unit length along loaded edges  $N_{xy}^o$  and the flexural plate rigidity  $D$ .

$$\nabla^4 w - \left[ \frac{2N_{xy}^o}{D} \right] \frac{\partial^2 w}{\partial x \partial y} = 0 \quad D = \frac{Eh^3}{12(1-v^2)} \quad (1)$$

The general solution to the above differential equation (1) is assumed in terms of the plate width or stiffener spacing  $b$  to be:

$$w(x,y) = C e^{(i\lambda y)/b} e^{(ikx)/b} \quad (2)$$

The coordinates  $x$  and  $y$  in the above formula correspond to the respective length and width directions of the plate. The constant  $C$  is an arbitrary constant that varies with the flexural plate bending stiffness. The constants  $k$  and  $\lambda$  correspond to spatial buckling wave numbers in the  $x$  and  $y$  directions, respectively.

Substitution of the general solution (2) into the Differential Equation (1) yields the following fourth order polynomial in  $\lambda$  and  $k$ .

$$\lambda^4 + 2k^2\lambda^2 + \frac{2N_{xy}^o}{D} k\lambda + k^4 = 0 \quad (3)$$

---

<sup>5</sup>R. Southwell and S. Skan, "On the Stability under Shearing Forces of a flat Elastic Strip", National Physical Laboratory, Aerodynamics Department, *Proceedings of The Royal Society of London*, ser. A, vol. 105, 1924, pp. 582-607.

Equation (3) has four roots  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ , and  $\lambda_4$  which are described as four transverse buckling wave numbers. These roots can be obtained for a given critical shear force buckling load per unit length  $N_{xy}^o$  and the longitudinal buckling mode wave number  $k$ . Southwell and Skan<sup>5</sup> determined the exact solution for the out-of-plane displacement of a flat elastic strip under uniform shear applied along its edges of infinite length. Solutions were obtained for two types of boundary conditions, clamped and simply-supported.

The complex form of the general solution for out of plane displacement  $w(x,y)$  is:

$$w(x,y) = [C_1 e^{i\lambda_1 y/b} + C_2 e^{i\lambda_2 y/b} + C_3 e^{i\lambda_3 y/b} + C_4 e^{i\lambda_4 y/b}] e^{ikx/b} \quad (4)$$

Substitution of the general solution (4) into four homogeneous boundary conditions leads to a determinantal equation from which the critical values of  $N_{xy}^o$  may be found. The four boundary condition equations below correspond to the conditions for clamped edges which most closely approximate the model for a longitudinally stiffened hull plate.

$$w(y=0) = 0 \quad w(y=b) = 0 \quad w'(y=0) = 0 \quad w'(y=b) = 0$$

Southwell and Skan carried out such an analysis and determined a result for the critical buckling load in shear which is:

$$N_{xy}^o = K_s \frac{\pi^2 D}{b^2} \quad (5)$$

where  $K_s$  is defined as the critical shear stress buckling coefficient. The critical shear buckling coefficients for an infinitely long flat plate are 5.35 for simply-supported edges and 8.98 for clamped edges. Since the bottom hull plating on a longitudinally framed ship extends for several hundred feet and is welded to the longitudinals, we can assume that the inter-longitudinal plating spans are clamped rectangular plates of approximate infinite length for which  $K_s = 8.98$ . The critical shear stress is obtained by substituting into Equation (5) the expression for the flexural plate rigidity and dividing by the thickness  $h$ .

$$\sigma_{cr} = \frac{N_{xy}^o}{h} = \frac{8.98 \pi^2 E h^2}{12(1-v^2)b^2} \quad (6)$$

Southwell and Skan also determined the real form of the general solution for the deformed shape  $w(x,y)$ .

$$w(x,y) = C \left[ \cos\left(\frac{kx}{b} + (\alpha+\beta)\frac{y}{b}\right) + K \cos\left(\frac{kx}{b} + \frac{(\alpha-\beta)y}{b}\right) + L \sinh\left(\frac{\delta y}{b}\right) \sin\left(\frac{Kx}{b} - \frac{\alpha y}{b}\right) + M \cosh\left(\frac{\delta y}{b}\right) \cos\left(\frac{Kx}{b} - \frac{\alpha y}{b}\right) \right] \quad (7)$$

Reference 5 gives the values for the constant coefficients  $K$ ,  $L$ , and  $M$  as well as the buckling parameters  $k$ ,  $\beta$ , and  $\alpha$ , which depend upon the boundary conditions. For clamped edges these constants are:

$$\begin{aligned} K &= 0.6728 & L &= -0.0131 & M &= -0.0008925 & k &= 2\pi/3.2 \\ \alpha &= -1.977 & \beta &= 1.804 & \delta &= 4.334 \end{aligned}$$

A surface plot of the out-of-plane displacement  $w(x,y)$ , given by Equation (7) for clamped edges, is shown in Figure 1.

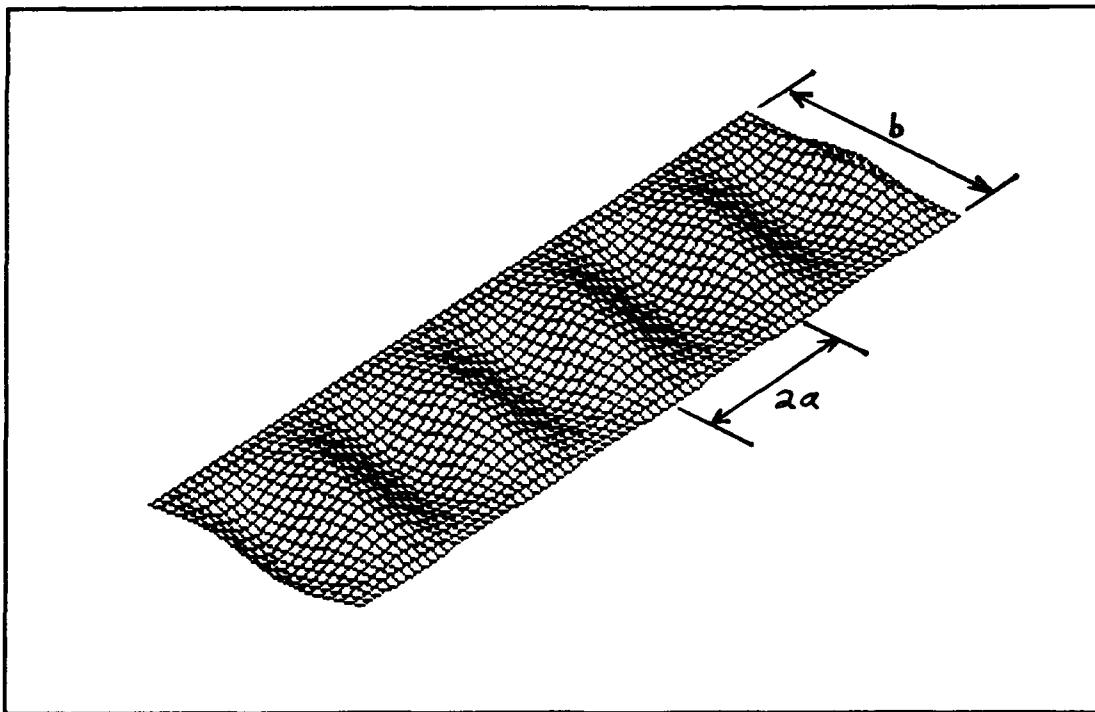


Figure 1: Plot of elastic shear buckling solution  $w(x,y)$

The longitudinal buckling wavelength  $2a$  shown in Figure 1 has also been determined by Southwell and Skan for both clamped and simply-supported boundary conditions. For a dimensionless wavelength defined by  $\Lambda \equiv 2a/b$  :

$$\Lambda = 2.67 \quad \text{--- for simply supported boundaries}$$

$$\Lambda = 1.60 \quad \text{--- for clamped boundaries}$$

After publication of the Southwell and Skan solution, additional analyses were conducted by structural engineers in the 1930s through the 1970s to determine the critical shear stress buckling coefficient  $K_s$  for plates of finite length with various degrees of restraint applied to their edges. In 1936 Timoshenko<sup>6</sup> solved for an approximate critical shear stress buckling coefficient for buckling of a finite length rectangular flat plate with simply-supported edges. Two years later Iguchi<sup>7</sup> presented an analysis of a flat rectangular plate with three different edge conditions.

- (1) All edges rigidly clamped
- (2) Long edges clamped, short edges simply-supported
- (3) Long edges simply-supported, short edges clamped

Up to this point, the solutions obtained by Southwell and Skan, Timoshenko, and Iguchi referred to the buckling mode shape as symmetric due to the symmetry of the mode shape with respect to a diagonal across the plate at the node line slope. In 1947 Stein and Neff<sup>8</sup> examined both a symmetric and an asymmetric buckling mode for a rectangular plate with all edges simply-supported and in 1948 Budiansky and Connor<sup>9</sup> investigated symmetric and asymmetric buckling modes for a fully clamped rectangular plate. Gerard and Becker<sup>10</sup> presented a comprehensive review of this work in 1957 showing that the

---

<sup>6</sup> Timoshenko, S. P., *Theory of elastic stability*, McGraw Hill, New York and London, First edition, 1936, pp 357-363.

<sup>7</sup> Iguchi, S., "Buckling of rectangular plates under shear stresses", *Ingenier-Archiv*, Part IX, p 1, 1938.

<sup>8</sup> Stein, M., Neff, J., "Buckling stress of simply-supported rectangular flat plates in shear", *NACA TN 1222*, 1947.

<sup>9</sup> Budiansky, B., Connor, R.W., "Buckling stresses of clamped rectangular flat plates in shear", *NACA TN 1559*, 1948.

<sup>10</sup> Gerard, G., Becker, H., "Handbook of structural stability, Part 1-Buckling of flat plates", *NACA TN 3781*, 1957.

asymmetric buckling mode has a lower  $K_s$  than the symmetric mode for a small range of plate aspect ratios. Gerard and Becker also presented a series of plots and tables from which  $K_s$  could be calculated for an intermediate level of edge restraint between the simply-supported and clamped conditions.

In 1963 Cook and Rockey<sup>11</sup> presented a solution for the in-plane shear buckling of a rectangular plate which is clamped along one edge and simply-supported on the other three edges. They also refined the solution for the case of two opposite edges simply-supported and the other edges clamped for various plate aspect ratios. A variational approach was utilized with the following approximation for the plate out-of-plane displacement  $w(x,y)$ .

$$w(x,y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cos\left(\frac{m\pi x}{d}\right) \sin\left(\frac{n\pi y}{b}\right)$$

The constants  $d$  and  $b$  are the respective length and width dimensions of the rectangular plate. The accuracy of this Fourier series solution was enhanced by carrying out the calculations using up to 45 coefficients "a<sub>mn</sub>".

Additional research conducted in the 1970s by Rockey<sup>12,13,14</sup> focused on the ultimate load plastic behavior of plate girders loaded in shear and developed a design method for predicting the collapse behavior of plate girders. Although the girder problem differs somewhat from the hull plate shear buckling problem in the application of the external load, the web response to the shear load near the ends of a simply-supported girder is similar to that of a flat rectangular plate to in-plane shear applied along its clamped edges. This is because the bending stresses induced within the web are small compared to the shear stresses close to the ends of the girder. Figure 2 illustrates, however, that the shear stress

---

<sup>11</sup> Cook, I.T., Rockey, K.C., "Shear buckling of rectangular plates with mixed boundary conditions", *The Aeronautical Quarterly*, vol. XIV, March 1963, pp 349-356.

<sup>12</sup> Rockey, K.C., Skaloud, M., "The ultimate load behavior of plate girders loaded in shear", *The Structural Engineer*, vol. 50, no. 1, Jan 1972, pp 29-48.

<sup>13</sup> Rockey, K.C., Evans, H.R., Porter, D.M., "The ultimate load behavior of longitudinally reinforced plate girders", structural analysis, non-linear behavior and techniques, *Transport and Road Research Lab Report SR 164UC*, Crowthorne 1975, pp 162-174.

<sup>14</sup> Rockey, K.C., Evans, H.R., Porter, D.M., "A design method for predicting the collapse behavior of plate girders", *Proc. Instn. Civ. Engrs.*, part 2, vol. 65, Mar 1978, pp 85-112.

through the depth of the web is not uniform as is the case in our problem. The flanges of the symmetric girder shown in Figure 2 act to restrain the web from buckling in a manner similar to the longitudinal stiffeners of a flat plate.

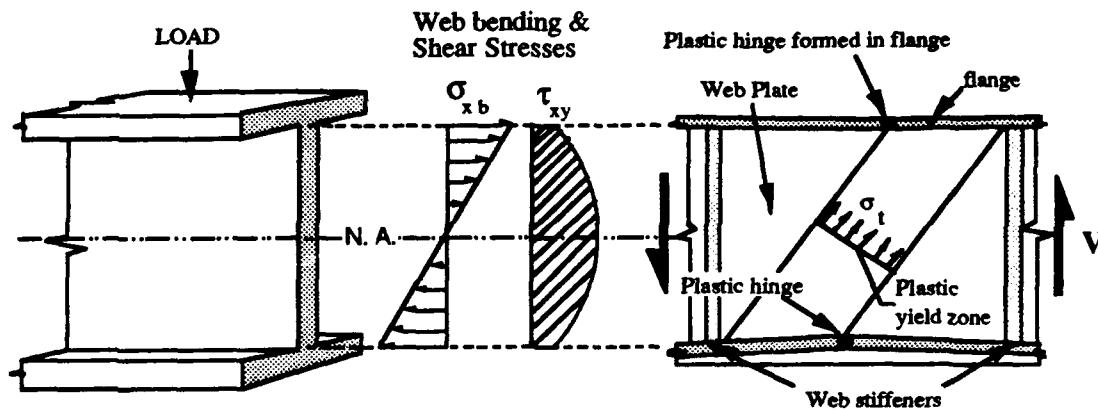


Figure 2: Failure Model for Symmetrical Girder Loaded in Shear

The plate girder experiments and analyses conducted by Rockey, Evans, Porter, and Skaloud give some insight into the pure shear problem by illustrating with numerous photographs and contour plots how buckling of the web is induced. The web buckling mode which develops when the flanges are strong enough to resist plastic hinge formation resembles the pure shear problem although the mode shape is strongly affected by the placement of web stiffeners unlike the hull plate of a longitudinally framed ship.

The research conducted after Southwell and Skan's initial solution of the infinite plate and discussed up to this point has focused on developing an accurate value for the critical shear stress buckling coefficient  $K_s$  for plates of finite length. As discussed earlier it is proposed that for a longitudinally framed ship hull the plate length may be assumed to be infinite. Therefore, the Southwell and Skan solution for a clamped plate of infinite length is sufficient to provide the insight necessary for this hull plate buckling analysis.

### 3. Formulation of Analytical Models

#### 3.1 Plastic Buckling Model

The derivation of an analytical model for plastic shear buckling is based upon the principle that the rate of external work is equal to the rate of internal energy dissipation  $\dot{E}_{\text{internal}} = \dot{E}_{\text{external}}$  for a stress field satisfying equilibrium and a strain field derived from a continuous displacement field. The rate of external work applied per unit length of the plate is equal to the shearing force per unit length multiplied by the translation velocity  $\dot{s}$  of one plate edge relative to the other.

$$\dot{E}_{\text{external}} = F_s \dot{s}$$

The rate of internal plastic work absorbed within the plate can be expressed in terms of the internal stress and strain fields integrated over the volume.

$$\dot{E}_{\text{int}} = \int_V \sigma_{\alpha\beta} \dot{\epsilon}_{\alpha\beta} dV$$

$$\dot{E}_{\text{int}} = \int_S \left[ \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \dot{\epsilon}_{\alpha\beta} dz \right] dS$$

The Love-Kirchhoff hypothesis can be applied to this problem, giving a linear strain distribution through the thickness of the plate that depends on the curvature.

$$\epsilon_{\alpha\beta} = \overset{\circ}{\epsilon}_{\alpha\beta} + z K_{\alpha\beta}$$

The strains  $\overset{\circ}{\epsilon}_{\alpha\beta}$  and curvatures  $K_{\alpha\beta}$  correspond to mean values through the thickness of the material. The strain rate becomes:

$$\dot{\epsilon}_{\alpha\beta} = \overset{\circ}{\epsilon}_{\alpha\beta} + z \dot{K}_{\alpha\beta}$$

Therefore, the rate of internal plastic work can be expressed as:

$$\dot{E}_{int} = \int_S \left[ \int_{-h/2}^{h/2} \sigma_{\alpha\beta} (\dot{\varepsilon}_{\alpha\beta}^o + z \dot{K}_{\alpha\beta}) dz \right] dS$$

Since  $\dot{\varepsilon}_{\alpha\beta}^o$  and  $\dot{K}_{\alpha\beta}$  are constant through the thickness of the material they can be brought outside the integral.

$$\dot{E}_{int} = \int_S \left[ \dot{\varepsilon}_{\alpha\beta}^o \int_{-h/2}^{h/2} \sigma_{\alpha\beta} dz + \dot{K}_{\alpha\beta} \int_{-h/2}^{h/2} \sigma_{\alpha\beta} z dz \right] dS$$

The internal plastic work can now be expressed in terms of the membrane forces and bending moments.

$$N_{\alpha\beta} \equiv \int_{-h/2}^{h/2} \sigma_{\alpha\beta} dz \quad M_{\alpha\beta} \equiv \int_{-h/2}^{h/2} \sigma_{\alpha\beta} z dz$$

$$\dot{E}_{int} = \int_S [\dot{\varepsilon}_{\alpha\beta}^o N_{\alpha\beta} + \dot{K}_{\alpha\beta} M_{\alpha\beta}] dS$$

The rate of plastic work for the prismatic buckling element can be written in an expanded form in terms of in-plane mean strain rates ( $\dot{\varepsilon}_{xx}$ ,  $\dot{\varepsilon}_{xy}$ ,  $\dot{\varepsilon}_{yy}$ ) and mean curvature rates ( $\dot{K}_{xx}$ ,  $\dot{K}_{xy}$ ,  $\dot{K}_{yy}$ ). The corresponding generalized stresses are membrane forces ( $N_{xx}$ ,  $N_{xy}$ ,  $N_{yy}$ ) and bending moments ( $M_{xx}$ ,  $M_{xy}$ ,  $M_{yy}$ ). For an element with plastic hinges the rate of internal plastic work is:

$$\begin{aligned} \dot{E}_{internal} = & \int_S (N_{xx}\dot{\varepsilon}_{xx} + 2N_{xy}\dot{\varepsilon}_{xy} + N_{yy}\dot{\varepsilon}_{yy}) dS + \\ & \int_S M_{xx}\dot{K}_{xx} + 2M_{xy}\dot{K}_{xy} + M_{yy}\dot{K}_{yy} + \sum_i M_o^{(i)} \theta^{(i)} L^{(i)} \end{aligned} \quad (8)$$

The last term in Equation (8) corresponds to the energy dissipation in the buckling element plastic hinges where  $M_o = \frac{\sigma_{ot}^2}{4}$  is the fully plastic bending moment,  $\sigma_o$  is the constant flow strength, and  $\dot{\theta}^{(i)}$  is the rate of rotation at the  $i^{\text{th}}$  hinge line of length  $L^{(i)}$ . The plastic hinge moment  $M_o$  is actually an approximation for the von Mises plastic hinge moment assuming plane strain.

$$M_o = \frac{2 \sigma_{ot}^2}{\sqrt{3} 4}$$

The large out-of-plane distortions associated with buckling in the plastic region make it difficult to determine an exact solution for the continuous, non-linear distribution of strains within the plate. Therefore, certain assumptions are made to simplify the geometry which make it possible to formulate an approximate solution for the plate resistance in shear during plastic buckling.

### 3.1.1 Geometry of Buckling Model

To formulate a geometry for plastic shear buckling of the plate, a simplified buckling shape is assumed based on intuition provided from the elastic buckling solution and from experiments. Prismatic buckling elements are chosen to replace the elastic elements corresponding to one half of the longitudinal buckling wavelength  $2a$  as shown in Figure 3.

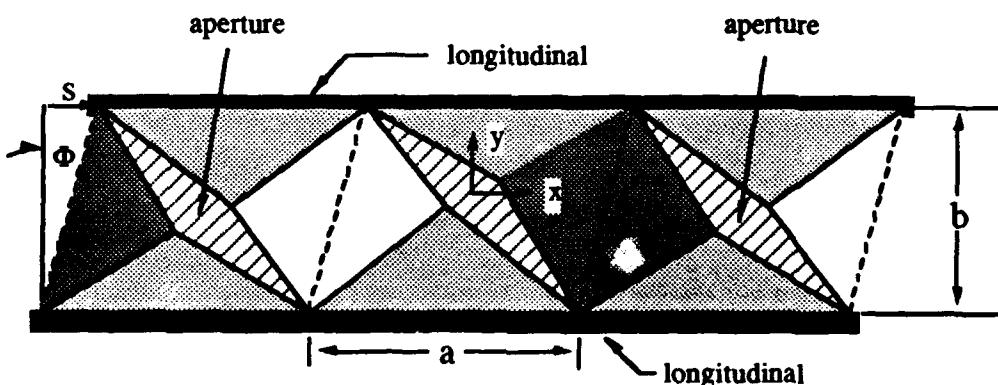


Figure 3: Prismatic buckling element model

The crossed hatched areas shown in Figure 3 indicate apertures, which do not really exist but are used to conceptually visualize the membrane stretching and shearing which must occur as the element buckles out-of-plane. The derivation of this analytical model is

based upon the geometry of a single buckling element of width  $b$  and length  $a$  and an initial angular orientation  $\beta$  relative to the transverse direction. The half wavelength "a" must be chosen later after analysis of both experimental and analytical buckling results. This buckling element, which is the foundation for the analytical model, is shown in Figures 4A and 4B.

The formulation for the plastic work due to element shear assumes that the out-of-plane deformations are small enough so that the relative displacements between the opposing edges of the aperture can be projected onto a horizontal plane and still accurately represent the actual relative displacements which induce shear in the element. In other words, the three dimensional nature of the aperture is not considered so that displacements which induce out-of-plane bending and torsion across the aperture can be neglected.

To obtain the deformed geometry in Figure 4B assume that the four triangular sub-elements which comprise a single buckling element retain their original shape which opens an aperture along the compressed diagonal. The entire element undergoes in-plane rotation by an angle  $\Phi$  and the flat sub-elements undergo out-of-plane rotations  $\theta_1$  and  $\theta_3$  along their respective longitudinal and transverse edges. This hypothetical element with the aperture can absorb plastic work only within its plastic hinges. Then assume that to account for membrane and shear strains within the element the aperture does not exist resulting in membrane and shear strains being induced within the element which can be measured from the displacement discontinuities which exist across the aperture.

The geometric construction of this deformed geometry for the aperture rests on maintaining the bisector angles  $\alpha_1$  and  $\alpha_2$ , shown in both Figures 4A and B, constant while translating the upper longitudinal edge a distance  $s$  relative to the lower longitudinal edge. The angles  $\alpha_1$  and  $\alpha_2$  are equal to  $(\pi/2 - \beta)$ . The points  $P_1$  and  $P_2$  are the apex points of the aperture and also exhibit the maximum out-of-plane displacement in the deformed geometry. These points diverge from the centroid of the undeformed element during plastic shear deformation forming the aperture. Later on it will be shown that the trajectories of these "apex" points during progressive buckling must be calculated so that the total shear strain and membrane strain within the element can be determined. These trajectories are equal but opposite in sign and are shown as the vectors  $+u_0$  and  $-u_0$  in Figure 4.

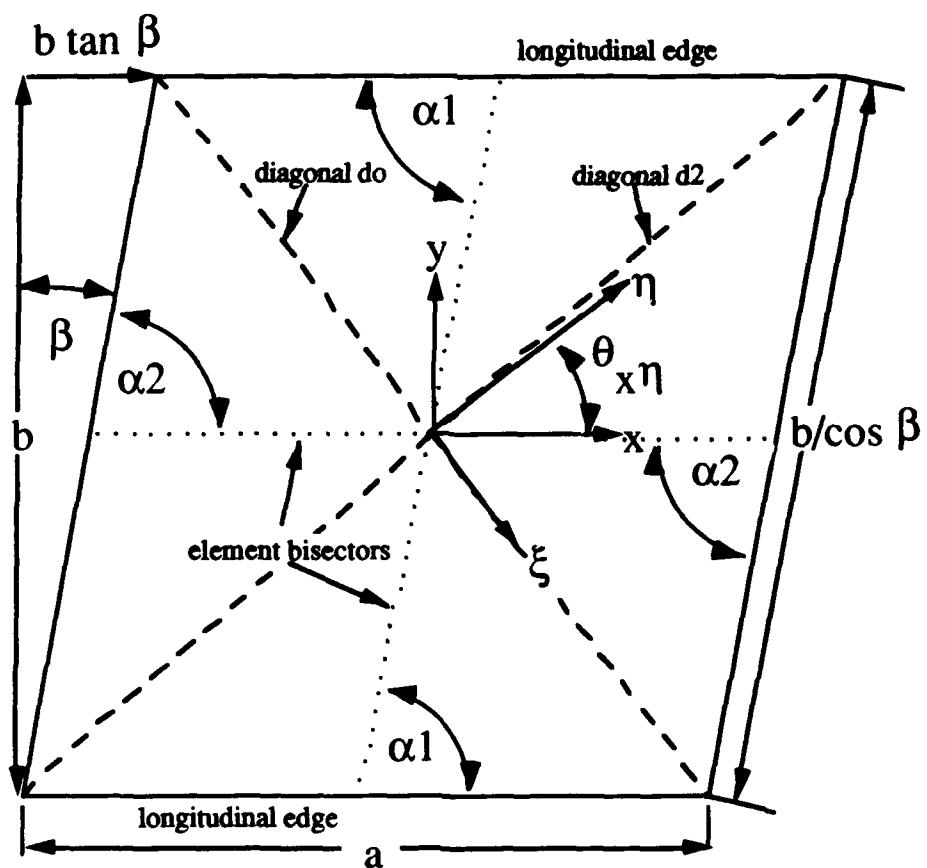


Figure 4A: Prismatic buckling element undeformed geometry

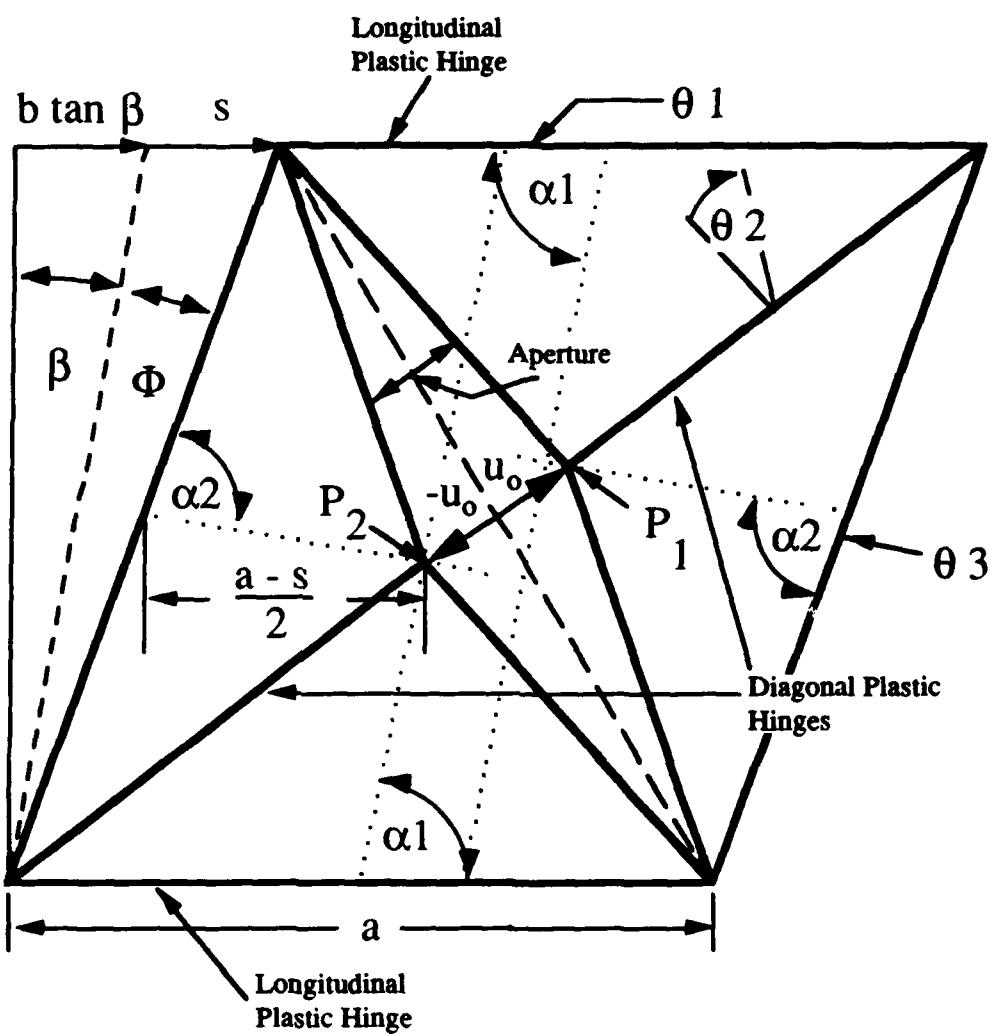


Figure 4B: Prismatic buckling element deformed geometry

Complete definitions for all parameters shown in Figures 4A and 4B are given below.

### Definitions

- a ----- one half of the longitudinal buckling wavelength
- b ----- width of plate element corresponding to longitudinal spacing
- ( $\alpha_1, \alpha_2$ ) -- element bisector angles equal to ( $\pi/2 - \beta$ )
- $\beta$  ----- initial angular orientation in radians of buckling element with respect to transverse direction
- s ----- relative translation in the longitudinal direction between element longitudinal edges
- $\Phi$  ----- element shear rotation angle, from Figure 3 it can be seen that:

$$\Phi = \tan^{-1} \left[ \frac{s \cos(\beta)}{b/\cos(\beta)} \right] = \tan^{-1} \left[ \frac{s}{b} \cos 2(\beta) \right]$$

- $\pm\theta_1$  ----- out-of-plane rotation about longitudinal axis for longitudinal edge plastic hinges
- $\pm\theta_2$  ----- out-of-plane angular rotation for element diagonal plastic hinges
- $\pm\theta_3$  ----- out-of-plane angular rotation for element transverse edges
- (x, y) ---- Cartesian coordinates with origin at element centroid and aligned with element longitudinal and transverse directions
- ( $\xi, \eta$ ) ---- local coordinates with origin at element centroid and aligned parallel and normal to aperture center axis projected onto the horizontal plane
- ( $u_o, -u_o$ ) --- displacement vectors which define the respective trajectories of apex points P1 and P2 in the horizontal plane, these vectors can be decomposed into two components in either the Cartesian or local coordinate systems,  $u_o = (u_x, u_y)$  or  $(u_\xi, u_\eta)$
- $\theta_{x\eta}$  ----- angle measured between x and  $\eta$  coordinate axes, from Figure 3:

$$\theta_{x\eta} = \tan^{-1} \left( \frac{a - b \tan \beta - s}{b} \right)$$

Other variables that are not indicated in Figures 4A and B that will be used in our formulation include:

- $\lambda$  ----- longitudinal non-dimensional buckling half-wavelength ( $\lambda = a/b$ )
- $\gamma_{\xi\eta}$  ----- shear strain relative to local coordinates
- $\tau_{\xi\eta}$  ----- shear stress relative to local coordinates

$t$  ----- plate thickness  
 $S$  ----- surface area of element

$A_o$  ----- maximum amplitude of aperture or distance between points  $P_1$  and  $P_2$ ,  
from Figures 4A and B:

$$A_o = |2 u_o|$$

$w_o$  ----- maximum out-of-plane displacement of plate at apex points  $P_1$  and  $P_2$

$\epsilon_\eta$  ----- strain along the  $\eta$  axis

$M_o$  ----- plastic hinge moment based on von Mises criterion

$$M_o = \frac{2 \sigma_o t^2}{\sqrt{3} 4} \quad \text{approximated by} \quad M_o = \frac{\sigma_o t^2}{4}$$

NDSF -- non-dimensional or "normalized" shear force obtained by dividing the shear force on an element of specified length by  $M_o$

### 3.1.2 Shear Strain Plastic Work

The shear strain in the buckled element has been approximated by measurement of a shear discontinuity across the aperture of the element. The relative motion in the  $\xi$  direction between the opposing edges of the aperture is utilized to measure a shear strain across the aperture which in turn can be used to approximate the shear work across the aperture that is actually distributed in a continuous fashion throughout the element.

In general, membrane deformations of thin plates involve both stretching and shear. The relative motion in the  $\xi$  direction between the opposing triangular sub-element edges which form the aperture boundaries is utilized to visualize and quantify the shear strains present within the buckled element for which there is no aperture. For the buckled element which has no aperture this motion can not occur thus inducing shear strains across the element aperture axis  $\xi$ . The first step in this formulation is to check whether or not there are any initial element orientations  $\beta$  such that the shear strain is zero after deformation occurs. This process will allow us to better understand the mechanism by which shear strains develop in the prismatic element.

For the shear strain to be zero, the aperture apex points  $P_1$  and  $P_2$  must move along a trajectory that is at all times normal to the local aperture center axis  $\xi$ . This means that

only membrane stretching must occur for the aperture edges to remain connected. For this to be true  $\theta_{x\eta}$  must equal the arctangent of  $(u_y/u_x)$ .

$$\theta_{x\eta} = \tan^{-1}\left(\frac{a - b \tan \beta - s}{b}\right) = \tan^{-1}\left(\frac{u_y}{u_x}\right) \quad (9)$$

The displacements  $u_x$  and  $u_y$  can be obtained from Figure 4B:

$$u_x = \frac{s}{2}$$

$$u_y = \left(\frac{a-s}{2}\right) \tan \Phi = \left(\frac{a-s}{2}\right) \left(\frac{s}{b}\right) \cos^2 \beta$$

The following relationship for zero shear can be obtained by substituting for  $u_x$  and  $u_y$  into Equation (9).

$$\frac{a - b \tan \beta - s}{b} = \frac{\left(\frac{a-s}{2}\right)\left(\frac{s}{b}\right) \cos^2 \beta}{s/2} = \left(\frac{a-s}{b}\right) \cos^2 \beta$$

Algebraic simplification of this no-shear condition yields:

$$(a - b \tan \beta - s) = (a - s) \cos^2 \beta \quad (10)$$

In this formulation,  $\beta$  can assume any value between 0 and  $\pi/2$ . Since the no shear condition, Equation (10), is only satisfied if  $\beta = 0$ , shear exists within the aperture for all  $(0 < \beta < \pi/2)$ .

An approximate shear strain  $\gamma_{\xi\eta}$  can be defined as the angular difference between the opposing trajectories of points  $P_1$  and  $P_2$  shown in Figure 4B and the axis  $\eta$  which is normal to the aperture centerline.

$$\gamma_{\xi\eta} \equiv \left[ \tan^{-1}\left(\frac{u_y}{u_x}\right) - \theta_{x\eta} \right]$$

Substituting for  $u_x$  and  $u_y$  gives:

$$\gamma_{\xi\eta} = \left[ \tan^{-1} \left( \frac{(a - s) \cos^2 \beta}{b} \right) - \tan^{-1} \left( \frac{a - b \tan \beta - s}{b} \right) \right]$$

In terms of the non-dimensional half-wavelength  $\lambda$  and element shear rotation  $\Phi$ , the shear strain is:

$$\gamma_{\xi\eta} = \left[ \tan^{-1} (\lambda \cos^2 \beta - \tan \Phi) - \tan^{-1} \left( \lambda - \tan \beta - \frac{\tan \Phi}{\cos^2 \beta} \right) \right] \quad (11)$$

The expression for total shear strain plastic work  $E_s$  within the element is given by:

$$E_s = t \int_S \tau_{\xi\eta} \gamma_{\xi\eta} dS$$

If the additional assumption is made that the plate has isotropic material properties it can be stated that for plastic behavior:

$$\tau_{\xi\eta} = \frac{\sigma_0}{\sqrt{3}}$$

The absolute value of  $\gamma_{\xi\eta}$  must be taken when making the above substitution for  $\tau_{\xi\eta}$  to keep the contribution to the total work positive. The shear strain  $\gamma_{\xi\eta}$  can be taken outside the integral since it is assumed to be constant across the element aperture.

$$E_s = \frac{\sigma_0 t}{\sqrt{3}} |\gamma_{\xi\eta}| \int_S dS$$

To complete the derivation, the surface area integral must be calculated. This surface integral is the area of the aperture gap opening which is the length of diagonal  $d_0$  shown in Figure 4A multiplied by the average gap width  $A_0/2$ .

$$\int_S dS = \frac{A_0 \sqrt{(a - b \tan \beta)^2 + b^2}}{2}$$

Therefore, the shear strain plastic work is:

$$E_s = \frac{\sigma_0 t A_0}{2\sqrt{3}} |\gamma_{\xi n}| \sqrt{(a - b \tan\beta)^2 + b^2} \quad (12)$$

In terms of  $\lambda$  and  $\Phi$ , Equation (12) becomes:

$$E_s = \frac{\sigma_0 t b A_0}{2\sqrt{3}} |\gamma_{\xi n}| \sqrt{(\lambda - \tan\beta)^2 + 1} \quad (13)$$

The only unknown in Equation (13) is  $A_0$  which can easily be expressed in terms of  $u_x$  and  $u_y$  as:

$$A_0 = 2 \sqrt{(u_x)^2 + (u_y)^2}$$

$$A_0 = 2 \sqrt{\left(\frac{s}{2}\right)^2 + \left(\frac{a-s}{2}\right)^2 \left(\frac{s}{b} \cos^2\beta\right)^2}$$

$$A_0 = 2 \sqrt{\left(\frac{s}{2}\right)^2 + \left(\frac{(as-s^2)}{2b} \cos^2\beta\right)^2}$$

In terms of  $\lambda$  and  $\Phi$ ,  $A_0$  becomes:

$$A_0 = b \sqrt{\left(\frac{\tan\Phi}{\cos^2\beta}\right)^2 + \left[\left(\lambda \frac{\tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta}\right) \cos^2\beta\right]^2}$$

$$A_0 = \frac{b \tan\Phi}{\cos^2\beta} \sqrt{1 + (\lambda \cos^2\beta - \tan\Phi)^2} \quad (14)$$

At this point the plastic work on a single element due to shear strains measured across a closed aperture has been calculated in terms of constant parameters and the non-dimensional variables  $\lambda$  and  $\Phi$ . Returning to Equation (13) the rate of shear strain plastic work can be expressed as:

$$\dot{E}_s = \left( \frac{\sigma_o t b}{2\sqrt{3}} \right) \sqrt{(\lambda - \tan\beta)^2 + 1} \frac{d}{dt} [A_o |\dot{\gamma}_{\xi_1}|]$$

$$\dot{E}_s = \left( \frac{\sigma_o t b}{2\sqrt{3}} \right) \sqrt{(\lambda - \tan\beta)^2 + 1} [A_o |\dot{\gamma}_{\xi_1}| + A_o |\dot{\gamma}_{\xi_1}|]$$

The component of normalized shear force per unit length in the longitudinal direction due to aperture shear strain is obtained by dividing  $\dot{E}_s$  by the shear velocity  $s$ , the plastic hinge moment  $M_o$ , and the element length  $a$ , substituting the above and rearranging:

$$\frac{NDSF_s}{\text{unit length}} \equiv \frac{\dot{E}_s}{s M_o a} = \frac{\dot{E}_s \cos^2\beta \cos^2\Phi}{\Phi b M_o a}$$

$$\frac{NDSF_s}{\text{unit length}} = \left[ \frac{\sigma_o t b \sqrt{(\lambda - \tan\beta)^2 + 1} \cos^2\beta \cos^2\Phi}{2\sqrt{3} \Phi b \left( \frac{\sigma_o t^2}{4} \right) a} \right] [A_o |\dot{\gamma}_{\xi_1}| + |\dot{\gamma}_{\xi_1}| \dot{A}_o]$$

$$\frac{NDSF_s}{\text{unit length}} = \left( \frac{2 \cos^2\beta \cos^2\Phi}{\sqrt{3} b t \lambda \Phi} \right) \sqrt{(\lambda - \tan\beta)^2 + 1} [A_o |\dot{\gamma}_{\xi_1}| + |\dot{\gamma}_{\xi_1}| \dot{A}_o]$$

$$\frac{NDSF_s}{\text{unit length}} = \frac{2 \cos^2\beta \cos^2\Phi}{\sqrt{3} t} \sqrt{\left( 1 - \frac{\tan\beta}{\lambda} \right)^2 + \frac{1}{\lambda^2}} \left[ \left( \frac{A_o}{b} \right) \frac{|\dot{\gamma}_{\xi_1}|}{\Phi} + |\dot{\gamma}_{\xi_1}| \left( \frac{\dot{A}_o}{\Phi b} \right) \right] \quad (15)$$

Equation (15) expresses the shear strain component of the non-dimensional shear force per unit length in a form suitable for calculations using a spreadsheet. In order to perform

these calculations, the three non-dimensional parameters inside the brackets of Equation (15) must be determined by separate calculations. These parameters are defined as:

(1)  $\frac{A_o}{b}$  --- ratio of maximum aperture opening to plate width

(2)  $\frac{|\dot{\gamma}_{\xi\eta}|}{\Phi}$  --- ratio of shear strain rate to shear rotation rate

(3)  $\frac{\dot{A}_o}{\Phi b}$  --- ratio of aperture opening velocity to the shear translation velocity difference between plate longitudinal edges at  $\beta = 0$  and small  $\Phi$

From Equation (14):

$$\frac{A_o}{b} = \frac{\tan\Phi}{\cos^2\beta} \sqrt{1 + (\lambda \cos^2\beta - \tan\Phi)^2} \quad (16)$$

The time rate of change of the maximum aperture gap (14) is:

$$\dot{A}_o = \frac{d}{dt}(A_o) = \frac{b \dot{\Phi}}{\cos^2\beta \cos^2\Phi} \left[ \frac{2 \tan^2\Phi - 3\lambda \tan\Phi \cos^2\beta + \lambda^2 \cos^4\beta + 1}{\sqrt{\tan^2\Phi - 2\lambda \tan\Phi \cos^2\beta + \lambda^2 \cos^4\beta + 1}} \right]$$

Therefore:

$$\frac{\dot{A}_o}{\Phi b} = \frac{1}{\cos^2\beta \cos^2\Phi} \left[ \frac{2 \tan^2\Phi - 3\lambda \tan\Phi \cos^2\beta + \lambda^2 \cos^4\beta + 1}{\sqrt{\tan^2\Phi - 2\lambda \tan\Phi \cos^2\beta + \lambda^2 \cos^4\beta + 1}} \right] \quad (17)$$

Equation (11) gives:

$$|\dot{\gamma}_{\xi\eta}| = \left| \frac{d}{dt} \dot{\gamma}_{\xi\eta} \right| = \Phi \sec^2\Phi \left| \frac{1}{\cos^2\beta \left[ 1 + \left( \lambda - \tan\beta - \frac{\tan\Phi}{\cos^2\beta} \right)^2 \right]} - \frac{1}{1 + (\lambda \cos^2\beta - \tan\Phi)^2} \right|$$

$$\frac{|\dot{\gamma}_{\xi\eta}|}{\Phi} = \sec^2 \Phi \left| \frac{1}{\cos^2 \beta \left[ 1 + \left( \lambda - \tan \beta - \frac{\tan \Phi}{\cos^2 \beta} \right)^2 \right]} - \frac{1}{1 + (\lambda \cos^2 \beta - \tan \Phi)^2} \right| \quad (18)$$

Equations (15), (16), (17), and (18) are utilized to calculate the shear strain component of the normalized shear force on a spreadsheet. The results of these calculations are presented in Chapter 5 and the calculations themselves are included in Appendix B.

### 3.1.3 Membrane Strain Plastic Work

To determine the membrane strain contribution to the internal plastic work, a principal coordinate system ( $\xi, \eta$ ) is chosen such that  $\varepsilon_\eta$  is aligned with the element's stretched diagonal  $d_2$  shown in Figure 4A. The component  $\varepsilon_\xi = 0$  so that the membrane plastic work becomes:

$$\dot{E}_m = \int_S N_o \dot{\varepsilon}_\eta dS$$

To determine  $\varepsilon_\eta$ , the cross section of the element along the stretched diagonal  $d_2$  must be examined as shown in Figure 5.

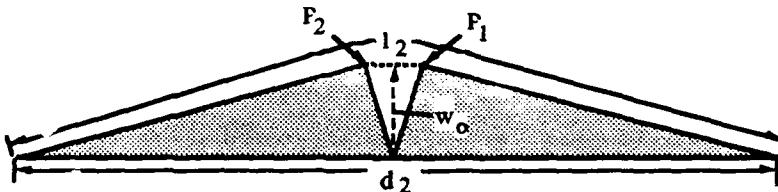


Figure 5 Element stretched diagonal cross section

The membrane strain  $\varepsilon_\eta$  can be determined in terms of the original stretched diagonal length  $d_0$  and the final arc length of the membrane  $l_2$  along the stretched diagonal. Figure 4B gives the dimensions necessary to calculate  $d_0$  and  $d_2$ .

$$d_0^2 = (a + b \tan \beta)^2 + b^2$$

$$d_2^2 = (a + s + b \tan \beta)^2 + b^2$$

The arc length of the out-of-plane membrane  $l_2$  can be determined from Figure 5.

$$\left(\frac{l_2}{2}\right)^2 = w_o^2 + \left(\frac{d_2}{2}\right)^2$$

$$l_2^2 = 4w_o^2 + (a + s + b \tan\beta)^2 + b^2$$

The Green-Lagrange strain then becomes:

$$\epsilon_{\eta} = \frac{1}{2} \left[ \frac{l_2^2 - d_o^2}{d_o^2} \right] = \frac{1}{2} \left[ \frac{4w_o^2 + (a + s + b \tan\beta)^2 - (a + b \tan\beta)^2}{(a + b \tan\beta)^2 + b^2} \right] \quad (19)$$

The terms in the Equation (19) can be expanded and then simplified to obtain the following expression for the membrane strain.

$$\epsilon_{\eta} = \frac{1}{2} \left[ \frac{4w_o^2 + 2as + s^2 + 2bs \tan\beta}{a^2 + 2ab \tan\beta + b^2 \tan^2 \beta + b^2} \right]$$

The time rate of change of the membrane strain becomes:

$$\dot{\epsilon}_{\eta} = \frac{1}{2} \left[ \frac{8w_o \dot{w}_o + 2a\dot{s} + 2s\dot{s} + 2btan\beta \dot{s}}{a^2 + 2ab \tan\beta + b^2 \tan^2 \beta + b^2} \right]$$

The rate of membrane strain plastic work can now be calculated. For a rigid-perfectly plastic material the normal force to the applied stress per unit length  $N_0 = \sigma_0 t$ . An additional observation can be made that at any cross section cut parallel to the stretched diagonal  $d_2$  the triangular sections are all similar triangles. Due to this fact, the ratio of the plate surface arc length to the original flat diagonal length at every cross section is the same. Therefore, the membrane strain is spatially constant throughout the plate and only varies with time. The rate of incremental strain energy absorbed by a small incremental area  $dS$

within the aperture due to membrane stresses is  $d\dot{E}_m = \sigma_o \dot{\varepsilon}_\eta t dS$ . The rate of membrane strain plastic work per unit length of plate is:

$$\dot{E}_m = \int_S N_o \dot{\varepsilon}_\eta dS = t \sigma_o \dot{\varepsilon}_\eta \int_S dS$$

For a per unit length strip of element:  $\int_S dS = 1 * b$  and the rate of membrane plastic work is:

$$\dot{E}_m = t \sigma_o b \dot{\varepsilon}_\eta = \frac{t \sigma_o b}{2} \left[ \frac{8w_o \dot{w}_o + 2a\dot{s} + 2s\dot{s} + 2b \tan\beta \dot{s}}{a^2 + 2ab \tan\beta + b^2 \tan^2\beta + b^2} \right]$$

The per unit length normalized membrane resistance  $NDSF_m/\text{unit length}$  of the buckling element is obtained by dividing  $\dot{E}_m$  by the shear translation velocity  $\dot{s}$  and the plastic hinge moment  $M_o$ .

$$\frac{NDSF_m}{\text{unit length}} = \frac{\dot{E}_m}{\dot{s} M_o} = \frac{\dot{E}_m \cos^2\beta \cos^2\Phi}{\dot{\Phi} b M_o} = \frac{t \sigma_o b \dot{\varepsilon}_\eta \cos^2\beta \cos^2\Phi}{\dot{\Phi} b \left( \frac{\sigma_o t^2}{4} \right)}$$

Substituting for  $\dot{\varepsilon}_\eta$  yields:

$$\frac{NDSF_m}{\text{unit length}} = \frac{2b \cos^2\beta \cos^2\Phi}{t} \left[ \frac{8w_o \left( \frac{\dot{w}_o}{\dot{\Phi} b} \right) + 2a \left( \frac{\dot{s}}{\dot{\Phi} b} \right) + 2s \left( \frac{\dot{s}}{\dot{\Phi} b} \right) + 2b \tan\beta \left( \frac{\dot{s}}{\dot{\Phi} b} \right)}{a^2 + 2ab \tan\beta + b^2 \tan^2\beta + b^2} \right]$$

Noting that  $\dot{s} = \frac{\dot{\Phi} b}{\cos^2\beta \cos^2\Phi}$  and dividing the numerator by  $b$ , the denominator by  $b^2$  and then the entire membrane non-dimensional shear expression by  $b$  yields an expression in terms of the non-dimensional wavelength  $\lambda$ .

$$\frac{\text{NDSF}_m}{\text{unit length}} = \frac{4}{t} \left[ \frac{4 \left( \frac{w_0}{b} \right) \left( \frac{\dot{w}_0}{\Phi b} \right) \cos^2 \beta \cos^2 \Phi + \frac{a}{b} + \frac{s}{b} + \tan \beta}{\frac{a^2}{b^2} + 2 \frac{a}{b} \tan \beta + \tan^2 \beta + 1} \right]$$

$$\frac{\text{NDSF}_m}{\text{unit length}} = \frac{4}{t} \left[ \frac{4 \left( \frac{w_0}{b} \right) \left( \frac{\dot{w}_0}{\Phi b} \right) \cos^2 \beta \cos^2 \Phi + \lambda + \frac{\tan \Phi}{\cos^2 \beta} + \tan \beta}{\lambda^2 + 2 \lambda \tan \beta + \tan^2 \beta + 1} \right] \quad (20)$$

To complete this derivation, two unknowns in Equation (20),  $w_0$  and  $\dot{w}_0$ , must first be determined. The buckled element geometry shown in Figure 6 gives the information necessary to solve for  $w_0$ .

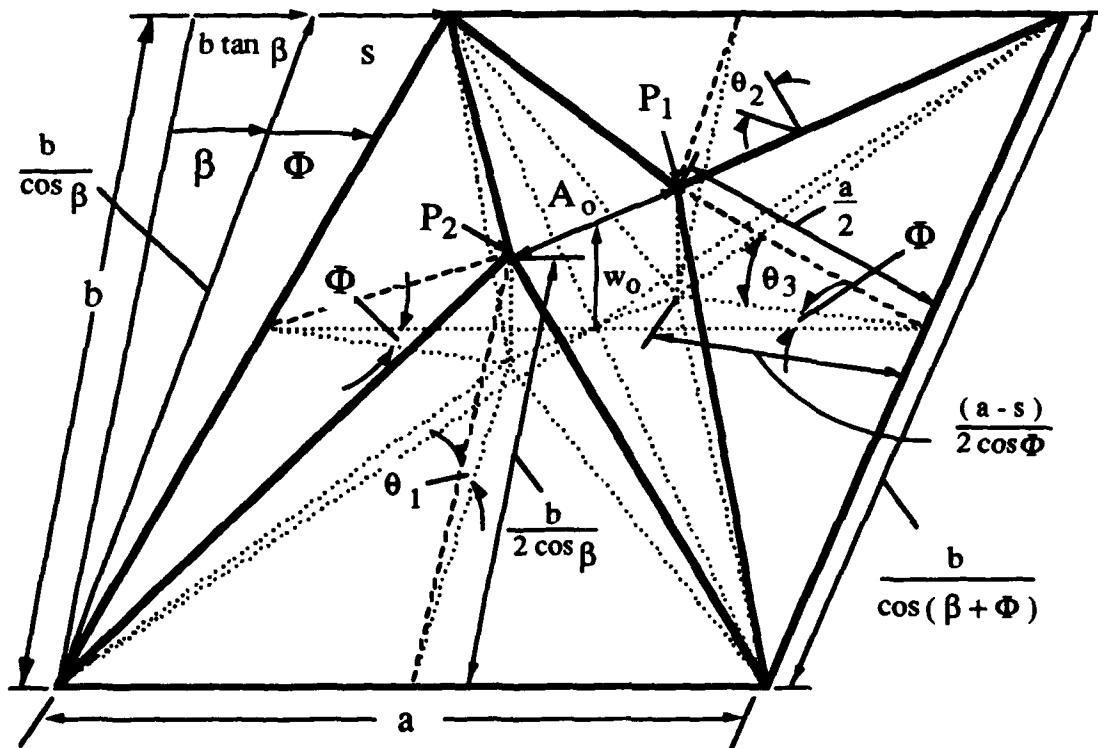


Figure 6: Upward buckled element geometry

From Figure 6:

$$w_o^2 = \left(\frac{a}{2}\right)^2 - \left(\frac{a-s}{2\cos\Phi}\right)^2$$

$$w_o = \frac{1}{2} \sqrt{a^2 - \left(\frac{a-s}{\cos\Phi}\right)^2} = \frac{1}{2} \sqrt{a^2 - \left(\frac{a^2 - 2sa + s^2}{\cos^2\Phi}\right)}$$

Multiplying and dividing by  $b$  permits the following expressions for  $w_o$  and  $\dot{w}_o$  to be obtained in terms of  $\lambda$  and  $\Phi$ .

$$w_o = \frac{b}{2} \sqrt{\lambda^2 + \left( \frac{\frac{2\lambda \tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta} - \lambda^2}{\cos^2\Phi} \right)} \quad (21)$$

$$\dot{w}_o = \frac{b\Phi}{4} \left[ \frac{\left( \frac{2\lambda}{\cos^2\beta} - \frac{2\tan\Phi}{\cos^4\beta} \right) + \left( \frac{2\lambda \tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta} - \lambda^2 \right) \sin(2\Phi)}{\sqrt{\lambda^2 \cos^8\Phi + \left( \frac{2\lambda \tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta} - \lambda^2 \right) \cos^6\Phi}} \right] \quad (22)$$

Equation (20) introduces two new non-dimensional parameters for the determination of the non-dimensional membrane resistance.

(1)  $\frac{w_o}{b}$  --- ratio of maximum out-of-plane displacement to plate width

(2)  $\frac{\dot{w}_o}{\Phi b}$  --- normalized out-of-plane displacement velocity

These non-dimensional parameters are easily calculated from Equations (21) and (22).

$$\frac{w_o}{b} = \frac{1}{2} \sqrt{\lambda^2 + \left( \frac{\frac{2\lambda \tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta} - \lambda^2}{\cos^2\Phi} \right)} \quad (23)$$

$$\frac{\dot{w}_o}{\Phi b} = \frac{1}{4} \left[ \frac{\left( \frac{2\lambda}{\cos^2 \beta} - \frac{2\tan \Phi}{\cos^4 \beta} \right) + \left( \frac{2\lambda \tan \Phi}{\cos^2 \beta} - \frac{\tan^2 \Phi}{\cos^4 \beta} - \lambda^2 \right) \sin(2\Phi)}{\sqrt{\lambda^2 \cos^8 \Phi + \left( \frac{2\lambda \tan \Phi}{\cos^2 \beta} - \frac{\tan^2 \Phi}{\cos^4 \beta} - \lambda^2 \right) \cos^6 \Phi}} \right] \quad (24)$$

Equations (20), (23) and (24) are presented in a form that is particularly well suited for calculations utilizing a spreadsheet. The non-dimensional parameters, Equations (23) and (24) are calculated first for various  $\lambda$  and  $\Phi$ . Once determined these parameters are input into Equation (20) to calculate the non-dimensional membrane resistance per unit length. The results of these calculations are presented in Chapter 5 and the calculations themselves are included in Appendix B.

### 3.1.4 Plastic Work due to Bending

The final component of plastic work absorbed by the prismatic buckling element is due to bending stresses and strains within the plastic hinges. It is assumed that plastic hinges are formed within an out-of-plane hinge above the stretching diagonal and along the longitudinal edges adjacent to the stiffeners. The stretching diagonal hinge is assumed to connect above the centroid of the aperture as the element deforms. However, this results in a rolling hinge due to the fact that the two halves of this hinge that are split by the aperture move in opposing directions. The plastic work associated with this rolling hinge is neglected. The concentrated twisting and bending that occurs across the aperture is also neglected.

No plastic hinges are formed at the transverse edges of the element because adjacent elements buckle out-of-plane in opposite directions, thus preventing hinge formation. This intuition is developed from observation of the elastic buckling solution presented earlier.

Note that this computational model consists of sub-elements with flat surfaces such that plate curvatures are zero.

$$\dot{K}_{xx} = \dot{K}_{xy} = \dot{K}_{yy} = 0$$

The contribution from the second term in Equation (8) to the continuous rate of plastic work dissipation vanishes. Therefore, the only contribution for the rate of plastic work due to bending is the third term of Equation (8).

$$\dot{E}_b = \sum_i M_o^{(i)} \dot{\theta}^{(i)} L^{(i)}$$

The rate of bending work for two longitudinal hinges and one diagonal hinge becomes:

$$\dot{E}_b = M_o [2\theta_1 L_1 + \theta_2 L_2]$$

where  $L_1$  and  $L_2$  are the plastic hinge lengths and  $\theta_1$  and  $\theta_2$  are the hinge rotations of the longitudinal and diagonal plastic hinges. Examination of the geometry in Figure 6 yields the following expressions for the longitudinal hinge rotation  $\theta_1$  and the transverse element boundary rotation  $\theta_3$ .

$$\theta_1 = \sin^{-1} \left( \frac{\frac{w_o}{b}}{2 \cos \beta} \right) = \sin^{-1} \left( \frac{2 w_o \cos \beta}{b} \right)$$

$$\theta_3 = \sin^{-1} \left( \frac{\frac{w_o}{a}}{\frac{2}{2}} \right) = \sin^{-1} \left( \frac{2 w_o}{a} \right)$$

The rotation of the diagonal hinge  $\theta_2$  can be solved for in terms of  $\theta_1$  and  $\theta_3$  for the geometry where  $\beta = 0$  and small rotations are assumed so that the small angle approximation can be applied.

$$\theta_2 = \sqrt{\theta_1^2 + \theta_3^2} = \sqrt{\left[ \sin^{-1} \left( \frac{2 w_o \cos \beta}{b} \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2 w_o}{a} \right) \right]^2}$$

The hinge rotation rates can now be expressed in terms of  $\dot{w}_o$ .

$$\dot{\theta}_1 = \frac{d}{dt} \left[ \sin^{-1} \left( \frac{2 w_o \cos \beta}{b} \right) \right] = \frac{\frac{2 \dot{w}_o}{b} \cos \beta}{\sqrt{1 - 4 \left( \frac{w_o}{b} \right)^2 \cos^2 \beta}} = \frac{2 \dot{w}_o \cos \beta}{\sqrt{b^2 - 4 w_o^2 \cos^2 \beta}}$$

$$\dot{\theta}_3 = \frac{d}{dt} \left[ \sin^{-1} \left( \frac{2 w_o}{a} \right) \right] = \frac{\frac{2 \dot{w}_o}{a}}{\sqrt{1 - \left( \frac{2 w_o}{a} \right)^2}} = \frac{2 \dot{w}_o}{\sqrt{a^2 - 4 w_o^2}}$$

$$\dot{\theta}_2 = \frac{d}{dt} \left[ \sqrt{\left[ \sin^{-1} \left( \frac{2 w_o \cos \beta}{b} \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2 w_o}{a} \right) \right]^2} \right]$$

Several algebraic steps are necessary to derive a simplified expression for  $\theta_2$ .

$$\theta_2 = \frac{\left( \frac{\dot{w}_o}{2} \right) \left[ 2 \sin^{-1} \left( \frac{2w_o \cos \beta}{b} \right) \frac{\frac{2 \cos \beta}{b}}{\sqrt{1 - \left( \frac{2w_o \cos \beta}{b} \right)^2}} + 2 \sin^{-1} \left( \frac{2w_o}{a} \right) \frac{\frac{2}{a}}{\sqrt{1 - \left( \frac{2w_o}{a} \right)^2}} \right]}{\sqrt{\left[ \sin^{-1} \left( \frac{2w_o \cos \beta}{b} \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2w_o}{a} \right) \right]^2}}$$

$$\theta_2 = 2\dot{w}_o \left[ \frac{\left( \cos \beta / b \right) \sin^{-1} \left( 2w_o \cos \beta / b \right)}{\sqrt{1 - \left( 2w_o \cos \beta / b \right)^2}} + \frac{(1/a) \sin^{-1} (2w_o / a)}{\sqrt{1 - (2w_o / a)^2}} \right] / \sqrt{\left[ \sin^{-1} \left( \frac{2w_o \cos \beta}{b} \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2w_o}{a} \right) \right]^2}$$

$$\theta_2 = 2\dot{w}_o \left[ \frac{\cos \beta \sin^{-1} \left( 2w_o \cos \beta / b \right)}{\sqrt{b^2 - (2w_o \cos \beta)^2}} + \frac{\sin^{-1} (2w_o / a)}{\sqrt{a^2 - (2w_o)^2}} \right] / \sqrt{\left[ \sin^{-1} \left( \frac{2w_o \cos \beta}{b} \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2w_o}{a} \right) \right]^2}$$

$$\theta_2 = 2 \left( \frac{\dot{w}_o}{b} \right) \left[ \frac{\cos \beta \sin^{-1} \left[ 2 \cos \beta \left( \frac{w_o}{b} \right) \right]}{\sqrt{1 - 4 \left( \frac{w_o}{b} \right)^2 \cos^2 \beta}} + \frac{\sin^{-1} \left( \frac{2 w_o}{\lambda b} \right)}{\sqrt{\lambda^2 - 4 \left( \frac{w_o}{b} \right)^2}} \right] / \sqrt{\left[ \sin^{-1} \left[ 2 \cos \beta \left( \frac{w_o}{b} \right) \right] \right]^2 + \left[ \sin^{-1} \left( \frac{2 w_o}{\lambda b} \right) \right]^2}$$

The rate of plastic work due to bending at the plastic hinges can be expressed in terms of the hinge rotation rates by Equation (25) since the longitudinal hinge length  $L_1 = a$  and the diagonal hinge length  $L_2$  can be expressed as:

$$L_2 = \sqrt{(a + b \tan \beta)^2 + b^2}$$

The total rate of bending work dissipation by the buckling element is:

$$\dot{E}_b = 2M_o a |\dot{\theta}_1| + M_o |\dot{\theta}_2| \sqrt{(a + b \tan \beta)^2 + b^2} \quad (25)$$

The first term in Equation (25) is the rate of plastic work due to bending in the longitudinal hinges while the second term is the rate of plastic work in the diagonal hinge.

The per unit length normalized shear force component due to bending  $NDSF_b$  can be obtained by dividing the rate of bending energy absorption by the shear translation velocity  $\dot{s}$ , the plastic hinge moment  $M_o$ , and the element length  $a$ .

$$\frac{NDSF_b}{\text{unit length}} = \frac{\dot{E}_b}{\dot{s} M_o a} = \frac{\dot{E}_b \cos^2 \beta \cos^2 \Phi}{\dot{\Phi} b M_o a}$$

$$\frac{NDSF_b}{\text{unit length}} = \cos^2 \beta \cos^2 \Phi \left[ \frac{2 |\dot{\theta}_1|}{\dot{\Phi} b} + \sqrt{\left( 1 + \frac{\tan \beta}{\lambda} \right)^2 + \frac{1}{\lambda^2} \left( \frac{|\dot{\theta}_2|}{\dot{\Phi} b} \right)} \right]$$

Substituting for  $|\dot{\theta}_1|$  and  $|\dot{\theta}_2|$  into the expression for  $NDSF_b$  yields the following result.

$$\frac{NDSF_b}{\text{unit length}} = \left( \frac{4}{b} \right) \left| \frac{\dot{w}_o}{\dot{\Phi} b} \right| \left[ \frac{\cos^3 \beta \cos^2 \Phi}{\sqrt{1 - 4 \left( \frac{w_o}{b} \right)^2 \cos^2 \beta}} \right] + \dots \quad (26)$$

$$\left( \frac{2 \cos^2 \Phi \cos^2 \beta}{b} \right) \sqrt{\left( 1 + \frac{\tan \beta}{\lambda} \right)^2 + \frac{1}{\lambda^2}} \left| \frac{\dot{w}_o}{\dot{\Phi} b} \right| \left[ \frac{\cos \beta \sin^{-1} \left( \frac{2 w_o}{b} \cos \beta \right)}{\sqrt{1 - 4 \left( \frac{w_o}{b} \right)^2 \cos^2 \beta}} + \frac{\sin^{-1} \left( \frac{2 w_o}{\lambda b} \right)}{\sqrt{\lambda^2 - 4 \left( \frac{w_o}{b} \right)^2}} \right] \\ \left[ \sqrt{\left[ \sin^{-1} \left( \frac{2 w_o}{b} \cos \beta \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2 w_o}{\lambda b} \right) \right]^2} \right]$$

It is interesting to note that Equation (26) contains the same non-dimensional parameters that were calculated earlier in Equations (23) and (24) for the membrane strain plastic work.

$$(1) \frac{w_o}{b} \quad \text{and} \quad (2) \frac{\dot{w}_o}{\Phi b}$$

### 3.1.5 Total Internal Plastic Work

The shear strain, membrane strain, and bending components of the normalized shear force are added together to calculate the total non-dimensional shear force or NDSF per unit length in the longitudinal direction.

$$\frac{\text{NDSF}}{\text{unit length}} = \frac{\text{NDSF}_s}{\text{unit length}} + \frac{\text{NDSF}_m}{\text{unit length}} + \frac{\text{NDSF}_b}{\text{unit length}}$$

Equations (15), (20), and (26) form the three components of the normalized shear force per unit length of the plastic buckling model that will be extensively used in Chapter 5 for comparison with non-dimensionalized experimental results. The purpose of normalization of the shear force is to formulate a result which depends solely on the dimensional parameters of plate thickness and width as well as the non-dimensional parameters  $\lambda$  and  $\Phi$ . Experimental results from materials of different yield strengths and plate lengths can be compared directly once the data have been normalized.

## 3.2 Elastic Buckling Models

The difficulty faced with determining the shear force associated with plastic buckling is that the non-dimensional plastic buckling half wavelength ( $\lambda = a/b$ ) is unknown. Another problem which must be faced is that the plastic buckling model does not accurately predict the shear force at small  $\Phi$  since the plate exhibits elastic rather than plastic behavior in this region. To overcome these difficulties it is necessary to examine the elastic behavior of the plate both before and after buckling occurs.

Plate elastic behavior at small  $\Phi$  is examined in this section. The shear force required to bring about in-plane plastic flow in the plate without buckling is also formulated at the end of this chapter. Finally, in Chapter 5 a comparison is made between the results from

all analytic models and experimental results resulting in the ability to predict how  $\lambda$  and the normalized shear force vary over the entire range of shear rotation  $\Phi$ .

The following derivations can be simplified by defining for  $\beta = 0$  that  $\Phi = \tan^{-1}(\frac{s}{b})$

or for small  $\Phi$ :  $\Phi = \frac{s}{b}$ . This is the same definition for  $\Phi$  that was used earlier for an initial buckling orientation angle of zero ( $\beta = 0$ ) and can be derived from the earlier definition by setting  $\beta = 0$ .

### 3.2.1 Elastic Pre-buckling Load

A variational approach is utilized to determine the elastic pre-buckling plate behavior. The goal of this analysis is to derive a simple relationship between the normalized shear force per unit length and the shear translation angle  $\Phi$ .

It is known from elementary mechanics that  $\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$  where the shear modulus,  $G = \frac{E}{2(1+\nu)}$ , and  $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$ . The principle of virtual work relates the external work done on the plate to the internal elastic strain energy stored in the plate for a given displacement field. The external work done on plate element of area  $S = ab$  by shear force  $F$  moving through a displacement  $\delta s$  is:  $\delta W = F \delta s$ . The internal strain energy stored within the plate element of surface area  $S = ab$  is:

$$\delta U = t \int_S \tau_{xy} \delta \varepsilon_{xy} dS$$

In the elastic regime, the principle of virtual work states that the incremental work done by an external force  $F_e$  on the element is equal to the internal elastic strain energy stored within the element.

$$F_e \delta s = t \int_S \tau_{xy} \delta \varepsilon_{xy} dS = t \int_S 2G\varepsilon_{xy} \delta \varepsilon_{xy} dS \quad (27)$$

Two dimensional shear strain is defined as:

$$\varepsilon_{xy} = \frac{1}{2} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]$$

For this analysis it is assumed that the transverse displacement  $v = 0$  and that  $\frac{\partial v}{\partial x} = 0$ .

Therefore, the shear strain is:

$$\epsilon_{xy} = \frac{1}{2} \frac{\partial u}{\partial y}$$

The first variation of the shear strain is given by:

$$\delta \epsilon_{xy} = \frac{1}{2} \delta \left( \frac{\partial u}{\partial y} \right)$$

From the problem geometry already established,  $\frac{\partial u}{\partial y} = \frac{x}{b} = \Phi$  for  $\beta = 0$  and small  $\frac{x}{b}$  and although  $\Phi$  varies with time it is a constant within the spatial domain of the plate at any particular instant. Therefore, substituting for  $\frac{\partial u}{\partial y}$  into the above expressions for  $\epsilon_{xy}$  and  $\delta \epsilon_{xy}$  gives:  $\epsilon_{xy} = \frac{1}{2} \Phi$  and  $\delta \epsilon_{xy} = \frac{1}{2} \delta(\Phi)$ . The shear strain and its first variation,  $\epsilon_{xy}$  and  $\delta \epsilon_{xy}$ , can then be substituted into Equation (27) yielding:

$$F_e \delta s = \frac{G t}{2} \int_S \Phi \delta \Phi dS = \frac{G t}{2} \Phi \delta \Phi \int_S dS$$

Although the plate has not yet buckled, consider a plate element of the same length  $a$  and width  $b$  as the plastic buckling element defined in the previous chapter. Therefore, the above integral for the element becomes:

$$F_e \delta s = \frac{G t a b}{2} \Phi \delta \Phi$$

The elemental shear force  $F_e$  can be expressed in terms of  $\lambda$  and  $\Phi$  by noting that

$$\lambda = \frac{a}{b} \text{ and } \delta \Phi = \frac{\delta s}{b}.$$

$$F_e \delta s = \left( \frac{G t \lambda b \Phi}{2} \right) \delta s \quad F_e = \left( \frac{G t \lambda b}{2} \right) \Phi \quad (28)$$

where  $K_\Phi = \left( \frac{G t \lambda b}{2} \right)$  is the elastic pre-buckling stiffness coefficient

Equation (28) relates the shear force within the elastic range on an element of width  $b$ , length  $a$ , and thickness  $t$  to the in-plane shear rotation  $\Phi$ . The term in parentheses is the element's elastic stiffness coefficient  $K_\Phi$  for the pre-buckling mode of shear.

The normalized shear force per unit length can be obtained by dividing the element force  $F_e$  by the plastic hinge moment  $M_o$  and the element length  $a$ . The result is a normalized shear force  $NDSF_{\text{elastic}}$  per unit length in the longitudinal direction for the elastic pre-buckled condition.

$$\frac{NDSF_{\text{elastic}}}{\text{unit length}} = \frac{F_e}{M_o a} = \left[ \frac{G t \lambda b}{2 M_o a} \right] \Phi = \left[ \frac{G t}{2 M_o} \right] \Phi \quad (29)$$

The plastic hinge moment  $M_o = \frac{\sigma_o t^2}{4}$  was stated earlier to be an approximation of the hinge moment for von Mises  $M_o = \frac{2 \sigma_o t^2}{\sqrt{3} 4}$ . The shear modulus  $G = \frac{E}{2(1 + v)}$ . Substituting for these constants into (29) yields:

$$\frac{NDSF_{\text{elastic}}}{\text{unit length}} = \left[ \frac{E}{\sigma_o t (1 + v)} \right] \Phi \quad (30)$$

The above expression (30) gives the final result which will be used for comparison with experimental results in Chapter 7. The stiffness coefficient for non-dimensional elastic shear per unit length becomes:

$$K_{\Phi_{\text{ND}}} = \frac{E}{\sigma_o t (1 + v)}$$

### 3.2.2 Elastic Post-buckling Load

From this point on, to simplify notation, the element shear force and shear rotation angle at which buckling occurs are designated  $F_{cr}$  and  $\Phi_{cr}$  respectively. Also, designate  $F_{eB}$  as the elastic elemental shear force after buckling. To derive the shear force for the elastic post buckling regime, the assumption is made that the relationship between this force and the shear rotation  $\Phi$  remains linear. This is reasonable to assume given the fact that the plate remains in the elastic range and the shear rotation  $\Phi$  is small. The next assumption is that the elastic stiffness coefficient after buckling (for  $\Phi > \Phi_{cr}$ ) is reduced to

half of its original value. This assumption is based on the fact that a clamped plate will lose approximately half of its elastic stiffness when it distorts out-of-plane under uniform compression. Hence it is assumed that the loss of stiffness due to buckling is the same in compression buckling and shear buckling. The above two assumptions lead to the following formulas for the buckling element shear force in the elastic range.

$$F_e = \left[ \frac{G t b \lambda}{2} \right] \Phi \quad (\text{element shear force prior to buckling}) \quad (31)$$

$$F_{eB} = F_{cr} + \left[ \frac{G t b \lambda}{4} \right] (\Phi - \Phi_{cr}) \quad (\text{element shear force after buckling}) \quad (32)$$

The shear rotation angle  $\Phi_{cr}$  and critical buckling load  $F_{cr}$  must be substituted into (32) to solve for  $F_{eB}$ . The critical shear force per unit length from Equation (5) is:

$$N_{xy_o} = 8.98 \left[ \frac{\pi^2 D}{b^2} \right]$$

The critical normalized shear force per unit length is:

$$\text{NDSF}_{cr}/\text{unit length} = N_{xy_o}/M_0 = \frac{8.98 \pi^2 E t}{3(1 - v^2) \sigma_o b^2}$$

In order to determine the force on a single buckled element  $F_{cr}$ , the shear force per unit length  $N_{xy_o}$  must be multiplied by the element length  $a$ . Therefore,  $F_{cr} = N_{xy_o} * a$ .

$$F_{cr} = \frac{8.98 \pi^2 E t^3 \lambda}{12(1 - v^2) b} \quad (\text{critical buckling force on a single buckled element})$$

The critical shear rotation for buckling  $\Phi_{cr}$  can be solved for by setting the elastic pre-buckling shear force equal to the critical buckling load.

$$F_e = \left[ \frac{G t b \lambda}{2} \right] \Phi = \left[ \frac{E t b \lambda}{4(1 + v)} \right] \Phi_{cr} = \frac{8.98 \pi^2 E t^3 \lambda}{12(1 - v^2) b}$$

Therefore:

$$\Phi_{cr} = \frac{8.98 \pi^2 (1 + v) t^2}{3 (1 - v^2) b^2}$$

Now that  $F_{cr}$  and  $\Phi_{cr}$  have been determined these expressions can be substituted back into Equation (32) to determine the shear force  $F_{eB}$  over the entire elastic post-buckling range.

$$F_{eB} = \frac{8.98 \pi^2 E t^3 \lambda}{12(1-v^2)b} + \left[ \frac{E t b \lambda}{8(1+v)} \right] \left[ \Phi - \left( \frac{8.98 \pi^2 (1+v) t^2}{3(1-v^2) b^2} \right) \right]$$

The post buckled normalized shear force,  $NDSF_{elB}$ , per unit length in the elastic regime can be determined from the post-buckled shear force by dividing by  $M_o$  and the element length  $a$ .

$$\frac{NDSF_{elB}}{\text{unit length}} = \frac{F_{eB}}{M_o a}$$

$$\frac{NDSF_{elB}}{\text{unit length}} = \frac{8.98 \pi^2 E t}{3(1-v^2) \sigma_o b^2} + \left[ \frac{E}{2 t \sigma_o (1+v)} \right] \left[ \Phi - \frac{8.98 \pi^2 (1+v) t^2}{3(1-v^2) b^2} \right] \quad (33)$$

Equation (33) is the approximate solution for the elastic post-buckling load over the entire range of  $\Phi > \Phi_{cr}$ . The above equation is only good as long as the plate behaves elastically. It does not predict the onset of plastic behavior. Therefore, development of a complete analytical model rests on combining the results of both the elastic and plastic models and then comparing them with experimental results to determine the range of  $\Phi$  over which each model is valid. This will be addressed further in Chapter 5.

### 3.3 Analytical Model for Plastic Flow without Buckling

To complete this analysis the in-plane plastic flow or "no buckling plastic flow" (NBPF) mode must be addressed. This behavior is expected to occur in plates with a relatively small width to thickness ( $b/t$ ) ratio for which the critical angle  $\Phi_{cr}$  for elastic buckling approaches infinity as shown in Figure 7. A plot of  $\Phi_{cr}$  versus  $b/t$  ratio is shown in Figure 7.

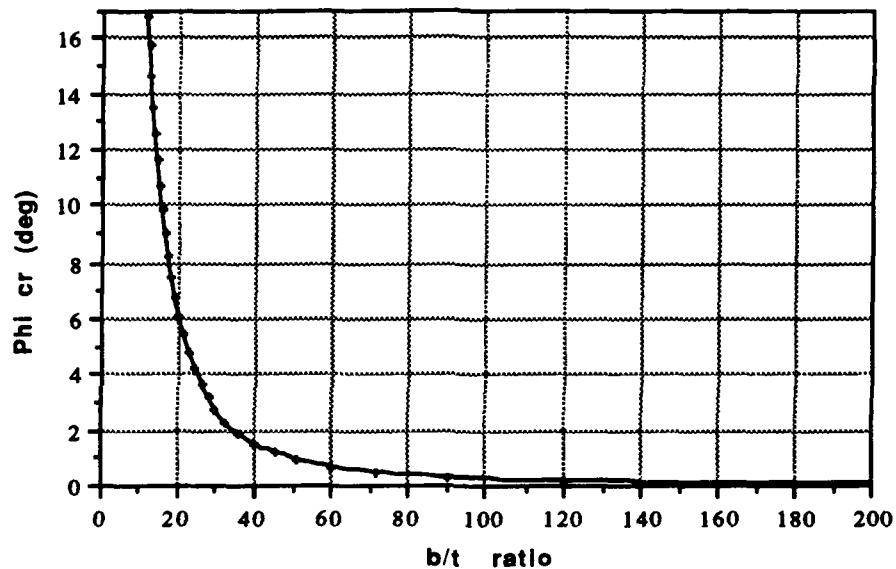


Figure 7: Critical shear rotation for elastic buckling versus b/t ratio

From Figure 7 it can be seen that  $\Phi_{cr}$  increases rapidly as the b/t ratio drops. This makes sense intuitively since it is expected that thick stocky plates will have a much higher resistance to out-of-plane distortion due to their higher bending rigidity. A greater force is required to achieve the larger critical shear translation angles for buckling at low b/t ratios. Therefore, it is expected that plastic failure will occur prior to the onset of buckling for plates with relatively low b/t ratios.

The purpose of this formulation is to determine the shear force required to cause plastic failure without buckling. There are four assumptions which must be made in order to determine this stress.

- (1) The boundary conditions on the plate are such that it is being loaded in uniform shear and tension along its longitudinal edges and no out-of-plane displacement or rotation occurs along the clamped edges.

\* *Note to reader:* For pure in-plane shear of an isotropic plate with edges maintained a constant distance apart the strain  $\epsilon_y$  in the transverse direction is zero. This follows from showing that for a displacement  $u$  in the x direction that satisfies strain-displacement, stress-strain, and equilibrium relations and is dependent only on  $y$ , with  $v = w = 0$ ,  $\epsilon_y = 0$  is the only solution. This condition was not noticed until a more general derivation was completed and is introduced later in this formulation on page 45.

(2) This derivation assumes that for small shear strains  $\gamma_{xy}$  the incremental change in shear strain equals the incremental change in the shear rotation  $\Phi$ . This is in contrast to the classical definition of shear strain which relates the shear strain increment to the tangent of the shear rotation  $\Phi$ :

$$\Delta\gamma_{xy} = \Delta(\tan \Phi)$$

Therefore the shear strain within the plate is a constant equal to the shear translation  $s$  divided by the plate width  $b$  and the shear rotation angle  $\Phi$  is small so that  $\Phi \approx s/b$ .

$$\varepsilon_{xy} = \frac{1}{2} \frac{\partial u}{\partial y} = \frac{s}{2b}$$

(3) The strain in the  $x$  direction is zero.

(4) The plate material is assumed to obey the von Mises yield condition in a plane stress (bi-axial) stress field.

$$F = \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 - \sigma_o^2 = 0 \quad (34)$$

The strain rates are determined from the associated flow rule.

$$\dot{\varepsilon}_{\alpha\beta} = \dot{\kappa} \frac{\partial F}{\partial \sigma_{\alpha\beta}}$$

$$\dot{\varepsilon}_x = \dot{\kappa} \frac{\partial F}{\partial \sigma_x} = \dot{\kappa} (2\sigma_x - \sigma_y) \quad (35)$$

$$\dot{\varepsilon}_y = \dot{\kappa} \frac{\partial F}{\partial \sigma_y} = \dot{\kappa} (2\sigma_y - \sigma_x) \quad (36)$$

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \frac{1}{2} \dot{\kappa} \frac{\partial F}{\partial \tau_{xy}} = 3\dot{\kappa}\tau_{xy} \quad (37)$$

The stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  can be solved for from the three flow rule equations. First manipulate (35) to solve for  $\sigma_y$  and then substitute for  $\sigma_y$  into (36) to solve for  $\sigma_x$  solely in terms of the strain rates.

$$\sigma_y = 2\sigma_x - \frac{\dot{\varepsilon}_x}{\dot{\kappa}}$$

$$\dot{\varepsilon}_y = \dot{\kappa} \left[ 2 \left( 2\sigma_x - \frac{\dot{\varepsilon}_x}{\dot{\kappa}} \right) - \sigma_x \right]$$

$$\dot{\varepsilon}_y = \dot{\kappa} \left[ 3\sigma_x - \frac{2\dot{\varepsilon}_x}{\dot{\kappa}} \right] = 3\dot{\kappa}\sigma_x - 2\dot{\varepsilon}_x$$

$$\sigma_x = \frac{1}{3\dot{\kappa}} (\dot{\varepsilon}_y + 2\dot{\varepsilon}_x) \quad (38)$$

The next step is to solve for  $\sigma_y$  in terms of the strain rates by substituting (38) into (35).

$$\dot{\varepsilon}_x = \dot{\kappa} \left[ 2 \left( \frac{1}{3\dot{\kappa}} (\dot{\varepsilon}_y + 2\dot{\varepsilon}_x) \right) - \sigma_y \right]$$

$$\frac{\dot{\varepsilon}_x}{\dot{\kappa}} = \frac{2}{3\dot{\kappa}} (\dot{\varepsilon}_y + 2\dot{\varepsilon}_x) - \sigma_y$$

$$\sigma_y = \frac{1}{3\dot{\kappa}} (2\dot{\varepsilon}_y + \dot{\varepsilon}_x) \quad (39)$$

From (37)  $\tau_{xy}$  can be expressed in terms of the shear strain rate.

$$\tau_{xy} = \frac{\dot{\varepsilon}_{xy}}{3\dot{\kappa}} \quad (40)$$

The constant  $\dot{\kappa}$  can now be determined by substituting for  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  into the von Mises yield condition.

$$F = \left[ \frac{1}{3\dot{\kappa}} (\dot{\varepsilon}_y + 2\dot{\varepsilon}_x) \right]^2 + \left[ \frac{1}{3\dot{\kappa}} (2\dot{\varepsilon}_y + \dot{\varepsilon}_x) \right]^2 - \left[ \frac{1}{3\dot{\kappa}} \right]^2 [(\dot{\varepsilon}_y + 2\dot{\varepsilon}_x)(2\dot{\varepsilon}_y + \dot{\varepsilon}_x)] + 3 \left( \frac{\dot{\varepsilon}_{xy}}{3\dot{\kappa}} \right)^2 - \sigma_o^2 = 0$$

$$\left( \frac{1}{3\dot{\kappa}} \right)^2 [(\dot{\varepsilon}_y + 2\dot{\varepsilon}_x)^2 + (2\dot{\varepsilon}_y + \dot{\varepsilon}_x)^2 - (\dot{\varepsilon}_y + 2\dot{\varepsilon}_x)(2\dot{\varepsilon}_y + \dot{\varepsilon}_x) + 3(\dot{\varepsilon}_{xy})^2] = \sigma_o^2$$

Expanding and then combining terms yields:

$$(3\sigma_o\dot{\kappa})^2 = 3\dot{\varepsilon}_y^2 + 3\dot{\varepsilon}_x^2 + 3\dot{\varepsilon}_x\dot{\varepsilon}_y + 3\dot{\varepsilon}_{xy}^2$$

and gives the following expression for  $\kappa$  .

$$\kappa = \frac{1}{\sqrt{3} \sigma_0} \sqrt{\dot{\epsilon}_y^2 + \dot{\epsilon}_x^2 + \dot{\epsilon}_x \dot{\epsilon}_y + \dot{\epsilon}_{xy}^2} \quad (41)$$

In order to solve for the shear force applied to the plate the rate of external work applied must be equated to the rate of total internal plastic work absorbed within the plate. If  $F_s$  is defined as the shear force applied per unit length of plating then:

$$F_s \dot{s} = t \int_S \bar{\epsilon} dS \quad (S = 1 * b) \quad (42)$$

$$\bar{\epsilon} \equiv \sigma_{\alpha\beta} \dot{\epsilon}_{\alpha\beta} = (\text{rate of plastic work dissipation per unit volume of material})$$

This integral will be taken over a plate surface area of unit length so that  $S = 1 * b$  . For a plane stress field the rate of work dissipation can be expanded into the following form.

$$\bar{\epsilon} = \sigma_x \dot{\epsilon}_x + 2\tau_{xy} \dot{\epsilon}_{xy} + \sigma_y \dot{\epsilon}_y$$

The rate of plastic work dissipation  $\bar{\epsilon}$  can be solved for in terms of the strain rates by substituting for  $\sigma_x$  ,  $\sigma_y$  , and  $\tau_{xy}$  from Equations (38), (39), and (40).

$$\bar{\epsilon} = \frac{1}{3\kappa} (\dot{\epsilon}_y + 2\dot{\epsilon}_x) \dot{\epsilon}_x + 2 \left( \frac{\dot{\epsilon}_{xy}}{3\kappa} \right) \dot{\epsilon}_{xy} + \frac{1}{3\kappa} (2\dot{\epsilon}_y + \dot{\epsilon}_x) \dot{\epsilon}_y$$

$$\bar{\epsilon} = \frac{1}{3\kappa} (\dot{\epsilon}_x \dot{\epsilon}_y + 2\dot{\epsilon}_x^2 + 2\dot{\epsilon}_{xy}^2 + 2\dot{\epsilon}_y^2 + \dot{\epsilon}_x \dot{\epsilon}_y)$$

$$\bar{\epsilon} = \frac{2}{3\kappa} (\dot{\epsilon}_x^2 + \dot{\epsilon}_x \dot{\epsilon}_y + \dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2)$$

Substituting for  $\kappa$  from Equation (41) into the expression for plastic work dissipation rate yields following expression.

$$\bar{\epsilon} = \frac{2 \sigma_0}{\sqrt{3}} \sqrt{\dot{\epsilon}_x^2 + \dot{\epsilon}_x \dot{\epsilon}_y + \dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2} \quad (43)$$

Further simplification of the expression for rate of work dissipation is achieved by applying the initial assumption that  $\dot{\epsilon}_x = 0$ .

$$\bar{\dot{\epsilon}} = \frac{2 \sigma_o}{\sqrt{3}} \sqrt{\dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2} \quad (44)$$

The rate of work dissipation per unit volume can now be substituted into the integral (42) for calculating the total rate of plastic work absorbed per unit length of plate.

$$F_s \dot{s} = t \int_S \bar{\dot{\epsilon}} dS = t \int_S \frac{2 \sigma_o}{\sqrt{3}} \sqrt{\dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2} dS$$

The strain rates are constant over the entire region of the plate for any instant in time and can be taken outside the surface integral.

$$F_s \dot{s} = \frac{2 t \sigma_o}{\sqrt{3}} \sqrt{\dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2} \int_S dS$$

$$F_s \dot{s} = \frac{2 t \sigma_o}{\sqrt{3}} \sqrt{\dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2} \int_0^b \int_0^1 dx dy$$

The surface area integral is equal to  $(1 * b)$  so that the rate of internal plastic work absorbed per unit length becomes:

$$F_s \dot{s} = \frac{2 t b \sigma_o}{\sqrt{3}} \sqrt{\dot{\epsilon}_y^2 + \dot{\epsilon}_{xy}^2}$$

**\* Note to reader:** As noted earlier the strain rate in the y-direction turns out to be zero when the plate stiffeners are constrained to have a constant separation distance between them. Since this condition is imposed during the buckling experiments, the total internal plastic work for the in-plane plastic flow deformation mode is:

$$F_s \dot{s} = \frac{2 t b \sigma_o}{\sqrt{3}} \dot{\epsilon}_{xy} \quad (45)$$

To proceed further with this formulation requires establishment of an approximate geometric relationship between the strain rate  $\dot{\epsilon}_{xy}$  and the non-dimensional geometric parameter  $\Phi$ . One of the initial assumptions of this formulation was that the

shear strain  $\gamma_{xy}$  was equal to  $s/b$ .  $\dot{\epsilon}_{xy} = \frac{1}{2} \frac{\partial u}{\partial y} = \frac{s}{2b}$  The shear strain rate becomes:

$$\dot{\epsilon}_{xy} = \frac{1}{2} \frac{\partial u}{\partial y} = \frac{\dot{s}}{2b} \approx \frac{\Phi}{2}$$

The rate of external plastic work applied per unit length can now be obtained by substituting for  $\dot{\epsilon}_{xy}$  into (45).

$$F_s \dot{s} = \frac{t b \sigma_0}{\sqrt{3}} \Phi \quad (46)$$

Dividing both sides by  $\dot{s}$  and noting that  $\dot{s} = \Phi b$  yields an expression for shear force per unit length.

$$F_s = \frac{t \sigma_0}{\sqrt{3}} \quad (47)$$

The normalized shear force per unit length for plastic deformation without buckling becomes:

$$\frac{NDSF_{NBPF}}{\text{per unit length}} = \frac{4}{\sqrt{3} t} \quad (48)$$

The subscript notation "NBPF" stands for "no buckling plastic flow". The above expression can now be plotted with the non-dimensional shear force for both elastic and plastic buckling and completes the series of analytical models that are used to predict actual plate behavior. A comparison with experimental results will be presented in Chapter 5.

**\* Note to reader:** It was noted earlier that  $\epsilon_y = 0$ , however, the NBPF curves representing in-plane plastic flow in Figures 8, 9, and 27 as well all Figures in Appendix B were calculated assuming that the strain rate in the y-direction  $\dot{\epsilon}_y$  was not zero. Therefore, the curve exhibits strengthening due to additional plastic work in tension while Equation (48) predicts a constant force which originates at the same level for  $\Phi = 0$ .

### 3.4 The Complete Analytical Model

In Sections 3.1 through 3.3 various analytical models have been formulated to describe the behavior of the plate as it is subjected to an in-plane shear force uniformly distributed along its clamped edges. To complete this analysis, these models must be combined in such a way that they approximate the actual plate behavior over the complete range of  $\Phi$ . To accomplish this, the previously derived solutions for non-dimensional shear force (NDSF) have been entered into a computer program and plotted against the shear rotation  $\Phi$  through a range of  $\Phi$  from 0 - 30 degrees and for initial buckling orientation angles  $\beta$  from 0 to 0.6 radians. The material parameters and geometry assumed for this plate are actual parameters from the experimental setup described in chapter 4. This facilitates comparison of this analysis with experimental results.

A plot of NDSF versus  $\Phi$  from 0 - 3 degrees is shown in Figure 8. The selected range for  $\Phi$  is small so that the elastic pre-buckling and post-buckling behavior can be shown. Another plot over an expanded range of  $\Phi$  from 0 - 30 degrees is shown in Figure 9. The following plate parameters were entered for this analysis and are the only inputs necessary for these calculations.

- (1) Young's Modulus -----  $E = 29,000$  ksi
- (2) Plate Width -----  $b = 4$  inches
- (3) Plate Thickness -----  $t = .0284$  inches
- (4) Plastic Flow Stress (mean over interval from  $\Phi = 0$  - 17 degrees) --  $\sigma_0 = 43.4$  ksi
- (5) Poisson's Ratio -----  $\nu = 0.3$

From Figure 8 it can be seen that the plate first behaves elastically without buckling up to an NDSF of 40 and  $\Phi = 0.1$  degrees. The sudden change in slope of the linear elastic curve at  $\Phi = 0.1$  degrees occurs when the critical load for elastic buckling is reached. This ideal buckling analysis assumes that the plate is perfectly flat and has no initial imperfections, therefore, the actual critical buckling load is going to be lower than this ideal value. In-plane plastic flow, shown as the "NBPF" curve, occurs at an NDSF slightly above 80. Elastic buckling occurs earlier due to its lower critical buckling load. Figure 8 illustrates that the plate will proceed into plastic buckling at wavelengths that are shorter than the initial elastic buckling wavelength solved for by Southwell and Skan which corresponds to a non-dimensional half-wavelength  $\lambda = (1/2*\Lambda) = 0.8$ . It also appears that

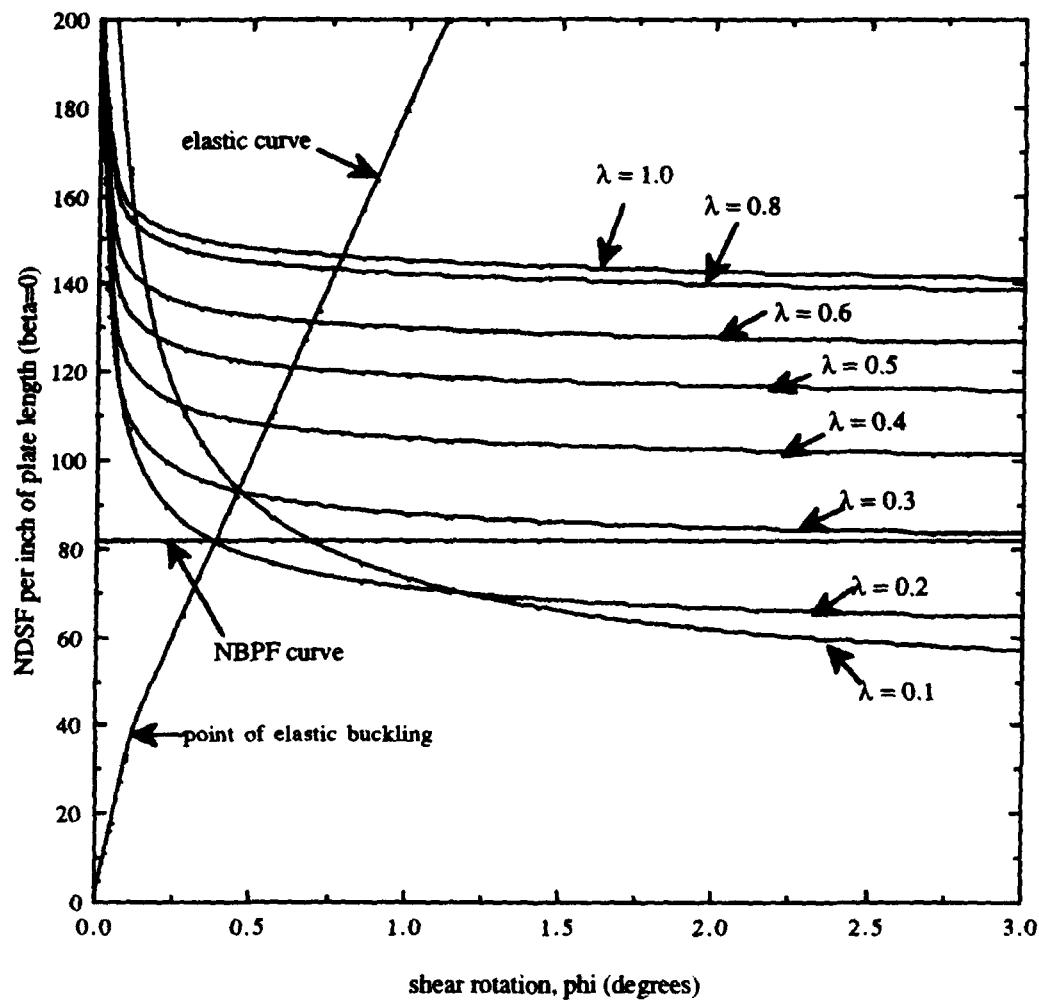


Figure 8: Non-dimensional shear force versus shear rotation angle (0-3 deg)

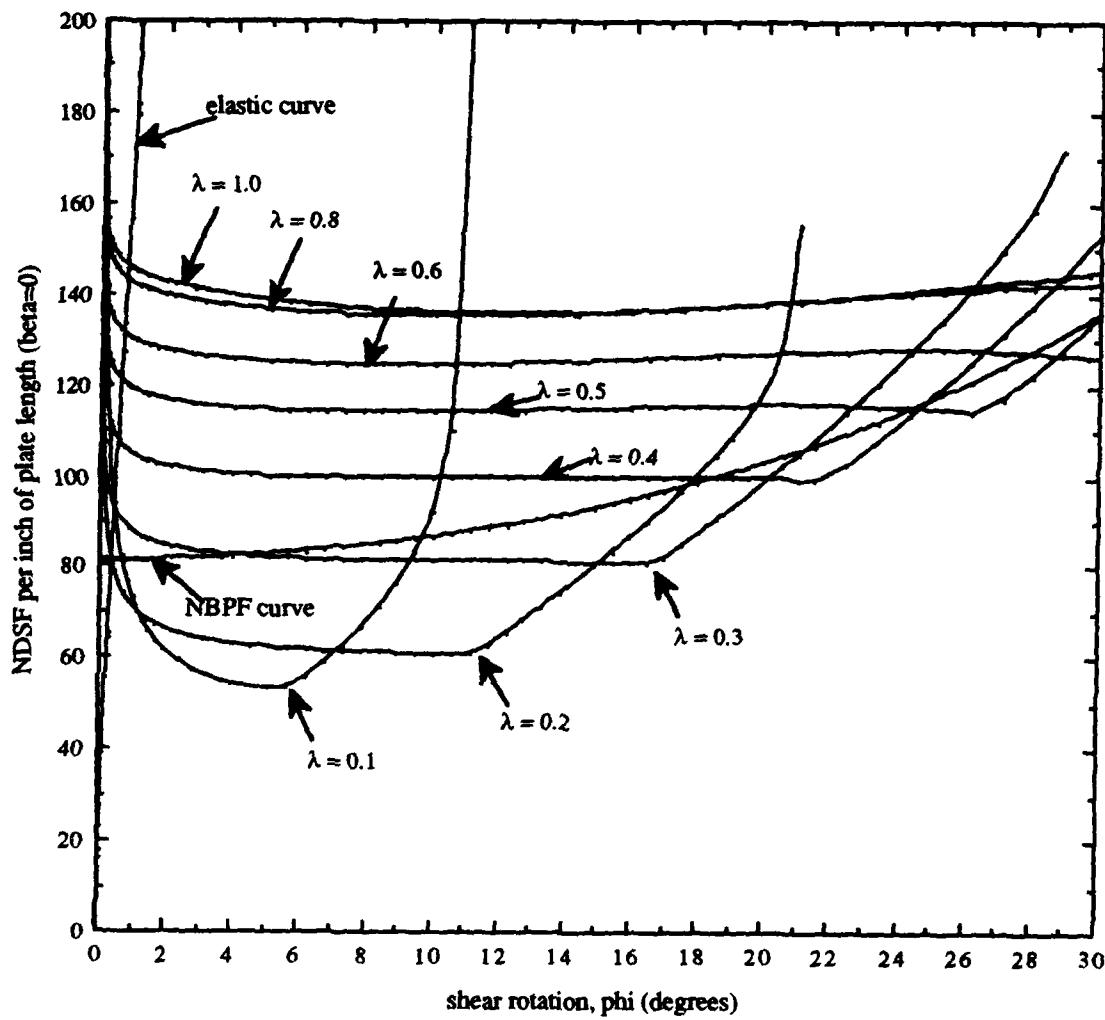


Figure 9: Non-dimensional shear force versus shear rotation angle (0-30 deg)

the initial plastic buckling half-wavelength  $\lambda$  for minimum resistance should be approximately 0.1, however, the plate must undergo a transition from the larger elastic buckling mode to a shorter wavelength plastic buckling mode.

Figure 9 shows the behavior of the plate over an expanded range of  $\Phi$ . As  $\Phi$  increases the non-dimensional shear force for the shorter wavelengths increases more rapidly than the longer wavelengths. Specifically, the wavelength corresponding to the minimum resistance buckling mode increases as the shear angle  $\Phi$  increases. The in-plane plastic flow curve that assumes a buildup in tensile membrane stresses runs adjacent to but slightly above the successive minima of the plastic buckling curves corresponding to increasing values of  $\lambda$ . This is interesting to note since the formulation of the in-plane failure result is entirely independent of the plastic buckling formulation.

In summary, the loading trajectory initially follows the elastic pre- and post-buckling curves. It then intersects and closely follows the in-plane plastic flow curve at a normalized load slightly above 80 and then bifurcates into the specific plastic buckling curve which intersects and drops below the NBPF curve. To determine the exact behavior during these transitions, buckling experiments must be conducted that allow measurement of the actual  $\lambda$  observed during progressive shear buckling.

## **4. Discussion of Plate Shear Buckling Apparatus and Experiments**

### **4.1 Experimental Apparatus and Setup**

A reusable buckling apparatus was designed and built for the purpose of conducting in-plane shear load tests. Sheet steel specimens are bolted into a mild steel frame and yoke assembly shown in Figures 10A, 10B, and 11. The entire assembled apparatus is then attached through two 1/2 inch pin connections to an INSTRON 8501 mechanical testing machine. In-plane shear is applied to the plate by holding the yoke assembly bolted to the outer steel bar stiffeners with the upper head and pulling on the center bar stiffener with the lower head actuator.

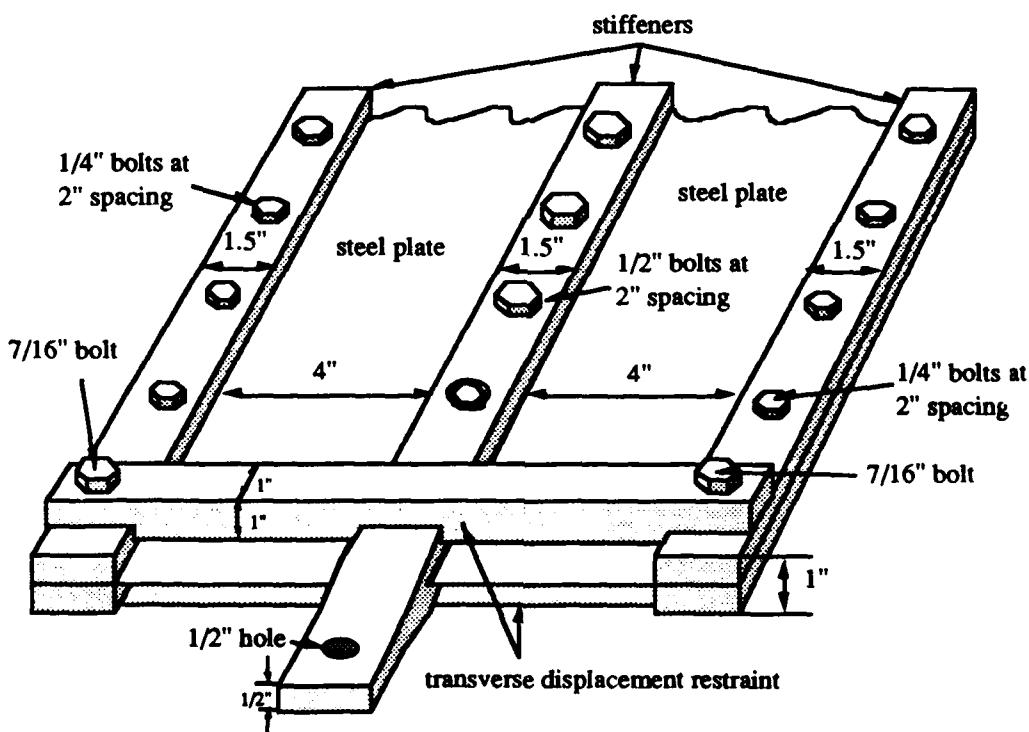


Figure 10A: Buckling apparatus (end angle view)

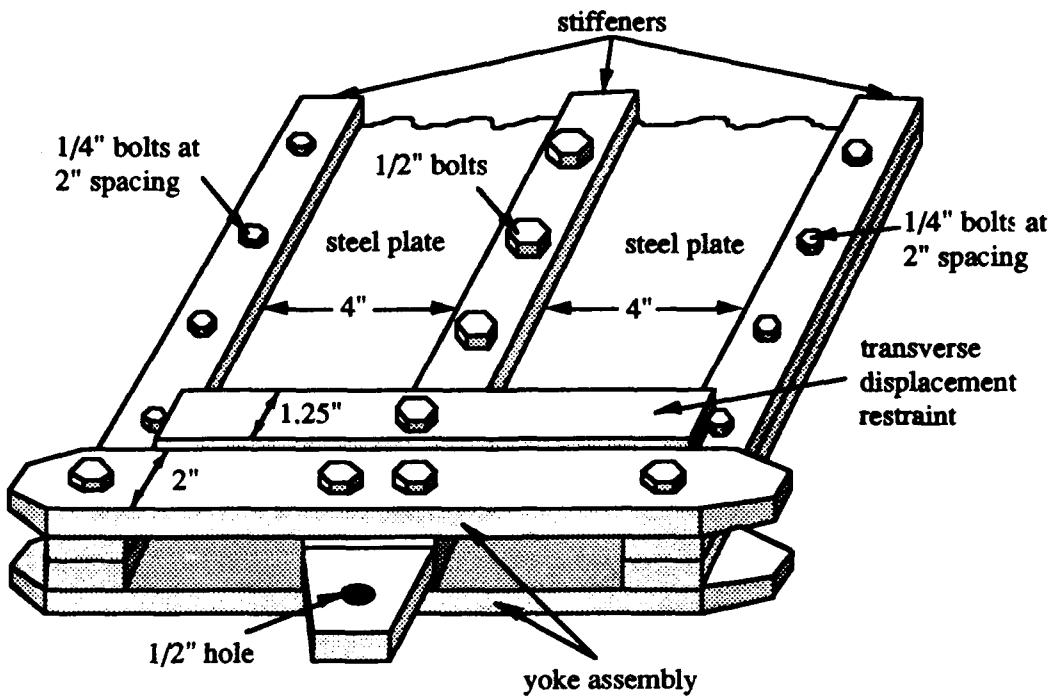


Figure 10B: Buckling apparatus (end angle view)

The outer steel stiffeners are prevented from moving closer together by cross bars bolted to the inner or outer stiffeners at each end of the apparatus. These cross members allow transverse tensile stresses to build up in the plate during buckling, while also permitting relative motion parallel to the machine loading axis between the center and outer bar stiffeners. These cross members are labeled as "transverse displacement restraints" in Figures 10A, 10B, and 11.

This steel frame is specifically designed to model the behavior of a ship's longitudinally stiffened hull plate structure subjected to in-plane shearing due to unequal longitudinal displacement of the stiffeners along each edge of the plate. This difference in longitudinal displacement between the two adjacent stiffeners causes in-plane shear to develop within the plate which can be induced by global indentation of the ship's hull structure. This hull indentation is modeled to be the result of a ship running into an underwater obstacle, such as a rocky bottom, and will be discussed further in Chapter 6.

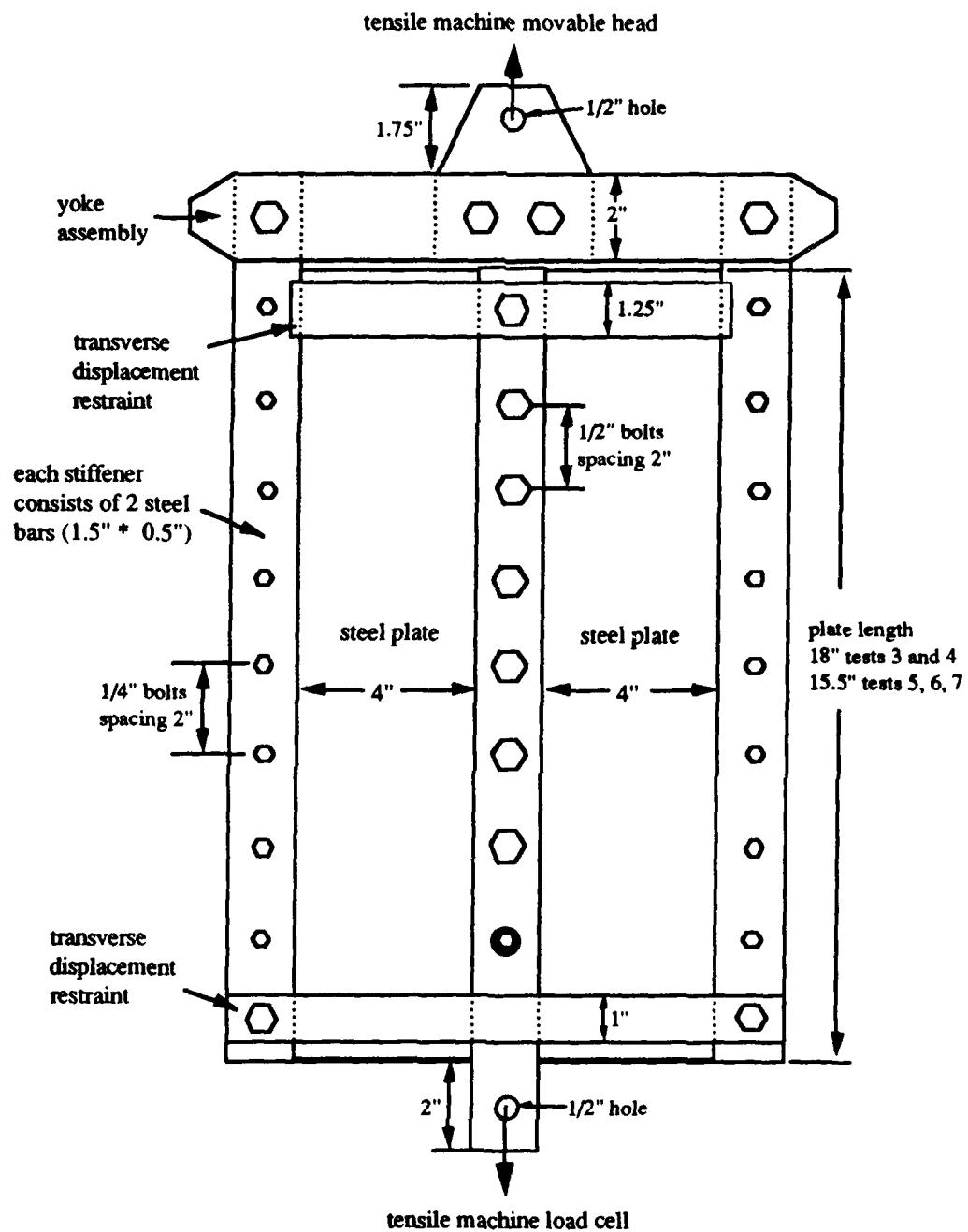


Figure 11: Buckling apparatus (plane view)

## **4.2 Design of Buckling Apparatus**

### **4.2.1 Metal Test Frame**

The priority for the design of the buckling test frame is that it be reusable so that various tests can be conducted on thin steel sheet metal. Another important factor is the load testing machine into which the test frame is to be fixed. The size and load carrying capacity of the tensile testing machine has dictated that the overall length of the apparatus should be no more than 24 inches for the plate thickness utilized. A final constraint is that the deflections within the test frame not be excessive at full load. This is necessary to approximate the rigid boundary conditions of the analytical model. A non-rigid frame can introduce errors in the load versus displacement test results which increase the displacements by a small amount and make the plate appear to have increased flexibility in shear buckling by allowing the plate to buckle more easily.

### **4.2.2 Steel Plating**

The thickness of the steel plating was driven more by the thinnest the supplier had in inventory than by design concerns. Rough calculations assured that the plate would fail long before the metal frame and bolts. Another consideration is that if the plate were too thick, plastic failure would occur without buckling. This is undesirable because the purpose of the experiment is to validate the analytical model which assumes buckling. The thickness of plate chosen for the initial series of tests was 0.284 inches, which results in a theoretical critical buckling load that is well below the load for no-buckling plastic flow (NBPF).

### **4.2.3 Fasteners**

The bolts used in this design were hardened, high strength, grade 8 fasteners. Failure of any fastener was considered unacceptable in this design due to the fact the it would interrupt the buckling test and might cause damage to the metal test frame. The high strength bolts can also take the torque necessary to apply significant pressure and increase friction between the plate surfaces and the stiffener bars. Friction between the 1/2 inch stiffener bars and the plate sandwiched in between them is considered important. Otherwise, localized failure can occur where the bolts penetrate the plate, since not enough of the shear load is distributed to the plate through friction.

#### 4.2.4 Ball Bearing Slippage Restraint

After the first two buckling tests revealed that localized failure at the bolt holes was occurring in the center stiffener, a modification was made to the center stiffener to distribute the shear load more uniformly. This was achieved through the use of a dozen 1/4 inch ball bearings placed in 1/4 inch holes drilled into one stiffener bar to a depth of 1/8 inch while slightly larger (5/16") holes of the same depth are drilled in the other bar. The assembly of the bars with the ball bearings and steel plate sandwiched in between results in local plastic deformation of the steel plate, forcing it to conform to the surface of the ball bearings, thereby locking the steel plate in position and preventing slippage. The locations of the 1/4 inch holes drilled to a depth of 1/8 inch for the ball bearings are shown below in Figure 12.

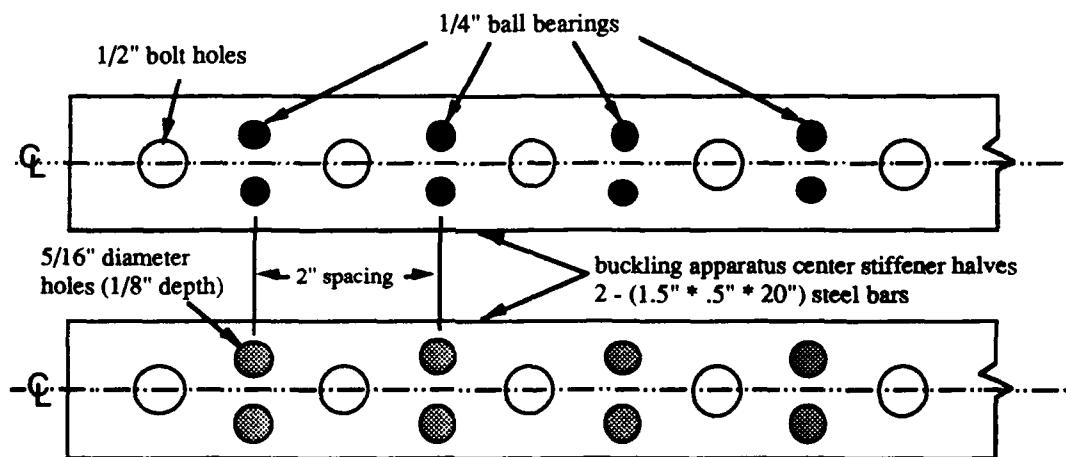


Figure 12: Ball bearing slippage restraint

Two important factors must be considered in the design of this slippage restraint. First, the twelve ball bearings should carry at least half of the load. The reasoning behind this is that the plating sandwiched between the outer stiffeners did not fail locally in the vicinity of the bolts during the first two tests. This plating is subjected to half of the load of the plating bolted to the center stiffener. Therefore, reducing the localized loading around the center stiffener bolts by one half should prevent localized failure of the plating. Second, the seven 1/2 inch bolts must be able to be torqued down with a sufficient force to plastically deform the plate over the ball bearings and cause both of the stiffener bars to come in contact with the plate.

Some preliminary calculations have been done to estimate the effectiveness of the ball bearing slippage restraints. The allowable shear of 10 kips to be held by the 12 ball

bearings gives a load of 0.833 kips/bearing. This force is transmitted from the bars through the ball bearings to the steel plating in the local vicinity surrounding the ball bearing. The diameter of the hemispherically deformed plating surrounding the ball bearing is assumed to be 5/16 inch, the diameter of the center stiffener holes into which the plate is to be pressed.

Therefore, the local stress on the plating surrounding the bolt is the shear force on the bearing divided by the circumference of the hemispherical region of plating and the plate thickness.

$$\text{Local Plate Normal Stress} \approx \frac{0.833 \text{ kips}}{\pi \left( \frac{5}{16} \text{ in} \right) (0.0284 \text{ in})} = 29.9 \text{ ksi}$$

Since this local plate normal stress is less than the plate yield stress it is assumed that no yielding will occur around the ball bearings.

The next step is to determine the mean force necessary to press the 12 ball bearings into the steel plate until both center stiffener bars are in contact with the plate. This has been accomplished by equating the external work of this mean force moving through a distance necessary to make the bars come into contact with the flat portion of plate (1/8 inch) to the plastic work necessary to obtain the local plating deformation. In order to see the mechanisms by which plastic work is done on the plate it is necessary to examine the deformed geometry of the plating surrounding the ball bearing as shown in Figure 13.

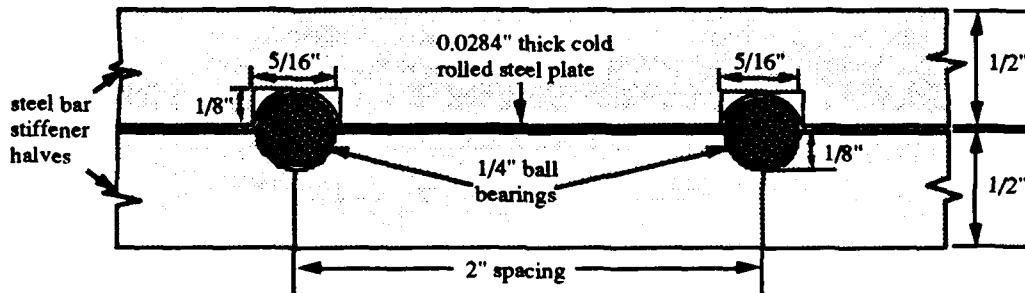


Figure 13: Cross section of ball bearing plate restraining mechanism

The plastic work done on the plating consists of two components, membrane work and bending work. The membrane work can be calculated by considering the fact that the original surface area of the plating is equal to the ball bearing maximum cross-sectional area and the final surface area is roughly equivalent to the surface area of a hemisphere of 5/16 inch diameter. If a plane stress condition is assumed everywhere in the plate, except in a 5/16 inch diameter circular plastic hinge where the plate initially deforms out-of-plane, both the bending and membrane energy contributions can be calculated.

The membrane work for plating of thickness  $t$  measured in inches becomes:

$$E_m = \sigma_0 t [\text{increase in plate surface area}]$$

$$E_m = \sigma_0 t \left[ \frac{1}{2} \pi \left( \frac{1}{4} \text{ in} \right)^2 - \pi \left( \frac{1}{8} \text{ in} \right)^2 \right]$$

$$E_m = \sigma_0 t \left[ \frac{\pi}{64} \text{ in}^2 \right]$$

Assuming a plastic hinge rotation of approximately 75 degrees, the bending work becomes:

$$E_b = M_0 \theta \pi \left( \frac{5}{16} \text{ in} \right) \quad \text{where } \theta = \text{plastic hinge rotation angle} = 75^\circ$$

$$E_b = \left( \frac{\sigma_0 t^2}{4} \right) \left( \frac{75\pi}{180} \right) \left( \pi \frac{5}{16} \text{ in} \right) = \sigma_0 t^2 \left( \frac{25\pi^2}{768} \text{ in} \right)$$

The total strain energy absorbed for the plating deformed by twelve ball bearings becomes:

$$E_{int} = E_m + E_b = 12 \sigma_0 t \left[ \frac{\pi}{64} \text{ in}^2 + \left( \frac{25\pi^2}{768} \text{ in} \right) t \right]$$

The mean force  $F_m$  required to press the center steel stiffeners together can be obtained by equating the external work to the total strain energy .

$$F_m s = E_{int} = 12 \sigma_0 t \left[ \frac{\pi}{64} \text{ in}^2 + \left( \frac{25\pi^2}{768} \text{ in} \right) t \right]$$

$$F_m = \frac{12 \sigma_0 t}{s} \left[ \frac{\pi}{64} \text{ in}^2 + \left( \frac{25\pi^2}{768} \text{ in} \right) t \right]$$

$F_m$  can be solved for by substituting  $\sigma_0 = 43.4$  ksi,  $t = 0.0284$  inch, and  $s = 1/8$  inch.

$$F_m = 6888 \text{ lbf}$$

The force per bolt  $F_b$  for seven 1/2 inch bolts becomes:

$$F_b = \frac{F_m}{7} = 984 \text{ lbf}$$

The bolt tensile stress necessary to press the center bars together and deform the plate is obtained by dividing the force per bolt  $F_b$  by the bolt cross-sectional area.

$$\text{Bolt Stress} = \frac{984 \text{ lbf/bolt}}{\frac{\pi (.38")^2}{4}} = 8676 \text{ psi}$$

The above stress is much less than the maximum stress allowed for the high strength bolts, however the bolt torque necessary to achieve this stress must also be calculated to see if the bars can be pressed together solely by torquing down on them with a torque wrench.

The geometry of the 1/2 inch diameter bolt threads must be examined to calculate bolt torque. The lead of the bolt threads is the distance that the bolt moves for each turn of the threads. These bolts have 13 threads per inch so the lead is (1in/13 threads) or 0.07692 in/turn. The lead angle is defined as:

$$\theta = \text{lead angle} = \arctan \left[ \frac{\text{lead}}{\pi (\text{thread diameter})} \right]$$

$$\theta = \arctan \left[ \frac{.07692 \text{ in/thread}}{\pi (0.45 \text{ in})} \right] = 3.1 \text{ degrees}$$

The coefficient of friction between the bolt and nut surfaces can be assumed conservatively as  $\mu_s = 0.6$ . The friction angle is therefore defined as:

$$\phi_s = \arctan[\mu_s] = \arctan[0.6] = 31.0 \text{ degrees}$$

The forces acting on the bolt threads are oriented approximately 30 degrees off the bolt rotational axis for triangular shaped threads. Therefore, the axial force on the threads is equal to:

$$\text{Axial Force on Bolt Threads} = \frac{F_b}{\cos(30^\circ)}$$

A free body diagram of the forces acting on the bolt threads is shown below in Figure 14.

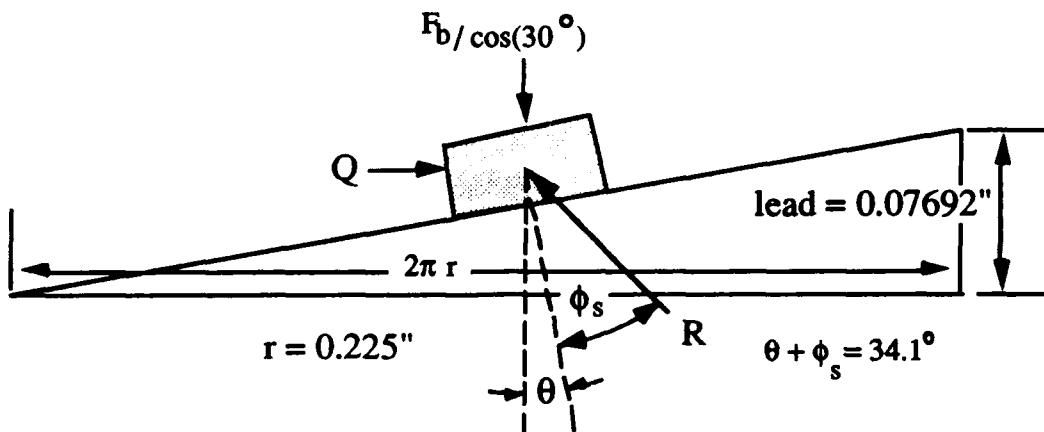


Figure 14: Bolt free body diagram

Figure 14 shows that the resultant force  $R$  and the radial force  $Q$  can be solved for by setting the summation of forces in the axial and radial directions equal to zero. The resultant force  $R$  can first be determined by setting the sum of the forces in the axial direction equal to zero.

$$R \cos(\theta + \phi_s) = \frac{F_b}{\cos(30^\circ)} \rightarrow R = \frac{984 \text{ lbf}}{\cos(34.08^\circ) \cos(30^\circ)} = 1372 \text{ lbf}$$

The radial force  $Q$  can now calculated for by setting the sum of forces in the radial direction equal to zero.

$$Q = R \sin(\theta + \phi_s) = (1372 \text{ lbf}) \sin(34.08^\circ) = 769 \text{ lbf}$$

Now that the radial force  $Q$  has been calculated, the bolt torque can be solved for using the following formula.

$$\tau = Q \text{ (Thread effective radius)} = 769 \text{ lbf (0.225 in)} = 173 \text{ in-lbf} = 14.4 \text{ ft-lbf}$$

The bolt torque of 14.5 ft-lbf is relatively easy to achieve with a medium sized torque wrench. Therefore, it can be concluded that the center stiffener bars can be pressed together by torquing of the bolts alone.

#### 4.3 Tensile Specimen Test Results

The stress-strain curves of both 0.0284" and 0.0262" sheet steel plates used in the buckling tests have been measured by conducting two series of tensile tests on seventeen 1/2 inch wide sheet type tensile test specimens fabricated in accordance with the ASTM, section A370 specifications and shown in Figure 15.

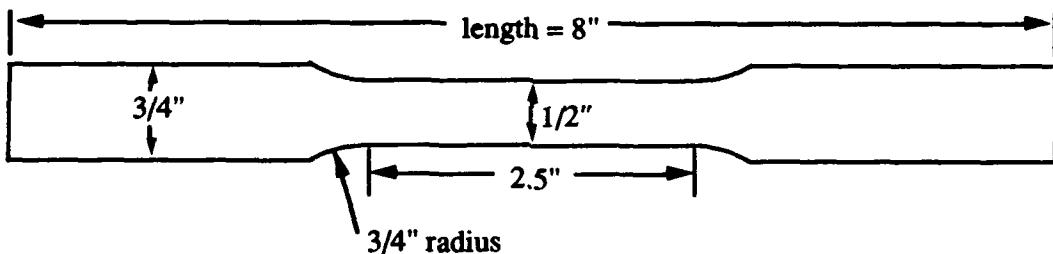


Figure 15: Tensile test specimen

Since the plate used for these experiments is cold rolled steel, a check for anisotropic mechanical behavior must be conducted. This has been accomplished on the first series of buckling tests by cutting the half the specimens parallel to and the other half perpendicular to the cold-rolled axis. For the second series of buckling tests specimens were cut parallel, perpendicular, and 45 degrees to the cold rolled axis to provide for complete data on in-plane orthotropy.

The IBM software program SYMPHONY was utilized for machine control and data recording and analysis during all phases of testing. A sampling interval of 0.1 sec was selected for the tensile tests and 1 sec for the buckling test. These sampling intervals were selected due to the fact that higher accuracy is needed during the tensile tests to accurately

determine the yield point and because the buckling tests lasted too long (approx. 1 hour) to make the higher sampling rate feasible due to insufficient computer memory. The buckling load tests were conducted by using displacement rate control with a displacement ramp of 0.01 mm/sec. This equates to a total displacement of 36 mm in one hour which results in a large enough shear translation to observe the full shear load capacity of the steel plate. The tensile tests were also conducted with displacement rate control but with an increased positive displacement ramp of 0.05 mm/sec.

The load cell was balanced for zero load prior to each buckling test. Balancing of both the load cell and extensometer strain gage was done before each tensile test. These calibrations have ensured that the load and strain signals are as close to zero as possible in the unloaded condition.

The output from the SYMPHONY data collection is two columns of load versus percent strain data for the tensile tests. A correction factor of 0.4 had to be applied to the raw strain data to take into account the extensometer gauge length for each of the tensile tests to give the actual percent strain. The load vs strain data was then verified by determining Young's Modulus E for the first specimen tested. Young's Modulus E for the first specimen was measured to be 30,500 ksi providing an initial check of the data.

Two quantities are desired from each of the tensile tests. These are the 0.2% yield strength, and the average plastic flow stress measured over a strain range corresponding to the particular buckling test series. The plastic flow stress,  $\sigma_0$ , is that stress that is postulated to be the most representative of the stresses in the plate during buckling and is therefore used to normalize the experimental buckling load data. A strain range of (0 - 6 %) was assumed for the first four buckling tests which approximately corresponds to the membrane strain along the stretched diagonal  $\epsilon_\eta$  of the element due to a maximum shear rotation  $\Phi_{max} \approx 20^\circ$ .

$$\epsilon_\eta = \frac{b(1/\cos\Phi - 1)}{b} = 1/\cos\Phi - 1$$

This calculation assumes that compressive normal stresses which build up in the element compressed diagonal to offset the tensile stresses are negligible due to the out-of-plane

buckling of the element along this diagonal. The maximum shear strain value  $\tau_{xy} = \tan(\Phi) = \frac{s}{b}$  corresponding to the shear rotation  $\Phi$  was not used due to the fact that this strain only exists throughout the plate surface if the plate remains in-plane and is not representative of the strain induced in a buckled element.

A strain range of (0 - 3 %) is assumed for the last three buckling tests corresponding to a maximum shear rotation  $\Phi_{max} \approx 14^\circ$ . The assumed maximum strains of 6 percent and 3 percent are close to the maximum stretched diagonal strains from the buckling tests shown in Table 1.

Table 1: Buckling tests maximum strain

| Buckling Test | $\Phi$ Maximum (deg) | Stretched Diagonal Strain % |
|---------------|----------------------|-----------------------------|
| 1             | 17.6                 | 4.91                        |
| 2             | 20.5                 | 6.76                        |
| 3             | 21                   | 7.11                        |
| 4             | 22.6                 | 8.32                        |
| 5             | 13.9                 | 3.02                        |
| 6             | 14.4                 | 3.24                        |
| 7             | 10.9                 | 1.84                        |

Average (tests 1-4): 6.78

Average (tests 5-7): 2.70

The average load over the assumed strain range is determined first for each specimen by calculating the area under the load versus strain curve for the assumed range of strain and then dividing this area by the strain range. The flow stress is then determined for each specimen by dividing this average load by the specimen cross-sectional area.

The flow stress and yield strength for both series of tensile tests are presented in Tables 2.1 and 2.2. The average yield strength and plastic flow stress are shown at the bottom of each table.

The standard deviation of the flow stress from the mean for the first series of tests is 0.83 ksi and the average flow stresses at zero degrees and 90 degrees to the cold rolled

axis are 44.15 ksi and 42.60 ksi respectively indicating some anisotropy. This level of anisotropy is considered small enough so that the mean flow stress from this series of tests can be used for normalization of the buckling test data.

Table 2.1: Tensile test results (0.0284" steel sheet)

| Specimen | Knoop Hardness HK | Orientation to cold roll direction | Yield Stress (ksi) | Plastic Flow Stress (ksi) |
|----------|-------------------|------------------------------------|--------------------|---------------------------|
| 2        | 129               | 0                                  | 38.70              | 44.43                     |
| 4        | 124               | 0                                  | 38.13              | 43.68                     |
| 6        | 123               | 0                                  | 38.52              | 44.40                     |
| 8        | 123               | 0                                  | 38.47              | 44.10                     |
| 1        | 122               | 90                                 | 37.17              | 42.39                     |
| 3        | 120               | 90                                 | 37.92              | 43.03                     |
| 5        | 119               | 90                                 | 37.01              | 42.30                     |
| 7        | 116               | 90                                 | 37.70              | 42.68                     |

Table 2.2: Tensile test results (0.0262" steel sheet)

| Specimen | Knoop Hardness HK | Orientation to cold roll axis | Yield Stress (ksi) | Plastic Flow Stress (ksi) |
|----------|-------------------|-------------------------------|--------------------|---------------------------|
| 1        | 89.7              | 0                             | 32.86              | 34.85                     |
| 2        | 94.3              | 0                             | 33.09              | 35.05                     |
| 3        | 94.3              | 0                             | 33.04              | 34.96                     |
| 4        | 92.7              | 90                            | 31.41              | 34.60                     |
| 5        | 110.7             | 90                            | 51.96              | 49.02                     |
| 6        | 112               | 90                            | 49.43              | 46.60                     |
| 7        | 95                | 45                            | 34.03              | 35.97                     |
| 8        | 95                | 45                            | 34.00              | 36.10                     |
| 9        | 94.7              | 45                            | 33.74              | 36.23                     |

Specimens 5 and 6 from the second test series show a yield strength and plastic flow stress that are significantly higher than all other specimens. The Rockwell Superficial Hardness of these specimens was also correspondingly greater by roughly 5 hardness units. These specimens were taken from the left over material from fabrication of the plates used in buckling tests 5 and 6 while all the other specimens were taken from a single plate that was reported by the supplier as being from the same lot of material. The measured strengths of the specimens taken from this plate were different enough to preclude use of these results for normalization of the buckling data. Therefore, only specimens 5 and 6, of the nine tensile tests using 0.0262" plate, can be utilized for normalization of the buckling test data that uses plating of this thickness. The mean values for yield strength and flow stress from Table 2.2 are unacceptable for use in the buckling tests because they include all the tests. The mean flow stress from specimens 5 and 6 to be used in buckling tests 5, 6, and 7 calculations is 47.81 ksi.

#### 4.4 Buckling Load Test Results

Four shear buckling load tests on 0.0284" thick cold rolled sheet steel specimens and three shear buckling load tests on 0.0262" steel sheet have been conducted for comparison with the predicted load from the analysis performed in chapter 3. The output from the SYMPHONY data collection is in the form of two columns of load versus displacement data for the load tests. The displacement data from the buckling tests must be corrected due to the fact that some initial slippage always occurs within the buckling apparatus at the onset of loading. This data is corrected by determining the displacement at which the linear load curve intercepts the horizontal displacement axis and then subtracting this displacement from the raw displacement data to obtain a displacement corrected for slippage.

The first two buckling tests were conducted without a ball bearing plate slippage restraint installed in the center stiffener bars. Local plate tearing failure surrounding the center stiffener bolts during these first two tests demonstrated that torquing down on the center stiffener bolts to increase friction between the stiffener bars and the plate was not sufficient to prevent plate slippage. Some other mechanical means were needed to prevent this localized failure due to the fact that slippage, which allowed local tearing of the plate in the vicinity of the bolts, reduced the plate load carrying capacity. Therefore, although the first two buckling tests are useful in providing an indication that some kind of slippage restraint is necessary, the data itself is not useful in validating the analytical model results.

Tests 3 and 4 are conducted with the ball bearing slippage restraint in the center stiffener and the results show an increase in the overall shear resistance of the plate. Reinforcement of the center stiffener with the ball bearing slippage restraint results in reduced local tearing in the center of the plate. However, the plate corners that are sandwiched between the two outer edge stiffeners have started to undergo some plastic deformation and tearing that was not observed in the first two tests.

A comparison between the results from buckling tests 3 and 4 and the analytical model is made after dividing the measured force by the plastic hinge moment  $M_o$  and then dividing again by the plate length of 18 inches to obtain a normalized shear force per unit length in inches<sup>-1</sup>. The non-dimensional results from tests 3 and 4 are plotted along with the results of the analytical models for comparison in Chapter 5.

Tests 5, 6, and 7 are conducted with a reduced plate length of 15.5" and a slightly reduced plate thickness of 0.0262" in an attempt to both reduce the load on the testing machine and revalidate the analytical model with a somewhat different plate geometry. Additionally, ball bearing slippage restraints are added to the side stiffeners in an attempt to prevent plastic deformation at the plate corners.

Tests 5, 6, and 7 demonstrated that the side stiffener slippage restraints indeed reduced the local plastic deformation at the plate corners, but increased tearing in the corners. These cracks form in the vicinity of the bolt holes or ball bearing holes and propagate longitudinally, in a direction perpendicular to the tensile stresses that build up in the transverse direction. Therefore, the additional ball bearing slippage restraints within the edge stiffener are ineffective in accomplishing their intended purpose due to a reduction of one local failure mode but increase of another.

The normalized loads per inch of plate length parallel to the machine loading axis from buckling tests 3 through 7 are shown in Figure 16. The shorter plates used in tests 5, 6, and 7 lost their load resisting capacity at a lower  $\Phi$  due to local cracking in the corners caused by the increased rigidity from adding ball bearing slippage restraints to the side stiffeners. The subsequent propagation of cracks from these corners further reduced the load capacity of these plates.

Two other observations made during the buckling tests are the wavelength  $2a$  and the initial angle  $\beta$  relative to the transverse direction. The final orientation  $\beta$  of the buckling waves was measured on the plates after removal from the testing machine and is between

40 and 50 degrees for all tests. The initial angle  $\beta$  is the difference between this final measured angular orientation and the final angle of shear rotation  $\Phi$ . For the first two tests the maximum shear rotation achieved is approximately 20 degrees which means that the initial buckling orientation  $\beta = 25$  degrees or 0.4 radians. The final plastic buckling wavelength  $2a$  observed is approximately 2.1 inches. This equates to a  $\lambda = a/b$  of 0.26.

Shortening of the plastic buckling wavelength was observed during all of the tests as the shear deformation progressed. In an attempt to better understand the wavelength shortening phenomenon, the buckling wavelength has been measured during tests 5, 6, 7 at various shear translation values  $s$  and plotted in non-dimensional form in Figure 17.

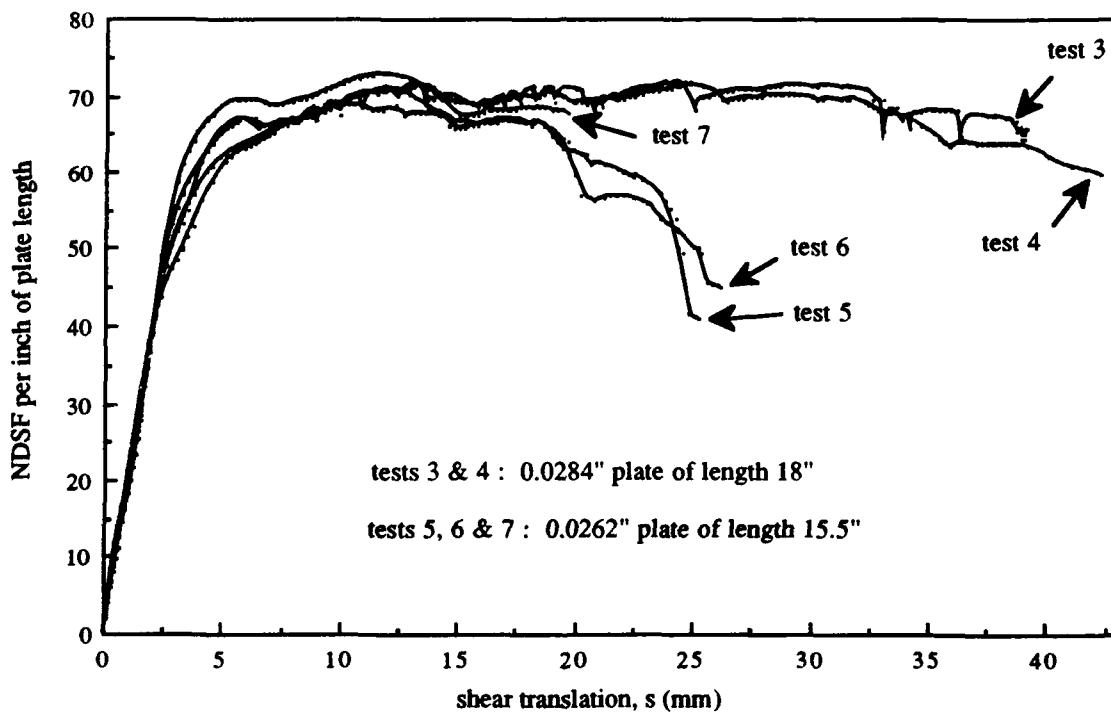


Figure 16: Normalized shear force buckling test results

For all tests the initial wavelength is between 6.5 inches and 7.0 inches. This equates to a full non-dimensional wavelength  $\Lambda = 2a/b$  of 1.69 which is just slightly greater than the elastic buckling wavelength  $\Lambda = 1.6$  predicted by Southwell and Skan<sup>5</sup> for an infinitely long flat plate with clamped edges. The fact that the initial wavelength is slightly greater than the Southwell and Skan solution can be justified by the fact that the clamped

edges of the plate are not perfectly rigid. The Southwell and Skan solution predicts a longer elastic wavelength  $\Lambda = 2.67$  for simply-supported edges. Since the longitudinal supports for the plate are not perfectly rigid, the wavelength  $\Lambda$  should assume a value greater than the clamped but less than the simply-supported plate.

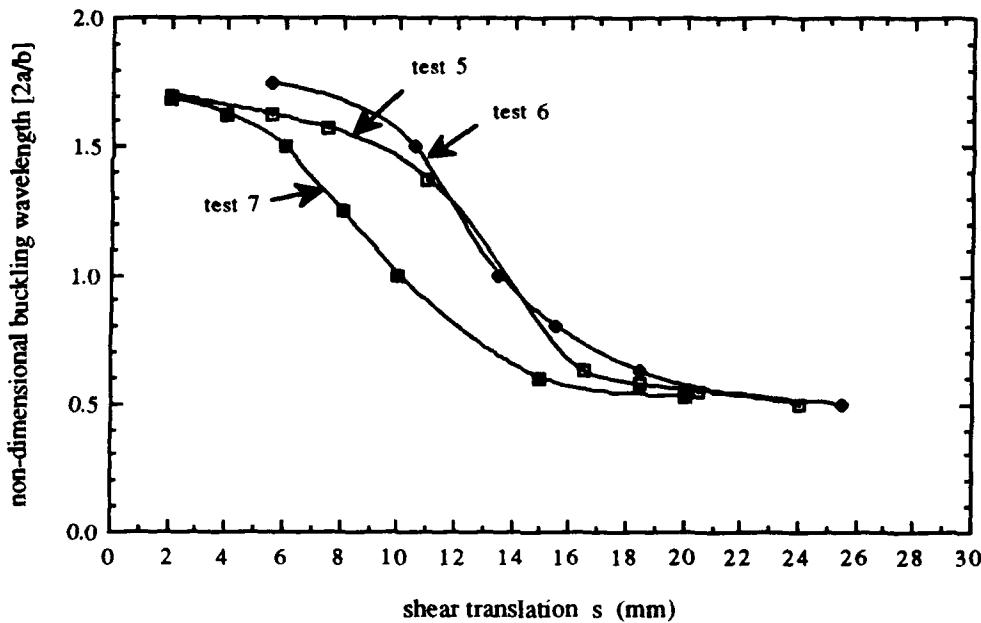


Figure 17: Observed non-dimensional buckling wavelength [2a/b]

Figure 17 shows the observed change in the non-dimensional buckling wavelength  $\Lambda = 2a/b$  during tests 5, 6, and 7. Figure 17 also illustrates that a transition to a shorter buckling wavelength occurs between  $s$  values between 6 mm and 16 mm which correspond to  $\Phi$  between 3.4 degrees and 9.0 degrees. The plastic buckling wavelength is initially nearly equal to the predicted elastic wavelength but slowly develops into a shorter wavelength buckling mode as  $s$  increases. This process starts when the peaks of the initial buckling waveform start to flatten out. As the shear translation continues, two new waves form on the flattened out maxima and minima of the original mode. These newly generated waves then move away from each other until a uniformly spaced buckling mode is formed with a wavelength that is one third of the original wavelength. The phases of this process are shown in Figure 18.

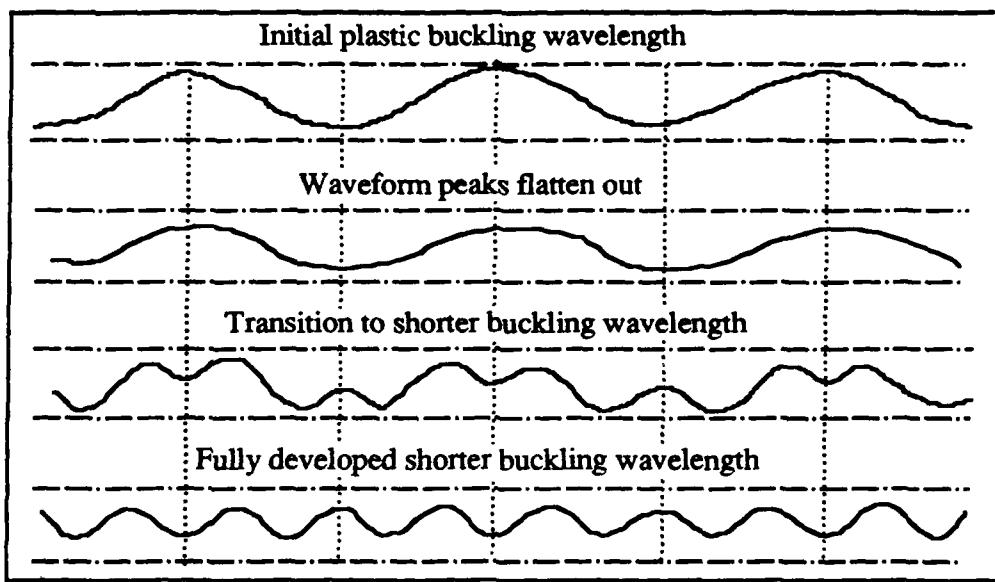


Figure 18: Buckling wavelength transition process

## 5. Comparison of Experimental and Analytical Results

The normalized shear force ( $F_s/M_0$ ) per inch of plate length from buckling tests 3 and 4 has been plotted against the analytical models and is shown in Figures 20 and 21. The results from tests 5, 6, and 7 have already been compared with tests 3 and 4 in the previous chapter and are not plotted here but are presented in Appendix B. Figure 20 includes the plastic buckling model for  $(0.1 < \lambda < 1.0)$  and  $\beta = 0$  radians. Figure 21 includes the plastic buckling model for  $\beta = 0.4$  radians. The model for  $\beta = 0.4$  is included here because the measured orientation of the buckling waves at the end of the tests is approximately  $45^\circ$  and subtracting the maximum shear rotation  $\Phi = 20^\circ$ , obtained during tests 3 and 4, yields an initial angle  $\beta$  of  $25^\circ$  or 0.4 radians. The plastic model results calculated for  $\beta$  from 0.1 to 0.6 are included in Appendix B.

Figure 20 shows that the plastic buckling model for  $\beta = 0$  is a good approximation for the non-dimensional shear force actually observed during the experiments. This model also predicts the transition of the plastic buckling wavelength to a shorter mode and comes very close to predicting the actual wavelength itself. Figure 20 illustrates that the NDSF curves for tests 3, and 4 fall between the NDSF curves corresponding to a half wavelength  $\lambda$  between 0.2 and 0.3. As stated before, the measured half-wavelength  $\lambda$  from these tests was approximately 0.26.

The plastic buckling curves for  $\beta = 0.4$  in Figure 21 are significantly greater than the experimental results for tests 3 and 4. These results, however, have not taken into account the fact that the combination of membrane and shear stresses working together actually reduces the shear and membrane yield stresses. Up to this point the analysis has assumed that yield always occurred when membrane stresses equal  $\sigma_o$  and shear stresses

$\tau_{xy} = \frac{\sigma_o}{\sqrt{3}}$ . This is not an entirely accurate approximation because the actual yield condition is:

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = \sigma_o^2$$

Since  $\dot{\epsilon}_x = k(2\sigma_x - \sigma_y)$  we can say that for  $\dot{\epsilon}_x = 0$   $\sigma_x = \frac{\sigma_y}{2}$ . Therefore, the yield condition becomes:

$$\frac{3}{4}\sigma_y^2 + 3\tau_{xy}^2 = \sigma_o^2$$

Figure 19 graphically shows the relationship between membrane and shear stresses at yield. When  $\tau_{xy} = 0$ ,  $\sigma_y = \sqrt{\frac{4}{3}} \sigma_o$  and when  $\sigma_y = 0$   $\tau_{xy} = \frac{\sigma_o}{\sqrt{3}}$ .

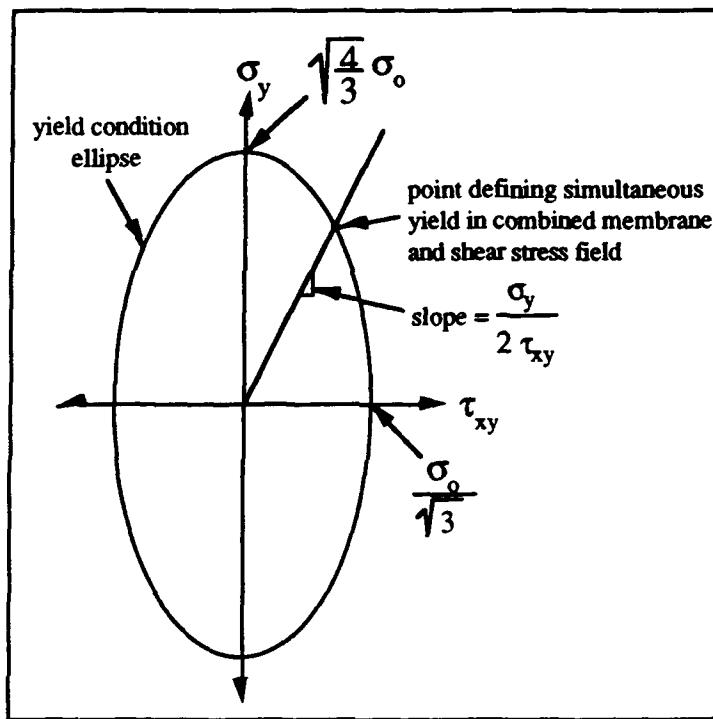


Figure 19: von Mises yield condition

The slope of the line which defines simultaneous yield in membrane and shear is:

$$\sigma_y = 2 \tau_{xy}$$

The intersection of this line and the yield curve is representative of the stresses during buckling and can be obtained by substituting for  $\sigma_y = 2 \tau_{xy}$  into the yield condition.

$$\frac{3}{4} (2 \tau_{xy})^2 + 3 \tau_{xy}^2 = \sigma_o^2 \quad \tau_{xy} = \frac{1}{\sqrt{6}} \sigma_o \quad \sigma_y = \frac{2}{\sqrt{6}} \sigma_o$$

The actual values for  $\tau_{xy}$  and  $\sigma_y$  are less than the assumed values used in the NDSF calculations by the following amounts.

$$\frac{\tau_{xy}(\text{actual})}{\tau_{xy}(\text{assumed})} = \frac{1}{\sqrt{2}} = 0.71 \quad \frac{\sigma_y(\text{actual})}{\sigma_y(\text{assumed})} = \frac{2}{\sqrt{6}} = 0.82$$

A rough approximation for the amount of reduction in the NDSF curves for  $\beta = 0.4$  of Figure 21 is approximately 25%. Even with this reduction in the NDSF buckling curves, they are still higher than the experimental data by approximately 40 NDSF units. Therefore, the  $\beta = 0$  curves more accurately predict the non-dimensional shear force and associated buckling wavelength  $\lambda$ .

The predicted slopes of the linear elastic pre and post-buckling curves are much steeper than the actual test data due to the fact that the analytical model assumes a perfectly rigid steel frame. In fact, the buckling apparatus does have a finite amount of flexibility. This causes the actual shear translation  $s$  to increase above the values predicted by analysis. The elastic load curves for each of the tests display a break in slope between an NDSF of 40 and 50 at a shear translation of approximately 2.3 mm. This is probably the point where plastic deformation starts to occur and not the critical point of elastic buckling due to the fact that the observed onset of elastic buckling should not occur after the theoretically predicted buckling load. The predicted critical buckling normalized shear forces for elastic buckling formulated in Chapter 3 are given below for both series of tests.

$$\text{NDSF}_{cr}/\text{unit length} = N_{xy,o}/M_o = \frac{8.98 \pi^2 E t}{3(1 - v^2) \sigma_o b^2}$$

$$\text{NDSF/inch tests 3 and 4 : } 38.8 \text{ in}^{-1}$$

$$\text{NDSF/inch tests 5, 6, and 7 : } 37.1 \text{ in}^{-1}$$

The no buckling plastic failure (NBPF) curve forms an upper envelope for all experimental test data since the normalized critical shear force required for plastic buckling is less than for in-plane deformation for the  $b/t$  ratio of the plating tested.

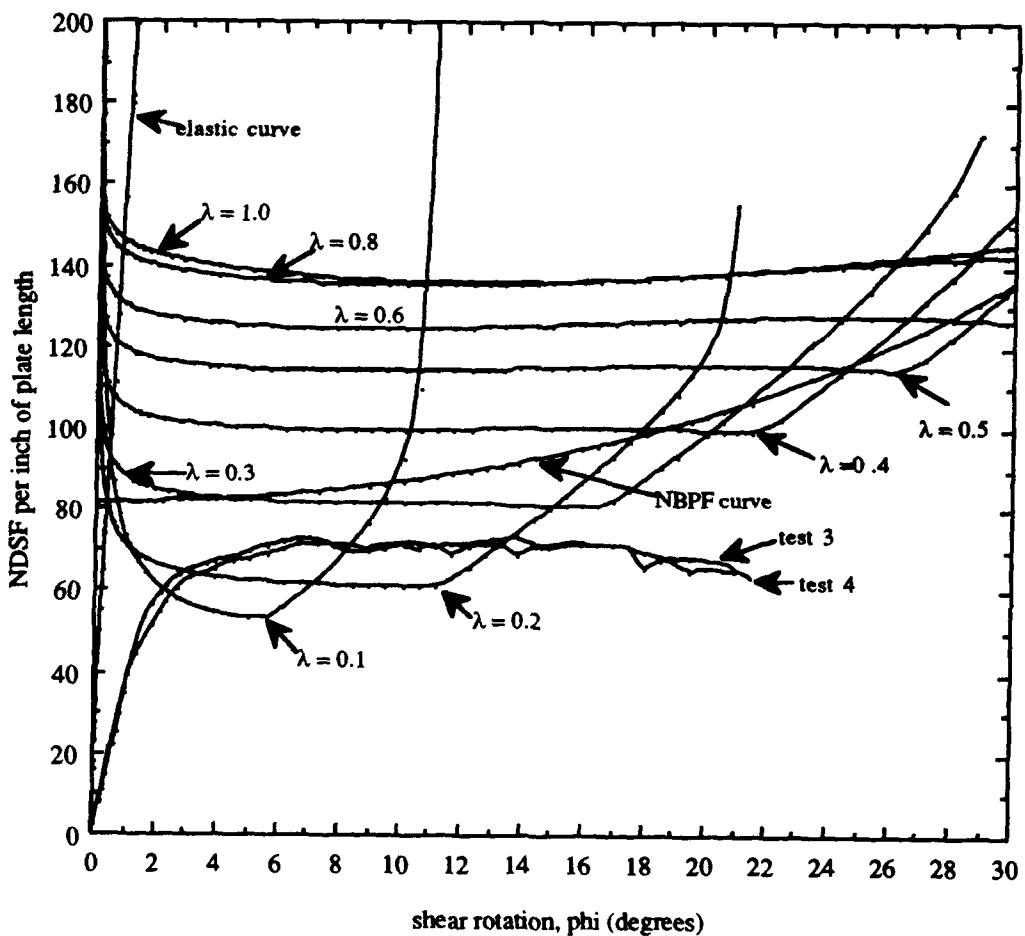


Figure 20: Comparison of analytical and experimental results ( $\beta = 0$ )

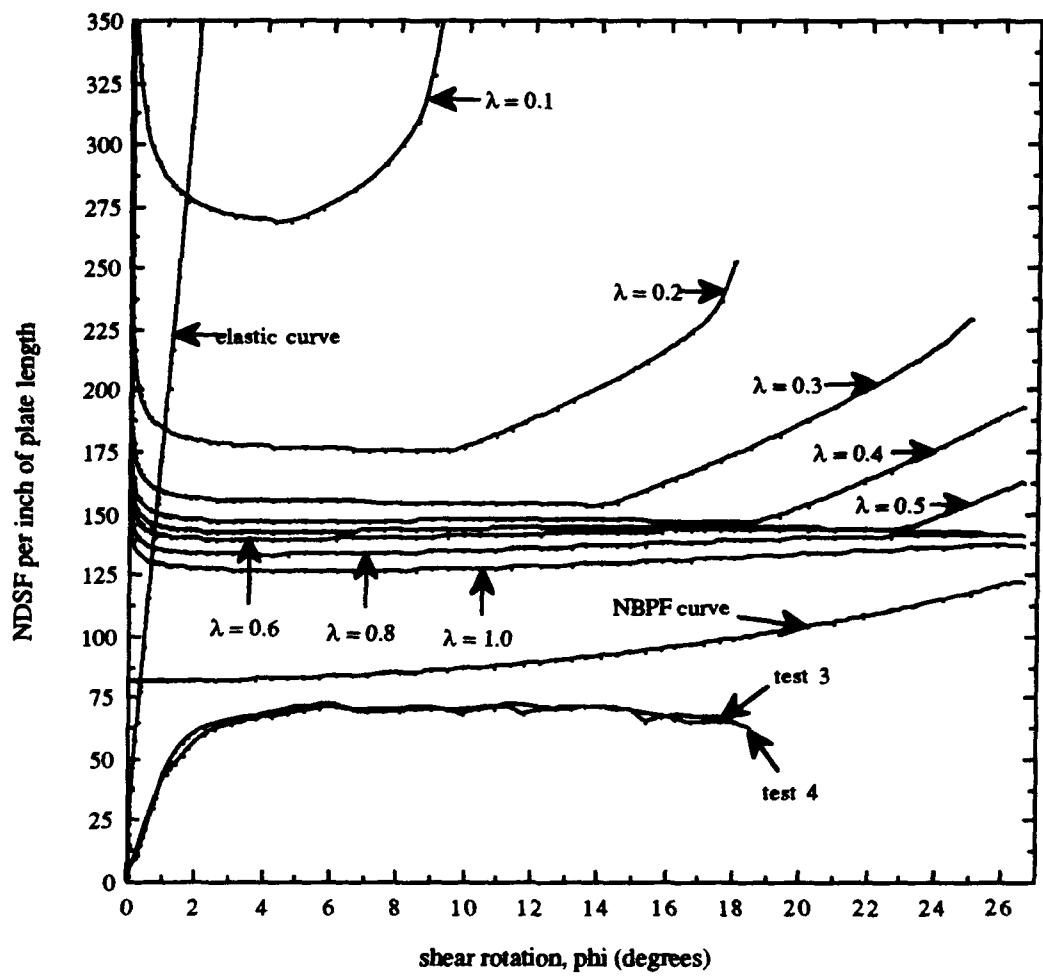


Figure 21: Comparison of analytical and experimental results ( $\beta = 0.4$ )

## **6. Formulation of total grounding resistance due to shear**

### **6.1 The Grounding Model**

The previous five chapters have focused on the development and experimental validation of a plastic shear buckling model for a plate element between two longitudinals. This chapter seeks to incorporate this local model into a global model for deformation of a VLCC hull bottom induced by grounding. The final goal of this new formulation is to determine the total resistance to ship motion over a rock due to in-plane shear buckling mechanisms.

To solve for the total grounding resistance due to shear a global hull deformation geometry must be assumed that will allow use of the plastic buckling element model. The energy dissipation mechanisms other than shear have already been analyzed by Peer<sup>4</sup> and will be presented for comparison with the shear grounding results in chapter 7 to determine the relative importance of in-plane plastic shearing as an energy dissipation mechanism during grounding.

Before proceeding further, the parameters of the global hull deformation problem are defined as follows.

#### **Parameters characterizing hull deformation**

- (x, y, z) --- longitudinal, transverse, and vertical coordinates
- (u, v, w) --- longitudinal, transverse, and vertical displacements
  - L --- overall length of ship
  - B --- beam of ship
  - b --- longitudinal stiffener spacing
  - l --- transverse web frame spacing
  - $l_1$  --- transverse extent of damage and longitudinal bulkhead spacing
  - $L_d$  --- longitudinal extent of fully developed damage cross section
  - $\eta$  --- longitudinal extent of partially developed damage transition zone
  - n --- transition zone parameter, number of transverse web frames included in damage transition zone
- $\Delta_t$  --- height of rock above ship keel prior to grounding
- $\Delta_L$  --- vertical lift of ship as it rides over rock during grounding
- $\Delta_o$  --- maximum amplitude of rock penetration into hull at the keel
- $\Delta_i$  --- maximum amplitude of rock penetration into hull at stiffener i

- $s_i$  --- total longitudinal displacement of hull plating in the vicinity of i-th stiffener occurring after stiffener has passed through damage transition region (assumes rigid stiffener and flexible shell plating)  
 $s_i(x)$  --- longitudinal displacement along deformed stiffener i measured at a distance  $x$  from the damage transition zone leading edge  
 $\psi_i$  --- total arc length along deformed stiffener i within the transition region  
 $\psi_i(x)$  --- arc length along deformed stiffener i measured at a distance  $x$  from the damage transition zone leading edge  
 $m$  --- total number of inter-longitudinal plate spans between the ship's centerline and the first longitudinal bulkhead

It is worth reminding the reader at this point that this formulation does not apply to grounding situations resulting in hull membrane rupture in the shear deformation region due to the fact that shear forces can only build up in the hull plating if it remains a continuous surface. Otherwise, the hull plating boundary conditions and geometry will not remain as previously assumed and the shear buckling problem will become extremely difficult to analyze.

There are several other assumptions that specifically apply to this global hull deformation model. First, the hull damage is assumed to be symmetric about the ship's keel or centerline. This allows external moments about the ship's yaw axis to be excluded which would otherwise lead to transverse components of grounding resistance causing damage to progress in both the transverse and longitudinal directions. This problem is extremely complex due to the progressive non-symmetry of the damage as well as the increased effect of complicated hull hydrodynamics on the problem.

The second assumption in this model involves the shape of the hull deformation. Figure 22 shows an assumed cross-sectional view and a longitudinal profile of the damage. The length of the damage progresses a distance ( $L_d + \eta$ ) where  $L_d$  is the longitudinal extent of the fully developed damage zone and  $\eta$  is the length of a damage transition zone. The length of this transition zone is assumed to be a multiple,  $n$ , of the transverse frame spacing,  $l$ , and the shape is assumed to be cosinusoidal with amplitude  $\Delta_0/2$  at the keel.

$$\eta = n l$$

$$w_0(x) = \Delta_0/2 [ 1 + \cos(\pi + \pi x/\eta) ]$$

The transverse cross section of the damage is assumed to be wedge shaped and is limited by the spacing between longitudinal bulkheads  $l_1$ . The vertical penetration of the damage  $\Delta_0$  is assumed to be twice the penetration of the rock above the ship's baseline in order to be consistent with the geometry analyzed by Peer. The rock height  $\Delta_r$  is the sum of the rock's vertical penetration into the ship's keel  $\Delta_0/2$  and the lift  $\Delta_l$  of the ship's hull as it rides over the rock.

$$\Delta_r = \Delta_0/2 + \Delta_l$$

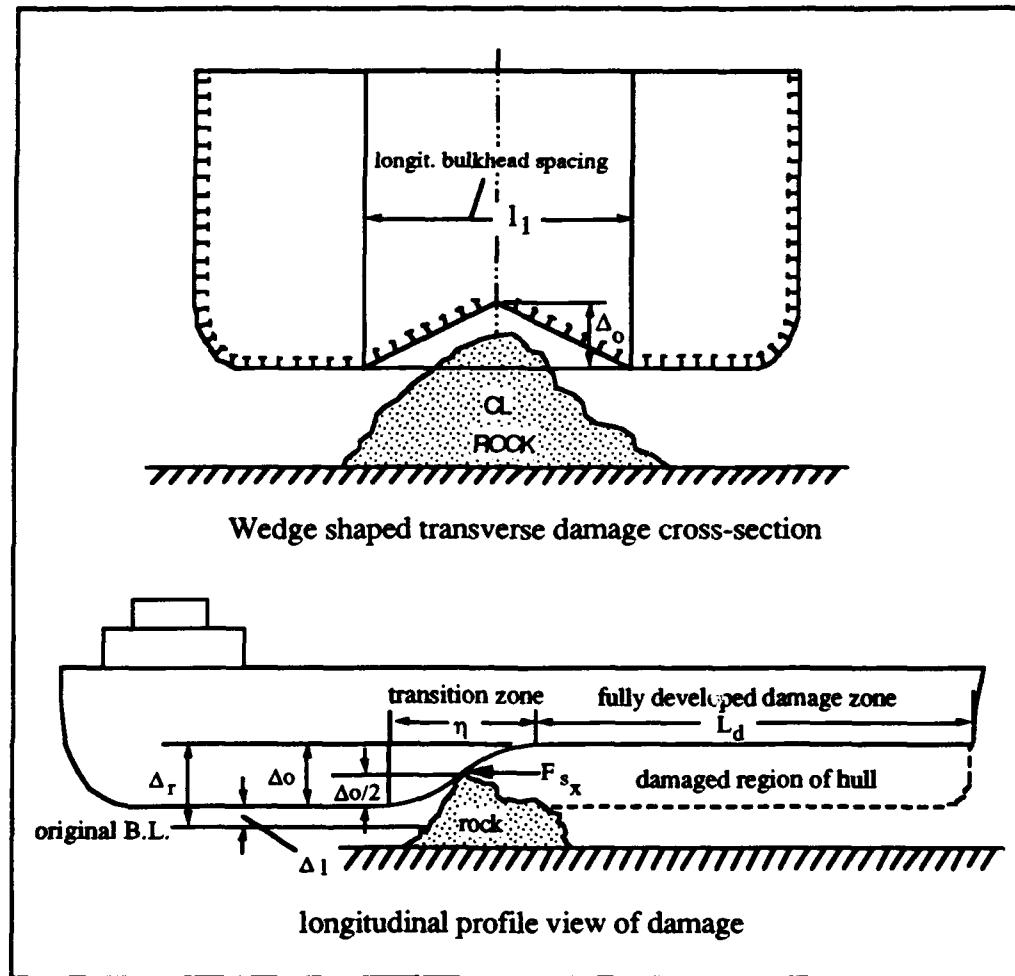


Figure 22: Grounding hull damage geometry

Figure 22 shows the overall extent of the transverse and longitudinal damage but fails to show the details of the damage transition zone. A more detailed profile view of the transition zone is given in Figure 23. The assumed longitudinal shape of the damage transition zone is chosen to be cosinusoidal to make the formulation easier while

maintaining some similarity to the shape assumed by Peer<sup>4</sup> to facilitate comparison with his results later on. The cosinusoidal damage profile can be extended to the longitudinal stiffeners by substituting  $\Delta_i$  for  $\Delta_0$  and realizing that, for the assumed wedge shaped cross section,  $\Delta_i$  can be expressed as a linear function of  $\Delta_0$ .

$$w_i(x) = \Delta_i/2 [ 1 + \cos(\pi + \pi x/\eta) ] \quad (49)$$

$$\Delta_i = \left( \frac{m-i}{m} \right) \Delta_0$$

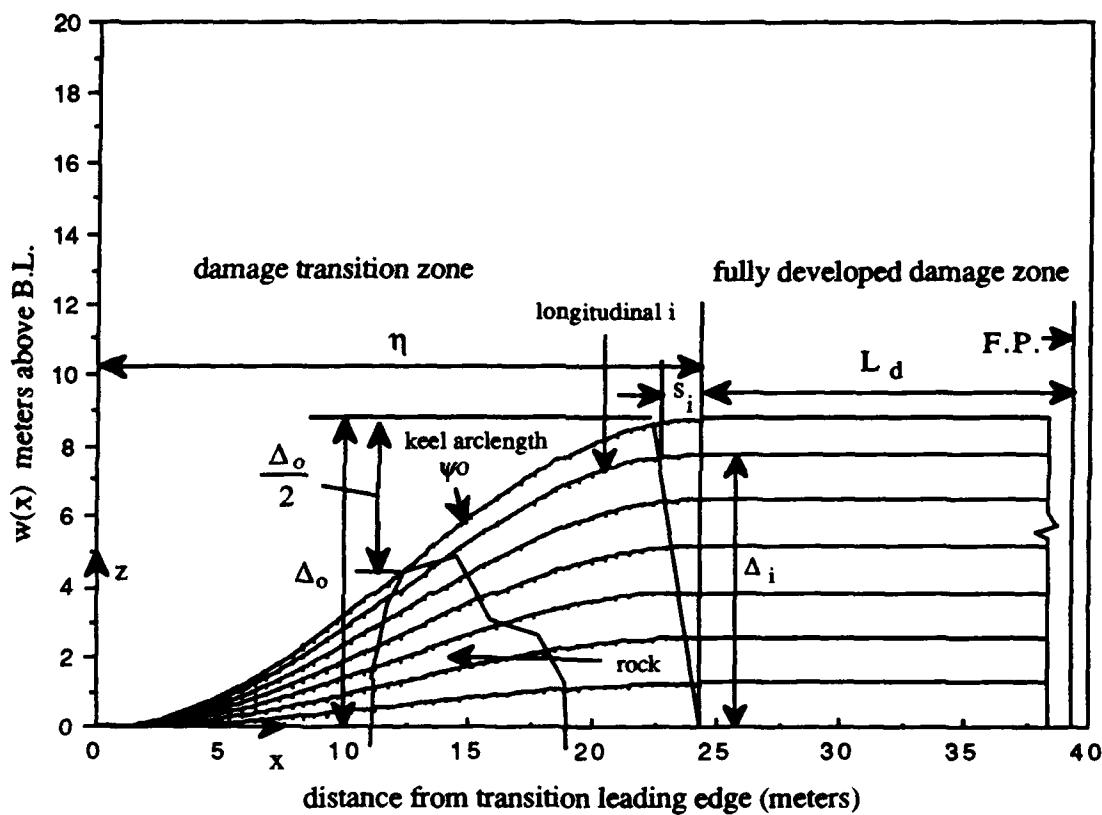


Figure 23: Damage transition zone for 5 m rock height

The constant  $m$ , which is the total number of inter-longitudinal plate spans between the keel and the first longitudinal bulkhead, can be determined from:

$$m = \frac{l_1}{2 b}$$

The subscript "i" is an integer designating the longitudinal stiffener that also gives the number of inter-longitudinal plate spans between that stiffener and the keel. In summary,

the shape of the damage transition zone profile is defined by the keel profile and deformed longitudinal profiles with identical shape but of linearly decreasing amplitude as the stiffener transverse distance from the keel increases.

The next step in the development of the grounding model is to relate the global deformation in the damage transition zone to in-plane shearing of the inter-longitudinal plating. This is accomplished by observing that in-plane shear in the hull plating is induced by differences in the damage profile arc lengths of adjacent longitudinals.

The arc length of each of the longitudinals has been solved for numerically using a spreadsheet. Only half of the damage transition region needs to be analyzed due to symmetry about the keel. The damage transition zone has been partitioned into a series of plate elements of length  $\eta/50$  and width  $b$ . For the purpose of this analysis partitioning the length of the transition zone into 50 elements provides sufficient accuracy in measuring the arc length of the deformed longitudinals. The out-of-plane deformations of the keel  $w_0(x)$  and all of the stiffeners  $w_i(x)$  are determined first. The arc length,  $\psi_i$ , of each stiffener through the transition zone is then calculated numerically by summing up the incremental arc lengths for each partition along the  $x$  axis,  $\Delta x = \eta/50$ . The relationships used in this formulation are given below.

$$\Delta\psi_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$\psi_i = \sum_{j=1}^n \Delta\psi_i(\Delta x_j)$$

The subscript "i" designates the particular longitudinal along which the arc length is measured. The subscript "j" provides the designation of the  $\Delta x$  increment where the arc length measurement is taking place where  $j = 1$  is defined as the increment closest to the transition zone leading edge. As an example  $\Delta\psi_1(\Delta x_1)$  corresponds to the arc length from  $x = 0$  to  $x = \eta/50$  of the first longitudinal outboard of the keel.

The arc length  $\psi_i$  is the total arc length of longitudinal i measured through the transition zone length to the coordinate  $x = \eta$ . This change in arc length corresponds to a stretching of the hull plating surrounding the i-th longitudinal through the transition zone.

From Figure 23 the parameter  $S_i$  measures the total longitudinal displacement of a point along the deformed arclength of stiffener i from its original undeformed position to its

final position after passing through the transition zone assuming an inextensional stiffener. Figure 24 shows a more detailed view of how a single stiffener deforms and causes stretching of the hull plating as it passes through the damage transition zone.

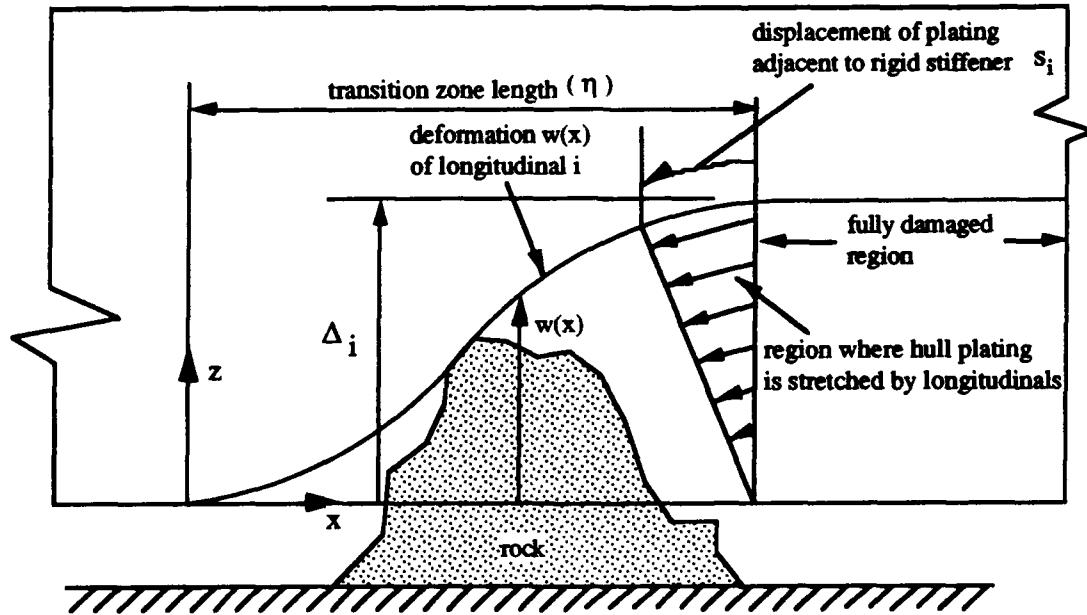


Figure 24: In-plane displacement of hull plating for rigid stiffeners

The total change in arc length for each longitudinal varies from a maximum value at the keel to zero at the first longitudinal bulkhead. The difference in the arc length profiles of adjacent longitudinals sets up differential longitudinal displacements,  $\Delta s_i$ , between the edges of the inter-longitudinal hull plating which results in in-plane shear deformation of the plating. Figure 25 shows how the longitudinal displacement of the hull plating surrounding each longitudinal varies as we progress off the ship's centerline axis.

It has been shown that in-plane shear forces are induced in the plating by a relative translation between adjacent stiffeners. This relative translation,  $\Delta s_i(x)$ , is just the difference in the plating displacement between adjacent stiffeners at a distance  $x$  from the transition zone leading edge. The maximum value for the relative translation,  $\Delta s_i$ , occurs after the damage transition has passed and is shown in Figure 25 for longitudinals 1 and 2. The relative translation,  $\Delta s_i(x)$ , is a general form of the shear translation,  $s$ , used in the chapter 3 formulation. The chapter 3 buckling model assumed that one edge was stationary while the other edge translated a distance  $s$ . The current model assumes that both edges translate and that the relative translation,  $\Delta s_i(x)$ , induces the shear forces. In the current

model the shear translation varies in both space and time and must be numerically mapped over the transition zone surface; otherwise, the formulation is equivalent to the earlier formulation in chapter 3. Figure 26 illustrates the geometric similarity of the two models.

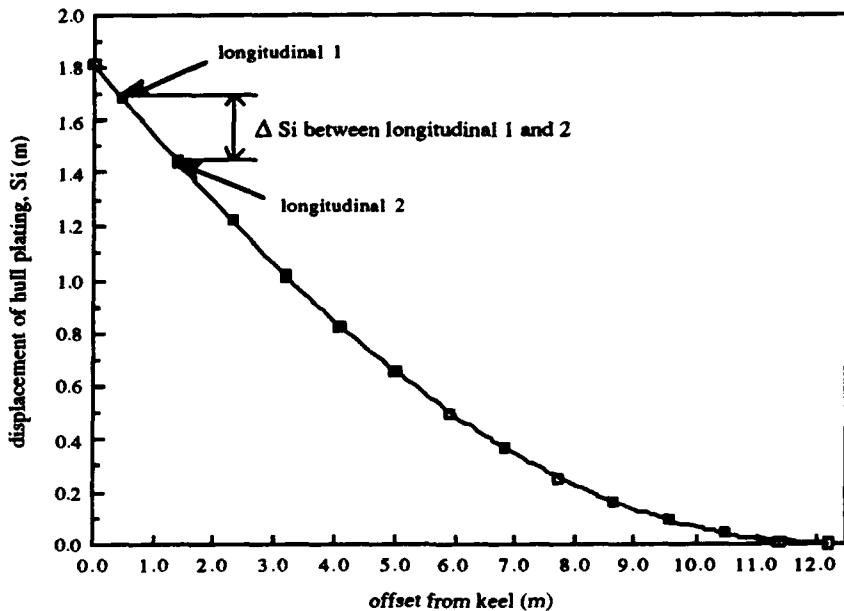


Figure 25: Maximum longitudinal displacement vs offset from keel

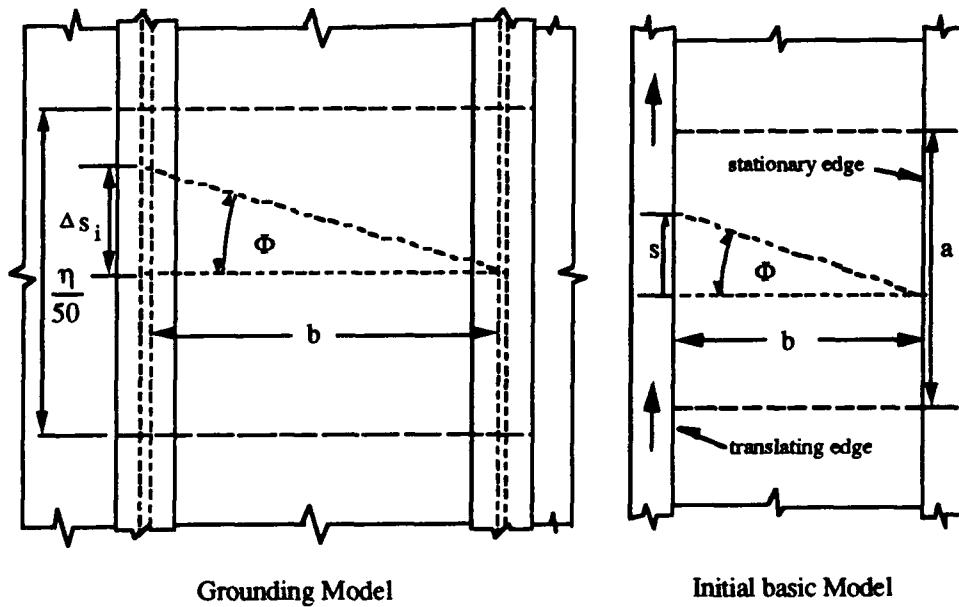


Figure 26: Plate element shear rotation geometry

Once the relative translation  $\Delta s_i(x)$  is determined for each of the plate elements over the surface of the transition zone the in-plane rotation  $\Phi$  can be calculated for each of these elements by the following relationship.

$$\Phi_i(x) = \arctan\left(\frac{\Delta s_i(x)}{b}\right)$$

The non-dimensional shear force per unit length can now be mapped over the surface by using a plot of NDSF vs  $\Phi$  for the representative VLCC dimensions and characteristic data shown in Table 3.

Table 3: VLCC dimensions and characteristic data

|  |   |
|--|---|
| $L = 313 \text{ m (1030 ft)}$              | <b>hull thickness = 19.1mm (3/4 in)</b> |
| $B = 56.6 \text{ m (185 ft)}$              | $b = 0.91 \text{ m (3 ft)}$             |
| $T_0 = 18.9 \text{ m (62 ft)}$             | <b>longitudinal T-stiffeners:</b>       |
| $\sigma_0 = 393 \text{ MPa (57 ksi)}$      | flange: (203 * 25.4 mm)<br>(8" * 1")    |
| rupture strain = 0.12                      | web: (711 * 12.7 mm)<br>(28" * 1/2")    |
| $l = 4.88 \text{ m (16 ft)}$               |   |
| $l_1 = 24.4 \text{ m (80 ft)}$             |   |
| Young's Modulus = 200,000 MPa (29,000 ksi) |   |

The NDSF model developed in chapter 3 was revised with new values from Table 3 for  $\sigma_0$ , hull plate thickness  $t$ , and longitudinal spacing  $b$ , and the results are plotted in Figure 27. The NDSF curves for plastic shear buckling in Figure 27 show an approximate lower bound of 3 for  $0 < \Phi < 17$  degrees. The non-dimensional shear force per unit length can therefore be approximated by the elastic NDSF curve up to NDSF = 3 and by a constant value of three up to  $\Phi = 17$  degrees as shown in Figure 28. Calculations of in-plane shear deformation are provided in Appendix A which show that the maximum value for  $\Phi$  that occurs for a rock heights less than 5 meters and transition zone parameters greater than 5 is less than 17 degrees.

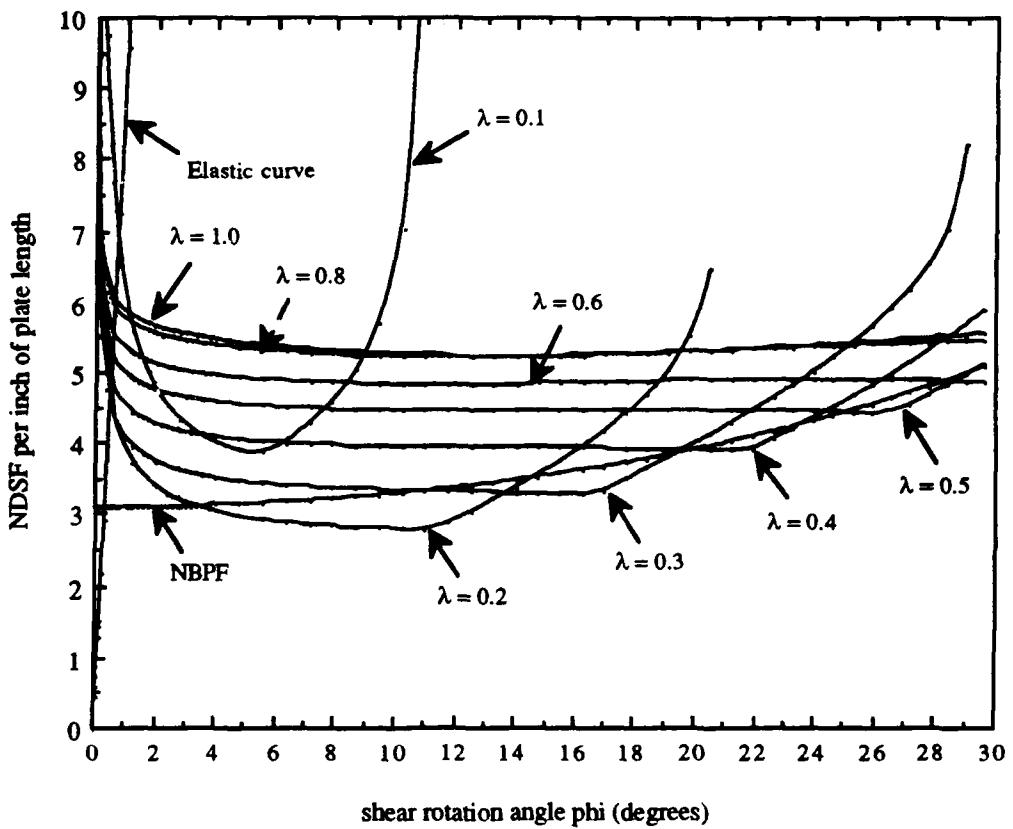


Figure 27: VLCC non-dimensional shear per unit length

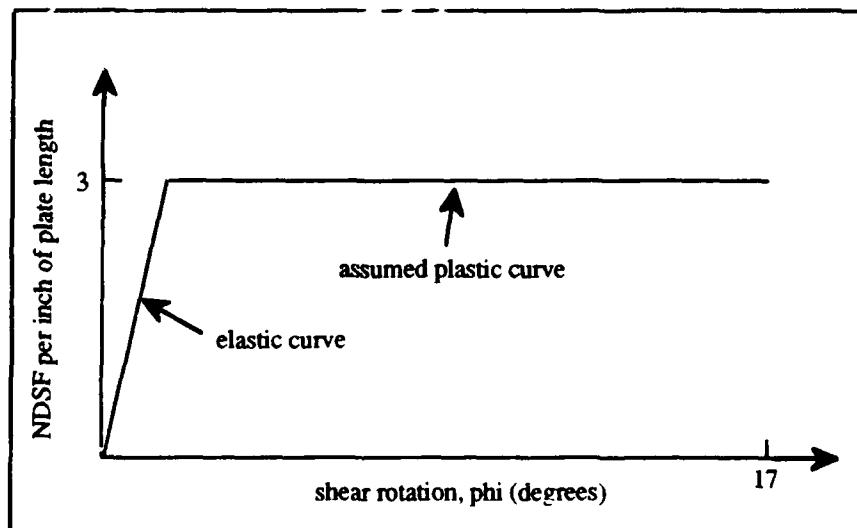


Figure 28: Approximation for VLCC NDSF( $\Phi$ ) per inch of plate length

Use of Figure 28 provides a quick means to determine the non-dimensional shear force per unit length since  $\Phi$  for each plate element on the damage transition surface has already been determined. The in-plane shear resistance for each plate element can now be calculated by multiplying the NDSF/unit length by the plate incremental length and the plastic hinge moment.

$$\frac{\text{shear resistance}}{\text{plate element}} = f_s = \left( \frac{\text{NDSF}}{\text{unit length}} \right) \times \left( \frac{\eta}{50} \right) \times M_o$$

Since the transition zone surface is not horizontal, the induced in-plane shear has both horizontal and vertical components.

\* *Note to reader:* An alternative approach in which the force is not projected in the horizontal direction is discussed at the end of this chapter.

The component of shear resistance in the  $x$ -direction is therefore given by:

$$f_{s_x} = f_s \cos[\arctan(w'(x))]$$

The slope of the plate element surface in the  $x$  direction is given by  $w'(x)$  and is just the mean value of its longitudinal edge slopes which have already been computed.

$$w'(x)_{1-2} = \frac{w_1(x) + w_2(x)}{2} = \frac{\partial z}{\partial x} \text{ of plate element between longitudinals 1 and 2}$$

We can finally compute the total horizontal grounding resistance,  $F_{s_x}$  in shear by numerically summing all of the plate element horizontal shear resistances over the transition zone surface.

$$F_{s_x} = 2 \sum_{i=1}^m \sum_{j=1}^{50} f_{s_x}(i,j)$$

## 6.2 Results of Grounding Model Calculations

The total horizontal grounding resistance  $F_{sx}$  has been computed on a spreadsheet for penetration depths from 1 to 10 meters and for transition zone parameters  $n$  from 1 to 15. The results are tabulated in Appendix A and plotted in Figures 29 and 30, from which several observations should be noted.

Figure 29 illustrates that, although the horizontal resistance increases as the damage penetration increases, it does so at a steadily decreasing rate. This is due to the fact that the surface slope of the deformed plating increases as out-of-plane deformations increase, thus reducing the horizontal component of the otherwise increasing in-plane shear force.

Figure 30 illustrates that large penetration depths produce a horizontal resistance that varies almost linearly with the transition zone parameter "n" while small penetration depths result in a horizontal resistance that approaches a constant value with  $n$ . The curve associated with a damage depth of one meter shows that the resistance is roughly constant for values of  $n$  greater than two meters.

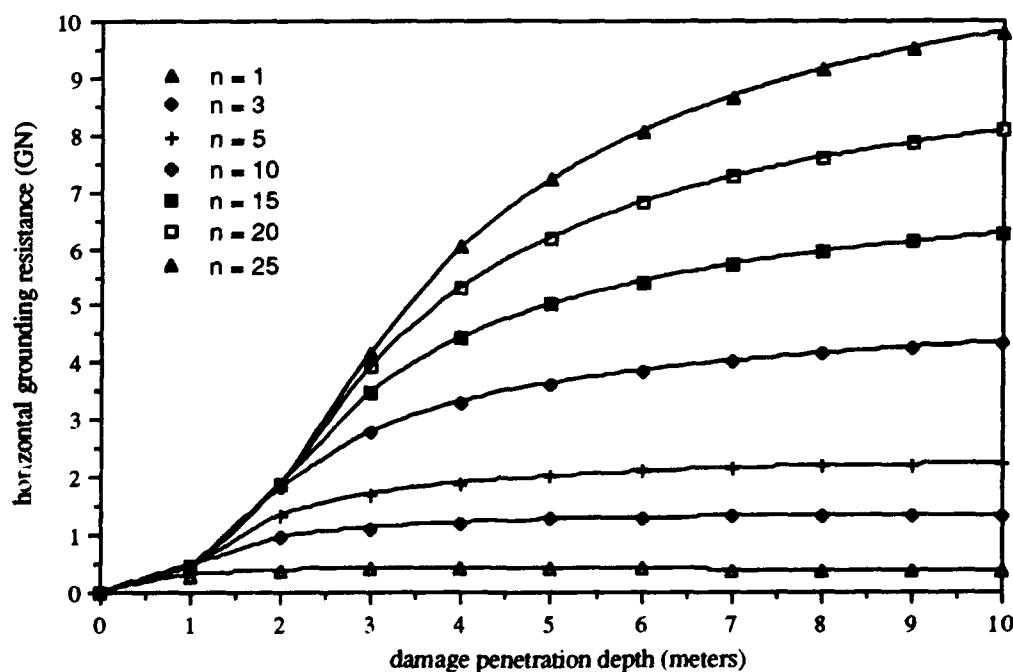


Figure 29: Horizontal resistance vs penetration depth vs transition parameter "n"

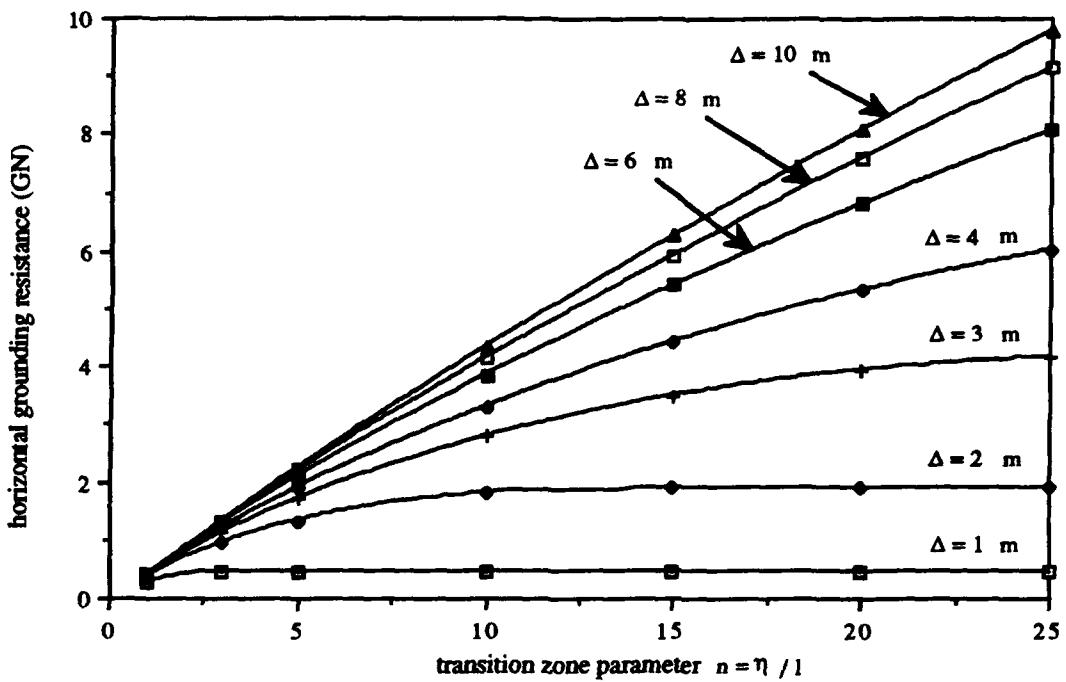


Figure 30: Horizontal resistance vs transition parameter "n" and penetration depth

Since the transition zone length is undetermined it is difficult to predict the horizontal resistance for large penetration depths using this model. However, since the resistance is independent of transition zone length for small penetration depths, the horizontal force for minor hull deformations can be calculated from this analytical model without knowing the transition zone parameter n.

Another point worth noting is that this model does not take into account the fact that for large penetration depths and short transition zone lengths (small n), the rupture strain for the hull material may be exceeded in certain locations on the plate surface. An initial assumption associated with this formulation was the no hull tearing or rupture would occur since plate rupture would result in a significant and unpredictable loss of load carrying capacity within the hull plating. In order to partially address this issue the model was used to calculate the uniaxial membrane strain in the longitudinal direction through the damage transition region at the keel. The damage penetration which induced a longitudinal strain in the keel of 0.1 was calculated for transition zone lengths corresponding to n = 1 to n = 9.

It should be emphasized that this is a rough method of determining the parameters which will result in hull membrane rupture and does not take into account the complicated two dimensional nature of the strain field. However, this method does provide a means to

quickly estimate the combinations of grounding parameters,  $\Delta_0$  and  $n$ , which may risk inducing membrane rupture. Figure 31 shows how the penetration depth for inducing longitudinal membrane rupture of the keel hull plating varies depending on the transition zone length.

The most obvious conclusion that can be taken from Figure 31 is that the penetration depth required to induce keel membrane rupture increases linearly with  $n$ . The results show that for a transition zone parameter greater than five, the damage penetration must exceed 10 meters to induce rupture and that for damage penetration less than two meters, the rupture strain will not be exceeded for  $n \geq 1$ . The preceding results show that although hull membrane rupture is a concern it will not occur for small damage penetration depths where we can calculate the hull resistance since it is independent of  $n$ .

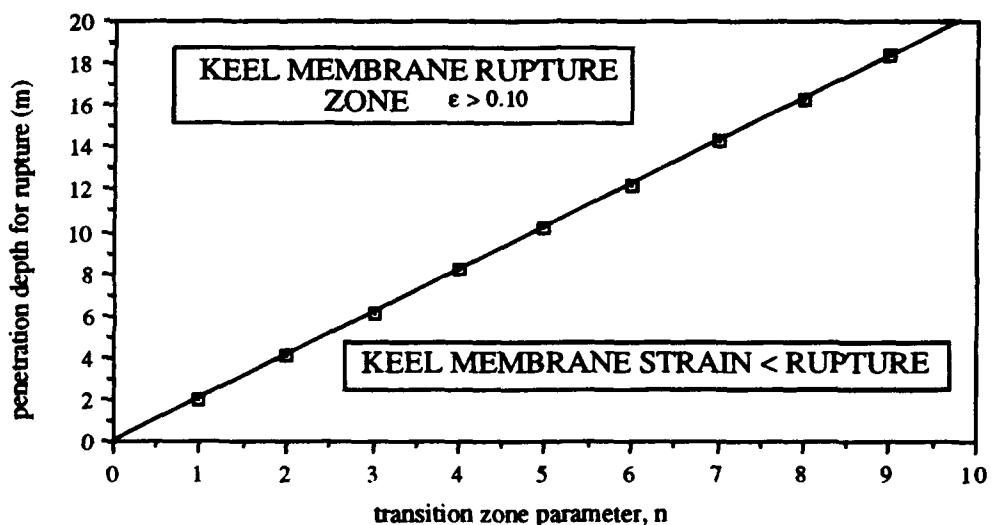


Figure 31: Limiting penetration depth vs "n" for prevention of keel membrane rupture assumed to be at a membrane strain of 0.1

### 6.3 Comparison with other Hull Failure Modes

Although the results show that for minor hull deformations the horizontal grounding force can be calculated independently of the transition zone length, no mention has yet been made of the fact that the structure may not fail in a shear buckling mode. The results discussed earlier are not valid unless the hull plating fails by the assumed geometry. It has already been stated that tearing or rupture of the hull plating will invalidate this model, however, no attempt has been made to compare the horizontal grounding resistance calculated from these other failure modes to that of the shear buckling model.

An attempt has been undertaken to compare the shear buckling failure mode with other modes of hull plate failure by comparing the results of the present model with those of Peer<sup>4</sup>. Peer considered internal work contributions from friction, crushing of transverse webs or bulkheads, and bending of longitudinal stiffeners and associated hull plating and performed an outer dynamic analysis on the ship hull to calculate the lift and penetration of the hull for a given rock height at various points along the ship length.

To perform this comparison the results of Peer's outer dynamic analysis are used for assumed rock heights of 3.5 meters and 5.0 meters which give damage penetration depths along the length of the VLCC for transition zone parameters of 5, 10, 15, and 20. The penetration depths used are shown in Tables 4-1 and 4-2.

Table 4-1: Penetration depths used in comparison (rock height = 3.5 m)

| Distance aft<br>FP<br>x (meters) | $\Delta o$ (meters)<br>n=5 | $\Delta o$ (meters)<br>n=10 | $\Delta o$ (meters)<br>n=15 | $\Delta o$ (meters)<br>n=20 |
|----------------------------------|----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0                                | 0.00                       | 0.00                        | 0.00                        | 0.00                        |
| 50                               | 3.76                       | 1.24                        | 0.30                        | 0.05                        |
| 100                              | 5.15                       | 3.00                        | 1.58                        | 1.00                        |
| 150                              | 5.60                       | 3.90                        | 2.50                        | 1.65                        |
| 200                              | 5.20                       | 3.20                        | 1.80                        | 1.17                        |
| 250                              | 3.95                       | 1.50                        | 0.55                        | 0.15                        |
| 313                              | 0.00                       | 0.00                        | 0.00                        | 0.00                        |

Table 4-2: Penetration depths used in comparison (rock height = 5 m)

| Distance aft<br>FP<br>x (meters) | $\Delta o$ (meters)<br>n=5 | $\Delta o$ (meters)<br>n=10 | $\Delta o$ (meters)<br>n=15 | $\Delta o$ (meters)<br>n=20 |
|----------------------------------|----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0                                | 0.00                       | 0.00                        | 0.00                        | 0.00                        |
| 50                               | 7.00                       | 3.30                        | 1.32                        | 0.60                        |
| 100                              | 8.35                       | 5.65                        | 3.35                        | 1.95                        |
| 150                              | 8.70                       | 6.50                        | 4.55                        | 3.10                        |
| 200                              | 8.35                       | 5.90                        | 3.65                        | 2.30                        |
| 250                              | 7.20                       | 3.75                        | 1.70                        | 1.00                        |
| 313                              | 0.00                       | 0.00                        | 0.00                        | 0.00                        |

The horizontal grounding resistance has been calculated for each penetration depth and transition zone parameter shown in Tables 4-1 and 4-2 using the grounding model and the results are plotted in Figures 32 and 33.

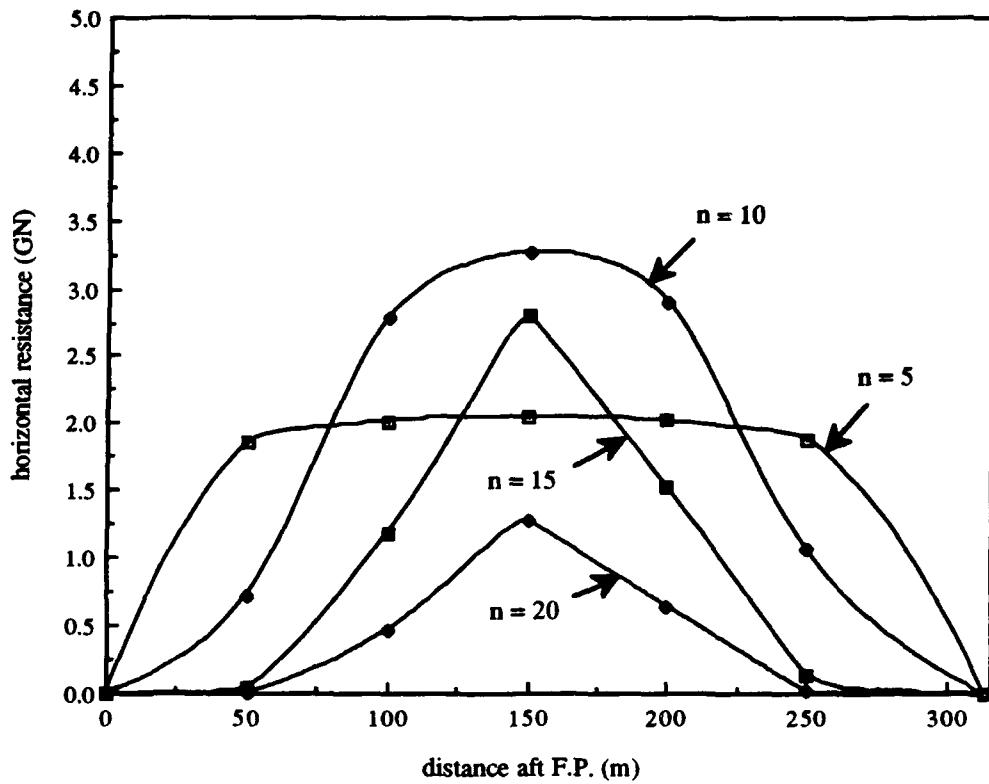


Figure 32: Horizontal grounding resistance for 3.5 m rock height

Several observations can be made from Figures 32 and 33. First of all, small transition zone parameters result in a horizontal grounding resistance that is more nearly constant over the length of the ship. The lifting of the ship at its bow and stern as it rides the obstacle results in a reduction in the damage penetration in these locations which in turn reduces the horizontal grounding resistance. It was shown by Peer<sup>4</sup> that short transition zone lengths generate less lift on the ship at its ends. This reduction in the lifting of the ship at its ends results in a more uniform grounding resistance along its length.

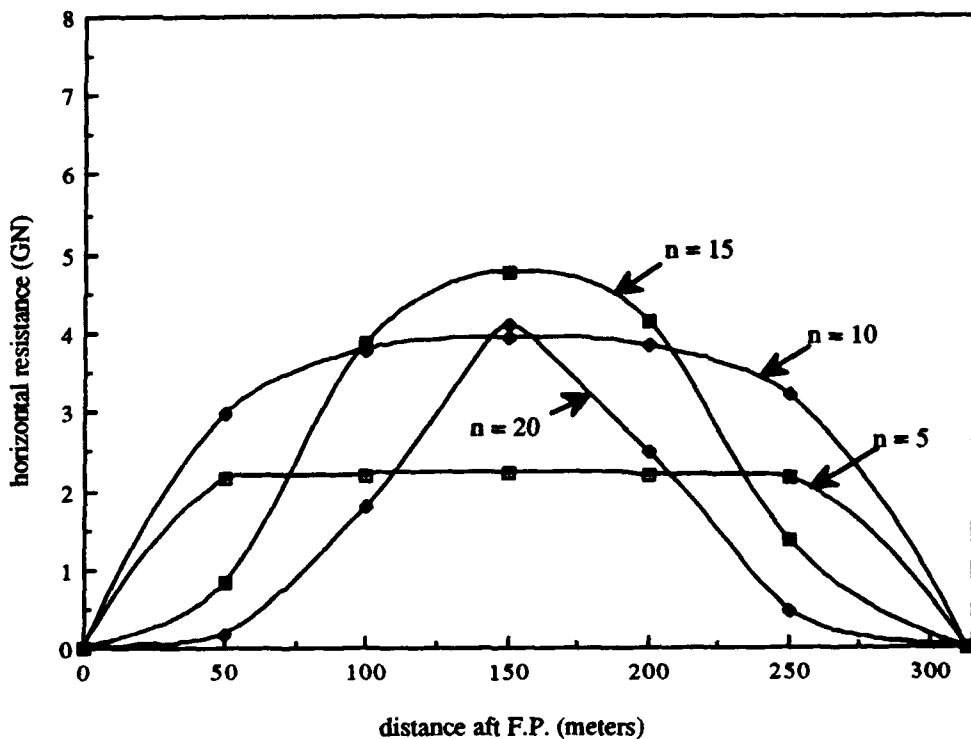


Figure 33: Horizontal grounding resistance for 5 m rock height

The larger rock height of 5 meters shown in Figure 33 results in a more uniform grounding resistance for all transition zone parameters less than 15. This is due to the fact that the ship does not ride over large obstacles as easily as smaller ones and a larger fraction of the total energy is dissipated within the hull structure. Since the fraction of energy dissipation associated with lifting of the ship is reduced, its effect on the total grounding resistance is also reduced.

Another interesting point is that the horizontal resistance for a short transition parameter ( $n = 5$ ) does not change as the penetration depth increases. This can also be seen in Figure 29 which illustrates that for small transition lengths the resistance does not change as the penetration depth increases.

Finally, the point is reached where these results can be compared to Peer's grounding model. The resistances shown in Figures 32 and 33 are roughly an order of magnitude greater than those calculated by Peer<sup>4</sup>. To illustrate this, the grounding resistances, from Peer's model which neglects shear, are plotted for a rock height of 5 meters in Figure 34.

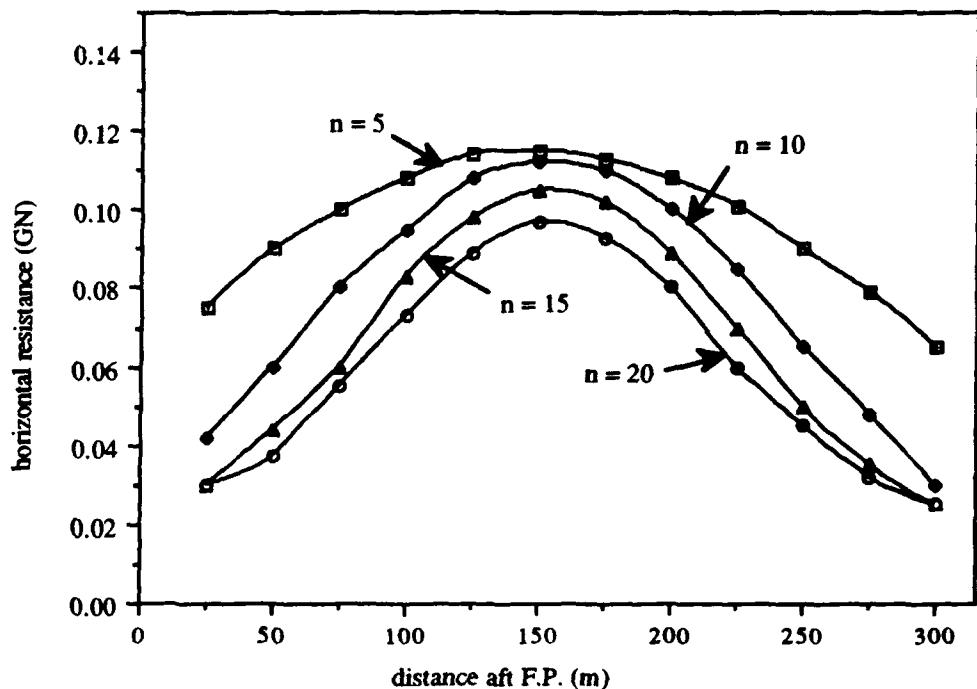


Figure 34: Horizontal resistance form Peer's grounding model  
for a 5 m rock height

The fact that Peer's results are much lower than those of the shear buckling model supports the conclusion that fracture may occur in the hull deformation process to pre-empt development of the high resistance associated with in-plane shear. The resistance of the hull structure to other structural failure modes is, under most circumstances, much less than that of shear buckling. Under these circumstances, the hull structure will fail by mechanisms other than shear. Further investigation of this phenomenon is beyond the scope of this paper.

Figures 29 and 30 show that if either the transition zone parameter or the penetration depth are small then the horizontal resistance force will also be small which may lead to a circumstance where the horizontal resistance force due to shear buckling is of the same order of magnitude as Peer's results. Another area that must be addressed is how increasing the transition zone parameter reduces the penetration depth for a given rock height. The effect on the horizontal grounding resistance has been observed by entering

the midships penetration depths for a 5 meter rock height from Peer's results into the shear buckling resistance model and plotting the calculated horizontal resistance. The result is shown in Figure 35.

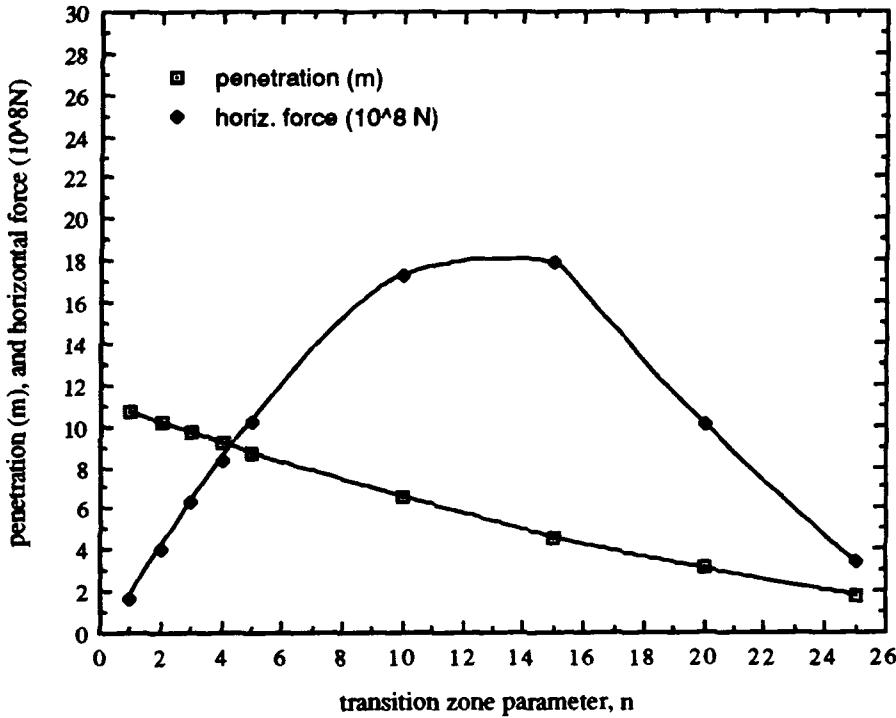


Figure 35: Horizontal resistance (based on penetration vs "n")

Figure 35 illustrates dramatically how the transition zone length affects the penetration depth which in turn affects the horizontal grounding resistance. It may be concluded from Figure 35 that either a very small or a very large transition zone length will reduce the horizontal shear resistance. The combination of a small penetration depth with a large transition zone length may be induced by minor grounding event such as the ship riding over a sand bar. In this situation, the resistance forces associated with shear buckling may be of the same order of magnitude as the forces associated with the damage mechanism's analyzed by Peer<sup>4</sup>. However, further study using both this model and the one developed by Peer to analyze grounding geometries with small penetration depths and large transition zone lengths will be necessary to confirm this.

The fact that the horizontal resistance shown in Figure 35 decreases for shorter transition zone lengths requires additional explanation. This grounding model formulation projects the tangential force on the transition zone surface into the longitudinal direction, however, the vertical component has been ignored. For short transition zones this vertical component will be quite large and, according to Peer, will entirely change the geometry of the transition zone. Therefore the validity of this result should be questioned.

An alternative approach may be considered which assumes all of the shear deformation occurs in the wake of the transition zone where the plate is horizontal. In this approach the horizontal force component will be much larger and the total grounding resistance will not drop as the transition zone length increases.

## **7. Conclusion**

The first part of the thesis focused on developing a set of simple analytical models to describe the in-plane shear load carrying capacity of a flat rectangular plate in both the elastic and plastic regimes. The plastic buckling model was found by experimentation to give a reasonable prediction of both the resistance of the plate and its longitudinal plastic buckling wavelength. This model also predicted the observed shortening in the longitudinal wavelength as plastic buckling progresses. The model for in-plane failure provided an upper envelope for the load observed during the buckling tests.

A grounding resistance model was then developed in the second part of the thesis by incorporating the buckling element model into an assumed ship hull deformation induced by a grounding event. The ship grounding model yielded results from which the following conclusions can be drawn.

- (1) The horizontal resistance of the hull to deformation induced by grounding is roughly an order of magnitude greater than the resistance of the energy dissipation mechanisms previously analyzed by Peer<sup>4</sup> which ignore skin effects. This supports the conclusion that the shear buckling mode of failure will be preceded and thus pre-empted other "weaker" failure modes such as fracture. Under these circumstances a majority of the total energy dissipation will be due to these "weaker" failure modes within the VLCC's hull structure.
- (2) Larger rock heights result in a more uniform hull resistance along the length of the ship due to grounding. The reduction in the resistance at the ends of the ship is due to lifting of the hull as it attempts to ride over the obstacle. Since the ratio of this lift to the damage penetration is less for large obstacles, less of a reduction in the resistance occurs at the ends of the ship, resulting in a more uniform resistance.
- (3) Short damage transition zones result in less lifting of the ship hull which in turn causes the grounding resistance to be nearly constant as the damage progresses down the length of the ship.
- (4) The penetration depth required to induce a longitudinal strain in the keel hull plating of 0.1 increases linearly with the transition zone length. Therefore, if the goal is to build a ship hull to resist membrane rupture due to grounding then some means must be engineered into the bottom hull structure so that it will

"diffuse" the damage out away from the local point of contact with the obstacle. This will reduce the probability of localized strain concentrations in the portion of the hull being impinged upon by the obstacle by distributing them in a more uniform manner over a larger volume of the structure which will in turn reduce the probability of hull rupture.

(5) The horizontal grounding resistance is independent of transition zone length for small penetration depths (i.e. < 1 meter). Since the transition zone length is unknown, the horizontal grounding force can only be predicted for small penetration depths.

Further study is needed in the area of minor groundings to determine if the resistance associated with hull plate shear buckling is of the same order of magnitude as the resistance associated with other hull failure mechanisms. Specifically, more results are needed from Peer's model for damage penetration depths less than 1 meter in combination with transition zone lengths greater than 10 for comparison with this model. The study of minor groundings should already be of interest when attempting to design a tanker bottom to resist hull plating rupture since it appears impractical to develop a design that is capable of remaining unbreached following impact with a large rock. Finally, the alternative formulation discussed earlier which considers all of the shear deformation to occur in the transition zone wake should also be analyzed and compared with this grounding model.

**APPENDIX A**

**Ship Total Shear Resistance Calculations**

## Total Horizontal Resistance vs Penetration Depth and "n"

| Penetration [m]<br>$\Delta o$ | Total horizontal resistance for transition zone parameter, n |        |        |         |         |         |         |
|-------------------------------|--|--------|--------|---------|---------|---------|---------|
|                               | 1  | 3      | 5      | 10      | 15      | 20      | 25      |
| 1                             | 0.2846   | 0.465  | 0.469  | 0.46934 | 0.46942 | 0.46944 | 0.46946 |
| 2                             | 0.3776   | 0.9404 | 1.3244 | 1.8046  | 1.8768  | 1.8774  | 1.8776  |
| 3                             | 0.4042   | 1.116  | 1.6926 | 2.776   | 3.4888  | 3.9264  | 4.1528  |
| 4                             | 0.4076   | 1.2016 | 1.886  | 3.3026  | 4.4194  | 5.3068  | 6.0062  |
| 5                             | 0.403  | 1.2598 | 2.0008 | 3.6294  | 5.0054  | 6.1854  | 7.2084  |
| 6                             | 0.3952   | 1.2902 | 2.0864 | 3.8506  | 5.4052  | 6.7924  | 8.0424  |
| 7                             | 0.3858   | 1.306  | 2.1486 | 4.0114  | 5.7016  | 7.2402  | 8.6496  |
| 8                             | 0.3762   | 1.3138 | 2.1866 | 4.1434  | 5.9192  | 7.583   | 9.1306  |
| 9                             | 0.3666   | 1.3162 | 2.212  | 4.2588  | 6.1004  | 7.844   | 9.5     |
| 10                            | 0.3572   | 1.3148 | 2.2296 | 4.3422  | 6.2586  | 8.063   | 9.7924  |

| transition zone<br>parameter | Total horizontal resistance for rock height in meters |        |        |        |        |
|------------------------------|---|--------|--------|--------|--------|
|                              | n   | 1      | 2      | 3      | 4      |
| 1                            | 0.2846  | 0.3776 | 0.4042 | 0.4076 | 0.403  |
| 3                            | 0.465   | 0.9404 | 1.116  | 1.2016 | 1.2598 |
| 5                            | 0.469   | 1.3244 | 1.6926 | 1.886  | 2.0008 |
| 10                           | 0.46934   | 1.8046 | 2.776  | 3.3026 | 3.6294 |
| 15                           | 0.46942   | 1.8768 | 3.4888 | 4.4194 | 5.0054 |
| 20                           | 0.46944   | 1.8774 | 3.9264 | 5.3068 | 6.1854 |
| 25                           | 0.46946   | 1.8776 | 4.1528 | 6.0062 | 7.2084 |

| transition zone<br>parameter | Total horizontal resistance for rock height in meters |        |        |        |        |
|------------------------------|---|--------|--------|--------|--------|
|                              | n   | 6      | 7      | 8      | 9      |
| 1                            | 0.3952  | 0.3858 | 0.3762 | 0.3666 | 0.3572 |
| 3                            | 1.2902  | 1.306  | 1.3138 | 1.3162 | 1.3148 |
| 5                            | 2.0864  | 2.1486 | 2.1866 | 2.212  | 2.2296 |
| 10                           | 3.8506  | 4.0114 | 4.1434 | 4.2588 | 4.3422 |
| 15                           | 5.4052  | 5.7016 | 5.9192 | 6.1004 | 6.2586 |
| 20                           | 6.7924  | 7.2402 | 7.583  | 7.844  | 8.063  |
| 25                           | 8.0424  | 8.6496 | 9.1306 | 9.5    | 9.7924 |

## Rock Penetration and Resistance Calculation Results

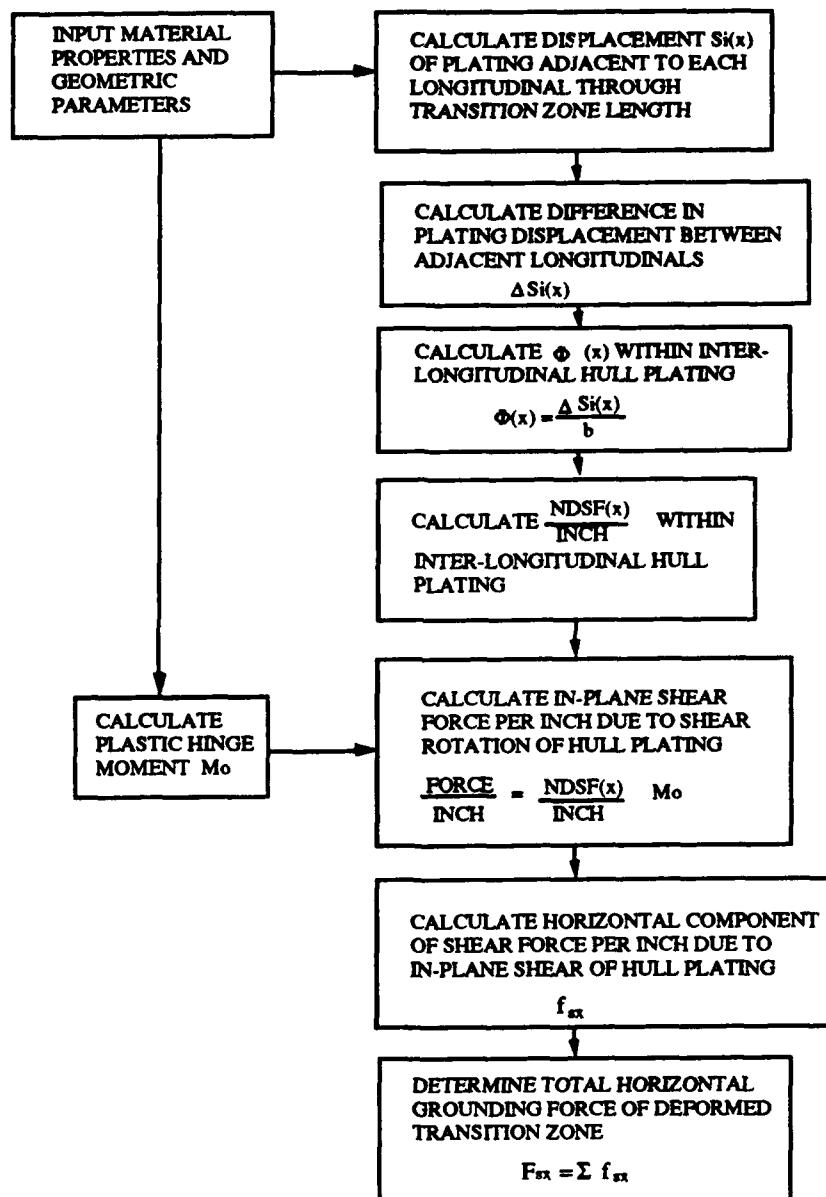
| Damage penetration for rock height = 3.5 meters |                    |                     |                     |                     |
|---|--------------------|---------------------|---------------------|---------------------|
| Distance aft FP<br>x (meters)                   | Δo<br>(meters) n=5 | Δo<br>(meters) n=10 | Δo<br>(meters) n=15 | Δo<br>(meters) n=20 |
| 0   | 0.00               | 0.00                | 0.00                | 0.00                |
| 50  | 3.76               | 1.24                | 0.30                | 0.05                |
| 100   | 5.15               | 3.00                | 1.58                | 1.00                |
| 150   | 5.60               | 3.90                | 2.50                | 1.65                |
| 200   | 5.20               | 3.20                | 1.80                | 1.17                |
| 250   | 3.95               | 1.50                | 0.55                | 0.15                |
| 313   | 0.00               | 0.00                | 0.00                | 0.00                |

| Damage penetration for rock height = 5 meters |                    |                     |                     |                     |
|---|--------------------|---------------------|---------------------|---------------------|
| Distance aft FP<br>x (meters)                 | Δo<br>(meters) n=5 | Δo<br>(meters) n=10 | Δo<br>(meters) n=15 | Δo<br>(meters) n=20 |
| 0   | 0.00               | 0.00                | 0.00                | 0.00                |
| 50  | 7.00               | 3.30                | 1.32                | 0.60                |
| 100   | 8.35               | 5.65                | 3.35                | 1.95                |
| 150   | 8.70               | 6.50                | 4.55                | 3.10                |
| 200   | 8.35               | 5.90                | 3.65                | 2.30                |
| 250   | 7.20               | 3.75                | 1.70                | 1.00                |
| 313   | 0.00               | 0.00                | 0.00                | 0.00                |

| Horizontal grounding resistance for rock height = 3.5 meters |                               |                                |                                |                                |
|--|-------------------------------|--------------------------------|--------------------------------|--------------------------------|
| Distance aft FP<br>x (meters)                                | Horiz. Force<br>(Newtons) n=5 | Horiz. Force<br>(Newtons) n=10 | Horiz. Force<br>(Newtons) n=15 | Horiz. Force<br>(Newtons) n=20 |
| 0  | 0                             | 0                              | 0                              | 0                              |
| 50   | 1.8504                        | 0.7216                         | 0.0422                         | 0.0012                         |
| 100  | 2.0148                        | 2.776                          | 1.1716                         | 0.4694                         |
| 150  | 2.0544                        | 3.2608                         | 2.7898                         | 1.2778                         |
| 200  | 2.0194                        | 2.9044                         | 1.5204                         | 0.6426                         |
| 250  | 1.879                         | 1.0556                         | 0.142                          | 0.0106                         |
| 313  | 0                             | 0                              | 0                              | 0                              |

| Horizontal grounding resistance for rock height = 5 meters |                               |                                |                                |                                |
|--|-------------------------------|--------------------------------|--------------------------------|--------------------------------|
| Distance aft FP<br>x (meters)                              | Horiz. Force<br>(Newtons) n=5 | Horiz. Force<br>(Newtons) n=10 | Horiz. Force<br>(Newtons) n=15 | Horiz. Force<br>(Newtons) n=20 |
| 0  | 0                             | 0                              | 0                              | 0                              |
| 50   | 2.1486                        | 2.9654                         | 0.8178                         | 0.169                          |
| 100  | 2.1966                        | 3.7836                         | 3.8696                         | 1.7846                         |
| 150  | 2.2054                        | 3.9352                         | 4.7678                         | 4.0966                         |
| 200  | 2.1966                        | 3.832                          | 4.146                          | 2.4826                         |
| 250  | 2.1576                        | 3.1928                         | 1.3562                         | 0.4694                         |
| 313  | 0                             | 0                              | 0                              | 0                              |

## Grounding Resistance Calculation Flow Diagram



Determination of total horizontal grounding resistance due to in-plane shear of hull plating

Basic Problem Parameters: (\* All dimensions in meters)

|  |              |             |
|--|--------------|-------------|
| transverse web frame spacing =         | $l =$        | 4.88 meters |
| # transverse webs in transition zone = | $n =$        | 5           |
| length of damage transition zone =     | $\eta =$     | 24.4 meters |
| assumed rock height =                  | $\Delta r =$ | 5 meters    |
| penetration depth =                    | $\Delta o =$ | 8.7 meters  |
| distance between longit. bulkheads =   | $ll =$       | 24.4 meters |

| Longitudinal distance $x$ | Tabulation of longitudinal elongation as function of $x$ [meters] |              |              |              |              |              |              |
|---------------------------|---|--------------|--------------|--------------|--------------|--------------|--------------|
|                           | Longitud. #1  | Longitud. #2 | Longitud. #3 | Longitud. #4 | Longitud. #5 | Longitud. #6 | Longitud. #7 |
| 0                         | 0   | 0            | 0            | 0            | 0            | 0            | 0            |
| 0.488                     | 6.99613E-05   | 5.95407E-05  | 4.99599E-05  | 4.12189E-05  | 3.33176E-05  | 2.62562E-05  | 2.00348E-05  |
| 0.976                     | 0.000697599   | 0.000593738  | 0.000498234  | 0.000411089  | 0.000332306  | 0.000261891  | 0.000199844  |
| 1.464                     | 0.002429913   | 0.002068402  | 0.001735894  | 0.001432423  | 0.00115802   | 0.000912714  | 0.000696528  |
| 1.952                     | 0.005792872   | 0.004931887  | 0.004139715  | 0.003416502  | 0.002762381  | 0.002177475  | 0.001661891  |
| 2.44                      | 0.011281668   | 0.009606968  | 0.008065548  | 0.006657645  | 0.005383868  | 0.004244517  | 0.003239922  |
| 2.928                     | 0.01935185  | 0.016483386  | 0.013841807  | 0.011428184  | 0.009243497  | 0.007288644  | 0.00556443   |
| 3.416                     | 0.030411537   | 0.025911248  | 0.021764635  | 0.017973926  | 0.014541174  | 0.011468258  | 0.00675687   |
| 3.904                     | 0.044814876   | 0.038195401  | 0.032092463  | 0.026510212  | 0.021452493  | 0.016922829  | 0.012924407  |
| 4.392                     | 0.062856818   | 0.0535990868 | 0.04504243   | 0.037218612  | 0.03012602   | 0.023770726  | 0.018158242  |
| 4.88                      | 0.084769261   | 0.072299377  | 0.060787271  | 0.050244312  | 0.040681109  | 0.032107444  | 0.024532225  |
| 5.368                     | 0.110718538   | 0.094467009  | 0.079453163  | 0.065694211  | 0.053206268  | 0.042004251  | 0.032101781  |
| 5.856                     | 0.140804189   | 0.120182901  | 0.101118453  | 0.08363572   | 0.067758063  | 0.053507261  | 0.040903151  |
| 6.344                     | 0.175058943   | 0.149478984  | 0.12581323   | 0.104096259  | 0.084360692  | 0.066636939  | 0.050952951  |
| 6.832                     | 0.21344978  | 0.182330648  | 0.153519698  | 0.127063409  | 0.103005794  | 0.081388021  | 0.06224806   |
| 7.32                      | 0.25587998  | 0.218658277  | 0.184173284  | 0.152485691  | 0.123653167  | 0.097729848  | 0.074765814  |
| 7.808                     | 0.302192021   | 0.258329539  | 0.217664419  | 0.18027391   | 0.146231672  | 0.115607077  | 0.0884645    |
| 8.296                     | 0.352171211   | 0.301162355  | 0.253840916  | 0.210303032  | 0.170640682  | 0.134940762  | 0.103284141  |
| 8.784                     | 0.405549951   | 0.346928446  | 0.292510874  | 0.242414516  | 0.196751927  | 0.155629757  | 0.119147537  |
| 9.272                     | 0.462012516   | 0.395357375  | 0.333446047  | 0.276419063  | 0.224411686  | 0.177352434  | 0.135961556  |
| 9.76                      | 0.521200263   | 0.446141011  | 0.376385599  | 0.312099726  | 0.253443302  | 0.200568651  | 0.153618638  |
| 10.248                    | 0.5827172   | 0.498938329  | 0.421040187  | 0.349215311  | 0.283649962  | 0.224521971  | 0.171998494  |
| 10.736                    | 0.646135823   | 0.5533380504 | 0.467096328  | 0.387504053  | 0.314817714  | 0.249242072  | 0.190969973  |
| 11.224                    | 0.711003182   | 0.609076216  | 0.514220958  | 0.426687479  | 0.346718674  | 0.274547324  | 0.210393067  |
| 11.712                    | 0.776847102   | 0.663617135  | 0.562066237  | 0.466474483  | 0.379114372  | 0.300247503  | 0.230121026  |
| 12.2                      | 0.843182532   | 0.722583518  | 0.610274318  | 0.506365391  | 0.411759229  | 0.3261466    | 0.25000236   |
| 12.688                    | 0.909517962   | 0.779549902  | 0.658482398  | 0.5466563    | 0.444404086  | 0.352045696  | 0.269884094  |
| 13.176                    | 0.975361882   | 0.83609082   | 0.706327667  | 0.5864643293 | 0.476799785  | 0.377745875  | 0.289612054  |
| 13.664                    | 1.040229241   | 0.891786533  | 0.753452308  | 0.625626728  | 0.508700744  | 0.403051127  | 0.309035148  |
| 14.152                    | 1.103647864   | 0.946228708  | 0.799508448  | 0.663915471  | 0.539868496  | 0.427771228  | 0.328006627  |
| 14.64                     | 1.165164801   | 0.999026026  | 0.844163037  | 0.701031057  | 0.570075156  | 0.451724548  | 0.346386483  |
| 15.128                    | 1.224352548   | 1.049809661  | 0.887102588  | 0.736711719  | 0.599106772  | 0.474740765  | 0.364043564  |
| 15.616                    | 1.280815113   | 1.098238591  | 0.928037761  | 0.770716267  | 0.626766331  | 0.496663442  | 0.380857583  |
| 16.104                    | 1.334193853   | 1.144004682  | 0.96670772   | 0.802827751  | 0.652877776  | 0.517352437  | 0.39672098   |
| 16.592                    | 1.384173043   | 1.186837498  | 1.002884216  | 0.832856873  | 0.677286786  | 0.536686122  | 0.411540621  |
| 17.08                     | 1.430485084   | 1.22658076   | 1.036375352  | 0.860645092  | 0.699863291  | 0.554563351  | 0.425239307  |
| 17.568                    | 1.472915284   | 1.262836389  | 1.067028938  | 0.888067373  | 0.720512665  | 0.570905178  | 0.437775706  |
| 18.056                    | 1.511306121   | 1.295688053  | 1.094735406  | 0.909034524  | 0.739157766  | 0.58563626   | 0.44905217   |
| 18.544                    | 1.545560875   | 1.324984135  | 1.119430183  | 0.929495063  | 0.755760373  | 0.598785938  | 0.45910197   |
| 19.032                    | 1.575646526   | 1.350700028  | 1.141095473  | 0.947436572  | 0.77031219   | 0.610288949  | 0.467903339  |
| 19.52                     | 1.601595803   | 1.37228766   | 1.159761365  | 0.96288647   | 0.78283733   | 0.620185755  | 0.475472896  |
| 20.008                    | 1.623508246   | 1.391576169  | 1.175506206  | 0.975912171  | 0.793392438  | 0.628522473  | 0.481846879  |
| 20.496                    | 1.641550187   | 1.406971636  | 1.188456172  | 0.986620571  | 0.802065963  | 0.63537037   | 0.487080713  |
| 20.984                    | 1.655953526   | 1.419255789  | 1.198784     | 0.995156857  | 0.808977284  | 0.640824941  | 0.491248251  |
| 21.472                    | 1.667013214   | 1.428683651  | 1.206706828  | 0.1001702599 | 0.814274961  | 0.645004535  | 0.494440691  |
| 21.96                     | 1.675083396   | 1.435560069  | 1.212483156  | 1.006473138  | 0.81813459   | 0.648048682  | 0.496765198  |
| 22.448                    | 1.680572192   | 1.44023515   | 1.21640892   | 1.009714281  | 0.820756077  | 0.650115724  | 0.498343229  |
| 22.936                    | 1.683935151   | 1.443098635  | 1.218812741  | 1.01169836   | 0.822360438  | 0.651380485  | 0.499308592  |
| 23.424                    | 1.685667465   | 1.444573298  | 1.220050402  | 1.012719694  | 0.823186152  | 0.652031308  | 0.499805276  |
| 23.912                    | 1.686295103   | 1.445107496  | 1.220498676  | 1.013089564  | 0.823485141  | 0.6522666943 | 0.499985086  |
| 24.4                      | 1.686365064   | 1.445167037  | 1.220548635  | 1.013130783  | 0.823518458  | 0.652293199  | 0.500005121  |

| Longitudinal<br>distance x | Tabulation of longitudinal elongation as function of x [meters] |              |               |               |               |               |
|----------------------------|---|--------------|---------------|---------------|---------------|---------------|
|                            | Longitud. #6  | Longitud. #9 | Longitud. #10 | Longitud. #11 | Longitud. #12 | Longitud. #13 |
| 0                          | 0   | 0            | 0             | 0             | 0             | 0             |
| 0.488                      | 1.46532E-05   | 1.01116E-05  | 6.41002E-06   | 3.5484E-06    | 1.5268E-06    | 3.45216E-07   |
| 0.976                      | 0.00014617  | 0.00010067   | 6.39455E-05   | 3.53991E-05   | 1.52317E-05   | 3.44398E-06   |
| 1.464                      | 0.000509487   | 0.000351609  | 0.00022291    | 0.000123403   | 5.30997E-05   | 1.20064E-05   |
| 1.952                      | 0.001215727   | 0.000439065  | 0.00031976    | 0.000294517   | 0.000126733   | 2.86562E-05   |
| 2.44                       | 0.002370376   | 0.001636132  | 0.001037406   | 0.000574372   | 0.000247168   | 5.58896E-05   |
| 2.928                      | 0.004071571   | 0.002810689  | 0.001782311   | 0.00098687    | 0.000424699   | 9.60358E-05   |
| 3.416                      | 0.006408517   | 0.004424512  | 0.002805968   | 0.0015538     | 0.000668716   | 0.00015122    |
| 3.904                      | 0.009460068   | 0.006532293  | 0.004143192   | 0.002294498   | 0.000987599   | 0.00022333    |
| 4.392                      | 0.01329349  | 0.009180781  | 0.005823791   | 0.00322554    | 0.001388382   | 0.000313986   |
| 4.88                       | 0.01796343  | 0.012408061  | 0.007872103   | 0.004360483   | 0.001877044   | 0.000424518   |
| 5.368                      | 0.023511098   | 0.016242969  | 0.010306617   | 0.005709653   | 0.002458017   | 0.000555939   |
| 5.856                      | 0.029963682   | 0.020704672  | 0.013139698   | 0.007279985   | 0.003134313   | 0.000708934   |
| 6.344                      | 0.037333989   | 0.025802397  | 0.016377399   | 0.009074923   | 0.00390744    | 0.000883849   |
| 6.832                      | 0.045620321   | 0.031535324  | 0.020019389   | 0.011094362   | 0.00477738    | 0.001080682   |
| 7.32                       | 0.054806585   | 0.037892639  | 0.024058966   | 0.013334661   | 0.005742586   | 0.001299086   |
| 7.808                      | 0.064862618   | 0.04485374   | 0.028483184   | 0.015788705   | 0.006800012   | 0.001538375   |
| 8.296                      | 0.075744734   | 0.052388395  | 0.033273067   | 0.018446016   | 0.00794516    | 0.001797533   |
| 8.784                      | 0.087396469   | 0.060458243  | 0.038403923   | 0.021292928   | 0.00917215    | 0.00207523    |
| 9.272                      | 0.099749505   | 0.069015426  | 0.043845735   | 0.024312802   | 0.010473815   | 0.002369846   |
| 9.76                       | 0.112724775   | 0.078005351  | 0.049563645   | 0.027486288   | 0.011841814   | 0.002679492   |
| 10.248                     | 0.126233709   | 0.087366346  | 0.055518505   | 0.030791635   | 0.013266763   | 0.003002043   |
| 10.736                     | 0.140179615   | 0.097031833  | 0.061667486   | 0.03420303    | 0.014738385   | 0.003335171   |
| 11.224                     | 0.15445917  | 0.10692936   | 0.067964749   | 0.037700972   | 0.016245667   | 0.003676381   |
| 11.712                     | 0.168964  | 0.116983719  | 0.07436216    | 0.041252673   | 0.017777038   | 0.004023051   |
| 12.2                       | 0.183582324   | 0.12717097   | 0.080810037   | 0.044832471   | 0.01932055    | 0.004372472   |
| 12.688                     | 0.198200648   | 0.137250476  | 0.087257913   | 0.048412269   | 0.020864061   | 0.004721894   |
| 13.176                     | 0.212705478   | 0.147304834  | 0.093655325   | 0.051963969   | 0.022395432   | 0.005068564   |
| 13.664                     | 0.226985033   | 0.157202362  | 0.099952588   | 0.055459911   | 0.023902715   | 0.005409774   |
| 14.152                     | 0.240930939   | 0.166867648  | 0.106101568   | 0.058873307   | 0.025374336   | 0.005742902   |
| 14.64                      | 0.254439873   | 0.176228844  | 0.112036428   | 0.062178654   | 0.026799285   | 0.006065453   |
| 15.128                     | 0.267415143   | 0.185218768  | 0.117774339   | 0.065335214   | 0.028167284   | 0.006375099   |
| 15.616                     | 0.279768179   | 0.193775952  | 0.123216151   | 0.068372013   | 0.029468949   | 0.006669714   |
| 16.104                     | 0.291419913   | 0.201845999  | 0.128347006   | 0.071218925   | 0.030695939   | 0.006947412   |
| 16.592                     | 0.30230203  | 0.209380454  | 0.133136889   | 0.073876236   | 0.031841087   | 0.00720657    |
| 17.08                      | 0.312358063   | 0.216341555  | 0.137561107   | 0.07633028    | 0.032898513   | 0.007443859   |
| 17.568                     | 0.321544327   | 0.222698871  | 0.141600685   | 0.07857058    | 0.03386372    | 0.007664263   |
| 18.056                     | 0.329830659   | 0.228431798  | 0.145242674   | 0.080590019   | 0.034733639   | 0.007861096   |
| 18.544                     | 0.337200966   | 0.233529522  | 0.148480375   | 0.082384956   | 0.035506786   | 0.008036011   |
| 19.032                     | 0.34365355  | 0.237991225  | 0.151313456   | 0.083955289   | 0.036183082   | 0.008189006   |
| 19.52                      | 0.349201218   | 0.241826133  | 0.15374797    | 0.085304458   | 0.036764055   | 0.008320427   |
| 20.008                     | 0.353871158   | 0.245053413  | 0.155796282   | 0.086439401   | 0.037252717   | 0.008430959   |
| 20.496                     | 0.35770458  | 0.247701902  | 0.157476882   | 0.087370443   | 0.03765354    | 0.008521615   |
| 20.984                     | 0.36075613  | 0.249809683  | 0.158814105   | 0.088111142   | 0.037972383   | 0.008593725   |
| 21.472                     | 0.363093077   | 0.251423503  | 0.159837762   | 0.088678072   | 0.0382164     | 0.008648909   |
| 21.96                      | 0.364794272   | 0.252598062  | 0.160582668   | 0.089090569   | 0.038393931   | 0.008689055   |
| 22.448                     | 0.365948921   | 0.25339513   | 0.161088098   | 0.089370424   | 0.038514366   | 0.008716289   |
| 22.936                     | 0.366655161   | 0.253882586  | 0.161397164   | 0.089541538   | 0.038587999   | 0.008732939   |
| 23.424                     | 0.367018478   | 0.254133325  | 0.161536128   | 0.089629542   | 0.038625867   | 0.008741501   |
| 23.912                     | 0.367149995   | 0.254224083  | 0.161613663   | 0.089661393   | 0.038639572   | 0.00874466    |
| 24.4                       | 0.367164648   | 0.254234194  | 0.161620073   | 0.089664941   | 0.038641099   | 0.008744945   |

**Absolute Value of delta s difference between adjacent longitudinals**

| (long. 1 & 2) | (long. 2 & 3) | (long. 3 & 4) | (long. 4 & 5) | (long. 5 & 6) | (long. 6 & 7) | (long. 7 & 8) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| 1.042058E-05  | 9.58082E-06   | 8.74105E-06   | 7.90123E-06   | 7.06137E-06   | 6.22148E-06   | 5.38155E-06   |
| 0.00010386    | 9.55047E-05   | 8.71453E-05   | 7.87822E-05   | 7.04158E-05   | 6.20464E-05   | 5.36745E-05   |
| 0.000361511   | 0.000332508   | 0.000303471   | 0.000274403   | 0.000245307   | 0.000216185   | 0.000187041   |
| 0.000860985   | 0.000792172   | 0.000723213   | 0.00065412    | 0.000584906   | 0.000515584   | 0.000446165   |
| 0.001674701   | 0.001541488   | 0.001407835   | 0.001273777   | 0.001139351   | 0.001004595   | 0.000869546   |
| 0.002868464   | 0.002641578   | 0.002413624   | 0.002184687   | 0.001954853   | 0.001724214   | 0.001492859   |
| 0.00430029    | 0.004146612   | 0.003790709   | 0.003432752   | 0.003072917   | 0.002711387   | 0.002348353   |
| 0.006619475   | 0.006102938   | 0.005582251   | 0.003057719   | 0.004529664   | 0.003998422   | 0.003464339   |
| 0.009263951   | 0.008548438   | 0.007823818   | 0.007092592   | 0.006355294   | 0.005612484   | 0.004864752   |
| 0.012469884   | 0.011512107   | 0.010542958   | 0.009563204   | 0.008573665   | 0.007575219   | 0.006568795   |
| 0.016251529   | 0.015013846   | 0.013758952   | 0.012487943   | 0.011202017   | 0.009902469   | 0.008590683   |
| 0.020621288   | 0.019064449   | 0.017482732   | 0.015877637   | 0.014250823   | 0.01260411    | 0.010939469   |
| 0.023579959   | 0.023665754   | 0.021716971   | 0.019735567   | 0.017723754   | 0.015683988   | 0.013618962   |
| 0.031119132   | 0.028110951   | 0.026456288   | 0.024057616   | 0.021617772   | 0.019139961   | 0.016627739   |
| 0.037221704   | 0.034484993   | 0.031687593   | 0.028832523   | 0.025992319   | 0.022964034   | 0.019959229   |
| 0.043862482   | 0.040665119   | 0.037390509   | 0.034042238   | 0.030624594   | 0.027142577   | 0.023601882   |
| 0.051006836   | 0.047321439   | 0.043537884   | 0.03966235    | 0.03569992    | 0.03163662    | 0.027539407   |
| 0.058621505   | 0.054417572   | 0.050096358   | 0.045662389   | 0.04112217    | 0.03648222    | 0.031751068   |
| 0.066635314   | 0.061911328   | 0.057026984   | 0.052007377   | 0.046859253   | 0.041590877   | 0.036212051   |
| 0.075059253   | 0.069755412   | 0.064285873   | 0.059656424   | 0.052874651   | 0.046950013   | 0.040693863   |
| 0.083778871   | 0.077898141   | 0.071824876   | 0.065563535   | 0.05912799    | 0.052523478   | 0.045764785   |
| 0.09275532    | 0.086284176   | 0.079592273   | 0.072686341   | 0.065575642   | 0.058272099   | 0.050790358   |
| 0.101926965   | 0.094855248   | 0.087533478   | 0.079968816   | 0.07217135    | 0.064154257   | 0.05933897    |
| 0.111229967   | 0.103550898   | 0.0925591755  | 0.08736011    | 0.07886687    | 0.070126476   | 0.061157027   |
| 0.120599014   | 0.112309201   | 0.103708926   | 0.094806162   | 0.08561263    | 0.076144039   | 0.066420236   |
| 0.12996806    | 0.121067504   | 0.111826098   | 0.102252215   | 0.09235839    | 0.082161602   | 0.071683446   |
| 0.139271062   | 0.129763153   | 0.119884374   | 0.109643308   | 0.09905391    | 0.088133821   | 0.076906576   |
| 0.148442707   | 0.138334226   | 0.12782558    | 0.116925984   | 0.105649617   | 0.094015979   | 0.082050114   |
| 0.157419156   | 0.14672026    | 0.135592977   | 0.124046975   | 0.112097269   | 0.097646601   | 0.087075688   |
| 0.166138774   | 0.15486299    | 0.143131979   | 0.130955901   | 0.118350608   | 0.105338065   | 0.09194661    |
| 0.174542887   | 0.162707073   | 0.150390869   | 0.137604947   | 0.124366007   | 0.110697201   | 0.096628421   |
| 0.182576522   | 0.170200829   | 0.157321494   | 0.143949736   | 0.130103089   | 0.115805859   | 0.101089404   |
| 0.190189171   | 0.177296962   | 0.163879969   | 0.149949975   | 0.135525339   | 0.120631458   | 0.105301066   |
| 0.197335545   | 0.183953282   | 0.170027343   | 0.155570087   | 0.140600665   | 0.125145501   | 0.10923859    |
| 0.203976324   | 0.190133409   | 0.175730259   | 0.160779801   | 0.14530194    | 0.129324044   | 0.112881244   |
| 0.210078895   | 0.195807451   | 0.180961564   | 0.165554709   | 0.149607487   | 0.133148117   | 0.116212734   |
| 0.215618068   | 0.200952647   | 0.185700882   | 0.169876758   | 0.153501506   | 0.136604091   | 0.119221511   |
| 0.220576739   | 0.205553953   | 0.18993512    | 0.173734688   | 0.150974437   | 0.139683969   | 0.121901004   |
| 0.224946448   | 0.209604555   | 0.193658901   | 0.177124382   | 0.160023242   | 0.142385609   | 0.12424979    |
| 0.228728143   | 0.213106295   | 0.196874894   | 0.180049121   | 0.162851594   | 0.144712859   | 0.126271678   |
| 0.231932076   | 0.216069964   | 0.199594035   | 0.182519732   | 0.164869965   | 0.144675594   | 0.127975721   |
| 0.234578552   | 0.218513463   | 0.201835602   | 0.184554606   | 0.166695595   | 0.148289657   | 0.129376134   |
| 0.236697737   | 0.220471789   | 0.203627143   | 0.186179573   | 0.168152343   | 0.149576691   | 0.13049212    |
| 0.238329563   | 0.221976823   | 0.205004229   | 0.187427638   | 0.169270406   | 0.150563865   | 0.131347614   |
| 0.239523327   | 0.223076913   | 0.206010018   | 0.188338548   | 0.170085908   | 0.151283483   | 0.131970926   |
| 0.240337042   | 0.223826229   | 0.206694639   | 0.188958204   | 0.170640353   | 0.151772495   | 0.132394308   |
| 0.240836516   | 0.224285893   | 0.207114382   | 0.189337922   | 0.170979953   | 0.152071893   | 0.132653431   |
| 0.241094167   | 0.224522897   | 0.207330707   | 0.189533542   | 0.171154843   | 0.152226032   | 0.132786798   |
| 0.241187607   | 0.224608821   | 0.207409112   | 0.189604423   | 0.171218198   | 0.152281857   | 0.132835091   |
| 0.241198027   | 0.224618401   | 0.207417853   | 0.189612325   | 0.171225259   | 0.152288078   | 0.132840473   |

| Absolute Value of delta s difference between adjacent longitudinals |                |                 |                 |                 |             | longitudinal 13<br>and side shell |
|---|----------------|-----------------|-----------------|-----------------|-------------|-----------------------------------|
| (long. 8 & 9)   | (long. 9 & 10) | (long. 10 & 11) | (long. 11 & 12) | (long. 12 & 13) |             |                                   |
| 0   | 0              | 0               | 0               | 0               | 0           | 0                                 |
| 4.54159E-06   | 3.70161E-06    | 2.86162E-06     | 2.0216B-06      | 1.18158E-06     | 3.45216B-07 |                                   |
| 4.53002B-05   | 3.69241E-05    | 2.85464E-05     | 2.01674E-05     | 1.17877E-05     | 3.44398E-06 |                                   |
| 0.000157878   | 0.000128699    | 9.95064E-05     | 7.03035E-05     | 4.10933E-05     | 1.20064E-05 |                                   |
| 0.000376662   | 0.000307089    | 0.000237459     | 0.000167784     | 9.80778E-05     | 2.86362E-05 |                                   |
| 0.000734244   | 0.000398727    | 0.000463033     | 0.000327204     | 0.000191278     | 5.58896E-05 |                                   |
| 0.001260882   | 0.001028378    | 0.000795442     | 0.000562171     | 0.000328663     | 9.60358E-05 |                                   |
| 0.001984006   | 0.001618544    | 0.001252168     | 0.000885084     | 0.000517496     | 0.00015122  |                                   |
| 0.002927776   | 0.002389101    | 0.001848694     | 0.001306939     | 0.000764229     | 0.00022333  |                                   |
| 0.004112709   | 0.003356991    | 0.002398251     | 0.001837158     | 0.001074396     | 0.000313986 |                                   |
| 0.005555369   | 0.004353958    | 0.00351162      | 0.002483439     | 0.001452526     | 0.000424518 |                                   |
| 0.007268129   | 0.005936352    | 0.004596965     | 0.003251636     | 0.001902078     | 0.000555939 |                                   |
| 0.00925901  | 0.007564974    | 0.005859712     | 0.004145673     | 0.002425378     | 0.000708934 |                                   |
| 0.011531592   | 0.009424997    | 0.007302477     | 0.005167483     | 0.003023591     | 0.000813849 |                                   |
| 0.014084997   | 0.011515935    | 0.008925027     | 0.006316982     | 0.003696698     | 0.001080682 |                                   |
| 0.016913945   | 0.013833673    | 0.010724304     | 0.007392075     | 0.0044435       | 0.001299086 |                                   |
| 0.020008877   | 0.016370556    | 0.012694479     | 0.008988692     | 0.005261637     | 0.001538375 |                                   |
| 0.023356139   | 0.019115528    | 0.014827051     | 0.010300856     | 0.006147627     | 0.001797533 |                                   |
| 0.026938226   | 0.02205432     | 0.017110994     | 0.012120778     | 0.00709692      | 0.00207523  |                                   |
| 0.030734079   | 0.025169692    | 0.019532933     | 0.013838986     | 0.008103969     | 0.002369846 |                                   |
| 0.034719424   | 0.028441705    | 0.022077358     | 0.015644474     | 0.009162322     | 0.002679492 |                                   |
| 0.038867162   | 0.031848041    | 0.02472687      | 0.017524872     | 0.01026472      | 0.003002043 |                                   |
| 0.043147782   | 0.035364347    | 0.027462456     | 0.019466645     | 0.011403214     | 0.003335171 |                                   |
| 0.04752981  | 0.038964611    | 0.030263776     | 0.021455306     | 0.012569285     | 0.003676381 |                                   |
| 0.051980281   | 0.042621558    | 0.033109488     | 0.023475635     | 0.013753987     | 0.004023051 |                                   |
| 0.056465227   | 0.046307061    | 0.035977566     | 0.025511921     | 0.014948077     | 0.004372472 |                                   |
| 0.060950172   | 0.049992563    | 0.038845644     | 0.027548204     | 0.016142167     | 0.004721894 |                                   |
| 0.065400643   | 0.05364951     | 0.041691356     | 0.029568536     | 0.017326869     | 0.005068564 |                                   |
| 0.069782671   | 0.057249774    | 0.044492676     | 0.031557197     | 0.018492941     | 0.005409774 |                                   |
| 0.074063291   | 0.060766608    | 0.047228261     | 0.033498971     | 0.019631434     | 0.005742902 |                                   |
| 0.078211029   | 0.064172416    | 0.049877774     | 0.035379366     | 0.020733832     | 0.006065453 |                                   |
| 0.082196375   | 0.067444429    | 0.052422199     | 0.037184856     | 0.021792185     | 0.006375099 |                                   |
| 0.085992227   | 0.070559801    | 0.054844138     | 0.038903064     | 0.022799234     | 0.006669714 |                                   |
| 0.089574314   | 0.073498594    | 0.057128081     | 0.040522987     | 0.023748527     | 0.006947412 |                                   |
| 0.092921576   | 0.076243565    | 0.059260653     | 0.04203515      | 0.024634517     | 0.00720657  |                                   |
| 0.096016508   | 0.078780448    | 0.061230827     | 0.043431767     | 0.025452654     | 0.007445859 |                                   |
| 0.098845456   | 0.081098186    | 0.063030105     | 0.04470586      | 0.026199456     | 0.007664263 |                                   |
| 0.101398861   | 0.083189124    | 0.064652655     | 0.04585636      | 0.026872563     | 0.007861096 |                                   |
| 0.103671443   | 0.085049147    | 0.066095419     | 0.04687817      | 0.027470776     | 0.008036011 |                                   |
| 0.105662325   | 0.086677769    | 0.067358167     | 0.047772207     | 0.027994076     | 0.008189006 |                                   |
| 0.107375083   | 0.088078163    | 0.068443512     | 0.048540404     | 0.028443628     | 0.008320427 |                                   |
| 0.108817743   | 0.089257131    | 0.069356881     | 0.049186684     | 0.028821759     | 0.008430959 |                                   |
| 0.110002678   | 0.09022502     | 0.070106438     | 0.049716903     | 0.029131925     | 0.008521515 |                                   |
| 0.110946448   | 0.090995578    | 0.070702964     | 0.050138759     | 0.029378658     | 0.008593725 |                                   |
| 0.111669571   | 0.091585743    | 0.07115969      | 0.050461671     | 0.029567491     | 0.008648909 |                                   |
| 0.112196209   | 0.092015395    | 0.071492099     | 0.050696638     | 0.029704876     | 0.008689055 |                                   |
| 0.112553791   | 0.092307032    | 0.071717673     | 0.050856059     | 0.029798077     | 0.008716289 |                                   |
| 0.112772575   | 0.092485422    | 0.071855626     | 0.050953539     | 0.029855061     | 0.008732939 |                                   |
| 0.112885153   | 0.092577197    | 0.071926586     | 0.051003675     | 0.029884366     | 0.008741501 |                                   |
| 0.112925912   | 0.09261042     | 0.07195227      | 0.051021821     | 0.029894972     | 0.0087446   |                                   |
| 0.112930453   | 0.092614121    | 0.071955132     | 0.051023842     | 0.029896154     | 0.008744945 |                                   |

| Shear rotation angle (Phi) for hull plate element [radians] |               |               |               |               |               |               |
|---|---------------|---------------|---------------|---------------|---------------|---------------|
| (long. 1 & 2)   | (long. 2 & 3) | (long. 3 & 4) | (long. 4 & 5) | (long. 5 & 6) | (long. 6 & 7) | (long. 7 & 8) |
| 0   | 0             | 0             | 0             | 0             | 0             | 0             |
| 1.14511E-05   | 1.05284E-05   | 9.60555E-06   | 8.68267E-06   | 7.75975E-06   | 6.83679E-06   | 5.91379E-06   |
| 0.000114132   | 0.00010495    | 9.5764E-05    | 8.65738E-05   | 7.738E-05     | 6.81829E-05   | 5.89829E-05   |
| 0.000397265   | 0.000365393   | 0.000333485   | 0.000301542   | 0.000269568   | 0.000237566   | 0.00020554    |
| 0.000946137   | 0.000670519   | 0.00079474    | 0.000718814   | 0.000642754   | 0.000566575   | 0.000490291   |
| 0.001840328   | 0.001693941   | 0.00154707    | 0.001399754   | 0.001252033   | 0.00110395    | 0.000955543   |
| 0.003152148   | 0.002902825   | 0.002652328   | 0.00240075    | 0.002148187   | 0.001894738   | 0.001640503   |
| 0.004945333   | 0.004556685   | 0.004165591   | 0.003772237   | 0.003376819   | 0.002979538   | 0.002580602   |
| 0.00727402  | 0.006706425   | 0.006134265   | 0.00557876    | 0.004977612   | 0.004393842   | 0.003906948   |
| 0.010182012   | 0.009393611   | 0.00859739    | 0.007793899   | 0.006983726   | 0.006167487   | 0.00534583    |
| 0.013702312   | 0.012649992   | 0.01158515    | 0.010508628   | 0.009421331   | 0.008324224   | 0.007218331   |
| 0.017856925   | 0.016497235   | 0.015118575   | 0.013722153   | 0.012309288   | 0.010881405   | 0.009440031   |
| 0.022656878   | 0.02094688    | 0.019209431   | 0.017446182   | 0.015658965   | 0.013849785   | 0.012020815   |
| 0.028102445   | 0.026000463   | 0.023860274   | 0.021684037   | 0.01947419    | 0.017233445   | 0.014964775   |
| 0.034183528   | 0.031649813   | 0.029064657   | 0.026430784   | 0.023751326   | 0.021029824   | 0.018270208   |
| 0.040880183   | 0.037877472   | 0.034807467   | 0.031673495   | 0.028479461   | 0.025229848   | 0.021929703   |
| 0.048163254   | 0.044657235   | 0.041065373   | 0.037391617   | 0.033640704   | 0.029818167   | 0.025930321   |
| 0.055995091   | 0.051954784   | 0.047807373   | 0.043557432   | 0.039210574   | 0.034773472   | 0.03025385    |
| 0.064330348   | 0.059728401   | 0.054995432   | 0.050136617   | 0.045158475   | 0.040068894   | 0.034877135   |
| 0.073116832   | 0.067929747   | 0.062585174   | 0.057088863   | 0.051448243   | 0.045672477   | 0.039772479   |
| 0.082296401   | 0.076504689   | 0.07052665    | 0.064368561   | 0.058038756   | 0.051547715   | 0.044906098   |
| 0.091805898   | 0.085394176   | 0.078765145   | 0.071925547   | 0.064884605   | 0.057654141   | 0.050248639   |
| 0.101578111   | 0.094535148   | 0.087242022   | 0.079705878   | 0.071936799   | 0.063947963   | 0.055755733   |
| 0.111542744   | 0.10386146    | 0.095895599   | 0.08765265    | 0.079143517   | 0.070382734   | 0.061388389   |
| 0.121627409   | 0.113304824   | 0.104662041   | 0.095706828   | 0.086450873   | 0.076910057   | 0.067104617   |
| 0.131758599   | 0.122795754   | 0.113476258   | 0.103808099   | 0.09380371    | 0.083480303   | 0.072860068   |
| 0.141862657   | 0.132264503   | 0.122272803   | 0.111895718   | 0.101146389   | 0.090043332   | 0.078610688   |
| 0.151866724   | 0.141641979   | 0.130986757   | 0.119909352   | 0.108423579   | 0.096549228   | 0.084312371   |
| 0.161699649   | 0.1508660641  | 0.139554592   | 0.127789904   | 0.115581034   | 0.102949013   | 0.089921807   |
| 0.171292869   | 0.159855358   | 0.147915009   | 0.135480316   | 0.12256635    | 0.109195348   | 0.095397121   |
| 0.180581243   | 0.168564231   | 0.156009736   | 0.142926335   | 0.129329693   | 0.115243209   | 0.100698477   |
| 0.189503835   | 0.176929369   | 0.163784283   | 0.150077244   | 0.135824488   | 0.121050529   | 0.105788668   |
| 0.198004659   | 0.184897621   | 0.171188665   | 0.156688653   | 0.142008072   | 0.126578801   | 0.110633658   |
| 0.206033367   | 0.19242126    | 0.178178057   | 0.163312629   | 0.147842296   | 0.131793637   | 0.115203088   |
| 0.213545895   | 0.199458613   | 0.184713414   | 0.169319293   | 0.153294074   | 0.136665273   | 0.119470731   |
| 0.220505068   | 0.205974653   | 0.190762036   | 0.174876346   | 0.158335883   | 0.141169026   | 0.123414894   |
| 0.226881161   | 0.211941532   | 0.196298071   | 0.179960045   | 0.162946196   | 0.145285681   | 0.127018766   |
| 0.232652411   | 0.21733907    | 0.201302968   | 0.184553513   | 0.167109845   | 0.149001814   | 0.130270695   |
| 0.237805494   | 0.222155177   | 0.205765859   | 0.188647078   | 0.170818327   | 0.152310053   | 0.1331644     |
| 0.24233594  | 0.226386226   | 0.209683874   | 0.19223854    | 0.174070014   | 0.155209249   | 0.135699117   |
| 0.246248478   | 0.230037332   | 0.21306237    | 0.195333344   | 0.176870291   | 0.157704572   | 0.137879656   |
| 0.249557303   | 0.23312255    | 0.215915062   | 0.197944667   | 0.17923159    | 0.159807518   | 0.139716386   |
| 0.25228622  | 0.235664949   | 0.218264045   | 0.200093384   | 0.181173332   | 0.161535816   | 0.141225134   |
| 0.254468659   | 0.237696562   | 0.220139681   | 0.20180792    | 0.18272174    | 0.162913244   | 0.14242699    |
| 0.256147505   | 0.239258156   | 0.221580337   | 0.203123963   | 0.183909553   | 0.163969329   | 0.143348029   |
| 0.257374733   | 0.240398836   | 0.222631967   | 0.204084036   | 0.1847756     | 0.16473895    | 0.144018944   |
| 0.258210805   | 0.241175435   | 0.223347505   | 0.204736919   | 0.185364248   | 0.165261827   | 0.144474585   |
| 0.25872382  | 0.241651689   | 0.223786087   | 0.20513691    | 0.185724734   | 0.165581915   | 0.144753422   |
| 0.258988401   | 0.241897202   | 0.224012088   | 0.205342949   | 0.185910362   | 0.165746692   | 0.144896927   |
| 0.259084344   | 0.241986204   | 0.224093993   | 0.205417601   | 0.185977602   | 0.165806367   | 0.144948889   |
| 0.259095044   | 0.241996128   | 0.224103124   | 0.205425922   | 0.185983097   | 0.165813018   | 0.14495468    |

| Shear rotation angle (Phi) for hull plate element [radians] |                |                 |                 |                 |                              |
|---|----------------|-----------------|-----------------|-----------------|------------------------------|
| (long. 8 & 9)   | (long. 9 & 10) | (long. 10 & 11) | (long. 11 & 12) | (long. 12 & 13) | longitudinal.13 & side shell |
| 0   | 0              | 0               | 0               | 0               | 0                            |
| 4.99076E-06   | 4.06771E-06    | 3.14463E-06     | 2.22154E-06     | 1.29844E-06     | 4.18443E-07                  |
| 4.97805E-05   | 4.05759E-05    | 3.13696E-05     | 2.2162E-05      | 1.29535E-05     | 4.17452E-06                  |
| 0.000173493   | 0.000141428    | 0.000109348     | 7.72566E-05     | 4.51575E-05     | 1.45332E-05                  |
| 0.000413914   | 0.000337461    | 0.000260944     | 0.000184378     | 0.000107777     | 3.47348E-05                  |
| 0.000806861   | 0.0006357941   | 0.000508828     | 0.000395953     | 0.000210196     | 6.7745E-05                   |
| 0.001385584   | 0.001130065    | 0.000874112     | 0.00061777      | 0.000361168     | 0.000116407                  |
| 0.002180222   | 0.001778618    | 0.001376008     | 0.000972619     | 0.000568677     | 0.000183297                  |
| 0.003217325   | 0.00262538     | 0.002031529     | 0.001436196     | 0.000839812     | 0.000270703                  |
| 0.004519429   | 0.003688984    | 0.002855213     | 0.002018853     | 0.001180654     | 0.00039059                   |
| 0.006104725   | 0.004984528    | 0.003858904     | 0.002729047     | 0.001596182     | 0.000514567                  |
| 0.007986785   | 0.006323371    | 0.005051567     | 0.003573211     | 0.002090193     | 0.000673865                  |
| 0.010174385   | 0.008312967    | 0.006439155     | 0.004555653     | 0.002665245     | 0.000859314                  |
| 0.012671401   | 0.01035677     | 0.008024527     | 0.005678491     | 0.003322615     | 0.001071331                  |
| 0.015476783   | 0.012634198    | 0.009807408     | 0.006941627     | 0.004062283     | 0.001309916                  |
| 0.018584613   | 0.015200668    | 0.011784404     | 0.008342747     | 0.004882928     | 0.001574648                  |
| 0.021984235   | 0.017987682    | 0.013949072     | 0.009877363     | 0.005781955     | 0.001864695                  |
| 0.025660453   | 0.021002986    | 0.016292021     | 0.01153889      | 0.006755532     | 0.002178824                  |
| 0.029593804   | 0.024230773    | 0.018801075     | 0.013318749     | 0.007798655     | 0.002515426                  |
| 0.033760688   | 0.027631952    | 0.021461466     | 0.015206305     | 0.008905225     | 0.002872533                  |
| 0.038134717   | 0.031244445    | 0.024256075     | 0.017190036     | 0.010068146     | 0.003247858                  |
| 0.042685224   | 0.034983369    | 0.027165701     | 0.01925572      | 0.011279434     | 0.003638824                  |
| 0.04737966  | 0.038842374    | 0.030169366     | 0.021388656     | 0.012530348     | 0.004042609                  |
| 0.052183142   | 0.042792115    | 0.033244644     | 0.023572892     | 0.013811523     | 0.00445619                   |
| 0.057059184   | 0.046802674    | 0.03636801      | 0.02579168      | 0.015113121     | 0.004876387                  |
| 0.061970249   | 0.050843024    | 0.039515207     | 0.028027737     | 0.016424981     | 0.005299917                  |
| 0.066878323   | 0.054881715    | 0.042661621     | 0.030263513     | 0.017736785     | 0.005723445                  |
| 0.071745484   | 0.058887343    | 0.045782662     | 0.032481469     | 0.019038214     | 0.006143637                  |
| 0.076534469   | 0.062829037    | 0.048854147     | 0.034664347     | 0.020319116     | 0.006557208                  |
| 0.081209236   | 0.066676925    | 0.051852666     | 0.036795441     | 0.021569658     | 0.006960981                  |
| 0.085735498   | 0.07040259     | 0.0547535952    | 0.038858856     | 0.022780489     | 0.007351932                  |
| 0.090081234   | 0.073979499    | 0.057543215     | 0.040839759     | 0.02394288      | 0.007727239                  |
| 0.094217175   | 0.07738341     | 0.060195472     | 0.042724605     | 0.025048863     | 0.008084326                  |
| 0.098117237   | 0.080592742    | 0.062699384     | 0.044501355     | 0.02609136      | 0.008420906                  |
| 0.101758927   | 0.08358891     | 0.065029774     | 0.04615966      | 0.027064288     | 0.008735014                  |
| 0.105123686   | 0.086356611    | 0.067185352     | 0.047691027     | 0.027962659     | 0.009025039                  |
| 0.108197186   | 0.088884071    | 0.069153405     | 0.049088949     | 0.02878266      | 0.009289749                  |
| 0.110969563   | 0.091163233    | 0.070927695     | 0.050349015     | 0.02952171      | 0.009528313                  |
| 0.113435596   | 0.093189896    | 0.072505009     | 0.051468976     | 0.030178501     | 0.009740311                  |
| 0.115594812   | 0.094963797    | 0.073885223     | 0.052448784     | 0.030753022     | 0.009925742                  |
| 0.117451526   | 0.096488632    | 0.075071306     | 0.053290599     | 0.03124656      | 0.010065024                  |
| 0.119014808   | 0.097772017    | 0.076069287     | 0.053998755     | 0.031661678     | 0.010218988                  |
| 0.120298377   | 0.098825387    | 0.076888167     | 0.054579698     | 0.032002175     | 0.010328863                  |
| 0.121320424   | 0.099663841    | 0.077539789     | 0.055041884     | 0.03227303      | 0.01041626                   |
| 0.122103351   | 0.100305913    | 0.078038636     | 0.055395653     | 0.032480322     | 0.010483142                  |
| 0.122673449   | 0.100773301    | 0.078401709     | 0.055653063     | 0.032631134     | 0.010531799                  |
| 0.123060494   | 0.101090527    | 0.078648068     | 0.055827706     | 0.032733444     | 0.010564805                  |
| 0.123297286   | 0.10128456     | 0.078798726     | 0.055934493     | 0.032795996     | 0.010584985                  |
| 0.123419125   | 0.10138438     | 0.078876221     | 0.055989415     | 0.032828165     | 0.010595362                  |
| 0.123463236   | 0.101420514    | 0.07890427      | 0.056009293     | 0.032839808     | 0.010599118                  |
| 0.123468151   | 0.10142454     | 0.078907396     | 0.056011506     | 0.032841105     | 0.010599536                  |

### In-plane [NDSF/in] for hull plate element



### In-plane shear resistance [Newtons] for hull plate elements

**In-plane shear resistance [Newtons] for hull plate elements**

**total in-plane force [Newtons] =**

**Horizontal shear resistance [Newtons] for hull plate elements**

| (long. 1 & 2) | (long. 2 & 3) | (long. 3 & 4) | (long. 4 & 5) | (long. 5 & 6) | (long. 6 & 7) | (long. 7 & 8) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| 4091162899    | 3761790708    | 3432321597    | 3102764029    | 2773126478    | 2443417425    | 2113645359    |
| 4071179898    | 3744876832    | 3418110849    | 3090921855    | 2763349939    | 2435435361    | 2107218531    |
| 141341.2183   | 130095.9125   | 118813.8225   | 107498.0492   | 96151.7267    | 84778.02015   | 73380.12206   |
| 335431.1961   | 309013.3149   | 282441.6854   | 255728.8642   | 228887.6554   | 201931.0895   | 174872.4022   |
| 649563.1186   | 599056.2725   | 548091.71     | 496707.1539   | 444939.4463   | 392826.4615   | 340407.0136   |
| 1106800.962   | 1022037.928   | 936187.6388   | 849331.1134   | 761553.0923   | 672941.7788   | 583588.5518   |
| 1726224.66    | 1596307.475   | 1464159.161   | 1329936.186   | 1193805.012   | 1055941.486   | 916530.1403   |
| 1993871.024   | 2002984.828   | 2011443.802   | 1952045.406   | 1754365.104   | 1553458.802   | 1349658.943   |
| 1980142.805   | 1991202.772   | 2001498.183   | 2010989.371   | 2019638.768   | 2027411.289   | 1891568.23    |
| 1965882.554   | 1978926.364   | 1991105.357   | 2002364.81    | 2012652.472   | 2021919.184   | 2030119.481   |
| 1951349.954   | 1966376.005   | 1980449.102   | 1993497.252   | 2005451.14    | 2016245.053   | 2025817.79    |
| 1936793.457   | 1953765.132   | 1969709.42    | 1984535.303   | 1998154.412   | 2010482.313   | 2021439.818   |
| 1922446.047   | 1941296.43    | 1959059.278   | 1975623.341   | 1990879.703   | 2004723.488   | 2017055.649   |
| 1908522.291   | 1929159.001   | 1948662.018   | 1966899.25    | 1983740.31    | 1999058.699   | 2012734.118   |
| 1895216.561   | 1917526.419   | 1938669.423   | 1958492.62    | 1976843.833   | 1993574.342   | 2008541.772   |
| 1882702.242   | 1906555.573   | 1929220.375   | 1950523.373   | 1970290.876   | 1988351.957   | 2004541.939   |
| 1871131.709   | 1896386.155   | 1920440.037   | 1943100.791   | 1964174.047   | 1983467.284   | 2000793.92    |
| 1860363.884   | 1887140.663   | 1912439.451   | 1936322.869   | 1958577.199   | 197889.508    | 1997352.303   |
| 1851330.177   | 1878924.757   | 1905315.477   | 1930275.951   | 1953574.897   | 1974980.653   | 1994266.381   |
| 1843305.648   | 1871827.873   | 1899150.95    | 1925034.581   | 1949232.056   | 1971495.127   | 1991579.681   |
| 1836640.248   | 1865923.949   | 1894015       | 1920661.494   | 1945603.729   | 1968579.382   | 1989329.588   |
| 1831395.03    | 1861272.191   | 1889963.443   | 1917207.716   | 1942734.986   | 1966271.672   | 1987547.043   |
| 1827616.238   | 1857917.798   | 1887039.183   | 1914712.711   | 1940660.873   | 1964601.883   | 1986256.327   |
| 1825336.219   | 1855892.578   | 1885272.57    | 1913204.529   | 1939406.406   | 1963591.427   | 1985474.887   |
| 1824574.103   | 1855215.417   | 1884681.697   | 1912699.942   | 1938986.584   | 1963253.178   | 1985213.237   |
| 1825336.219   | 1855892.578   | 1885272.57    | 1913204.529   | 1939406.406   | 1963591.427   | 1985474.887   |
| 1827616.238   | 1857917.798   | 1887039.183   | 1914712.711   | 1940660.873   | 1964601.883   | 1986256.327   |
| 1831395.03    | 1861272.191   | 1889963.443   | 1917207.716   | 1942734.986   | 1966271.672   | 1987547.043   |
| 1836640.248   | 1865923.949   | 1894015       | 1920661.494   | 1945603.729   | 1968579.382   | 1989329.588   |
| 1843305.648   | 1871827.873   | 1899150.95    | 1925034.581   | 1949232.056   | 1971495.127   | 1991579.681   |
| 1851330.177   | 1878924.757   | 1905315.477   | 1930275.951   | 1953574.897   | 1974980.653   | 1994266.381   |
| 1860363.884   | 1887140.663   | 1912439.451   | 1936322.869   | 1958577.199   | 197889.508    | 1997352.303   |
| 1871131.709   | 1896386.155   | 1920440.037   | 1943100.791   | 1964174.047   | 1983467.284   | 2000793.92    |
| 1882702.242   | 1906555.573   | 1929220.375   | 1950523.373   | 1970290.876   | 1988351.957   | 2004541.939   |
| 1895216.561   | 1917526.419   | 1938669.423   | 1958492.62    | 1976843.833   | 1993574.342   | 2008541.772   |
| 1908522.291   | 1929159.001   | 1948662.018   | 1966899.25    | 1983740.31    | 1999058.699   | 2012734.118   |
| 1922446.047   | 1941296.43    | 1939039.278   | 1973623.341   | 1990879.703   | 2004723.488   | 2017055.649   |
| 1936793.457   | 1953765.132   | 1969709.42    | 1984535.303   | 1998154.412   | 2010482.313   | 2021439.818   |
| 1951349.954   | 1966376.005   | 1980449.102   | 1993497.252   | 2005451.14    | 2016245.053   | 2025817.79    |
| 1965882.554   | 1978926.364   | 1991105.357   | 2002364.81    | 2012652.472   | 2021919.184   | 2030119.481   |
| 1980142.805   | 1991202.772   | 2001498.183   | 2010989.371   | 2019638.768   | 2027411.289   | 2034274.707   |
| 1993871.024   | 2002984.828   | 2011443.802   | 2019220.806   | 2026290.333   | 2032628.734   | 2038214.421   |
| 2006801.903   | 2014049.918   | 2020758.577   | 2026910.595   | 2032489.846   | 2037481.475   | 2041872.007   |
| 2018671.404   | 2024178.841   | 2029263.484   | 2033915.292   | 2038124.957   | 2041883.961   | 2045184.613   |
| 2029224.768   | 2033162.14    | 2036789.009   | 2040100.216   | 2043091.006   | 2045757.055   | 2048094.47    |
| 2038225.279   | 2040806.861   | 2043180.252   | 2045343.225   | 2047293.736   | 2049029.935   | 2050550.165   |
| 2045463.302   | 2046943.39    | 2048301.993   | 2049538.376   | 2050651.869   | 2051641.864   | 2052507.823   |
| 2050764.985   | 2051431.915   | 2052043.404   | 2052599.3     | 2053099.469   | 2053543.787   | 2053932.145   |
| 2053999.96    | 2054168.084   | 2054322.125   | 2054462.072   | 2054587.917   | 2054699.651   | 2054797.269   |
| 2055087.413   | 2055087.413   | 2055087.413   | 2055087.413   | 2055087.413   | 2055087.413   | 2055087.413   |
| column total  |
| 86495617.41   | 86998922.42   | 87458641.54   | 8780207.92    | 87953043.21   | 88041168.98   | 87918620.9    |

| Horizontal shear resistance [Newtons] for hull plate elements |                |                 |                 |                 |                              |
|---|----------------|-----------------|-----------------|-----------------|------------------------------|
| (long. 8 & 9)   | (long. 9 & 10) | (long. 10 & 11) | (long. 11 & 12) | (long. 12 & 13) | longitudinal.13 & side abell |
| 0   | 0              | 0               | 0               | 0               | 0                            |
| 1783.818775   | 1453.946173    | 1124.036059     | 794.0969405     | 464.1373308     | 149.5766534                  |
| 17787.39989   | 1450.40395     | 11211.60504     | 7921.411507     | 4630.232317     | 1492.220063                  |
| 61961.24956   | 50524.64067    | 39073.55233     | 27611.25499     | 16141.03134     | 5202.145149                  |
| 147725.0116   | 120502.495     | 93218.56541     | 65887.04669     | 38521.8492      | 12416.16291                  |
| 287720.7579   | 234808.0879    | 181710.0274     | 128468.1193     | 75124.31116     | 24215.72294                  |
| 493587.6534   | 403035.8522    | 312032.0864     | 220677.0876     | 129072.9903     | 41609.83579                  |
| 775763.3995   | 633840.6986    | 490967.5285     | 347354.4109     | 203215.8164     | 65519.14588                  |
| 1143316.579   | 934799.4203    | 724489.6637     | 512781.6011     | 300079.0614     | 96761.40724                  |
| 1603812.625   | 1312295.632    | 1017670.005     | 720614.3217     | 421827.6178     | 136038.5916                  |
| 2037212.183   | 1771429.712    | 1374603.065     | 973826.3929     | 570231.1904     | 183925.8314                  |
| 2034113.575   | 2041082.907    | 1798349.134     | 1274665.479     | 746636.9141     | 240862.3723                  |
| 2030954.3   | 2088960.946    | 2045403.937     | 1624620.991     | 951948.7959     | 307144.6751                  |
| 2027784.766   | 2036828.804    | 2044116.775     | 2024404.497     | 1186614.261     | 382921.773                   |
| 2024654.842   | 2034720.07     | 2042842.162     | 2048948.685     | 1450617.97      | 468192.9693                  |
| 2021613.096   | 2032667.657    | 2041600.078     | 2048322.933     | 1743482.942     | 562807.8666                  |
| 2018706.078   | 2030703.307    | 2040409.887     | 2047722.792     | 2052562.65      | 666468.7959                  |
| 2015977.685   | 2028857.126    | 2039290.046     | 2047157.646     | 2052368.165     | 778735.5341                  |
| 2013468.596   | 2027157.173    | 2038257.83      | 2046636.312     | 2052188.663     | 899032.2753                  |
| 2011215.773   | 2025629.075    | 2037329.088     | 2046166.902     | 2052026.962     | 1026636.733                  |
| 2009252.05  | 2024295.706    | 2036518.018     | 2045756.706     | 2051885.6       | 1160791.222                  |
| 2007605.76  | 2023176.894    | 2035836.971     | 2045412.081     | 2051766.791     | 1300515.563                  |
| 2006300.463   | 2022289.183    | 2035296.283     | 2045138.358     | 2051672.398     | 1444821.58                   |
| 2005354.717   | 2021645.643    | 2034904.139     | 2044939.768     | 2051603.898     | 1592628.983                  |
| 2004781.886   | 2021255.713    | 2034666.462     | 2044619.375     | 2051562.365     | 1742802.385                  |
| 2004590.043   | 2021125.099    | 2034586.836     | 2044779.037     | 2051548.448     | 1894169.171                  |
| 2004781.886   | 2021255.713    | 2034666.462     | 2044819.375     | 2051562.365     | 2045537.956                  |
| 2005354.717   | 2021645.643    | 2034904.139     | 2044939.768     | 2051603.898     | 2054724.72                   |
| 2006300.463   | 2022289.183    | 2035296.283     | 2045138.358     | 2051672.398     | 2054731.87                   |
| 2007605.76  | 2023176.894    | 2035836.971     | 2045412.081     | 2051766.791     | 2054741.721                  |
| 2009252.05  | 2024295.706    | 2036518.018     | 2045756.706     | 2051885.6       | 2054734.118                  |
| 2011215.773   | 2025629.075    | 2037329.088     | 2046166.902     | 2052026.962     | 2054768.866                  |
| 2013468.596   | 2027157.173    | 2038257.83      | 2046636.312     | 2052188.663     | 2054785.732                  |
| 2015977.685   | 2028857.126    | 2039290.046     | 2047157.646     | 2052368.165     | 2054804.45                   |
| 2018706.078   | 2030703.307    | 2040409.887     | 2047722.792     | 2052562.65      | 2054824.725                  |
| 2021613.096   | 2032667.657    | 2041600.078     | 2048322.933     | 2052769.062     | 2054846.237                  |
| 2024654.842   | 2034720.07     | 2042842.162     | 2048948.685     | 2052984.156     | 2054868.647                  |
| 2027784.766   | 2036828.804    | 2044116.775     | 2049590.234     | 2053204.545     | 2054891.601                  |
| 2030954.3   | 2038960.946    | 2045403.937     | 2050237.492     | 2053426.758     | 2054914.738                  |
| 2034113.575   | 2041082.907    | 2046683.354     | 2050880.252     | 2053647.291     | 2054937.693                  |
| 2037212.183   | 2043160.942    | 2047934.745     | 2051508.351     | 2053862.661     | 2054960.103                  |
| 2040200.004   | 2045161.702    | 2049138.159     | 2052111.828     | 2054069.465     | 2054981.616                  |
| 2043028.068   | 2047052.792    | 2050274.312     | 2052681.088     | 2054264.434     | 2055001.891                  |
| 2045649.439   | 2048803.335    | 2051324.902     | 2053207.056     | 2054444.482     | 2055020.61                   |
| 2048020.101   | 2050384.523    | 2052272.925     | 2053681.329     | 2054606.754     | 2055037.476                  |
| 2050099.821   | 2051770.147    | 2053102.973     | 2054096.312     | 2054748.68      | 2055052.225                  |
| 2051852.97  | 2052937.097    | 2053801.501     | 2054445.346     | 2054868.008     | 2055064.623                  |
| 2053249.271   | 2053865.801    | 2054357.075     | 2054722.823     | 2054962.843     | 2055074.474                  |
| 2054264.448   | 2054540.614    | 2054760.575     | 2054924.277     | 2055031.679     | 2055081.624                  |
| 2054880.764   | 2054950.131    | 2055005.365     | 2055046.464     | 2055073.424     | 2055085.96                   |
| 2055087.413   | 2055087.413    | 2055087.413     | 2055087.413     | 2055087.413     | 2055087.413                  |
| column total  | column total   | column total    | column total    | column total    | column total                 |
| 87372372.39   | 86874570.89    | 85715722.75     | 83738669.13     | 79692484.21     | 66399463.63                  |

total horizontal force [Newtons] = 2205322651  
 [GNewtons] = 2.205322651

## **APPENDIX B**

### **Non-dimensional Shear Force Plots and Calculations**

## Summary of Analytical Model Results

### **1. Elastic pre-buckling range of shear deformation**

a. elemental shear force:

$$F_e = \left( \frac{G t \lambda b}{2} \right) \Phi$$

$$G = \frac{E}{2(1 + v)}$$

b. non-dimensional shear per unit length:

$$\text{NDSF}_{el}/\text{per unit length} = \left[ \frac{E}{\sigma_o t (1 + v)} \right] \Phi$$

### **2. Elastic post-buckling range of deformation**

a. elemental shear force:

$$F_{eB} = \frac{8.98 \pi^2 E t^3 \lambda}{12(1 - v^2) b} + \left[ \frac{E t b \lambda}{8(1 + v)} \right] \left[ \Phi - \left( \frac{8.98 \pi^2 (1 + v) t^2}{3(1 - v^2) b^2} \right) \right]$$

b. non-dimensional shear per unit length:

$$\frac{\text{NDSF}_{elB}}{\text{unit length}} = \frac{8.98 \pi^2 E t}{3(1 - v^2) \sigma_o b^2} + \left[ \frac{E}{2 t \sigma_o (1 + v)} \right] \left[ \Phi - \frac{8.98 \pi^2 (1 + v) t^2}{3(1 - v^2) b^2} \right]$$

### 3. Plastic buckling range of deformation

a. non-dimensional shear force per unit length due to shear strain in aperture:

$$\frac{\text{NDSF}_s}{\text{unit length}} = \frac{2 \cos^2 \beta \cos^2 \Phi}{\sqrt{3} t} \sqrt{\left(1 - \frac{\tan \beta}{\lambda}\right)^2 + \frac{1}{\lambda^2}} \left[ \left( \frac{A_o}{b} \right) \frac{|\dot{\gamma}_{\xi_1}|}{\Phi} + |\gamma_{\xi_1}| \left( \frac{\dot{A}_o}{\Phi b} \right) \right]$$

b. non-dimensional shear force per unit length due to membrane strain:

$$\frac{\text{NDSF}_m}{\text{unit length}} = \frac{4}{t} \left[ \frac{4 \left( \frac{w_o}{b} \right) \left( \frac{\dot{w}_o}{\Phi b} \right) \cos^2 \beta \cos^2 \Phi + \lambda + \frac{\tan \Phi + \tan \beta}{\cos^2 \beta}}{\lambda^2 + 2 \lambda \tan \beta + \tan^2 \beta + 1} \right]$$

c. non-dimensional shear force per unit length due to plastic hinge bending:

$$\frac{\text{NDSF}_b}{\text{unit length}} = \left( \frac{4}{b} \right) \left| \frac{\dot{w}_o}{\Phi b} \right| \left[ \frac{\cos^3 \beta \cos^2 \Phi}{\sqrt{1 - 4 \left( \frac{w_o}{b} \right)^2 \cos^2 \beta}} \right] + \dots$$

$$\left( \frac{2 \cos^2 \Phi \cos^2 \beta}{b} \right) \sqrt{\left(1 + \frac{\tan \beta}{\lambda}\right)^2 + \frac{1}{\lambda^2}} \left| \frac{\dot{w}_o}{\Phi b} \right| \left[ \frac{\cos \beta \sin^{-1} \left( \frac{2 w_o}{b} \cos \beta \right)}{\sqrt{1 - 4 \left( \frac{w_o}{b} \right)^2 \cos^2 \beta}} + \frac{\sin^{-1} \left( \frac{2 w_o}{\lambda b} \right)}{\sqrt{\lambda^2 - 4 \left( \frac{w_o}{b} \right)^2}} \right] \\ \left[ \frac{\sqrt{\lambda^2 - 4 \left( \frac{w_o}{b} \right)^2}}{\sqrt{\left[ \sin^{-1} \left( \frac{2 w_o}{b} \cos \beta \right) \right]^2 + \left[ \sin^{-1} \left( \frac{2 w_o}{\lambda b} \right) \right]^2}} \right]$$

### 3. Plastic buckling model non-dimensional parameters

a. ratio of maximum aperture opening to plate width:

$$\frac{A_o}{b} = \frac{\tan\Phi}{\cos^2\beta} \sqrt{1 + (\lambda \cos^2\beta - \tan\Phi)^2}$$

b. ratio of aperture opening velocity to shear translation velocity difference between plate longitudinal edges:

$$\frac{\dot{A}_o}{\Phi b} = \frac{1}{\cos^2\beta \cos^2\Phi} \left[ \frac{2 \tan^2\Phi - 3\lambda \tan\Phi \cos^2\beta + \lambda^2 \cos^4\beta + 1}{\sqrt{\tan^2\Phi - 2\lambda \tan\Phi \cos^2\beta + \lambda^2 \cos^4\beta + 1}} \right]$$

c. shear strain:

$$\gamma_{\xi\eta} = \left[ \tan^{-1}(\lambda \cos^2\beta - \tan\Phi) - \tan^{-1}\left(\lambda - \tan\beta - \frac{\tan\Phi}{\cos^2\beta}\right) \right]$$

d. ratio of aperture shear strain rate to shear rotation rate:

$$\frac{\dot{\gamma}_{\xi\eta}}{\Phi} = \sec^2\Phi \left| \frac{1}{\cos^2\beta \left[ 1 + \left( \lambda - \tan\beta - \frac{\tan\Phi}{\cos^2\beta} \right)^2 \right]} - \frac{1}{1 + (\lambda \cos^2\beta - \tan\Phi)^2} \right|$$

e. ratio of maximum out-of-plane displacement to plate width:

$$\frac{w_o}{b} = \frac{1}{2} \sqrt{\lambda^2 + \left( \frac{2\lambda \tan\Phi - \frac{\tan^2\Phi}{\cos^2\beta} \cdot \lambda^2}{\cos^2\beta \cos^4\beta} \right)}$$

### 3. Plastic buckling model non-dimensional parameters (continued)

f. ratio of plate out-of-plane displacement rate to shear translation velocity difference between plate longitudinal edges:

$$\frac{\dot{w}_o}{\dot{\Phi}_b} = \frac{1}{4} \left[ \frac{\left( \frac{2\lambda}{\cos^2\beta} - \frac{2\tan\Phi}{\cos^4\beta} \right) + \left( \frac{2\lambda \tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta} - \lambda^2 \right) \sin(2\Phi)}{\sqrt{\lambda^2 \cos^8\Phi + \left( \frac{2\lambda \tan\Phi}{\cos^2\beta} - \frac{\tan^2\Phi}{\cos^4\beta} - \lambda^2 \right) \cos^6\Phi}} \right]$$

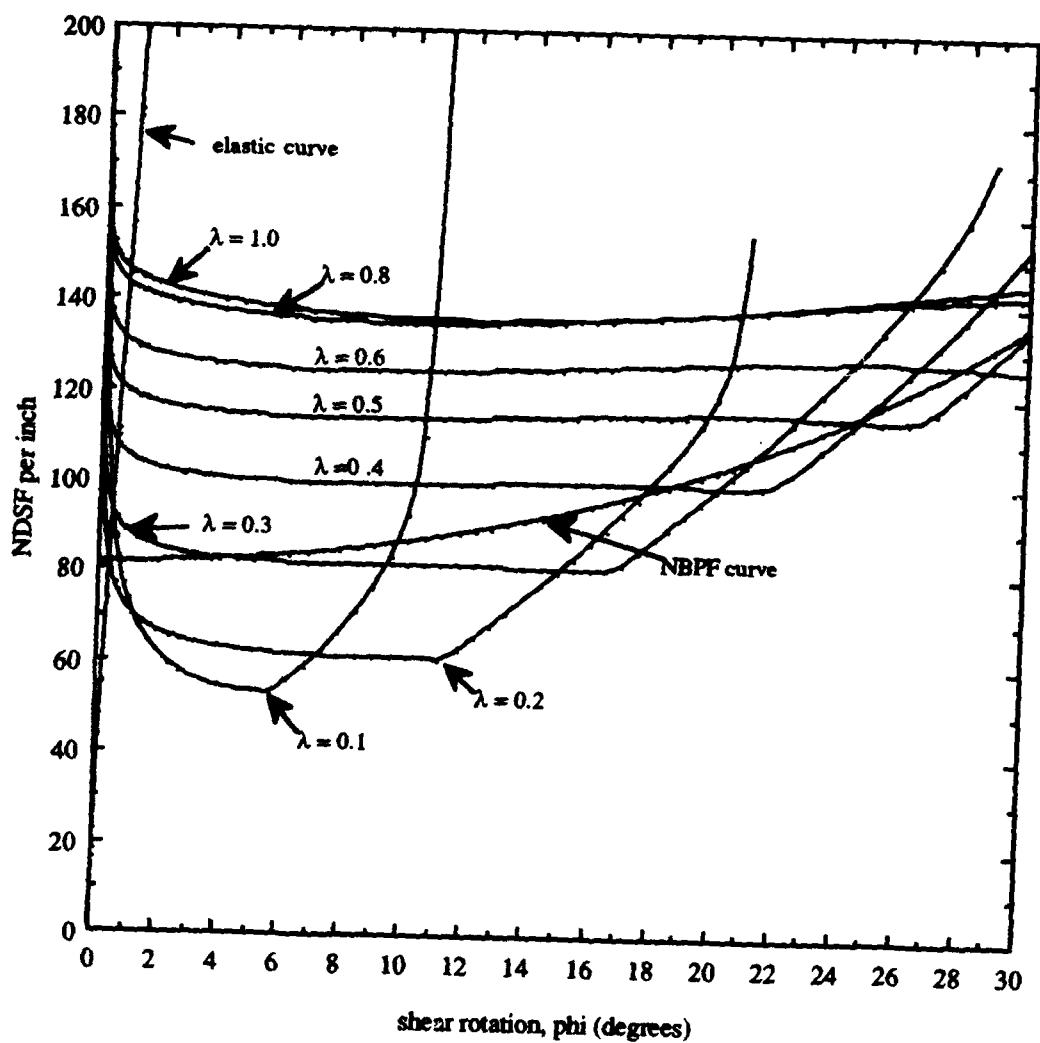


Figure B-1: Analytical results for beta = 0

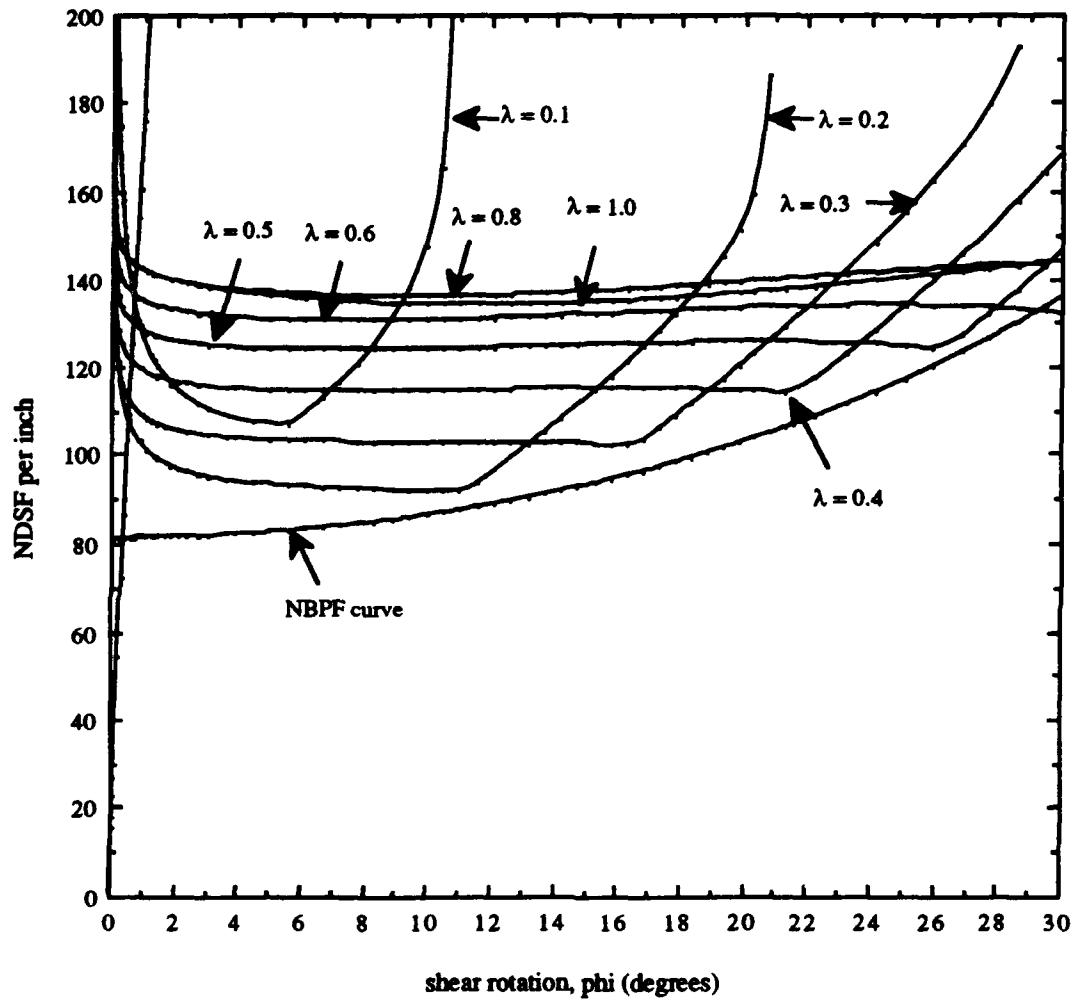


Figure B-2: Analytical results for beta = 0.1

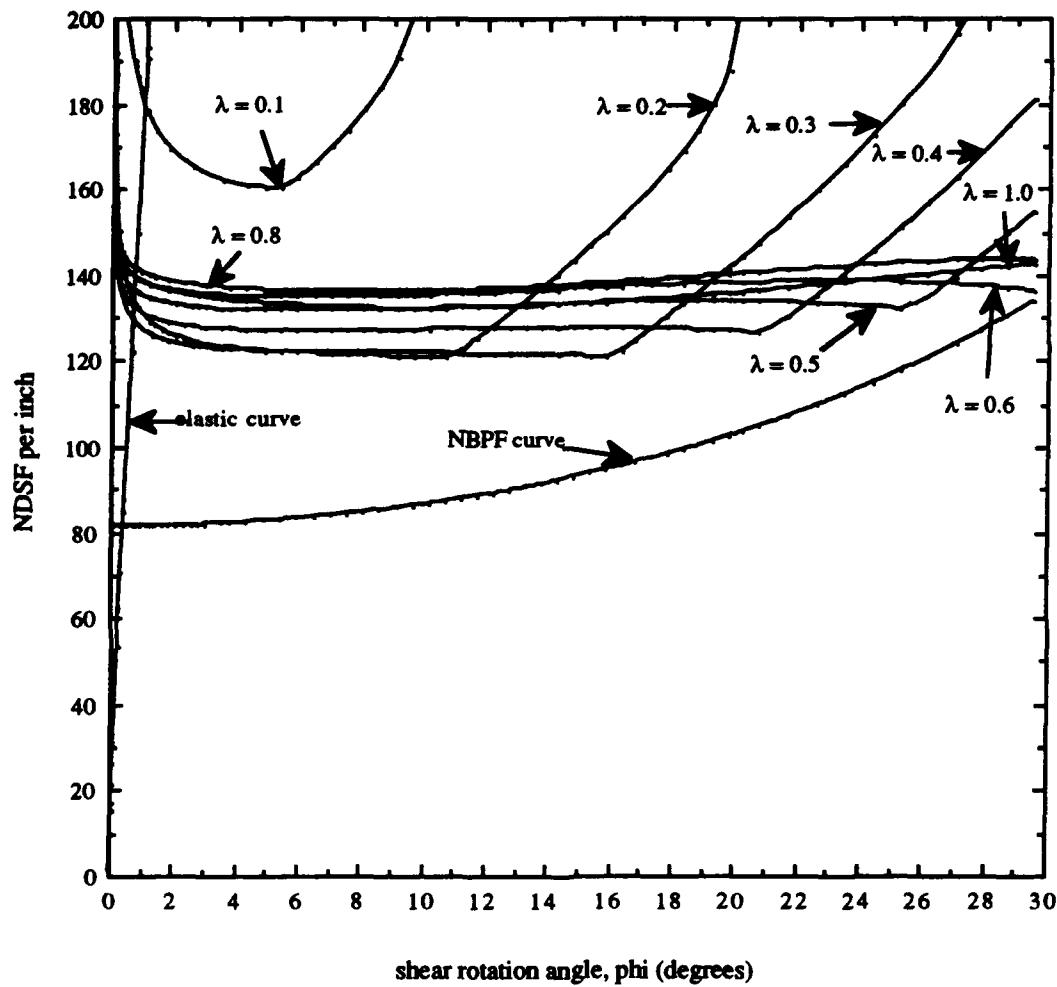


Figure B-3: Analytical results for  $\beta = 0.2$

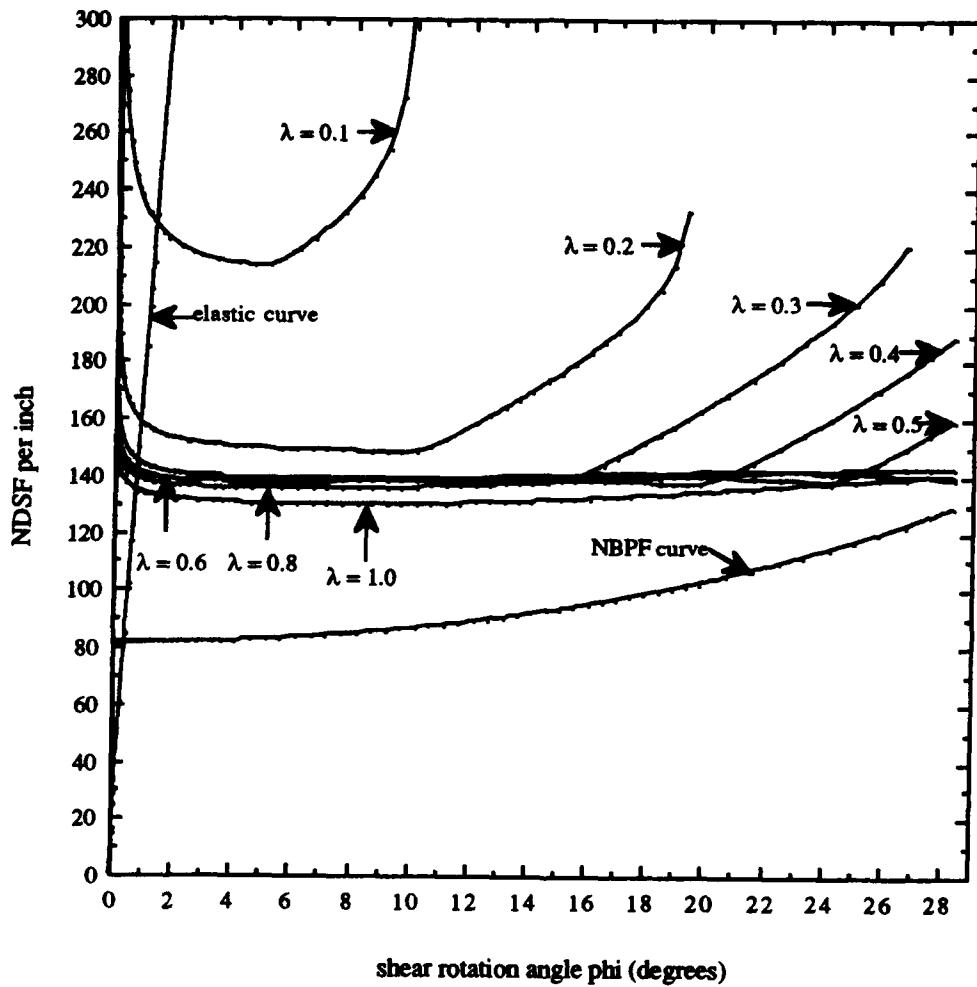


Figure B-4: Analytical results for beta=0.3

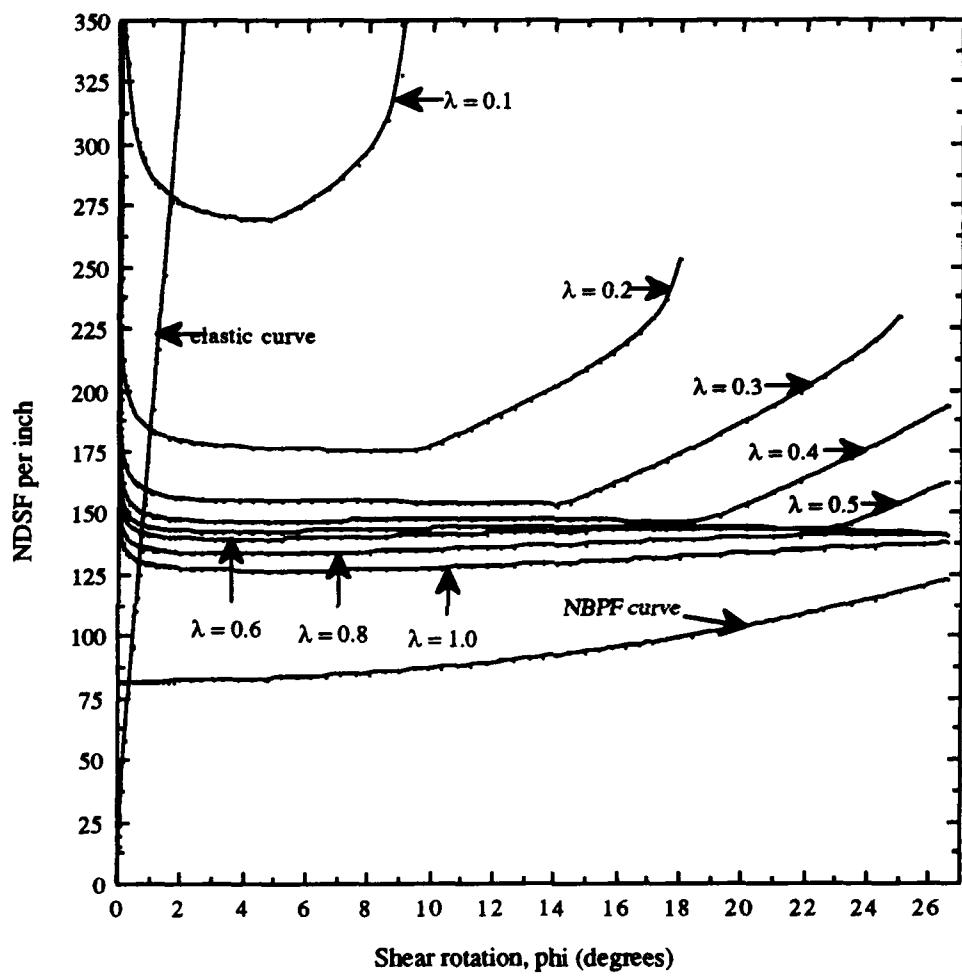


Figure B-5: Analytical results for beta = 0.4

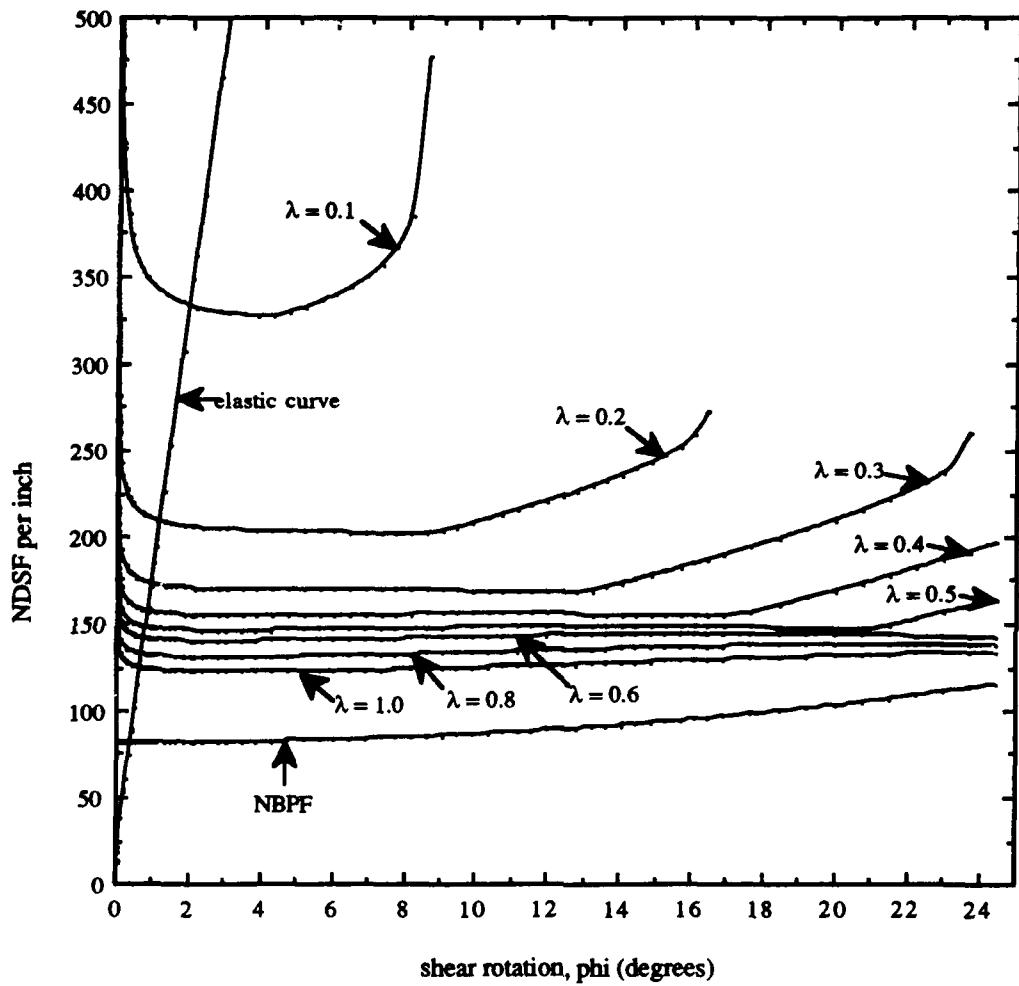


Figure B-6: Analytical results for beta = 0.5

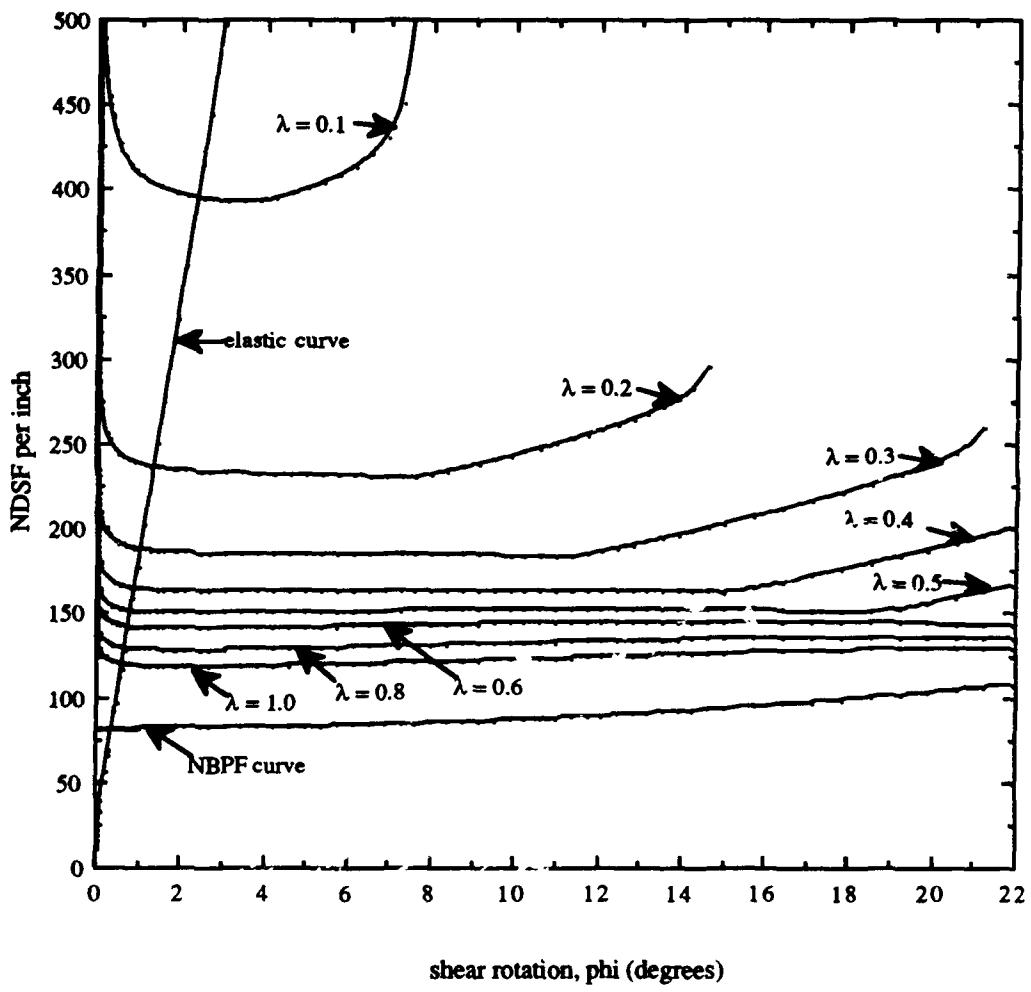


Figure B-7: Analytical results for beta = 0.6

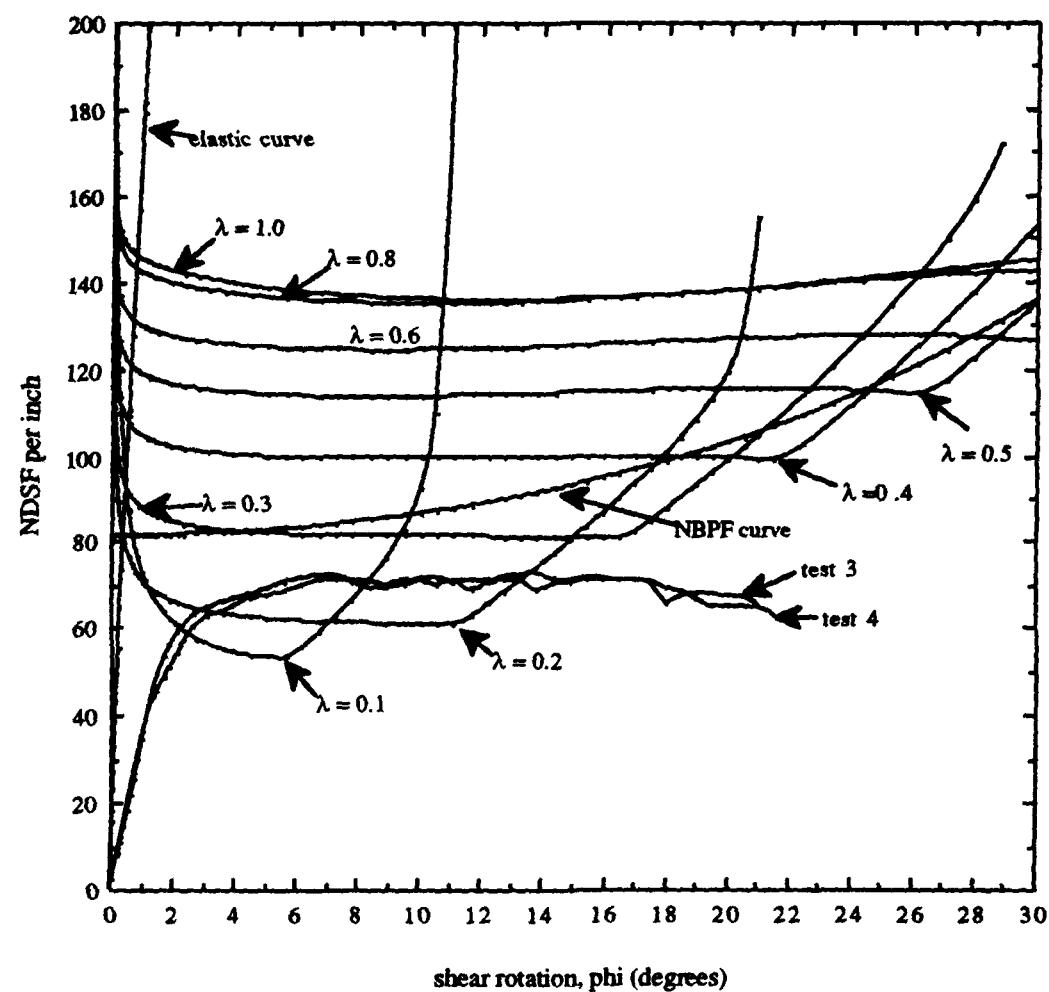


Figure B-8: Comparison of analytical models vs tests 3 and 4 (beta = 0)

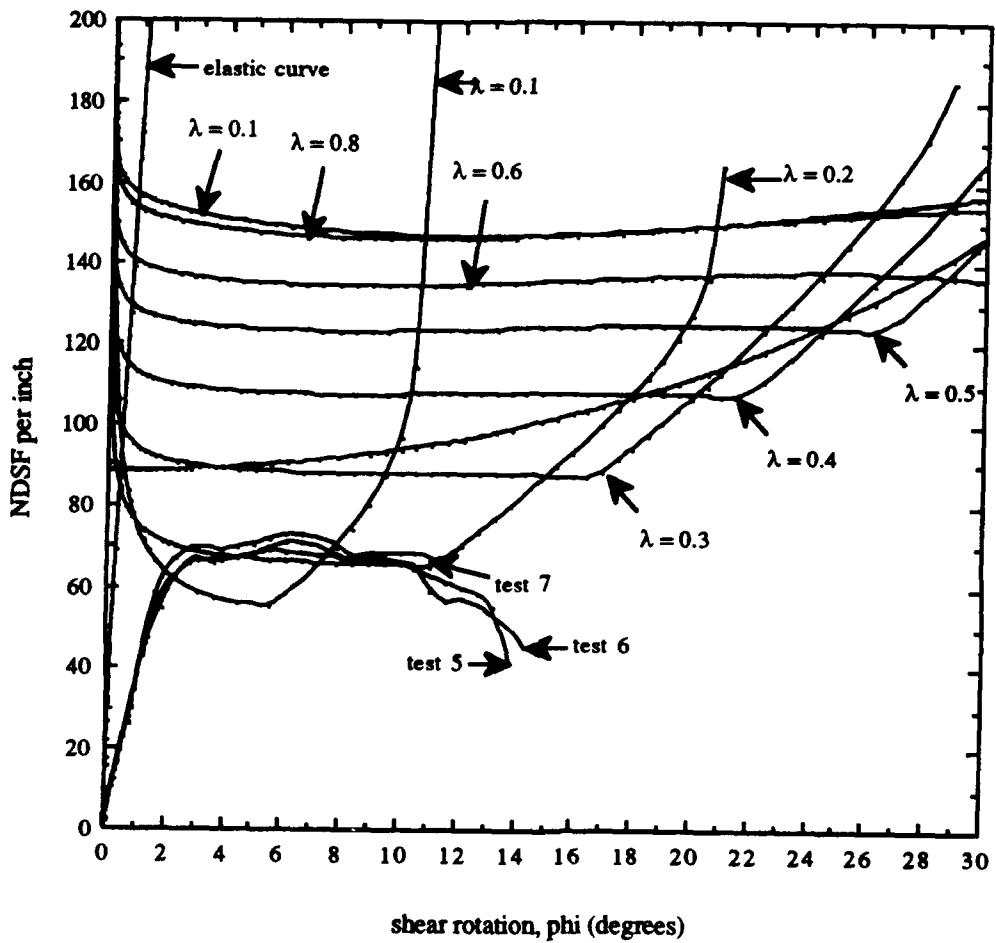
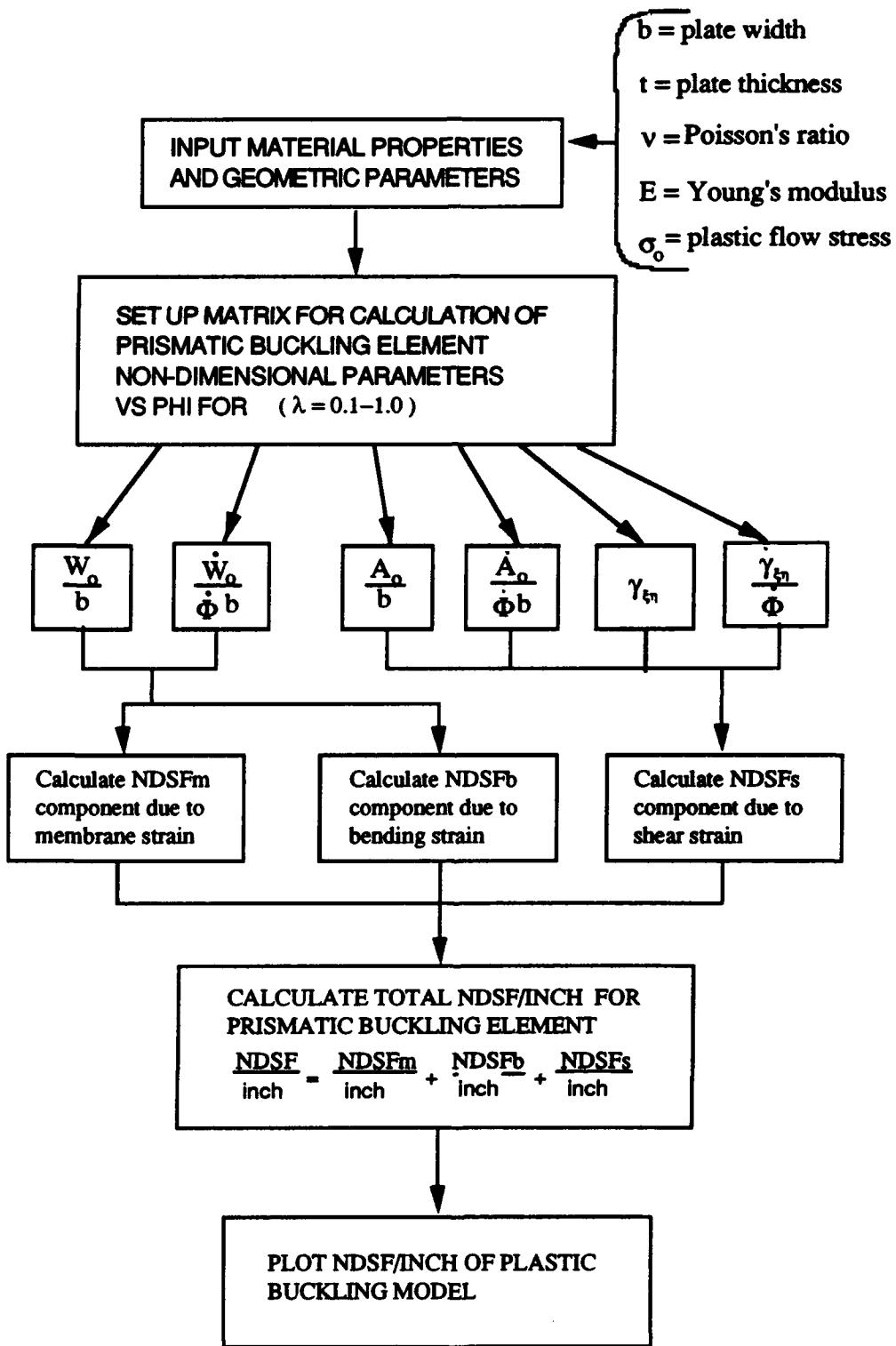


Figure B-9: Comparison of analytical models vs tests 5, 6, and 7 (beta = 0)

## Buckling Model Spreadsheet Logic Flow Diagram



### Sample Analytical Calculations (beta=0.1)

| S (mm)      | S (inches)  | Plate Shear Geometry |             | TAN(Phi)    | Test Material and Geometric Parameters |                     |                     |
|-------------|-------------|----------------------|-------------|-------------|--|---------------------|---------------------|
|             |             | Phi (rads)           | Phi (deg)   |             | Parameter                              | Tests 3 & 4 Average | Tests 5,6,7 Average |
| 0.00254     | 0.0001      | 2.47508E-05          | 0.001418118 | 2.47508E-05 | b (in) =                               | 4                   | 4                   |
| 0.0254      | 0.001       | 0.000247508          | 0.014181182 | 0.000247508 | t (inches) =                           | 0.02835             | 0.0262              |
| 0.0508      | 0.002       | 0.000495017          | 0.028362362 | 0.000495017 | b/t =                                  | 141.0934744         | 152.6717557         |
| 0.0635      | 0.0025      | 0.000618771          | 0.035452951 | 0.000618771 | Poisson's Ratio =                      | 0.3                 | 0.3                 |
| 0.0889      | 0.0035      | 0.000866279          | 0.049634125 | 0.000866279 | Young's Modulus (psi) =                | 29000000            | 29000000            |
| 0.1         | 0.000937008 | 0.000974442          | 0.055831409 | 0.000974442 | Plastic Flow Stress (psi) =            | 43380               | 47810               |
| 0.127       | 0.005       | 0.001237541          | 0.070905875 | 0.001237542 | Phi crit =                             | 0.002120035         | 0.00181067          |
| 0.1524      | 0.006       | 0.001485049          | 0.085087031 | 0.00148505  | S crit (in) =                          | 0.008565521         | 0.007315602         |
| 0.1778      | 0.007       | 0.001732557          | 0.099268176 | 0.001732558 | NDSF crit =                            | 38.45519038         | 32.24586251         |
| 0.181166406 | 0.007132536 | 0.00176536           | 0.101147683 | 0.001765362 | Plastic Hinge Mom. (Mo) =              | 0.008716371         | 0.008204674         |
| 0.185816285 | 0.007315602 | 0.00181067           | 0.103743768 | 0.001810672 | Plate Length =                         | 18                  | 15.5                |
| 0.217564229 | 0.008563521 | 0.002120035          | 0.12146903  | 0.002120038 |  |                     |                     |
| 0.4         | 0.015748031 | 0.003897749          | 0.223324574 | 0.003897769 |  |                     |                     |
| 0.5         | 0.019685039 | 0.004872173          | 0.279154922 | 0.004872211 |  |                     |                     |
| 0.6         | 0.023620407 | 0.005846587          | 0.33498474  | 0.005846653 |  |                     |                     |
| 0.8         | 0.031496063 | 0.00779538           | 0.446642362 | 0.007795538 |  |                     |                     |
| 1           | 0.039370079 | 0.009744114          | 0.558296592 | 0.009744422 |  |                     |                     |
| 1.2         | 0.047244094 | 0.011692774          | 0.669946381 | 0.011693307 |  |                     |                     |
| 1.4         | 0.055118111 | 0.013641345          | 0.781591482 | 0.013642191 |  |                     |                     |
| 1.6         | 0.062992126 | 0.015589812          | 0.893230448 | 0.015591075 |  |                     |                     |
| 1.8         | 0.070866142 | 0.017538161          | 1.004862632 | 0.01753996  |  |                     |                     |
| 2           | 0.078740157 | 0.019486377          | 1.116487186 | 0.019488844 |  |                     |                     |
| 2.2         | 0.086614173 | 0.021434446          | 1.228103264 | 0.021437729 |  |                     |                     |
| 2.4         | 0.094488189 | 0.023382351          | 1.339710021 | 0.023386613 |  |                     |                     |
| 2.6         | 0.102362205 | 0.025330079          | 1.45130661  | 0.025335498 |  |                     |                     |
| 3           | 0.118110236 | 0.029224943          | 1.674465905 | 0.029233266 |  |                     |                     |
| 3.4         | 0.133858268 | 0.033118921          | 1.897574394 | 0.033131035 |  |                     |                     |
| 4.2         | 0.165354331 | 0.040903745          | 2.34361198  | 0.040926573 |  |                     |                     |
| 4.8         | 0.188976378 | 0.046739162          | 2.67795671  | 0.046773226 |  |                     |                     |
| 5           | 0.196850394 | 0.048683613          | 2.789365533 | 0.048722111 |  |                     |                     |
| 5.5         | 0.216535433 | 0.053543096          | 3.067793428 | 0.053594322 |  |                     |                     |
| 6.5         | 0.255905512 | 0.063254246          | 3.624201358 | 0.063338744 |  |                     |                     |
| 7           | 0.275590551 | 0.06810546           | 3.902135428 | 0.068210955 |  |                     |                     |
| 8           | 0.31496063  | 0.077798038          | 4.457499215 | 0.077955377 |  |                     |                     |
| 9           | 0.354330709 | 0.087475991          | 5.012005072 | 0.087699799 |  |                     |                     |
| 10          | 0.393700787 | 0.097137543          | 5.565571274 | 0.097444221 |  |                     |                     |
| 11          | 0.433070866 | 0.106780939          | 6.118097145 | 0.107188643 |  |                     |                     |
| 12          | 0.472440495 | 0.11640444           | 6.669483156 | 0.116933066 |  |                     |                     |
| 13          | 0.511811024 | 0.126006332          | 7.219631036 | 0.126677488 |  |                     |                     |
| 14          | 0.551181102 | 0.135584923          | 7.768443868 | 0.13642191  |  |                     |                     |
| 15          | 0.590551181 | 0.145138547          | 8.315826182 | 0.146166332 |  |                     |                     |
| 16          | 0.62992126  | 0.1544665564         | 8.86168405  | 0.155910754 |  |                     |                     |
| 17          | 0.669291339 | 0.164164363          | 9.405925174 | 0.165655176 |  |                     |                     |
| 18          | 0.708661417 | 0.173633364          | 9.948458968 | 0.175399598 |  |                     |                     |
| 19          | 0.748031496 | 0.183071017          | 10.48919664 | 0.185144021 |  |                     |                     |
| 20          | 0.787401575 | 0.192475804          | 11.02805126 | 0.194888443 |  |                     |                     |
| 22          | 0.866141732 | 0.211180884          | 12.09977337 | 0.214377287 |  |                     |                     |
| 24          | 0.94488189  | 0.229737161          | 13.16296975 | 0.233866131 |  |                     |                     |
| 26          | 1.023622047 | 0.248133786          | 14.21701869 | 0.253354976 |  |                     |                     |
| 28          | 1.102362205 | 0.266360524          | 15.26133387 | 0.27284382  |  |                     |                     |
| 30          | 1.181102362 | 0.284407782          | 16.2953656  | 0.292332664 |  |                     |                     |

| Plate Shear Geometry |             |             |             |             |
|----------------------|-------------|-------------|-------------|-------------|
| S (mm)               | S (inches)  | Phi (rads)  | Phi (deg)   | TAN(Phi)    |
| 32                   | 1.25984252  | 0.302266621 | 17.31860169 | 0.311821508 |
| 34                   | 1.338582677 | 0.319928768 | 18.33056816 | 0.331310353 |
| 36                   | 1.417322835 | 0.337386621 | 19.33082946 | 0.390799197 |
| 38                   | 1.496062992 | 0.354633252 | 20.31898861 | 0.370288041 |
| 40                   | 1.57480315  | 0.3716624   | 21.29468693 | 0.389776885 |
| 42                   | 1.653543307 | 0.388468467 | 22.25760362 | 0.40926573  |
| 44                   | 1.732283465 | 0.405046502 | 23.20745509 | 0.428754574 |
| 46                   | 1.811023622 | 0.421392191 | 24.14399409 | 0.448243418 |
| 48                   | 1.88976378  | 0.437501835 | 25.0670087  | 0.467732262 |
| 50                   | 1.968503937 | 0.453372332 | 25.97632114 | 0.487221107 |
| 52                   | 2.047244094 | 0.46900115  | 26.87178647 | 0.506709951 |
| 54                   | 2.125984252 | 0.484386309 | 27.79329115 | 0.526198795 |
| 56                   | 2.204724409 | 0.49952635  | 28.6207516  | 0.54568764  |
| 58                   | 2.283464567 | 0.51442031  | 29.47411263 | 0.565176484 |
| 60                   | 2.362204724 | 0.529067692 | 30.31334584 | 0.584665328 |

| PM (deg)    | lambda > Max. Out-of-Plane Displacement [wo] divided by pit width [b] |              |             |             |             |              |             |             |
|-------------|---|--------------|-------------|-------------|-------------|--------------|-------------|-------------|
|             | 0.1   | 0.2          | 0.3         | 0.4         | 0.5         | 0.6          | 0.8         | 1           |
| 0.001418118 | 0.001117963   | 0.001581087  | 0.001936448 | 0.002236028 | 0.002499961 | 0.002738574  | 0.003162237 | 0.00353549  |
| 0.014181182 | 0.003533302   | 0.00498376   | 0.006122336 | 0.00706979  | 0.007904464 | 0.008659034  | 0.009998729 | 0.011178956 |
| 0.028362362 | 0.004993685   | 0.007066474  | 0.008656327 | 0.009996385 | 0.011176861 | 0.012243998  | 0.014138541 | 0.015807476 |
| 0.033452951 | 0.005581344   | 0.007899275  | 0.009676971 | 0.011175289 | 0.012495138 | 0.013688242  | 0.015806365 | 0.017672203 |
| 0.049634125 | 0.006599754   | 0.009343509  | 0.011447349 | 0.01322039  | 0.014782147 | 0.016193863  | 0.018699968 | 0.020907446 |
| 0.053831409 | 0.00699772  | 0.00990826   | 0.012139785 | 0.014020357 | 0.015676788 | 0.017174057  | 0.01983197  | 0.022173112 |
| 0.070905875 | 0.007880713   | 0.011162179  | 0.013677544 | 0.015797104 | 0.017663922 | 0.019351282  | 0.022346478 | 0.024984544 |
| 0.083087031 | 0.008627407   | 0.012223575  | 0.014979599 | 0.017301732 | 0.019346847 | 0.021195283  | 0.024476233 | 0.027365815 |
| 0.099268176 | 0.00931274  | 0.01319867   | 0.016176144 | 0.018684629 | 0.020893733 | 0.022890299  | 0.026433997 | 0.029554805 |
| 0.101147683 | 0.009399696   | 0.013322458  | 0.016328071 | 0.018860232 | 0.021090168 | 0.02310555   | 0.026682621 | 0.029832796 |
| 0.103743768 | 0.009518451   | 0.013491539  | 0.016353597 | 0.019100103 | 0.021358499 | 0.023999586  | 0.027022246 | 0.030212537 |
| 0.12146909  | 0.010291349   | 0.014592726  | 0.017887453 | 0.020662812 | 0.023106713 | 0.025315332  | 0.029235089 | 0.032686769 |
| 0.223324574 | 0.013890296   | 0.019740308  | 0.024214526 | 0.0279809   | 0.031296043 | 0.034291076  | 0.03960456  | 0.044281606 |
| 0.279154922 | 0.015490469   | 0.022041902  | 0.02704843  | 0.031261333 | 0.034968638 | 0.038317381  | 0.044257177 | 0.04948433  |
| 0.33498474  | 0.016925754   | 0.024114528  | 0.029603528 | 0.0342207   | 0.038282802 | 0.041951383  | 0.048457167 | 0.054181103 |
| 0.446642362 | 0.019444069   | 0.027773022  | 0.034121926 | 0.039458422 | 0.044151154 | 0.048387787  | 0.055897884 | 0.062502428 |
| 0.558296692 | 0.021626702   | 0.030970547  | 0.038008974 | 0.044050504 | 0.049302209 | 0.054039517  | 0.062433709 | 0.069412374 |
| 0.669946581 | 0.023567144   | 0.033830095  | 0.041640399 | 0.048188908 | 0.05394195  | 0.05912104   | 0.068324946 | 0.076401938 |
| 0.781591482 | 0.02532117  | 0.036453675  | 0.044895976 | 0.051975921 | 0.058192899 | 0.063799687  | 0.073726397 | 0.082444169 |
| 0.893230448 | 0.026925268   | 0.038868238  | 0.047909231 | 0.055485435 | 0.062134991 | 0.068129768  | 0.078739035 | 0.08805194  |
| 1.004862632 | 0.028404879   | 0.04111726   | 0.050723591 | 0.058767322 | 0.06582382  | 0.072183199  | 0.083433071 | 0.093303737 |
| 1.116487186 | 0.029778588   | 0.043226629  | 0.053370587 | 0.061857891 | 0.069299918 | 0.076004338  | 0.087859684 | 0.098256767 |
| 1.228103264 | 0.031060472   | 0.045215936  | 0.058574663 | 0.06478459  | 0.072593921 | 0.079626715  | 0.092057552 | 0.102954264 |
| 1.339710021 | 0.032261501   | 0.047100433  | 0.058252575 | 0.067568776 | 0.075729663 | 0.083076407  | 0.096056758 | 0.107429854 |
| 1.45130661  | 0.033390418   | 0.048892277  | 0.060520905 | 0.070227458 | 0.078726122 | 0.086374181  | 0.099881254 | 0.111710307 |
| 1.674465905 | 0.035459099   | 0.052235792  | 0.064772952 | 0.075221139 | 0.084360128 | 0.092578452  | 0.107080476 | 0.119768923 |
| 1.897574394 | 0.037310109   | 0.053307113  | 0.068703864 | 0.079893034 | 0.089390425 | 0.098342887  | 0.113774412 | 0.127263337 |
| 2.34361198  | 0.04047553  | 0.060795538  | 0.075799415 | 0.088241885 | 0.099092554 | 0.108824859  | 0.125963063 | 0.140915687 |
| 2.67795671  | 0.042457996   | 0.064448786  | 0.080583323 | 0.093929544 | 0.10550498  | 0.115966019  | 0.134274798 | 0.150225066 |
| 2.789365533 | 0.043053794   | 0.0655911893 | 0.082091988 | 0.095728905 | 0.107596853 | 0.118229748  | 0.13691255  | 0.153180751 |
| 3.067933428 | 0.044413552   | 0.068804381  | 0.085697442 | 0.100041256 | 0.112508189 | 0.123667176  | 0.143253065 | 0.160286818 |
| 3.624201358 | 0.046629102   | 0.073183581  | 0.092289442 | 0.107975861 | 0.121573735 | 0.133721728  | 0.154996705 | 0.1734538   |
| 3.902155428 | 0.047508268   | 0.075386007  | 0.095319468 | 0.111648054 | 0.12578359  | 0.13839968   | 0.160469935 | 0.179993079 |
| 4.457499215 | 0.048849955   | 0.079383081  | 0.100928904 | 0.118495691 | 0.133661612 | 0.147170796  | 0.170750452 | 0.191129929 |
| 5.012005072 | 0.049670572   | 0.082901602  | 0.106017613 | 0.124773142 | 0.14092013  | 0.159274509  | 0.180272527 | 0.20182262  |
| 5.565571274 | 0.049993741   | 0.086000066  | 0.110659137 | 0.130565174 | 0.147653675 | 0.162814231  | 0.189155427 | 0.211804556 |
| 6.118097145 | 0.049826849   | 0.088725006  | 0.114909058 | 0.135936049 | 0.153934012 | 0.16986849   | 0.197489773 | 0.221177143 |
| 6.669483156 | 0.049161811   | 0.091107593  | 0.118810507 | 0.140935847 | 0.159817154 | 0.176498628  | 0.205346419 | 0.230019718 |
| 7.219631036 | 0.047974224   | 0.093174429  | 0.122397656 | 0.145604472 | 0.165347823 | 0.182753678  | 0.212782066 | 0.238395815 |
| 7.708443868 | 0.0462194   | 0.094945456  | 0.125698018 | 0.149974294 | 0.170562398 | 0.18867358   | 0.21984297  | 0.246357311 |
| 8.315826182 | 0.043823956   | 0.096436018  | 0.128734024 | 0.154071962 | 0.175409931 | 0.194291382  | 0.226367476 | 0.253947257 |
| 8.86168403  | 0.0406568473  | 0.09765773   | 0.131524138 | 0.157919678 | 0.180158568 | 0.199634801  | 0.23298781  | 0.26120188  |
| 9.405925174 | 0.0363548805  | 0.098619091  | 0.134083657 | 0.161536128 | 0.184586582 | 0.20472734   | 0.239131373 | 0.26815203  |
| 9.948458968 | 0.031073775   | 0.0993259    | 0.136425306 | 0.164937146 | 0.188793136 | 0.209589129  | 0.245021705 | 0.274824252 |
| 10.48919664 | 0.023292329   | 0.099781335  | 0.138559681 | 0.168136326 | 0.192793858 | 0.214237549  | 0.250679207 | 0.281241597 |
| 11.02805126 | 0.008119956   | 0.099867128  | 0.140495588 | 0.171145223 | 0.196602281 | 0.21868772   | 0.256121691 | 0.287424237 |
| 12.09977337 |   | 0.099641876  | 0.143799783 | 0.176630868 | 0.203687902 | 0.227044645  | 0.266422458 | 0.299154524 |
| 13.16296975 |   | 0.098251183  | 0.146381085 | 0.181458801 | 0.210127828 | 0.234748045  | 0.276029211 | 0.310135034 |
| 14.21701869 |   | 0.095752228  | 0.148265701 | 0.185677148 | 0.215982279 | 0.2418672    | 0.285025801 | 0.320462005 |
| 15.26133387 |   | 0.092006388  | 0.149465558 | 0.189321202 | 0.221298126 | 0.248457209  | 0.293480013 | 0.330214012 |
| 16.2953656  |   | 0.086814196  | 0.149979809 | 0.192415929 | 0.226111796 | 0.254562187  | 0.301447767 | 0.339458092 |
| 17.31860169 |   | 0.07**342082 | 0.149795208 | 0.194977658 | 0.230451311 | 0.260217531  | 0.308974734 | 0.348242669 |
| 18.33056816 |   | 0.070499177  | 0.148885797 | 0.197015214 | 0.234337738 | 0.265451547  | 0.31099229  | 0.356619671 |
| 19.33082946 |   | 0.057356475  | 0.147211516 | 0.198530635 | 0.237786238 | 0.270286656  | 0.322853122 | 0.364626108 |
| 20.31898861 |   | 0.037304583  | 0.144715685 | 0.199519574 | 0.240006822 | 0.274740286  | 0.329263001 | 0.372295257 |
| 21.29468693 |   | 0.141320082  | 0.19997143  | 0.243404902 | 0.278825555 | 0.33353099   | 0.379655585 |             |
| 22.25760362 |   | 0.136921726  | 0.199869212 | 0.245981661 | 0.282551784 | 0.3411135385 | 0.38673146  |             |
| 23.20745509 |   | 0.131374371  | 0.199189147 | 0.247334298 | 0.285924888 | 0.346631154  | 0.393543724 |             |
| 24.14399409 |   | 0.124476144  | 0.197899982 | 0.248656156 | 0.288947668 | 0.351850342  | 0.400110146 |             |
| 25.0670067  |   | 0.115929611  | 0.199561893 | 0.249536741 | 0.291620025 | 0.356802389  | 0.406445792 |             |
| 25.97632114 |   | 0.105267729  | 0.193324905 | 0.249961638 | 0.29393911  | 0.36149439   | 0.412563327 |             |
| 26.87178647 |   | 0.091674776  | 0.189926578 | 0.249912326 | 0.295899411 | 0.365931302  | 0.418473267 |             |
| 27.75329115 |   | 0.073435649  | 0.185688634 | 0.249365859 | 0.297492793 | 0.370116122  | 0.424184185 |             |
| 28.6207516  |   | 0.045057878  | 0.180512057 | 0.248294418 | 0.298708484 | 0.374050015  | 0.429702885 |             |
| 29.47411263 |   |              |             | 0.174269252 | 0.24666467  | 0.29953301   | 0.377732428 | 0.435034551 |
| 30.31334584 |   |              |             | 0.166792033 | 0.244436911 | 0.29995008   | 0.381161174 | 0.440182871 |

| Phi (deg)   | Lambda > Ratio of absolute vertical apex velocity [w0 dot] to shear velocity [Phi dot * b] |             |             |             |             |             |             |             |
|-------------|--|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
|             | 0.1  | 0.2         | 0.3         | 0.4         | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118 | 22.58150926  | 31.93801266 | 39.11706786 | 43.16912213 | 50.50099445 | 55.32131177 | 63.87981277 | 71.41987414 |
| 0.014181182 | 7.128724006  | 10.09082932 | 12.36233164 | 14.27676111 | 15.96310823 | 17.48748103 | 20.19363965 | 22.57740408 |
| 0.028362362 | 5.031195013  | 7.128334391 | 8.73554135  | 10.08972139 | 11.28235474 | 12.36029257 | 14.27361888 | 15.95871439 |
| 0.034532951 | 4.495753299  | 6.372663502 | 7.81064655  | 9.02207437  | 10.0888904  | 11.0504695  | 12.76428137 | 14.27126196 |
| 0.049634123 | 3.792358673  | 5.380621997 | 6.596707202 | 7.620910219 | 8.522690842 | 9.33758541  | 10.78367959 | 12.05694818 |
| 0.055831409 | 3.57270592   | 5.071047925 | 6.217972534 | 7.1838092   | 8.034133931 | 8.802465874 | 10.16589484 | 11.3662698  |
| 0.070905875 | 3.163811294  | 4.495147113 | 5.5135631   | 6.370925572 | 7.125606746 | 7.807439282 | 9.017125137 | 10.08196396 |
| 0.085067031 | 2.882605257  | 4.099469372 | 5.02974128  | 5.812678947 | 6.501726143 | 7.124177743 | 8.222341693 | 9.200126208 |
| 0.099268176 | 2.663634867  | 3.791649167 | 4.653460617 | 5.378576568 | 6.016622479 | 6.59925537  | 7.615069227 | 8.514511789 |
| 0.101147683 | 2.638096129  | 3.73576734  | 4.609606133 | 5.327987225 | 5.960092063 | 6.531018914 | 7.54360635  | 8.434619812 |
| 0.103743768 | 2.603958236  | 3.707810775 | 4.530996737 | 5.260378494 | 5.88454445  | 6.448287082 | 7.448104659 | 8.327853272 |
| 0.12146903  | 2.400665583  | 3.422411808 | 4.202272334 | 4.858146935 | 5.435104874 | 5.956123043 | 6.879989072 | 7.692731725 |
| 0.223324574 | 1.745769819  | 2.506203873 | 3.084005292 | 3.568960394 | 3.995017292 | 4.379897523 | 5.060221004 | 5.658409154 |
| 0.279154922 | 1.549281905  | 2.232867008 | 2.750984312 | 3.185360427 | 3.565711944 | 3.910578979 | 4.519272862 | 5.05371843  |
| 0.33498474  | 1.403134994  | 2.030327753 | 2.504513635 | 2.901613556 | 3.249992632 | 3.563962683 | 4.119393531 | 4.606738671 |
| 0.446642362 | 1.195718928  | 1.744462155 | 2.11244885  | 2.502146053 | 2.804301283 | 3.076325072 | 3.536958812 | 3.978092799 |
| 0.558296592 | 1.051977107  | 1.547889118 | 1.919027172 | 2.22843     | 2.499909876 | 2.742519289 | 3.172080934 | 3.547941676 |
| 0.669946581 | 0.944219067  | 1.401684192 | 1.742281041 | 2.025575917 | 2.273051054 | 2.495373492 | 2.887218089 | 3.229988007 |
| 0.781591482 | 0.859157006  | 1.287195379 | 1.604217715 | 1.867299563 | 2.0967865   | 2.302728092 | 2.663248305 | 2.981558145 |
| 0.893230448 | 0.789507599  | 1.194213531 | 1.492369476 | 1.792239597 | 1.95424421  | 2.14699572  | 2.485871822 | 2.78113209  |
| 1.004862632 | 0.730897076  | 1.116616499 | 1.399263141 | 1.632733342 | 1.835799765 | 2.01763868  | 2.336926518 | 2.614722593 |
| 1.116487186 | 0.680517252  | 1.050479128 | 1.320109506 | 1.542307463 | 1.735287367 | 1.907906557 | 2.210621894 | 2.473620774 |
| 1.228103264 | 0.636470354  | 0.993152625 | 1.251678141 | 1.464223567 | 1.648549404 | 1.81324784  | 2.101705498 | 2.351959093 |
| 1.339710021 | 0.597420765  | 0.94277557  | 1.19169991  | 1.39586725  | 1.572666462 | 1.7304667   | 2.006489573 | 2.245603198 |
| 1.451306611 | 0.562397002  | 0.897996125 | 1.138527585 | 1.333541043 | 1.50519794  | 1.657244111 | 1.922298307 | 2.151573989 |
| 1.674465905 | 0.501692784  | 0.821432695 | 1.04797616  | 1.232453293 | 1.39149006  | 1.532967173 | 1.779481743 | 1.992091126 |
| 1.897574394 | 0.450320561  | 0.757866223 | 0.973212211 | 1.147717706 | 1.297705877 | 1.430836135 | 1.662202537 | 1.861151325 |
| 2.34361198  | 0.366329679  | 0.636972528 | 0.85554265  | 1.01486149  | 1.150964114 | 1.271225446 | 1.479125763 | 1.656811673 |
| 2.67795671  | 0.314369227  | 0.596891544 | 0.786217337 | 0.936965033 | 1.085149044 | 1.178025686 | 1.372377337 | 1.537712216 |
| 2.789365533 | 0.298539921  | 0.579030876 | 0.765740215 | 0.914022286 | 1.039913126 | 1.150642892 | 1.341040902 | 1.502798558 |
| 3.067793428 | 0.261639341  | 0.538174095 | 0.719160591 | 0.861964951 | 0.982729789 | 1.08864376  | 1.270143687 | 1.42369467  |
| 3.624201358 | 0.196209459  | 0.469097544 | 0.641389095 | 0.775531492 | 0.88806974  | 0.986191756 | 1.153185763 | 1.293329205 |
| 3.902155428 | 0.166508887  | 0.43933697  | 0.608338788 | 0.739023691 | 0.848218304 | 0.943124992 | 1.104133635 | 1.238684987 |
| 4.457499215 | 0.111055405  | 0.386796891 | 0.55081407  | 0.675879825 | 0.779524351 | 0.868908534 | 1.019912591 | 1.144920152 |
| 5.012005072 | 0.058882104  | 0.341307315 | 0.502022901 | 0.622808275 | 0.722071066 | 0.807325566 | 0.949878122 | 1.067022577 |
| 5.565371274 | 0.008104835  | 0.30098563  | 0.459727127 | 0.577253045 | 0.673017003 | 0.754761268 | 0.890459343 | 1.001003819 |
| 6.118097145 | 0.042905364  | 0.264538644 | 0.422404681 | 0.537480342 | 0.630436911 | 0.709291709 | 0.839239741 | 0.944163447 |
| 6.669483156 | 0.095781101  | 0.231036469 | 0.388976254 | 0.502264663 | 0.592971802 | 0.669435097 | 0.794316015 | 0.894608007 |
| 7.219631036 | 0.152415983  | 0.199783263 | 0.358649597 | 0.470710109 | 0.559629821 | 0.634110885 | 0.755047138 | 0.850940597 |
| 7.768443768 | 0.215323338  | 0.170239035 | 0.330825876 | 0.442143015 | 0.529666339 | 0.60250902  | 0.719903681 | 0.81213249  |
| 8.315826182 | 0.288279183  | 0.141969976 | 0.305040778 | 0.41604448  | 0.502510331 | 0.57400784  | 0.688373216 | 0.777387429 |
| 8.86168403  | 0.377709888  | 0.114615464 | 0.280926032 | 0.392006366 | 0.477712392 | 0.548120559 | 0.659898667 | 0.746084747 |
| 9.408925174 | 0.496290456  | 0.087865155 | 0.258183451 | 0.369701666 | 0.454914783 | 0.52445926  | 0.634036847 | 0.717731037 |
| 9.948458968 | 0.674912241  | 0.061442292 | 0.236566919 | 0.348863995 | 0.433827129 | 0.502710002 | 0.610429788 | 0.691928125 |
| 10.48919664 | 1.024604691  | 0.035090856 | 0.215869553 | 0.329273051 | 0.414210569 | 0.482615182 | 0.588784445 | 0.668350396 |
| 11.02805126 | 3.30343861   | 0.008564998 | 0.195914363 | 0.31074408  | 0.395866073 | 0.463960793 | 0.56885027  | 0.646728408 |
| 12.09977337 | 0.045998833  | 0.157631859 | 0.276266073 | 0.362346791 | 0.430281288 | 0.533379941 | 0.60848536  |             |
| 13.16296975 | 0.10437125   | 0.120674432 | 0.244411621 | 0.3321956   | 0.400527955 | 0.502709578 | 0.575777457 |             |
| 14.21701869 | 0.169168141  | 0.084165452 | 0.214392771 | 0.304602487 | 0.373847933 | 0.475896978 | 0.547599238 |             |
| 15.26133387 | 0.244405238  | 0.047315238 | 0.185571393 | 0.278933329 | 0.349583213 | 0.452218831 | 0.523039255 |             |
| 16.2953656  | 0.334813002  | 0.009360173 | 0.157407035 | 0.254674153 | 0.327210674 | 0.431109544 | 0.501603181 |             |
| 17.31860169 | 0.451886922  | 0.030490451 | 0.129421197 | 0.23139423  | 0.306302429 | 0.412116113 | 0.48276352  |             |
| 18.33056816 | 0.617604129  | 0.073109487 | 0.101171215 | 0.20872051  | 0.286498902 | 0.394867596 | 0.466125619 |             |
| 19.33082946 | 0.894751001  | 0.119518737 | 0.072229669 | 0.186319849 | 0.267489886 | 0.379053853 | 0.451364063 |             |
| 20.31898861 | 1.608531797  | 0.170980926 | 0.042166628 | 0.16388431  | 0.249000815 | 0.364410376 | 0.438205862 |             |
| 21.29468693 | 0.229139122  | 0.010532714 | 0.141121387 | 0.230782514 | 0.35070718  | 0.426418212 |             |             |
| 22.25760362 | 0.296245349  | 0.023158745 | 0.117744536 | 0.212603267 | 0.337740522 | 0.415799406 |             |             |
| 23.20745509 | 0.37556457   | 0.059452313 | 0.093465334 | 0.194242484 | 0.325326562 | 0.406171966 |             |             |
| 24.14399409 | 0.47214984   | 0.09897609  | 0.067985975 | 0.175485417 | 0.313296444 | 0.397377354 |             |             |
| 25.0670087  | 0.594488201  | 0.142475622 | 0.040991569 | 0.156118563 | 0.301492369 | 0.389271843 |             |             |
| 25.97632114 | 0.758501942  | 0.190860988 | 0.012141918 | 0.135925466 | 0.289764407 | 0.381723236 |             |             |
| 26.87178647 | 0.999399701  | 0.245275127 | 0.018937996 | 0.114682679 | 0.277967843 | 0.374606224 |             |             |
| 27.75329115 | 1.420448906  | 0.307197152 | 0.052669394 | 0.092153682 | 0.263960906 | 0.367810221 |             |             |
| 28.6207516  | 2.618762183  | 0.378605576 | 0.069533926 | 0.068094565 | 0.253602782 | 0.361217564 |             |             |
| 29.47411263 |  | 0.462249123 | 0.13009203  | 0.042229253 | 0.24075181  | 0.354721992 |             |             |
| 30.31334584 |  | 0.562122791 | 0.175007325 | 0.01426403  | 0.227263801 | 0.348217335 |             |             |

| Phi (deg)    | lambda>      |             |              |              |             |             |             |             |
|--------------|--------------|-------------|--------------|--------------|-------------|-------------|-------------|-------------|
|              | 0.1          | 0.2         | 0.3          | 0.4          | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118  | 2.51222E-05  | 2.54853B-05 | 2.60792E-05  | 2.68887B-05  | 2.78951E-05 | 2.90778E-05 | 3.18911E-05 | 3.51792E-05 |
| 0.014181182  | 0.000251216  | 0.000254842 | 0.000260776  | 0.000268867  | 0.000278926 | 0.00029075  | 0.000318876 | 0.000351752 |
| 0.026362362  | 0.00050242   | 0.000509659 | 0.000521517  | 0.000537688  | 0.000557797 | 0.000581436 | 0.000637676 | 0.000703418 |
| 0.035452951  | 0.000628018  | 0.000637059 | 0.000651875  | 0.000672062  | 0.000697212 | 0.000726756 | 0.000797047 | 0.000879218 |
| 0.049634125  | 0.000879203  | 0.00091841  | 0.000912563  | 0.000940835  | 0.000976001 | 0.001017347 | 0.001115731 | 0.001230753 |
| 0.055831409  | 0.00098987   | 0.001003175 | 0.001026475  | 0.00105266   | 0.001097817 | 0.001144319 | 0.001254975 | 0.001384349 |
| 0.070905875  | 0.00125596   | 0.001273969 | 0.00130353   | 0.00134388   | 0.001394082 | 0.001453117 | 0.001593614 | 0.001757892 |
| 0.085087031  | 0.001507115  | 0.001528691 | 0.00156413   | 0.001612519  | 0.001672734 | 0.001743551 | 0.001912106 | 0.002109209 |
| 0.099268176  | 0.001758259  | 0.00178339  | 0.001824696  | 0.001881114  | 0.001951332 | 0.002039222 | 0.002230522 | 0.002460439 |
| 0.101147683  | 0.001791544  | 0.001817144 | 0.001859228  | 0.001916708  | 0.001988252 | 0.002072402 | 0.002272718 | 0.002506983 |
| 0.103743768  | 0.001837518  | 0.001863768 | 0.001906924  | 0.001965873  | 0.002039246 | 0.00212555  | 0.002330999 | 0.00257127  |
| 0.121469093  | 0.002151407  | 0.002182078 | 0.002232547  | 0.002301512  | 0.002387372 | 0.002488377 | 0.002728855 | 0.003010123 |
| 0.223324574  | 0.003954773  | 0.004010492 | 0.004102647  | 0.004228857  | 0.004386183 | 0.004571414 | 0.005012767 | 0.005529308 |
| 0.279154922  | 0.004943015  | 0.005012203 | 0.005126962  | 0.005244323  | 0.005400617 | 0.005711832 | 0.006262991 | 0.006908268 |
| 0.33498474   | 0.005931081  | 0.006013555 | 0.006150743  | 0.006339093  | 0.006574209 | 0.006851279 | 0.007512031 | 0.008285884 |
| 0.446642632  | 0.007906699  | 0.008015192 | 0.008196714  | 0.008446599  | 0.00875888  | 0.009127268 | 0.010006365 | 0.011037067 |
| 0.558296592  | 0.00988163   | 0.010015424 | 0.01024058   | 0.010551271  | 0.010940212 | 0.011399397 | 0.012496378 | 0.013782924 |
| 0.669946581  | 0.011855956  | 0.012014271 | 0.012282361  | 0.01265325   | 0.013118222 | 0.013667678 | 0.014981482 | 0.016523403 |
| 0.781591482  | 0.013829639  | 0.014011756 | 0.014322076  | 0.014752512  | 0.015292925 | 0.015932128 | 0.017461888 | 0.019258533 |
| 0.893230448  | 0.0158002721 | 0.016007899 | 0.016359746  | 0.016849077  | 0.017464339 | 0.01819276  | 0.019937606 | 0.021988322 |
| 1.004862632  | 0.017775224  | 0.018002722 | 0.018395391  | 0.018942962  | 0.019632479 | 0.020449588 | 0.022408648 | 0.024712778 |
| 1.116487186  | 0.019747171  | 0.019996247 | 0.020429031  | 0.021034187  | 0.021797363 | 0.022702629 | 0.024875026 | 0.027431909 |
| 1.228103264  | 0.021718584  | 0.021988494 | 0.022466086  | 0.023122769  | 0.023959007 | 0.024951896 | 0.027336749 | 0.030145725 |
| 1.339710021  | 0.023689485  | 0.023979487 | 0.024490376  | 0.025208727  | 0.026117428 | 0.027197404 | 0.029793831 | 0.032854232 |
| 1.45130661   | 0.025659897  | 0.025969245 | 0.026518122  | 0.02729208   | 0.028272642 | 0.029439168 | 0.032246281 | 0.035557441 |
| 1.674465903  | 0.02959934   | 0.029945145 | 0.030567861  | 0.031451045  | 0.032573517 | 0.033911525 | 0.037137333 | 0.040947993 |
| 1.897574394  | 0.033537092  | 0.033916367 | 0.034610066  | 0.035599812  | 0.036861768 | 0.038369084 | 0.042009996 | 0.04631745  |
| 2.34361198   | 0.04140624   | 0.041845472 | 0.042672526  | 0.043867358  | 0.04540094  | 0.047240294 | 0.051700526 | 0.056993352 |
| 2.67795671   | 0.047308491  | 0.047781313 | 0.048700812  | 0.050042372  | 0.051773194 | 0.05385576  | 0.058920842 | 0.064945416 |
| 2.799363533  | 0.049274769  | 0.049757995 | 0.050706816  | 0.052095927  | 0.053891252 | 0.056053771 | 0.061318613 | 0.067985701 |
| 3.067793428  | 0.054189641  | 0.054695424 | 0.055714529  | 0.057219533  | 0.059173375 | 0.061333134 | 0.067293463 | 0.07416374  |
| 3.624201358  | 0.064017053  | 0.064533891 | 0.065697972  | 0.067423708  | 0.069682753 | 0.072426871 | 0.079159905 | 0.087223123 |
| 3.902155428  | 0.068930293  | 0.069475571 | 0.070677996  | 0.072504883  | 0.074910558 | 0.077841377 | 0.08308187  | 0.093704749 |
| 4.457499213  | 0.078757597  | 0.079305541 | 0.080067193  | 0.082626938  | 0.085313792 | 0.088607089 | 0.09675425  | 0.106572589 |
| 5.012005072  | 0.088588336  | 0.089119967 | 0.090502314  | 0.092697327  | 0.095649072 | 0.099290083 | 0.106349191 | 0.119314182 |
| 5.563571274  | 0.098425316  | 0.098921622 | 0.100366027  | 0.102718542  | 0.105918675 | 0.109892403 | 0.119838274 | 0.131930716 |
| 6.118097143  | 0.106271343  | 0.106713291 | 0.110201014  | 0.112693104  | 0.116124914 | 0.120416128 | 0.131223111 | 0.144423398 |
| 6.669483156  | 0.118129219  | 0.118497765 | 0.12000998   | 0.122623557  | 0.126270128 | 0.130863366 | 0.142505339 | 0.156793463 |
| 7.219631036  | 0.128001743  | 0.128277845 | 0.129795641  | 0.132512472  | 0.136356685 | 0.141236255 | 0.153686626 | 0.169042164 |
| 7.768443868  | 0.137891709  | 0.138056335 | 0.139560735  | 0.14236244   | 0.146386984 | 0.151536967 | 0.164768671 | 0.181170784 |
| 8.315826182  | 0.147801903  | 0.147836045 | 0.149308009  | 0.152176079  | 0.156363449 | 0.161767702 | 0.175753201 | 0.193180625 |
| 8.861684045  | 0.157735105  | 0.157619786 | 0.159040226  | 0.161956024  | 0.166288535 | 0.171930694 | 0.186641973 | 0.205073017 |
| 9.405925174  | 0.167694086  | 0.167410373 | 0.168760163  | 0.171704935  | 0.176164722 | 0.182028204 | 0.197436776 | 0.216849314 |
| 9.948458968  | 0.177681605  | 0.177210621 | 0.178470603  | 0.18142549   | 0.18594519  | 0.192062526 | 0.208139427 | 0.228510897 |
| 10.48919664  | 0.187700411  | 0.187023343 | 0.188174344  | 0.191120388  | 0.195780458 | 0.202033984 | 0.218751776 | 0.24005917  |
| 11.020805126 | 0.19775324   | 0.196981351 | 0.197874189  | 0.200792343  | 0.205525098 | 0.21195093  | 0.229275704 | 0.231495567 |
| 12.099773337 | 0.217971837  | 0.216564447 | 0.2217273445 | 0.220078379  | 0.22490084  | 0.231614843 | 0.250065968 | 0.274038593 |
| 13.162946975 | 0.238358979  | 0.236372302 | 0.236690927  | 0.239305645  | 0.244124697 | 0.251073671 | 0.270525884 | 0.296151969 |
| 14.21701869  | 0.25895397   | 0.256297187 | 0.256149243  | 0.258496414  | 0.263271974 | 0.270347265 | 0.29067162  | 0.317848135 |
| 15.26133387  | 0.279723801  | 0.27636121  | 0.275671012  | 0.277673137  | 0.282310307 | 0.289455909 | 0.310519847 | 0.339139983 |
| 16.2953656   | 0.300743115  | 0.296386273 | 0.29527882   | 0.296858407  | 0.301279627 | 0.308420299 | 0.330067744 | 0.36004087  |
| 17.31860169  | 0.32201418   | 0.316994038 | 0.314995176  | 0.31607492   | 0.320202128 | 0.32726152  | 0.349392992 | 0.380564626 |
| 18.33056816  | 0.343556859  | 0.337605882 | 0.334842469  | 0.335354543  | 0.339100231 | 0.346001022 | 0.368453771 | 0.400725565 |
| 19.33082946  | 0.363590589  | 0.358442871 | 0.354842927  | 0.354692706  | 0.357969655 | 0.364660589 | 0.387288757 | 0.420538492 |
| 20.31898861  | 0.387534355  | 0.379525719 | 0.375018573  | 0.3747439487 | 0.376913843 | 0.383262311 | 0.409917111 | 0.440018706 |
| 21.29468693  | 0.410006673  | 0.400874746 | 0.395391193  | 0.393708443  | 0.395874982 | 0.401828552 | 0.424358472 | 0.439182015 |
| 22.25760362  | 0.432825573  | 0.422509921 | 0.415982283  | 0.414322126  | 0.414902901 | 0.420381909 | 0.442632941 | 0.478044732 |
| 23.20745509  | 0.456008586  | 0.444430695 | 0.436813025  | 0.433302928  | 0.434020559 | 0.438945183 | 0.460761064 | 0.496623682 |
| 24.14399409  | 0.479572733  | 0.466716115 | 0.457904249  | 0.45337304   | 0.45325089  | 0.457541329 | 0.47876382  | 0.514936205 |
| 25.0670087   | 0.503534515  | 0.489324742 | 0.4792764    | 0.473654412  | 0.472616764 | 0.476193425 | 0.496662593 | 0.533000153 |
| 25.97632114  | 0.527909909  | 0.51229464  | 0.50094951   | 0.494168709  | 0.494140937 | 0.494924621 | 0.514479154 | 0.550833888 |
| 26.87178647  | 0.552714364  | 0.535643363 | 0.522943167  | 0.514937277  | 0.511846013 | 0.513758098 | 0.53223563  | 0.568456283 |
| 27.75329115  | 0.577962799  | 0.559387948 | 0.545276498  | 0.535981106  | 0.531754394 | 0.532717025 | 0.549954476 | 0.585886709 |
| 28.6207516   | 0.603669606  | 0.583544902 | 0.567968143  | 0.557320797  | 0.551888241 | 0.55182451  | 0.567658444 | 0.603145031 |
| 29.47411263  | 0.629848649  | 0.608130194 | 0.591036238  | 0.578976532  | 0.572269434 | 0.571103558 | 0.585370548 | 0.620251594 |
| 30.31334584  | 0.656513275  | 0.633159257 | 0.614498398  | 0.600968043  | 0.592919528 | 0.59057702  | 0.603114025 | 0.637227214 |

| Phi (deg)    | Absolute Aperture Shear Strain (Gamma) |             |              |             |             |              |              |              |
|--------------|--|-------------|--------------|-------------|-------------|--------------|--------------|--------------|
|              | 0.1                                    | 0.2         | 0.3          | 0.4         | 0.5         | 0.6          | 0.8          | 1            |
| 0.001418118  | 0.099016914                            | 0.096141806 | 0.091638969  | 0.08591668  | 0.079436827 | 0.072632694  | 0.059360444  | 0.04776104   |
| 0.014181182  | 0.099021313                            | 0.096150235 | 0.091650639  | 0.085930578 | 0.07945192  | 0.072648085  | 0.059374613  | 0.047772898  |
| 0.028362362  | 0.099026189                            | 0.09615959  | 0.091663599  | 0.085946017 | 0.079468689 | 0.072665187  | 0.059390358  | 0.047786077  |
| 0.035452951  | 0.099028622                            | 0.096164264 | 0.091670076  | 0.085953734 | 0.079477073 | 0.072673738  | 0.059398231  | 0.047792667  |
| 0.049634125  | 0.099033481                            | 0.096173603 | 0.091683024  | 0.085969165 | 0.079493839 | 0.072690841  | 0.059413981  | 0.047803851  |
| 0.055831409  | 0.0990356                              | 0.096177682 | 0.09168868   | 0.085975907 | 0.079501165 | 0.072698315  | 0.059420864  | 0.047811614  |
| 0.070905875  | 0.099040746                            | 0.096187594 | 0.091702432  | 0.085992303 | 0.079518983 | 0.0727116494 | 0.05943761   | 0.047825634  |
| 0.085087031  | 0.099043574                            | 0.096196906 | 0.091715361  | 0.086007723 | 0.079535744 | 0.072733597  | 0.059453366  | 0.047838826  |
| 0.099268176  | 0.09905039                             | 0.096206212 | 0.091728282  | 0.086023137 | 0.079552503 | 0.0727507    | 0.059469125  | 0.047852022  |
| 0.101147683  | 0.099051028                            | 0.096207445 | 0.091729994  | 0.08602518  | 0.079554724 | 0.072752967  | 0.059471214  | 0.047853772  |
| 0.103743768  | 0.099051908                            | 0.096209146 | 0.091732395  | 0.086028001 | 0.079557792 | 0.072756098  | 0.059474099  | 0.047856188  |
| 0.12146903   | 0.099057907                            | 0.096220757 | 0.091748495  | 0.08604726  | 0.079578735 | 0.072777476  | 0.059493802  | 0.047872688  |
| 0.223324574  | 0.099092013                            | 0.096287164 | 0.091840989  | 0.086157781 | 0.079699022 | 0.072900325  | 0.059607103  | 0.047967607  |
| 0.279134922  | 0.099110449                            | 0.096323338 | 0.091891519  | 0.086218257 | 0.07976491  | 0.072967668  | 0.059669268  | 0.04801971   |
| 0.33498474   | 0.099128697                            | 0.096359352 | 0.091941928  | 0.086278657 | 0.079830766 | 0.073035012  | 0.059731476  | 0.048071867  |
| 0.446642362  | 0.099164633                            | 0.096430895 | 0.092042383  | 0.086399229 | 0.079962378 | 0.073169705  | 0.059856017  | 0.048176338  |
| 0.558296592  | 0.099199827                            | 0.096501793 | 0.092142349  | 0.086519495 | 0.080093854 | 0.073304404  | 0.059980728  | 0.048281021  |
| 0.669946581  | 0.099234274                            | 0.096572042 | 0.092241823  | 0.08663945  | 0.080225193 | 0.073439105  | 0.060105605  | 0.048385915  |
| 0.781591482  | 0.099267974                            | 0.09664164  | 0.092340803  | 0.086759092 | 0.08035639  | 0.073573806  | 0.060230649  | 0.048491021  |
| 0.893230448  | 0.099300924                            | 0.096710583 | 0.092439284  | 0.086878416 | 0.080487442 | 0.073708504  | 0.060355899  | 0.048596339  |
| 1.004862632  | 0.099333126                            | 0.09677887  | 0.092537264  | 0.086997419 | 0.080618347 | 0.073843198  | 0.060481234  | 0.048701867  |
| 1.116487186  | 0.099364577                            | 0.096846498 | 0.092634738  | 0.087116096 | 0.0807491   | 0.073977885  | 0.060605772  | 0.048807607  |
| 1.228103264  | 0.099395276                            | 0.096913464 | 0.092731703  | 0.087234449 | 0.08067797  | 0.074112562  | 0.060732473  | 0.048913558  |
| 1.339710021  | 0.099425223                            | 0.096979766 | 0.092828156  | 0.087352468 | 0.081010142 | 0.074247226  | 0.060858336  | 0.04901972   |
| 1.451306661  | 0.099454417                            | 0.097045402 | 0.092924094  | 0.087470151 | 0.081140424 | 0.074381876  | 0.060984359  | 0.049126094  |
| 1.674465905  | 0.099510541                            | 0.097174663 | 0.093114409  | 0.087704498 | 0.081404093 | 0.074651123  | 0.061236884  | 0.049339473  |
| 1.897574394  | 0.099563641                            | 0.097301230 | 0.093302622  | 0.08793746  | 0.081659881 | 0.07492028   | 0.061490039  | 0.049553695  |
| 2.34361196   | 0.099660748                            | 0.097546204 | 0.093672634  | 0.088399116 | 0.082176511 | 0.075458245  | 0.061998208  | 0.049984664  |
| 2.67795671   | 0.099725599                            | 0.097722724 | 0.093944427  | 0.088741509 | 0.082562002 | 0.075861331  | 0.062380925  | 0.050310096  |
| 2.789365533  | 0.099745694                            | 0.097780179 | 0.094033919  | 0.088854888 | 0.082690103 | 0.075995606  | 0.062508794  | 0.050418992  |
| 3.067793428  | 0.099792593                            | 0.097920766 | 0.094255198  | 0.089136655 | 0.083009461 | 0.076331073  | 0.062829106  | 0.050692149  |
| 3.624201358  | 0.099872083                            | 0.098188762 | 0.094687087  | 0.089692798 | 0.083644156 | 0.077000929  | 0.063472402  | 0.051242376  |
| 3.902155428  | 0.099904661                            | 0.098316111 | 0.094897959  | 0.089967062 | 0.083959387 | 0.077335232  | 0.063795347  | 0.05151944   |
| 4.457499215  | 0.099955476                            | 0.098557361 | 0.099307517  | 0.090507693 | 0.084583548 | 0.078002363  | 0.064443731  | 0.05207745   |
| 5.012005072  | 0.099987167                            | 0.09878051  | 0.095702318  | 0.091037424 | 0.085204948 | 0.078667229  | 0.063095299  | 0.05264061   |
| 5.563571274  | 0.099999756                            | 0.098985359 | 0.096081638  | 0.091555812 | 0.085817747 | 0.079329462  | 0.065749874  | 0.053208881  |
| 6.118097145  | 0.099993284                            | 0.099171733 | 0.096445127  | 0.092062415 | 0.086423307 | 0.079988684  | 0.06640727   | 0.053782222  |
| 6.669483156  | 0.099967813                            | 0.099339476 | 0.096792448  | 0.092556799 | 0.087021181 | 0.080644511  | 0.067067292  | 0.054360585  |
| 7.219631036  | 0.099923429                            | 0.099488454 | 0.097123281  | 0.093038534 | 0.087619024 | 0.081296546  | 0.067729737  | 0.054943918  |
| 7.768443868  | 0.099860237                            | 0.099618552 | 0.097437318  | 0.093507196 | 0.088152085 | 0.081944391  | 0.06839439   | 0.05532163   |
| 8.315826182  | 0.099778362                            | 0.099729679 | 0.097734271  | 0.09396237  | 0.088764215 | 0.082587638  | 0.06906103   | 0.056125256  |
| 8.86168405   | 0.099877792                            | 0.099821763 | 0.098013864  | 0.094403649 | 0.089326863 | 0.083225872  | 0.069729423  | 0.056723127  |
| 9.405925174  | 0.099955193                            | 0.099894756 | 0.0982735844 | 0.094830634 | 0.089879579 | 0.083858672  | 0.0703999329 | 0.0573257    |
| 9.948458968  | 0.099942209                            | 0.099948631 | 0.098519972  | 0.095242937 | 0.090421913 | 0.0864485611 | 0.071070494  | 0.057932894  |
| 10.489196664 | 0.099926765                            | 0.099983381 | 0.098746031  | 0.095640181 | 0.090953418 | 0.085106257  | 0.071742658  | 0.0581544618 |
| 11.02805126  | 0.099094563                            | 0.099999024 | 0.09895382   | 0.096022002 | 0.09147365  | 0.085720175  | 0.07241555   | 0.059160779  |
| 12.09977337  | 0.098696862                            | 0.099973162 | 0.099313895  | 0.096737977 | 0.092478336 | 0.086926056  | 0.073762383  | 0.060405989  |
| 13.16296975  | 0.098231222                            | 0.098971607 | 0.099599009  | 0.097388213 | 0.093433106 | 0.088099694  | 0.075108626  | 0.061667619  |
| 14.21701869  | 0.097700004                            | 0.099695257 | 0.099808312  | 0.097970305 | 0.094333994 | 0.089237495  | 0.076451739  | 0.062944639  |
| 15.26133387  | 0.097105789                            | 0.099445337 | 0.099941296  | 0.098482121 | 0.09517797  | 0.090335856  | 0.07778911   | 0.064235894  |
| 16.2953656   | 0.096451354                            | 0.099123379 | 0.099997804  | 0.098921809 | 0.095961962 | 0.091391189  | 0.079117857  | 0.065340094  |
| 17.31860169  | 0.095739642                            | 0.098731208 | 0.099978026  | 0.099287834 | 0.096683084 | 0.092399952  | 0.080435039  | 0.066855811  |
| 18.33056816  | 0.094973734                            | 0.098270919 | 0.099882497  | 0.099578981 | 0.097338672 | 0.093338672  | 0.081737351  | 0.068181474  |
| 19.33082946  | 0.094156823                            | 0.097744853 | 0.099712091  | 0.099794372 | 0.097926299 | 0.094263976  | 0.083022208  | 0.069515361  |
| 20.31898861  | 0.093292187                            | 0.09715558  | 0.099465804  | 0.099933472 | 0.098443807 | 0.09511262   | 0.084283654  | 0.0708556    |
| 21.29468693  | 0.092838159                            | 0.096505857 | 0.099151746  | 0.099960995 | 0.098889326 | 0.095901519  | 0.08552448   | 0.072200163  |
| 22.25760362  | 0.091433105                            | 0.095798615 | 0.098765118  | 0.099982405 | 0.099261293 | 0.096627772  | 0.086735189  | 0.073546866  |
| 23.20745509  | 0.090445399                            | 0.095036926 | 0.098310195  | 0.099892909 | 0.099558468 | 0.097288695  | 0.087914227  | 0.074893369  |
| 24.14399409  | 0.0894234                              | 0.094223976 | 0.097789301  | 0.099728452 | 0.099779943 | 0.097881842  | 0.089058004  | 0.076237175  |
| 25.0670087   | 0.088370429                            | 0.093363036 | 0.097204984  | 0.099490208 | 0.099925158 | 0.0984405033 | 0.090162916  | 0.077575637  |
| 25.97632114  | 0.087289756                            | 0.092457435 | 0.096559991  | 0.099179661 | 0.099993898 | 0.098856373  | 0.091223374  | 0.078903956  |
| 26.87178647  | 0.086184575                            | 0.09151054  | 0.095857239  | 0.098798591 | 0.099986299 | 0.099234274  | 0.092241823  | 0.080225193  |
| 27.75329115  | 0.085057997                            | 0.090525722 | 0.095099791  | 0.098349049 | 0.09990284  | 0.099537469  | 0.09320878   | 0.081530274  |
| 28.6207516   | 0.083913031                            | 0.089506343 | 0.094290822  | 0.097833342 | 0.099744342 | 0.099765026  | 0.094122852  | 0.082818002  |
| 29.47411263  | 0.082752576                            | 0.088455729 | 0.093433599  | 0.097254    | 0.09951195  | 0.099916354  | 0.094980774  | 0.084065067  |
| 30.31334584  | 0.081579412                            | 0.087377153 | 0.092531448  | 0.096613756 | 0.099207126 | 0.099991212  | 0.095779432  | 0.085328066  |

| Phi (deg)    | lambda >     |              |              |             |             |              |             |             |
|--------------|--------------|--------------|--------------|-------------|-------------|--------------|-------------|-------------|
|              | 0.1          | 0.2          | 0.3          | 0.4         | 0.5         | 0.6          | 0.8         | 1           |
| 0.011418118  | 0.019768629  | 0.03785588   | 0.052404037  | 0.062401931 | 0.067761499 | 0.069095128  | 0.063599148 | 0.053227042 |
| 0.014181182  | 0.019725218  | 0.037818601  | 0.052376216  | 0.062384768 | 0.067754285 | 0.069095956  | 0.063609235 | 0.05323946  |
| 0.028362362  | 0.019676975  | 0.037777163  | 0.052345282  | 0.062365678 | 0.067746254 | 0.069096867  | 0.063620443 | 0.053253264 |
| 0.035452951  | 0.01965285   | 0.037756437  | 0.052329807  | 0.062356125 | 0.067742232 | 0.069097318  | 0.063626047 | 0.053260169 |
| 0.049634125  | 0.019604595  | 0.037714972  | 0.052298839  | 0.062337003 | 0.067734175 | 0.069098212  | 0.063637256 | 0.053273982 |
| 0.055831409  | 0.019583504  | 0.037696846  | 0.052285299  | 0.062328639 | 0.067730649 | 0.069098599  | 0.063642155 | 0.053280021 |
| 0.070905875  | 0.019532196  | 0.037632741  | 0.052252345  | 0.062308278 | 0.067722058 | 0.069099533  | 0.063654071 | 0.053294714 |
| 0.085087031  | 0.01948392   | 0.037611232  | 0.052221321  | 0.062289101 | 0.067713958 | 0.0691004    | 0.063665282 | 0.053308544 |
| 0.099268176  | 0.019435635  | 0.037569704  | 0.052190275  | 0.062269902 | 0.067705841 | 0.069101257  | 0.063676493 | 0.053322379 |
| 0.101147683  | 0.019429235  | 0.037564199  | 0.052186158  | 0.062267356 | 0.067704764 | 0.06910137   | 0.063677797 | 0.053324214 |
| 0.103743768  | 0.019420395  | 0.037556594  | 0.052180472  | 0.062263839 | 0.067703276 | 0.069101525  | 0.063680031 | 0.053326747 |
| 0.121466903  | 0.019360028  | 0.037504656  | 0.052141625  | 0.062239802 | 0.0676931   | 0.069102576  | 0.063694046 | 0.053344052 |
| 0.223324574  | 0.019012892  | 0.037205657  | 0.051917716  | 0.06210102  | 0.067634102 | 0.069108288  | 0.063774592 | 0.053443685 |
| 0.279154922  | 0.018822437  | 0.037041374  | 0.05179449   | 0.062024469 | 0.067601384 | 0.069111118  | 0.063818753 | 0.053498436 |
| 0.33498474   | 0.018631857  | 0.0368766815 | 0.051670913  | 0.061947577 | 0.067568398 | 0.069113902  | 0.063862922 | 0.053553285 |
| 0.446642362  | 0.018250328  | 0.036546873  | 0.051422708  | 0.06179277  | 0.067501613 | 0.069118831  | 0.063951279 | 0.053663278 |
| 0.558296592  | 0.017868314  | 0.036215835  | 0.051173101  | 0.061636592 | 0.06743374  | 0.069123066  | 0.064039657 | 0.053773662 |
| 0.669946581  | 0.017485825  | 0.035883705  | 0.050922091  | 0.061479038 | 0.067364769 | 0.069126599  | 0.064128054 | 0.053884438 |
| 0.781591482  | 0.017102867  | 0.035550487  | 0.050669677  | 0.061320101 | 0.067294692 | 0.069129423  | 0.064216463 | 0.053995605 |
| 0.893230448  | 0.01671945   | 0.035216186  | 0.050415856  | 0.061159777 | 0.067223501 | 0.069131528  | 0.064304886 | 0.054107162 |
| 1.004862632  | 0.016335582  | 0.034880808  | 0.05016063   | 0.060998058 | 0.067151189 | 0.069132907  | 0.064393311 | 0.054219111 |
| 1.116487186  | 0.015951271  | 0.0345444355 | 0.049903995  | 0.06083494  | 0.067077746 | 0.069133552  | 0.064481738 | 0.054331447 |
| 1.228103264  | 0.015566526  | 0.034206833  | 0.049645952  | 0.060670416 | 0.067003165 | 0.069133454  | 0.064570161 | 0.054444174 |
| 1.339710021  | 0.015181355  | 0.033868247  | 0.049386499  | 0.060504481 | 0.066927436 | 0.069132604  | 0.064658575 | 0.054357289 |
| 1.45130661   | 0.014795767  | 0.033528601  | 0.049125636  | 0.060337129 | 0.066850552 | 0.069130993  | 0.064746978 | 0.054670792 |
| 1.6744653905 | 0.014023373  | 0.032846151  | 0.048599677  | 0.059998151 | 0.066693286 | 0.069125464  | 0.064923725 | 0.05489896  |
| 1.897574394  | 0.013249414  | 0.032159523  | 0.04806807   | 0.059653438 | 0.066531298 | 0.069116791  | 0.065100366 | 0.055128671 |
| 2.34361198   | 0.011697075  | 0.030773899  | 0.046987899  | 0.058946628 | 0.066192889 | 0.069089745  | 0.06545317  | 0.053592704 |
| 2.67795671   | 0.010529229  | 0.029724029  | 0.046162928  | 0.058401114 | 0.06592619  | 0.069060694  | 0.065717197 | 0.053944734 |
| 2.789365533  | 0.01013931   | 0.029372074  | 0.045885111  | 0.058216353 | 0.06583479  | 0.069049293  | 0.065805069 | 0.056062836 |
| 3.067793428  | 0.009163202  | 0.028487867  | 0.045184384  | 0.057747865 | 0.065600739 | 0.069016949  | 0.066024397 | 0.056359739 |
| 3.624201358  | 0.007205845  | 0.026701262  | 0.043756459  | 0.056782681 | 0.065108404 | 0.06893531   | 0.066461253 | 0.056960551 |
| 3.902155428  | 0.006224876  | 0.025799058  | 0.043029282  | 0.056285832 | 0.064849856 | 0.068885725  | 0.066678602 | 0.057264422 |
| 4.457499215  | 0.004259063  | 0.023977347  | 0.041548574  | 0.055263254 | 0.064307343 | 0.068768287  | 0.067110678 | 0.057878988 |
| 5.012003072  | 0.002289026  | 0.022133266  | 0.040032874  | 0.054201705 | 0.063730065 | 0.068625495  | 0.067538607 | 0.058502488 |
| 5.565571274  | 0.000315908  | 0.020267691  | 0.038482398  | 0.053100668 | 0.063116983 | 0.068456129  | 0.067961568 | 0.059134706 |
| 6.118097145  | 0.001659147  | 0.01838154   | 0.036897423  | 0.051959662 | 0.062467063 | 0.066258951  | 0.068378699 | 0.059775396 |
| 6.669483156  | 0.003634993  | 0.01647577   | 0.03527828   | 0.050778254 | 0.061779289 | 0.068032702  | 0.068789093 | 0.060424282 |
| 7.219631036  | 0.005610484  | 0.014551375  | 0.033625359  | 0.049556061 | 0.061052659 | 0.067776106  | 0.069191798 | 0.061081058 |
| 7.768443868  | 0.007584474  | 0.012609385  | 0.031939109  | 0.048292745 | 0.060286191 | 0.067487875  | 0.069585815 | 0.061745386 |
| 8.315826182  | 0.009555826  | 0.010650866  | 0.030220037  | 0.046988028 | 0.059478929 | 0.067166707  | 0.0699701   | 0.062416891 |
| 8.86168405   | 0.011523406  | 0.008676916  | 0.028468711  | 0.045641683 | 0.058629942 | 0.066811292  | 0.070343562 | 0.063095165 |
| 9.405925174  | 0.013486091  | 0.006688664  | 0.026685757  | 0.044223546 | 0.057738331 | 0.066420314  | 0.070705061 | 0.063779762 |
| 9.948458968  | 0.015442768  | 0.004687266  | 0.024871862  | 0.042823513 | 0.056803234 | 0.065992454  | 0.071053412 | 0.064470196 |
| 10.48919664  | 0.017392337  | 0.002673905  | 0.023027771  | 0.041351544 | 0.055823824 | 0.065526395  | 0.071387381 | 0.065165942 |
| 11.02805126  | 0.019333713  | 0.000649789  | 0.021154286  | 0.039837665 | 0.054799321 | 0.065020823  | 0.071705686 | 0.063866432 |
| 12.09977337  | 0.023187639  | 0.003425777  | 0.017322646  | 0.036684625 | 0.052612143 | 0.0638883929 | 0.072289944 | 0.06727915  |
| 13.16296975  | 0.026996241  | 0.007529394  | 0.013384511  | 0.033366004 | 0.050323645 | 0.062577496  | 0.072795001 | 0.068702901 |
| 14.21701869  | 0.030751575  | 0.011650804  | 0.009348318  | 0.029884578 | 0.047667931 | 0.061083576  | 0.073208947 | 0.070131459 |
| 15.261333387 | 0.034446105  | 0.015779625  | 0.005223312  | 0.02624433  | 0.044903321 | 0.059400691  | 0.073519184 | 0.071557769 |
| 16.2953656   | 0.0380772747 | 0.019905431  | 0.001019479  | 0.022450464 | 0.041940523 | 0.057513978  | 0.073712479 | 0.072973905 |
| 17.31860169  | 0.041624907  | 0.024017848  | 0.003252534  | 0.018509405 | 0.038778707 | 0.055417352  | 0.073775028 | 0.074371022 |
| 18.33056816  | 0.045096513  | 0.028106636  | 0.007581518  | 0.014428779 | 0.035418396 | 0.053103649  | 0.073692545 | 0.075739324 |
| 19.33082946  | 0.048482039  | 0.032161779  | 0.011955794  | 0.010217374 | 0.031861537 | 0.050566789  | 0.073450358 | 0.077068035 |
| 20.3189861   | 0.051776519  | 0.036173564  | 0.016363329  | 0.005885084 | 0.028111559 | 0.047801923  | 0.073033536 | 0.078345371 |
| 21.294658693 | 0.05497556   | 0.040132653  | 0.020791849  | 0.001442832 | 0.024173399 | 0.044805569  | 0.072427021 | 0.07958538  |
| 22.25760362  | 0.058075347  | 0.044040315  | 0.02522895   | 0.003097523 | 0.020053513 | 0.041575744  | 0.071615786 | 0.080693728 |
| 23.20745509  | 0.061072635  | 0.047857666  | 0.02966222   | 0.007723291 | 0.01575987  | 0.038112074  | 0.070585007 | 0.081736141 |
| 24.14399409  | 0.063964745  | 0.051607364  | 0.034079352  | 0.012421079 | 0.011301906 | 0.034415889  | 0.06932025  | 0.082670006 |
| 25.0670087   | 0.066749547  | 0.055272005  | 0.038468256  | 0.017176917 | 0.006690464 | 0.030490292  | 0.067807668 | 0.083478653 |
| 25.97632114  | 0.069425447  | 0.058844981  | 0.042817163  | 0.021976402 | 0.00193771  | 0.026340207  | 0.06603421  | 0.084144533 |
| 26.87178647  | 0.07199136   | 0.06232034   | 0.047114725  | 0.026804845 | 0.012942975 | 0.021972393  | 0.063987842 | 0.084649434 |
| 27.75329115  | 0.074446685  | 0.063692798  | 0.051350102  | 0.031647413 | 0.007937121 | 0.017395439  | 0.061657757 | 0.084974384 |
| 28.6207516   | 0.076791281  | 0.06895775   | 0.055513044  | 0.036489286 | 0.013029316 | 0.01261972   | 0.059034596 | 0.085099988 |
| 29.47411263  | 0.079025432  | 0.072111268  | 0.0509593954 | 0.041315791 | 0.018203364 | 0.007657331  | 0.056110636 | 0.085006492 |
| 30.31334584  | 0.081149817  | 0.075150091  | 0.06358395   | 0.046112548 | 0.023442457 | 0.002521986  | 0.052880088 | 0.084673981 |

| Phi (deg)    | Rate of Aperture Opening [Ao dot] divided by shear transits [Phi dot^2 b] |             |              |             |             |             |             |             |
|--------------|---|-------------|--------------|-------------|-------------|-------------|-------------|-------------|
|              | 0.1   | 0.2         | 0.3          | 0.4         | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118  | 1.015000217   | 1.029667636 | 1.053662869  | 1.086368003 | 1.12702504  | 1.174808683 | 1.288470193 | 1.421314688 |
| 0.014181182  | 1.014956035   | 1.02958088  | 1.053534891  | 1.086202458 | 1.126825538 | 1.174578993 | 1.288190923 | 1.420998207 |
| 0.028362362  | 1.014907235   | 1.029483714 | 1.053392971  | 1.086018784 | 1.126604127 | 1.174324031 | 1.287880855 | 1.42054679  |
| 0.035432951  | 1.01488293  | 1.029435494 | 1.053322121  | 1.085927054 | 1.126493524 | 1.174196648 | 1.287725914 | 1.420471171 |
| 0.049634125  | 1.014834611   | 1.02933928  | 1.053180641  | 1.085743804 | 1.12627252  | 1.173942078 | 1.287416218 | 1.420120115 |
| 0.055831409  | 1.014813583   | 1.029297329 | 1.053118904  | 1.085663811 | 1.126176025 | 1.173830911 | 1.287280955 | 1.419966776 |
| 0.070905873  | 1.014762679   | 1.029195525 | 1.052968968  | 1.085469459 | 1.125941524 | 1.173560714 | 1.286952136 | 1.419593981 |
| 0.085087031  | 1.014715108   | 1.029100065 | 1.052828219  | 1.085286914 | 1.125721199 | 1.17306799  | 1.286643057 | 1.419243525 |
| 0.099268176  | 1.014667845   | 1.029004906 | 1.052687762  | 1.085104652 | 1.125501146 | 1.173053146 | 1.286334224 | 1.41889331  |
| 0.101147683  | 1.014661604   | 1.028992316 | 1.052669168  | 1.085080517 | 1.125472002 | 1.173019547 | 1.286293312 | 1.418846912 |
| 0.103743768  | 1.014652992   | 1.028974936 | 1.052643494  | 1.085047189 | 1.125431753 | 1.172973146 | 1.286236808 | 1.418782831 |
| 0.12146903   | 1.01459447  | 1.028856538 | 1.052468462  | 1.084819884 | 1.125157194 | 1.172656571 | 1.285851237 | 1.418345523 |
| 0.223324574  | 1.014267485   | 1.028185303 | 1.051471516  | 1.083522249 | 1.123587688 | 1.170845326 | 1.283643065 | 1.415839842 |
| 0.279154922  | 1.014094979   | 1.027823969 | 1.050931454  | 1.082817139 | 1.122733321 | 1.169858233 | 1.282438072 | 1.414471622 |
| 0.334984747  | 1.013927236   | 1.027467307 | 1.050395926  | 1.0821164   | 1.121883157 | 1.168875186 | 1.281236891 | 1.413107104 |
| 0.4466642362 | 1.013606039   | 1.026767994 | 1.049338469  | 1.080728031 | 1.12019543  | 1.166921225 | 1.278845946 | 1.410389154 |
| 0.558296592  | 1.013303897   | 1.026087365 | 1.048299142  | 1.079357134 | 1.118524496 | 1.164983422 | 1.276470198 | 1.407685946 |
| 0.669946581  | 1.01302081  | 1.025425419 | 1.047277941  | 1.078003702 | 1.116870342 | 1.163061739 | 1.274109612 | 1.404997432 |
| 0.781591482  | 1.012757678   | 1.024782158 | 1.046274866  | 1.076666773 | 1.115232956 | 1.161156216 | 1.271764135 | 1.402323569 |
| 0.893230448  | 1.012511809   | 1.024157581 | 1.045289914  | 1.075349211 | 1.113612326 | 1.159266774 | 1.269433796 | 1.39966431  |
| 1.004862632  | 1.012285899   | 1.023551691 | 1.044323085  | 1.074048139 | 1.112008442 | 1.157934318 | 1.267118502 | 1.39701961  |
| 1.116487186  | 1.012079053   | 1.022964488 | 1.043374376  | 1.072764511 | 1.110421291 | 1.155536127 | 1.264818241 | 1.394389424 |
| 1.228103264  | 1.011891272   | 1.022395975 | 1.042443789  | 1.071498319 | 1.108850864 | 1.153694886 | 1.262532981 | 1.391773709 |
| 1.339710021  | 1.011722561   | 1.021846154 | 1.041531321  | 1.070249562 | 1.107297151 | 1.151869677 | 1.260262691 | 1.389172418 |
| 1.45130661   | 1.011572923   | 1.021315027 | 1.040636975  | 1.069018233 | 1.105760141 | 1.150060483 | 1.25800734  | 1.386585509 |
| 1.674465905  | 1.011330881   | 1.020308868 | 1.038902646  | 1.06660785  | 1.102736196 | 1.14649008  | 1.253541332 | 1.381454639 |
| 1.897574394  | 1.011165179   | 1.019377525 | 1.03724081   | 1.064267146 | 1.099778959 | 1.142983551 | 1.249134717 | 1.376380812 |
| 2.34361198   | 1.011062956   | 1.017739424 | 1.034134679  | 1.059794705 | 1.094064359 | 1.13616165  | 1.240498727 | 1.366402757 |
| 2.67795671   | 1.011187006   | 1.01670752  | 1.031995323  | 1.056623188 | 1.089953087 | 1.131212059 | 1.234175927 | 1.359066325 |
| 2.789365533  | 1.011266623   | 1.016401046 | 1.031318765  | 1.056600836 | 1.088615904 | 1.129593907 | 1.232097549 | 1.356648649 |
| 3.067793428  | 1.011549447   | 1.015716941 | 1.029706313  | 1.053121136 | 1.085345618 | 1.125617785 | 1.226965279 | 1.350664887 |
| 3.624201358  | 1.012474597   | 1.014700921 | 1.026822175  | 1.048488325 | 1.079116306 | 1.117961815 | 1.216972266 | 1.338954229 |
| 3.902155428  | 1.013117209   | 1.014369278 | 1.025550702  | 1.046335331 | 1.076157203 | 1.114281676 | 1.212110754 | 1.33226115  |
| 4.457499215  | 1.014763316   | 1.0140595   | 1.023349585  | 1.042356412 | 1.07055003  | 1.107216481 | 1.2026559   | 1.322021355 |
| 5.012005072  | 1.016891818   | 1.01422225  | 1.02160525   | 1.038814336 | 1.063537653 | 1.100544106 | 1.193556355 | 1.311148075 |
| 5.565571274  | 1.01950434  | 1.014859126 | 1.020319107  | 1.035710071 | 1.060580402 | 1.094264031 | 1.184809617 | 1.300601864 |
| 6.118097145  | 1.022602698   | 1.015971928 | 1.019492786  | 1.030448448 | 1.056218858 | 1.088375986 | 1.176413424 | 1.290378511 |
| 6.669483156  | 1.026188897   | 1.017562657 | 1.0191128141 | 1.030620138 | 1.052273866 | 1.062879972 | 1.168365757 | 1.280474001 |
| 7.219631036  | 1.030265127   | 1.019633515 | 1.019227244  | 1.029037658 | 1.048746525 | 1.077776254 | 1.160664845 | 1.270884526 |
| 7.768443868  | 1.034833763   | 1.022186898 | 1.019792387  | 1.027699365 | 1.045638197 | 1.073065366 | 1.153309168 | 1.261606487 |
| 8.315826182  | 1.039897363   | 1.025225401 | 1.020626077  | 1.026807458 | 1.042950504 | 1.068748111 | 1.146297466 | 1.2526365   |
| 8.86168405   | 1.04545666  | 1.028751809 | 1.022331038  | 1.026364374 | 1.040685332 | 1.064825569 | 1.13962874  | 1.243971402 |
| 9.405925174  | 1.051520597   | 1.032769102 | 1.024310205  | 1.026372792 | 1.038844824 | 1.061299091 | 1.13302257  | 1.235608254 |
| 9.948458968  | 1.058086248   | 1.037280445 | 1.026766725  | 1.026835626 | 1.037431388 | 1.058170307 | 1.127317555 | 1.227544348 |
| 10.48919664  | 1.065158892   | 1.042289191 | 1.029703951  | 1.027756026 | 1.036447694 | 1.055441126 | 1.121674449 | 1.219777213 |
| 11.02805126  | 1.072741976   | 1.047798878 | 1.033125442  | 1.029137377 | 1.03589667  | 1.053113736 | 1.116373032 | 1.212304618 |
| 12.09977337  | 1.089454102   | 1.06033612  | 1.04143647   | 1.03297622  | 1.036103654 | 1.04967448  | 1.106797067 | 1.198235366 |
| 13.16296975  | 1.106253576   | 1.074942043 | 1.051732275  | 1.039348013 | 1.038086953 | 1.047875668 | 1.098596166 | 1.18532378  |
| 14.21701869  | 1.129173977   | 1.091597239 | 1.0604048412 | 1.0473277   | 1.041873271 | 1.047744934 | 1.091781566 | 1.173561255 |
| 15.26133387  | 1.15225143  | 1.110393015 | 1.078423405  | 1.057263576 | 1.047501327 | 1.04931441  | 1.086369354 | 1.162943571 |
| 16.2953656   | 1.177524529   | 1.131351199 | 1.09489863   | 1.069209453 | 1.055011771 | 1.052620694 | 1.082380564 | 1.153471061 |
| 17.31860169  | 1.209034251   | 1.154514078 | 1.113518186  | 1.083206633 | 1.064449073 | 1.057704798 | 1.079841264 | 1.145148783 |
| 18.33056816  | 1.234823871   | 1.179926288 | 1.134328762  | 1.099303408 | 1.075861395 | 1.064612071 | 1.078782628 | 1.137986693 |
| 19.33029646  | 1.266938878   | 1.207634704 | 1.157379509  | 1.117550981 | 1.089300453 | 1.07392106  | 1.079240987 | 1.131999799 |
| 20.31898861  | 1.301426886   | 1.237688332 | 1.182721865  | 1.138003299 | 1.104821353 | 1.084098619 | 1.081257863 | 1.127208325 |
| 21.29468693  | 1.338337555   | 1.270138205 | 1.210409488  | 1.160716885 | 1.122482417 | 1.096789306 | 1.084879976 | 1.123637849 |
| 22.25760362  | 1.377722499   | 1.305037279 | 1.240498044  | 1.185750663 | 1.142344998 | 1.111525685 | 1.090159234 | 1.121319445 |
| 23.20745509  | 1.419635206   | 1.342440329 | 1.273045107  | 1.213165785 | 1.164473279 | 1.128372915 | 1.097152693 | 1.120289806 |
| 24.14399409  | 1.464130956   | 1.382403853 | 1.30811002   | 1.243025448 | 1.18893407  | 1.147399589 | 1.105922491 | 1.120591347 |
| 25.0670087   | 1.511266737   | 1.424985975 | 1.345753764  | 1.275394723 | 1.215796589 | 1.168677526 | 1.116353756 | 1.122272301 |
| 25.97632114  | 1.561101171   | 1.470246357 | 1.386038832  | 1.310340382 | 1.245132248 | 1.192281533 | 1.129064484 | 1.125386786 |
| 26.87178647  | 1.613694437   | 1.518246106 | 1.429029114  | 1.347930724 | 1.277014434 | 1.218289162 | 1.143585388 | 1.129994856 |
| 27.75329115  | 1.669108192   | 1.569047697 | 1.474789785  | 1.388235421 | 1.311518287 | 1.246780457 | 1.160179721 | 1.136162521 |
| 28.6207516   | 1.727405508   | 1.622714892 | 1.523387198  | 1.431325358 | 1.348720486 | 1.277837693 | 1.178933067 | 1.143961749 |
| 29.47411263  | 1.788650795   | 1.679312667 | 1.574888786  | 1.477272487 | 1.38869904  | 1.31154511  | 1.199935104 | 1.153470432 |
| 30.31334584  | 1.852909741   | 1.738907143 | 1.629362978  | 1.526149695 | 1.431533087 | 1.34798651  | 1.223279343 | 1.164772322 |

| Phi (deg)    | Aperture Shear Energy contrib. to NDSF (tests 3,4) |              |             |              |             |             |             |             |
|--------------|--|--------------|-------------|--------------|-------------|-------------|-------------|-------------|
|              | 0.1  | 0.2          | 0.3         | 0.4          | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118  | 40.52693817  | 20.0583866   | 13.23492861 | 9.82294321   | 7.775702051 | 6.410933275 | 4.705264134 | 3.682214761 |
| 0.014181182  | 40.52876982  | 20.06020122  | 13.23669037 | 9.824623155  | 7.777279509 | 6.412396118 | 4.706489838 | 3.683222863 |
| 0.028362362  | 40.53079772  | 20.06221409  | 13.23864596 | 9.826488651  | 7.779031642 | 6.414021233 | 4.707851811 | 3.684343174 |
| 0.035452951  | 40.53180881  | 20.0632192   | 13.239623   | 9.82742096   | 7.779907471 | 6.414833684 | 4.708532829 | 3.684903407 |
| 0.049634125  | 40.53382524  | 20.06522673  | 13.24157556 | 9.829284702  | 7.781658656 | 6.416458376 | 4.709894928 | 3.68602403  |
| 0.053831409  | 40.53470403  | 20.06610292  | 13.24242821 | 9.830098804  | 7.782423738 | 6.417168291 | 4.710490202 | 3.686513816 |
| 0.070905875  | 40.536383554                                       | 20.06823136  | 13.2445006  | 9.832078117  | 7.784284245 | 6.418094884 | 4.711938231 | 3.687705354 |
| 0.085087031  | 40.53883285  | 20.07022999  | 13.24644809 | 9.833938928  | 7.786033845 | 6.420518866 | 4.713300536 | 3.688826495 |
| 0.099268176  | 40.54062251  | 20.07222505  | 13.24839356 | 9.835798564  | 7.787782809 | 6.422142567 | 4.714662924 | 3.689947845 |
| 0.101147683  | 40.54108564  | 20.0724892   | 13.24865125 | 9.836044944  | 7.788014561 | 6.422357744 | 4.714843495 | 3.690096479 |
| 0.103743768  | 40.54144886  | 20.07285396  | 13.24900713 | 9.836385224  | 7.788334653 | 6.42265495  | 4.715092912 | 3.690301787 |
| 0.12146903   | 40.543922  | 20.07534121  | 13.25143515 | 9.8387075    | 7.790519569 | 6.424683928 | 4.716795933 | 3.691703756 |
| 0.223324574  | 40.55790183  | 20.08952579  | 13.26532585 | 9.852016425  | 7.803055521 | 6.436334474 | 4.726584571 | 3.699766265 |
| 0.279154922  | 40.56339712  | 20.09722261  | 13.27289519 | 9.859285585  | 7.80991283  | 6.442714195 | 4.73195182  | 3.704190148 |
| 0.33498474   | 40.5727737   | 20.10486389  | 13.28043277 | 9.866536252  | 7.816760052 | 6.449089329 | 4.73732026  | 3.708617217 |
| 0.446642362  | 40.58717054  | 20.11997946  | 13.2954123  | 9.880981778  | 7.830423969 | 6.461825636 | 4.748060618 | 3.717480881 |
| 0.558296592  | 40.60109205  | 20.13487183  | 13.3102637  | 9.895352348  | 7.844046737 | 6.47454299  | 4.758805453 | 3.726357191 |
| 0.669946581  | 40.61453799  | 20.14954032  | 13.32498626 | 9.909647308  | 7.857627823 | 6.487240988 | 4.769554576 | 3.73524608  |
| 0.781591482  | 40.62750812  | 20.16398427  | 13.33957925 | 9.923866005  | 7.87116669  | 6.499919222 | 4.780307791 | 3.744147431 |
| 0.893230448  | 40.64000223  | 20.17820304  | 13.35404197 | 9.938007787  | 7.884662802 | 6.512577284 | 4.791064905 | 3.753061323 |
| 1.004862632  | 40.65202016  | 20.192196    | 13.36837377 | 9.952072003  | 7.898115621 | 6.525214762 | 4.80182572  | 3.761987536 |
| 1.116487186  | 40.66356174  | 20.20596252  | 13.38257375 | 9.966058005  | 7.911524609 | 6.537831244 | 4.812590039 | 3.77092605  |
| 1.228103264  | 40.67462687  | 20.219502    | 13.39664141 | 9.979965144  | 7.924889226 | 6.550426317 | 4.82335766  | 3.779876793 |
| 1.339710021  | 40.68521545  | 20.23281383  | 13.410576   | 9.993792773  | 7.938208932 | 6.562999564 | 4.834128383 | 3.78883969  |
| 1.45130661   | 40.69532742  | 20.24589743  | 13.42437682 | 10.00754025  | 7.951483186 | 6.575550567 | 4.844902003 | 3.797814669 |
| 1.674465905  | 40.7141214   | 20.27137769  | 13.45157446 | 10.03479216  | 7.977893172 | 6.600584164 | 4.866457111 | 3.815800567 |
| 1.897574394  | 40.73100687  | 20.29593835  | 13.47822896 | 10.06171575  | 8.004114842 | 6.625523733 | 4.888021322 | 3.833833873 |
| 2.343611198  | 40.7590667   | 20.34228422  | 13.52988751 | 10.1145576   | 8.055975819 | 6.67510715  | 4.931170258 | 3.870040172 |
| 2.67795671   | 40.7751124   | 20.37449981  | 13.56716708 | 10.15328941  | 8.094344801 | 6.712022395 | 4.963543068 | 3.897314528 |
| 2.789365533  | 40.77951021  | 20.38490378  | 13.57931135 | 10.16602533  | 8.107031213 | 6.72427329  | 4.974334939 | 3.906428315 |
| 3.067793428  | 40.78842776  | 20.40963622  | 13.60904884 | 10.19747712  | 8.138516201 | 6.754777838 | 5.001314516 | 3.929260804 |
| 3.624201358  | 40.7975813   | 20.4546775   | 13.66582172 | 10.25868518  | 8.2004669   | 6.815237613 | 5.05526095  | 3.975126757 |
| 3.902155428  | 40.7974313   | 20.47497568  | 13.69283931 | 10.28842251  | 8.230915495 | 6.845178892 | 5.082220425 | 3.998157278 |
| 4.457499215  | 40.78872068  | 20.5110444   | 13.74409439 | 10.34611723  | 8.29071642  | 6.904448984 | 5.136092914 | 4.044405711 |
| 5.012005072  | 40.76833188  | 20.54125254  | 13.79158948 | 10.40137805  | 8.3489988   | 6.962854909 | 5.189877334 | 4.090893191 |
| 5.565571274  | 40.73636244  | 20.56539893  | 13.83526502 | 10.45413406  | 8.405694697 | 7.020339029 | 5.243541359 | 4.13760609  |
| 6.118097145  | 40.83615555  | 20.58353138  | 13.87506644 | 10.50431665  | 8.460736648 | 7.076843113 | 5.297051543 | 4.184529996 |
| 6.669483156  | 40.9797835   | 20.59564648  | 13.91094435 | 10.55185975  | 8.514057826 | 7.132308442 | 5.350373327 | 4.231649698 |
| 7.219631036  | 41.14221341  | 20.6017496   | 13.94285475 | 10.59669999  | 8.565392205 | 7.186675905 | 5.40347104  | 4.278949167 |
| 7.768443868  | 41.323221882                                       | 20.60185496  | 13.97075904 | 10.638776991 | 8.615274726 | 7.239886107 | 5.456307917 | 4.326411531 |
| 8.315826182  | 41.52254006  | 20.59598547  | 13.9946244  | 10.67803317  | 8.663041468 | 7.291879486 | 5.508846108 | 4.374019061 |
| 8.86168405   | 41.7398852   | 20.5841727   | 14.01442369 | 10.7144147   | 8.708829819 | 7.342596428 | 5.561046697 | 4.42175315  |
| 9.405925174  | 41.9749311   | 20.566645678 | 14.03013562 | 10.74787094  | 8.752578647 | 7.391977391 | 5.612869724 | 4.469594294 |
| 9.948458968  | 42.22732467  | 20.54288617  | 14.04174483 | 10.77835492  | 8.794228478 | 7.439963037 | 5.664274209 | 4.517522077 |
| 10.48919664  | 42.49668402  | 20.5135176   | 14.049242   | 10.8058235   | 8.833721664 | 7.486494359 | 5.715218182 | 4.563515156 |
| 11.028051216 | 42.7825991   | 20.47841577  | 14.05262374 | 10.83023748  | 8.871002559 | 5.731512824 | 5.765658718 | 4.613551243 |
| 12.09977337  | 43.40233589  | 20.67874819  | 14.04705784 | 10.86976544  | 8.938715905 | 7.616780246 | 5.864853231 | 4.709658497 |
| 13.16296975  | 44.08276637  | 20.96605899  | 14.02514105 | 10.8967092   | 8.996969731 | 7.695311873 | 5.961496792 | 4.805646197 |
| 14.21701869  | 44.8197843   | 21.28906525  | 13.98706257 | 10.91090998  | 9.045407784 | 7.766671114 | 6.055216074 | 4.901300098 |
| 15.26133387  | 45.60889942  | 21.64617533  | 13.93310452 | 10.91228235  | 9.083721161 | 7.830443039 | 6.145627286 | 4.996393156 |
| 16.2953656   | 46.44577043  | 22.03557151  | 13.86363827 | 10.90081579  | 9.111652682 | 7.886239231 | 6.232338385 | 5.090677465 |
| 17.31860169  | 47.32530531  | 22.45523368  | 14.03509055 | 10.8765753   | 9.12900726  | 7.933702585 | 6.314951618 | 5.183892355 |
| 18.33056816  | 48.24267103  | 22.90296508  | 14.30717113 | 10.83970086  | 9.135622414 | 7.97251194  | 6.393066362 | 5.275760637 |
| 19.33082946  | 49.19286951  | 23.37641964  | 14.60269412 | 10.79040602  | 9.131436085 | 8.002386479 | 6.466282268 | 5.365989979 |
| 20.31898861  | 50.17088415  | 23.8713038   | 14.92034008 | 10.72897532  | 9.116422971 | 8.023089778 | 6.534202674 | 5.454273381 |
| 21.29468693  | 51.17172801  | 24.39058323  | 15.25864636 | 10.63576091  | 9.090628042 | 8.034433425 | 6.596438264 | 5.540290304 |
| 22.25760362  | 52.19048832  | 24.92602073  | 15.61602492 | 10.80204488  | 9.054159983 | 8.036280108 | 6.652610936 | 5.623707739 |
| 23.20745509  | 53.22236701  | 25.47692019  | 15.99078149 | 11.07069635  | 9.007190291 | 8.028546111 | 6.702357812 | 5.704181673 |
| 24.14399409  | 54.26271675  | 26.04057071  | 16.38113541 | 11.35683741  | 8.949951513 | 8.011203134 | 6.745335361 | 5.781358812 |
| 25.0670087   | 55.30707252  | 26.61432378  | 16.78524008 | 11.65926686  | 8.882734652 | 7.984279392 | 6.781223542 | 5.854878589 |
| 25.97632114  | 56.35117833  | 27.19557211  | 17.20120336 | 11.97668159  | 8.805885802 | 7.947859955 | 6.809729915 | 5.924375463 |
| 26.87178647  | 57.39100935  | 27.78177124  | 17.62710771 | 12.3076907   | 8.928001861 | 7.902066302 | 6.830593628 | 5.989481512 |
| 27.7529115   | 58.42278939  | 28.37045902  | 18.06102965 | 12.65083161  | 9.199079383 | 7.847155106 | 6.843589207 | 6.049829284 |
| 28.6207516   | 59.44300403  | 28.9592726   | 18.50105828 | 13.00458656  | 9.4841953   | 7.78331625  | 6.848530069 | 6.105054927 |
| 29.47411263  | 60.44840953  | 29.54596291  | 18.94531254 | 13.36739914  | 9.782203154 | 7.710870125 | 6.845271663 | 6.154801537 |
| 30.31334584  | 61.4360379   | 30.12840666  | 19.39195708 | 13.73769077  | 10.09187745 | 7.630164257 | 6.833714179 | 6.198722712 |

| Phi (deg)   | Bending Energy Contribution from 2 longitudinal hinges |             |             |              |             |             |             |             |
|-------------|--|-------------|-------------|--------------|-------------|-------------|-------------|-------------|
|             | 0.1  | 0.3         | 0.5         | 0.6          | 0.8         | 1           | lambda>     |             |
| 0.001418118 | 22.24481181  | 31.46188466 | 38.53401075 | 44.49596662  | 49.74850116 | 54.49712793 | 62.92843482 | 70.35654846 |
| 0.014181182 | 7.022588268  | 9.940838442 | 12.17887842 | 14.06524649  | 15.72699916 | 17.22925158 | 19.89643467 | 22.24621689 |
| 0.028362362 | 4.956409563  | 7.022723979 | 8.606545315 | 9.941220625  | 11.11685118 | 12.17957954 | 14.06632519 | 15.72850611 |
| 0.035452951 | 4.428980737  | 6.278402902 | 7.695592411 | 8.889725559  | 9.941507295 | 10.8922532  | 12.58015094 | 14.0671344  |
| 0.049634125 | 3.736123595  | 5.301294601 | 6.500014043 | 7.50985524   | 8.399222614 | 9.203109398 | 10.63022025 | 11.47743034 |
| 0.055831409 | 3.519764988  | 4.996391237 | 6.12702782  | 7.079428112  | 7.918169469 | 8.676276911 | 10.02209183 | 11.2076727  |
| 0.070905875 | 3.117008715  | 4.429198149 | 5.433345373 | 6.279011289  | 7.023674205 | 7.696706152 | 8.891437358 | 9.943897284 |
| 0.085067031 | 2.840030008  | 4.03952137  | 4.956925931 | 5.729380268  | 6.409505479 | 7.024172116 | 8.115248852 | 9.076375881 |
| 0.099268176 | 2.624355567  | 3.736383065 | 4.586427878 | 5.302017678  | 5.932010092 | 6.501335922 | 7.51188558  | 8.402056237 |
| 0.101147683 | 2.599201515  | 3.701047893 | 4.543249063 | 5.252216484  | 5.876370122 | 6.440415095 | 7.441585495 | 8.323491536 |
| 0.103743768 | 2.565577933  | 3.653822261 | 4.485543283 | 5.185662023  | 5.802013881 | 6.3590025   | 7.347640117 | 8.218502717 |
| 0.121446903 | 2.363530515  | 3.372781506 | 4.142211381 | 4.789727992  | 5.359697686 | 5.874734    | 6.788857849 | 7.594060515 |
| 0.223324574 | 1.720366822  | 2.470699072 | 3.041501357 | 3.521146858  | 3.943035193 | 4.324102733 | 5.000237253 | 5.595710449 |
| 0.279154922 | 1.526867245  | 2.201635606 | 2.713829477 | 3.143872931  | 3.521976932 | 3.863418046 | 4.469138423 | 5.002549782 |
| 0.33498474  | 1.382947842  | 2.002288432 | 2.471372329 | 2.864895556  | 3.210747787 | 3.522992411 | 4.076822321 | 4.564493838 |
| 0.446642362 | 1.1786598602   | 1.72097346  | 2.129861151 | 2.47231607   | 2.773037647 | 3.044411461 | 3.522558203 | 3.949197396 |
| 0.558296592 | 1.037151799  | 1.527565014 | 1.895680831 | 2.203473077  | 2.473531482 | 2.717121243 | 3.148878098 | 3.52892164  |
| 0.669946381 | 0.931035466  | 1.383733225 | 1.721986496 | 2.004333554  | 2.251859082 | 2.479502010 | 2.870429569 | 3.218416128 |
| 0.781591482 | 0.847263446  | 1.271112838 | 1.586344332 | 1.849028019  | 2.079121539 | 2.286469023 | 2.65373056  | 2.976886386 |
| 0.893230448 | 0.778663426  | 1.179652692 | 1.476482553 | 1.723408916  | 1.93951706  | 2.134169831 | 2.478824895 | 2.782033981 |
| 1.004862632 | 0.720928619  | 1.103326022 | 1.385046497 | 1.618999532  | 1.823579258 | 2.007760821 | 2.333760433 | 2.620505214 |
| 1.116487186 | 0.671294076  | 1.038269262 | 1.307323601 | 1.530369718  | 1.725244941 | 1.900660671 | 2.210883787 | 2.483750042 |
| 1.228103264 | 0.627891409  | 0.981875482 | 1.240135624 | 1.453858183  | 1.640426625 | 1.808232468 | 2.105036733 | 2.366006454 |
| 1.339710021 | 0.589405814  | 0.932312915 | 1.181250439 | 1.386894186  | 1.566254762 | 1.72750008  | 2.012598936 | 2.263230725 |
| 1.45130661  | 0.55488105   | 0.8882514   | 1.129047943 | 1.32761232   | 1.500647442 | 1.65613107  | 1.930944429 | 2.172490064 |
| 1.674465905 | 0.495023142  | 0.812895762 | 1.040143598 | 1.226860571  | 1.389285596 | 1.535093494 | 1.792620289 | 2.018889348 |
| 1.897574394 | 0.444345104  | 0.750304044 | 0.966725742 | 1.143896504  | 1.297743292 | 1.435715133 | 1.679225937 | 1.893102203 |
| 2.34361198  | 0.361437628  | 0.650874056 | 0.851113332 | 1.013808728  | 1.154575056 | 1.280565441 | 1.502606354 | 1.697484025 |
| 2.67795671  | 0.310114007  | 0.591592412 | 0.782934446 | 0.937500875  | 1.07086368  | 1.190047153 | 1.399860741 | 1.58390759  |
| 2.78 155533 | 0.294493933  | 0.573955544 | 0.762781802 | 0.915016115  | 1.046243981 | 1.163459858 | 1.36973363  | 1.550643078 |
| 3.067793428 | 0.258009095  | 0.533582228 | 0.716908882 | 0.863973179  | 0.990445435 | 1.103268858 | 1.301629506 | 1.475521837 |
| 3.624201358 | 0.193345311  | 0.465215568 | 0.640188931 | 0.779112027  | 0.898007669 | 1.003796587 | 1.189444043 | 1.352050341 |
| 3.902155428 | 0.164000725  | 0.435711824 | 0.607521194 | 0.743207364  | 0.859046554 | 0.961979981 | 1.14244775  | 1.300450653 |
| 4.457499215 | 0.109255922  | 0.383543129 | 0.550541285 | 0.680982784  | 0.791784029 | 0.889977404 | 1.061812419 | 1.212134604 |
| 5.012005072 | 0.057844563  | 0.338286366 | 0.502062581 | 0.628520972  | 0.735382008 | 0.829825363 | 0.994737961 | 1.17987469  |
| 5.565571274 | 0.007948305  | 0.298106408 | 0.459901297 | 0.583330081  | 0.687073888 | 0.778507662 | 0.937914064 | 1.07715794  |
| 6.118097145 | 0.041992379  | 0.26174581  | 0.422575525 | 0.543723017  | 0.64498866  | 0.733985993 | 0.888854685 | 1.024048246 |
| 6.669483156 | 0.093528692  | 0.228304835 | 0.389037073 | 0.508509745  | 0.607808231 | 0.694825551 | 0.845967301 | 0.977833589 |
| 7.219631036 | 0.148449781  | 0.197113673 | 0.358518805 | 0.476822258  | 0.574572578 | 0.659982241 | 0.808058156 | 0.937189235 |
| 7.768443868 | 0.209123237  | 0.167636504 | 0.3304429   | 0.448009459  | 0.544562812 | 0.628674986 | 0.774235465 | 0.901126856 |
| 8.315826182 | 0.279103335  | 0.139520709 | 0.304363203 | 0.421571336  | 0.517227801 | 0.60305654  | 0.743817361 | 0.868891941 |
| 8.86168405  | 0.364442786  | 0.112369219 | 0.279927603 | 0.397115664  | 0.492136373 | 0.574406823 | 0.716271973 | 0.839897027 |
| 9.405925174 | 0.477095387  | 0.085913814 | 0.256852693 | 0.374329393  | 0.468945182 | 0.550606874 | 0.691177172 | 0.813676864 |
| 9.948458968 | 0.646240622  | 0.059901318 | 0.234906203 | 0.352958499  | 0.447376512 | 0.528605637 | 0.668192774 | 0.789857457 |
| 10.48919664 | 0.976923739  | 0.034100943 | 0.21389453  | 0.332793866  | 0.42720257  | 0.508157338 | 0.647040686 | 0.768134192 |
| 11.02805126 | 3.135508825  | 0.008294387 | 0.193635669 | 0.313661048  | 0.408234137 | 0.489058222 | 0.627491723 | 0.748256044 |
| 12.09977337 | 0.044199529  | 0.154937676 | 0.277923075 | 0.373301806  | 0.454250295 | 0.592462966 | 0.713232235 |             |
| 13.16296975 | 0.099401762  | 0.117819966 | 0.244799027 | 0.341567941  | 0.423098726 | 0.561893811 | 0.68347539  |             |
| 14.21701869 | 0.15951649   | 0.061539392 | 0.213571077 | 0.312276803  | 0.394799152 | 0.53489002  | 0.658030618 |             |
| 15.26133387 | 0.227603192  | 0.045436657 | 0.183672231 | 0.284844265  | 0.368734431 | 0.51077068  | 0.636182293 |             |
| 16.2953656  | 0.308491764  | 0.006903038 | 0.154637318 | 0.258804789  | 0.344417147 | 0.489003281 | 0.617381013 |             |
| 17.31860169 | 0.410920668  | 0.028677892 | 0.126070177 | 0.233776728  | 0.32145213  | 0.46915873  | 0.601196048 |             |
| 18.33056816 | 0.553694048  | 0.067946644 | 0.097620512 | 0.209438826  | 0.299511239 | 0.45088438  | 0.587283413 |             |
| 19.33082946 | 0.790026192  | 0.109645818 | 0.068966521 | 0.185513787  | 0.27831593  | 0.433882964 | 0.575363822 |             |
| 20.31898861 | 1.397335367  | 0.154674503 | 0.039800849 | 0.1617564    | 0.257624913 | 0.417898543 | 0.565207115 |             |
| 21.29468693 | 0.204192965  | 0.009818093 | 0.137944666 | 0.237225217  | 0.402706162 | 0.556621007 |             |             |
| 22.25760362 | 0.259789376  | 0.021297493 | 0.113872885 | 0.216925613  | 0.388104207 | 0.549442815 |             |             |
| 23.20745509 | 0.3237744861   | 0.053885505 | 0.089346006 | 0.196551654  | 0.373906651 | 0.54333257  |             |             |
| 24.14399409 | 0.399751449  | 0.086325583 | 0.064174704 | 0.175941872  | 0.359948693 | 0.538771721 |             |             |
| 25.0670087  | 0.493822725  | 0.125058472 | 0.038170786 | 0.154944791  | 0.346063407 | 0.535072589 |             |             |
| 25.97632114 | 0.617548863  | 0.16461503  | 0.01114255  | 0.133416496  | 0.332099184 | 0.532253222 |             |             |
| 26.87178647 | 0.796734086  | 0.207658286 | 0.017110282 | 0.111218568  | 0.317907799 | 0.530377085 |             |             |
| 27.75329115 | 1.107735594  | 0.255047198 | 0.046802466 | 0.088216243  | 0.303344998 | 0.52260752  |             |             |
| 28.6207516  | 1.995833253  | 0.307937823 | 0.078170521 | 0.064276623  | 0.288269544 | 0.528863163 |             |             |
| 29.47411263 |  |             | 0.367951986 | 0.1111482102 | 0.039266828 | 0.27254267  | 0.52911851  |             |
| 30.31334584 |  |             | 0.437475309 | 0.147048512  | 0.013051917 | 0.256027935 | 0.529963752 |             |

| Phi (deg)   | Bending Energy Contribution from diagonal hinge |             |             |              |             |             |             |             |
|-------------|---|-------------|-------------|--------------|-------------|-------------|-------------|-------------|
|             | 0.1   | 0.2         | 0.3         | 0.4          | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118 | 1145.946124                                     | 420.8299959 | 241.8748354 | 168.193388   | 130.2727281 | 108.1638254 | 84.97718887 | 74.15634844 |
| 0.014181182 | 362.5701341                                     | 133.1057044 | 76.49381879 | 53.1879415   | 41.19431666 | 34.20205006 | 26.86940771 | 23.44771179 |
| 0.028362362 | 256.5255121                                     | 94.14164753 | 54.09424428 | 37.60997068  | 29.12757702 | 24.18265437 | 18.99739054 | 16.57800704 |
| 0.035452951 | 229.5104577                                     | 84.21256736 | 48.38557935 | 33.63956896  | 26.05196005 | 21.62879859 | 16.99082192 | 14.82691348 |
| 0.049634125 | 194.0852882                                     | 71.18905779 | 40.897032   | 28.43094729  | 22.01701985 | 18.27827835 | 14.3582427  | 12.52949728 |
| 0.055831409 | 183.043662                                      | 67.128668   | 38.56205094 | 26.80676378  | 20.75877075 | 17.23342992 | 13.53725892 | 11.81303087 |
| 0.070905875 | 162.5263005                                     | 59.38172691 | 34.22163817 | 23.78744971  | 18.41962966 | 15.29096059 | 12.01093371 | 10.48100934 |
| 0.085087031 | 148.4527419                                     | 54.40304449 | 31.24282434 | 21.71512723  | 16.81405937 | 13.95761412 | 10.96319589 | 9.566641442 |
| 0.099268176 | 137.52145                                       | 50.37913478 | 28.92791335 | 20.10454177  | 15.56616215 | 12.92126065 | 10.14880364 | 8.855909187 |
| 0.101147683 | 136.2483954                                     | 49.9104094  | 28.65823828 | 19.916908    | 15.42077718 | 12.80051887 | 10.05391969 | 8.773102105 |
| 0.103743768 | 134.5473642                                     | 49.28407065 | 28.2978743  | 19.66617171  | 15.22649664 | 12.63916856 | 9.927123125 | 8.662444092 |
| 0.12146903  | 124.4352963                                     | 45.55972571 | 26.15486006 | 18.17500754  | 14.07104028 | 11.67953664 | 9.172980451 | 8.004283781 |
| 0.223324574 | 92.16380702                                     | 33.65703303 | 19.3023017  | 13.40537008  | 10.37446912 | 8.609049391 | 6.759635108 | 5.898038286 |
| 0.279154922 | 82.62861245                                     | 30.13178828 | 17.27097617 | 11.99078663  | 9.277783894 | 7.697913314 | 6.043364801 | 5.272856262 |
| 0.33498474  | 75.60866546                                     | 27.53219226 | 15.77214017 | 10.94667271  | 8.468139257 | 7.025157315 | 5.514398476 | 4.811154494 |
| 0.446642362 | 65.79394649                                     | 23.88865799 | 13.66956616 | 9.481270939  | 7.331457378 | 6.08046132  | 4.771453988 | 4.162649663 |
| 0.558296592 | 59.13465697                                     | 21.4075456  | 12.23600207 | 8.481451916  | 6.355577761 | 5.435440554 | 4.264031919 | 3.719695532 |
| 0.669946581 | 54.24884849                                     | 19.58016116 | 11.17879815 | 7.743603729  | 5.982738383 | 4.959075928 | 3.889173445 | 3.392438243 |
| 0.781591482 | 50.47596379                                     | 18.16321913 | 10.35796231 | 7.170313993  | 5.537456113 | 4.58867626  | 3.597611007 | 3.137880557 |
| 0.893230448 | 47.45534015                                     | 17.02381242 | 9.697000169 | 6.708347985  | 5.178476897 | 4.289976831 | 3.36241519  | 2.93252005  |
| 1.004862632 | 44.97123051                                     | 16.08239171 | 9.15011745  | 6.325832107  | 4.88109839  | 4.042459285 | 3.16745923  | 2.762282024 |
| 1.116487186 | 42.88565293                                     | 15.28805132 | 8.688002939 | 6.002362726  | 4.629506294 | 3.832987177 | 3.002417719 | 2.618154816 |
| 1.228103264 | 41.10572614                                     | 14.60650382 | 8.290909874 | 5.724192354  | 4.413044447 | 3.652708288 | 2.860332313 | 2.494065335 |
| 1.339710021 | 39.56634408                                     | 14.01369967 | 7.944984713 | 5.481675113  | 4.224235387 | 3.495410651 | 2.736319781 | 2.38575155  |
| 1.45130661  | 38.22034545                                     | 13.49220595 | 7.640183198 | 5.267816571  | 4.057636614 | 3.356589238 | 2.62683852  | 2.290122347 |
| 1.674465903 | 35.97644847                                     | 12.61429376 | 7.125794464 | 4.906465489  | 3.779585542 | 3.121742424 | 2.441538057 | 2.12824911  |
| 1.897574394 | 34.18003477                                     | 11.90092158 | 6.706327477 | 4.611290249  | 3.545661737 | 2.929580918 | 2.289817053 | 1.995690255 |
| 2.34361198  | 31.49076169                                     | 10.80484816 | 6.058173267 | 4.1539813    | 3.188268183 | 2.631106874 | 2.05392553  | 1.789547633 |
| 2.67795671  | 30.01309026                                     | 10.17845443 | 5.684924321 | 3.889727075  | 2.981338299 | 2.458078375 | 1.917006105 | 1.669865444 |
| 2.789365533 | 29.59616396                                     | 9.996773813 | 5.576143293 | 3.812550036  | 2.920830866 | 2.407445806 | 1.876911372 | 1.634813486 |
| 3.067793428 | 28.6879087                                      | 9.590032438 | 5.3315379   | 3.638686639  | 2.784377994 | 2.293188643 | 1.786377533 | 1.555657255 |
| 3.624201358 | 27.32773547                                     | 8.934688951 | 4.933340264 | 3.354453577  | 2.560782859 | 2.105697635 | 1.63761386  | 1.425559477 |
| 3.902155428 | 26.82689137                                     | 8.666991007 | 4.76868655  | 3.236362241  | 2.467645304 | 2.027478026 | 1.575460689 | 1.371193099 |
| 4.457499215 | 26.10626513                                     | 8.219982884 | 4.489952824 | 3.035435737  | 2.308753078 | 1.893825008 | 1.46910617  | 1.278146447 |
| 5.012005072 | 25.6997399                                      | 7.863569125 | 4.262783807 | 2.870424748  | 2.177751993 | 1.78338223  | 1.381041832 | 1.201086422 |
| 5.565571274 | 25.56700423                                     | 7.575438883 | 4.074136998 | 2.732201902  | 2.067546415 | 1.690244472 | 1.306617787 | 1.135954181 |
| 6.118097145 | 25.71127546                                     | 7.340688166 | 3.915238356 | 2.614617909  | 1.973352802 | 1.610429297 | 1.242697303 | 1.080011936 |
| 6.669483156 | 26.13072543                                     | 7.148989785 | 3.779944488 | 2.513359015  | 1.891812931 | 1.54113977  | 1.187076737 | 1.031336367 |
| 7.219631036 | 26.86024264                                     | 6.992991298 | 3.663813255 | 2.425298374  | 1.820490362 | 1.480345736 | 1.138156055 | 0.988531987 |
| 7.768443868 | 27.97538377                                     | 6.867359471 | 3.563547102 | 2.348107336  | 1.757568601 | 1.426532794 | 1.09474126  | 0.950556658 |
| 8.315826182 | 29.61507154                                     | 6.768186495 | 3.476644505 | 2.280011908  | 1.701661942 | 1.37854495  | 1.055920553 | 0.916615467 |
| 8.86168405  | 32.04273943                                     | 6.692609442 | 3.401173521 | 2.219634381  | 1.631692421 | 1.335482256 | 1.02098373  | 0.886089358 |
| 9.405925174 | 35.81081698                                     | 6.63856145  | 3.335619983 | 2.165887033  | 1.606807207 | 1.29663206  | 0.989368067 | 0.858488078 |
| 9.948458968 | 42.31822835                                     | 6.604608123 | 3.278782999 | 2.117898816  | 1.566321594 | 1.261421576 | 0.960620947 | 0.833417549 |
| 10.48919664 | 56.73804093                                     | 6.58984193  | 3.229701357 | 2.074963591  | 1.529678738 | 1.229384358 | 0.934373466 | 0.810556817 |
| 11.02805126 | 163.6169749                                     | 6.593818581 | 3.187600692 | 2.036502806  | 1.496420603 | 1.200136113 | 0.910321366 | 0.789641408 |
| 12.09977337 |   | 6.676070436 | 3.121957848 | 1.971170869  | 1.438597747 | 1.148777973 | 0.867829008 | 0.752800755 |
| 13.16296975 |   | 6.843163429 | 3.078161489 | 1.918944003  | 1.390477084 | 1.103542889 | 0.831551141 | 0.721515007 |
| 14.21701869 |   | 7.106233006 | 3.053872959 | 1.877723858  | 1.350342087 | 1.068373147 | 0.800323956 | 0.694778558 |
| 15.26133387 |   | 7.492888597 | 3.047762242 | 1.846001588  | 1.316922415 | 1.036779595 | 0.773272009 | 0.671838033 |
| 16.2953656  |   | 8.05442016  | 3.059327913 | 1.822690392  | 1.289260721 | 1.009730208 | 0.74972008  | 0.652115317 |
| 17.31860169 |   | 8.892114016 | 3.100946865 | 1.807019585  | 1.266626567 | 0.986577777 | 0.729135817 | 0.635157468 |
| 18.33056816 |   | 10.23513272 | 3.165965    | 1.798467161  | 1.248459144 | 0.966811542 | 0.711091231 | 0.620603082 |
| 19.33082946 |   | 12.75343463 | 3.25281087  | 1.7965717902 | 1.234328535 | 0.95002405  | 0.695236139 | 0.608159069 |
| 20.31898861 |   | 20.03458246 | 3.364918137 | 1.801639701  | 1.223909072 | 0.935887978 | 0.68127943  | 0.597584364 |
| 21.29468693 |   |             | 3.507483331 | 1.813274218  | 1.216961076 | 0.924139693 | 0.668975593 | 0.588677911 |
| 22.25760362 |   |             | 3.688387741 | 1.841474709  | 1.213318532 | 0.914567467 | 0.658114862 | 0.581270188 |
| 23.20745509 |   |             | 3.919895505 | 1.882087414  | 1.212881215 | 0.907003019 | 0.648515924 | 0.575216621 |
| 24.14399409 |   |             | 4.221992715 | 1.931412533  | 1.215610338 | 0.901315469 | 0.640020478 | 0.570392693 |
| 25.0670087  |   |             | 4.629598063 | 1.99048525   | 1.221527207 | 0.897407121 | 0.632489148 | 0.566690148 |
| 25.97632114 |   |             | 5.210333459 | 2.060723311  | 1.230714714 | 0.895210635 | 0.625798443 | 0.564014033 |
| 26.87178647 |   |             | 6.11782074  | 2.144066298  | 1.251715337 | 0.89468747  | 0.619838511 | 0.562280325 |
| 27.75329115 |   |             | 7.813798462 | 2.243194646  | 1.282500645 | 0.895826991 | 0.614511547 | 0.561414012 |
| 28.6207516  |   |             | 13.03562996 | 2.361884738  | 1.317972453 | 0.89864686  | 0.609730739 | 0.561347478 |
| 29.47411263 |   |             |             | 2.505609     | 1.358605393 | 0.903193977 | 0.605419667 | 0.562019105 |
| 30.31334584 |   |             |             | 2.682606168  | 1.405010135 | 0.909546413 | 0.601512122 | 0.563372005 |

| Phi (deg)   | Membrane Energy Contribution (tests 3,4) |             |             |              |             |             |             |             |
|-------------|--|-------------|-------------|--------------|-------------|-------------|-------------|-------------|
|             | 0.1                                      | 0.2         | 0.3         | 0.4          | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118 | 40.74016291                              | 64.75303591 | 85.163388   | 101.5974246  | 114.1200982 | 123.0938282 | 132.4994918 | 134.0458593 |
| 0.014181182 | 40.73986622                              | 64.75189894 | 85.16098098 | 101.593451   | 114.1143894 | 123.0863228 | 132.4885044 | 134.0317958 |
| 0.028362362 | 40.73954127                              | 64.75064466 | 85.15831925 | 101.5890516  | 114.1080644 | 123.0780034 | 132.4763178 | 134.016192  |
| 0.035452951 | 40.7393065                               | 64.75002107 | 85.15699339 | 101.5868581  | 114.104909  | 123.0738515 | 132.4702331 | 134.0063988 |
| 0.049634125 | 40.7390631                               | 64.74878097 | 85.15435169 | 101.5824835  | 114.0986125 | 123.0655632 | 132.4580809 | 133.9928301 |
| 0.055831409 | 40.73892586                              | 64.74824201 | 85.15320143 | 101.580577   | 114.0958668 | 123.0619478 | 132.4527775 | 133.9860338 |
| 0.070905875 | 40.73859594                              | 64.74693851 | 85.15041412 | 101.5759526  | 114.0892034 | 123.05317   | 132.4398954 | 133.969521  |
| 0.085087031 | 40.73829059                              | 64.74572195 | 85.14780571 | 101.5716193  | 114.0829543 | 123.0469338 | 132.4278003 | 133.9540108 |
| 0.099268176 | 40.73799008                              | 64.74451476 | 85.14521057 | 101.5673024  | 114.0767242 | 123.0367183 | 132.4157281 | 133.938524  |
| 0.101147683 | 40.73795051                              | 64.74435546 | 85.14486761 | 101.5667315  | 114.0758999 | 123.035631  | 132.4141296 | 133.9364732 |
| 0.103743768 | 40.73789623                              | 64.74413571 | 85.14439429 | 101.5659435  | 114.0747619 | 123.0341298 | 132.4119228 | 133.9336412 |
| 0.12146903  | 40.73752926                              | 64.74264364 | 85.14117443 | 101.5605773  | 114.0670089 | 123.0238986 | 132.3968745 | 133.9143261 |
| 0.223324574 | 40.73556336                              | 64.73435015 | 85.1230708  | 101.5302374  | 114.023028  | 122.9657325 | 132.3110907 | 133.8040405 |
| 0.279154922 | 40.73458721                              | 64.73000479 | 85.11343385 | 101.5139634  | 113.9993315 | 122.9343009 | 132.2645662 | 133.7440986 |
| 0.33498474  | 40.73368054                              | 64.72579964 | 85.10399792 | 101.4979403  | 113.9759247 | 122.9031875 | 132.2183926 | 133.6845166 |
| 0.446642362 | 40.73207015                              | 64.71780416 | 85.08572368 | 101.4666415  | 113.9299752 | 122.8419114 | 132.1270943 | 133.5664399 |
| 0.558296592 | 40.7307202                               | 64.7103521  | 85.06823717 | 101.4363308  | 113.8851703 | 122.7818956 | 132.0371887 | 133.4497747 |
| 0.669946581 | 40.72961853                              | 64.70343185 | 85.05152747 | 101.4069981  | 113.8415006 | 122.7231318 | 131.9486688 | 133.3345453 |
| 0.781591482 | 40.72875296                              | 64.6970318  | 85.03558368 | 101.3786333  | 113.7989569 | 122.6656114 | 131.8615277 | 133.2207361 |
| 0.893230448 | 40.72811133                              | 64.69114034 | 85.02039488 | 101.3152263  | 113.7575298 | 122.6093259 | 131.7757582 | 133.1083411 |
| 1.004862632 | 40.72768147                              | 64.68574584 | 85.0095017  | 101.3247669  | 113.71721   | 122.5542669 | 131.6913536 | 132.9973548 |
| 1.116487186 | 40.72745121                              | 64.68083671 | 84.99223863 | 101.299245   | 113.6779882 | 122.5004257 | 131.6083067 | 132.8877714 |
| 1.228103264 | 40.72740837                              | 64.67640133 | 84.97924936 | 101.2746506  | 113.6398552 | 122.4477941 | 131.5266106 | 132.7795851 |
| 1.339710021 | 40.72754078                              | 64.67242809 | 84.96697144 | 101.2509734  | 113.6028016 | 122.3963633 | 131.4462582 | 132.6727903 |
| 1.45130661  | 40.72783628                              | 64.66890538 | 84.95539397 | 101.2282034  | 113.566818  | 122.346125  | 131.3672426 | 132.5673812 |
| 1.674465905 | 40.72886785                              | 64.6631651  | 84.93429671 | 101.1853443  | 113.4980241 | 122.2491918 | 131.2131939 | 132.3606973 |
| 1.897754394 | 40.73040571                              | 64.65908759 | 84.91587031 | 101.1459922  | 113.4333989 | 121.1562625 | 131.0644084 | 132.1594875 |
| 2.34361198  | 40.73461085                              | 64.65554932 | 84.88668093 | 101.0774855  | 113.3163571 | 121.9861277 | 130.7824033 | 131.7733072 |
| 2.67795671  | 40.73836958                              | 64.65656951 | 84.87115071 | 101.0347092  | 113.2389651 | 121.8697765 | 130.5842996 | 131.4977009 |
| 2.789365533 | 40.73967279                              | 64.65754743 | 84.86712756 | 101.0220353  | 113.2150971 | 121.8331852 | 130.5207801 | 131.4064738 |
| 3.067793428 | 40.7429255                               | 64.6612775  | 84.85948973 | 101.09937212 | 113.1595591 | 121.7464222 | 130.3674168 | 131.1911304 |
| 3.624201358 | 40.74876155                              | 64.67362899 | 84.8540061  | 100.9510014  | 113.0656979 | 121.5926535 | 130.0836139 | 130.7806721 |
| 3.902155428 | 40.75096456                              | 64.68188756 | 84.85581935 | 100.9362794  | 113.0270639 | 121.5253825 | 129.9529557 | 130.5873783 |
| 4.457499215 | 40.75298972                              | 64.70166318 | 84.86770362 | 100.91932    | 112.9657621 | 121.4094035 | 129.7135797 | 130.2242141 |
| 5.012005072 | 40.75057267                              | 64.72457393 | 84.88946175 | 100.9179521  | 112.9246793 | 121.3172906 | 129.5027292 | 129.8916846 |
| 5.565571274 | 40.74219208                              | 64.74916835 | 84.91972995 | 100.9309104  | 112.9026723 | 121.247982  | 129.3195303 | 129.589074  |
| 6.118097145 | 42.99708596                              | 64.77399499 | 84.95714443 | 100.9569292  | 112.8957779 | 121.2004161 | 129.163109  | 129.3156666 |
| 6.669948156 | 45.69921987                              | 64.79760243 | 85.00034138 | 100.9947429  | 112.9112331 | 121.1735313 | 129.0325914 | 129.0707466 |
| 7.219631036 | 48.39788124                              | 64.81853921 | 85.04795702 | 101.043086   | 112.9394747 | 121.1662659 | 128.9271036 | 128.8535982 |
| 7.768443868 | 51.11559138                              | 64.83533389 | 85.09862756 | 101.1006929  | 112.9821394 | 121.1775583 | 128.8457716 | 128.6635056 |
| 8.315826182 | 53.84687159                              | 64.84659503 | 85.1509892  | 101.1662982  | 113.0380641 | 121.2063467 | 128.7877215 | 128.4997532 |
| 8.86168405  | 56.59324318                              | 64.85081118 | 85.20367816 | 101.2386362  | 113.1060857 | 121.2515696 | 128.7520792 | 128.3616251 |
| 9.405925174 | 59.35622746                              | 64.84655091 | 85.25533063 | 101.3164414  | 113.185041  | 121.3121652 | 128.737971  | 128.2484056 |
| 9.948458968 | 62.13734575                              | 64.83236276 | 85.30485828 | 101.3984482  | 113.2737668 | 121.387072  | 128.7445228 | 128.1593789 |
| 10.48919664 | 64.93811934                              | 64.80679529 | 85.35007095 | 101.4833911  | 113.3711001 | 121.4752282 | 128.7708606 | 128.0938293 |
| 11.02805126 | 67.76006955                              | 64.76839707 | 85.39043121 | 101.5700045  | 113.4758775 | 121.5755722 | 128.8161107 | 128.0510409 |
| 12.09977337 |  | 69.13904442 | 85.45031299 | 101.7431808  | 113.7031125 | 121.8065771 | 128.9598514 | 128.030885  |
| 13.16296975 |  | 74.42414302 | 85.47331781 | 101.9078525  | 113.9461664 | 122.0775934 | 129.1687533 | 128.0931851 |
| 14.21701869 |  | 79.78959354 | 85.44853533 | 102.0538953  | 114.195734  | 122.3741278 | 129.4358249 | 128.232215  |
| 15.26133387 |  | 85.2470075  | 85.36350521 | 102.1711847  | 114.4425101 | 122.689687  | 129.7540747 | 128.4422487 |
| 16.2953656  |  | 90.80799647 | 85.21196711 | 102.2495964  | 114.6771892 | 123.0157779 | 130.1165111 | 128.7175601 |
| 17.31860169 |  | 96.48417198 | 88.9875032  | 102.2790058  | 114.8904662 | 123.3439072 | 130.5161426 | 129.052423  |
| 18.33056816 |  | 102.2871456 | 94.10010628 | 102.2492886  | 115.0793358 | 123.6655815 | 130.9459775 | 129.4411113 |
| 19.33082946 |  | 108.2285288 | 99.31504836 | 102.1503203  | 115.2155927 | 123.9723078 | 131.3990244 | 129.877899  |
| 20.31898861 |  | 114.3199333 | 104.6432398 | 101.971966   | 115.3088317 | 124.2555927 | 131.8682917 | 130.3570399 |
| 21.29468693 |  |             | 110.0955909 | 101.7041329  | 115.3434473 | 124.5069493 | 132.3467878 | 130.8728678 |
| 22.25760362 |  |             | 115.6830121 | 104.8801044  | 115.3101344 | 124.7178655 | 132.8275213 | 131.4195968 |
| 23.20745509 |  |             | 121.4164136 | 109.8000202  | 115.1995877 | 124.8798668 | 133.3035004 | 131.9915206 |
| 24.14399409 |  |             | 127.3067059 | 114.8398092  | 115.0025018 | 124.9844538 | 133.7677338 | 132.5829133 |
| 25.0670087  |  |             | 133.3647992 | 120.0095958  | 114.7095716 | 125.0231332 | 134.2132298 | 133.1880486 |
| 25.97632114 |  |             | 139.601604  | 125.3195044  | 114.3114917 | 124.9874117 | 134.632997  | 133.8012004 |
| 26.87178647 |  |             | 146.0280304 | 130.7796595  | 116.8924604 | 124.8687962 | 135.0200436 | 134.4166427 |
| 27.75329115 |  |             | 152.654989  | 136.4001855  | 121.6120062 | 124.6387932 | 135.3673783 | 135.0286493 |
| 28.6207516  |  |             | 159.49339   | 142.1912067  | 126.4646174 | 124.3489097 | 135.6680093 | 135.6149451 |
| 29.47411263 |  |             |             | 148.1628477  | 131.4595994 | 123.9306524 | 135.9149453 | 136.2194511 |
| 30.31334584 |  |             |             | 154.3252328  | 136.6062575 | 123.395528  | 136.1011946 | 136.786794  |

| Phi (deg)   | lambda>     | Total Analytical NDSF (tests 3,4) |             |             |             |             |             |             |             |
|-------------|-------------|-----------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
|             |             | 0.1                               | 0.2         | 0.3         | 0.4         | 0.5         | 0.6         | 0.8         | 1           |
| 0.001418118 | 1249.458037 | 537.103303                        | 378.8071627 | 324.1097224 | 301.9170295 | 292.1657148 | 285.1103797 | 282.240971  |             |
| 0.014181182 | 450.8613584 | 227.858643                        | 187.0703686 | 178.6712621 | 178.8129847 | 180.9300206 | 183.9608366 | 183.4069474 |             |
| 0.028362362 | 342.7522606 | 185.9772303                       | 161.0977548 | 158.9667315 | 162.1315242 | 165.8542585 | 170.2478854 | 170.0070483 |             |
| 0.035452951 | 315.2106279 | 175.3042105                       | 154.4777882 | 153.9435736 | 157.8782838 | 162.0097369 | 166.7497388 | 166.5873501 |             |
| 0.049634125 | 279.0943001 | 161.3043601                       | 145.7929733 | 147.3525708 | 152.2965136 | 156.9634094 | 162.1564388 | 162.0957818 |             |
| 0.055831409 | 267.8370569 | 156.9394042                       | 143.0847084 | 145.2968677 | 150.5552308 | 155.3888229 | 160.7226184 | 160.6932512 |             |
| 0.070905875 | 246.9187407 | 148.8260949                       | 138.0498983 | 141.4744917 | 147.3167915 | 152.4597316 | 158.0542047 | 158.082133  |             |
| 0.085087031 | 232.5698954 | 143.285178                        | 134.5940041 | 138.8500657 | 145.092553  | 150.4472389 | 156.2195456 | 156.2858546 |             |
| 0.099268176 | 221.4246182 | 138.9322577                       | 131.9079454 | 136.8096605 | 143.3626793 | 148.8814575 | 154.7910802 | 154.8864373 |             |
| 0.101147683 | 220.1266331 | 138.428302                        | 131.5950062 | 136.571901  | 143.1610618 | 148.5998227 | 154.6244784 | 154.7231634 |             |
| 0.103743768 | 218.3922872 | 137.7548826                       | 131.176819  | 136.2541624 | 142.8916071 | 148.4549558 | 154.4017789 | 154.5048898 |             |
| 0.12146903  | 208.082098  | 133.7504921                       | 128.689681  | 134.3640204 | 141.2882664 | 147.0028531 | 153.0755087 | 153.2043741 |             |
| 0.223324574 | 175.1776392 | 120.951608                        | 120.7321997 | 128.3087708 | 136.1435878 | 142.3352191 | 148.7975576 | 148.9975555 |             |
| 0.279154922 | 165.455464  | 117.1606513                       | 118.3711347 | 126.5079086 | 134.6090052 | 140.9383464 | 147.5090213 | 147.7236947 |             |
| 0.33498474  | 158.2980675 | 114.3651442                       | 116.6279432 | 125.1760448 | 133.4715715 | 139.9004265 | 146.5469337 | 146.7687821 |             |
| 0.446642362 | 148.2918858 | 110.4474151                       | 114.1805633 | 123.3012102 | 131.8648942 | 138.4286098 | 145.1721909 | 145.3957378 |             |
| 0.558296592 | 141.503621  | 107.7803346                       | 112.5101838 | 122.0166081 | 130.7583263 | 137.4090004 | 144.2089042 | 144.4247491 |             |
| 0.669946581 | 136.5240405 | 105.8168666                       | 111.2772984 | 121.0645827 | 129.9337259 | 136.6444688 | 143.4778264 | 143.6806458 |             |
| 0.781591482 | 132.6794883 | 104.295348                        | 110.3194696 | 120.3218413 | 129.2867012 | 136.0406759 | 142.893177  | 143.0796305 |             |
| 0.893230448 | 129.6021171 | 103.0728085                       | 109.5479196 | 119.7209909 | 128.7601865 | 135.5460499 | 142.4080632 | 142.5795655 |             |
| 1.004862632 | 127.0718608 | 102.0636596                       | 108.9094878 | 119.2216705 | 128.3200032 | 135.1297017 | 141.994399  | 142.1421296 |             |
| 1.116487186 | 124.94796   | 101.2131198                       | 108.3701389 | 118.7980355 | 127.9442641 | 134.7718502 | 141.6341982 | 141.7606023 |             |
| 1.228103264 | 123.1356528 | 100.4842826                       | 107.9069363 | 118.4326663 | 127.6182155 | 134.4591611 | 141.3153373 | 141.4195337 |             |
| 1.339710021 | 121.5685061 | 99.85125451                       | 107.5037826 | 118.1133355 | 127.3315006 | 134.1822736 | 141.0293053 | 141.1106123 |             |
| 1.45130661  | 120.1983902 | 99.29526017                       | 107.1490019 | 117.8311725 | 127.0766053 | 133.9343959 | 140.7699276 | 140.8278083 |             |
| 1.674465905 | 117.9144609 | 98.36173231                       | 106.5518092 | 117.3534625 | 126.6411884 | 133.5066119 | 140.3138093 | 140.3236363 |             |
| 1.897574394 | 116.0857945 | 97.60625157                       | 106.0671525 | 116.9628947 | 126.2809188 | 133.1477463 | 139.9214727 | 139.8821138 |             |
| 2.34361198  | 113.3458769 | 96.45355575                       | 105.325855  | 116.3598332 | 125.7151762 | 132.572909  | 139.2701055 | 139.130379  |             |
| 2.67795671  | 111.8366862 | 95.80121616                       | 104.9061766 | 116.0152266 | 125.3855119 | 132.2299244 | 138.8647095 | 138.6487885 |             |
| 2.789365533 | 111.4098409 | 95.61318057                       | 104.785364  | 115.9156268 | 125.2892031 | 132.1283641 | 138.74176   | 138.5003587 |             |
| 3.067793428 | 110.4772711 | 95.19452838                       | 104.5169853 | 115.6938581 | 125.0728987 | 131.8976575 | 138.4567384 | 138.1515703 |             |
| 3.624201358 | 109.0672236 | 94.52821101                       | 104.093357  | 115.3432522 | 124.7249553 | 131.5173854 | 137.9659327 | 137.5334087 |             |
| 3.902155428 | 108.539288  | 94.25956607                       | 103.9248664 | 115.2042715 | 124.5846912 | 131.3600194 | 137.7530845 | 137.2571792 |             |
| 4.457499215 | 107.7572314 | 93.81629364                       | 103.6322921 | 114.9818557 | 124.3570156 | 131.0976549 | 137.3805912 | 136.7589009 |             |
| 5.012005072 | 107.276489  | 93.46768215                       | 103.4458976 | 114.8182759 | 124.1868121 | 130.8933531 | 137.0684363 | 136.3226317 |             |
| 5.565571274 | 107.0535071 | 93.18811257                       | 103.2890333 | 114.7005764 | 124.0629873 | 130.7370732 | 136.8076035 | 135.9397922 |             |
| 6.118097145 | 109.5865093 | 92.95996036                       | 103.1700247 | 114.6195867 | 123.9776561 | 130.6216745 | 136.5917125 | 135.6042568 |             |
| 6.669483156 | 112.8962575 | 92.77054353                       | 103.0802673 | 114.5684714 | 123.9249121 | 130.5418051 | 136.4160088 | 135.3115664 |             |
| 7.219631036 | 116.5487871 | 92.61039379                       | 103.0131438 | 114.5419066 | 123.9001298 | 130.4932698 | 136.2767889 | 135.0582685 |             |
| 7.768443868 | 120.6233172 | 92.47222437                       | 102.9633766 | 114.5355867 | 123.8995455 | 130.4726521 | 136.1710563 | 134.8416007 |             |
| 8.315826182 | 125.2635865 | 92.3502877                        | 102.9266213 | 114.5459146 | 123.9199953 | 130.4770768 | 136.0963055 | 134.6592797 |             |
| 8.86168405  | 130.7403106 | 92.23996255                       | 102.899203  | 114.5698008 | 123.9587443 | 130.5040551 | 136.0503816 | 134.5093647 |             |
| 9.405925174 | 137.6190709 | 92.13748295                       | 102.8779389 | 114.6045287 | 124.013372  | 130.5513815 | 136.031386  | 134.3901648 |             |
| 9.948458968 | 147.3291394 | 92.03973837                       | 102.8600169 | 114.6476604 | 124.0816934 | 130.6170622 | 136.0376107 | 134.300176  |             |
| 10.48919664 | 165.149768  | 91.94425577                       | 102.8429088 | 114.6969721 | 124.161703  | 130.6992642 | 136.0674932 | 134.2380354 |             |
| 11.02805126 | 277.2951532 | 91.8489258                        | 102.8243093 | 114.7504059 | 124.2515348 | 130.7962794 | 136.1195825 | 134.2024896 |             |
| 12.09977337 |             | 96.53806257                       | 102.7742664 | 114.8620402 | 124.4537279 | 131.0283856 | 136.2849966 | 134.2065765 |             |
| 13.16296975 |             | 102.3327672                       | 102.6944403 | 114.9683047 | 124.6751812 | 131.3013469 | 136.523695  | 134.3038217 |             |
| 14.21701869 |             | 108.3444083                       | 102.5710102 | 115.0561002 | 124.9037607 | 131.6039712 | 136.826255  | 134.4863251 |             |
| 15.26133387 |             | 114.6136746                       | 102.3913586 | 115.1131409 | 125.1279979 | 131.9256441 | 137.1837449 | 134.7466622 |             |
| 16.2953656  |             | 121.2064799                       | 102.1438336 | 115.1277399 | 125.3369074 | 132.2561645 | 137.5875729 | 135.0777339 |             |
| 17.31860169 |             | 128.2424404                       | 106.1522185 | 115.0886708 | 125.5198703 | 132.3586396 | 138.0293887 | 135.4726689 |             |
| 18.33056816 |             | 135.9789374                       | 111.641189  | 114.9850771 | 125.6665562 | 132.9044163 | 138.5010195 | 135.9247585 |             |
| 19.33082946 |             | 145.1484093                       | 117.2801992 | 114.8064108 | 125.7668711 | 133.2030343 | 138.9944258 | 136.4274119 |             |
| 20.3189861  |             | 159.6249815                       | 123.0831725 | 114.5423924 | 125.8109201 | 133.4721954 | 139.5016723 | 136.9741247 |             |
| 21.29468693 |             |                                   | 129.0659136 | 114.1829861 | 125.7889811 | 133.7027414 | 140.0149079 | 137.5584571 |             |
| 22.25760362 |             |                                   | 135.2472141 | 117.5449214 | 125.6914858 | 133.8856387 | 140.5263513 | 138.1740175 |             |
| 23.20745509 |             |                                   | 141.6508655 | 122.8066897 | 125.5090052 | 134.0119676 | 141.0282828 | 138.8144522 |             |
| 24.14399409 |             |                                   | 148.3095855 | 128.2163847 | 125.2322384 | 134.0729143 | 141.5130384 | 139.4734365 |             |
| 25.0670087  |             |                                   | 155.2734601 | 133.7844064 | 124.8520042 | 134.0597645 | 141.9730059 | 140.1446699 |             |
| 25.97632114 |             |                                   | 162.6306896 | 139.5215244 | 124.3592348 | 133.9638988 | 142.4006245 | 140.8218722 |             |
| 26.87178647 |             |                                   | 170.569693  | 145.4390748 | 127.0892879 | 133.7767885 | 142.7883835 | 141.4987816 |             |
| 27.75329115 |             |                                   | 179.6375527 | 151.5492589 | 132.1403889 | 133.4899916 | 143.128824  | 142.1691533 |             |
| 28.6207516  |             |                                   | 193.0259115 | 157.8656158 | 137.3449557 | 133.0951495 | 143.4145397 | 142.8267597 |             |
| 29.47411263 |             |                                   |             |             | 164.4038078 | 142.71189   | 132.5839833 | 143.6381793 | 143.4653902 |
| 30.31334584 |             |                                   |             |             | 171.1830051 | 148.2501936 | 131.9482906 | 143.7924489 | 144.0788525 |

Tabulation of Elastic and NBPF non-dimensional shear

| Phi (deg)   | NBPF for Test 3,4 | NBPF for Tests 5,6,7 | Phi (deg)   | Elastic Model NDSF |             |
|-------------|-------------------|----------------------|-------------|--------------------|-------------|
|             |                   |                      |             | Test 3,4           | Test 5,6,7  |
| 0.001418118 | 81.46035554       | 88.14507937          | 0.001418118 | 0.448953995        | 0.440782574 |
| 0.014181182 | 81.46036542       | 88.14509007          | 0.014181182 | 4.499539863        | 4.407825651 |
| 0.028362362 | 81.46039536       | 88.14512246          | 0.028362362 | 8.979079176        | 8.815650763 |
| 0.035452951 | 81.46041782       | 88.14514676          | 0.035452951 | 11.22384845        | 11.01956295 |
| 0.049634125 | 81.4604777        | 88.14521156          | 0.049634125 | 15.71338591        | 15.42738624 |
| 0.055831409 | 81.46051014       | 88.14524666          | 0.055831409 | 17.67534856        | 17.35363916 |
| 0.070905875 | 81.46060496       | 88.14534926          | 0.070905875 | 22.44768831        | 22.08911746 |
| 0.085087031 | 81.46071474       | 88.14546805          | 0.085087031 | 26.93721993        | 26.44693501 |
| 0.099268176 | 81.46084449       | 88.14560845          | 0.099268176 | 31.42674824        | 30.85474932 |
| 0.101147683 | 81.46086318       | 88.14562867          | 0.101147683 | 32.0217706         | 31.43894167 |
| 0.103743768 | 81.46088958       | 88.14565724          | 0.103743768 | 32.8436505         | 32.24586251 |
| 0.12146903  | 81.46104877       | 88.14587161          | 0.12146903  | 38.45519038        | 35.00056457 |
| 0.223324574 | 81.46283063       | 88.14775757          | 0.223324574 | 54.57812448        | 50.83004479 |
| 0.279154922 | 81.46422292       | 88.14926412          | 0.279154922 | 63.41563093        | 59.5066996  |
| 0.33498474  | 81.46592462       | 88.15110545          | 0.33498474  | 72.25305346        | 68.18327201 |
| 0.446642362 | 81.4702562        | 88.15579249          | 0.446642362 | 89.92757967        | 85.53610379 |
| 0.558296592 | 81.47582538       | 88.16181868          | 0.558296592 | 107.6015689        | 102.8884083 |
| 0.669946581 | 81.48263215       | 88.16918402          | 0.669946581 | 125.2748869        | 120.2400539 |
| 0.781591482 | 81.49067651       | 88.17788852          | 0.781591482 | 142.9473993        | 137.5909087 |
| 0.893230448 | 81.49995847       | 88.18793217          | 0.893230448 | 160.6189726        | 154.9408412 |
| 1.004862632 | 81.51047803       | 88.19931497          | 1.004862632 | 178.2894721        | 172.2897195 |
| 1.116487186 | 81.52223518       | 88.21203692          | 1.116487186 | 195.9587639        | 189.6374122 |
| 1.228103264 | 81.53522992       | 88.22609803          | 1.228103264 | 213.6267141        | 206.9837876 |
| 1.339710021 | 81.54946226       | 88.24149829          | 1.339710021 | 231.2931887        | 224.3287144 |
| 1.45130661  | 81.5649322        | 88.2582377           | 1.45130661  | 248.9580539        | 241.6720609 |
| 1.674465905 | 81.59958485       | 88.29573399          | 1.674465905 | 284.2824208        | 276.3534883 |
| 1.897574394 | 81.63918788       | 88.33858689          | 1.897574394 | 319.5987456        | 311.0270199 |
| 2.34361198  | 81.73324509       | 88.44036253          | 2.34361198  | 390.2029988        | 380.3462036 |
| 2.67795671  | 81.81678273       | 88.53075536          | 2.67795671  | 443.1271484        | 432.3070796 |
| 2.789365533 | 81.8471038        | 88.56356461          | 2.789365533 | 460.7622917        | 449.6212453 |
| 3.067793428 | 81.92832096       | 88.65144654          | 3.067793428 | 504.8352447        | 492.8920255 |
| 3.624201358 | 82.11396018       | 88.8523195           | 3.624201358 | 592.9102581        | 579.3639837 |
| 3.902155428 | 82.21838223       | 88.96531055          | 3.902155428 | 636.9082083        | 622.5611262 |
| 4.457499215 | 82.45043125       | 89.21640175          | 4.457499215 | 724.8147762        | 708.8677048 |
| 5.012005072 | 82.71342014       | 89.50097179          | 5.012005072 | 812.5887062        | 795.0440596 |
| 5.565571274 | 83.00734889       | 89.81902063          | 5.565571274 | 900.2138964        | 881.0743818 |
| 6.118097145 | 83.33221752       | 90.17054835          | 6.118097145 | 987.6744101        | 966.9430248 |
| 6.669483156 | 83.68802601       | 90.55555487          | 6.669483156 | 1074.954493        | 1052.634521 |
| 7.219631036 | 84.07477438       | 90.97404021          | 7.219631036 | 1162.03859         | 1138.133598 |
| 7.768443868 | 84.49246261       | 91.42600439          | 7.768443868 | 1248.911358        | 1223.425194 |
| 8.315826182 | 84.94109071       | 91.91144739          | 8.315826182 | 1335.557687        | 1308.494471 |
| 8.86168405  | 85.42065868       | 92.43036922          | 8.86168405  | 1421.962709        | 1393.326833 |
| 9.405925174 | 85.93116652       | 92.98276988          | 9.405925174 | 1508.111812        | 1477.907934 |
| 9.948458968 | 86.47261423       | 93.56864937          | 9.948458968 | 1593.990658        | 1562.223698 |
| 10.48919664 | 87.04500181       | 94.18800768          | 10.48919664 | 1679.585192        | 1646.260324 |
| 11.02805126 | 87.64832925       | 94.84064482          | 11.02805126 | 1764.881654        | 1730.004303 |
| 12.09977337 | 88.94780375       | 96.24695559          | 12.09977337 | 1934.526862        | 1896.561795 |
| 13.16296975 | 90.37103773       | 97.78698166          | 13.16296975 | 2102.822515        | 2061.794295 |
| 14.21701869 | 91.91803118       | 99.46092305          | 14.21701869 | 2269.670199        | 2225.605181 |
| 15.26133387 | 93.58878411       | 101.2687798          | 15.26133387 | 2434.977106        | 2387.903333 |
| 16.2953656  | 95.38329652       | 103.2105518          | 16.2953656  | 2598.656221        | 2548.603321 |
| 17.31860169 | 97.3015684        | 105.2862391          | 17.31860169 | 2760.626473        | 2707.625549 |
| 18.33056816 | 99.34359976       | 107.4958417          | 18.33056816 | 2920.812831        | 2864.896352 |
| 19.33082946 | 101.5093906       | 109.8393597          | 19.33082946 | 3079.146354        | 3020.348043 |
| 20.31898861 | 103.7989409       | 112.3167929          | 20.31898861 | 3235.564201        | 3173.918925 |
| 21.29468693 | 106.2122507       | 114.9281415          | 21.29468693 | 3390.009596        | 3325.553256 |
| 22.25760362 | 108.7493199       | 117.6734054          | 22.25760362 | 3542.431759        | 3475.20118  |
| 23.20745509 | 111.4101487       | 120.5525846          | 23.20745509 | 3692.7858          | 3622.818624 |
| 24.14399409 | 114.1947369       | 123.5656791          | 24.14399409 | 3841.032581        | 3768.367163 |
| 25.0670087  | 117.1030846       | 126.7126889          | 25.0670087  | 3987.138559        | 3911.813862 |
| 25.97632114 | 120.1351918       | 129.993614           | 25.97632114 | 4131.07559         | 4053.131092 |
| 26.87178647 | 123.2910584       | 133.4084544          | 26.87178647 | 4272.82073         | 4192.296326 |
| 27.75329115 | 126.5706845       | 136.9572102          | 27.75329115 | 4412.356011        | 4329.291922 |
| 28.6207516  | 129.9740701       | 140.6398812          | 28.6207516  | 4549.6682          | 4464.104889 |
| 29.47411263 | 133.5012152       | 144.4564676          | 29.47411263 | 4684.748562        | 4596.72665  |
| 30.31334584 | 137.1521197       | 148.4069693          | 30.31334584 | 4817.592599        | 4727.15279  |

| Experimental Results (KN) |        |        |        |        |        |
|---------------------------|--------|--------|--------|--------|--------|
| Pbi (deg)                 | Test 3 | Test 4 | Test 5 | Test 6 | Test 7 |
| 0.001418118               | 0      | 0      | 0      | 0      | 0      |
| 0.014181182               |        |        |        |        |        |
| 0.028362362               |        |        |        |        |        |
| 0.035452951               |        |        |        |        |        |
| 0.049634125               |        |        |        |        |        |
| 0.055831409               | 3.4    | 3.3    | 2.3    | 1.9    | 3.1    |
| 0.070905875               |        |        |        |        |        |
| 0.085087031               |        |        |        |        |        |
| 0.099268176               |        |        |        |        |        |
| 0.101147683               |        |        |        |        |        |
| 0.103743768               |        |        |        |        |        |
| 0.12146903                |        |        |        |        |        |
| 0.223324574               | 11.7   | 10.8   | 10.4   | 9.7    | 11.8   |
| 0.279154922               | 14.4   | 13.7   | 13     | 12.1   | 13.3   |
| 0.33498474                | 16.9   | 15.4   | 16     | 14.2   | 15.9   |
| 0.446642362               | 23.1   | 20     | 20.4   | 18.4   | 19.2   |
| 0.558296592               | 29     | 24.9   | 23.6   | 22     | 22.8   |
| 0.669946581               | 34.1   | 30     | 27.2   | 25.6   | 26     |
| 0.7815911482              | 39.5   | 35     | 31.3   | 30.3   | 29.3   |
| 0.893230448               | 44.8   | 41.2   | 35.9   | 34.5   | 33.9   |
| 1.004862632               | 50.1   | 48     | 40.3   | 38.9   | 39     |
| 1.116487186               | 54.8   | 54.5   | 44.35  | 44     | 44     |
| 1.228103264               | 58.7   | 60     | 49.5   | 48.7   | 49.6   |
| 1.339710021               | 61.3   | 65     | 54.2   | 52.2   | 55     |
| 1.45130661                | 63.5   | 68.8   | 57.6   | 53.8   | 59.3   |
| 1.674465905               | 67     | 74     | 62.35  | 58.8   | 66     |
| 1.897574394               | 70.6   | 78.1   | 65.75  | 61.4   | 70.6   |
| 2.34361198                | 79.3   | 84.4   | 70.75  | 70.8   | 75.35  |
| 2.67795671                | 83.5   | 87.4   | 73.4   | 74.1   | 77.51  |
| 2.789365533               | 84.7   | 88.2   | 74     | 74.8   | 78.08  |
| 3.067793428               | 86.6   | 89.4   | 75.25  | 75.85  | 78.75  |
| 3.624201358               | 88.6   | 91.5   | 74.7   | 74.49  | 78.58  |
| 3.902155428               | 90     | 92.3   | 74.9   | 74.95  | 77.8   |
| 4.457499215               | 92.4   | 93.8   | 75.6   | 75.31  | 78.57  |
| 5.012005072               | 93.3   | 95.7   | 76.75  | 76.56  | 79.5   |
| 5.565571274               | 94.8   | 97.8   | 78.4   | 77.72  | 80.98  |
| 6.118097145               | 96.5   | 99.2   | 79.8   | 77.5   | 82.05  |
| 6.669483156               | 98.35  | 100    | 80     | 77.35  | 82.4   |
| 7.219631036               | 98.8   | 100.2  | 79     | 76.6   | 81.35  |
| 7.768443868               | 97.9   | 98.4   | 77     | 76.5   | 79.65  |
| 8.315826182               | 95.5   | 98.35  | 75.4   | 74.35  | 76.4   |
| 8.86168405                | 95.9   | 97     | 75.3   | 75.08  | 76.7   |
| 9.405925174               | 96.5   | 98.1   | 75.75  | 75.4   | 77.12  |
| 9.948458968               | 98.1   | 98.75  | 75     | 75.22  | 77.31  |
| 10.48919664               | 97     | 99.25  | 73     | 74.2   | 77.2   |
| 11.02805126               | 98.1   | 97.7   | 71     | 67     | 76.23  |
| 12.09977337               | 97.4   | 98.25  | 67.8   | 64.35  |        |
| 13.16296975               | 98.8   | 100.2  | 62.2   | 60     |        |
| 14.21701869               | 97.2   | 98.9   |        | 51.3   |        |
| 15.26133387               | 98.03  | 98.85  |        |        |        |
| 16.2953656                | 98.5   | 98.65  |        |        |        |
| 17.31860169               | 98.1   | 98.1   |        |        |        |
| 18.33056816               | 93     | 94.9   |        |        |        |
| 19.33082946               | 94.1   | 90     |        |        |        |
| 20.31898861               | 93.2   | 90.05  |        |        |        |
| 21.29468693               |        | 87.6   |        |        |        |

| Experimental NDSF |             |             |             |             |             |
|-------------------|-------------|-------------|-------------|-------------|-------------|
| Phi (deg)         | Test 3      | Test 4      | Test 5      | Test 6      | Test 7      |
| 0.001418118       | 0           | 0           | 0           | 0           | 0           |
| 0.014181182       |             |             |             |             |             |
| 0.028362362       |             |             |             |             |             |
| 0.035452951       |             |             |             |             |             |
| 0.049634125       |             |             |             |             |             |
| 0.055831409       | 2.435871181 | 2.364227911 | 2.032911118 | 1.679361358 | 2.740010637 |
| 0.070905875       |             |             |             |             |             |
| 0.085087031       |             |             |             |             |             |
| 0.099268176       |             |             |             |             |             |
| 0.101147683       |             |             |             |             |             |
| 0.103743768       |             |             |             |             |             |
| 0.12146903        |             |             |             |             |             |
| 0.223324574       | 8.382262595 | 7.737473164 | 9.192293751 | 8.573581672 | 10.42971791 |
| 0.279154922       | 10.31663089 | 9.815127995 | 11.49036719 | 10.69488023 | 11.75552951 |
| 0.33498474        | 12.10771264 | 11.03306359 | 14.14199039 | 12.55101647 | 14.05360295 |
| 0.446642362       | 16.54959538 | 14.32865401 | 18.03103774 | 16.26328894 | 16.97038846 |
| 0.558296592       | 20.77654831 | 17.83917424 | 20.85943582 | 19.44523678 | 20.1523363  |
| 0.669946581       | 24.43035508 | 21.49298101 | 24.04138366 | 22.62718462 | 22.98073438 |
| 0.781591482       | 28.29909167 | 25.07514451 | 27.66526869 | 26.78139429 | 25.89751989 |
| 0.893230448       | 32.09618498 | 29.51702726 | 31.73109093 | 30.49366677 | 29.96334213 |
| 1.004862632       | 35.89327829 | 34.38876962 | 35.62013829 | 34.38271413 | 34.47110157 |
| 1.116487186       | 39.26051198 | 39.04558217 | 39.1998296  | 38.89047356 | 38.89047356 |
| 1.228103264       | 42.05459951 | 42.98596202 | 43.75178276 | 43.04468324 | 43.8401702  |
| 1.339710021       | 43.91732453 | 46.56812553 | 47.90599243 | 46.13824364 | 48.61309195 |
| 1.45130661        | 45.49347647 | 49.29056979 | 50.91116539 | 47.55244267 | 52.41375187 |
| 1.674465905       | 48.00099093 | 53.01601983 | 55.10956879 | 51.97181467 | 58.33571034 |
| 1.897574394       | 50.58014865 | 55.9533939  | 58.11474174 | 54.26988811 | 62.40153258 |
| 2.34361198        | 56.81311314 | 60.46691991 | 62.53411374 | 62.57830746 | 66.59993398 |
| 2.67795671        | 59.82213048 | 62.61621801 | 64.8763809  | 65.49509298 | 68.50910468 |
| 2.789365533       | 60.68184972 | 63.18936417 | 65.40670554 | 66.11380506 | 69.01291309 |
| 3.067793428       | 62.04307185 | 64.04908341 | 66.51154854 | 67.04187318 | 69.60510893 |
| 3.624201358       | 63.47593725 | 65.55359209 | 66.02541762 | 65.83980399 | 69.45485028 |
| 3.902155428       | 64.47894304 | 66.12673825 | 66.2021925  | 66.24638622 | 68.76542825 |
| 4.457499215       | 66.19838152 | 67.2013873  | 66.82090458 | 66.564581   | 69.44601154 |
| 5.012005072       | 66.84317095 | 68.36260943 | 67.83736013 | 67.669424   | 70.26801473 |
| 5.565571274       | 67.91782    | 70.0671181  | 69.29575289 | 68.6947183  | 71.57614884 |
| 6.118097145       | 69.13575559 | 71.07012388 | 70.53317705 | 68.50026593 | 72.52189445 |
| 6.669483156       | 70.46115608 | 71.64327004 | 70.70995193 | 68.36768477 | 72.83125049 |
| 7.219631036       | 70.7835508  | 71.78655658 | 69.82607753 | 67.70477897 | 71.90318237 |
| 7.768443868       | 70.13876137 | 70.49697772 | 68.05832873 | 67.61639153 | 70.40059589 |
| 8.315826182       | 68.41932289 | 70.46115608 | 66.6441297  | 65.71606158 | 67.52800409 |
| 8.86168405        | 68.70589597 | 69.49397194 | 66.55574226 | 66.36128989 | 67.79316641 |
| 9.405925174       | 69.13575559 | 70.28204791 | 66.95348574 | 66.6441297  | 68.16439366 |
| 9.948458968       | 70.28204791 | 70.74772916 | 66.29057994 | 66.4850323  | 68.3323298  |
| 10.48919664       | 69.49397194 | 71.10594551 | 64.52283114 | 65.58348042 | 68.23510361 |
| 11.02805126       | 70.28204791 | 69.99547483 | 62.75508234 | 59.21958474 | 67.37774545 |
| 12.09977337       | 69.78054502 | 70.38951281 | 59.92668426 | 56.87731758 |             |
| 13.16296975       | 70.7835508  | 71.78655658 | 54.97698763 | 53.03246395 |             |
| 14.21701869       | 69.63725848 | 70.85519407 |             |             |             |
| 15.26133387       | 70.23189762 | 70.81937243 |             |             |             |
| 16.2953656        | 70.56862099 | 70.67608589 |             |             |             |
| 17.31860169       | 70.28204791 | 70.28204791 |             |             |             |
| 18.33056816       | 66.62824114 | 67.98946327 |             |             |             |
| 19.33082946       | 67.41631711 | 64.47894304 |             |             |             |
| 20.31898861       | 66.77152768 | 64.51476467 |             |             |             |
| 21.29468693       |             | 62.75950455 |             |             |             |

## **APPENDIX C**

### **Experimental Data**

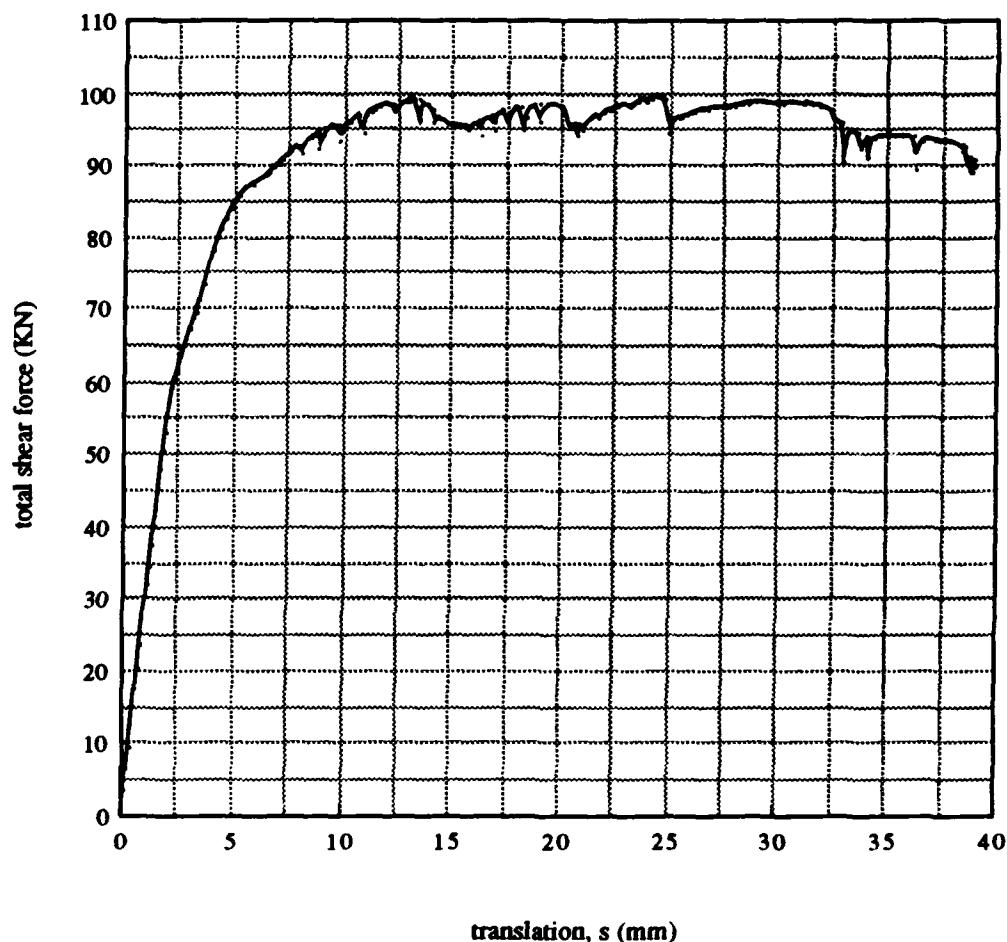


Figure C-1: Shear buckling load test 3 data

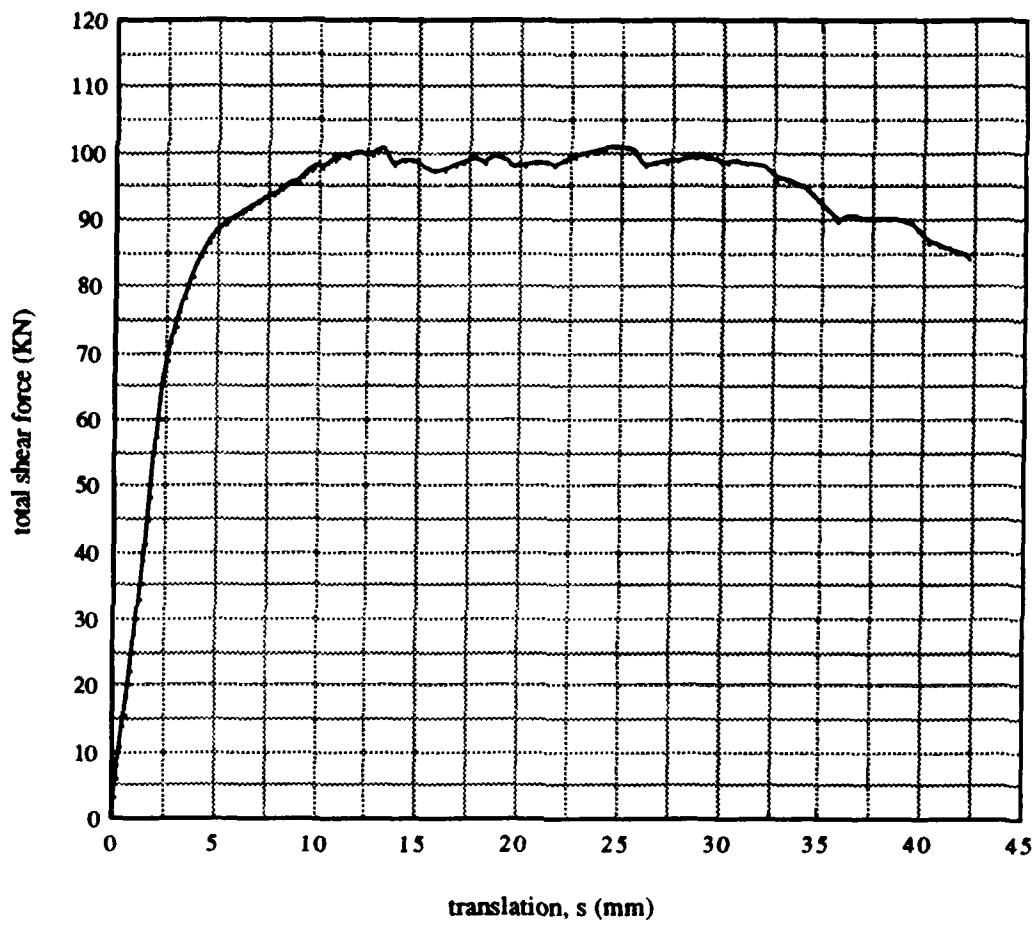


Figure C-2: Shear buckling load test 4 data

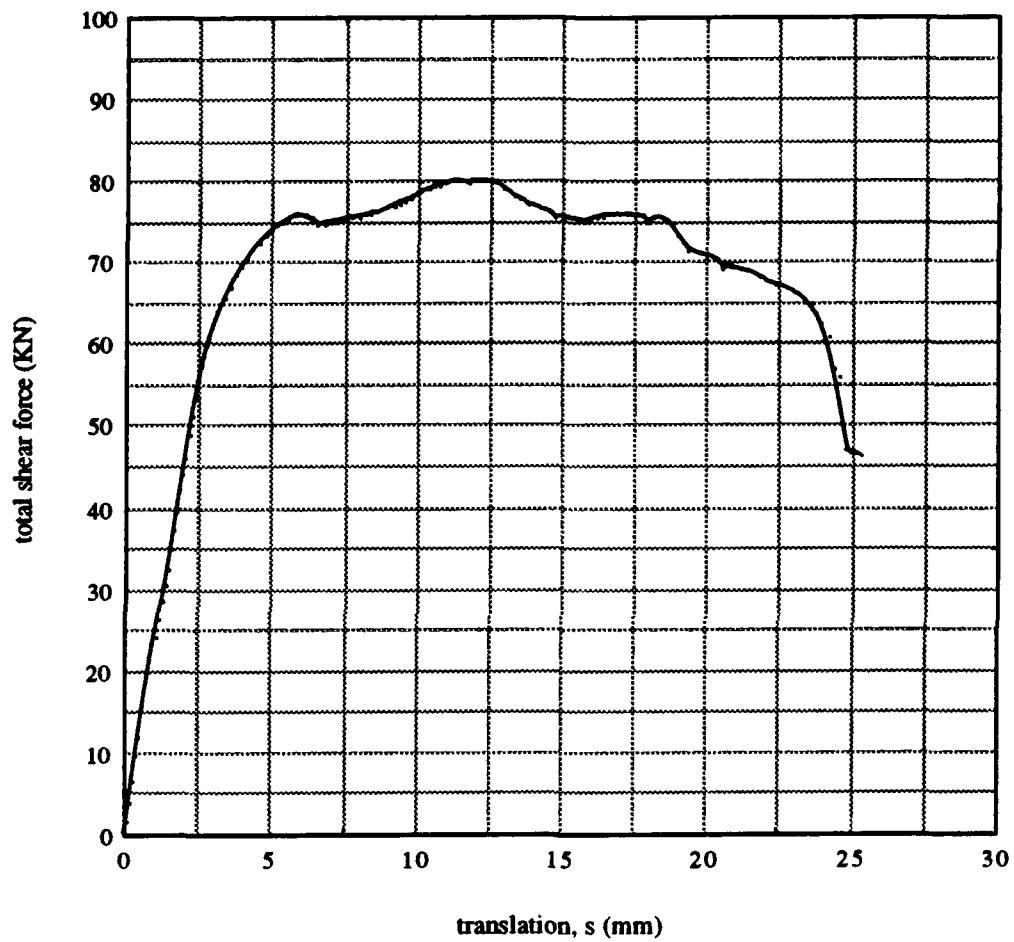


Figure C-3: Shear buckling load test 5 data

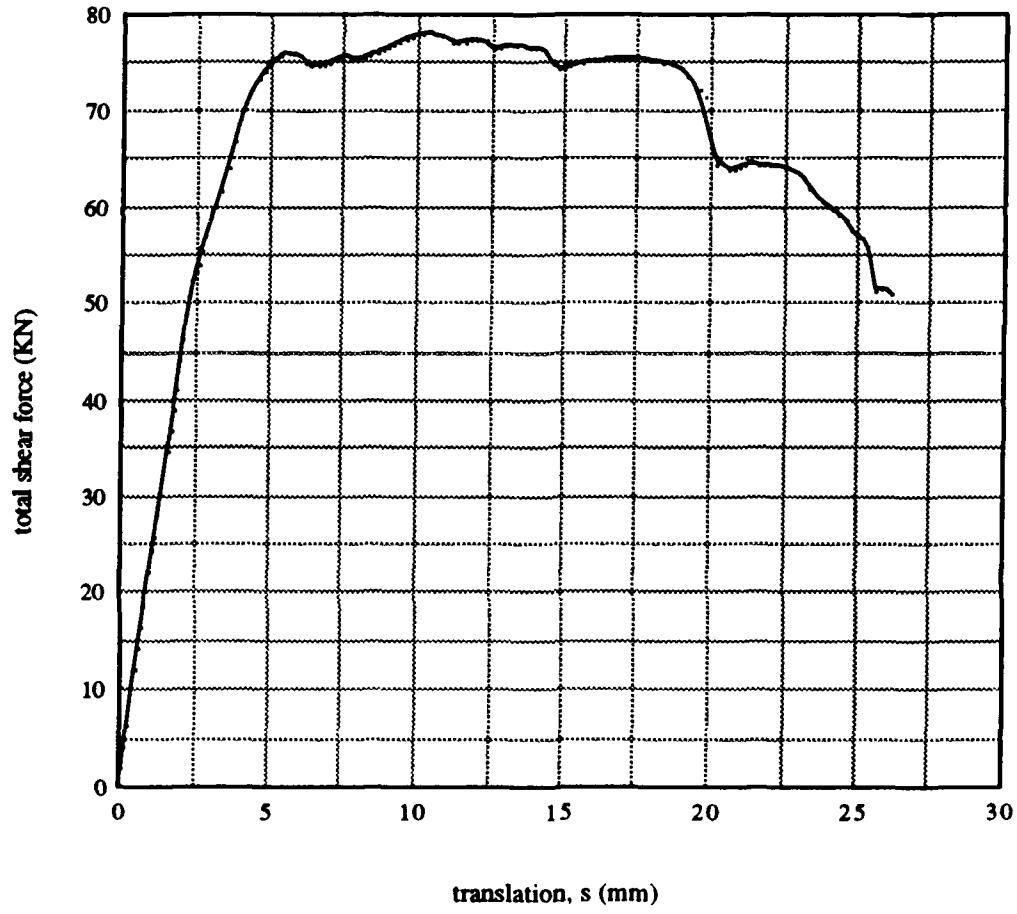


Figure C-4: Shear buckling load test 6 data

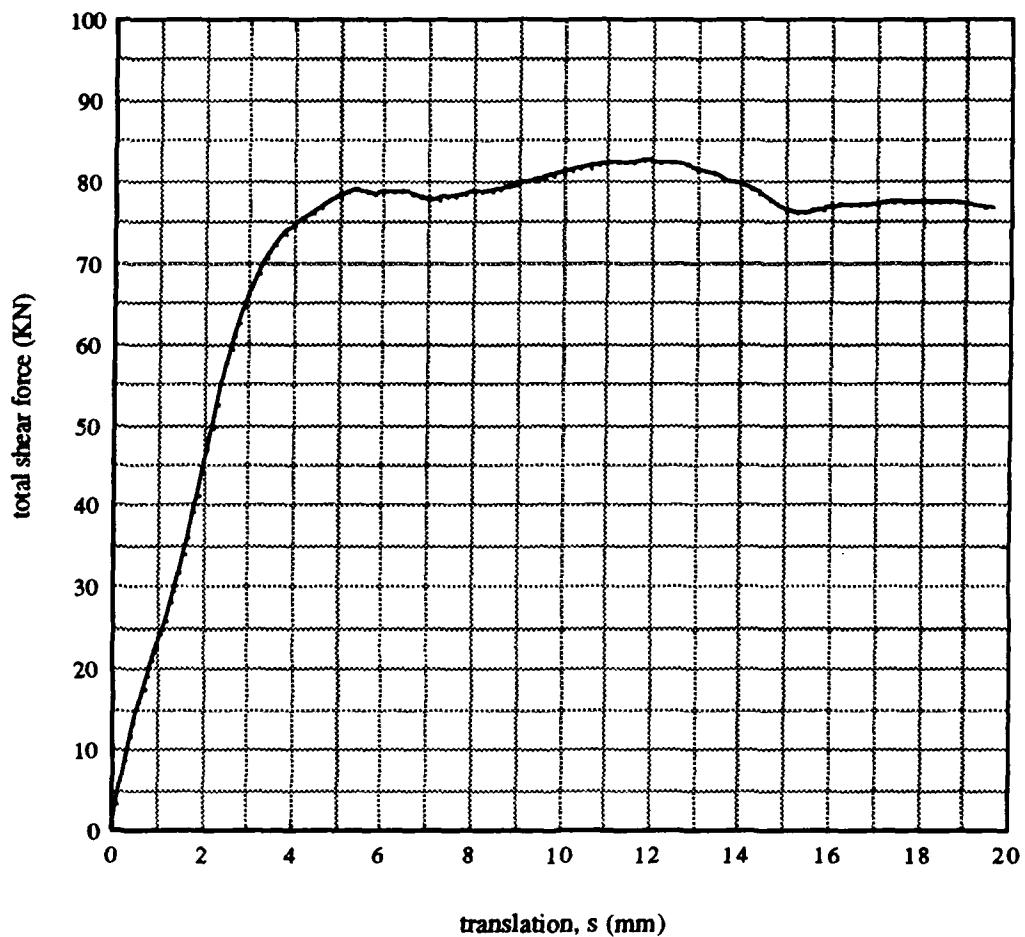


Figure C-5: Shear buckling load test 7 data

**Specimen Tensile Test Data for Buckling Tests 3, 4**

| Specimen             | Orientation to cold roll axis | Rockwell Superficial Hardness | Knoop Hardness | Approximate Yield Strength (ksi) = UTS * .65 | Specimen thickness t (in) | Specimen width w (in) | Specimen X-Sectional Area (sq-in) | Specimen Yield Pt. (kN) | Specimen Yield Stress (ksi) | Specimen Avg. Flow Force (kN) | Specimen Plastic Flow Stress (ksi) |
|----------------------|-------------------------------|-------------------------------|----------------|--|---------------------------|-----------------------|-----------------------------------|-------------------------|-----------------------------|-------------------------------|------------------------------------|
| 1                    | 90                            | 57.16                         | 122            | 38.3286208                                   | 0.0282                    | 0.444                 | 0.0123208                         | 2.07                    | 37.0836766                  | 2.361                         | 42.59348602                        |
| 2                    |                               | 59.68                         | 129            | 40.52780396                                  | 0.028                     | 0.444                 | 0.012432                          | 2.14                    | 38.69973519                 | 2.457                         | 44.43239552                        |
| 3                    |                               | 56.63                         | 120            | 37.70028275                                  | 0.0281                    | 0.441                 | 0.0123921                         | 2.09                    | 37.91722958                 | 2.372                         | 43.03333424                        |
| 4                    |                               | 57.6                          | 124            | 38.95695884                                  | 0.0272                    | 0.453                 | 0.0123216                         | 2.09                    | 38.15417905                 | 2.394                         | 43.68096574                        |
| 5                    |                               | 56.07                         | 119            | 37.386111573                                 | 0.0283                    | 0.44                  | 0.012432                          | 2.05                    | 37.0126321                  | 2.343                         | 42.30273025                        |
| 6                    |                               | 57.35                         | 123            | 38.64278982                                  | 0.0281                    | 0.437                 | 0.0128417                         | 2.2                     | 38.515486                   | 2.536                         | 44.39785113                        |
| 7                    |                               | 54.76                         | 116            | 36.44360666                                  | 0.0282                    | 0.442                 | 0.0124644                         | 2.09                    | 37.69728994                 | 2.366                         | 42.67549665                        |
| 8                    |                               | 58.04                         | 123            | 38.64278982                                  | 0.0279                    | 0.442                 | 0.0123318                         | 2.11                    | 38.46725568                 | 2.419                         | 44.10061208                        |
| Average = 37.9618219 |                               |                               |                |  |                           |                       |                                   |                         |                             |                               | Average = 43.37710483              |

**Specimen Tensile Test Data for Buckling Tests 5, 6, 7**

| Specimen   | Orientation to cold roll axis | Rockwell Superficial Hardness | Knoop Hardness | Approximate Yield Strength (ksi) = UTS * .65 | Specimen thickness t (in) | Specimen width w (in) | Specimen X-Sectional Area (sq-in) | Specimen Yield Pt. (kN) | Specimen Yield Stress (ksi) | Specimen Avg. Flow Force (kN) | Specimen Plastic Flow Stress (ksi) |  |
|--|-------------------------------|-------------------------------|----------------|--|---------------------------|-----------------------|-----------------------------------|-------------------------|-----------------------------|-------------------------------|------------------------------------|--|
| 1  | 0                             | 71.3                          | 89.7           | 34.68426013                                  | 0.0284                    | 0.489                 | 0.0138876                         | 2.03                    | 32.84276189                 | 2.133                         | 34.83395387                        |  |
| 2  |                               | 72.8                          | 94.3           | 36.46294014                                  | 0.0283                    | 0.485                 | 0.0137253                         | 2.02                    | 33.08707811                 | 2.14                          | 35.03264711                        |  |
| 3  |                               | 72.8                          | 94.3           | 36.46294014                                  | 0.0282                    | 0.485                 | 0.013677                          | 2.01                    | 33.04002992                 | 2.127                         | 34.96525554                        |  |
| 4  |                               | 72.2                          | 92.7           | 35.84426883                                  | 0.0281                    | 0.484                 | 0.0136004                         | 1.9                     | 31.40777281                 | 2.093                         | 34.59814132                        |  |
| 5  |                               | 78.1                          | 110.7          | 42.88422193                                  | 0.0261                    | 0.469                 | 0.0127629                         | 2.95                    | 51.94463378                 | 2.783                         | 49.62290705                        |  |
| 6  |                               | 78.5                          | 112            | 43.30699147                                  | 0.0258                    | 0.463                 | 0.0124614                         | 2.74                    | 49.43322834                 | 2.883                         | 46.000773761                       |  |
| 7  |                               | 73                            | 95             | 36.73360884                                  | 0.0281                    | 0.482                 | 0.0135442                         | 2.05                    | 34.02794517                 | 2.167                         | 35.97002789                        |  |
| 8  |                               | 73                            | 95             | 36.73360884                                  | 0.0282                    | 0.483                 | 0.0136205                         | 2.06                    | 34.00213621                 | 2.187                         | 36.09838441                        |  |
| 9  |                               | 72.9                          | 94.7           | 36.61760797                                  | 0.0283                    | 0.478                 | 0.0135274                         | 2.03                    | 33.73781304                 | 2.18                          | 36.23075489                        |  |
| Average = 41.93271743                            |                               |                               |                |  |                           |                       |                                   |                         |                             |                               | Average = 41.68386801              |  |
| Average flow stress for tests 5 & 6 = 47.8182233 |                               |                               |                |  |                           |                       |                                   |                         |                             |                               |                                    |  |