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ABSTRACT

The problem of linearly constrained least squares has many applications in signal processing. In this paper, we present a perturbation analysis of a linearly constrained least squares algorithm for adaptive beamforming. The perturbation bounds for the solution as well as for the latest residual element are derived. We also propose an error estimation scheme for the residual element, which can be incorporated into a systolic array implementation of the algorithm.

1. INTRODUCTION

The least squares problem with linear equality constraints has important applications in signal processing, e.g., adaptive beamforming. To solve this problem, McWhirter and Shepherd [5] proposed a systolic algorithm and architecture. In this paper, we present a perturbation analysis of the problem and propose an error estimation scheme for the McWhirter-Shepherd (MS) algorithm [5]. This paper is organized as follows. The least squares problem is defined in Section 2 and error bounds are derived in Section 3. An error estimation algorithm is given in Section 4, and in Section 5 a numerical example is presented to illustrate how well our new algorithm works.

2. PROBLEM DEFINITION

Given an $n \times q$ complex data matrix $X(n)$, the least squares problem with linear equality constraints is to find a q-element complex vector $w(n)$ such that

$$\|X(n)w(n)\| = \min \tag{2.1a}$$

subject to the linear constraints

$$Sw(n) = b, \tag{2.1b}$$

where S is a $k \times q$ $(k < q)$ complex matrix and b is a k-element complex vector. Throughout this paper, we use the 2-norm:

$$\|\cdot\| = \|\cdot\|_2.$$

In signal processing, new data arrives continuously. Define the data matrix $X(n)$ recursively by

$$X(n) \equiv \begin{pmatrix} X(n-1) \\ x(n)^T \end{pmatrix},$$

i.e., the nth row $x(n)^T$ represents a snapshot at time n. Our goal is to compute the n-th residual element

$$r_n = x(n)^T w(n). \tag{2.2}$$

Is the solution vector $w(n)$ unique? Define a $(k+n) \times q$ matrix $S_X(n)$ by

$$S_X(n) \equiv \begin{pmatrix} S \\ X(n) \end{pmatrix}.$$

We assume that $k + n \geq q$. The solution is unique if and only if the matrix $S_X(n)$ has full column rank; that is, the overdetermined matrix equation

$$S_X(n)w(n) = 0 \tag{2.3}$$

has a unique solution $w(n) = 0$.

Next, we wish to transform (2.1) into a familiar unconstrained problem; see [3] and [4]. Let

$$p = q - k$$

and partition the matrix S as

$$S = (\, S_1 \quad S_2 \,),$$

where S_1 is $k \times k$ and S_2 is $k \times p$. For simplicity, we assume that S_1 is nonsingular and upper triangular; for example, S_1 may be the result of an initial QR decomposition of S. Accordingly, we also partition $X(n)$ as

$$X(n) = (\, X_1(n) \quad X_2(n) \,),$$

so that X_1 is $n \times k$ and X_2 is $n \times p$. Then (2.3) becomes

$$\begin{pmatrix} S_1 & S_2 \\ X_1(n) & X_2(n) \end{pmatrix} w(n) = 0,$$

which is equivalent to

$$\begin{pmatrix} S_1 & S_2 \\ 0 & C(n) \end{pmatrix} w(n) = 0,$$

where

$$C(n) \equiv X_2(n) - X_1(n)S_1^{-1}S_2.$$

The matrix $C(n)$ is called the Schur complement of S_1 in S_X. The equation (2.3) has the trivial solution if and only if $C(n)$ has full column rank. We proceed to eliminate the constraints. Let

$$w(n) = \begin{pmatrix} w_1(n) \\ w_2(n) \end{pmatrix},$$

so that $w_1(n)$ is $k \times 1$ and $w_2(n)$ is $p \times 1$. Since

$$S_1 w_1(n) + S_2 w_2(n) = b,$$

we get

$$w_1(n) = S_1^{-1}b - S_1^{-1}S_2 w_2(n). \tag{2.4}$$

Let

$$v(n) = -X_1(n)S_1^{-1}b.$$

We derive

$$\|C(n)w_2(n) - v(n)\| = \min, \tag{2.5}$$

an unconstrained problem analyzed in [3], [4]. Now, what about the residual element r_n? Define the Schur complement matrix $C(n)$ recursively by

$$C(n) \equiv \begin{pmatrix} C(n-1) \\ c(n)^T \end{pmatrix}.$$

Partition the row vector $x(n)^T$ so that

$$x(n)^T = (x_1(n)^T \quad x_2(n)^T),$$

where $x_1(n)^T$ is $1 \times k$ and $x_2(n)^T$ is $1 \times p$. We get

$$c(n)^T = x_2(n)^T - x_1(n)^T S_1^{-1} S_2.$$

Let v_n denote the n-th element of $v(n)$. The last residual element of (2.5) is then

$$c(n)^T w_2(n) - v_n = x_2(n)^T w_2(n) + x_1(n)^T w_1(n) + v_n - v_n = r_n,$$

i.e., the same residual element as desired by the constrained problem (2.1).

How do we calculate r_n recursively? Suppose that we have available a QR decomposition of the $(n-1) \times p$ matrix $C(n-1)$:

$$C(n-1) = Q(n-1)R(n-1),$$

where $Q(n-1)$ is $(n-1) \times p$ with orthonormal columns and the matrix $R(n-1)$ is $p \times p$ upper triangular. The problem (2.5) is reduced to

$$\left\| \begin{pmatrix} R(n-1) \\ c(n)^T \end{pmatrix} w_2(n) - \begin{pmatrix} u(n-1) \\ v_n \end{pmatrix} \right\| = \min,$$

where $u(n-1) = Q(n-1)^H v(n-1)$. We triangularize the coefficient matrix by a unitary matrix P. Then

$$P^H \begin{pmatrix} R(n-1) & u(n-1) \\ c(n)^T & v_n \end{pmatrix} = \begin{pmatrix} R(n) & u(n) \\ 0^T & \gamma \end{pmatrix},$$

so that $R(n)$ is $p \times p$ upper triangular. The matrix P consists of p Givens matrices. From P and $Q(n-1)$ we can construct an $n \times p$ orthonormal matrix $Q(n)$ such that $C(n) = Q(n)R(n)$ and $u(n) = Q(n)^H v(n)$. The desired element r_n is given by

$$r_n = -(c_1 \ldots c_p)\gamma,$$

where $c_1, \ldots, c_p$ denote cosines of the p rotations that make up P.

3. PERTURBATION ANALYSIS

Eldén [1] presented a perturbation analysis of the linearly constrained least squares problem. Since his theory is general, it involves weighted pseudoinverses and their corresponding condition numbers. In this section, we derive simpler perturbation bounds for the solution $w(n)$ as well as for the residual element r_n. To simplify our presentation, we will drop the argument (n) for the matrices and vectors, and let $\kappa(M)$ denote the condition number of a matrix M with respect to the 2-norm.

Let $\hat{w}$ solve the perturbed least squares problem

$$\| (X_1 + \epsilon E_{X_1} \quad X_2 + \epsilon E_{X_2}) \hat{w} \| = \min \tag{3.1a}$$

subject to the perturbed linear equality constraints

$$(S_1 + \epsilon E_{S_1} \quad S_2 + \epsilon E_{S_2}) \hat{w} = b + \epsilon f_b. \tag{3.1b}$$

Suppose that $t \geq 0$ is a real variable and let

$$C + tE_C = (X_2 + tE_{X_2}) - (X_1 + tE_{X_1})(S_1 + tE_{S_1})^{-1}(S_2 + tE_{S_2})$$

and

$$v + tf_v = -(X_1 + tE_{X_1})(S_1 + tE_{S_1})^{-1}(b + tf_b).$$

Recall that S_1 is nonsingular and that C has full column rank. Suppose ϵ is sufficiently small so that for $t \in [0, \epsilon]$ we have $S_1 + tE_{S_1}$ is nonsingular and $C + tE_C$ has full column rank. Let $w(t)$ solve the matrix equation

$$\begin{pmatrix} S_1 + tE_{S_1} & S_2 + tE_{S_2} \\ 0 & (C + tE_C)^H (C + tE_c) \end{pmatrix} w(t) = \begin{pmatrix} b + tf_b \\ (C + tE_C)^H (v + tf_v) \end{pmatrix}. \tag{3.2}$$

Then $w(0)$ and $w(\epsilon)$ are solutions to problems (2.1) and (3.1), respectively. Define $w \equiv w(0)$ and $\hat{w} \equiv w(\epsilon)$. Then

$$\hat{w} = w(0) + \epsilon \dot{w}(0) + O(\epsilon^2).$$

Differentiate (3.2) with respect to t and set $t = 0$. We get

$$\begin{pmatrix} E_{S_1} & E_{S_2} \\ 0 & E_C^H C + C^H E_C \end{pmatrix} w(0) + \begin{pmatrix} S_1 & S_2 \\ 0 & C^H C \end{pmatrix} \dot{w}(0) = \begin{pmatrix} f_b \\ E_C^H v + C^H f_v \end{pmatrix}. \tag{3.3}$$

Let

$$\tilde{S} \equiv \begin{pmatrix} S_1 & S_2 \\ 0 & I \end{pmatrix}, \quad \tilde{C} \equiv \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}, \quad d \equiv \begin{pmatrix} b \\ v \end{pmatrix} \quad \text{and} \quad f_d \equiv \begin{pmatrix} f_b \\ f_v \end{pmatrix}.$$

Then

$$\begin{pmatrix} S_1 & S_2 \\ 0 & C^H C \end{pmatrix}^{-1} = \tilde{S}^{-1} (\tilde{C}^H \tilde{C})^{-1} \quad \text{and} \quad \|C\| \le \|\tilde{C}\|.$$

Solving for $\dot{w}(0)$ in (3.3), we obtain

$$\begin{aligned} \dot{w}(0) &= \begin{pmatrix} S_1 & S_2 \\ 0 & C^H C \end{pmatrix}^{-1} \left[\begin{pmatrix} f_b \\ C^H f_v \end{pmatrix} - \begin{pmatrix} E_{S_1} & E_{S_2} \\ 0 & C^H E_C \end{pmatrix} w + \begin{pmatrix} 0 \\ E_C^H v \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & E_C^H C \end{pmatrix} w \right] \\ &= \tilde{S}^{-1} (\tilde{C}^H \tilde{C})^{-1} \tilde{C}^H \left[f_d - \begin{pmatrix} E_{S_1} & E_{S_2} \\ 0 & E_C \end{pmatrix} w \right] - \tilde{S}^{-1} (\tilde{C}^H \tilde{C})^{-1} \begin{pmatrix} 0 \\ E_C^H r \end{pmatrix}, \end{aligned} \tag{3.4}$$

where $r \equiv Cw_2 - v$ denotes the residual vector. Furthermore, by assuming

$$\|f_b\| \le \|b\|, \quad \|f_v\| \le \|v\|, \quad \|E_C\| \le \|C\| \tag{3.5a}$$

and

$$\left\| \begin{pmatrix} E_{S_1} & E_{S_2} \\ 0 & E_C \end{pmatrix} \right\| \le \left\| \begin{pmatrix} S_1 & S_2 \\ 0 & C \end{pmatrix} \right\| \le \|\tilde{S}\| \, \|\tilde{C}\|, \tag{3.5b}$$

we derive the inequality

$$\|\dot{w}(0)\| \le \|\tilde{S}^{-1}\| \, \|(\tilde{C}^H \tilde{C})^{-1} \tilde{C}^H\| \left[\|d\| + \|\tilde{C}\| \, \|\tilde{S}\| \, \|w\| \right] + \|\tilde{S}^{-1}\| \, \|(\tilde{C}^H \tilde{C})^{-1}\| \, \|\tilde{C}\| \, \|r\|.$$

Consequently, we obtain the following perturbation result.

Lemma. *Using the notations defined above and assuming that ϵ in (3.1) is sufficiently small so that the inequalities (3.5) are satisfied, we get*

$$\frac{\|\hat{w} - w\|}{\|w\|} \le \epsilon \left\{ \kappa(\tilde{S}) \kappa(\tilde{C}) \left(\frac{\|d\|}{\|\tilde{C}\| \, \|\tilde{S}\| \, \|w\|} + 1 \right) + \kappa(\tilde{S}) \kappa(\tilde{C})^2 \frac{\|r\|}{\|\tilde{C}\| \, \|\tilde{S}\| \, \|w\|} \right\} + O(\epsilon^2). \ \square \tag{3.6}$$

To illustrate the effect of $\kappa(\tilde{S})$ on the solution of (2.1), consider a simple example in which $S = (\, S_1 \quad 0 \,)$ and $X = (\, I_n \quad I_n \,)$, where I_n is an $n \times n$ identity matrix and $n \le k < 2n$. By observation, $w_1 = S_1^{-1} b$ and $w_2 = -w_1$. Since $\kappa(\tilde{S}) = \kappa(S_1)$ in this example, we see why the presence of $\kappa(\tilde{S})$ is necessary in (3.6).

We proceed to derive a bound for the error in the residual. Let

$$\begin{pmatrix} 0 \\ r(t) \end{pmatrix} = \begin{pmatrix} S_1 + tE_{S_1} & S_2 + tE_{S_2} \\ 0 & C + tE_C \end{pmatrix} w(t) - \begin{pmatrix} b + tf_b \\ v + tf_v \end{pmatrix},$$

differentiate the equation, and then set $t = 0$. Using (3.4) to substitute for $\dot{w}(0)$, we get

$$\begin{aligned} \begin{pmatrix} 0 \\ \dot{r}(0) \end{pmatrix} &= \begin{pmatrix} E_{S_1} & E_{S_2} \\ 0 & E_C \end{pmatrix} w + \begin{pmatrix} S_1 & S_2 \\ 0 & C \end{pmatrix} \dot{w}(0) - f_d \\ &= (I - \tilde{C}\tilde{C}^\dagger) \left[\begin{pmatrix} E_{S_1} & E_{S_2} \\ 0 & E_C \end{pmatrix} w - f_d \right] - \tilde{C}(\tilde{C}^H \tilde{C})^{-1} \begin{pmatrix} 0 \\ E_C^H r \end{pmatrix}, \end{aligned}$$

where $\tilde{C}^\dagger = (\tilde{C}^H \tilde{C})^{-1} \tilde{C}^H$. Consequently,

$$\dot{r}(0) = (I - CC^\dagger)(E_C w_2 - f_v) - C(C^H C)^{-1} E_C^H r.$$

As for the residual element we have $r_n = e_n^T r$, where $e_n \equiv (0, \ldots, 0, 1)$ denotes the n-th unit coordinate vector. Using the assumptions (3.5a) and noticing that $w_2 = C^\dagger v$ and $r = (I - CC^\dagger)v$, we derive our major result.

Theorem. *Under the same conditions as in the Lemma, we get*

$$\frac{\|\hat{r} - r\|}{\|v\|} \leq \epsilon \left[\|I - CC^\dagger\| (2\kappa(C) + 1) \right] + O(\epsilon^2) \tag{3.7}$$

and

$$\frac{|\hat{r}_n - r_n|}{\|v\|} \leq \epsilon \left[\|I - CC^\dagger\| (\kappa(C) + \|C\| \, \|C^\dagger e_n\| + 1) \right] + O(\epsilon^2). \quad \square \tag{3.8}$$

Here are some additional remarks. If we set $\tilde{S} = I$ and $b = 0$, then (3.6) leads to a perturbation bound for the standard least squares problem [2]. We also note that $\|\tilde{C}\tilde{S}w\|^2 + \|r\|^2 = \|d\|^2$. Thus, we can define

$$\cos\theta = \|\tilde{C}\tilde{S}w\| / \|d\|$$

and use $(1/\cos\theta)$ and $\tan\theta$ in (3.6). The bound (3.7) is similar to a result derived in [2]. The inequality (3.8) indicates that $|\hat{r}_n - r_n|$ depends on $\kappa(C)$ as well as on $\|v\|$. Both (3.7) and (3.8) can be simplified by using the relation that $\|I - CC^\dagger\| = \min\{1, \; n - p\}$.

4. ERROR ESTIMATION

Although the error bound (3.8) is simple, it requires $C^\dagger e_n$, whose computation involves at least a back-solve. In this section, we present an error estimation scheme for the desired residual element. When the new data vector $x(n)^T$ arrives, it is first processed by S so that $x_1(n)^T$ is annihilated. In particular, let

$$(z_1^{(0)} \; \ldots \; z_q^{(0)}) = x(n)^T \quad \text{and} \quad u^{(0)} = 0.$$

Then the preprocessing proceeds as follows:

$$\begin{pmatrix} s_{l,l} & s_{l,l+1} & \cdots & s_{l,q} \\ 0 & z_{l+1}^{(l)} & \cdots & z_q^{(l)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -g_l & 1 \end{pmatrix} \begin{pmatrix} s_{l,l} & s_{l,l+1} & \cdots & s_{l,q} \\ z_l^{(l-1)} & z_{l+1}^{(l-1)} & \cdots & z_q^{(l-1)} \end{pmatrix}$$

and

$$\begin{pmatrix} b_l \\ u^{(l)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -g_l & 1 \end{pmatrix} \begin{pmatrix} b_l \\ u^{(l-1)} \end{pmatrix},$$

for $l = 1, 2, \ldots, k$, where $g_l = z_l^{(l-1)}/s_{l,l}$. Writing in algorithmic form, we have

for $l = 1, 2, \ldots, k$

begin

$g_l = z_l^{(l-1)}/s_{l,l}$;

for $j = l+1, \ldots, q$

$z_j^{(l)} = z_j^{(l-1)} - g_l s_{l,j}$;

$u^{(l)} = u^{(l-1)} - g_l b_l$

end.

The above process shows that

$$(z_{k+1}^{(k)} \ \ldots \ z_q^{(k)}) = x_2(n)^T - x_1(n)^T S_1^{-1} S_2 \quad \text{and} \quad u^{(k)} = -x_1(n)^T S_1^{-1} b.$$

These two variables are then used for updating the QR decomposition of $C(n-1)$ and computing the residual element. We present below the algorithm derived in [4].

for $l = 1, 2, \ldots, p$

begin

$c_{l,l}^{(n)} = \sqrt{|c_{l,l}^{(n-1)}|^2 + |z_{k+l}^{(k+l-1)}|^2}$;

$\cos\theta_l = c_{l,l}^{(n-1)}/c_{l,l}^{(n)}$;

$\sin\theta_l = z_{k+l}^{(k+l-1)}/c_{l,l}^{(n)}$;

for $j = l+1, \ldots, p$

begin

$c_{l,j}^{(n)} = c_{l,j}^{(n-1)} \cos\theta_l + z_{k+j}^{(k+l-1)} \sin\theta_l$;

$z_{k+j}^{(k+l)} = -c_{l,j}^{(n-1)} \sin\theta_l + z_{k+j}^{(k+l-1)} \cos\theta_l$

end;

$v_l^{(n)} = v_l^{(n-1)} \cos\theta_l + u^{(k+l-1)} \sin\theta_l$;

$u^{(k+l)} = -v_l^{(n-1)} \sin\theta_l + u^{(k+l-1)} \cos\theta_l$

end;

$r_n = u^{(k+p)} \prod_{i=1}^{p} \cos\theta_i$.

In the above, $c_{i,j}^{(k)}$ (for $k = n-1, n$) denotes the (i,j)-element of $C(k)$ and $v_i^{(k)}$ the i-th element of $v(k)$.

Now, we discuss an error estimation scheme for the preprocessing. Let $\hat{}$ denote the corresponding computed value and fl the floating point computation. In the above procedure we calculate

$$\begin{aligned}
\hat{g}_l &= fl(\hat{z}_l^{(l-1)}/\hat{s}_{l,l}),\\
\hat{z}_j^{(l)} &= fl(\hat{z}_j^{(l-1)} - fl(\hat{g}_l\hat{s}_{l,j})),\\
\hat{u}^{(l)} &= fl(\hat{u}^{(l-1)} - fl(\hat{g}_l\hat{m}_l)).
\end{aligned}$$

Define the relations between the exact and computed quantities as follows:

$$\begin{aligned}
\hat{s}_{i,j} &= s_{i,j}(1+\sigma_{i,j}\phi_{i,j}(\epsilon)),\\
\hat{z}_j^{(l)} &= z_j^{(l)}(1+\zeta_j^{(l)}\psi_j^{(l)}(\epsilon)),\\
\hat{g}_l &= g_l(1+\alpha_l\xi_l(\epsilon)),\\
\hat{u}^{(l)} &= u^{(l)}(1+\eta^{(l)}\theta^{(l)}(\epsilon)),\\
\hat{b}_l &= b_l(1+\mu_l\delta_l(\epsilon)),
\end{aligned}$$

where $|\phi_{i,j}(\epsilon)| = O(\epsilon)$, $|\psi_j^{(l)}(\epsilon)| = O(\epsilon)$, $|\xi_l(\epsilon)| = O(\epsilon)$, $|\theta^{(l)}(\epsilon)| = O(\epsilon)$ and $|\delta_l(\epsilon)| = O(\epsilon)$. The five quantities $\sigma_{i,j}$, $\zeta_j^{(l)}$, α_l, $\eta^{(l)}$ and μ_l are all real and nonnegative. We also assume that the errors such as $\sigma_{i,j}\phi_{i,j}(\epsilon)$ and $\zeta_j^{(l)}\psi_j^{(l)}(\epsilon)$ are so small that higher order terms like $(\sigma_{l,l}\phi_{l,l}(\epsilon))^2$ and $(\sigma_{l,l}\phi_{l,l}(\epsilon))(\zeta_l^{(l-1)}\psi_l^{(l-1)}(\epsilon))$ can be ignored. Using the lemma in [3], we obtain the following algorithm for estimating the errors in preprocessing.

for $l = 1, 2, \ldots, k$

begin

$\alpha_l = \max\{\zeta_j^{(l-1)}, \sigma_{l,l}\}$;

for $j = l+1, \ldots, q$

$$\zeta_j^{(l)} = \frac{|z_j^{(l-1)}\zeta_j^{(l-1)}| + |g_l s_{l,j}\max\{\alpha_l, \sigma_{l,j}\}|}{|z_j^{(l)}|};$$

$$\eta^{(l)} = \frac{|u^{(l-1)}\eta^{(l-1)}| + |g_l b_l \max\{\alpha_l, \mu_l\}|}{|u^{(l)}|}$$

end.

As explained in [3], the above estimation scheme can be incorporated with the preprocessing procedure and implemented on the same systolic architecture. Additional time is minimal because the calculations can be carried out during the otherwise idle time of the processors.

The error estimate for r_n can be obtained by the algorithm presented in [3] using $(\zeta_{k+1}^{(k)} \cdots \zeta_q^{(k)})$ and $\eta^{(k)}$ as the error estimates for $(z_{k+1}^{(k)} \cdots z_q^{(k)})$ and $u^{(k)}$, respectively. Again, we list the error estimation algorithm and refer the details to [3]. Define the relations between the exact and computed quantities as follows:

$$\begin{aligned}
\hat{c}_{i,j}^{(n-1)} &= c_{i,j}^{(n-1)}(1+\xi_{i,j}\alpha_{i,j}(\epsilon)), & \hat{c}_{i,j}^{(n)} &= c_{i,j}^{(n)}(1+\sigma_{i,j}\phi_{i,j}(\epsilon)),\\
\hat{z}_j^{(k)} &= z_j^{(k)}(1+\zeta_j^{(k)}\psi_j^{(k)}(\epsilon)), & \cos\hat{\theta}_l &= \cos\theta_l(1+\sigma_{l,l}\phi_{l,l}^{(c)}(\epsilon)),\\
\hat{u}^{(k)} &= u^{(k)}(1+\eta^{(k)}\theta^{(k)}(\epsilon)), & \sin\hat{\theta}_l &= \sin\theta_l(1+\sigma_{l,l}\phi_{l,l}^{(s)}(\epsilon)),\\
\hat{v}_i^{(n-1)} &= v_i^{(n-1)}(1+\xi_{i,p+1}\alpha_{i,p+1}(\epsilon)), & \hat{v}_i^{(n)} &= v_i^{(n)}(1+\sigma_{i,p+1}\phi_{i,p+1}(\epsilon)),
\end{aligned}$$

and

$$\hat{r}_n = r_n(1+\eta\theta(\epsilon)).$$

The following algorithm estimates the error in the last element of the residual vector:

for $l = 1, 2, \ldots, p$

begin

$\sigma_{l,l} = \max\{\xi_{l,l}, \zeta_{k+l}^{(k+l-1)}\}$;

for $j = l+1, \ldots, p$

begin

$$\sigma_{l,j} = \frac{|c_{l,j}^{(n-1)} \cos\theta_l \max\{\xi_{l,j}, \sigma_{l,l}\}| + |z_{k+j}^{(k+l-1)} \sin\theta_l \max\{\zeta_{k+j}^{(k+l-1)}, \sigma_{l,l}\}|}{|c_{l,j}^{(n-1)} \cos\theta_l + z_{k+j}^{(k+l-1)} \sin\theta_l|};$$

$$\zeta_{k+j}^{(l+k)} = \frac{|c_{l,j}^{(n-1)} \sin\theta_l \max\{\xi_{l,j}, \sigma_{l,l}\}| + |z_{k+j}^{(k+l-1)} \cos\theta_l \max\{\zeta_{k+j}^{(k+l-1)}, \sigma_{l,l}\}|}{|c_{l,j}^{(n-1)} \sin\theta_l - z_{k+j}^{(k+l-1)} \cos\theta_l|}$$

end;

$$\sigma_{l,p+1} = \frac{|v_l^{(n-1)} \cos\theta_l \max\{\xi_{l,p+1}, \sigma_{l,l}\}| + |u^{(k+l-1)} \sin\theta_l \max\{\eta^{(k+l-1)}, \sigma_{l,l}\}|}{|v_l^{(n-1)} \cos\theta_l + u^{(k+l-1)} \sin\theta_l|};$$

$$\eta^{(k+l)} = \frac{|v_l^{(n-1)} \sin\theta_l \max\{\xi_{l,p+1}, \sigma_{l,l}\}| + |u^{(k+l-1)} \cos\theta_l \max\{\eta^{(k+l-1)}, \sigma_{l,l}\}|}{|v_l^{(n-1)} \sin\theta_l - u^{(k+l-1)} \cos\theta_l|}$$

end;

$$\eta = \eta^{(k+p)} \max\{\sigma_{1,1}, \cdots, \sigma_{p,p}\}.$$

5. AN EXAMPLE

The example in this section shows that the computed residual element may be accurate even when the matrix C is ill-conditioned. In this case, the proposed scheme gives a better error estimate than (3.8). Both the MS algorithm and the error estimation algorithm were implemented using MATLAB and run on a VAX 8300 with machine precision $\epsilon = 1.1102 \times 10^{-16}$ in the Communications Research Laboratory at McMaster University.

Example. Suppose the exact constraint matrix and corresponding right side vector are

$$S = \begin{pmatrix} 10^3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 10^{-3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -120000\sqrt{2}/7 \\ \sqrt{10}/700 \\ 6\sqrt{5}/7 \end{pmatrix}.$$

Thus we set the error estimates as $\sigma_{i,j} = \mu_l = 1$ and $\phi_{i,j}(\epsilon) = \delta_l(\epsilon) = \epsilon$. The data matrix at time $n-1$ is

$$X(n-1) = \begin{pmatrix} -1 & -\sqrt{5} & -2\sqrt{10} & 0 & 0 & 0 \\ 0 & -1 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose we know the exact $R(n-1)$ and $u(n-1)$:

$$R(n-1) = \begin{pmatrix} 0.001 & 1000\sqrt{5} & 2\sqrt{10} \\ 0 & 1000 & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad u(n-1) = \begin{pmatrix} -10\sqrt{2}/7 \\ 4\sqrt{10}/7 \\ 6\sqrt{5}/7 \end{pmatrix}.$$

Similarly, the error estimates of their elements are all initialized as ϵ. Now the new data

$$x(n)^T = (-1 \quad -\sqrt{5} \quad -2\sqrt{10} \quad 0.001 \quad 0 \quad 0) \quad \text{and} \quad u^{(0)} = 0$$

are available and their error estimates are initialized as $\zeta_j^{(0)} = 1$, for $j = 1, \ldots, 6$, and $\eta^{(0)} = 1$, respectively. After preprocessing, we get $c(n)^T = (\,0.002 \quad 1000\sqrt{5} \quad 2\sqrt{10}\,)$ and $v_n = -10\sqrt{2}/7$. The corresponding error estimates are $\zeta_j^{(3)} = 1$, for $j = 4, 5, 6$, and $\eta^{(3)} = 23$. The QR updating scheme and its error estimation algorithm are then applied to $R(n-1)$, $u(n-1)$, $c(n)$, v_n and their error estimates. The exact residual element $r_n = 6\sqrt{2}/35$. The computed error is

$$|\hat{r}_n - r_n| = 1.11 \times 10^{-16}.$$

The condition number of $C(n)$ is 4.6×10^6 and the error bound as given by (3.8) equals 3.40×10^{-9}. The estimation algorithm gives a much more accurate value of 9.62×10^{-16}.

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