Naval Command,
Control and Ocean
Surveillance Center RDT&E Division San Diego, CA
92152-5000

(2)

Technical Document 2320
July 1992

A New Matrix Formulation of Classical Electrodynamics

Part II: Wave Propagation in Optical Materials of Infinite Extent

R. P. Bocker

Technical Document 2320
July 1992

A New Matrix Formulation of Classical Electrodynamics

Part II: Wave Propagation in Optical Materials of Infinite Extent

R. P. Bocker

NAVAL COMMAND, CONTROL AND
OCEAN SURVEILLANCE CENTER
RDT&E DIVISION
San Diego, California 92152-5000

J. D. FONTANA, CAPT, USN
Commanding Officer

R. T. SHEARER
Executive Director

ADMINISTRATIVE INFORMATION

The work in this report was performed in the Processing Research and Development Branch (Code 761) as part of the Independent Research Program. Sponsorship was provided by the Office of Chief of Naval Research, Arlington, VA 22217-5000.

Released by
G. W. Byram, Head
Processing Research
and Development
Branch

Under authority of
J. R. Wrangler, Head
Space Systems and
Technology Division

PK

CONTENTS

1.0 BACKGROUND

In an earlier publication (Bocker & Frieden, 1992) a new covariant matrix representation of classical electromagnetic theory for vacuum was presented. The quintessential basis of this representation is a skew-Hermitian space-time 8-by-8 differential matrix operator

$$[M] = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}, \tag{1}$$

where

$$[M_1] \equiv \begin{bmatrix} -\frac{\partial}{\partial \tau} & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & -\frac{\partial}{\partial \tau} & 0 & -\frac{\partial}{\partial y} \\ 0 & 0 & -\frac{\partial}{\partial \tau} & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & -\frac{\partial}{\partial \tau} \end{bmatrix} \quad \text{and} \quad [M_2] \equiv \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

and

$$\tau \equiv ict, \qquad i \equiv \sqrt{-1} \ . \tag{3}$$

The symbol c represents the speed of light in vacuum. Use of the square matrix operator [M] allows Maxwell's equations to be placed in a compact matrix form. From the matrix form of Maxwell's equations, other landmark effects of electromagnetic theory are easily derived with use of the simple matrix multiply operation: (a) the electromagnetic wave and charge continuity equations; (b) the Lorentz-gauge and Coulomb-gauge definitions of the electromagnetic potentials; (c) the wave equations for the potentials; and (d) Poynting's theorem on energy conservation. Taking the four-dimensional Fourier transform of the matrix form of Maxwell's equations leads to: (e) a Fourier representation of Maxwell's equations; (f) their inversion, for the fields directly in terms of the sources in Fourier-space; and (g) corresponding inversion formulae in direct-space through the use of the convolution theorem.

Because of the power of matrix operations, we were able to derive the key effects of electromagnetic theory without the need for the usual plethora of vector calculus identities that have become the standard in these derivations: Stokes' theorem, the Divergence theorem, Green's theorem, and the formula for the curl of a curl, etc. Instead, the simple matrix operations of matrix multiplication and matrix inversion were used.

In this document, the matrix representation of classical electromagnetic theory for vacuum will be extended to include the presence of matter. Emphasis is placed on electromagnetic wave propagation in linear, homogeneous, anisotropic optical media of infinite extent without boundaries. A subsequent document treating wave propagation in optical media will include boundaries.

2.0 EXTENSION OF THE MATRIX FORMULATION

2.1 MAXWELL FIELD EQUATIONS

The fundamental equations of classical electromagnetic phenomena, namely the Maxwell field equations, serve as our starting point. In the Gaussian system of units, the four Maxwell field equations in vector form are given by the following (Jackson, 1962):

Ampere-Maxwell law

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r}, t) + \frac{4\pi}{c}\boldsymbol{J}^{e}(\boldsymbol{r}, t), \tag{4a}$$

Gauss' law for electricity

$$\nabla \bullet \boldsymbol{D}(\boldsymbol{r}, t) = 4\pi\rho^{e}(\boldsymbol{r}, t), \tag{4b}$$

Faraday's law of induction

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}, t) - \frac{4\pi}{c}\boldsymbol{J}^{m}(\boldsymbol{r}, t), \tag{4c}$$

and

Gauss' law for magnetism

$$\nabla \bullet \boldsymbol{B}(\boldsymbol{r}, t) = 4\pi\rho^{m}(\boldsymbol{r}, t). \tag{4d}$$

In rectangular coordinates,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{5}$$

and

$$\boldsymbol{r} = (x, y, z)\,. \tag{6}$$

The physical quantities appearing in the Maxwell field equations are $\boldsymbol{E}(\boldsymbol{r}, t)$ the electric field; $\boldsymbol{D}(\boldsymbol{r}, t)$ the electric displacement; $\boldsymbol{B}(\boldsymbol{r}, t)$ the magnetic induction; $\boldsymbol{H}(\boldsymbol{r}, t)$ the magnetic field; $J^{e}(\boldsymbol{r}, t)$ the electric current density; $J^{m}(\boldsymbol{r}, t)$ the magnetic current density; $\rho^{e}(\boldsymbol{r}, t)$ the electric charge density; $\rho^{m}(\boldsymbol{r}, t)$ the magnetic charge density; and $(\boldsymbol{r}, t)$ a space-time point. Both magnetic charge and current densities (Magid, 1972) have been included in Maxwell's equations for purposes of completeness. They, of course, can be set equal to zero since magnetic charge has not been discovered in nature.

The electric displacement and electric field, as well as the magnetic induction and magnetic field, are related (Jackson, 1962) through the expressions

$$\boldsymbol{D}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, t) + 4\pi\, \boldsymbol{P}(\boldsymbol{r}, t) \tag{7a}$$

and

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{H}(\boldsymbol{r}, t) + 4\pi\, \boldsymbol{M}(\boldsymbol{r}, t)\,, \tag{7b}$$

where $\boldsymbol{P}(\boldsymbol{r}, t)$ is the macroscopic polarization and $\boldsymbol{M}(\boldsymbol{r}, t)$ is the macroscopic magnetization. By using the approach adopted previously by Bocker and Frieden (1992), the four Maxwell field equations (4) with the use of equation (7) can be cast into the following matrix form

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 4\pi \begin{bmatrix} M_1 & O \\ O & M_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{8}$$

The 4-by-4 matrix operators $\left[M_1\right]$ and $\left[M_2\right]$ are defined by equation (2). The matrix $[O]$ represents the 4-by-4 null matrix. The 4-by-1 field vectors appearing in equation (8) are defined by

$$[f_1] \equiv \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \end{bmatrix}, \qquad [f_2] \equiv \begin{bmatrix} H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}, \qquad [d_1] \equiv \begin{bmatrix} iP_x \\ iP_y \\ iP_z \\ 0 \end{bmatrix}, \qquad \text{and} \qquad [d_2] \equiv \begin{bmatrix} M_x \\ M_y \\ M_z \\ 0 \end{bmatrix}. \tag{9}$$

The source vectors in equation (8) are relativistic 4-vectors defined by

$$[s_1] \equiv \begin{bmatrix} J_x^e \\ J_y^e \\ J_z^e \\ ic\rho^e \end{bmatrix} \qquad \text{and} \qquad [s_2] \equiv \begin{bmatrix} -iJ_x^m \\ -iJ_y^m \\ -iJ_z^m \\ c\rho^m \end{bmatrix}. \tag{10}$$

2.2 ELECTROMAGNETIC FIELD WAVE AND CHARGE CONTINUITY EQUATIONS

Multiply both sides of matrix equation (8) by the complex conjugate of the space-time operator [M]. This gives

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 4\pi \begin{bmatrix} D_1 & D_2 \\ D_2 & D_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{11}$$

The matrix $[D]$ is defined by

$$[D] \equiv [M_1]^* [M_1] + [M_2]^* [M_2] = \begin{bmatrix} \Box^2 & 0 & 0 & 0 \\ 0 & \Box^2 & 0 & 0 \\ 0 & 0 & \Box^2 & 0 \\ 0 & 0 & 0 & \Box^2 \end{bmatrix}. \tag{12}$$

The D'Alembertian operator $\Box^2$ appearing in equation (12) is defined by (Ohanian, 1988)

$$\Box^2 \equiv -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} . \tag{13}$$

The matrices $[D_1]$ and $[D_2]$ are defined by

$$[D_1] \equiv [M_1]^* [M_1] \qquad \text{and} \qquad [D_2] \equiv [M_2]^* [M_1] . \tag{14}$$

Again, the matrix $[O]$ represents the 4-by-4 null matrix. Matrix equation (11) is equivalent to eight scalar equations. These scalar equations are equivalent to the vector electromagnetic field wave equations:

$$\begin{aligned} \nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} = 4\pi \ \nabla\rho^e + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\boldsymbol{J}^e + \frac{4\pi}{c}\nabla\times \boldsymbol{J}^m \\ - 4\pi \ \nabla\ (\nabla\bullet \boldsymbol{P}) + \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{P} + \frac{4\pi}{c}\frac{\partial}{\partial t}(\nabla\times \boldsymbol{M}) \end{aligned} \tag{15a}$$

and

$$\begin{aligned} \nabla^2 \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{H} = 4\pi \ \nabla\rho^m + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\boldsymbol{J}^m - \frac{4\pi}{c}\nabla\times \boldsymbol{J}^e \\ - 4\pi \ \nabla\ (\nabla\bullet \boldsymbol{M}) + \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{M} - \frac{4\pi}{c}\frac{\partial}{\partial t}(\nabla\times \boldsymbol{P}) , \end{aligned} \tag{15b}$$

and the electric and magnetic charge continuity equations

$$\nabla\bullet \boldsymbol{J}^e + \frac{\partial}{\partial t}\rho^e = 0 \tag{16a}$$

and

$$\nabla\bullet \boldsymbol{J}^m + \frac{\partial}{\partial t}\rho^m = 0. \tag{16b}$$

2.3 ELECTROMAGNETIC POTENTIALS AND LORENTZ CONDITIONS

We found before that the space-time operator [M] defines the eight scalar Maxwell field equations. We now observe the complex conjugate of [M] also provides the definition of the electromagnetic fields in terms of the familiar vector and scalar potentials. In particular, for the Lorentz gauge we have

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \tag{17}$$

where the relativistic 4-vector potentials are defined by

$$\begin{bmatrix} p_1 \end{bmatrix} \equiv \begin{bmatrix} A_x^e \\ A_y^e \\ A_z^e \\ i\varphi^e \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_2 \end{bmatrix} \equiv \begin{bmatrix} -iA_x^m \\ -iA_y^m \\ -iA_z^m \\ \varphi^m \end{bmatrix}. \tag{18}$$

Equation (17) is equivalent to eight scalar equations. Six of these are equivalent to the two vector equations defining the relationship between the electromagnetic potentials and the electromagnetic fields:

$$\boldsymbol{E} = -\nabla\varphi^e - \frac{1}{c}\frac{\partial}{\partial t}A^e - \nabla\times A^m \tag{19a}$$

and

$$\boldsymbol{H} = -\nabla\varphi^m - \frac{1}{c}\frac{\partial}{\partial t}A^m + \nabla\times A^e. \tag{19b}$$

The remaining two scalar equations correspond to the Lorentz conditions

$$\nabla\bullet A^e + \frac{1}{c}\frac{\partial}{\partial t}\phi^e = 0 \tag{20a}$$

and

$$\nabla\bullet A^m + \frac{1}{c}\frac{\partial}{\partial t}\phi^m = 0. \tag{20b}$$

Equations (19) and (20) are easily verified by explicit multiplication of the right-hand side of matrix equation (17).

2.4 ELECTROMAGNETIC POTENTIAL WAVE EQUATIONS

We know the electromagnetic vector and scalar potentials satisfy inhomogeneous wave equations (Jackson, 1962). Once again, we show this by simple matrix multiplication. First, substitute the matrix expression (equation 17) into the matrix representation (equation 8) of the Maxwell field equations. This gives

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + 4\pi \begin{bmatrix} M_1 & O \\ O & M_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{21}$$

For the Lorentz gauge choice of [M],

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}^* \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} = \begin{bmatrix} D & O \\ O & D \end{bmatrix} \tag{22}$$

by direct evaluation of the matrix product. Then, by equations (21) and (22), the following matrix representation of the electromagnetic potential wave equations is obtained

$$\begin{bmatrix} D & O \\ O & D \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + 4\pi \begin{bmatrix} M_1 & O \\ O & M_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{23}$$

By explicit evaluation, this single matrix equation is equivalent to eight scalar equations. Six of these scalar equations are equivalent to the following pair of inhomogeneous wave equations for the electric and magnetic vector potentials:

$$\nabla^2 A^e - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} A^e = -\frac{4\pi}{c} J^e - \frac{4\pi}{c}\frac{\partial}{\partial t} P \tag{24a}$$

and

$$\nabla^2 A^m - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} A^m = -\frac{4\pi}{c} J^m - \frac{4\pi}{c}\frac{\partial}{\partial t} M. \tag{24b}$$

The remaining two represent the inhomogeneous wave equations for the electric and magnetic scalar potentials:

$$\nabla^2 \varphi^e - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi^e = -4\pi\ \rho^e + 4\pi\ \nabla \bullet \boldsymbol{P} \tag{25a}$$

and

$$\nabla^2 \varphi^m - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi^m = -4\pi\ \rho^m + 4\pi\ \nabla \bullet \boldsymbol{M}. \tag{25b}$$

2.5 ENERGY CONSERVATION: POYNTING'S THEOREM

The law of conservation of energy, often called Poynting's theorem (Jackson, 1962), is an important milestone of electromagnetic theory. Again, simple matrix manipulation of equation (8) will accomplish the derivation. Multiply both sides of equation (8) by the Hermitian conjugate of the electromagnetic field vector. Directly,

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^H \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 4\pi \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^H \begin{bmatrix} M_1 & O \\ O & M_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^H \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{26}$$

This is a single scalar equation representing Poynting's theorem

$$\nabla \bullet \boldsymbol{S} + \frac{\partial}{\partial t}u + \boldsymbol{E} \bullet \boldsymbol{J}^e + \boldsymbol{H} \bullet \boldsymbol{J}^m + \boldsymbol{E} \bullet \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{H} \bullet \frac{\partial \boldsymbol{M}}{\partial t} = 0. \tag{27}$$

We define quantity u, the total energy density of the electromagnetic fields, to obey

$$u \equiv \frac{1}{8\pi}(\boldsymbol{E} \bullet \boldsymbol{E} + \boldsymbol{H} \bullet \boldsymbol{H}) \tag{28}$$

and define $\boldsymbol{S}$, the Poynting vector representing energy flow, by

$$\boldsymbol{S} \equiv \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{H}). \tag{29}$$

Again, this is verified by explicit evaluation of the matrix products in equation (26).

3.0 OPTICAL MATERIALS

The study of the propagation of electromagnetic radiation through matter comprises an important branch of modern optics (Fowles, 1968). Many of the optical properties of matter can be understood on the basis of classical electrodynamics. In this section, the matrix representations of both the Maxwell field equations (8) and electromagnetic wave and charge continuity equations (11) will be used to mathematically describe wave propagation in linear, homogeneous, anisotropic optical media of infinite extent. Monochromatic plane-wave solutions in the absence of charges and currents will be considered.

3.1 MAXWELL FIELD EQUATIONS

For linear, homogeneous, anisotropic optical media, both the macroscopic polarization and the magnetization vectors are related to the electric and the magnetic field vectors through 4-by-4 electric and magnetic susceptibility tensors $[\chi_e]$ and $[\chi_m]$. Mathematically, we can express these relationships in the compact matrix form

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \chi_e & O \\ O & \chi_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{30}$$

In a medium with neither free charges nor currents, we also have

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{31}$$

Substitution of equations (30) and (31) into the Maxwell field equations (8) gives

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 4\pi \begin{bmatrix} M_1 & O \\ O & M_1 \end{bmatrix} \begin{bmatrix} \chi_e & O \\ O & \chi_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{32}$$

For monochromatic plane-wave fields, the electromagnetic field vector in equation (32) can be expressed in the form

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} e^{i[k]^T[r]}, \tag{33}$$

where the relativistic 4-vectors $[r]$ and $[k]$ are defined by

$$[r] \equiv \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} \quad \text{and} \quad [k] \equiv \begin{bmatrix} k_x \\ k_y \\ k_z \\ i(\omega/c) \end{bmatrix}. \tag{34}$$

The superscript, T, appearing in equation (33) denotes transpose. The 4-by-1 vectors $[f_{o1}]$ and $[f_{o2}]$ appearing in equation (33) are both constant and describe the polarization properties of the plane-wave field. With the use of equation (34), the argument of the exponential function in equation (33) is given by

$$[k]^T[r] = \mathbf{k} \bullet \mathbf{r} - \omega t = xk_x + yk_y + zk_z - \omega t. \tag{35}$$

The wave vector $\mathbf{k}$ in equation (35) defines the direction of wave propagation within the medium. Its magnitude k, commonly referred to as the wavenumber, is related to the wavelength λ of the electromagnetic wave through the equation

$$k = \frac{2\pi}{\lambda}. \tag{36}$$

The angular frequency ω appearing in equation (34) is related to the frequency f through the equation

$$\omega = 2\pi f. \tag{37}$$

Note the monochromatic plane-wave representation (equation 33) is equivalent to the vector equations

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_o e^{i(\mathbf{k} \bullet \mathbf{r} - \omega t)} \quad \text{and} \quad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}_o e^{i(\mathbf{k} \bullet \mathbf{r} - \omega t)}. \tag{38}$$

Substitution of equation (33) into equation (32) gives the following monochromatic plane-wave matrix representation of the Maxwell field equations for an infinite anisotropic optical medium:

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} + 4\pi \begin{bmatrix} K_1 & O \\ O & K_1 \end{bmatrix} \begin{bmatrix} \chi_e & O \\ O & \chi_m \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{39}$$

The 4-by-4 matrices $\left[K_1\right]$ and $\left[K_2\right]$ are defined by

$$\left[K_1\right] \equiv i \begin{bmatrix} -k_\tau & 0 & 0 & -k_x \\ 0 & -k_\tau & 0 & -k_y \\ 0 & 0 & -k_\tau & -k_z \\ k_x & k_y & k_z & -k_\tau \end{bmatrix} \quad \text{and} \quad \left[K_2\right] \equiv i \begin{bmatrix} 0 & -k_z & k_y & 0 \\ k_z & 0 & -k_x & 0 \\ -k_y & k_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{40}$$

where, by definition,

$$k_\tau \equiv \frac{i\omega}{c}. \tag{41}$$

Note the matrix equation (39) represents a system of algebraic equations in contrast to equation (32), which describes a system of first order partial differential equations. Through the use of matrix multiplication, equation (39) can be rewritten in the form

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} + 4\pi \begin{bmatrix} K_1\chi_e & O \\ O & K_1\chi_m \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{42}$$

The 4-by-4 dielectric tensor $[\varepsilon]$ is related to the electric susceptibility tensor $[\chi_e]$ and the 4-by-4 identity matrix $[\mathrm{I}]$ through the defining equation

$$[\varepsilon] \equiv [\mathrm{I}] + 4\pi\left[\chi_e\right]. \tag{43}$$

Similarly, the 4-by-4 permeability tensor $[\mu]$ is related to the magnetic susceptibility tensor $[\chi_m]$ and the identity matrix $[\mathrm{I}]$ through the defining equation

$$[\mu] \equiv [\mathrm{I}] + 4\pi\left[\chi_m\right]. \tag{44}$$

Using definitions (43) and (44) with equation (42) gives

$$\begin{bmatrix} K_1\varepsilon & K_2 \\ K_2 & K_1\mu \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{45}$$

Matrix equation (45) is equivalent to the following transversality conditions in vector form:

$$\boldsymbol{k} \times \boldsymbol{H}_o = -\frac{\omega}{c}\boldsymbol{D}_o, \tag{46a}$$

$$\boldsymbol{k} \bullet \boldsymbol{D}_o = 0, \tag{46b}$$

$$\boldsymbol{k} \times \boldsymbol{E}_o = +\frac{\omega}{c}\boldsymbol{B}_o, \tag{46c}$$

and

$$\boldsymbol{k} \bullet \boldsymbol{B}_o = 0. \tag{46d}$$

The next important step in this development is to cast equation (45) into an eigenvalue equation. The 4-by-4 matrices $[K_1]$ and $[K_2]$, first defined in equation (40), can be rewritten in the form

$$[K_1] = ik[\alpha_1] + \frac{\omega}{c}[\mathrm{I}] \qquad \text{and} \qquad [K_2] = ik[\alpha_2], \tag{47}$$

where the matrices $[\alpha_1]$ and $[\alpha_2]$ are defined by

$$[\alpha_1] \equiv \begin{bmatrix} 0 & 0 & 0 & -\alpha_x \\ 0 & 0 & 0 & -\alpha_y \\ 0 & 0 & 0 & -\alpha_z \\ \alpha_x & \alpha_y & \alpha_z & 0 \end{bmatrix} \qquad \text{and} \qquad [\alpha_2] \equiv \begin{bmatrix} 0 & -\alpha_z & \alpha_y & 0 \\ \alpha_z & 0 & -\alpha_x & 0 \\ -\alpha_y & \alpha_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{48}$$

Again, the matrix $[\mathrm{I}]$ represents the 4-by-4 identity matrix. The matrix elements α_x, α_y, and α_z appearing in equation (48) are direction cosines (Morrill, 1961) defining the direction of the wavevector $\boldsymbol{k}$. These direction cosines are related to the components of $\boldsymbol{k}$ and the wavenumber k through the equation

$$k_x = k\alpha_x, \qquad k_y = k\alpha_y, \qquad \text{and} \qquad k_z = k\alpha_z. \tag{49}$$

Also,

$$k^2 = k_x^2 + k_y^2 + k_z^2 \qquad \text{and} \qquad \alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1. \tag{50}$$

With the use of equations (47) and (48), matrix equation (45) can be rewritten as a generalized eigenvalue equation

$$\begin{bmatrix} \alpha_1 \varepsilon & \alpha_2 \\ \alpha_2 & \alpha_1 \mu \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \frac{i}{n} \begin{bmatrix} \varepsilon & O \\ O & \mu \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix}, \tag{51}$$

where the index of refraction n is related to the temporal frequency f, the wavelength λ, and the speed of light in vacuum c through the equation

$$n\lambda f = c. \tag{52}$$

The matrix eigenvalue representation (equation 51) of the Maxwell field equations, hereafter referred to as the Maxwell eigenvalue matrix representation, is the most important result of subsection 3.1. Knowledge of both the dielectric and permeability tensors $[\varepsilon]$ and $[\mu]$ of the optical medium of interest, as well as the direction of wave propagation within the optical medium, completely specifies the two 8-by-8 matrices appearing on both sides of equation (51). Once these 8-by-8 matrices have been specified, then the unknown eigenvalues i/n (hence, eigenindices of refraction n) and corresponding unknown eigenvectors $[f_{01}\ f_{02}]^T$ (hence, eigenstates of polarization) can, in principle, be determined. Off-the-shelf computer software packages, like MATLAB for numerical computations and MATHEMATICA for symbolic manipulations, can facilitate the process of finding the eigenvalues and eigenvectors of equation (51).

3.2 ELECTROMAGNETIC FIELD WAVE AND CHARGE CONTINUITY EQUATIONS

As previously demonstrated, multiplication of both sides of the matrix form of the Maxwell field equations (8) by the complex conjugate of the space-time operator [M] leads to the matrix form of the electromagnetic field wave and charge continuity equation (11). For the case of monochromatic plane-wave fields (equation 33) propagating in linear, homogeneous, anisotropic optical media (equation 30) in the absence of charges and currents (equation 31), the matrix equation (11) simplifies to

$$\begin{bmatrix} K_1 & K_2 \\ K_2 & K_1 \end{bmatrix}^* \begin{bmatrix} K_1 \varepsilon & K_2 \\ K_2 & K_1 \mu \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{53}$$

Matrix multiplication further yields

$$\begin{bmatrix} \overset{*}{K_1}K_1\varepsilon + \overset{*}{K_2}K_2 & \overset{*}{K_1}K_2 + \overset{*}{K_2}K_1\mu \\ \overset{*}{K_2}K_1\varepsilon + \overset{*}{K_1}K_2 & \overset{*}{K_2}K_2 + \overset{*}{K_1}K_1\mu \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}. \tag{54}$$

Substituting the mathematical expressions for $[K_1]$ and $[K_2]$ from equation (47) into equation (54), with the help of equation (52), leads to the following important result for subsection 3.2:

$$\begin{bmatrix} I - \alpha_1\alpha_1(\varepsilon - I) & \frac{i}{n}\alpha_2(\mu - I) \\ \frac{i}{n}\alpha_2(\varepsilon - I) & I - \alpha_1\alpha_1(\mu - I) \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \frac{1}{n^2}\begin{bmatrix} \varepsilon & O \\ O & \mu \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix}. \tag{55}$$

Hereafter, equation (55) will be referred to as the electromagnetic wave equation matrix representation. Equation (55) greatly simplifies in structure when either the dielectric tensor $[\varepsilon]$ or the permeability tensor $[\mu]$ is equal to the identity matrix $[I]$. This fact will become quite evident in section 4 where monochromatic plane-wave propagation in a variety of nonmagnetic optical media is considered in greater detail.

4.0 SPECIFIC APPLICATIONS

The utility of using the Maxwell eigenvalue matrix representation (equation 51) or the electromagnetic wave equation matrix representation (equation 55) in solving wave-propagation problems for a variety of nonmagnetic optical media will be pursued in this section. Examples of nonmagnetic optical materials considered include crystalline materials, optically active materials, and electrooptic materials. For nonmagnetic optical materials, the permeability tensor, defined by equation (44), simplifies to the following form

$$[\mu] = [I]. \tag{56}$$

At this point, we introduce the optical impermeability tensor $[\eta]$ (Yariv & Yeh, 1984), which is defined as the multiplicative inverse of the dielectric tensor $[\varepsilon]$. That is,

$$[\eta] \equiv [\varepsilon]^{-1}. \tag{57}$$

With the use of equations (56) and (57), the Maxwell eigenvalue matrix representation (equation 51) can be rewritten in the form

$$\begin{bmatrix} \eta\alpha_1\varepsilon & \eta\alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \frac{i}{n} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix}. \tag{58}$$

Matrix multiplication involving the left-hand side of equation (58) leads to the following pair of matrix equations:

$$[\alpha_2]\,[f_{o1}] + [\alpha_1]\,[f_{o2}] = \frac{i}{n}\,[f_{o2}] \tag{59a}$$

and

$$[\eta]\,[\alpha_1]\,[\varepsilon]\,[f_{o1}] + [\eta]\,[\alpha_2]\,[f_{o2}] = \frac{i}{n}\,[f_{o1}]\,. \tag{59b}$$

Similarly, with the use of equations (56) and (57), the electromagnetic wave equation matrix representation (equation 55) simplifies to

$$\begin{bmatrix} \eta - \eta\alpha_1\alpha_1(\varepsilon - \mathrm{I}) & O \\ (i/n)\,\alpha_2(\varepsilon - \mathrm{I}) & \mathrm{I} \end{bmatrix} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix} = \frac{1}{n^2} \begin{bmatrix} f_{o1} \\ f_{o2} \end{bmatrix}. \tag{60}$$

Matrix multiplication involving the left-hand side of equation 60 leads to the following pair of matrix equations:

$$[\eta]\,(\,[\mathrm{I}] - [\alpha_1]\,[\alpha_1]\,[\varepsilon - \mathrm{I}]\,)\,[f_{o1}] = \frac{1}{n^2}\,[f_{o1}] \tag{61a}$$

and

$$[f_{o2}] = \frac{-ni}{n^2 - 1}\,[\alpha_2]\,[\varepsilon - \mathrm{I}]\,[f_{o1}]\,. \tag{61b}$$

Equation (61a), like equation (58), is an eigenvalue equation. However, equation (61a) involves the unknown vector $[f_{o1}]$ only. Equation (61b) allows $[f_{o2}]$ to be determined directly in terms of the solution $[f_{o1}]$ obtained from equation (61a). Thus, for non-magnetic materials, the electromagnetic wave equation matrix representation (equation 55) reduces to the simplify pair of matrix equations given by equation (61).

At this time, it is important to emphasize the following point. Either the Maxwell eigenvalue matrix representation (equation 58 or 59) or the electromagnetic wave equation matrix representation (equation 60 or 61) can be used to determine the eigenvalues (hence, eigenindices of refraction) and corresponding eigenvectors (hence, eigenstates of polarization) associated with the propagation of monochromatic electromagnetic plane-wave radiation in nonmagnetic media.

4.1 CRYSTALLINE MATERIALS

A basic feature of a crystalline material, as far as optical properties are concerned, is that crystals are generally electrically anisotropic. Hence, the macroscopic polarization produced in the crystal by an applied electric field varies in a manner that depends on the direction of the applied electric field in relation to the crystal lattice. It is a simple matter to show that for ordinary nonabsorbing crystals (Born & Wolf, 1965), there always exists a set of coordinate axes, called principal axes, such that the dielectric tensor $[\varepsilon]$ assumes the diagonal form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & 0 & 0 & 0 \\ 0 & \varepsilon_{22} & 0 & 0 \\ 0 & 0 & \varepsilon_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{62}$$

Using equation (57), we find the optical impermeability tensor is given by

$$[\eta] = \begin{bmatrix} \varepsilon_{11}^{-1} & 0 & 0 & 0 \\ 0 & \varepsilon_{22}^{-1} & 0 & 0 \\ 0 & 0 & \varepsilon_{33}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{63}$$

Substituting these expressions for $[\varepsilon]$ and $[\eta]$ back into the Maxwell eigenvalue matrix representation (equation 58) yields the following 8-by-8 eigenvalue equation:

$$\begin{bmatrix} 0 & 0 & 0 & -\alpha_x\varepsilon_{11}^{-1} & 0 & -\alpha_z\varepsilon_{11}^{-1} & \alpha_y\varepsilon_{11}^{-1} & 0 \\ 0 & 0 & 0 & -\alpha_y\varepsilon_{22}^{-1} & \alpha_z\varepsilon_{22}^{-1} & 0 & -\alpha_x\varepsilon_{22}^{-1} & 0 \\ 0 & 0 & 0 & -\alpha_z\varepsilon_{33}^{-1} & -\alpha_y\varepsilon_{33}^{-1} & \alpha_x\varepsilon_{33}^{-1} & 0 & 0 \\ \alpha_x\varepsilon_{11} & \alpha_y\varepsilon_{22} & \alpha_z\varepsilon_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_z & \alpha_y & 0 & 0 & 0 & 0 & -\alpha_x \\ \alpha_z & 0 & -\alpha_x & 0 & 0 & 0 & 0 & -\alpha_y \\ -\alpha_y & \alpha_x & 0 & 0 & 0 & 0 & 0 & -\alpha_z \\ 0 & 0 & 0 & 0 & \alpha_x & \alpha_y & \alpha_z & 0 \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \frac{i}{n} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} . \tag{64}$$

Substituting the expressions for $[\varepsilon]$ and $[\eta]$ back into the electromagnetic wave equation matrix representation (equations 61a and 61b) yields the following 3-by-3 eigenvalue equation:

$$\begin{bmatrix} \dfrac{1+\alpha_x^2(\varepsilon_{11}-1)}{\varepsilon_{11}} & \dfrac{\alpha_x\alpha_y(\varepsilon_{22}-1)}{\varepsilon_{11}} & \dfrac{\alpha_z\alpha_x(\varepsilon_{33}-1)}{\varepsilon_{11}} \\ \dfrac{\alpha_x\alpha_y(\varepsilon_{11}-1)}{\varepsilon_{22}} & \dfrac{1+\alpha_y^2(\varepsilon_{22}-1)}{\varepsilon_{22}} & \dfrac{\alpha_y\alpha_z(\varepsilon_{33}-1)}{\varepsilon_{22}} \\ \dfrac{\alpha_z\alpha_x(\varepsilon_{11}-1)}{\varepsilon_{33}} & \dfrac{\alpha_y\alpha_z(\varepsilon_{22}-1)}{\varepsilon_{33}} & \dfrac{1+\alpha_z^2(\varepsilon_{33}-1)}{\varepsilon_{33}} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{1}{n^2} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} , \tag{65a}$$

as well as the solution for the magnetic field in terms of the electric field

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \frac{n}{n^2-1} \begin{bmatrix} 0 & -\alpha_z(\varepsilon_{22}-1) & \alpha_y(\varepsilon_{33}-1) \\ \alpha_z(\varepsilon_{11}-1) & 0 & -\alpha_x(\varepsilon_{33}-1) \\ -\alpha_y(\varepsilon_{11}-1) & \alpha_x(\varepsilon_{22}-1) & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{65b}$$

A crystalline medium in which all principal dielectric tensor elements are different has two optic axes (Fowles, 1[illegible]8). For this case, the crystal is said to be biaxial. On the other hand, two of the principal dielectric tensor elements of uniaxial crystal are equal. A uniaxial crystal has a single optic axis. A linear, homogeneous, isotropic optical medium is commonly referred to as a Class A dielectric. The principal dielectric tensor elements of a dielectric are all equal. Vacuum is a special case of a dielectric. For vacuum, the principal dielectric tensor elements are unity. To illustrate the use of equations (64) and (65), let us consider wave propagation in a vacuum, a dielectric, and a uniaxial crystalline medium, respectively.

4.1.1 Vacuum

For a vacuous medium, the elements of the dielectric tensor $[\varepsilon]$ are given by

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 1. \tag{66}$$

Substituting these values of ε_{11}, ε_{22}, and ε_{33} back into equation (65a) gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{1}{n^2} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{67}$$

The 3-by-3 matrix on the left-side of equation (67a) is independent of the direction cosines. Inspection of equation (67) indicates the only physically acceptable value for the index of refraction is

$$n = 1 \tag{68}$$

for all possible states of polarization.

Substituting the values of $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ and n from equations (66) and (68), respectively, into the Maxwell eigenvalue matrix representation (equation 64) gives

$$\begin{bmatrix} 0 & 0 & 0 & -\alpha_x & 0 & -\alpha_z & \alpha_y & 0 \\ 0 & 0 & 0 & -\alpha_y & \alpha_z & 0 & -\alpha_x & 0 \\ 0 & 0 & 0 & -\alpha_z & -\alpha_y & \alpha_x & 0 & 0 \\ \alpha_x & \alpha_y & \alpha_z & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_z & \alpha_y & 0 & 0 & 0 & 0 & -\alpha_x \\ \alpha_z & 0 & -\alpha_x & 0 & 0 & 0 & 0 & -\alpha_y \\ -\alpha_y & \alpha_x & 0 & 0 & 0 & 0 & 0 & -\alpha_z \\ 0 & 0 & 0 & 0 & \alpha_x & \alpha_y & \alpha_z & 0 \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = i \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}. \tag{69}$$

Matrix equation (69) is equivalent to eight scalar equations. Six of these scalar equations correspond to the vector equations

$$\boldsymbol{E_o} = -(\hat{\alpha} \times \boldsymbol{H_o}) \tag{70a}$$

and

$$\boldsymbol{H_o} = +(\hat{\alpha} \times \boldsymbol{E_o}), \tag{70b}$$

and the remaining two scalar equations are equivalent to

$$\hat{\alpha} \bullet \boldsymbol{E_o} = 0 \tag{71a}$$

and

$$\hat{\alpha} \bullet \boldsymbol{H_o} = 0. \tag{71b}$$

$\hat{\alpha}$ is the unit vector whose components are α_x, α_y, and α_z . Equations (70) and (71) reveal the three vectors $\boldsymbol{E_o}, \boldsymbol{H_o}$, and $\hat{\alpha}$ are mutually orthogonal to one another. Also, the magnitudes of the electric and magnetic field vectors are equal. That is

$$H_o = E_o. \tag{72}$$

Thus, monochromatic electromagnetic plane-waves in vacuum propagate at the speed of light, namely c. The electric and magnetic field vectors associated with these waves are orthogonal to each other as well as to the direction of wave propagation. Such waves are called transverse waves. Any state of polarization is allowed as long as it satisfies the above transversality conditions.

4.1.2 Dielectrics

As previously mentioned, the principal dielectric tensor elements of a dielectric material are all equal

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon > 1. \tag{73}$$

Substituting these values of the principal dielectric tensor elements back into the 3-by-3 eigenvalue equation (65a) gives

$$\begin{bmatrix} \alpha_x^2 & \alpha_x\alpha_y & \alpha_z\alpha_x \\ \alpha_x\alpha_y & \alpha_y^2 & \alpha_y\alpha_z \\ \alpha_z\alpha_x & \alpha_y\alpha_z & \alpha_z^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{\varepsilon - n^2}{n^2(\varepsilon - 1)} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{74}$$

The only physically acceptable value for the index of refraction satisfying equation (75) is

$$n = \sqrt{\varepsilon}. \tag{75}$$

This value of the index of refraction applies to all possible states of polarization. Substituting the values of $\varepsilon_{11}, \varepsilon_{22}$, and ε_{33} back into the 8-by-8 eigenvalue equation (64) gives

$$\begin{bmatrix} 0 & 0 & 0 & -\alpha_x\varepsilon^{-1} & 0 & -\alpha_z\varepsilon^{-1} & \alpha_y\varepsilon^{-1} & 0 \\ 0 & 0 & 0 & -\alpha_y\varepsilon^{-1} & \alpha_z\varepsilon^{-1} & 0 & -\alpha_x\varepsilon^{-1} & 0 \\ 0 & 0 & 0 & -\alpha_z\varepsilon^{-1} & -\alpha_y\varepsilon^{-1} & \alpha_x\varepsilon^{-1} & 0 & 0 \\ \alpha_x\varepsilon & \alpha_y\varepsilon & \alpha_z\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_z & \alpha_y & 0 & 0 & 0 & 0 & -\alpha_x \\ \alpha_z & 0 & -\alpha_x & 0 & 0 & 0 & 0 & -\alpha_y \\ -\alpha_y & \alpha_x & 0 & 0 & 0 & 0 & 0 & -\alpha_z \\ 0 & 0 & 0 & 0 & \alpha_x & \alpha_y & \alpha_z & 0 \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \frac{i}{n} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}. \tag{76}$$

Matrix equation (76) is equivalent to eight scalar equations. Six of these scalar equations correspond to the vector equations

$$E_o = -\frac{1}{\sqrt{\varepsilon}}(\hat{\alpha} \times H_o) \tag{77a}$$

and

$$H_o = +\sqrt{\varepsilon}(\hat{\alpha} \times E_o), \tag{77b}$$

and the remaining two scalar equations are equivalent to

$$\hat{\alpha} \bullet E_o = 0 \tag{78a}$$

and

$$\hat{\alpha} \bullet H_o = 0. \tag{78b}$$

Equations (77) and (78) tell us the three vectors E_o, H_o, and $\hat{\alpha}$ are mutually orthogonal to one another. Also, the magnitudes of the electric and magnetic field vectors are related by the equation

$$H_o = \sqrt{\varepsilon}\, E_o. \tag{79}$$

So monochromatic electromagnetic plane-waves in a dielectric optical medium propagate at the speed of c/n, where the index of refraction n is given by equation (75). The electric and magnetic field vectors associated with these waves are orthogonal to each other as well as the direction of wave propagation. Any state of polarization is allowed providing it satisfies the transversality conditions (equations 77 and 78).

4.1.3 Uniaxial Crystals

As previously indicated, two of the principal dielectric tensor elements of a uniaxial crystal are equal. Also, a uniaxial crystal has a single optic axis. Without loss of generality, we will take the z-axis as the optic axis of the uniaxial crystal. The dielectric tensor elements then have the form

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_o \quad \text{and} \quad \varepsilon_{33} = \varepsilon_e. \tag{80}$$

ε_o is referred to as the ordinary dielectric constant and ε_e is called the extraordinary dielectric constant.

For both uniaxial and biaxial crystals, the speed of propagation of a light wave in the crystal is a function of the direction of propagation and the polarization of the light. There exists, in general, two possible values of the phase velocity for a given direction of propagation. These two values are associated with mutually orthogonal polarizations of the light waves. Crystals are thus said to be doubly refracting or birefringent. The 8-by-8 eigenvalue equation (64) or the 3-by-3 eigenvalue equation (65a), in conjunction with equation (65b), can be used to find these phase velocities and corresponding polarizations states. Let us now look at some specific examples to illustrate these ideas.

Example 1: Wave Propagation Along the x-Axis. The direction cosines describing wave propagation in the positive x-direction are given by

$$\alpha_x = 1, \qquad \alpha_y = 0, \qquad \alpha_z = 0. \tag{81}$$

Substituting the dielectric tensor elements from equation (80) and the direction cosines from equation (81) back into the 8-by-8 eigenvalue equation (64) gives

$$\begin{bmatrix} 0 & 0 & 0 & -\varepsilon_o^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_o^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_e^{-1} & 0 & 0 \\ \varepsilon_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \frac{i}{n} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}. \tag{82}$$

It is a simple matter to show that only two physically acceptable eigenvalues satisfy equation (82). The corresponding eigenindices of refraction are given by

$$n = \sqrt{\varepsilon_o} \qquad \text{and} \qquad n = \sqrt{\varepsilon_e}. \tag{83}$$

Associated with each value of n is a corresponding eigenvector solution (eigenstate of polarization). The eigenvector solutions are

$$\begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = C \begin{bmatrix} 0 \\ i \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\varepsilon_o} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = C \begin{bmatrix} 0 \\ 0 \\ i \\ 0 \\ 0 \\ -\sqrt{\varepsilon_e} \\ 0 \\ 0 \end{bmatrix}, \tag{84}$$

where C is an arbitrary constant. These two eigenvector solutions represent orthogonal states of linearly polarized electromagnetic radiation.

Substituting the dielectric tensor elements from equation (80) and the direction cosines from equation (81) back into equations (65a) and (65b), respectively, gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_o^{-1} & 0 \\ 0 & 0 & \varepsilon_e^{-1} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{1}{n^2} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{85a}$$

and

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \frac{n}{n^2 - 1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -(\varepsilon_e - 1) \\ 0 & (\varepsilon_o - 1) & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{85b}$$

It can be shown that only two physically acceptable eigenvalues satisfy equation (85a). The corresponding eigenindices of refraction are again given by equation (83). The corresponding eigenvectors (electric fields) obtained from equation (85a), and subsequent solutions (magnetic fields) obtained from equation (85b), are identical to the electric and magnetic field solutions summarized in equation (84). We see the Maxwell eigenvalue matrix representation (equation 64) and the electromagnetic wave equation matrix representation (equation 65) yield identical results. As previously indicated, either representation can be used in solving wave-propagation problems in crystalline media.

Example 2: Wave Propagation in the xz-Plane. Our second example concerns wave propagation in the xz-plane. Suppose the direction of wave propagation is such that the wavevector $\boldsymbol{k}$ makes an angle of 45 degrees with respect to both the positive x- and z-axes. The direction cosines associated with this example are given by

$$\alpha_x = \frac{\sqrt{2}}{2}, \qquad \alpha_y = 0, \qquad \alpha_z = \frac{\sqrt{2}}{2}. \tag{86}$$

Substituting the dielectric tensor elements from equation (80) and the direction cosines from equation (86) back into the 8-by-8 eigenvalue equation (64) gives

$$\begin{bmatrix} 0 & 0 & 0 & -\varepsilon_o^{-1} & 0 & -\varepsilon_o^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_o^{-1} & 0 & -\varepsilon_o^{-1} & 0 \\ 0 & 0 & 0 & -\varepsilon_e^{-1} & 0 & \varepsilon_e^{-1} & 0 & 0 \\ \varepsilon_o & 0 & \varepsilon_e & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \frac{i\sqrt{2}}{n} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}. \tag{87}$$

It can be easily shown that only two physically acceptable eigenvalues satisfy equation (87). The corresponding eigenindices of refraction are given by the following formula:

$$n = \sqrt{\varepsilon_o} \qquad \text{and} \qquad n = \sqrt{\frac{2\varepsilon_o \varepsilon_e}{\varepsilon_o + \varepsilon_e}}. \tag{88}$$

Associated with each value of n is a corresponding eigenvector solution (eigenstate of polarization). The eigenvector solutions are

$$\begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = C \begin{bmatrix} 0 \\ i \\ 0 \\ 0 \\ -\sqrt{\varepsilon_o/2} \\ 0 \\ +\sqrt{\varepsilon_o/2} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = C \begin{bmatrix} i \\ 0 \\ -i\,(\varepsilon_o/\varepsilon_e) \\ 0 \\ 0 \\ \sqrt{(\varepsilon_o/\varepsilon_e)\,(\varepsilon_o+\varepsilon_e)} \\ 0 \\ 0 \end{bmatrix}, \tag{89}$$

where C is an arbitrary constant. These two eigenvector solutions represent orthogonal states of linearly polarized electromagnetic radiation.

Example 3: Wave Propagation Along the Optic Axis. Our third example deals with wave propagation along the optic axis. Recall the optic axis lies along the z-axis. The direction cosines describing wave propagation in the positive z-direction are given by

$$\alpha_x = 0, \qquad \alpha_y = 0, \qquad \alpha_z = 1. \tag{90}$$

Substituting the dielectric tensor elements from equation (80) and the direction cosines from equation (90) back into the 8-by-8 eigenvalue equation (64) gives the following eigenvalue equation:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\varepsilon_o^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_o^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_e^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_e & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \frac{i}{n} \begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix}. \qquad (91)$$

It can be easily shown that only one physically acceptable eigenvalue satisfies equation (91). The corresponding index of refraction is given by the following:

$$n = \sqrt{\varepsilon_o}. \qquad (92)$$

The corresponding eigenvector solution is given by

$$\begin{bmatrix} iE_x \\ iE_y \\ iE_z \\ 0 \\ H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \begin{bmatrix} iC_x \\ iC_y \\ 0 \\ 0 \\ -\sqrt{\varepsilon_o}C_y \\ +\sqrt{\varepsilon_o}C_x \\ 0 \\ 0 \end{bmatrix}, \qquad (93)$$

where C_x and C_y are arbitrary constants. This eigenvector solution describes monochromatic electromagnetic plane-waves propagating along the z-direction (optic axis) with speed of c/n. The electric and magnetic field vectors of this wave are orthogonal to each other as well as the direction of wave propagation. Any state of polarization is allowed providing it satisfies the transversality conditions summarized in equation (93).

4.2 OPTICALLY ACTIVE MATERIALS

Certain optical materials are found to possess the ability to rotate the plane of polarization of electromagnetic radiation passing through them. This phenomenon is commonly referred to as optical activity. Optical activity can be explained on the basis of the assumption that the speed of propagation for left circularly polarized light in the material is different from that of right circularly polarized light. If a dielectric material is placed in a static magnetic field and a beam of linearly polarized light is sent through the dielectric material in the direction of the applied magnetic field, then a rotation of the plane of polarization of the emerging light is found to occur. In other words, the presence of the magnetic field causes the dielectric material to become optically active. This is known as the Faraday effect.

It is a simple matter to show that if the dielectric tensor has conjugate imaginary off-diagonal elements (Fowles, 1968), namely

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & +i\varepsilon_{12} & 0 & 0 \\ -i\varepsilon_{12} & \varepsilon_{11} & 0 & 0 \\ 0 & 0 & \varepsilon_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{94}$$

where ε_{12} is real, then the tensor describes an optically active medium. For the case of the Faraday effect, the off-diagonal dielectric tensor element ε_{12} is proportional to the applied static magnetic field strength assumed to lie along the z-axis. The optic axis of the material also lies along the direction of the applied magnetic field. Note the dielectric tensor (equation 94) is identical to that of a uniaxial crystal aside from the off-diagonal elements.

Using equation (63), we find that the optical impermeability tensor has the form

$$[\eta] = \begin{bmatrix} \dfrac{\varepsilon_{11}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & \dfrac{+i\varepsilon_{12}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & 0 & 0 \\ \dfrac{-i\varepsilon_{12}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & \dfrac{\varepsilon_{11}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\varepsilon_{33}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{95}$$

Let us now look at another example to illustrate these ideas.

Example 4: Wave Propagation Along the Optic Axis. For this case we have

$$\alpha_x = 0, \qquad \alpha_y = 0, \qquad \alpha_z = 1. \tag{96}$$

Substituting the dielectric tensor (equation 94), the optical impermeability tensor (equation 95), and the direction cosines (equation 96) back into the electromagnetic wave equation matrix representation (equation 61a) gives the 3-by-3 eigenvalue equation

$$\begin{bmatrix} \dfrac{\varepsilon_{11}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & \dfrac{-i\varepsilon_{12}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & 0 \\ \dfrac{+i\varepsilon_{12}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & \dfrac{\varepsilon_{11}}{\varepsilon_{11}^2 - \varepsilon_{12}^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \left(\frac{1}{n^2}\right) \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{97}$$

It can be easily shown only two physically acceptable eigenvalues satisfy equation (97). The corresponding eigenindices of refraction are given by the following formula

$$n = \sqrt{\varepsilon_{11} - \varepsilon_{12}} \qquad \text{and} \qquad n = \sqrt{\varepsilon_{11} + \varepsilon_{12}}. \tag{98}$$

Associated with each value of the index of refraction n is a corresponding eigenvector solution (eigenstate of polarization). The eigenvector solutions are

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = C \begin{bmatrix} 1 \\ +i \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = C \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \tag{99}$$

where C is an arbitrary constant. These two eigenvector solutions represent orthogonal states of circularly polarized electromagnetic radiation.

4.3 ELECTROOPTICAL MATERIALS

In this subsection, we consider the propagation of electromagnetic radiation in a crystalline material in the presence of an applied static electric field. For certain types of crystals, the application of an electric field results in a change of the optical birefringent properties of the crystal. This is referred to as the electrooptic effect. The electrooptic effect affords a convenient means of controlling the phase or intensity of electromagnetic radiation. The electrooptic effect is used in a number of applications, including optical modulation, optical beam deflection, and spectral tunable filters (Yariv & Yeh, 1984).

There are primarily three electrooptic effects described to various degrees in the literature. They are (1) the Pockels effect, (2) the Kerr effect, and (3) the Stark effect. The Pockels effect is concerned with the alteration in the refractive properties in a piezoelectric crystal by the application of a strong electric field. The effect is proportional to the first power of the electric field. The Kerr effect deals with the occurrence of birefringence in a transparent isotropic medium when it is placed in an electric field. The medium then behaves like a uniaxial crystal with its optic axis lying in the direction of the applied electric field. The effect is proportional to the square of the electric field strength. The Stark effect concerns the displacement and splitting of the lines in atomic spectra, and the appearance of new lines, owing to the influence of a transverse electric field. For pedagogical purposes, we will only concern ourselves with the electrooptic Pockels effect. A detailed discussion on the various electrooptic effects can be found in the book of selected reprints on electrooptic devices by Kaminow (1974).

A popular method for determining the eigenindices of refraction and corresponding eigenvectors associated with wave propagation through electrooptical materials is through the use of the index ellipsoid (or optical indicatrix) describing the electrooptic crystal of interest (Yariv & Yeh, 1984). For the case of the Pockels effect, the mathematical form of the index ellipsoid equation is given by

$$\eta_{11}x^2 + \eta_{22}y^2 + \eta_{33}z^2 + 2\eta_{23}yz + 2\eta_{31}zx + 2\eta_{12}xy = 1, \tag{100}$$

where

$$\begin{bmatrix} \eta_{11} \\ \eta_{22} \\ \eta_{33} \\ \eta_{23} \\ \eta_{31} \\ \eta_{12} \end{bmatrix} = \begin{bmatrix} 1/n_x^2 \\ 1/n_y^2 \\ 1/n_z^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{bmatrix} \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix}. \tag{101}$$

The quantities η_{ij} are the elements of the optical impermeability tensor, r_{ij} are the linear (or Pockels) electrooptic coefficients, ξ_i are the components of the applied static electric field along the three principle crystallographic axes, and n_j are the principal indices of refraction of the electrooptic crystal in the absence of the applied electric field.

An alternative method for determining the eigenindices of refraction and corresponding eigenvectors is through the use of either the Maxwell eigenvalue matrix representation (equation 58) or the electromagnetic wave equation matrix representation (equation 61). The dielectric tensor $[\varepsilon]$ can be determined from the optical impermeability tensor $[\eta]$ with the help of equation (57), namely

$$[\varepsilon] \equiv [\eta]^{-1}. \tag{102}$$

With the help of equation (101), the optical impermeability tensor can be written in a form compatible with both equations (58) and (61), namely

$$[\eta] = [\eta_o] + [\Delta\eta], \tag{103}$$

where, by definition

$$[\eta_o] \equiv \begin{bmatrix} 1/n_x^2 & 0 & 0 & 0 \\ 0 & 1/n_y^2 & 0 & 0 \\ 0 & 0 & 1/n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{104}$$

and

$$[\Delta\eta] = \xi_x \begin{bmatrix} r_{11} & r_{61} & r_{51} & 0 \\ r_{61} & r_{21} & r_{41} & 0 \\ r_{51} & r_{41} & r_{31} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \xi_y \begin{bmatrix} r_{12} & r_{62} & r_{52} & 0 \\ r_{62} & r_{22} & r_{42} & 0 \\ r_{52} & r_{42} & r_{32} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \xi_z \begin{bmatrix} r_{13} & r_{63} & r_{53} & 0 \\ r_{63} & r_{23} & r_{43} & 0 \\ r_{53} & r_{43} & r_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{105}$$

Let us look at one final example, now illustrating wave propagation in an electrooptical medium.

Example 5: Bismuth Silicon Oxide ($Bi_{12}SiO_{20}$): Bismuth silicon oxide belongs to the cubic class of crystals (Yariv & Yeh, 1984), hence

$$[\eta_o] = \begin{bmatrix} 1/n_o^2 & 0 & 0 & 0 \\ 0 & 1/n_o^2 & 0 & 0 \\ 0 & 0 & 1/n_o^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{106}$$

In addition,

$$r_{41} = r_{52} = r_{63}. \tag{107}$$

All other Pockels (linear) electrooptic coefficients are equal to zero for bismuth silicon oxide, hence equation (105) simplifies to

$$[\Delta\eta] = \xi_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r_{41} & 0 \\ 0 & r_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \xi_y \begin{bmatrix} 0 & 0 & r_{41} & 0 \\ 0 & 0 & 0 & 0 \\ r_{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \xi_z \begin{bmatrix} 0 & r_{41} & 0 & 0 \\ r_{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{108}$$

Suppose, for this example, that the applied static electric field points only along the z-axis. That is,

$$\xi_x = 0, \qquad \xi_y = 0, \qquad \xi_z \neq 0. \tag{109}$$

Also, let us consider wave propagation along the z-axis. For this case we have

$$\alpha_x = 0, \qquad \alpha_y = 0, \qquad \alpha_z = 1. \tag{110}$$

Substituting the optical impermeability tensor information (equations 103, 106, 108 and 109), dielectric tensor (equation 102), and the direction cosines (equation 110) into the electromagnetic wave equation matrix representation (equation 61a) gives the following 3-by-3 eigenvalue equation:

$$\begin{bmatrix} \frac{1}{n_o^2} & \xi_z r_{41} & 0 \\ \xi_z r_{41} & \frac{1}{n_o^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \left(\frac{1}{n^2}\right) \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{111}$$

It can be shown only two physically acceptable eigenvalues satisfy equation (111). The corresponding eigenindices of refraction are given by the following formula

$$n = \frac{n_o}{\sqrt{1 + \xi_z r_{41} n_o^2}} \quad \text{and} \quad n = \frac{n_o}{\sqrt{1 - \xi_z r_{41} n_o^2}}. \tag{112}$$

Associated with each value of the index of refraction n is a corresponding eigenvector solution (eigenstate of polarization). The eigenvector solutions are

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = C \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = C \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \tag{113}$$

where C is an arbitrary constant. These two eigenvector solutions represent orthogonal states of linearly polarized electromagnetic radiation.

5.0 SUMMARY AND CONCLUSIONS

From the vector form of the Maxwell field equations (4), an 8-by-8 covariant matrix formulation of Maxwell's equations (8) was developed. Tantamount to this formulation is a skew-Hermitian space-time 8-by-8 differential matrix operator [M] defined in equation (1). In addition, other landmark effects of classical electromagnetic theory were easily derived with the use of the 8-by-8 *matrix* operator [M]: (a) the electromagnetic wave and charge continuity equations (11); (b) definition of the electromagnetic potentials (employing the Lorentz gauge) in terms of the electromagnetic fields, and the Lorentz conditions (equation 17); (c) the wave equations for the electromagnetic potentials (equation 23); and (d) Poynting's theorem (equation 26) on energy conservation. These matrix-based derivations completely avoid the need for vector calculus and attendant special-purpose identities involving multiple curls and divergences. The necessary mathematical baggage is strongly reduced by the approach. Note that the extension of the matrix [M] into a square form, described above, is essential to the above electromagnetic derivations.

The matrix form of the Maxwell field equations (8) were then cast into the Maxwell eigenvalue 8-by-8 matrix representation (equation 51). This eigenvalue matrix representation was used to solve for the eigenindices of refraction and corresponding polarization eigenstates for a variety of wave-propagation problems dealing with linear homogeneous anisotropic optical media of infinite extent in the presence of monochromatic plane-wave electromagnetic fields. In a similar manner, the electromagnetic wave and continuity equations (11) were used to formulate a wave equation eigenvalue 3-by-3 matrix representation (equation 61) for nonmagnetic materials. This eigenvalue representation can also be used to find the eigenindices of refraction and polarization eigenstates for monochromatic plane-wave electromagnetic fields.

A variety of examples were considered to illustrate the usefulness of these matrix eigenvalue representations in solving wave-propagation problems. In particular, the eigenindices of refraction and corresponding polarization eigenstates associated with wave propagation in vacuum, dielectrics, uniaxial crystalline media, optically active media and electrooptical media were easily obtained. Off-the-shelf computer software packages, like MATLAB for numerical computations and MATHEMATICA for symbolic manipulations, are well suited for finding these eigenindices and polarization eigenstates.

6.0 REFERENCES

Bocker, R.P., and B.R. Frieden. 1992. "A New Matrix Formulation of Classical Electrodynamics. Part I. Vacuum." NCCOSC TD 2259. Naval Command, Control, and Ocean Surveillance Center, RDT&E Division, San Diego, CA.

Born, M., and E. Wolf. 1965. *Principles of Optics*, Pergamon Press, Oxford, England.

Fowles, G.R. 1968. *Introduction to Modern Optics*, Holt, Rinehart and Winston, Inc., New York, NY.

Jackson, J.D. 1962. *Classical Electrodynamics*, John Wiley and Sons, Inc., New York, NY.

Kaminow, I.P., Ed., 1974. *An Introduction to Electrooptic Devices*, Academic Press, New York, NY.

Magid, L.M. 1972. *Electromagnetic Fields, Energy, and Waves*, John Wiley and Sons, Inc., New York, NY.

Morrill, W.K. 1961. *Analytic Geometry*, International Textbook Company, Scranton, PA.

Ohanian, H.C. 1988. *Classical Electrodynamics*, Allyn and Bacon, Inc., Boston, MA.

Yariv, A. 1975. *Quantum Electronics*, John Wiley & Sons, New York, NY.

Yariv, A., and P. Yeh. 1984. *Optical Waves in Crystals*, John Wiley & Sons, New York, NY.

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	July 1992	Final

4. TITLE AND SUBTITLE

A NEW MATRIX FORMULATION OF CLASSICAL ELECTRODYNAMICS
Part II: Wave Propagation in Optical Materials of Infinite Extent

5. FUNDING NUMBERS

PE: 0601152N
WU: DN302006

6. AUTHOR(S)

R. P. Bocker

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Command, Control and Ocean Surveillance Center (NCCOSC)
RDT&E Division (NRaD)
San Diego, CA 92152–5000

8. PERFORMING ORGANIZATION REPORT NUMBER

NRaD TD 2320

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Chief of Naval Research
Arlington, VA 22217–5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

Presented in this document is the development of a new matrix description of electromagnetic wave propagation in optical media of infinite extent. This material will interest individuals desiring a description of electromagnetic wave propagation that deviates from the traditional vector calculus approach. Our starting point will be with the fundamental equations of classical electrodynamics, namely the Maxwell field equations. From the vector form of Maxwell's equations, and 8-by-8 differential matrix operator formulation of Maxwell's equations will be developed. The matrix form of the Maxwell field equations allows for simple and direct derivation of matrix representations of the electromagnetic wave and charge continuity equations, the Lorentz conditions and definition of the electromagnetic potentials, the electromagnetic potential wave equations, and Poynting's conservation of energy theorem.

The matrix form of the Maxwell field equations and the electromagnetic wave and continuity equations will be used to solve a variety of wave-propagation problems dealing with linear, homogeneous, anisotropic optical media of infinite extent in the presence of monochromatic plane-wave electromagnetic fields. The indices of refraction as well as corresponding states of polarization, associated with wave propagation in crystalline, optically active, and electrooptical media, will be determined by using these matrix representations.

14. SUBJECT TERMS	15. NUMBER OF PAGES
electromagnetic wave propagation Maxwell field equations plane-wave electromagnetic fields	42
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAME AS REPORT

21a. NAME OF RESPONSIBLE INDIVIDUAL	21b. TELEPHONE (Include Area Code)	21c. OFFICE SYMBOL
R. P. Bocker	(619) 553–6404	Code 761

INITIAL DISTRIBUTION

Code 0012	T. Keough	(1)	Code 761	J. P. Loughlin	(1)
Code 0141	A. Gordon	(1)	Code 80	K. D. Regan	(1)
Code 0141	G. Dillard	(1)	Code 804	J. W. Rockway	(1)
Code 144	V. Ware	(1)	Code 804	E. J. Schimitschek	(1)
Code 414	T. Phillips	(1)	Code 804	J. R. Zeidler	(1)
Code 575	H. E. Vanbrundt	(1)	Code 804	E. Munn	(1)
Code 653	K. Swecker	(1)	Code 804	S. Hoff	(1)
Code 70	R. E. Shutters	(1)	Code 804	G. W. Geaghler	(1)
Code 743	H. J. Whitehouse	(1)	Code 805	M. S. Kvigne	(1)
Code 75	J. E. Griffin	(1)	Code 82	R. J. Kochanski	(1)
Code 7502	B. R. Hung	(1)	Code 824	J. C. Logan	(1)
Code 76	J. R. Wangler	(1)	Code 832	P. M. Hansen	(1)
Code 761	R. P. Bocker	(10)	Code 843	M. H. Berry	(1)
Code 761	G. W. Byram	(1)	Code 843	M. Lasher	(1)
Code 761	S. I. Chou	(1)	Code 844	B. James	(1)

Defense Technical Information Center
Alexandria, VA 22304-6145 (4)

NCCOSC Washington Liaison Office
Washington, DC 20363-5101

Center for Naval Analyses
Alexandria, VA 22302-0268

Navy Acquisition, Research & Development
Information Center (NARDIC)
Washington, DC 20360-5000