2-EXTENDABILITY
IN
TWO CLASSES OF CLAW-FREE GRAPHS

by

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dedicated to the memory of
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ABSTRACT

A graph G is **2-extendable** if it has at least six vertices and every pair of independent edges extends to (i.e., is a subset of) a perfect matching. In this paper two classes of claw-free graphs are discussed: those which are 3-regular and 3-connected and those which are 4-regular and 4-connected (as well as even). None of the first class is 2-extendable, whereas those of the second class which are 2-extendable are determined. More particularly, in the graphs belonging to these classes, those pairs of independent edges which extend to a perfect matching are determined.

1. Introduction

A graph G is **claw-free** if it contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. Claw-free graphs have been widely studied in graph theory in connection with such diverse concepts as independent sets, perfect graphs, Hamiltonian (and other traversability) properties, reconstruction and matching. (For a selected set of references in each of these areas, see the Introduction in [P4].)

The subject of this paper is matching in claw-free graphs. Sumner [Su1, Su2] and Las Vergnas [La] began the study of perfect matchings in claw-free graphs. In particular, they showed, independently, that any connected claw-free graph with an even number of vertices must contain a perfect matching.

Let G be a graph containing a perfect matching and let n be a positive integer, $1 \leq n \leq (|V(G)|-2)/2$. Graph G is said to be **n-extendable** if every matching of size n extends to (i.e., is a subset of) a perfect matching. A graph G is **bicritical** if $G-u-v$ contains a perfect matching for every pair of vertices u and v in $V(G)$. (Clearly, then, every bicritical graph is 1-extendable.)

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A 3-connected bicritical graph is called a **brick**. Bicritical graphs–and more especially bricks—have emerged as an important special class in the study of graphs with perfect matchings and their structure remains far from completely understood. (See [LP], [ELP], [Lo] and [LR].)

In [P1], it was shown that any non-bipartite 2-extendable graph is bicritical. Partly because of this implication, the author and others have continued the study of n-extendable graphs—and of 2-extendable graphs in particular. For a recent survey of much of this work, see [P3] and the references contained therein.

In [P4], the author focused upon matching extension in claw-free graphs. In particular, it was show there that any 3-connected claw-free graph (with an even number of vertices) must be a brick (and hence 1-extendable). More generally, it was shown that for any integer $n \geq 2$ if G is $(2n+1)$-connected, claw-free and even, then G must be n-extendable. (Thus for example, every 5-connected graph of this kind must be 2-extendable.)

In the present paper, we consider two special classes of claw-free graphs: those which are 3-regular (i.e., *cubic*) and 3-connected and those which are 4-regular and 4-connected. By the remark in the preceding paragraph, clearly all such graphs are bricks. On the other hand, it is easy to see that no graph in the first class is 2-extendable, but in the second class, some graphs are 2-extendable, while others are not. We characterise those that are 2-extendable. Perhaps the most important idea pursued in this paper is that for the first time a study is undertaken as to just which pairs of independent edges are extendable to a perfect matching and which are not. In the cases of these two families of claw-free graphs we are able to characterise precisely those pairs of edges which are 2-extendable. It is an easy matter to characterise all 3-connected cubic claw-free graphs; they are just those cubic graphs obtained from other 3-connected cubic graphs by inserting a single triangle at each vertex in such a manner so as to preserve 3-regularity. The 4-connected 4-regular claw-free graphs are not so easily characterised, but we are able to obtain a complete characterisation using matching extendability of pairs of edges. More particularly, it is shown that such graphs belong to one of precisely three classes of graphs and of these three classes, one consists of all those which are 2-extendable.

2. The 3-regular 3-connected Case

We begin with a property of general 3-connected claw-free graphs (i.e., those which are not necessarily cubic) having an even number of vertices. To deal with this, we introduce the concept of a *generalized moth* (or more succinctly, *gmoth*). (This concept is closely related to, but different from, that of a *generalized butterfly* first introduced in [P2].) Let G be a connected graph and let $e_i = a_ib_i$, $i = 1, 2$ be two independent edges in G. Then G is called a **generalised moth** (or **gmoth** for short) if $G - a_1 - b_1 - a_2 - b_2$ consists of precisely two components and both are of odd cardinality. For the sake of brevity, we shall call such components *odd*. The subgraph of G induced by $V(e_1) \cup V(e_2)$ is called the **body** of the gmoth at $\{e_1, e_2\}$ and the two odd components are called the **wings** of the gmoth at $\{e_1, e_2\}$. Of course a given graph can be viewed as a gmoth in many different ways; that is, there may be many pairs of independent edges which form the body of a gmoth structure for G.

It is easy to characterise 3-connected claw-free even graphs which are not 2-extendable in terms of gmoths.

Theorem 2.1. Let G be 3-connected, claw-free and even. Then G is 2-extendable if and only if G is not a gmoth.

Proof. The left-to-right implication is clear.

Let us therefore suppose that G is 3-connected, claw-free and even. but that G contains a pair of edges $\{e_1 = a_1b_1, e_2 = a_2b_2\}$ which do not extend to a perfect matching. Let $G' = G - a_1 - b_1 - a_2 - b_2$. Then by Tutte's Theorem on perfect matchings, there exists a set $S' \subseteq V(G')$ such that $|S'| < c_o(G' - S')$, where $c_o(G' - S')$ denotes the number of odd components of $G' - S'$. Let $s' = |S'|$ and let $C_1, \ldots, C_{s'+1}, \ldots$ be these odd components. By parity, since G is even, we must have $s' \leq c_o(G' - S') - 2$. By Theorem 1.1 of [P4], graph G is a brick and hence is 1-extendable. But then it follows that $G' - S'$ has exactly $s' + 2$ components.

Now among all sets S' with the above properties, choose one which is minimal. Suppose S' is not empty. Then suppose $u \in S'$. But then, by Corollary 1 of [Su2], u is a claw-center which is impossible in G' since G—and therefore G'—are claw-free. Thus $S' = \emptyset$ and hence G' has exactly two odd components.

It remains only to show that G' has no *even* components. Suppose G_e were such an even component and denote the two odd components of G' by C_1 and C_2. By 3-connectivity, there are at least three edges from C_1 to the set $S = \{a_1, b_1, a_2, b_2\}$. Without loss of generality, assume that there are edges from C_1 to vertices a_1, b_1 and a_2. (For the duration of this paper we will denote the relation of adjacency between two vertices by the symbol "$\sim$".) Suppose a_1 is adjacent to no vertex in C_2. Then each of b_1, a_2 and b_2 must be adjacent to C_2 by 3-connectivity. Also by 3-connectivity, component C_e has edges to at least three vertices of S and hence to at least one of b_1 and a_2. Without loss of generality, assume that there is an edge from b_1 to C_e. But then b_1 must be a claw-center in G which is impossible. So we may suppose that vertex a_1 is adjacent to a vertex of C_2 and by symmetry, so is vertex b_1. Again, by 3-connectivity, there must be an edge from C_e to at least one of a_1 and b_1 and hence again we get a claw-center, contradicting our claw-free hypothesis. Thus there are no even components C_e of G'. ∎

We now proceed to give a characterisation of those 3-connected claw-free graphs which are 3-regular. (We shall call 3-regular graphs *cubic*.) In order to do this, we shall need the following concept. Let G be an r-regular graph for any $r \geq 3$. The **r-inflation of G**, denoted by $G(r)$, is the graph obtained from G by inserting at each vertex of G a copy of the complete graph on r vertices, K_r, and joining each "half edge" resulting from the removal of the vertices of G to a different vertex of the K_r which replaced the vertex of G. Thus the r-inflation of G is also r-regular and the original graph G can easily be recovered from $G(r)$ by contracting each of the inserted K_r's to a single vertex. In the case $r = 3$, this use of the word "inflation" is due to V. Chvátal.

Finally, let us denote by R_3 the the six-vertex cubic graph formed by joining the vertices of two disjoint triangles with a perfect matching. (This graph is commonly called the *triangular prism*.)

Theorem 2.2. Graph G is cubic 3-connected and claw-free if and only if $G = K_4, R_3$ or is the 3-inflation of a cubic 3-connected graph H.

Proof. The right-to-left implication is clear.

Conversely, suppose $G \neq K_4, R_3$. Since G is cubic, 3-connected and claw-free, each vertex of G lies on precisely one triangle. For suppose two different triangles in G have a vertex in common. Since G is cubic, they must also have an edge in common. But then, since G is 3-connected, $G = K_4$, a contradiction.

Thus the triangles in G partition the set $V(G)$. Suppose there are only two triangles in G. Then $G = R_3$. So suppose there are at least three triangles in G. By parity, there cannot be exactly three such triangles, so we may assume that there are at least m such where $m \geq 4$ and even.

Denote by H the graph obtained from G by shrinking all triangles to single vertices. (Note that there are no parallel edges in H since $G \neq R_3$.)

Clearly, graph H is cubic.

It remains only to show that H is 3-connected. Let u' and v' be two non-adjacent vertices in H. Then in G there are two vertex-disjoint triangles $T(u')$ and $T(v')$ corresponding to vertices u' and v' respectively in H. Since $u' \not\sim v'$ in H, there is no edge in G joining $T(u')$ and $T(v')$. Choose any two vertices $u \in T(u')$ and $v \in T(v')$ in G. Then $u \not\sim v$ in G and since G is 3-connected, there are at least three openly disjoint u–v paths in G; say P_1, P_2 and P_3. Now if T is any triangle in G such that $T \neq T(u'), T(v')$, then at most one of the P_i's meets T. Thus if, for each $i = 1, 2, 3$, P_i' is the path in H resulting from shrinking triangles $T(u')$ and $T(v')$ in G, the paths P_1', P_2' and P_3' are openly disjoint u–v paths in H. Thus H is 3-connected. ∎

Let G be any cubic graph and let T be any triangle in G with $V(T) = \{a, b, c\}$. We will call any edge lying in such a triangle in a cubic graph **triangular**. A pair of independent edges $\{e_1 = a_1b_1, e_2 = a_2b_2\}$ in any connected graph G will be called a **singleton isolator** if graph $G - a_1 - b_1 - a_2 - b_2$ is disconnected and moreover one of the components of $G - a_1 - b_1 - a_2 - b_2$ is a single vertex. Clearly, if $\{e_1, e_2\}$ is a singleton isolator in any graph, then $\{e_1, e_2\}$ does not extend to a perfect matching. In a 3-connected cubic graph it is clear that if $\{e_1, e_2\}$ is a singleton isolator, then one of the e_i—say e_1—is triangular and if e_1 lies in triangle T where $V(T) = \{a, b, c\}$ and $e_1 = ab$, then edge e_2 does not lie in T and in fact $e_2 = de$ where $\{d, e\} \cap \{a, b, c\} = \emptyset$ and precisely one of the vertices d and e is adjacent to vertex c.

It is also obvious that every 3-connected cubic claw-free graph different from K_4 has singleton isolators and hence no such graph is 2-extendable.

On the other hand, it is an interesting consequence of the next theorem that if $\{e_1, e_2\}$ is *not* a singleton isolator in a 3-connected cubic claw-free graph, then, in fact, $\{e_1, e_2\}$ *does* extend to a perfect matching.

Theorem 2.3. Let G be a 3-connected cubic claw-free graph $\neq R_3$. Suppose G is a gmoth with body $\{e_1, e_2\}$. Then:
(a) no edges of G join edges e_1 and e_2,
(b) precisely one wing of the gmoth at $\{e_1, e_2\}$ is a singleton and
(c) the other wing of the gmoth has exactly four vertices of attachment to $\{e_1, e_2\}$.

Proof. As before, let $e_i = a_i b_i$ for $i = 1, 2$. Since each of the wings C_1 and C_2 send at least three edges to the body by 3-connectivity, then since G is cubic, there can be at most one additional edge in the body; that is, there can be at most one edge joining e_1 and e_2.

Suppose there is such an edge e_3. Then without loss of generality, we may assume that $e_3 = b_1 a_2$. But then each wing C_i sends *exactly* three edges to set S. If a_1 sends two edges to C_1 or two edges to C_2, then G is not 3-connected. So we may assume that a_1 sends one edge to C_1 and one edge to C_2. Similarly for b_2. Let a_1 be adjacent to $u_1 \in C_1$ and $u_2 \in C_2$. But a_1 is not a claw-center, so either $b_1 \sim u_1$ or $b_1 \sim u_2$. Without loss of generality, assume that $b_1 \sim u_1$.

Now suppose $b_2 \sim u_1$ also. Then $V(C_1) = \{u\}$. Suppose that $V(C_2)$ is also a singleton. Then it follows that $G = R_3$, a contradiction. So we may suppose that $|V(C_2)| \geq 3$. But then a_1, a_2 and b_2 must be matched into C_2 by 3-connectivity and it follows that a_2 is a claw-center, a contradiction. So we may assume that $b_2 \not\sim u_1$. More specifically, suppose $b_2 \sim u_3 \neq u_1$ where $u_3 \in V(C_1)$. Then by 3-connectivity, a_2 is also adjacent to a vertex $u_4 \in V(C_1)$, $u_4 \notin \{u_1, u_3\}$. But then $\{a_1, b_2\}$ is a vertex cut of size 2 in G, a contradiction. This proves part (a).

There are, then, eight edges from $\{e_1, e_2\}$ to wings C_1 and C_2. Since both wings are odd, and since G is 3-connected, parity dictates that precisely one wing is attached to the body by three edges and the other by five. Without loss of generality, assume that there are precisely three edges joining C_1 to the body. Clearly, then, wing $C_2 \neq K_1$.

We claim then that wing C_1 is a singleton.

Since G is 3-connected, there is a matching from S into C_2 of size $= 3$. Without loss of generality, let us assume it contains $\{b_1u_1, a_2u_2, b_2u_3\}$. Also without loss of generality, by 3-connectivity we may assume that $a_2 \sim v_1$ where $v_1 \in V(C_1)$. But a_2 is not a claw-center, so either $b_2 \sim v_1$ or $b_2 \sim u_2$.

Suppose first that $b_2 \sim v_1$. But then, again by 3-connectivity, the third edge into C_1 from S—that is, the edge different from a_2v_1 and b_2v_1—must be incident with vertex v_1. Thus C_1 is a singleton.

So suppose that $b_2 \not\sim v_1$ and hence $b_2 \sim u_2$. Suppose C_1 is not a singleton. Then there exists a matching from S into C_1 and, moreover, this matching must be incident with vertices a_1, b_1 and a_2 (and hence one of the edges of the matching is v_1a_2). Now if $a_1 \sim u_1$, then $\{u_1, a_2\}$ is a vertex cutset of size 2 in G, a contradiction. So a_1 is adjacent to a vertex of C_2 other than u_1. But then a_1 is a claw-center in G, a contradiction. Thus again C_1 is a singleton and part (b) is proved.

To prove part (c), we begin by observing that by 3-connectivity and an edge count, the non-trivial wing (C_2, say) has at least three vertices of attachment and no more than five. Assume $V(C_1) = \{v_1\}$. Without loss of generality, we may assume that v_1 is adjacent to all of a_1, b_1 and a_2. Also by 3-connectivity, there must be a matching of S into C_2 of size 3. Without loss of generality, we may assume that one edge of this matching is incident with a_1. Call this edge a_1u_1.

(i) Suppose first that the other two matching edges are u_2b_1 and u_3a_2. Since a_2 is not a claw-center, it follows that $b_2 \sim u_3$. Suppose $b_2 \sim u_1$. Let u_1w be the edge of C_2 incident with u_1. Then since u_1 is not a claw-center, $w = u_3$. But then C_2 cannot be connected; that is, u_2 is not joined to u_1 or to u_3 by any path in C_2. This is a contradiction. So $b_2 \not\sim u_1$. By symmetry, we may also suppose that $b_2 \not\sim u_2$. But then b_2 must be adjacent to a fourth vertex of attachment in C_2; call it u_4. But a_2 is not a claw center, so $b_2 \sim u_3$ and hence C_2 has exactly four vertices of attachment as claimed.

(ii) Suppose the other two edges of the matching are u_2b_1 and u_3b_2. Consider the third neighbor of a_2 in C_2. Suppose it is u_1. Then u_1 is a claw-center, contrary to hypothesis. So $a_2 \not\sim u_1$ and by symmetry we may also assume that $a_2 \not\sim u_2$. So suppose $a_2 \sim u_3$.

First suppose $b_2 \sim u_1$. Then since u_1 is not a claw-center, the third neighbor of u_1 must be u_3. But then again wing C_2 is not connected, a contradiction. So $b_2 \not\sim u_1$ and by symmetry, we may also suppose that $b_2 \not\sim u_2$. Thus b_2 must be adjacent to a fourth vertex of C_2 different from u_1, u_2 and u_3 and we have precisely four vertices of attachment on C_2 as claimed.

So suppose the third neighbor of a_2 in C_2 is none of u_1, u_2 or u_3. Say it is

u_4. Then a_2 is not a claw-center, so $b_2 \sim u_4$. So again we have four vertices of attachment as claimed.

(iii) So finally assume that the matching from C_2 to S is a_1u_1, a_2u_2 and b_2u_3. But a_2 is not a claw-center, so $b_2 \sim u_2$. Then if $b_1 \sim u_1$, set $\{u_1, a_2\}$ is a 2-cut, contrary to 3-connectivity. If $b_1 \sim u_2$ then C_2 is disconnected, a contradiction. Suppose $b_1 \sim u_3$. Then u_3 is a claw center, a contradiction. Thus b_1 is adjacent to a fourth point of attachment u_4 in C_2, $u_4 \notin \{u_1, u_2, u_3\}$. This proves part (c) and hence the theorem. ∎

The next result now follows from Theorems 2.1 and 2.3(b).

Corollary 2.4. If G is 3-connected, cubic, claw-free and even, and $\{e_1, e_2\}$ is any set of two independent edges in G, then $\{e_1, e_2\}$ extends to a perfect matching of G if and only if $\{e_1, e_2\}$ is not a singleton isolator. ∎

It is now obvious to see that by far the large proportion of independent pairs of edges in a 3-connected cubic claw-free graph *do* extend to perfect matchings. We can, in fact, determine such proportions. There are two types of pairs of *adjacent* edges. The first type consist of those pairs both of which share a triangle and the second type consist of one edge on a triangle and the other not on a triangle. Since there are three pairs per triangle of the first type, there are a total of $3 \cdot p/3 = p$ pairs of the first type and since there are six pairs per triangle of the second type, there are $6 \cdot p/3 = 2p$ of the second type in total. (Here p is the number of vertices in G.) Hence there are a total of $3p$ pairs of adjacent edges altogether.

On the other hand, there are $\binom{q}{2} = \binom{3p/2}{2}$ distinct pairs of edges in G and hence the number of pairs of *independent* edges is $\binom{3p/2}{2} - 3p = (9p^2 - 30p)/8$. (Here q denotes the number of edges in G.)

So the proportion of non-adjacent pairs of edges to all pairs of edges in G is:

$$\frac{\frac{1}{8}(9p^2 - 30p)}{\binom{3p/2}{2}} = 1 - \frac{8}{3p-2}.$$

If $G \neq R_3, K_4$, then the non-triangular edges form a perfect matching of G each edge of which joins a pair of triangles in G. Thus there are $p/2$ such pairings of triangles. Moreover, each edge of this perfect matching corresponds to four different singleton isolator pairs one edge of which belongs to each triangle joined by the matching edge. Thus there are $p/2 \cdot 4 = 2p$ different singleton isolator pairs. Thus the proportion of singleton isolator pairs to all pairs is $16/(9p-6)$ and the proportion of singleton isolator pairs to all *independent* pairs is $16/(9p-30)$.

3. The 4-regular 4-connected Case

Although no 3-connected cubic claw-free graph is 2-extendable, this is not true in general for 4-connected 4-regular claw-free graphs. Our first task in this section is to prove that those 4-connected 4-regular claw-free graphs which contain a K_4 are, except for one trivial exception, partitionable into vertex-disjoint K_4's which are then joined together by a perfect matching.

Theorem 3.1. If G is is 4-connected 4-regular claw-free and contains a K_4, then either $G = K_5$ or the vertex set $V(G)$ can be partitioned into disjoint sets of four vertices each such that each four-vertex set induces a K_4 in G.

Proof. Assume G is not K_5. First we prove that if two K_4's intersect, they must be identical. Suppose $K_4(1)$ and $K_4(2)$ are two K_4's in G having a vertex in common. Call the common vertex v. Now if they have *exactly* one vertex in common, $\deg v = 6$ which is impossible, while if they have exactly two vertices in common, the degree of each must be 5 which is also impossible. So suppose they have exactly three vertices in common. Then let a be the vertex of $V(K_4(1)) - V(K_4(2))$ and let b be the vertex of $V(K_4(2)) - V(K_4(1))$. Since $G \neq K_5$, $a \not\sim b$. But then it follows that $\{a, b\}$ must be a cutset of size two in G, a contradiction.

Now we prove that if one vertex of G lies on a K_4, they all do. It will suffice to show that if a vertex v lies in a K_4, then so do all of its neighbors. Clearly this is true for K_5, so suppose $G \neq K_5$.

So suppose $N(v) = \{a, b, c, d\}$ and that $\{v, b, c, d\}$ all lie on a common K_4. If a is adjacent to any of b, c or d, then, since G is not K_5, we get vertex cuts of size three in G, a contradiction. So a is adjacent to none of these three vertices. Hence let the neighbors of a different from v be $\{e, f, g\}$. But since a is not a claw-center, it follows that $e \sim f$, $f \sim g$ and $e \sim g$. In other words, the vertices $\{a, e, f, g\}$ induce a K_4. ∎

Let us denote the class of 4-connected 4-regular claw-free graphs which contain a K_4 by $\mathcal{G}_0$.

Theorem 3.2. Every graph in class $\mathcal{G}_0$, except K_5, is 2-extendable.

Proof. From the preceding theorem, we know that G is partitioned into K_4's. Thus there are two kinds of edges in G: type A edges which lie in a K_4 and type B edges which do not. All type B edges together form a perfect matching of G and hence any two type B edges (in fact, any number of type B edges) extend to a perfect matching of G.

Similarly, any two independent type A edges (in fact, any number of independent type A edges) also extend to a perfect matching of G. This perfect matching contains precisely two type A edges from each of the K_4's in G.

So it remains to treat the case of two independent edges, one type A, the other type B. Let $e_1 = ab$ be a type A edge and let e_2 be a type B edge. Let e_3 be the edge in the same K_4 as e_1, but not adjacent to e_1. Let $G' = G - a - b$. Then G' is 2-connected and hence there is a cycle C in G' which contains edges e_2 and e_3. Among all such cycles, choose C to be a shortest one. Then let $K_4(1), K_4(2), \ldots, K_4(r)$ be the K_4's in G through which cycle C passes, where $e_1 \in E(K_4(1))$.

Now since C was chosen to be shortest, we may assume that it alternates between A edges and B edges. To see this, it is necessary only to note how C can intersect each K_4 in G'. So let choose an arbitrary K_4 in G'. Let the four vertices of this K_4 be w, x, y and z and assume cycle C enters K_4 on a B edge at vertex w. Then C must pass through at least one vertex of the K_4 other than w. Suppose C passes through w, x and y of K_4, but not z. Then $C' = C - wx - xy + wy$ is another cycle passing through e_1 and e_2, but is shorter than C, a contradiction. If C passes through all four vertices of the K_4 before exiting, (say through wx, xy and yz), then $C'' = C - wx - xy - yz + wz$ still passes through the K_4 as well as edges E_1 and e_2, but is shorter than C, again a contradiction.

Thus we may assume that cycle C either

(a) encounters K_4 only in edge wx,

(b) encounters K_4 twice—say in edge wx and later through edge yz, (but then C is not a shortest cycle) or

(c) misses K_4 altogether.

Thus C is an even cycle the edges of which alternate between type A and type B. Also one of its type B edges is e_2.

We now build a perfect matching F for G as follows. Insert all type B edges of cycle C into F. (This includes e_2 of course.) The remaining edges of F will all be type A and are chosen as follows. From $K_4(1)$ choose edge e_1. For each $i = 1, \ldots, r$, if C meets $K_4(i)$ in a single edge—say wx—where $V(K_4(i)) = \{w, x, y, z\}$, add edge yz to F. If $V(C) \cap V(K_4) = \emptyset$ for any arbitrary K_4 in G, insert any two independent edges from this K_4 into F. ∎

In view of Theorem 3.1 the structure of the graphs in class $\mathcal{G}_0$ is quite clear. However, let us note that all the graphs in this class have an alternate description as line-graphs. More particularly, $K_5 = L(K_{1,5})$ and each of the other graphs in class $\mathcal{G}_0$ is a line-graph $L(H)$ of a bipartite graph H with bipartition $V(H) = A \cup B$, each vertex of A having degree 4 and each vertex of B having degree 2. Moreover, H must be 2-connected and every vertex cut of H of size 2 must consist of two vertices of degree 4, while each vertex cut of size 3 must contain at least one vertex of degree 4.

We mention this line-graph description of class $\mathcal{G}_0$ primarily in view of Theorem 3.4 below.

Next let us study those 4-connected 4-regular claw-free graphs which contain no K_4.

Let us denote by $\mathcal{G}_1$ the infinite class of graphs defined as follows. Consider two vertex-disjoint cycles C_1 and C_2 both of length $k \geq 3$ denoted by $u_1u_2\cdots u_ku_1$ and $v_1v_2\cdots v_kv_1$ respectively. Now join each u_i to each of v_{i-1} and v_i, where the subscripts are taken modulo k. Call the resulting graph on $2k$ vertices G_{2k} and let $\mathcal{G}_1$ be defined as $\mathcal{G}_1 = \{G_{2k}\}_{k=3}^{\infty}$.

Note that each member of $\mathcal{G}_1$ is 4-connected, 4-regular and claw-free. In fact each is also planar, even and for $k \geq 4$ each vertex lies on precisely three triangles. (Note also that G_6—the octahedron—has all of these properties except the last.)

Let us call all edges of the form u_iv_j *rungs*. Note that two independent rungs in G_{2k} which have the property that the deletion of all four of their endvertices results in a disconnected graph consisting of two odd components will not extend to a perfect matching. Such pairs are either of the form $\{v_iu_{i+1}, v_ju_j\}$ or $\{v_iu_i, v_ju_{j+1}\}$, where $j \neq i, i+1$ and again all subscripts are taken modulo k. For brevity below, let us call any such pair of edges which do not extend a **red pair**. Note also that, by the proof of Theorem 2.1, all other pairs of independent edges in G_{2k} *do* extend to perfect matchings.

Theorem 3.3. Suppose G is a 4-connected, 4-regular claw-free graph with an even number of vertices. Suppose further that G contains a non-extendable pair of edges $e_1 = a_1b_1, e_2 = a_2b_2$ which have a third edge joining them.

Then $G = G_{2k} \in \mathcal{G}_1$ for some $k \geq 3$ and $\{e_1, e_2\}$ is a red pair in G. Furthermore, then, a pair of independent edges in $G = G_{2k}$ extends to a perfect matching iff the pair is not a red pair.

Proof. The proof is by induction on $|V(G)|$. Without loss of generality, suppose that $b_1 \sim a_2$. Since $\{e_1, e_2\}$ does not extend, by Theorem 2.1 we know that graph $G' = G - \{a_1, b_1, a_2, b_2\}$ consists of precisely two odd components. Let us call them C_1 and C_2.

Suppose first that $|V(C_1)| = 1 = |V(C_2)|$. Then since G is 4-regular, $a_1 \sim b_2$. But then $G = G_6$, the octahedron.

Suppose next that $|V(C_1)| = 1$ and $|V(C_2)| = 3$. Let $V(C_1) = \{c_1\}$ and $V(C_2) = \{u_1, u_2, u_3\}$. since C_2 is connected, we may assume, without loss of generality, that $u_1 \sim u_2$ and $u_1 \sim u_3$.

1. Suppose $u_1 \sim a_1$. Since there is no claw at u_1, we may assume that either $u_2 \sim u_3$ or a_1 is adjacent to at least one of u_2 and u_3.

1.1. Suppose $u_2 \sim u_3$.

1.1.1. Suppose also that $u_1 \sim b_1$.

1.1.1.1. Suppose further that $u_2 \sim a_1$. Then $u_3 \sim a_2$, $u_3 \sim b_2$ and $u_2 \sim b_2$. But then $G = G_8 \in \mathcal{G}_1$.

1.1.1.2. So suppose that $u_2 \not\sim a_1$.

1.1.1.2.1. Suppose $u_2 \sim a_2$. Then since there is no claw at a_2, $u_2 \sim b_2$. Thus $u_3 \sim a_1$ and $u_3 \sim b_2$. But then again $G = G_8 \in \mathcal{G}_1$.

1.1.1.2.2. So we may suppose that $u_2 \not\sim a_2$. But then $\deg u_2 \leq 3$, a contradiction.

1.1.2. So suppose that $u_1 \not\sim b_1$.

1.1.2.1. Suppose $u_1 \sim a_2$. Then, since G is 4-connected, $b_1 \not\sim b_2$. So b_1 is adjacent to one of u_2 or u_3; without loss of generality, assume $b_1 \sim u_2$.

1.1.2.1.1. Suppose $u_2 \sim a_1$. Then $\deg u_3 \leq 3$, again a contradiction.

1.1.2.1.2. So we may assume that $u_2 \not\sim a_1$. But then we get a claw at u_1, a contradiction.

1.1.2.2. So suppose $u_1 \not\sim a_2$. Then $u_1 \sim b_2$.

1.1.2.2.1. Suppose $u_2 \sim a_1$. Since there is no claw at u_1, we may also suppose that $u_3 \sim b_2$.

1.1.2.2.1.1. Suppose $u_2 \sim b_1$. Then it follows that $u_3 \sim a_2$ and we see that $G = G_8 \in \mathcal{G}_1$ once again.

1.1.2.2.1.2. So we may suppose that $u_2 \not\sim b_1$. Thus $u_2 \sim a_2$ and $u_3 \sim b_1$. But then we have claws at both b_1 and a_2, a contradiction.

1.1.2.2.2. So suppose $u_2 \not\sim a_1$ and by symmetry, that $u_3 \not\sim a_1, u_3 \not\sim b_2$ and $u_2 \not\sim b_2$. Thus u_2 is adjacent to one of the two vertices a_2, b_1 and u_3 is adjacent to the other. But then $\deg u_2, \deg u_3 \leq 3$, a contradiction.

1.2. So suppose that $u_2 \not\sim u_3$. Since there is no claw at u_1, either $a_1 \sim u_2$ or $a_1 \sim u_3$. Without loss of generality, suppose that $a_1 \sim u_2$.

1.2.1. Suppose $u_2 \sim b_1$.

1.2.1.1. Suppose further that $u_2 \sim a_2$. Then $\deg u_1 = 4$ implies that $u_1 \sim b_2$. But then $\deg u_3 \leq 2$, a contradiction.

1.2.1.2. So suppose that $u_2 \not\sim a_2$. Thus $u_2 \sim b_2$. Then since there is no claw at u_2, it follows that $u_1 \sim b_2$. Thus again $\deg u_3 \leq 2$, a contradiction.

1.2.2. Suppose $u_2 \sim a_2$. Now if $b_1 \sim b_2$, then $\{a_1, a_2, b_2\}$ is a 3-cut in G, a contradiction. So $b_1 \not\sim b_2$. So $b_1 \sim u_1$ or $b_1 \sim u_3$.

1.2.2.1. Suppose $b_1 \sim u_1$. Then we have a claw at u_1, a contradiction.

1.2.2.2. Thus $b_1 \not\sim u_1$ and hence $b_1 \sim u_3$. Then $u_2 \sim b_2, u_1 \sim b_2$ and $u_3 \sim b_2$. But then $\deg b_2 \geq 5$, a contradiction.

2. So we may suppose that $u_1 \not\sim a_1$ and by symmetry, that $u_1 \not\sim b_2$ also. Thus $u_1 \sim b_1$ and $u_1 \sim a_2$. Moreover, since there is no claw at u_1, we have $u_2 \sim u_3$. Now without loss of generality, we may assume that $a_1 \sim u_2$. Suppose $b_2 \not\sim u_3$. Then $b_2 \sim a_1$ and $b_2 \sim u_2$. But then $\deg_G u_3 = 2$, a contradiction.

Thus $b_2 \sim u_3$. Now if $a_1 \sim b_2$, then $\deg u_2 = \deg u_3 = 3$, a contradiction. So $a_1 \not\sim b_2$. But then $a_1 \sim u_3$ and $b_2 \sim u_2$. But then both u_2 and u_3 are claw centers, a contradiction.

So if $|V(C_1)| = 1$ and $|V(C_2)| = 3$ we must have $G = G_8$.

Next suppose $|V(C_1)| = |V(C_2)| = 3$. Then each of C_1 and C_2 must send at least six edges to $\{a_1, b_1, a_2, b_2\}$, contradicting the 4-regularity of G. So there is no graph when $|V(C_1)| = |V(C_2)| = 3$. (We remark that there is such a graph on ten vertices in the case when edges e_1 and e_2 are *not* joined by an edge; namely, graph G_{10}.)

So now suppose that at least one of the components C_1, C_2 has at least five vertices. Without loss of generality, suppose that C_2 has at least five vertices. Then by 4-connectivity, there is a matching of $\{a_1, b_1, a_2, b_2\}$ into $V(C_2)$, say $a_1u_1, b_1u_2, a_2u_3, b_2u_4$.

1. First suppose that $|V(C_1)| = 1$. Since there is no claw at b_1, we must have $a_1 \sim u_2$ and since there is no claw at a_2, we also have $b_2 \sim u_3$. But then no a_1 claw implies $u_1 \sim u_2$ and no b_2 claw implies $u_3 \sim u_4$. Also since G is 4-connected, it follows that $u_1 \not\sim u_3, u_2 \not\sim u_3$ and $u_2 \not\sim u_4$.

1.1. Suppose that $u_1 \sim u_4$. Then since there is no claw at u_2, it follows that u_1 and u_2 have a common neighbor in C_2', where $C_2' = C_2 - \{u_1, u_2, u_3, u_4\}$. Call this common neighbor u_5. But then $\{u_3, u_4, u_5\}$ is not a 3-cut, so it follows that $V(C_2') = \{u_5\}$. Thus $G = G_{10} \in \mathcal{G}_1$.

1.2. So suppose that $u_1 \not\sim u_4$. Thus $|V(C_2)| \geq 7$.

1.2.1. Suppose $|V(C_2)| = 7$. Hence $|V(C_2')| = 3$. Let $V(C_2') = \{u_5, u_6, u_7\}$. Note that C_2' is connected, so without loss of generality, assume that $u_5 \sim u_6$ and $u_5 \sim u_7$.

1.2.1.1. Suppose $u_6 \not\sim u_7$.

1.2.1.1.1. Suppose $u_5 \sim u_1$. Then since there is no claw at u_5, by symmetry we may assume, without loss of generality, that $u_1 \sim u_6$. If $u_5 \sim u_2$, then it is impossible for all of u_3, u_4, u_6 and u_7 to have degree 4, a contradiction. So $u_5 \not\sim u_2$.

1.2.1.1.1.1. Suppose $u_5 \sim u_3$. Then we get a claw at u_5, a contradiction. So $u_5 \not\sim u_3$.

1.2.1.1.1.2. Suppose $u_5 \sim u_4$. Then since there is no claw at u_5, either $u_4 \sim u_6$ or $u_4 \sim u_7$.

1.2.1.1.1.2.1. Suppose $u_4 \sim u_6$. Then we get a claw at u_5, a contradiction.

1.2.1.1.1.2.2. So we may suppose that $u_4 \not\sim u_6$ and hence $u_4 \sim u_7$. Then if $u_6 \sim u_3$, we get a claw at u_6, so $u_6 \sim u_2$ and by symmetry $u_7 \sim u_3$. But then $\deg u_6 = \deg u_7 = 3$, contradictions both.

1.2.1.1.2. So we may suppose that $u_5 \not\sim u_1$ and by symmetry that $u_5 \not\sim u_4$ also. Thus $u_5 \sim u_2$ and $u_5 \sim u_3$. But then it is easy to see that both u_6 and u_7 have degree at most 3, a contradiction.

1.2.1.2. So suppose $u_6 \sim u_7$. Then without loss of generality, we may suppose that $u_1 \sim u_5$ and $u_1 \sim u_6$.

1.2.1.2.1. Suppose $u_5 \sim u_2$. If $u_6 \sim u_3$, we have a claw at u_6, so $u_6 \not\sim u_3$. So $u_6 \sim u_4$. But then $u_7 \sim u_3$ and $u_7 \sim u_4$ and $G = G_{12} \in \mathcal{G}_1$.

1.2.1.2.2. So suppose that $u_5 \not\sim u_2$. If $u_5 \sim u_3$, we get a claw at u_5, so $u_5 \not\sim u_3$. Thus $u_5 \sim u_4$. Since there is no claw at u_5, it follows that $u_4 \sim u_7$. If $u_6 \sim u_3$, we have a claw at u_6, so $u_6 \not\sim u_3$. Hence $u_6 \sim u_2$ and $u_7 \sim u_3$ and once again we have that $G = G_{12}$.

1.2.2. Suppose now that $|V(C_2)| \geq 9$. Hence $|V(C_2')| \geq 5$. Thus, since

G is 4-connected, we must have a matching $\{u_1v_1, u_2v_2, u_3v_3, u_4v_4\}$ for some set of vertices $\{v_1, v_2, v_3, v_4\} \subseteq V(C_2')$. Now since there is no claw at u_2, $u_1 \sim v_2$ and since there is no claw at u_3, $u_4 \sim v_3$. But then since there is no claw at u_1, $v_1 \sim v_2$ and since there is no claw at u_4, $v_3 \sim v_4$.

Now build a new graph G' from G by deleting vertices c_1, a_1, b_1, a_2 and b_2 and replacing them with one new vertex w_1 which we join to each of u_1, u_2, u_3 and u_4. Finally, join vertices u_2 and u_3. Then $\{u_1u_2, u_3u_4\}$ are a non-extendable pair in G' and G' is 4-connected, 4-regular, claw-free and even, But $|V(G')| = |V(G)| - 4$. So by the induction hypothesis, $G' = G_{2j} \in \mathcal{G}_1$ for some $j \geq 3$. But then clearly $G = G_{2j+4}$.

2. Now suppose $|V(C_1)| = 3$. Let $V(C_1) = \{w_1, w_2, w_3\}$. Recall that $|V(C_2)| \geq 5$ and so there must be a matching $\{a_1u_1, b_1u_2, a_2u_3, b_2u_4\}$ where $\{u_1, u_2, u_3, u_4\} \subseteq V(C_2)$. This is a consequence of the fact that G is 4-connected. Since C_1 is connected, we may assume that w_1 is adjacent to both w_2 and w_3.

2.1. Suppose $w_2 \not\sim w_3$.

2.1.1. Suppose also that $a_1 \sim w_1$.

2.1.1.1. Suppose even further that $w_1 \sim b_1$. Then we get a claw at w_1, a contradiction. So $w_1 \not\sim b_1$.

2.1.1.2. Suppose $w_1 \sim a_2$. Then we get a claw at w_1 again and again we have a contradiction. so $w_1 \not\sim a_2$.

2.1.1.3. So since $w_1 \not\sim b_1, a_2$, we must have $w_1 \sim b_2$. By symmetry and 4-connectivity, we may assume, without loss of generality, that $w_2 \sim b_1$. Moreover, $a_1 \not\sim a_2$ also by 4-connectivity, so $a_2 \sim w_2$ or $a_2 \sim w_3$.

2.1.1.3.1. Suppose $a_2 \sim w_2$. Then since there is no w_1 claw, $w_3 \sim b_2$. But then $\deg w_3 = 3$, a contradiction.

2.1.1.3.2. So we may suppose that $a_2 \not\sim w_2$ and hence $a_2 \sim w_3$. But then at least one of w_2 and w_3 has degree at most 3, a contradiction.

2.1.2. So assume $a_1 \not\sim w_1$ and by symmetry that $b_2 \not\sim w_1$. Thus w_1 is adjacent to both b_1 and a_2. But then $\{a_1, w_1, b_2\}$ is a 3-cut in G which is impossible.

2.2. So suppose that $w_2 \sim w_3$. Then without loss of generality, we may suppose that $b_1 \sim w_1$. But then if $w_1 \sim a_2$, $\{a_1, w_1, b_2\}$ is a 3-cut which is impossible. Thus $w_1 \not\sim a_2$. Now by 4-connectivity, a_2 is adjacent to at least one of w_2 and w_3. By symmetry, without loss of generality we may assume that $a_2 \sim w_3$.

2.2.1. Suppose $a_1 \sim w_1$. Then if $a_1 \sim w_3$ it follows that $\{w_1, w_3, b_2\}$ is a 3-cut which is impossible. So $a_1 \not\sim w_3$. On the other hand, if $a_1 \sim w_2$, then $w_2 \sim b_2$ and $w_3 \sim b_2$. But then we have a claw at b_1. So $a_1 \not\sim w_2$. But then $\deg w_3 \leq 3$, a contradiction.

2.2.2. So $a_1 \not\sim w_1$. By symmetry, we may also suppose that $b_2 \not\sim w_3$. But then $a_1 \sim w_3$ and by symmetry $w_1 \sim b_2$. But then w_2 is adjacent to both a_1 and b_2. But then we have claws at w_1 and at w_3, a contradiction.

So we have shown that it is impossible to have $|V(C_1)| = 3$.

3. So suppose that $|V(C_1)| \geq 5$. Recall that we also have $|V(C_2)| \geq 5$ as well. So by 4-connectivity, there must be a matching of vertex set $\{a_1, b_1, a_2, b_2\}$ into component C_1 and another into component C_2. But then we must have claws at both b_1 and a_2, a contradiction.

Now that we know that $G = G_{2k}$, it is easy to verify that the only pairs of independent edges which do not extend to a perfect matching are indeed the red pairs. ∎

Now let us proceed to characterise yet another class of 4-connected 4-regular claw-free graphs. These turn out to be a class of line-graphs. Denote by $\mathcal{G}_2$ the class of all 4-connected 4-regular claw-free graphs in which each vertex lies on *exactly two* triangles.

In order to formulate the next result, we need the concept of *cyclic connectivity*. Let us define the **cyclic (edge) connectivity** of a graph G to be minimum taken over the cardinalities of all edge cuts F of G which separate G and such that at least two components of $G - F$ contain cycles. Denote the cyclic connectivity of G by $c\lambda(G)$. Now let us say that G is **cyclically k-edge connected** for all $k \leq c\lambda(G)$.

If a graph has no cycle-separating edge cut, we shall define the cyclic connectivity to be 0. For example, both K_4 and $K_{3,3}$ are examples of cubic graphs with cyclic connectivity 0. (The reader is warned that some authors define the cyclic connectivity of such graphs to be $+\infty$ and others say that the cyclic connectivity of these graphs is not defined!)

Wormald [W] showed that if graph G is cubic, but different from either K_4 or $K_{3,3}$, then G is 3-connected if and only if G is cyclically 3-connected. Let us now call any vertex cut S in a connected graph G **star-like** if at least one component of $G - S$ is a singleton. Fouquet and Thuillier [FT] showed that, if G is 3-connected and cubic, then G is cyclically 4-connected if and only if all 3-edge cutsets are incident with one common vertex. For cubic graphs this is clearly equivalent to saying that all 3-vertex cuts are star-like.

It will also be helpful to recall that a cyclically 4-connected cubic graph is necessarily triangle-free.

Theorem 3.4. Graph G is a member of class $\mathcal{G}_2$ if and only if $G = L(H)$ where either H is a 3-connected cyclically 4-connected cubic graph, or else $H = K_{3,3}$.

Proof. Suppose first that graph G belongs to class $\mathcal{G}_2$. Clearly, $L(K_{3,3})$ is in $\mathcal{G}_2$, so suppose that $G \neq L(K_{3,3})$. Note immediately that graph G cannot contain a K_4. Let v be any vertex of G. We claim that the two triangles containing v must be edge-disjoint. Suppose not. Say, for example, that $N(v) = \{a, b, c, d\}$ and that $a \sim b \sim c$. Then by the condition on triangles given in the definition of class $\mathcal{G}_2$, $d \not\sim a$, $d \not\sim c$ and $a \not\sim c$. But then we have a claw at v, a contradiction.

So G cannot contain any of the nine induced subgraphs forbidden in line-graphs. (See the well-known theorem due independently to Beineke [Bei] and to Robertson (unpublished).) Thus $G = L(H)$ for some graph H. Note that the definition of class $\mathcal{G}_2$ implies that graph H is cubic and triangle-free.

Now if $\kappa(H) = 1$, clearly $\kappa(G) = 1$ also.

Suppose that $\kappa(H) = 2$. Let $\{u, v\}$ be a vertex cut of size two in H. First suppose that $\{u, v\}$ is independent. Then $H - u - v$ has either two or three components. Suppose one of these components (call it C) has only two edges e_1 and e_2 joining it to the cutset $\{u, v\}$. Then $\{e_1, e_2\}$ is a 2-cutset of *vertices* in $G = L(H)$, a contradiction.

So we may suppose that each component of H' has three edges incident with $\{u, v\}$. Thus H' has precisely two components C_1 and C_2 and we may assume that edges e_1 and e_2 join u to C_1 and e_3 joins u to C_2 while edges e_4 and e_5 join v to C_2 and edge e_6 joins v to C_1. But then $\{e_3, e_6\}$ is a 2-cut of *vertices* in G, a contradiction.

Now assume that vertices u and v are adjacent. But then H' has exactly two components and each is joined to the cutset with precisely two edges. Say, for example, that component C_1 is joined to the cutset via the two edges e_1 and e_2. Now H is cubic, so C_1 must contain an edge. Hence $\{e_1, e_2\}$ is a cut set of two *vertices* in G, a contradiction.

So $\kappa(H) \geq 3$.

By the result of Fouquet and Thuillier [FT1] mentioned just before this theorem, it remains only to show that all vertex cuts of size three in H are star-like. To this end, let $\{u, v, w\}$ be a 3-cut in H. Suppose no component of $H'' = H - u - v - w$ is a singleton. Since H is 3-connected and cubic, induced subgraph $H[u, v, w]$ either contains exactly one edge or none.

Suppose $H[u, v, w]$ contains an edge, say uv. Without loss of generality, assume that there are four edges attaching C_1 to $\{u, v, w\}$ and three edges attaching C_2 to this cutset. Let the three edges of attachment to C_2 be e_1, e_2 and e_3. Since C_2 is not a singleton, it must contain an edge, so $\{e_1, e_2, e_3\}$ is a cutset of three vertices in G, a contradiction.

So we may suppose that the cutset $\{u, v, w\}$ is independent. Suppose one of the components of H'' is joined to the cutset by exactly three edges. Say component C_1 is joined to the cutset via edges e_1, e_2 and e_3. But then since C_1 is not a singleton, it contains an edge and again $\{e_1, e_2, e_3\}$ is a cutset of three vertices in G, a contradiction.

So we may assume without loss of generality that H'' has exactly two components C_1 and C_2, that C_1 has five edges joining it to the cutset and that C_2 has four edges joining it to the cutset. Without loss of generality, we may suppose that u and v are joined by two edges each to C_1 and by edges e_1 and e_2 to C_2. Then vertex w must be joined to component C_1 by one edge e_3. Then since neither component is a singleton, $\{e_1, e_2, e_3\}$ is another cutset of three vertices in G and once more we have a contradiction.

Thus all 3-cuts of vertices in H are star-like.

To prove the converse, we note, to begin with, that $L(K_{3,3})$ is in class $\mathcal{G}_2$. Let us suppose that $G = L(H)$, where graph H is 3-connected, cubic and cyclically 4-connected, hence all the 3-cuts of vertices in H are star-like.

Since H is cubic, G is 4-regular and since H is triangle-free, each vertex of G lies on exactly two triangles which intersect only in this vertex. Since G is a line-graph, G is claw-free by the aforementioned result of Beineke and Robertson.

It remains only to show that G is 4-connected. Since H is 3-connected, so is G by a result of Chartrand and Stewart [CS]. So suppose $\kappa(G) = 3$ and that $\{e_1, e_2, e_3\}$ is a vertex cut in G. Let C_1 and C_2 be any two components of $G' = G - e_1 - e_2 - e_3$. So in H, if D_i denotes the component of H corresponding to C_i, for $i = 1, 2$, the edge set $\{e_1, e_2, e_3\}$ separates D_1 from D_2. But H is 3-connected and cubic, so at least one of D_1 and D_2 is a singleton or else $\{e_1, e_2, e_3\}$ is a matching in H. But since both C_1 and C_2 are non-empty in G, each of D_1 and D_2 contains at least one edge in H and hence neither is a singleton.

So $\{e_1, e_2, e_3\}$ is a matching in H. But then each component D_1 and D_2 has minimum degree 2 and hence each contains a cycle. Thus $\{e_1, e_2, e_3\}$ is a cyclic 3-cut in H, a contradiction.

So $\kappa(G) \geq 4$ and the proof of the theorem is complete. ∎

Note that class $\mathcal{G}_2$ contains some graphs with an *odd* number of vertices; for example, $L(K_{3,3})$ is such a graph. Another example is $G = L(R_5)$, where R_5 is the so-called "pentagonal prism"; i.e., the cubic graph on ten vertices obtained by taking two disjoint pentagons and joining them with a perfect matching. More generally, we observe that the odd members of class $\mathcal{G}_2$ arise as line-graphs of 3-connected cubic triangle-free graphs H which have only starlike 3-cuts and such that $|V(H)|$ is not divisible by 4.

In view of these observations, let us denote those graphs in $\mathcal{G}_2$ having an even number of vertices by $\mathcal{G}_{2E}$.

We now proceed to determine exactly which pairs of independent edges in a graph belonging to class $\mathcal{G}_{2E}$ extend to perfect matchings.

Let v be a vertex of any graph G of degree 4 and suppose v lies on exactly two triangles in G. More precisely, suppose $N(v) = \{a_1, b_1, a_2, b_2\}$ and suppose $a_1 \sim b_1$ and $a_2 \sim b_2$. Moreover, suppose that no other edges join any two of these four vertices. Then the 5-vertex (induced) subgraph on these five vertices will be called a **butterfly** in G and the two edges e_1 and e_2 will be called its **wingtips**. (The term "butterfly" was first employed in this manner in [HLP].)

Theorem 3.5. Let G belong to the class $\mathcal{G}_{2E}$ and let e_1 and e_2 be any two independent edges in G. Then $\{e_1, e_2\}$ extend to a perfect matching if and only if they are not the wingtips of a butterfly.

Proof. Clearly G contains a butterfly and the wingtips of any butterfly do not extend to a perfect matching.

Conversely, suppose $\{e_1 = a_1b_1, e_2 = a_2b_2\}$ do not extend. Then by the proof of Theorem 2.1, we know that $G' = G - a_1 - b_1 - a_2 - b_2$ consists of two odd components C_1 and C_2. Moreover, we know that if the edges e_1 and e_2 are joined by an edge, then $G \in \mathcal{G}_1$. But $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$, so since G is a member of $\mathcal{G}_2$, the edges e_1 and e_2 are not joined by any edge.

Suppose that neither C_1 nor C_2 is a singleton.

1. First suppose that $V(C_1) = 3$. Let $V(C_1) = \{u_1, u_2, u_3\}$. Then without loss of generality, we may assume that $b_1 \sim u_1$ and $u_2 \sim u_1 \sim u_3$.

1.1. Suppose that $u_2 \sim u_3$.

1.1.1. Further suppose that $a_1 \sim u_1$. Since $G \in \mathcal{G}_2$, it then follows that $a_1 \not\sim u_2$, $b_1 \not\sim u_2$, $a_1 \not\sim u_3$ and $u_3 \not\sim b_1$. So it follows that $u_2 \sim a_2$ and $u_2 \sim b_2$ and $u_3 \sim a_2$ and $u_3 \sim b_2$. But then $\{u_2, u_3, a_2, b_2\}$ induce a K_4, a contradiction.

1.1.2. So we may assume that $a_1 \not\sim u_1$. So without loss of generality, assume that $u_1 \sim a_2$. Thus $u_1 \not\sim b_2$. Since G is 4-connected, by symmetry we may assume that $a_1 \sim u_2$. Then since u_2 lies on only two triangles in G, $u_2 \not\sim b_1$. But then since there is no claw at u_1, $u_2 \sim a_2$ and $u_3 \sim b_1$. But then vertex u_1 lies on three triangles, a contradiction.

1.2. So suppose that $u_2 \not\sim u_3$. Since $\deg u_3 = 4$, we may assume by symmetry without loss of generality that $u_3 \sim b_2$. Now since there is no claw at u_1, either $u_2 \sim b_1$ or $u_3 \sim b_1$.

1.2.1. Suppose that $u_2 \sim b_1$. Then since $G \in \mathcal{G}_2$, we have that $u_2 \not\sim a_1$ and $u_1 \not\sim a_1$. But then 4-connectivity implies that $a_1 \sim u_3$. But since there is no claw at u_3, $u_1 \sim b_2$. But again since $G \in \mathcal{G}_2$, $u_3 \not\sim a_2$. Thus $\deg u_3 = 3$, a contradiction.

1.2.2. So $u_2 \not\sim b_1$ and hence $u_3 \sim b_1$. Thus $u_2 \sim a_1$, $u_2 \sim a_2$ and $u_2 \sim b_2$. But since there is no claw at u_2, either $u_1 \sim a_1$ or $u_1 \sim a_2$. But if $u_1 \sim a_1$, then $u_3 \sim a_2$. But then edge a_2b_2 lies on two triangles, contradicting the fact that $G \in \mathcal{G}_2$. So $u_1 \not\sim a_1$ and hence $u_1 \sim a_2$. Thus it follows that $u_3 \sim a_1$. But then there is a claw at u_2, a contradiction.

2. Thus $|V(C_1)| \geq 5$ and by symmetry $|V(C_2)| \geq 5$ also. Thus we may assume that a_1, b_1, a_2 and b_2 are matched into C_1 to, say, u_1, u_2, u_3 and u_4 respectively and also matched into C_2 to, say v_1, v_2, v_3 and v_4 respectively. Since there is no claw at a_1, we may assume without loss of generality that $u_1 \sim b_1$. But then since $G \in \mathcal{G}_2$, we have $a_1 \not\sim v_2$, $u_1 \not\sim u_2$ and $a_1 \not\sim u_2$. But then we have a claw at b_1, a contradiction. ∎

The *even* members of class $\mathcal{G}_2$ can be given an alternative characterisation to that of the preceding theorem.

Theorem 3.6. Let G be a 4-connected 4-regular claw-free even graph. Then $G \in \mathcal{G}_{2E}$ if and only if G contains a non-extendable pair of edges and no pair of non-extendable edges has an edge joining them.

Proof. Suppose first that $G \in \mathcal{G}_{2E}$. Then every vertex of G is the body of a butterfly the wingtips of which are non-extendable. Suppose now that $\{e_1, e_2\}$ are a non-extendable pair which are joined by an edge. Then by Theorem 3.3, graph $G \in \mathcal{G}_1$. But every graph in $\mathcal{G}_1$ has the property that each of its vertices lies on three triangles and hence $G \notin \mathcal{G}_2$, a contradiction.

To prove the converse, suppose that G has the property that it contains non-extendable pairs of edges, but no such pair is has an edge joining them.

Let $v \in V(G)$ and denote its set of neighbors by $N(v) = \{a, b, c, d\}$. Since G is claw-free, we may assume without loss of generality that $a \sim b$. Now suppose $c \not\sim d$. Then again since there is no claw at v, either $b \sim c$ or $b \sim d$. By symmetry we may assume without loss of generality that $b \sim c$. Then again since G is claw-free, either $a \sim c$ or $a \sim d$. But if $a \sim c$, G contains a K_4 and hence by Theorem 3.2, G is 2-extendable, a contradiction. So $a \not\sim c$ and hence $a \sim d$. But then $\{ad, bc\}$ are a non-extendable pair and they are joined by edge ab, contradicting the hypothesis.

So $c \sim d$. Moreover, by hypothesis it then follows that $a \not\sim c$, $a \not\sim d$, $b \not\sim c$ and $b \not\sim d$. Thus vertex v lies on exactly two triangles. But the selection of vertex v was arbitrary, so this property then holds for all $v \in V(G)$ and hence $G \in \mathcal{G}_{2E}$. ∎

Finally, we observe that we have partitioned the set of *all* 4-connected 4-regular claw-free graphs.

Theorem 3.7. The class of all 4-connected 4-regular claw-free graphs $= \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$, where the classes $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$ are pair-wise disjoint.

Proof. In class $\mathcal{G}_0$ every vertex lies on a K_4, so clearly this class is disjoint from $\mathcal{G}_1$ and $\mathcal{G}_2$. But a graph in class $\mathcal{G}_1$ has the property that each of its vertices lies on three triangles, whereas in class $\mathcal{G}_2$ vertices lie on precisely two triangles, so these classes are disjoint as well.

It remains only to show that every graph G which is 4-connected 4-regular and claw-free lies in one of these three classes. Let $v \in V(G)$ and again let $N(v) = \{a, b, c, d\}$. By the claw-free property there are at least two triangles at v. Call two such triangles T_1 and T_2. Either T_1 and T_2 share an edge or they do not.

First suppose that T_1 and T_2 do share an edge. Suppose without loss of generality that $T_1 = vabv$ and that $T_2 = vbcv$. Then if $a \sim c$, graph G lies in class $\mathcal{G}_0$. So suppose $a \not\sim c$. But then since there is no claw at v, either $a \sim d$ or $c \sim d$. But in either case G will then have a non-extendable pair of edges joined by an edge and thus $G \in \mathcal{G}_1$.

So suppose that G has the property that each of its vertices lies on two triangles which do not share an edge. But then it follows that v lies on *precisely* two triangles and hence by definition, G lies in class $\mathcal{G}_2$. ∎

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