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Finding an H-Function Distribution for the Sum of
Independent H-Function Variates

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FINDING AN H-FUNCTION DISTRIBUTION FOR THE SUM OF INDEPENDENT H-FUNCTION VARIATES

by

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DISSERTATION

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To God,

for the beauty and simplicity of the H-function

and for my ability to study it;

To my parents, Carl A. and Jody,

for teaching me the value of knowledge

and instilling in me a commitment to quality in

everything I attempt;

To my wife, Debbie,

for her continued love and friendship,

honesty, pureness of heart, and inner strength;

To my sons, Luke, John, and Paul,

for their future challenges, commitment to

excellence, and complete fulfillment.

FINDING AN H-FUNCTION DISTRIBUTION

FOR THE SUM OF INDEPENDENT

H-FUNCTION VARIATES

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C. D. B.

FINDING AN H-FUNCTION DISTRIBUTION

FOR THE SUM OF INDEPENDENT

H-FUNCTION VARIATES

Publication No._______

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The University of Texas at Austin, 1992

Supervisor: J. Wesley Barnes

A practical method of finding an H-function distribution for the sum of two or more independent H-function variates is presented. Simple formulas exist which immediately give the probability density function, as an H-function distribution, of the random variable defined as the product, quotient, or power of independent H-function variates. Unfortunately, there are no similar formulas for the sum or difference of independent H-function variates.

The new practical technique finds an H-function distribution whose moments closely match the moments of the

random variable defined as the sum of independent H-function variates. This allows an analyst to find the distribution of more complicated algebraic combinations of independent random variables. The method and implementing computer program are demonstrated through five examples. For comparison, the exact distribution of the general sum of independent Erlang variates with different scale parameters is derived using Laplace transforms and partial fractions decomposition.

The H-function is the most general special function, encompassing as a special case nearly every named mathematical function and continuous statistical distribution. The Laplace and Fourier transforms (and their inverses) and the derivatives of an H-function are readily-determined H-functions. The Mellin transform of an H-function is also easily obtained. The H-function exactly represents the probability density function and cumulative distribution function of nearly all continuous statistical distributions defined over positive values.

A previously unstated restriction on the variable in the H-function representations of power functions and beta-type functions is highlighted. Several ways of overcoming this limitation when representing mathematical functions are presented. The restriction, however, is an advantage when representing certain statistical distributions. Many new H-function representations of other mathematical functions are also given.

The hierarchical structure among classes of H-functions is given through seven new theorems. Every class of H-functions is wholly contained in many higher-order classes of H-functions through the application of the duplication, triplication, and multiplication formulas for the gamma function.

Four new theorems show when and how a generalizing constant may be present in an H-function representation. Many generalized H-function representations are given, including those of every cumulative distribution function of an H-function variate.

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* Indicates the section contains significant new material.

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CHAPTER 1

INTRODUCTION AND REVIEW

1.1. PURPOSE AND SCOPE

The primary purpose of this research effort was to develop a practical method of finding an H-function distribution for the sum of two or more independent H-function variates. Simple formulas exist which immediately give the probability density function (p.d.f.), as an H-function distribution, of the random variable defined as the product, quotient, or power of independent H-function variates. Unfortunately, there are no similar formulas for the sum or difference of independent H-function variates.

A related issue was whether the class of H-functions is closed under the operation of multiplication. In other words, is the product of two H-functions another H-function? It is important to make the distinction here between the product of two H-functions and the p.d.f. of the random variable defined as the product of two H-function variates. It is well known that the latter case is an H-function. But the former case was unproven. Of course, similar statements can be made about the quotient of two H-functions.

If the class of H-functions is closed under multiplication, one could easily find the p.d.f. (as an H-function) of the sum or difference of independent H-function variates. The Laplace (or Fourier) transforms of the H-function variates in the sum (or difference) are immediately available as H-functions of higher order. The product of these H-functions (in transform space) would yield the transform of the desired density. If this product was available as another H-function, it could be inverted from transform space analytically.

Because the H-function can exactly represent nearly every common mathematical function and statistical density, there was ample reason to suspect that the product of two H-functions was, in general, another H-function. Indeed, there are many cases where two individual functions and their product are all special cases of the H-function.

Throughout this thesis, a number of other new results are identified with an asterisk. Sufficient convergence conditions for the alternate definition of the H-function are given in Section 2.3. These show how the H-function may be evaluated by the sum of residues, without first changing the form of the alternate definition of the H-function to that of the primary definition.

The hierarchical structure among classes of H-functions is given through seven new theorems in Section 2.4.6. Every class of H-functions is wholly contained in many higher-order classes of H-functions through the application of the duplication, triplication, and multiplication formulas for the gamma function. Figure 1 in Section 3.5 illustrates this hierarchical structure with a venn diagram showing many common statistical distributions as first and second order H-function distributions.

Four new theorems in Section 2.4.7 show when and how a generalizing constant may be present in an H-function representation. Many generalized H-function representations are given, including those of every cumulative distribution function of an H-function variate. The generalizing constant is also possible in the H-function representations of power functions, the error function and its complement, the incomplete gamma function and its complement, the incomplete beta function and its complement, many inverse trigonometric and hyperbolic functions, and the logarithmic functions.

A number of new H-function representations of certain mathematical functions and statistical distributions are given in Sections 2.5, 2.6, and 3.6. Several of these expand upon a previously unstated limitation on the variable in the H-function representations of power functions and beta-type functions.

The exact distribution of the general sum of independent Erlang variates with different scale parameters, λ, is derived in Section 5.1.3. An Erlang variate is simply a gamma variate with an integer shape parameter r. The derivation uses partial fractions to decompose the product of Laplace transforms of the individual densities. This produces a sum of terms, each of which can easily be inverted from transform space, yielding the desired density of the sum of independent variates.

Since the H-function is not defined for zero or negative real arguments, the scope of this research effort was limited to continuous random variables defined only over positive values. Continuous and doubly infinite distributions such as the normal and Student's t are only represented as H-functions in their folded forms.

1.2. LITERATURE SURVEY

Regrettably, little research in the field of H-functions has been done in the United States. Much of what is known about the H-function is due to Indian mathematicians. Mathai and Saxena [1978] and Srivastava et al [1982] compiled many results of the early study of H-functions. In recent years, Soviet mathematicians [Prudnikov et al, 1990] have shown a considerable interest in the H-function and have developed significant new results.

The foundation of H-function theory is grounded in the gamma function, integral transform theory, complex analysis, and statistical distribution theory. Therefore, several landmark references such as Abramowitz and Stegun [1970], Erdélyi [1953], Erdélyi [1954], and Springer [1979], though somewhat dated, have timeless value.

Carter [1972] defined the H-function distribution and, using Mellin transform theory, gave startling and powerful results showing that products, quotients, and rational powers of independent H-function variates are themselves H-function variates. Further, the p.d.f. of the new random variable can immediately be written as an H-function distribution. The usual techniques of conditioning on one of the random variables and/or using the Jacobian of the transformation are no longer necessary.

The above results become especially useful when one realizes that nearly every common positive continuous random variable can be written as an H-function distribution. Therefore, the p.d.f. of any algebraic combination involving products, quotients, or powers of any number of independent positive continuous random variables can immediately be written as an H-function distribution.

Carter [1972] also wrote a FORTRAN computer program to calculate the moments of an algebraic combination of independent H-function variates and approximate the p.d.f. and cumulative distribution function (c.d.f.) from these moments. The approximation procedure was developed by Hill [1969] and, if possible, uses either a Gram-Charlier type A series (Hermite polynomial) or a Laguerre polynomial series. If a series approximation is not possible, the first four moments are used to fit a probability distribution from the Pearson family. As Carter [1972] himself notes "... there were many situations in which the methods did not work or in which the approximations were totally unsatisfactory."

Springer [1979] literally wrote the book on the algebra of random variables. He gives an excellent explanation of the value of integral transforms in finding the distribution of algebraic combinations of random variables. He also gave the known applications of the H-function in these problems.

Cook [1981] gave a very thorough survey and an extensive bibliography of the literature related to H-functions and H-function distributions. He also presented a technique for finding, in tabular form, the p.d.f. and c.d.f. of an algebraic combination (including sums and excluding differences) of independent H-function variates.

Cook's technique [1981; Cook and Barnes, 1981] first uses Carter's [1972] results to find the H-function distribution of any products, quotients, or powers of random variables. The Laplace transform of each term in the resulting sum of independent H-function variates is then obtained. These transform functions are evaluated and multiplied at corresponding values of the transform variable, yielding a tabular representation of the Laplace transform in transform space. This Laplace transform is then numerically inverted using Crump's method. His FORTRAN computer program implements this technique and will plot the resulting p.d.f. and c.d.f.

Bodenschatz and Boedigheimer [1983; Boedigheimer et al, 1984] developed a technique to fit the H-function to a set of data using the method of moments. The technique can be used to curve-fit a mathematical function or to estimate the density of a particular probability distribution. Their FORTRAN computer program will accept known moments, univariate data, ordered pair data from a relative frequency, or ordered pair data directly from the function.

Kellogg [1984; Kellogg and Barnes, 1987; Kellogg and Barnes, 1989] studied the distribution of products, quotients, and powers of dependent random variables with bivariate H-function distributions. Jacobs [1986; Jacobs et al, 1987] presented a method of obtaining parameter estimates for the H-function distribution using the method of maximum likelihood or the method of moments.

Prudnikov et al [1990] gave extensive tables of H-function results and Mellin transforms. Their books, although more terse than the series by Erdélyi [1953 and 1954], are at least as complete and, likely, will become the new standard reference for special functions.

1.3. INTEGRAL TRANSFORMS AND TRANSFORM PAIRS

Integral transforms are frequently encountered in several areas of mathematics, probability, and statistics. Although various integral transforms exist, certain characteristics are common among them. The function to be transformed is usually multiplied by another function (called the kernel) and then integrated over an appropriate range. What distinguishes the various transforms are the kernel function, the limits of integration, and the type of integration (e.g. Riemann or Lebesgue).

Often the use of integral transforms can simplify a difficult problem. Laplace transforms are usually first encountered in the solution of systems of linear differential equations. In probability and statistics, certain integral transforms are often known by other names such as the moment generating function, characteristic function, or probability generating function.

The definitions of most integral transforms are not standard. It is important, therefore, to explicitly state the form of the definition to be used. Listed below are the definitions of certain integral transforms and the corresponding inverse transforms as used in this thesis. Together, each transform and its corresponding inverse constitute a transform pair.

1.3.1. LAPLACE TRANSFORM

Consider a function $f(t)$ which is sectionally continuous and defined for all positive values of the variable t with $f(t)=0$ for $t \leq 0$. A sectionally continuous function may not have an infinite number of discontinuities nor any positive vertical asymptotes. If $f(t)$ grows no faster than an exponential function, then the Laplace transform of $f(t)$ will exist. There must exist two positive numbers M and T such that for all $t>T$ and for some real number α,

$$\left| \frac{f(t)}{e^{\alpha t}} \right| \leq M \tag{1.1}$$

The definition of the Laplace transform of the function $f(t)$, $\mathcal{L}_s\left\{ f(t) \right\}$, is

$$\mathcal{L}_s\left\{ f(t) \right\} = \int_0^{\infty} e^{-st} f(t)\ dt \tag{1.2}$$

In general, s is a complex variable. The Laplace transform of $f(t)$ will exist for the real part of s greater than α ($Re(s) > \alpha$).

The inversion integral or inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} e^{st}\ \mathcal{L}_s\left\{ f(t) \right\}\ ds \tag{1.3}$$

where $\mathcal{L}_s\left\{ f(t) \right\}$ is an analytic function for $Re(s) > \omega$. A function is analytic at $s=s_o$ if its derivative exists at s_o and at every point in some neighborhood of s_o. The Taylor series expansion of an analytic function of a complex variable will exist, converge, and equal the function evaluated at the argument. For all practical purposes, a function $f(t)$ and its Laplace transform (if it exists) uniquely determine each other.

In probability and statistics, if $f(t)$ is the p.d.f. of a random variable defined only for positive values, its moment generating function is simply the Laplace transform with r replacing -s in Eq (1.2).

1.3.2. FOURIER TRANSFORM

The form of the exponential Fourier transform used in this thesis is

$$\mathcal{F}_s\left\{ f(t) \right\} = \int_{-\infty}^{\infty} e^{ist} \; f(t) \; dt \tag{1.4}$$

The Fourier transform is a function of the complex variable s.

If f(t) is a p.d.f., this definition corresponds to the definition of the characteristic function in probability and statistics. The characteristic function of a p.d.f. will always exist but the moment generating function of a p.d.f. may or may not exist.

The inversion integral or inverse Fourier transform is given by

$$f(t)^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \; \mathcal{F}_s\left\{ f(t) \right\} \; ds \tag{1.5}$$

where

$$f(t_o)^* = \frac{1}{2} \left[\lim_{\substack{t \to t_o \\ t < t_o}} f(t) + \lim_{\substack{t \to t_o \\ t > t_o}} f(t) \right] \tag{1.6}$$

If f(t) exists and is continuous at t_o, the inverse Fourier transform of $\mathcal{F}_s\left\{ f(t) \right\}$ will give $f(t_o)$. If $f(t)$ is not continuous at t_o, the inverse Fourier transform of $\mathcal{F}_s\left\{ f(t) \right\}$ will produce the average of the limits of $f(t)$ from the left of t_o and the right of t_o.

1.3.3. MELLIN TRANSFORM

Because the Mellin transform is perhaps less well known than the Laplace or Fourier transforms and because the Mellin transform is so crucial in the study of H-functions, both common sets of transform pairs will be presented. The Mellin transform uses a power function instead of an exponential function as its kernel.

Again consider a function $f(t)$ which is sectionally continuous and defined for all positive values of the variable t with $f(t)=0$ for $t \le 0$. Using what will be regarded in this thesis as the primary definition of the Mellin transform, the Mellin transform of $f(t)$, $\mathcal{M}_s\left\{ f(t) \right\}$, is

$$\mathcal{M}_s\left\{ f(t) \right\} = \int_0^\infty t^{s-1} \; f(t) \; dt \tag{1.7}$$

The Mellin transform is related to the Fourier and Laplace transforms as follows [Erdélyi, 1954, p.305]:

$$\mathcal{M}_s\left\{ f(t) \right\} = \mathcal{F}_{-is}\left\{ f(e^t) \right\} \tag{1.8}$$

$$= \mathcal{L}_{-s}\left\{ f(e^t) \right\} + \mathcal{L}_s\left\{ f(e^{-t}) \right\} \tag{1.9}$$

Again, s is a complex variable. The Mellin transform inversion integral, or inverse Mellin transform, is given as

$$f(t) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} x^{-s} \; \mathcal{M}_s\left\{ f(t) \right\} \; ds \tag{1.10}$$

As mentioned earlier, there is another important transform pair also referred to as a Mellin transform pair. This alternate definition will arise later when an alternate definition of the H-function is given. The alternate definition is

$$\mathcal{M}_r^-\left\{ f(t) \right\} = \int_0^\infty t^{-r-1} \; f(t) \; dt \tag{1.11}$$

with inverse transform

$$f(t) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} x^r \; \mathcal{M}_r^-\left\{ f(t) \right\} \; dr \tag{1.12}$$

1.4. TRANSFORMATIONS OF INDEPENDENT RANDOM VARIABLES

A common problem in statistical distribution theory is to find the distribution of an algebraic combination of independent random variables. The algebraic combination could include sums, differences, products, and/or quotients of independent random variables or their powers. It is important to recognize that the algebraic combination is itself a random variable and, therefore, has a probability distribution. The task is to find this distribution.

Using the properties of mathematical expectation, it is relatively easy to find the mean, variance, and other moments of the algebraic combination of independent random variables. For example, the mean of the sum of two independent random variables is simply the sum of the means of the random variables. Finding the completely specified distribution of the algebraic combination is usually much more difficult.

For simple combinations of independent random variables, the method of Jacobians is often employed. An appropriate one-to-one transformation between the independent random variables in the algebraic combination and a set of new random variables is first created. After finding the inverse transformation functions, the Jacobian (the determinant of the matrix of first partial derivatives of the inverse functions) may be computed.

The joint p.d.f. of the newly defined random variables is the absolute value of the Jacobian multiplied by the product of the original densities with the inverse transformation functions substituted for the variables. A great deal of care must be used in determining the values of the new variables for which the joint density is nonzero. Once this is done, the desired marginal density can be obtained by integrating over the complete ranges of the other new variables.

An example of the method of Jacobians will be presented below. In most of the following sections, however, only the result using integral transforms will be given.

1.4.1. DISTRIBUTION OF A SUM

Let X_1 and X_2 be independent random variables with respective densities $f_1(x_1)$ and $f_2(x_2)$, each nonzero only for positive values of the variable. Suppose we want the density of $Y=X_1+X_2$. Using the method of Jacobians, we define $W=X_2$ so the inverse transformations are $X_1=Y-W$ and $X_2=W$. The Jacobian is

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \qquad (1.13)$$

The joint density of Y and W is

$$f_{YW}(y,w) = f_1(y-w)\ f_2(w) \qquad 0 < w < y < \infty \qquad (1.14)$$

The marginal p.d.f. of Y can be obtained by integrating the joint p.d.f. with respect to W over the range of W.

$$f_Y(y) = \int_0^y f_1(y-w)\, f_2(w)\, dw \tag{1.15}$$

Eq (1.15) may be recognized as the Fourier convolution integral [Springer, 1979, p. 47]. This is no accident or coincidence as the Fourier or Laplace transform could also have been used to find the distribution of Y. It is well known that the Laplace (or Fourier) transform of the p.d.f. of the sum of two independent random variables defined for positive values is the product of the Laplace (or Fourier) transforms of the individual densities. Further, the product of two transform functions, upon inversion, yields a convolution integral.

If the product of transform functions can be recognized as the transform of some function, then the convolution inversion is not necessary. The p.d.f. of Y is the function whose transform is the product. When statisticians use the moment generating function of each density to find the p.d.f. of Y, this recognition approach is usually taken. One advantage of using transform functions is that the procedure easily extends when the distribution of the sum of three or more independent random variables is desired.

Finding the distribution of the sum of independent random variables with certain special distributional forms is considerably simplified. Several of these cases are covered in the following subsections.

1.4.1.1. INFINITE DIVISIBILITY

Although a complete discussion of infinite divisibility is beyond the scope of this thesis, its definition is given below [Petrov, 1975, p. 25].

> A distribution function $F(x)$ and the corresponding characteristic function $f(t)$ are said to be infinitely divisible if for every positive integer n there exists a characteristic function $f_n(t)$ such that
>
> $$f(t) = \left(f_n(t)\right)^n$$
>
> In other words, the distribution F is infinitely divisible if for every positive integer n there exists a distribution function F_n such that $F=F_n^{*n}$. Here F_n^{*n} is the n-fold convolution of the function F_n.

Common examples of infinitely divisible distributions include the normal and Poisson distributions.

1.4.1.2. SPECIAL CASES

The distribution of the sum of independent random variables with certain distributional forms is well known and immediately available. For example, the sum of n independent and identically distributed random variables with a Bernoulli distribution with parameter p has a Binomial distribution with parameters n and p. Similarly, the sum of independent geometrically distributed random variables with a common parameter has a negative Binomial (or Pascal) distribution.

In continuous random variables, the sum of n independent and identically distributed random variables with an exponential distribution with parameter λ has a gamma distribution with parameters n and λ.

1.4.1.3. REPRODUCTIVE DISTRIBUTIONS

A probability distribution is "reproductive" if it replicates under positive addition of independent random variables with the same distributional form. The normal distribution is reproductive since given that

$$X_1 \sim \text{Normal}\left(\mu_1, \sigma_1^2\right) \qquad X_2 \sim \text{Normal}\left(\mu_2, \sigma_2^2\right)$$

X_1 and X_2 are statistically independent, and

$$Y = X_1 + X_2$$

then

$$Y \sim \text{Normal}\left(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2\right)$$

Other examples of reproductive distributions include the Chi-Square and Poisson distributions.

It is well known that the gamma distribution is reproductive provided the scale parameter, λ, is the same for each random variable in the sum. In particular, if

$$X_1 \sim \text{Gamma}\left(r_1, \lambda_1\right) \qquad X_2 \sim \text{Gamma}\left(r_2, \lambda_2\right)$$

X_1 and X_2 are statistically independent, $\lambda_1 = \lambda_2 = \lambda$

and $Y = X_1 + X_2$

then

$$Y \sim \text{Gamma}\left(r_1 + r_2, \lambda \right)$$

The result for gamma distributions above is readily verified by considering the product of the Laplace transforms of each p.d.f. Using the definition in Eq (1.2), the Laplace transform of the gamma p.d.f. with parameters r and λ is $\left(\frac{\lambda}{s+\lambda}\right)^r$. Clearly, then, if X_1 and X_2 are independent random variables with gamma distributions and a common scale parameter, λ,

$$\mathcal{L}_s\left\{f_Y(y)\right\} = \mathcal{L}_s\left\{f_1(x_1)\right\} \; \mathcal{L}_s\left\{f_2(x_2)\right\} \tag{1.16}$$

$$= \left(\frac{\lambda}{s+\lambda}\right)^{r_1} \left(\frac{\lambda}{s+\lambda}\right)^{r_2} \tag{1.17}$$

$$= \left(\frac{\lambda}{s+\lambda}\right)^{r_1+r_2} \tag{1.18}$$

which is recognized as the Laplace transform of a gamma p.d.f. with parameters r_1+r_2 and λ. This confirms the reproductive property for the gamma distribution when λ is common.

1.4.2. DISTRIBUTION OF A DIFFERENCE

The study of the distribution of the difference of independent random variables has received much less attention than for the sum. The normal distribution is one notable exception since given that

$$X_1 \sim \text{Normal}\left(\mu_1, \sigma_1^2\right) \qquad X_2 \sim \text{Normal}\left(\mu_2, \sigma_2^2\right)$$

X_1 and X_2 are statistically independent, and

$$Y = X_1 - X_2$$

then

$$Y \sim \text{Normal}\left(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

If X_1 and X_2 are independent random variables with respective densities $f_1(x_1)$ and $f_2(x_2)$, each nonzero only for positive values of the variable, then the density of $Y = X_1 - X_2$ is the inverse Fourier transform [Springer, 1979, p. 59]

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isy}\, \mathcal{F}_s\left\{ f_1(x_1) \right\} \mathcal{F}_s\left\{ f_2(-x_2) \right\} ds$$

for $-\infty < y < \infty$ (1.19)

1.4.3. <u>DISTRIBUTION OF A PRODUCT</u>

If X_1 and X_2 are independent random variables with respective densities $f_1(x_1)$ and $f_2(x_2)$, each nonzero only for positive values of the variable, then the density of $Y = (X_1)\cdot(X_2)$ is the inverse Mellin transform [Springer, 1979, p. 97]

$$f_Y(y) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} y^{-s}\, \mathcal{M}_s\left\{ f_1(x_1) \right\} \mathcal{M}_s\left\{ f_2(x_2) \right\} ds$$

for $0 < y < \infty$ (1.20)

1.4.4. DISTRIBUTION OF A QUOTIENT

If X_1 and X_2 are independent random variables with respective densities $f_1(x_1)$ and $f_2(x_2)$, each nonzero only for positive values of the variable, then the density of $Y=\frac{X_1}{X_2}$ is the inverse Mellin transform [Springer, 1979, p. 100]

$$f_Y(y) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} y^{-s}\, \mathcal{M}_s\left\{ f_1(x_1) \right\} \mathcal{M}_{2-s}\left\{ f_2(x_2) \right\} ds$$

for $0 < y < \infty$ (1.21)

One common example of the quotient of two independent random variables lies in the derivation of the Snedecor F distribution [Springer, 1979, pp. 328-9]. If

$X_1 \sim$ Chi-Square (ν) $X_2 \sim$ Chi-Square (ω)

X_1 and X_2 are statistically independent, and

$$Y = \frac{\frac{X_1}{\nu}}{\frac{X_2}{\omega}}$$

then

$Y \sim F\ (\nu,\omega)$

1.4.5. DISTRIBUTION OF A VARIATE TO A POWER

If X is a continuous random variable with density f(x), nonzero only for positive values of the variable, then the density of $Y=X^P$ is the inverse Mellin transform [Springer, 1979, p. 212]

$$f_Y(y) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} y^{-s} \mathcal{M}_{Ps-P+1}\left\{ f(x) \right\} ds$$

for $0 < y < \infty$ (1.22)

One common example of a variate to a power is finding the distribution of the square of a standard normal (zero mean, unit variance) random variable. If X ~ Normal (0,1) and $Y=X^2$ then Y ~ Chi-Square (1). The same result holds if X has a half-normal distribution with $\sigma^2=1$ [Springer, 1979, pp. 213-4].

1.4.6. MOMENTS OF A DISTRIBUTION

If X is a continuous random variable with density f(x), nonzero only for positive values of the variable, then the moments about the origin of f(x) are

$$\mu_r = E\left[X^r \right]$$

$$= \int_0^\infty x^r f(x)\, dx \qquad (1.23)$$

provided the integral in Eq (1.23) exists.

There is a natural relationship between the integral given in Eq (1.23) and the Mellin transform of the function f(x) given in Eq (1.7) or Eq (1.11). Using the Mellin transform in Eq (1.7) we can write

$$\mu_r = \mathcal{M}_{r+1}[f(x)] \tag{1.24}$$

This relationship simplifies the computation of moments for H-functions and H-function distributions.

CHAPTER 2

THE H-FUNCTION

The H-function is a very general function, encompassing as special cases nearly every named mathematical function and continuous statistical distribution defined over positive values. Although the H-function does not enjoy an extensive popularity and acceptance in the fields of mathematics, probability, and statistics, this is primarily because mathematicians and statisticians have not yet learned of its versatility and power. Most analysts are not familiar with the H-function, and many who have seen the H-function definition may have been disquieted by its overt complexity. This is unfortunate because practical use of the H-function does not require extensive knowledge of complex analysis and integral transform theory.

Mathematical functions defined by an integral which do not have a closed form representation are common. Examples include the gamma function, the error function, and the cumulative normal probability density function. In all of these cases, the function is usually evaluated with the help of tables. The

H-function is another example of such functions. Like all of the transcendental functions (e.g. e^x, sin x, cos x), it must be evaluated using an infinite series expansion. A FORTRAN computer program is available [Cook, 1981; Cook and Barnes, 1981] which will evaluate the H-function at desired values once the parameters are specified.

Fox [1961] first developed the H-function as a direct generalization of Meijer's G-function. Mathai and Saxena [1978] presented many useful properties of the H-function and listed the mathematical functions that are known to be special cases of the H-function. More recently, Prudnikov et al [1990] compiled extensive results of all special functions, including H-functions.

2.1. PRIMARY DEFINITION

The primary definition of the H-function as used in this thesis is:

$$H(cz) = H_{p\ q}^{m\ n}[cz] = H_{p\ q}^{m\ n}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\dots,q} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{i=1}^{n} \Gamma\left(1 - a_i - A_i s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i s\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j s\right)} (cz)^{-s}\, ds \qquad (2.1)$$

where z, c, and all a_i and b_j are real or complex numbers, all A_i and B_j are positive real numbers, and m, n, p, and q are integers such that $0 \le m \le q$ and $0 \le n \le p$. Empty products are defined to be unity (1). The path of integration, C, is a contour in the complex s-plane from $\omega - i\infty$ to $\omega + i\infty$ such that all Left Half-Plane (LHP) poles of $\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right)$ lie to the left of C and all Right Half-Plane (RHP) poles of $\prod_{i=1}^{n} \Gamma\left(1 - a_i - A_i s\right)$ lie to the right.

Although the H-function is defined for the complex variable z, we will often restrict our attention to the real variable x. Further, since the H-function is not valid for non-positive real values of z, we will often consider only positive values of the real variable x.

The definition of the H-function in Eq (2.1) may be recognized as the inverse Mellin transform where the transform pair is as given by Eq (1.7) and Eq (1.10).

H-functions are sometimes classified according to their order, the number of gamma terms in the integrand (p+q), and the placement of those terms. References will be made to certain classes of H-functions as $H_{p\,q}^{m\,n}$ with particular values for the parameters m, n, p, and q.

2.2. ALTERNATE DEFINITION

It should be noted that there is an alternate, but equivalent definition of the H-function:

$$H(cz) = H_{p\ q}^{m\ n}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q} \right]$$

$$= \frac{1}{2\pi i} \int_{C'} \frac{\prod_{j=1}^{m} \Gamma\left(b_j - B_j r\right) \prod_{i=1}^{n} \Gamma\left(1 - a_i + A_i r\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i - A_i r\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j + B_j r\right)} (cz)^r \, dr \tag{2.2}$$

where the restrictions on the variable and parameters are as above. Here, the path of integration, C', is a contour in the complex r-plane from $\nu - i\infty$ to $\nu + i\infty$ such that all RHP poles of $\prod_{j=1}^{m} \Gamma\left(b_j - B_j r\right)$ lie to the right of C' and all LHP poles of $\prod_{i=1}^{n} \Gamma\left(1 - a_i + A_i r\right)$ lie to the left.

Eq (2.2) above can be derived from Eq (2.1) by the substitution $r=-s$. Under this transformation, $\nu=-\omega$ since the substitution $r=-s$ rotates about the imaginary axis all poles of the integrand and the contour, C, into a contour, C', which also separates the new LHP poles from the new RHP poles in the standard manner and direction. Again, the definition in

Eq (2.2) corresponds to an inverse Mellin transform where the transform pair is as given by Eq (1.11) and Eq (1.12).

2.3. SUFFICIENT CONVERGENCE CONDITIONS

Cook [1981; Cook and Barnes, 1981] gave sufficient conditions for which the H-function could be evaluated as the sum of residues in the appropriate half-plane for certain values of the variable. In an unpublished working paper, Eldred et al [1979] applied the well-known convergence restrictions of Mellin-Barnes integrals to the H-function. These restrictions were originally developed by Dixon and Ferrar [1936] and are also given by Erdélyi [1953, Vol. 1] and Prudnikov et al [1990].

These conditions help determine how to evaluate the H-function in Eq (2.1) as the sum of residues and give the values of the complex variable z for which this evaluation is valid. These convergence conditions specify the restrictions on the argument, arg(z), and the modulus, $|z|$, of z to guarantee convergence. They indicate that the sum of residues for the H-function is always convergent for positive real values of the variable and sometimes convergent for values of z with a nonzero imaginary part.

For the H-function defined by Eq (2.1), the convergence conditions are based on the values of $\mathbf{D}$, $\mathbf{E}$, $\mathbf{L}$, and $\mathbf{R}$ defined as:

$$D = \sum_{i=1}^{n} A_i + \sum_{j=1}^{m} B_j - \sum_{i=n+1}^{p} A_i - \sum_{j=m+1}^{q} B_j \tag{2.3}$$

$$E = \sum_{i=1}^{p} A_i - \sum_{j=1}^{q} B_j \tag{2.4}$$

$$L = \text{Re}\left(\sum_{j=1}^{q} b_j - \frac{q}{2} - \sum_{i=1}^{p} a_i + \frac{p}{2} \right) \tag{2.5}$$

$$R = \frac{\prod_{i=1}^{p} \left(A_i\right)^{A_i}}{\prod_{j=1}^{q} \left(B_j\right)^{B_j}} \tag{2.6}$$

Once these values are determined, the H-function in Eq (2.1) may be evaluated by the positive sum of LHP residues, the negative sum of RHP residues, or both, depending on the value of the variable z. These criteria are based upon which semicircle satisfies the hypotheses of Jordan's Lemma. The types of convergence and the applicable ranges for the complex variable z were given by Eldred et al [1979], Cook [1981], and Cook and Barnes [1981] and are repeated below in Table 1:

<u>Table 1. Convergence Types for H-Functions of Eq (2.1)</u>

TYPE	$\mathbb{D}$	$\mathbb{E}$	$\mathbb{L}$	H(cz)	$\lvert z \rvert$	$\lvert \arg(z) \rvert$
I	>0	<0	$>\mathbb{E}\omega$	$+\sum$ LHP res	>0	$< \pi\ ,\ < \frac{\pi\mathbb{D}}{2}$
II	≥ 0	<0	$\leq\mathbb{E}\omega$	$+\sum$ LHP res	>0	$< \pi\ ,\ \leq \frac{\pi\mathbb{D}}{2}$
III	>0	>0	$>\mathbb{E}\omega$	$-\sum$ RHP res	>0	$< \pi\ ,\ < \frac{\pi\mathbb{D}}{2}$
IV	≥ 0	>0	$\leq\mathbb{E}\omega$	$-\sum$ RHP res	>0	$< \pi\ ,\ \leq \frac{\pi\mathbb{D}}{2}$
V	>0	$=0$	≥ 0	$+\sum$ LHP res	$< \frac{1}{c\mathbb{R}}$	$< \pi\ ,\ < \frac{\pi\mathbb{D}}{2}$
				$-\sum$ RHP res	$> \frac{1}{c\mathbb{R}}$	
VI	≥ 0	$=0$	<0	$+\sum$ LHP res	$< \frac{1}{c\mathbb{R}}$	$< \pi\ ,\ \leq \frac{\pi\mathbb{D}}{2}$
				$-\sum$ RHP res	$> \frac{1}{c\mathbb{R}}$	

where, if $\mathbb{L} < -1$ in Type VI convergence, one may use the sum of either LHP or RHP residues at $\lvert z \rvert = \frac{1}{c\mathbb{R}}$. Type VI convergent H-functions play a central role in several new results given in Sections 2.5, 2.6, 3.6, and 4.3.

Because there was some disagreement [Springer, 1987] about the validity of the convergence conditions given by Cook [1981; Cook and Barnes, 1981], it was necessary to develop the corresponding convergence conditions for the H-function in definition (2.2). By rewriting Eq (2.2) as

$$\frac{1}{2\pi i}\int_{C'} \frac{\prod_{i=1}^{n}\Gamma\left(1-a_i+A_i r\right)\prod_{j=1}^{m}\Gamma\left(b_j-B_j r\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j r\right)\prod_{i=n+1}^{p}\Gamma\left(a_i-A_i r\right)}\left(\frac{1}{cz}\right)^{-r} dr \tag{2.7}$$

and using Eq (2.1) by considering the "A" terms as "B" terms and the "B" terms as "A" terms, we have

$$H_{q\ p}^{n\ m}\left[\frac{1}{cz} : \left\{\left(1-b_j,B_j\right)\right\}_{j=1,\ldots,q} ; \left\{\left(1-a_i,A_i\right)\right\}_{i=1,\ldots,p}\right] \tag{2.8}$$

Evaluating $\mathbb{D}'$, $\mathbb{E}'$, $\mathbb{L}'$, and $\mathbb{R}'$ as in Eq (2.3) through Eq (2.6) yields

$$* \quad \mathbb{D}' = \sum_{j=1}^{m} B_j + \sum_{i=1}^{n} A_i - \sum_{j=m+1}^{q} B_j - \sum_{i=n+1}^{p} A_i \tag{2.9}$$

$$* \quad \mathbb{E}' = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \tag{2.10}$$

$$* \quad \mathbb{L}' = \mathrm{Re}\left(\sum_{i=1}^{p}(1-a_i) - \frac{p}{2} - \sum_{j=1}^{q}(1-b_j) + \frac{q}{2}\right)$$

$$= \mathrm{Re}\left(-\sum_{i=1}^{p} a_i + \frac{p}{2} + \sum_{j=1}^{q} b_j - \frac{q}{2}\right) \tag{2.11}$$

$$* \quad \mathbb{R}' = \frac{\prod_{j=1}^{q} \left(B_j\right)^{B_j}}{\prod_{i=1}^{p} \left(A_i\right)^{A_i}} \tag{2.12}$$

Comparing Eq (2.9) through Eq (2.12) to Eq (2.3) through Eq (2.6),

$$\mathbb{D}' \text{ in Eq (2.9)} = \mathbb{D} \text{ in Eq (2.3)} \tag{2.13}$$

$$\mathbb{E}' \text{ in Eq (2.10)} = (-1) \left[\mathbb{E} \text{ in Eq (2.4)} \right] \tag{2.14}$$

$$\mathbb{L}' \text{ in Eq (2.11)} = \mathbb{L} \text{ in Eq (2.5)} \tag{2.15}$$

$$\mathbb{R}' \text{ in Eq (2.12)} = \left[\mathbb{R} \text{ in Eq (2.6)} \right]^{-1} \tag{2.16}$$

These relationships allow the corresponding convergence types for H-functions defined by Eq (2.2) to be written. Table 2 lists these types of convergence.

* Table 2. Convergence Types for H-Functions of Eq (2.2)

TYPE	D'	E'	L'	$H(cz)$	$\lvert z \rvert$	$\lvert \arg(z) \rvert$
I	>0	<0	$>E'\nu$	$-\sum$ RHP res	>0	$< \pi\ ,\ < \frac{\pi D'}{2}$
II	≥0	<0	$\le E'\nu$	$-\sum$ RHP res	>0	$< \pi\ ,\ \le \frac{\pi D'}{2}$
III	>0	>0	$>E'\nu$	$+\sum$ LHP res	>0	$< \pi\ ,\ < \frac{\pi D'}{2}$
IV	≥0	>0	$\le E'\nu$	$+\sum$ LHP res	>0	$< \pi\ ,\ \le \frac{\pi D'}{2}$
V	>0	=0	≥0	$-\sum$ RHP res $+\sum$ LHP res	$< \frac{R'}{c}$ $> \frac{R'}{c}$	$< \pi\ ,\ < \frac{\pi D'}{2}$
VI	≥0	=0	<0	$-\sum$ RHP res $+\sum$ LHP res	$< \frac{R'}{c}$ $> \frac{R'}{c}$	$< \pi\ ,\ \le \frac{\pi D'}{2}$

where, if $L' < -1$ in Type VI convergence, one may use the sum of either LHP or RHP residues at $\lvert z \rvert = \frac{R'}{c}$.

As expected, there is a complete interchange of LHP and RHP poles. Type I convergence for the H-function of Eq (2.2) in Table 2 corresponds to Type III convergence for the H-function of Eq (2.1) in Table 1. Similar statements apply between Types II and IV, III and I, and IV and II. Even Types

V and VI in Table 2 correspond to Types V and VI in Table 1 when considering the relationship between $\mathbb{R}$ and $\mathbb{R}'$ in Eq (2.16).

Either set of conditions is sufficient, but not necessary for the appropriate H-function to converge. Still, nearly all of the many special cases of the H-function satisfy the convergence conditions. Further, the conditions consistently and correctly identify the valid range of the variable over which the H-function representation equals the special case.

One should not think of these conditions as constraints on the H-function, as Springer [1987] did. Instead, they should be viewed as a sufficient tool to determine how the H-function can be evaluated with the sum of residues - those in the LHP or those in the RHP. Depending on where Jordan's Lemma is satisfied, residues at the LHP poles, or RHP poles, or either are summed for different values of the variable. These choices are succinctly given in Table 1 (and Table 2) for most cases of interest.

To demonstrate that the convergence conditions are sufficient, but not necessary, consider the following representation of e^x for $x>0$ as a Meijer G-function [Prudnikov et al, 1990, p. 633]

$$e^{x} = \frac{\pi}{\sin(d\pi)} G_{1\ 2}^{1\ 0}\left[x : (1-d) ; (0), (1-d) \right] \tag{2.17}$$

$$= \frac{\pi}{\sin(d\pi)} H_{1\ 2}^{1\ 0}\left[x : (1-d,1) ; (0,1), (1-d,1) \right] \tag{2.18}$$

where d is an arbitrary constant. For this H-function, $D=E=-1$, $L= -\frac{1}{2}$, and $R=1$. It does not meet any of the types of convergence listed in Table 1 above. Using the definition in Eq (2.1), this H-function can be written as

$$= \frac{\pi}{\sin(d\pi)} \frac{1}{2\pi i} \int_C \frac{\Gamma(s)}{\Gamma(1-d+s)\,\Gamma(d-s)} (x)^{-s}\ ds \tag{2.19}$$

Using the reflection formula for the gamma function [Abramowitz and Stegun, 1970, p.256, 6.1.17] with z=d-s, this is

$$= \frac{\pi}{\sin(d\pi)} \frac{1}{2\pi i} \int_C \frac{\Gamma(s)}{\pi \ \csc(\pi(d-s))} (x)^{-s}\ ds \tag{2.20}$$

$$= \frac{1}{\sin(d\pi)} \frac{1}{2\pi i} \int_C \sin(\pi d-\pi s)\ \Gamma(s)\ (x)^{-s}\ ds \tag{2.21}$$

$$= \frac{1}{\sin(d\pi)} \frac{1}{2\pi i} \int_C \big(\sin(\pi d)\cos(\pi s) - \cos(\pi d)\sin(\pi s)\big) \Gamma(s)\ (x)^{-s}\ ds \tag{2.22}$$

Now, $\Gamma(s)$ has LHP poles of order 1 at $s_J= -J$, $J=0,1,\ldots.$ These are the only poles of the integrand. At these values of s, $\sin(\pi s)$ vanishes and $\cos(\pi s) = (-1)^J$. Using the residue theorem to evaluate the contour integral as the sum of residues produces

$$= \frac{1}{\sin(d\pi)} \frac{1}{2\pi i} 2\pi i \sum_{J=1}^{\infty} \left[(-1)^J \sin(\pi d)\right] \left[\frac{(-1)^J}{J!}\right] x^J \tag{2.23}$$

$$= \sum_{J=1}^{\infty} \frac{x^J}{J!} \quad = e^x \qquad \text{Q.E.D.} \tag{2.24}$$

This proves that the convergence conditions given in Table 1 are sufficient, but not necessary, for the H-function to converge.

2.4. PROPERTIES

Among the useful properties of the H-function are the identities dealing with the reciprocal of an argument, an argument to a power, and the multiplication of an H-function by the argument to a power [Carter and Springer, 1977].

2.4.1. RECIPROCAL OF AN ARGUMENT

$$H_{p\,q}^{m\,n}\left[\frac{1}{z} : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q}\right]$$

$$= H_{q\,p}^{n\,m}\left[z : \left\{\left(1-b_j, B_j\right)\right\}_{j=1,\ldots,q} ; \left\{\left(1-a_i, A_i\right)\right\}_{i=1,\ldots,p}\right] \qquad (2.25)$$

2.4.2. ARGUMENT TO A POWER

$$H_{p\,q}^{m\,n}\left[z^c : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q}\right]$$

$$= \frac{1}{c} H_{p\,q}^{m\,n}\left[z : \left\{\left(a_i, \frac{A_i}{c}\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, \frac{B_j}{c}\right)\right\}_{j=1,\ldots,q}\right] \qquad (2.26)$$

for $c > 0$

2.4.3. MULTIPLICATION BY THE ARGUMENT TO A POWER

$$z^c H_{p\ q}^{m\ n}\left[z : \underset{i=1,\ldots,p}{\left\{\left(a_i, A_i\right)\right\}} ; \underset{j=1,\ldots,q}{\left\{\left(b_j, B_j\right)\right\}} \right]$$

$$= H_{p\ q}^{m\ n}\left[z : \underset{i=1,\ldots,p}{\left\{\left(a_i + A_i c, A_i\right)\right\}} ; \underset{j=1,\ldots,q}{\left\{\left(b_j + B_j c, B_j\right)\right\}} \right] \tag{2.27}$$

2.4.4. FIRST REDUCTION PROPERTY

If a pair of "A" terms and a pair of "B" terms in an H-function are identical and one is in the numerator of the integrand and the other is in the denominator, then it is equivalent to an H-function of lower order. Specifically, [Mathai and Saxena, 1978, p. 4]

$$H_{p\ q}^{m\ n}\left[z : \underset{i=1,\ldots,p}{\left\{\left(a_i, A_i\right)\right\}} ; \underset{j=1,\ldots,q-1}{\left\{\left(b_j, B_j\right)\right\}}, \left(a_1, A_1\right) \right]$$

$$= H_{p-1\ q-1}^{m\ \ n-1}\left[z : \underset{i=2,\ldots,p}{\left\{\left(a_i, A_i\right)\right\}} ; \underset{j=1,\ldots,q-1}{\left\{\left(b_j, B_j\right)\right\}} \right] \tag{2.28}$$

provided $n>0$ and $q>m$. Also,

$$H_{p\ q}^{m\ n}\left[z : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p-1}, \left(b_1, B_1\right) ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\dots,q} \right]$$

$$= H_{p-1\ q-1}^{m-1\ n}\left[z : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p-1} ; \left\{\left(b_j, B_j\right)\right\}_{j=2,\dots,q} \right] \quad (2.29)$$

provided $m>0$ and $p>n$.

2.4.5. SECOND REDUCTION PROPERTY

Bodenschatz and Boedigheimer [1983, pp. 11-12] discovered another way in which the H-function can reduce to one of lower order, at least in the limit. If any A_i or B_j is close to zero, that gamma term in the integrand of Eq (2.1) is essentially a constant. Thus,

$$H_{p\ q}^{m\ n}[z] \approx \Gamma(b_1)\ H_{p\ \ q-1}^{m-1\ n}\left[z : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=2,\dots,q} \right] \quad (2.30)$$

for $B_1 \approx 0$ and $m>0$. Here, the symbol $\approx$ means the limit of $H_{p\ q}^{m\ n}[z]$ as $B_1 \to 0$ is given by the right side of Eq (2.30).

$$H_{p\ q}^{m\ n}[z] \approx \frac{1}{\Gamma(1-b_q)} H_{p\ q-1}^{m\ n}\left[z : \left\{(a_i, A_i)\right\}_{i=1,\dots,p} ; \left\{(b_j, B_j)\right\}_{j=1,\dots,q-1} \right] \quad (2.31)$$

for $B_q \approx 0$ and $m<q$.

$$H_{p\ q}^{m\ n}[z] \approx \Gamma(1-a_1)\ H_{p-1\ q}^{m\ \ n-1}\left[z : \left\{(a_i, A_i)\right\}_{i=2,\dots,p} ; \left\{(b_j, B_j)\right\}_{j=1,\dots,q} \right] \quad (2.32)$$

for $A_1 \approx 0$ and $n>0$.

$$H_{p\ q}^{m\ n}[z] \approx \frac{1}{\Gamma(a_p)} H_{p-1\ q}^{m\ \ n}\left[z : \left\{(a_i, A_i)\right\}_{i=1,\dots,p-1} ; \left\{(b_j, B_j)\right\}_{j=1,\dots,q} \right] \quad (2.33)$$

for $A_p \approx 0$ and $n<p$.

* 2.4.6. HIERARCHICAL RELATIONSHIPS AMONG ORDERS OF H-FUNCTIONS

One significant, but surprisingly simple, discovery was that whole classes of H-functions are embedded in other classes. For example, the $H_{0\ 1}^{1\ 0}$ class of H-functions is a proper subset of both the $H_{1\ 1}^{1\ 0}$ and $H_{0\ 2}^{2\ 0}$ classes of H-functions. These classes, in turn, are embedded in other, higher orders of H-function classes.

This newly discovered hierarchical relationship among classes of H-functions results in several new ways in which certain H-functions can reduce to H-functions of lower order. Further, while the first and second reduction properties listed above are readily apparent and easily understood, the new reduction properties are less transparent.

The new results are based on the Duplication Formula, Triplication Formula, and Gauss' Multiplication Formula for the gamma function [Abramowitz and Stegun, 1970, p. 256]. The Duplication Formula is

$$\Gamma(2w) = (2\pi)^{-\frac{1}{2}}\, 2^{2w-\frac{1}{2}}\, \Gamma(w)\, \Gamma\left(w+\frac{1}{2}\right) \tag{2.34}$$

which can be rewritten as

$$\Gamma(2w) = \frac{1}{2\sqrt{\pi}}\, 4^{w}\, \Gamma(w)\, \Gamma\left(\frac{1}{2}+w\right) \tag{2.35}$$

or

$$\Gamma\left(\frac{1}{2}+w\right) = 2\sqrt{\pi}\, \frac{\Gamma(2w)}{\Gamma(w)}\, 4^{-w} \tag{2.36}$$

or

$$\Gamma(w) = 2\sqrt{\pi}\,\frac{\Gamma(2w)}{\Gamma\left(\frac{1}{2}+w\right)}\,4^{-w} \tag{2.37}$$

Any gamma function present in the integrand of Eq (2.1) in the definition of the H-function can be replaced with an equivalent expression as in Eq (2.35), Eq (2.36), or Eq (2.37). Terms of the above equations which do not involve gamma functions can be combined with the parameters k or c in the definition of the H-function. Using Eq (2.35), a $H_{0\,1}^{1\,0}$ H-function may be rewritten as a $H_{0\,2}^{2\,0}$ H-function. Using Eq (2.36) or Eq (2.37), a $H_{0\,1}^{1\,0}$ H-function may be written as a $H_{1\,1}^{1\,0}$ H-function. The H-functions resulting from Eq (2.36) and Eq (2.37) appear distinct, but can be shown to be equivalent.

I will state these new upgrade and reduction results for first order H-functions as theorems and provide proofs. The proofs simply use the argument to a power property in Eq (2.26) to change B to unity, the definition of the H-function in Eq (2.1), and one of the forms of the duplication property in Eq (2.35) to Eq (2.37).

* Theorem 2.1. $H_{0\,1}^{1\,0}\left[\,cz : ; (b,B)\,\right]$

$$= \frac{\sqrt{\pi}}{4^{b-1}}\,H_{1\,1}^{1\,0}\left[\,4^{B}cz : \left(b-\frac{1}{2},B\right) ; (2b-1,2B)\,\right] \tag{2.38}$$

Proof:

$$H_{0\ 1}^{1\ 0}\left[\ cz\ :\ ;\ (b,B)\ \right] = \frac{1}{B}\ H_{0\ 1}^{1\ 0}\left[\ (cz)^{\frac{1}{B}}\ :\ ;\ (b,1)\ \right]$$

$$= \frac{1}{B2\pi i}\int_C \Gamma(b+s)\ (cz)^{-\frac{s}{B}}\ ds$$

$$= \frac{1}{B2\pi i}\int_C \frac{4\sqrt{\pi}}{4^b}\ \frac{\Gamma\left[2\left(b-\frac{1}{2}+s\right)\right]}{\Gamma\left(b-\frac{1}{2}+s\right)}\ 4^{-s}\ (cz)^{-\frac{s}{B}}\ ds$$

using Eq (2.36) with $w = b-\frac{1}{2}+s$

$$= \frac{\sqrt{\pi}}{B4^{b-1}}\ \frac{1}{2\pi i}\int_C \frac{\Gamma\left(2b-1+2s\right)}{\Gamma\left(b-\frac{1}{2}+s\right)}\ \left(4^B\ cz\right)^{-\frac{s}{B}}\ ds$$

$$= \frac{\sqrt{\pi}}{B4^{b-1}}\ H_{1\ 1}^{1\ 0}\left[\ \left(4^B cz\right)^{\frac{1}{B}}\ :\ \left(b-\frac{1}{2},1\right)\ ;\ (2b-1,2)\ \right]$$

$$= \frac{\sqrt{\pi}}{4^{b-1}}\ H_{1\ 1}^{1\ 0}\left[\ 4^B cz\ :\ \left(b-\frac{1}{2},B\right)\ ;\ (2b-1,2B)\ \right] \qquad (2.39)$$

Q.E.D.

* Theorem 2.2. $H_{0\ 1}^{1\ 0}\left[\ cz\ :\ ;\ (b,B)\ \right]$

$$= \frac{2\sqrt{\pi}}{4^{b}}\ H_{1\ 1}^{1\ 0}\left[\ 4^B\ cz\ :\ \left(b+\frac{1}{2},B\right)\ ;\ (2b,2B)\ \right] \qquad (2.40)$$

Proof:

$$H_{0\ 1}^{1\ 0}\left[\ cz\ :\ ;\ (b,B)\ \right] = \frac{1}{B}\, H_{0\ 1}^{1\ 0}\left[\ (cz)^{\frac{1}{B}}\ :\ ;\ (b,1)\ \right]$$

$$= \frac{1}{B2\pi i}\int_C \Gamma(b+s)\ (cz)^{-\frac{s}{B}}\ ds$$

$$= \frac{1}{B2\pi i}\int_C \frac{2\sqrt{\pi}}{4^b}\ \frac{\Gamma\left(2(b+s)\right)}{\Gamma\left(b+\frac{1}{2}+s\right)}\ 4^{-s}\ (cz)^{-\frac{s}{B}}\ ds$$

using Eq (2.37) with w=b+s

$$= \frac{2\sqrt{\pi}}{B4^b}\ \frac{1}{2\pi i}\int_C \frac{\Gamma(2b+2s)}{\Gamma\left(b+\frac{1}{2}+s\right)}\left(4^B\ cz\right)^{-\frac{s}{B}}\ ds$$

$$= \frac{2\sqrt{\pi}}{B4^b}\ H_{1\ 1}^{1\ 0}\left[\ \left(4^B\ cz\right)^{\frac{1}{B}}\ :\ \left(b+\frac{1}{2},1\right)\ ;\ (2b,2)\ \right]$$

$$= \frac{2\sqrt{\pi}}{4^b}\ H_{1\ 1}^{1\ 0}\left[\ 4^B\ cz\ :\ \left(b+\frac{1}{2},B\right)\ ;\ (2b,2B)\ \right] \qquad (2.41)$$

Q.E.D.

Equivalence between Eq (2.41) and Eq (2.39) can be proven by using the telescoping property of the gamma function $\Gamma(z+1) = z\Gamma(z)$. The core of the proof is provided below.

$$\frac{2\sqrt{\pi}}{4^b}\,\frac{\Gamma(2b+2Bs)}{\Gamma\left(b+\frac{1}{2}+Bs\right)} = \frac{2\sqrt{\pi}}{4^b}\,\frac{(2b-1+2Bs)\,\Gamma(2b-1+2Bs)}{\left(b-\frac{1}{2}+Bs\right)\Gamma\left(b-\frac{1}{2}+Bs\right)}$$

$$= \frac{\sqrt{\pi}}{4^{b-1}}\,\frac{\Gamma(2b-1+2Bs)}{\Gamma\left(b-\frac{1}{2}+Bs\right)} \tag{2.42}$$

* Theorem 2.3. $H_{0\ 1}^{1\ 0}\left[\ cz\ :\ ;\ (b,B)\ \right]$

$$= \frac{2^{b-1}}{\sqrt{\pi}}\,H_{0\ 2}^{2\ 0}\left[\ 2^{-B}cz\ :\ ;\ \left(\frac{b}{2},\frac{B}{2}\right),\ \left(\frac{b+1}{2},\frac{B}{2}\right)\ \right] \tag{2.43}$$

Proof:

$$H_{0\ 1}^{1\ 0}\left[\ cz\ :\ ;\ (b,B)\ \right] = \frac{1}{B}\,H_{0\ 1}^{1\ 0}\left[\ (cz)^{\frac{1}{B}}\ :\ ;\ (b,1)\ \right]$$

$$= \frac{1}{B2\pi i}\int_C \Gamma(b+s)\ (cz)^{-\frac{s}{B}}\,ds$$

$$= \frac{1}{B2\pi i}\int_C \frac{1}{2\sqrt{\pi}}\,2^{b+s}\,\Gamma\left(\frac{b}{2}+\frac{s}{2}\right)\Gamma\left(\frac{b+1}{2}+\frac{s}{2}\right)(cz)^{-\frac{s}{B}}\,ds$$

using Eq (2.35) with $w = \frac{b+s}{2}$

$$= \frac{2^{b-1}}{\sqrt{\pi}}\,\frac{1}{B2\pi i}\int_C \Gamma\left(\frac{b}{2}+\frac{s}{2}\right)\Gamma\left(\frac{b+1}{2}+\frac{s}{2}\right)\left(2^{-B}cz\right)^{-\frac{s}{B}}\,ds$$

$$= \frac{2^{b-1}}{\sqrt{\pi}}\,\frac{1}{B}\,H_{0\ 2}^{2\ 0}\left[\left(2^{-B}cz\right)^{\frac{1}{B}}\ :\ ;\ \left(\frac{b}{2},\frac{1}{2}\right),\ \left(\frac{b+1}{2},\frac{1}{2}\right)\right]$$

$$= \frac{2^{b-1}}{\sqrt{\pi}} H_{0\,2}^{2\,0}\left[2^{-B}cz \; : \; ; \left(\frac{b}{2},\frac{B}{2}\right), \left(\frac{b+1}{2},\frac{B}{2}\right) \right] \tag{2.44}$$

Q.E.D.

Upgrade and reduction results for the more general class of H-functions $H_{p\,q}^{m\,n}$ can be proven using the same steps as in the above proofs by working on any gamma function present in the integrand of Eq (2.1). Since the proofs are very similar to those already provided, Theorems 2.4 through 2.7 will be stated without proof. The generalized results are

* **Theorem 2.4.** $$H_{p\,q}^{m\,n}\left[cz \; : \; \left\{\left(a_i,A_i\right)\right\}_{i=1,\dots,p} \; ; \; \left\{\left(b_j,B_j\right)\right\}_{j=1,\dots,q} \right]$$

$$= \frac{2\sqrt{\pi}}{4^{b_1}} H_{p+1\,q}^{m\;\;\,n}\left[4^{B_1}cz \; : \; \left\{\left(a_i,A_i\right)\right\}_{i=1,\dots,p}, \left(b_1+\frac{1}{2},B_1\right) \; ; \; \left(2b_1,2B_1\right), \left\{\left(b_j,B_j\right)\right\}_{j=2,\dots,q} \right] \tag{2.45}$$

$$= \frac{2^{b_1-1}}{\sqrt{\pi}} H^{m+1\ n}_{p\ \ q+1}\left[2^{-B_1}cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \right.$$

$$\left. \left(\frac{b_1}{2}, \frac{B_1}{2}\right), \left(\frac{b_1+1}{2}, \frac{B_1}{2}\right), \left\{\left(b_j, B_j\right)\right\}_{j=2,\ldots,q} \right] \tag{2.46}$$

for $m>0$.

* Theorem 2.5. $H^{m\ n}_{p\ q}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q} \right]$

$$= \frac{4^{a_p}}{2\sqrt{\pi}} H^{m+1\ n}_{p\ \ q+1}\left[4^{-A_p}cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p-1} , \left(2a_p, 2A_p\right) ; \right.$$

$$\left. \left(a_p+\frac{1}{2}, A_p\right), \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q} \right] \tag{2.47}$$

$$= \frac{\sqrt{\pi}}{2^{a_p-1}} H^{m\ \ n}_{p+1\ q}\left[2^{A_p}cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p-1} , \left(\frac{a_p}{2}, \frac{A_p}{2}\right), \right.$$

$$\left. \left(\frac{a_p+1}{2}, \frac{A_p}{2}\right) ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q} \right] \tag{2.48}$$

for $p>n$.

* Theorem 2.6. $H_{p\ q}^{m\ n}\left[cz : \left\{\left(a_i,A_i\right)\right\}_{i=1,\dots,p} ; \left\{\left(b_j,B_j\right)\right\}_{j=1,\dots,q} \right]$

$$= 2\sqrt{\pi}\, 4^{a_1-1}\, H_{p\ q+1}^{m\ n}\left[\frac{cz}{4^{A_1}} : \left(2a_1-1,2A_1\right), \left\{\left(a_i,A_i\right)\right\}_{i=2,\dots,p} ; \left\{\left(b_j,B_j\right)\right\}_{j=1,\dots,q}, \left(a_1-\frac{1}{2},A_1\right) \right] \tag{2.49}$$

$$= \frac{1}{2^{a_1}\sqrt{\pi}}\, H_{p+1\ q}^{m\ n+1}\left[2^{A_1}cz : \left(\frac{a_1}{2},\frac{A_1}{2}\right), \left(\frac{a_1+1}{2},\frac{A_1}{2}\right), \left\{\left(a_i,A_i\right)\right\}_{i=2,\dots,p} ; \left\{\left(b_j,B_j\right)\right\}_{j=1,\dots,q} \right] \tag{2.50}$$

for n>0.

* Theorem 2.7. $H_{p\ q}^{m\ n}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q}\right]$

$$= \frac{4^{1-b_q}}{2\sqrt{\pi}} H_{p+1\ \ q}^{m\ \ n+1}\left[4^{B_q} cz : \left(b_q - \frac{1}{2}, B_q\right), \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(2b_q - 1, 2B_q\right)\right] \quad (2.51)$$

$$= 2\sqrt{\pi}\, 2^{b_q} H_{p\ \ q+1}^{m\ \ n}\left[\frac{cz}{2^{B_q}} : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(\frac{b_q}{2}, \frac{B_q}{2}\right), \left(\frac{b_q+1}{2}, \frac{B_q}{2}\right)\right] \quad (2.52)$$

for $q>m$.

Similar results are available by using the triplication formula or Gauss' multiplication formula. For example, using the triplication formula, it is possible to show the $H_{0\ 1}^{1\ 0}$ class of H-functions is also a proper subset of both the $H_{0\ 3}^{3\ 0}$ and $H_{2\ 1}^{1\ 0}$ classes of H-functions.

* 2.4.7. GENERALIZING CONSTANT

Bodenschatz and Boedigheimer [1983; Bodenschatz et al, 1990] gave the first H-function representations which recognized the existence of a positive generalizing constant, u. They showed that for certain mathematical functions and statistical distributions, the A_i and B_j parameters need not equal unity. They gave some new generalized H-function representations, but did not state general conditions under which a generalizing constant was possible. These general conditions will be given below as four theorems.

* Theorem 2.8. If $m>0$, $p>n$, $A_p=B_1$, and $a_p-b_1=1$, then

$$H_{p\,q}^{m\,n}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p-1}, \left(b_1+1, B_1\right) ; \left(b_1, B_1\right), \left\{\left(b_j, B_j\right)\right\}_{j=2,\dots,q} \right]$$

$$= u\, H_{p\,q}^{m\,n}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p-1}, \left(ub_1+1, uB_1\right) ; \left(ub_1, uB_1\right), \left\{\left(b_j, B_j\right)\right\}_{j=2,\dots,q} \right] \quad (2.53)$$

for $u>0$.

Proof:

$$H_{p\,q}^{m\,n}\left[cz : \left\{\left(a_i, A_i\right)\right\}_{i=1,\dots,p-1}, \left(b_1+1, B_1\right) ; \left(b_1, B_1\right), \left\{\left(b_j, B_j\right)\right\}_{j=2,\dots,q} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_j s\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_i s\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_i s\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_j s\right)} \frac{\Gamma\left(b_1+B_1 s\right)}{\Gamma\left(b_1+1+B_1 s\right)}$$

$$(cz)^{-s}\, ds$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_j s\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_i s\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_i s\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_j s\right)} \frac{1}{b_1+B_1 s}$$

$$(cz)^{-s}\, ds$$

$$= \frac{u}{2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_j s\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_i s\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_i s\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_j s\right)} \frac{1}{ub_1+uB_1 s}$$

$$(cz)^{-s}\, ds$$

$$= \frac{u}{2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_j s\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_i s\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_i s\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_j s\right)} \frac{\Gamma\left(ub_1+uB_1 s\right)}{\Gamma\left(ub_1+1+uB_1 s\right)}$$

$$(cz)^{-s}\, ds$$

$$= u\, H^{m\,n}_{p\,q}\left[\, cz \;:\; \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p-1},\ \left(ub_1+1, uB_1\right) \;;\; \left(ub_1, uB_1\right),\ \left\{\left(b_j, B_j\right)\right\}_{j=2,\ldots,q} \right] \tag{2.54}$$

Q.E.D.

* Theorem 2.9. If $m>0$, $p>n$, $B_1=A_p$, and $b_1-a_p=1$, then

$$H^{m\,n}_{p\,q}\left[\, cz \;:\; \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p-1},\ \left(a_p, A_p\right) \;;\; \left(a_p+1, A_p\right),\ \left\{\left(b_j, B_j\right)\right\}_{j=2,\ldots,q} \right]$$

$$= \frac{1}{u}\, H^{m\,n}_{p\,q}\left[\, cz \;:\; \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p-1},\ \left(ua_p, uA_p\right) \;;\; \left(ua_p+1, uA_p\right),\ \left\{\left(b_j, B_j\right)\right\}_{j=2,\ldots,q} \right] \tag{2.55}$$

for $u>0$.

Proof:

$$H^{m\,n}_{p\,q}\left[\, cz \;:\; \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p-1},\ \left(a_p, A_p\right) \;;\; \left(a_p+1, A_p\right),\ \left\{\left(b_j, B_j\right)\right\}_{j=2,\ldots,q} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_js\right)} \frac{\Gamma\left(a_p+1+A_ps\right)}{\Gamma\left(a_p+A_ps\right)}$$

$$(cz)^{-s}\, ds$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_js\right)} \left(a_p+A_ps\right)$$

$$(cz)^{-s}\, ds$$

$$= \frac{1}{u2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_js\right)} \left(ua_p+uA_ps\right)$$

$$(cz)^{-s}\, ds$$

$$= \frac{1}{u2\pi i} \int_C \frac{\prod_{j=2}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p-1} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_js\right)}$$

$$\frac{\Gamma\left(ua_p+1+uA_ps\right)}{\Gamma\left(ua_p+uA_ps\right)} (cz)^{-s}\, ds$$

$$= \frac{1}{u} H^{m\ n}_{p\ q}\left[cz : \left\{\left(a_i,A_i\right)\right\}_{i=1,\dots,p-1} , \left(ua_p,uA_p\right) ; \left(ua_p+1,uA_p\right), \left\{\left(b_j,B_j\right)\right\}_{j=2,\dots,q} \right] \tag{2.56}$$

Q.E.D.

* Theorem 2.10. If $n>0$, $q>m$, $A_1=B_q$, and $a_1-b_q=1$, then

$$H_{p\ q}^{m\ n}\left[cz : \left(b_q+1,B_q\right),\ \left\{\left(a_i,A_i\right)\right\}_{i=2,\ldots,p} ;\ \left\{\left(b_j,B_j\right)\right\}_{j=1,\ldots,q-1},\ \left(b_q,B_q\right) \right]$$

$$= u\, H_{p\ q}^{m\ n}\left[cz : \left(ub_q+1,uB_q\right),\ \left\{\left(a_i,A_i\right)\right\}_{i=2,\ldots,p} ;\ \left\{\left(b_j,B_j\right)\right\}_{j=1,\ldots,q-1},\ \left(ub_q,uB_q\right) \right] \tag{2.57}$$

for $u>0$.

Proof:

$$H_{p\ q}^{m\ n}\left[cz : \left(b_q+1,B_q\right),\ \left\{\left(a_i,A_i\right)\right\}_{i=2,\ldots,p} ;\ \left\{\left(b_j,B_j\right)\right\}_{j=1,\ldots,q-1},\ \left(b_q,B_q\right) \right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod_{i=2}^{n}\Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p}\Gamma\left(a_i+A_is\right)\prod_{j=m+1}^{q-1}\Gamma\left(1-b_j-B_js\right)}\frac{\Gamma\left(-b_q-B_qs\right)}{\Gamma\left(1-b_q-B_qs\right)}$$

$$(cz)^{-s}\,ds$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=2}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q-1} \Gamma\left(1-b_j-B_js\right)} \frac{1}{-b_q-B_qs}$$

$$(cz)^{-s}\, ds$$

$$= \frac{u}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=2}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q-1} \Gamma\left(1-b_j-B_js\right)} \frac{1}{-ub_q-uB_qs}$$

$$(cz)^{-s}\, ds$$

$$= \frac{u}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=2}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q-1} \Gamma\left(1-b_j-B_js\right)} \frac{\Gamma\left(-ub_q-uB_qs\right)}{\Gamma\left(1-ub_q-uB_qs\right)}$$

$$(cz)^{-s}\, ds$$

$$= u\, H_{p\,q}^{m\,n}\left[cz : \left(ub_q+1, uB_q\right), \left\{\left(a_i, A_i\right)\right\}_{i=2,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(ub_q, uB_q\right) \right] \quad (2.58)$$

Q.E.D.

* Theorem 2.11. If $n>0$, $q>m$, $B_q=A_1$, and $b_q-a_1=1$, then

$$H_{p\,q}^{m\,n}\left[cz : \left(a_1, A_1\right), \left\{\left(a_i, A_i\right)\right\}_{i=2,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(a_1+1, A_1\right) \right]$$

$$= \frac{1}{u}\, H_{p\,q}^{m\,n}\left[cz : \left(ua_1, uA_1\right), \left\{\left(a_i, A_i\right)\right\}_{i=2,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(ua_1+1, uA_1\right) \right] \quad (2.59)$$

for $u>0$.

Proof:

$$H_{p\,q}^{m\,n}\left[cz : \left(a_1, A_1\right), \left\{\left(a_i, A_i\right)\right\}_{i=2,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(a_1+1, A_1\right) \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{i=2}^{n} \Gamma\left(1 - a_i - A_i s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i s\right) \prod_{j=m+1}^{q-1} \Gamma\left(1 - b_j - B_j s\right)} \frac{\Gamma\left(1 - a_1 - A_1 s\right)}{\Gamma\left(-a_1 - A_1 s\right)} (cz)^{-s}\, ds$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{i=2}^{n} \Gamma\left(1 - a_i - A_i s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i s\right) \prod_{j=m+1}^{q-1} \Gamma\left(1 - b_j - B_j s\right)} \left(-a_1 - A_1 s\right) (cz)^{-s}\, ds$$

$$= \frac{1}{u2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{i=2}^{n} \Gamma\left(1 - a_i - A_i s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i s\right) \prod_{j=m+1}^{q-1} \Gamma\left(1 - b_j - B_j s\right)} \left(-ua_1 - uA_1 s\right) (cz)^{-s}\, ds$$

$$= \frac{1}{u2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=2}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q-1} \Gamma\left(1-b_j-B_js\right)} \frac{\Gamma\left(1-ua_1-uA_1s\right)}{\Gamma\left(-ua_1-uA_1s\right)} (cz)^{-s}\, ds$$

$$= \frac{1}{u} H_{p\ q}^{m\ n}\left[cz : \left(ua_1, uA_1\right), \left\{\left(a_i, A_i\right)\right\}_{i=2,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q-1}, \left(ua_1+1, uA_1\right) \right] \tag{2.60}$$

Q.E.D.

2.4.8. <u>DERIVATIVE</u>

It is well known that the derivative of an H-function is an H-function of higher order. Let the r^{th} derivative of $H_{p\ q}^{m\ n}[z]$ be denoted by $H^{(r)}[z]$. If all real b_j are such that $\frac{-b_j}{B_j} < 1$ for $j=1,\ldots,m$ then the r^{th} derivative of $H_{p\ q}^{m\ n}[z]$ is [Mathai and Saxena, 1978, p. 7; Cook, 1981, p. 83]

$$H^{(r)}[z] = H^{m\ \ n+1}_{p+1\ q+1}\left[z : (-r,1), \left\{\left(a_i - rA_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j - rB_j, B_j\right)\right\}_{j=1,\ldots,q}, (0,1) \right]$$

(2.61)

Cook [1981, p. 83] gave an improved formula for the r^{th} derivative of an H-function if any real b_j is such that $\frac{-b_j}{B_j} \geq 1$ for $j=1,\ldots,m$. The formula is based on the value of I, defined as

$$I = \underset{j=1,\ldots,m}{\text{maximum}}\left(0, \text{ largest integer less than } \frac{-b_j}{B_j} \right)$$

(2.62)

The r^{th} derivative of $H^{m\ n}_{p\ q}[z]$ is

$$H^{(r)}[z] = (-1)^I\, H^{m+1\ n+1}_{p+2\ q+2}\left[z : (-I-r,1), \left\{\left(a_i - rA_i, A_i\right)\right\}_{i=1,\ldots,p}, (-r,1) ; \left\{\left(b_j - rB_j, B_j\right)\right\}_{j=1,\ldots,q}, (0,1) \right]$$

(2.63)

2.4.9. LAPLACE TRANSFORM

It is well known that the Laplace transform of an H-function is an H-function of higher order. If all real b_j are such that $\frac{-b_j}{B_j} < 1$ for $j=1,\ldots,m$ then the Laplace transform of $H_{p\ q}^{m\ n}[cz]$ is [Springer, 1979, p. 200; Cook, 1981, p. 35]

$$\mathcal{L}_r\left\{ H(cz) \right\} = \int_0^\infty e^{-rz} \; H(cz) \; dz$$

$$= \frac{1}{c} H_{q\ \ p+1}^{n+1\ m}\left[\frac{1}{c} r : \left\{\left(1-b_j-B_j, B_j\right)\right\}_{j=1,\ldots,q} ; \right.$$

$$\left. (0,1), \left\{\left(1-a_i-A_i, A_i\right)\right\}_{i=1,\ldots,p} \right] \qquad (2.64)$$

Cook [1981, p. 82] gave an improved formula for the Laplace transform of an H-function if any real b_j is such that $\frac{-b_j}{B_j} \geq 1$ for $j=1,\ldots,m$.

$$\mathcal{L}_r\left\{ H(cz) \right\} = \frac{(-1)^I}{c} H_{q+1\ p+1}^{n+1\ m+1}\left[\frac{1}{c} r : (I,1), \left\{\left(1-b_j-B_j, B_j\right)\right\}_{j=1,\ldots,q} ; (I,1), \left\{\left(1-a_i-A_i, A_i\right)\right\}_{i=1,\ldots,p}, (0,1) \right] \tag{2.65}$$

where I is given by Eq (2.62).

2.4.10. FOURIER TRANSFORM

It is well known that the Fourier transform of an H-function is an H-function of higher order. If all real b_j are such that $\frac{-b_j}{B_j} < 1$ for $j=1,\ldots,m$ then the Fourier transform of $H_{p\ q}^{m\ n}[cz]$ is [Springer, 1979, p. 201; Cook, 1981, p. 35]

$$\mathcal{F}_t\left\{ H(cz) \right\} = \int_0^\infty e^{itz}\ H(cz)\ dz$$

$$= \frac{1}{c} H_{q\ \ p+1}^{n+1\ \ m}\left[-\frac{i}{c} t : \left\{\left(1-b_j-B_j, B_j\right)\right\}_{j=1,\ldots,q} ; (0,1), \left\{\left(1-a_i-A_i, A_i\right)\right\}_{i=1,\ldots,p} \right] \tag{2.66}$$

Cook [1981, p. 82] gave an improved formula for the Fourier transform of an H-function if any real b_j is such that

$$\frac{-b_j}{B_j} \geq 1 \text{ for } j=1,\ldots,m.$$

$$\mathcal{F}_t\left\{ H(cz) \right\} = \frac{(-1)^I}{c} H_{q+1\ p+1}^{n+1\ m+1}\left[-\frac{i}{c}\, t : (I,1), \left\{\left(1-b_j-B_j, B_j\right)\right\}_{j=1,\ldots,q} ; (I,1), \left\{\left(1-a_i-A_i, A_i\right)\right\}_{i=1,\ldots,p}, (0,1) \right] \qquad (2.67)$$

where I is given by Eq (2.62).

2.4.11. MELLIN TRANSFORM

Since the definition of the H-function in Eq (2.1) may be recognized as the inverse Mellin transform given by Eq (1.10), the Mellin transform of an H-function is readily obtained.

$$\mathcal{M}_s\left\{ H(cz) \right\} = \int_0^\infty z^{s-1}\ H(cz)\ dz$$

$$= \frac{1}{c^s} \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_js\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_is\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_is\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_js\right)} \qquad (2.68)$$

2.5. SPECIAL CASES - MATHEMATICAL FUNCTIONS

With the parameters given below, the H-function can represent all of the following mathematical functions as a special case [Mathai and Saxena, 1978; Cook, 1981; Bodenschatz and Boedigheimer, 1983; Bodenschatz et al 1990; Prudnikov et al, 1990; Cook and Barnes, 1991]. In some of the representations, u>0 is a generalizing constant.

Exponential and Power Functions:

$$e^{z} = \frac{\pi}{\sin(d\pi)} H_{1\ 2}^{1\ 0}[\ z : (1-d,1)\ ;\ (0,1),\ (1-d,1)\ ]$$

$$e^{-z} = H_{0\ 1}^{1\ 0}[\ z : \ ;\ (0,1)\ ]$$

$$z^{b}\ e^{-z} = H_{0\ 1}^{1\ 0}[\ z : \ ;\ (b,1)\ ]$$

$$\frac{1}{B}\ z^{\left(\frac{b}{B}\right)}\ e^{-z^{\left(\frac{1}{B}\right)}} = H_{0\ 1}^{1\ 0}[\ z : \ ;\ (b,B)\ ]$$

$$z^{b} = H_{1\ 1}^{1\ 0}[\ z : (b+1,1)\ ;\ (b,1)\ ] \qquad |z| < 1$$

$$* \quad = u\ H_{1\ 1}^{1\ 0}[\ z : (ub+1,u)\ ;\ (ub,u)\ ] \qquad |z| < 1$$

$$* \ z^{b} = M^{b}\ H_{1\ 1}^{1\ 0}\left[\ \frac{z}{M} : (b+1,1)\ ;\ (b,1)\ \right] \qquad |z| < M$$

$$* \quad = u\ M^{b}\ H_{1\ 1}^{1\ 0}\left[\ \frac{z}{M} : (ub+1,u)\ ;\ (ub,u)\ \right] \qquad |z| < M$$

$$z^{b} = H_{1\ 1}^{0\ 1}[\ z : (b+1,1)\ ;\ (b,1)\ ] \qquad |z| > 1$$

$$* \quad = u\ H_{1\ 1}^{0\ 1}[\ z : (ub+1,u)\ ;\ (ub,u)\ ] \qquad |z| > 1$$

$$* \; z^{b} = \varepsilon^{b} H_{1\,1}^{0\,1}\left[\frac{z}{\varepsilon} : (b+1,1) \; ; \; (b,1) \right] \qquad |z| > \varepsilon$$

$$* \quad = u\, \varepsilon^{b} H_{1\,1}^{0\,1}\left[\frac{z}{\varepsilon} : (ub+1,u) \; ; \; (ub,u) \right] \qquad |z| > \varepsilon$$

$$(1-z)^{b} = \Gamma(b+1)\, H_{1\,1}^{1\,0}[\, z : (b+1,1) \; ; \; (0,1) \,] \qquad |z| < 1$$

$$* \; (M-z)^{b} = \Gamma(b+1)\, M^{b} H_{1\,1}^{1\,0}\left[\frac{z}{M} : (b+1,1) \; ; \; (0,1) \right] \qquad |z| < M$$

$$(z-1)^{b} = \Gamma(b+1)\, H_{1\,1}^{0\,1}[\, z : (b+1,1) \; ; \; (0,1) \,] \qquad |z| > 1$$

$$* \; (z-1)^{b} = \Gamma(b+1)\, \varepsilon^{b} H_{1\,1}^{0\,1}\left[\frac{z}{\varepsilon} : (b+1,1) \; ; \; (0,1) \right] \qquad |z| > \varepsilon$$

$$z^{b}\, (1-z)^{+a} = \Gamma(a+1)\, H_{1\,1}^{1\,0}[\, z : (b+a+1,1) \; ; \; (b,1) \,] \qquad |z| < 1$$

$$* \; z^{b}\, (M-z)^{+a} = \Gamma(a+1)\, M^{a+b} H_{1\,1}^{1\,0}\left[\frac{z}{M} : (b+a+1,1) \; ; \; (b,1) \right] \qquad |z| < M$$

$$(1+z)^{-b} = \frac{1}{\Gamma(b)}\, H_{1\,1}^{1\,1}[\, z : (1-b,1) \; ; \; (0,1) \,]$$

$$z^{b}\, (1+z)^{-a} = \frac{1}{\Gamma(a)}\, H_{1\,1}^{1\,1}[\, z : (b-a+1,1) \; ; \; (b,1) \,]$$

$$\frac{1}{1-z} = \pi\, H_{2\,2}^{1\,1}\left[z : (0,1), \left(\frac{1}{2},1\right) \; ; \; (0,1), \left(\frac{1}{2},1\right) \right]$$

$$\frac{1}{|1-z|^{b}} = \frac{\pi}{\Gamma(b)\, \cos\left(\frac{b\pi}{2}\right)}\, H_{2\,2}^{1\,1}\left[z : (1-b,1), \left(\frac{1-b}{2},1\right) \; ; \; (0,1), \left(\frac{1-b}{2},1\right) \right]$$

$$\frac{1}{b-a}\left(z^{a}-z^{b}\right) = H_{2\,2}^{2\,0}[\ z : (b+1,1),\ (a+1,1)\ ;\ (a,1),\ (b,1)\ ]$$

$$|z| < 1$$

$$\frac{1}{b-a}\left(z^{b}-z^{a}\right) = H_{2\,2}^{0\,2}[\ z : (b+1,1),\ (a+1,1)\ ;\ (a,1),\ (b,1)\ ]$$

$$|z| > 1$$

$$\left\{\begin{array}{ll} \frac{1}{b-a}\, z^{a} & |z| < 1 \\ \frac{1}{b-a}\, z^{b} & |z| > 1 \end{array}\right\} = -\, H_{2\,2}^{1\,1}[\ z : (b+1,1),\ (a+1,1)\ ;$$

$$(a,1),\ (b,1)\ ]$$

$$\frac{1}{a+z^{n}} = \frac{1}{an}\, H_{1\,1}^{1\,1}\left[\frac{z}{a^{\left(\frac{1}{n}\right)}} : \left(0,\frac{1}{n}\right)\ ;\ \left(0,\frac{1}{n}\right)\right]$$

$$n > 0,\ |\arg a| < \pi$$

$$\frac{1-z^{n}}{1-z^{mn}} = \frac{1}{mn\,\Gamma\left(\frac{1}{m}\right)\,\Gamma\left(1-\frac{1}{m}\right)}\, H_{2\,2}^{2\,2}\Big[\ z:\ :\ \left(0,\frac{1}{mn}\right),$$

$$\left(\frac{1}{m},\frac{1}{mn}\right)\ ;\ \left(0,\frac{1}{mn}\right),\ \left(\frac{1}{m},\frac{1}{mn}\right)\ \Big]$$

$$n > 0,\ m > 1$$

$$\frac{(\ln z)^{2} + \pi^{2}}{1 + z} = 2\, H_{3\,3}^{3\,3}[\ z : (0,1),\ (0,1),\ (0,1)\ ;$$

$$(0,1),\ (0,1),\ (0,1)\ ]$$

$$\frac{1}{z^2 + 2az\cos\theta + a^2} = \frac{-1}{a^2}\,\Gamma\left(\frac{\theta}{\pi}\right)\,\Gamma\left(1-\frac{\theta}{\pi}\right)\,H_{2\ 2}^{1\ 1}\left[\frac{z}{a} : (0,1), \left(-\frac{\theta}{\pi},\frac{\theta}{\pi}\right) ; (0,1), \left(-\frac{\theta}{\pi},\frac{\theta}{\pi}\right)\right]$$

$$a > 0,\ |\theta| < \pi$$

Unit Step Function and its Complement:

$$S_k(x) = \begin{cases} 1 & x \geq k > 0 \\ 0 & 0 < x < k \end{cases}$$

* $$= H_{1\ 1}^{0\ 1}\left[\frac{x}{k} : (1,1) ; (0,1)\right]$$

* $$= u\,H_{1\ 1}^{0\ 1}\left[\frac{x}{k} : (1,u) ; (0,u)\right]$$

$$1-S_k(x) = \begin{cases} 0 & x \geq k > 0 \\ 1 & 0 < x < k \end{cases}$$

* $$= H_{1\ 1}^{1\ 0}\left[\frac{x}{k} : (1,1) ; (0,1)\right]$$

* $$= u\,H_{1\ 1}^{1\ 0}\left[\frac{x}{k} : (1,u) ; (0,u)\right]$$

Error Function and its Complement:

$$\text{erf } x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\theta^2}\,d\theta$$

$$= \frac{1}{\sqrt{\pi}}\,H_{1\ 2}^{1\ 1}\left[x : (1,1) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,1)\right]$$

$$* \qquad = \frac{u}{\sqrt{\pi}} H_{1\ 2}^{1\ 1}\left[x : (1,u) \ ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\theta^2} d\theta = 1 - \text{erf } x$$

$$* \qquad = \frac{1}{\sqrt{\pi}} H_{1\ 2}^{2\ 0}\left[x : (1,1) \ ; \left(\frac{1}{2},\frac{1}{2}\right), (0,1) \right]$$

$$* \qquad = \frac{u}{\sqrt{\pi}} H_{1\ 2}^{2\ 0}\left[x : (1,u) \ ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

Incomplete Gamma Function and its Complement:

$$\gamma(\alpha,x) = \int_0^x e^{-\theta} \theta^{\alpha-1} d\theta$$

$$= H_{1\ 2}^{1\ 1}[x : (1,1) \ ; (\alpha,1), (0,1)]$$

$$* \qquad = u\, H_{1\ 2}^{1\ 1}[x : (1,u) \ ; (\alpha,1), (0,u)]$$

$$\Gamma(\alpha,x) = \int_x^{\infty} e^{-\theta} \theta^{\alpha-1} d\theta = \Gamma(\alpha) - \gamma(\alpha,x)$$

$$* \qquad = H_{1\ 2}^{2\ 0}[x : (1,1) \ ; (\alpha,1), (0,1)]$$

$$* \qquad = u\, H_{1\ 2}^{2\ 0}[x : (1,u) \ ; (\alpha,1), (0,u)]$$

Incomplete Beta Function and its Complement:

$$B_x(\alpha,\beta) = \int_0^x \theta^{\alpha-1} (1-\theta)^{\beta-1}\, d\theta$$

$$= \Gamma(\beta)\, H_{2\ 2}^{1\ 1}[\ x : (1,1),\ (\alpha+\beta,1)\ ;\ (\alpha,1),\ (0,1)\ ]$$

* $$= u\,\Gamma(\beta)\, H_{2\ 2}^{1\ 1}[\ x : (1,u),\ (\alpha+\beta,1)\ ;\ (\alpha,1),\ (0,u)\ ]$$

$$\overline{B_x(\alpha,\beta)} = \int_x^1 \theta^{\alpha-1} (1-\theta)^{\beta-1}\, d\theta$$

* $$= \Gamma(\beta)\, H_{2\ 2}^{2\ 0}[\ x : (1,1),\ (\alpha+\beta,1)\ ;\ (\alpha,1),\ (0,1)\ ]$$

* $$= u\,\Gamma(\beta)\, H_{2\ 2}^{2\ 0}[\ x : (1,u),\ (\alpha+\beta,1)\ ;\ (\alpha,1),\ (0,u)\ ]$$

Trigonometric and Hyperbolic Functions and their Inverses:

$$\sin z = \frac{\sqrt{\pi}}{2}\, H_{0\ 2}^{1\ 0}\left[\ \frac{z}{2} : ;\ \left(\frac{1}{2},\frac{1}{2}\right),\ \left(0,\frac{1}{2}\right)\ \right]$$

$$\sinh z = -\frac{i\sqrt{\pi}}{2}\, H_{0\ 2}^{1\ 0}\left[\ \frac{iz}{2} : ;\ \left(\frac{1}{2},\frac{1}{2}\right),\ \left(0,\frac{1}{2}\right)\ \right]$$

$$\cos z = \frac{\sqrt{\pi}}{2}\, H_{0\ 2}^{1\ 0}\left[\ \frac{z}{2} : ;\ \left(0,\frac{1}{2}\right),\ \left(\frac{1}{2},\frac{1}{2}\right)\ \right]$$

$$\cosh z = \frac{\sqrt{\pi}}{2}\, H_{0\ 2}^{1\ 0}\left[\ \frac{iz}{2} : ;\ \left(0,\frac{1}{2}\right),\ \left(\frac{1}{2},\frac{1}{2}\right)\ \right]$$

$$\arcsin z = -\frac{i}{4\sqrt{\pi}} H^{1\,2}_{2\,2}\left[iz : \left(1,\frac{1}{2}\right), \left(1,\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = -\frac{iu}{2\sqrt{\pi}} H^{1\,2}_{2\,2}\left[iz : (1,u), \left(1,\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$\operatorname{arcsinh} z = \frac{1}{4\sqrt{\pi}} H^{1\,2}_{2\,2}\left[z : \left(1,\frac{1}{2}\right), \left(1,\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = \frac{u}{2\sqrt{\pi}} H^{1\,2}_{2\,2}\left[z : (1,u), \left(1,\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$\arctan z = \frac{1}{4} H^{1\,2}_{2\,2}\left[z : \left(1,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = \frac{u}{2} H^{1\,2}_{2\,2}\left[z : (1,u), \left(\frac{1}{2},\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$\operatorname{arctanh} z = -\frac{i}{4} H_{2\,2}^{1\,2}\left[iz : \left(1,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = -\frac{iu}{2} H_{2\,2}^{1\,2}\left[iz : (1,u), \left(\frac{1}{2},\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$\operatorname{arccot} z = \frac{1}{4} H_{2\,2}^{2\,1}\left[z : \left(\frac{1}{2},\frac{1}{2}\right), \left(1,\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = \frac{u}{2} H_{2\,2}^{2\,1}\left[z : \left(\frac{1}{2},\frac{1}{2}\right), (1,u) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$* \operatorname{arccoth} z = \frac{i}{4} H_{2\,2}^{2\,1}\left[iz : \left(\frac{1}{2},\frac{1}{2}\right), \left(1,\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = \frac{iu}{2} H_{2\,2}^{2\,1}\left[iz : \left(\frac{1}{2},\frac{1}{2}\right), (1,u) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$e^{-bz} \sin(bz) = \pi H_{1\,2}^{1\,0}\left[b\sqrt{2}\, z : \left(0,\frac{1}{4}\right) ; (0,1), \left(0,\frac{1}{4}\right) \right]$$

$$e^{-bz}\cos(bz) = \pi\, H^{1\,0}_{1\,2}\left[\, b\sqrt{2}\, z : \left(\frac{1}{2},\frac{1}{4}\right) ; (0,1),\ \left(\frac{1}{2},\frac{1}{4}\right) \right]$$

$$e^{-bz}\sin(cz) = \pi\, H^{1\,0}_{1\,2}\left[\, \sqrt{b^2+c^2}\, z : \left(0,\frac{\arctan\left(\frac{c}{b}\right)}{\pi}\right) ; (0,1),\ \left(0,\frac{\arctan\left(\frac{c}{b}\right)}{\pi}\right) \right]$$

$$e^{-bz}\cos(cz) = \pi\, H^{1\,0}_{1\,2}\left[\, \sqrt{b^2+c^2}\, z : \left(\frac{1}{2},\frac{\arctan\left(\frac{c}{b}\right)}{\pi}\right) ; (0,1),\ \left(\frac{1}{2},\frac{\arctan\left(\frac{c}{b}\right)}{\pi}\right) \right]$$

$$\sin^2(az) = \frac{1}{4}\, H^{1\,1}_{1\,2}\left[\, 2a\, z : \left(1,\frac{1}{2}\right) ; \left(1,\frac{1}{2}\right), (0,1) \right]$$

Logarithmic Functions:

$$\ln z = \begin{cases} -H^{2\,0}_{2\,2}[\, z : (1,1), (1,1) ; (0,1), (0,1) \,] & 0 < z \le 1 \\ H^{0\,2}_{2\,2}[\, z : (1,1), (1,1) ; (0,1), (0,1) \,] & z > 1 \end{cases}$$

$$= \begin{cases} -u^2 \; H_{2\;2}^{2\;0}[\; z : (1,u),\, (1,u) \;;\; (0,u),\, (0,u) \;] & 0 < z \le 1 \\ u^2 \; H_{2\;2}^{0\;2}[\; z : (1,u),\, (1,u) \;;\; (0,u),\, (0,u) \;] & z > 1 \end{cases}$$

$$\ln(1+z) = H_{2\;2}^{1\;2}[\; z : (1,1),\, (1,1) \;;\; (1,1),\, (0,1) \;]$$

$$* \qquad = u\, H_{2\;2}^{1\;2}[\; z : (1,u),\, (1,1) \;;\; (1,1),\, (0,u) \;]$$

$$\ln\left(1+\frac{1}{z}\right) = H_{2\;2}^{2\;1}[\; z : (0,1),\, (1,1) \;;\; (0,1),\, (0,1) \;]$$

$$* \qquad = u\, H_{2\;2}^{2\;1}[\; z : (0,1),\, (1,u) \;;\; (0,u),\, (0,1) \;]$$

$$\ln|1-z| = \pi\, H_{3\;3}^{1\;2}\left[\; z : (1,1),\, (1,1),\, \left(\frac{1}{2},1\right) \;; \; (1,1),\, (0,1),\, \left(\frac{1}{2},1\right) \;\right]$$

$$* \qquad = u\pi\, H_{3\;3}^{1\;2}\left[\; z : (1,u),\, (1,1),\, \left(\frac{1}{2},1\right) \;; \; (1,1),\, (0,u),\, \left(\frac{1}{2},1\right) \;\right]$$

$$\ln\left|1-\frac{1}{z}\right| = \pi\, H_{3\;3}^{2\;1}\left[\; z : (0,1),\, (1,1),\, \left(\frac{1}{2},1\right) \;; \; (0,1),\, (0,1),\, \left(\frac{1}{2},1\right) \;\right]$$

$$* \qquad = u\pi\, H_{3\ 3}^{2\ 1}\left[z : (0,1),\ (1,u),\ \left(\frac{1}{2},1\right) ;\ (0,u),\ (0,1),\ \left(\frac{1}{2},1\right) \right]$$

$$\ln\left(1 + 2z\cos\theta + z^2\right) = 2\pi\, H_{3\ 3}^{1\ 2}\left[z : (1,1),\ (1,1),\ \left(\frac{1}{2},\frac{\theta}{\pi}\right) ;\ (1,1),\ (0,1),\ \left(\frac{1}{2},\frac{\theta}{\pi}\right) \right]$$

$$* \qquad = 2u\pi\, H_{3\ 3}^{1\ 2}\left[z : (1,u),\ (1,1),\ \left(\frac{1}{2},\frac{\theta}{\pi}\right) ;\ (1,1),\ (0,u),\ \left(\frac{1}{2},\frac{\theta}{\pi}\right) \right]$$

$$\ln\left(1+z^2\right) = \frac{1}{2}\, H_{2\ 2}^{1\ 2}\left[z : \left(1,\frac{1}{2}\right),\ \left(1,\frac{1}{2}\right) ;\ \left(1,\frac{1}{2}\right),\ \left(0,\frac{1}{2}\right) \right]$$

$$* \qquad = u\, H_{2\ 2}^{1\ 2}\left[z : (1,u),\ \left(1,\frac{1}{2}\right) ;\ \left(1,\frac{1}{2}\right),\ (0,u) \right]$$

Bessel Functions:

$$J_v(z) = \frac{1}{2}\, H_{0\ 2}^{1\ 0}\left[\frac{z}{2} : ;\ \left(\frac{v}{2},\frac{1}{2}\right),\ \left(-\frac{v}{2},\frac{1}{2}\right) \right]$$

$$K_v(z) = \frac{1}{4}\, H_{0\ 2}^{2\ 0}\left[\frac{z}{2} : ;\ \left(\frac{v}{2},\frac{1}{2}\right),\ \left(-\frac{v}{2},\frac{1}{2}\right) \right]$$

$$Y_v(z) = \frac{1}{2} H^{2\,0}_{1\,3}\left[\frac{z}{2} : \left(-\frac{v+1}{2}, \frac{1}{2} \right) ; \left(\frac{v}{2}, \frac{1}{2} \right), \left(-\frac{v}{2}, \frac{1}{2} \right), \left(-\frac{v+1}{2}, \frac{1}{2} \right) \right]$$

$$J^u_v(z) = H^{1\,0}_{0\,2}[\, z : ; (0,1), (-v,u) \,]$$

(Maitland's generalized Bessel function)

Hypergeometric Functions:

$$M(a,b,-z) = {}_1F_1\,(a;b;-z)$$

$$= \frac{\Gamma(b)}{\Gamma(a)} H^{1\,1}_{1\,2}[\, z : (1-a,1) ; (0,1), (1-b,1) \,]$$

(Confluent Hypergeometric function)

$${}_2F_1\,(a,b;c;-z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H^{1\,2}_{2\,2}[\, z : (1-a,1), (1-b,1) ; (0,1), (1-c,1) \,]$$

(Hypergeometric function)

$${}_pF_q\left(\{a_i\};\{b_j\};-z\right) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} H^{1\,p}_{p\,q+1}\left[\, z : \{(1-a_i,1)\} ; (0,1), \{(1-b_j,1)\} \right]$$

for $p \le q$ or for $p=q+1$ and $|z| < 1$

(Generalized Hypergeometric functions)

$${}_p\Psi_q\left[\left\{(a_i,A_i)\right\};\left\{(b_j,B_j)\right\};-z\right] = H^{1\ \ p}_{p\ \ q+1}\left[\ z\ :\ \left\{\left(1-a_i,A_i\right)\right\}\ ;\right.$$

$$\left.(0,1),\ \left\{\left(1-b_j,B_j\right)\right\}\ \right]$$

(Maitland's or Wright's Generalized Hypergeometric function)

MacRobert's E-Function:

$$E\left(p;\ \left\{a_i\right\};\ q;\ \left\{b_j\right\};\ z\right) = H^{p\ \ 1}_{q+1\ \ p}\left[\ z\ :\ (1,1),\ \left\{\left(b_j,B_j\right)\right\}\ ;\right.$$

$$\left.\left\{\left(a_i,A_i\right)\right\}\ \right]$$

Meijer's G-Function:

$$G^{m\ n}_{p\ q}\left[\ z\ :\ \left\{\left(a_i\right)\right\}\ ;\ \left\{\left(b_j\right)\right\}\ \right] = H^{m\ n}_{p\ q}\left[\ z\ :\ \left\{\left(a_i,1\right)\right\}\ ;\ \left\{\left(b_j,1\right)\right\}\ \right]$$

* 2.6. FUNCTIONS REPRESENTED OVER A RESTRICTED RANGE

For some of the special cases of the H-function listed above, the H-function represents the special case only for certain values of the variable z. For other values of the variable, the H-function takes the value zero. These cases can be identified by a restriction on the variable such as $|z| < 1$.

These restrictions arise from the convergence conditions for the H-function given earlier. These H-functions are of

Convergence Type VI which means they can be evaluated by the sum of LHP residues for $|z| < \frac{1}{cR}$ and by the negative sum of RHP residues for $|z| > \frac{1}{cR}$. But the $H_{1\ 1}^{1\ 0}$ and $H_{2\ 2}^{2\ 0}$ classes of H-functions have no RHP poles so the value of the negative sum of RHP residues is zero. Therefore, $H(z)=0$ for $|z| > \frac{1}{cR}$. Similarly, the $H_{1\ 1}^{0\ 1}$ and $H_{2\ 2}^{0\ 2}$ classes of H-functions have no LHP poles so the value of the sum of LHP residues is zero. Therefore, $H(z)=0$ for $|z| < \frac{1}{cR}$.

Through scaling the variable with the parameter c, it is possible to change the value where the H-function changes from representing the special case to taking the value zero. For example, an $H_{1\ 1}^{1\ 0}$ H-function can exactly represent the power function z^b for $|z| < M$ where M is any finite positive constant. Provided M is finite, M may be as large as desired. Similarly, an $H_{1\ 1}^{0\ 1}$ H-function can exactly represent the same power function z^b for $|z| > \varepsilon$ where $\varepsilon>0$ may be as small as desired. These scaled H-function representations (given earlier in the list of special cases) allow a nearly complete representation of the special cases.

Another way to avoid this limitation of Convergence Type VI H-functions is to allow a slightly different function to be represented for certain values of the variable. The method basically involves introducing poles into the other half plane so that when the H-function is evaluated by summing the residues at these poles, a function very close to the desired special case is obtained.

An important consideration in this approach is to ensure the LHP and RHP poles may be properly divided by a contour C. I will give several examples of this technique for the power function x^b. In some of these representations, u>0 and v>0 are generalizing constants. In all of these representations, $\varepsilon>0$ may be as small as desired. The first two examples involve a horizontal shift of either the RHP poles of $H_{1\ 1}^{0\ 1}[x]$ or the LHP poles of $H_{1\ 1}^{1\ 0}[x]$ by ε.

$$* \begin{Bmatrix} x^b & 0 < x < 1 \\ x^{b-\varepsilon} & x > 1 \end{Bmatrix} = \varepsilon\, H_{2\ 2}^{1\ 1}\left[\, x : \begin{matrix} (b+1-\varepsilon,1),\ (b+1,1)\ ; \\ (b,1),\ (b-\varepsilon,1) \end{matrix} \right]$$

$$* \qquad = uv\varepsilon\, H_{2\ 2}^{1\ 1}\left[\, x : \begin{matrix} (ub+1-u\varepsilon,u),\ (vb+1,v)\ ; \\ (vb,v),\ (ub-u\varepsilon,u) \end{matrix} \right]$$

$$* \begin{Bmatrix} x^{b+\varepsilon} & 0 < x < 1 \\ x^{b} & x > 1 \end{Bmatrix} = \varepsilon\, H_{2\ 2}^{1\ 1}\left[\, x : \begin{matrix} (b+1,1),\ (b+1+\varepsilon,1)\ ; \\ (b+\varepsilon,1),\ (b,1) \end{matrix} \right]$$

$$* \qquad = uv\varepsilon \, H_{2\ 2}^{1\ 1}[\ x : (ub+1,u),\ (vb+v\varepsilon+1,v)\ ;\ (vb+v\varepsilon,v),\ (ub,u)\ ]$$

The next two examples involve a vertical shift of either the RHP poles of $H_{1\ 1}^{0\ 1}[x]$ or the LHP poles of $H_{1\ 1}^{1\ 0}[x]$ by ε. This introduces both complex poles and a complex function for certain values of the variable.

$$* \begin{Bmatrix} x^{b} & 0 < x < 1 \\ x^{b-i\varepsilon} & x > 1 \end{Bmatrix} = i\varepsilon \, H_{2\ 2}^{1\ 1}[\ x : (b+1-i\varepsilon,1),\ (b+1,1)\ ;\ (b,1),\ (b-i\varepsilon,1)\ ]$$

$$* \qquad = uvi\varepsilon \, H_{2\ 2}^{1\ 1}[\ x : (ub+1-ui\varepsilon,u),\ (vb+1,v)\ ;\ (vb,v),\ (ub-ui\varepsilon,u)\ ]$$

$$* \begin{Bmatrix} x^{b+i\varepsilon} & 0 < x < 1 \\ x^{b} & x > 1 \end{Bmatrix} = i\varepsilon \, H_{2\ 2}^{1\ 1}[\ x : (b+1,1),\ (b+1+i\varepsilon,1)\ ;\ (b+i\varepsilon,1),\ (b,1)\ ]$$

$$* \qquad = uvi\varepsilon \, H_{2\ 2}^{1\ 1}[\ x : (ub+1,u),\ (vb+vi\varepsilon+1,v)\ ;\ (vb+vi\varepsilon,v),\ (ub,u)\ ]$$

Although these restrictions on the variable z have always been present for Type VI convergent H-functions, this limitation of the H-function's ability to represent certain functions over all values of the variable was only recently discovered. The practical methods suggested above to minimize the impact of this limitation through scaling, a slightly different function, or use of complex parameters are also newly developed.

* 2.7. MOMENTS OF AN H-FUNCTION AND INFINITE SUMMABILITY

For a function of a real variable, $f(x)$, which is nonzero only for positive values of the variable, the r^{th} moment about the origin of $f(x)$ is given by

$$\mu_r = \int_0^{\infty} x^r \, f(x) \, dx \tag{2.69}$$

provided the integral in Eq (2.69) exists. If $f(x)$ can be represented as an H-function, it is often easier to find the r^{th} moment of $f(x)$ using the Mellin transform formula

$$\mu_r = \mathcal{M}_{r+1}[f(x)] \tag{2.70}$$

In particular, if

$$f(x) = H_{p\ q}^{m\ n}\left[cx : \underset{i=1,\ldots,p}{\left\{\left(a_i, A_i\right)\right\}} \; ; \; \underset{j=1,\ldots,q}{\left\{\left(b_j, B_j\right)\right\}} \right] \tag{2.71}$$

then using Eq (2.68),

$$\mu_r = \frac{1}{c^{r+1}} \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_j+B_j r\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_i-A_i r\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_i+A_i r\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_j-B_j r\right)} \tag{2.72}$$

Use of Eq (2.72) consistently produces the correct values for the moments of all the special cases of the H-function presented earlier. But there are some functions for which Eq (2.72) will produce a value for the r^{th} moment even when the integral in Eq (2.69) does not exist. Consider, for example, the sine function, sin x. The zeroth moment, μ_0, may be interpreted as the signed area under the sine function between zero and infinity.

$$\begin{aligned} \mu_0 &= \int_0^\infty \sin x \, dx \\ &= \lim_{\beta \to \infty} \int_0^\beta \sin x \, dx \end{aligned} \tag{2.73}$$

The value of the integral in Eq (2.73) oscillates between zero and two and does not approach a limit as $\beta \to \infty$. Similarly, all moments of sin x do not exist. However, using the H-function representation of sin x and Eq (2.72) to compute the moments of sin x does produce valid values. In particular,

$$\mu_r = \frac{\frac{\sqrt{\pi}}{2}}{\left(\frac{1}{2}\right)^{r+1}} \frac{\Gamma\left(1+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}r\right)}$$

$$= \begin{cases} (-1)^{\frac{r}{2}}\ r! & r \text{ an even integer} \\ 0 & r \text{ an odd integer} \end{cases} \qquad (2.74)$$

This apparent contradiction can be resolved with the concept of infinite summability of integrals. A thorough discussion of infinite summability is beyond the scope of this thesis, but a brief introduction to the concept through infinite sums will be given. Euler began the study of this topic by considering the value of the infinite sum $\sum_{i=1}^{\infty}(-1)^{i-1}$. The limit of partial sums $S_n = \sum_{i=1}^{n}(-1)^{i-1}$ does not exist, since S_n=1 if n is odd and S_n=0 if n is even. Euler believed the infinite sum should have the value $\frac{1}{2}$ since it is the limit of the average of the partial sums $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} S_i$ whether n is odd or even.

Since Euler's time, Cesàro and Hölder have developed well-accepted schemes to find the values of infinite sums whose partial sums do not converge to a limit. Hölder's schemes are based on the averages of the partial sums (H-1), the averages of those averages (H-2), etc. Cesàro's schemes (C-1, C-2, etc.) are less intuitive and slightly more complicated. The approach described in the previous paragraph defines both the C-1 and H-1 schemes, which are identical at the first level.

It is important to recognize that if an infinite sum does converge, any of the summability schemes will produce the same result. Similarly, an infinite sum which is C-m (H-m) summable will produce the same result using the C-n (H-n) scheme where $n>m$.

There is a direct analog to this approach for integrals over an infinite range. An integral which does not exist may still be summable under a summability scheme. The zeroth moment of the sine function is an example of such an integral. Under the C-1 or H-1 summability schemes for integrals, the integral for the zeroth moment of the sine function takes the value unity, the same as that produced with Eq (2.72).

There are several functions for which Eq (2.72) will produce a value for the r^{th} moment even when the integral in Eq (2.69) does not exist. In this case, however, Eq (2.72) always produces the "correct" value under an appropriate summability scheme. It is as though the H-function knows about infinite summability and uses it correctly when it is appropriate to do so. The first few moments of some trigonometric and hyperbolic functions are given in Table 3 below. Although the integrals for these moments as in Eq (2.69) do not exist, these values are widely accepted as "correct."

Table 3. Moments of Trigonometric and Hyperbolic Functions

Moment	sin x	cos x	sinh x	cosh x
Zeroth	1	0	-1	0
First	0	-1	0	1
Second	-2	0	-2	0
Third	0	6	0	6
Fourth	24	0	-24	0
Fifth	0	-120	0	120

2.8. EVALUATION OF THE H-FUNCTION

Like most contour integrals in the complex plane, the H-function is usually evaluated by summing the residues at the poles of the integrand. The contour C, which is sometimes referred to as the Bromwich path, is connected to a semi-circular arc to create a Bromwich contour, a closed curve in the complex plane. By the residue theorem, the value of the integral around the closed Bromwich contour in the positive (counter-clockwise) direction is the sum of the residues at the poles enclosed by the contour. Under very general conditions, the contribution of the semi-circular arc vanishes as the radius increases without bound. In this case, the desired integral over the Bromwich path C equals the integral around the closed Bromwich contour, the sum of residues at the poles interior to the closed contour.

Since the Bromwich path C (by definition) divides the LHP and RHP poles of the integrand of the H-function, connecting a semi-circular arc to the left will enclose all the LHP poles of the integrand as the radius increases without bound. Travelling around this Bromwich contour in the positive (counter-clockwise) direction, we cover the Bromwich path C in the desired direction from $\omega-i\infty$ to $\omega+i\infty$. Under the general conditions referred to earlier, the desired integral along the Bromwich path C equals the sum of residues at the LHP poles.

Conversely, connecting to C a semi-circular arc to the right will enclose all RHP poles of the integrand as the radius increases without bound. Travelling around this Bromwich contour in the positive (counter-clockwise) direction, we cover the Bromwich path C in the opposite of the desired direction from $\omega-i\infty$ to $\omega+i\infty$. Changing the direction travelled around the contour from counter-clockwise to clockwise simply reverses the sign of the integral. Under the general conditions referred to earlier, the desired integral along the Bromwich path C from

$\omega - i\infty$ to $\omega + i\infty$ equals the negative sum of residues at the RHP poles.

The convergence conditions given in Section 2.3 take care of these details and indicate where the H-function may be evaluated as the sum of LHP residues or the negative sum of RHP residues. Depending on the pattern of the poles of the integrand, the process will result in a finite or infinite series. For all of the special cases listed above, this evaluation method will produce a series which equals the Taylor series expansion of the special case.

While it is possible to verify the H-function representations of the special cases in this manner, it is often difficult and tedious to produce the series, especially with poles of multiple orders. Eldred [1979] wrote a computer program to evaluate the H-function by summing the residues at the appropriate poles. Cook [1981; Cook and Barnes, 1981] improved the program and added extra capabilities dealing with algebraic combinations of independent H-function variates. Still, Cook's program will evaluate a general H-function over a specified range (and with a specified interval) of the real variable x when the parameters are input.

CHAPTER 3

THE H-FUNCTION DISTRIBUTION

3.1. DEFINITION

Because the H-function in Eq (2.1) can exactly represent the kernel of many common probability density functions, it was natural to define an H-function distribution as the product of an H-function and a constant, k, which normalizes the area under the H-function (over the appropriate range) to unity. An H-function variate has the following probability density function (p.d.f.) [Carter, 1972; Carter and Springer, 1977, pp. 545-546; Springer, 1979, p. 200]:

$$f(x) = \begin{cases} k\, H_{p\ q}^{m\ n}\left[cx : \underset{i=1,\dots,p}{\left\{\left(a_i, A_i\right)\right\}} ; \underset{j=1,\dots,q}{\left\{\left(b_j, B_j\right)\right\}} \right] & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

In this case, the random variable X is called an H-function variate which follows an H-function probability law or H-function distribution.

Use of an H-function representation as in Eq (3.1) for a probability distribution has several advantages. First, it unifies nearly all common continuous probability distributions of positive random variables under one very general class of functions. The many named distributional forms which arise naturally in common problems in probability may be managed with only one function, the H-function. The H-function can also represent an infinite number of other, unnamed distributional forms.

The H-function also eliminates the need to specify the range of the random variable for which the density is nonzero. The H-function exactly represents the desired density over the appropriate range and is zero elsewhere.

Many characteristics of a probability distribution such as the moments about the origin, the cumulative distribution function, and the Laplace, Fourier, or Mellin transform are easily found from the H-function representation. Finally, if new random variables are defined by algebraic combinations of independent H-function variates, the densities of the new random variables are easily obtained.

3.2. MOMENTS OF AN H-FUNCTION DISTRIBUTION

For the real random variable X with p.d.f. f(x) which is nonzero only for positive values of the variable, the r^{th} moment about the origin of f(x) is given by

$$\mu_r = \int_0^\infty x^r \, f(x) \, dx \tag{3.2}$$

provided the integral in Eq (3.2) exists. If the density f(x) can be represented as an H-function, it is often easier to find the r^{th} moment of f(x) using the Mellin transform formula

$$\mu_r = \mathcal{M}_{r+1}[f(x)] \tag{3.3}$$

In particular, if

$$f(x) = k \, H_{p\,q}^{m\,n}\left[cx : \underset{i=1,\ldots,p}{\left\{\left(a_i, A_i\right)\right\}} \; ; \; \underset{j=1,\ldots,q}{\left\{\left(b_j, B_j\right)\right\}} \right] \tag{3.4}$$

as in Eq (3.1) then using Eq (2.68) [Carter, 1972; Carter and Springer, 1977, pp. 546-547; Springer, 1979, pp. 201-202],

$$\mu_r = \frac{k}{c^{r+1}} \; \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j + B_j r\right) \; \prod_{i=1}^{n} \Gamma\left(1 - a_i - A_i - A_i r\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i + A_i r\right) \; \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j - B_j r\right)} \tag{3.5}$$

Eq (3.5) is very useful in finding the moments of all H-function variates, including all of the special cases listed in Section 3.5.

3.3. EVALUATION OF THE H-FUNCTION DISTRIBUTION CONSTANT

Cook [1981, p. 109; Cook and Barnes, 1991] gave a practical method to determine the normalizing constant, k, for the H-function distribution. If the integrand of $H_{p\ q}^{m\ n}[cx]$ has no pole or zero at s=1, then since the zeroth moment of a valid statistical distribution must be unity,

$$k = \frac{1}{\mu_0} = \frac{1}{\mathcal{M}_s\left\{ H_{p\ q}^{m\ n}[cx] \right\}\Big|_{s=1}}$$

$$= c\,\frac{\prod\limits_{i=n+1}^{p} \Gamma\left(a_i+A_i\right) \prod\limits_{j=m+1}^{q} \Gamma\left(1-b_j-B_j\right)}{\prod\limits_{j=1}^{m} \Gamma\left(b_j+B_j\right) \prod\limits_{i=1}^{n} \Gamma\left(1-a_i-A_i\right)} \tag{3.6}$$

using Eq (3.5) with r=0.

The condition that the integrand of $H_{p\ q}^{m\ n}[cx]$ has no pole at s=1 comes from the need to evaluate the Mellin transform at s=1. A commonly met condition of nearly all H-function distributions which guarantees no pole at s=1 is that $\frac{-b_j}{B_j} < 1$ for j=1,...,m and $\frac{1-a_i}{A_i} > 1$ for i=1,...,n. For the first order H-function $H_{0\ 1}^{1\ 0}[cx]$, Jacobs [1986, p. 46] noted that this restriction corresponds to the region in his (B,b) plot [p. 53] above the line b=-B. He also shows [pp. 71-72] that first order H-function distributions in this region are uniquely determined by their moments.

The condition that the integrand of $H_{p\ q}^{m\ n}[cx]$ has no zero at s=1 arises from the need to have a nonzero Mellin transform at s=1. If the Mellin transform were zero at s=1, no constant k would exist to create a valid p.d.f.

3.4. CUMULATIVE DISTRIBUTION FUNCTION

The cumulative distribution function (c.d.f.) of an H-function distribution was available as simply one (unity) minus another H-function of higher order [Eldred, 1979, pp. 139-140; Springer, 1979, p. 243; Cook, 1981, p. 103]. If $f(x) = kH_{p\ q}^{m\ n}[cx]$ is the p.d.f. of the H-function variate X and F(x) represents the c.d.f. then

$$F(x) = 1 - \frac{k}{c} H_{p+1\ q+1}^{m+1\ n}\left[cx : \left\{\left(a_i+A_i, A_i\right)\right\}, (1,1) ; (0,1), \left\{\left(b_j+B_j, B_j\right)\right\} \right] \tag{3.7}$$

Cook [1981, p. 103] gave another equivalent representation which allows simultaneous computation of the p.d.f. and the c.d.f. of an H-function distribution by the sum of residues. The H-function in this representation of the c.d.f. has a nearly identical pattern of poles and residues as the H-function representing the p.d.f.

$$F(x) = 1 - k\,x\,H^{m+1\ \ n}_{p+1\ q+1}\left[\,cx : \left\{\left(a_i, A_i\right)\right\},\ (0,1)\ ;\ (-1,1),\ \left\{\left(b_j, B_j\right)\right\}\,\right] \tag{3.8}$$

Cook [1981, p. 104] also showed that the c.d.f. (in addition to the complementary c.d.f.) was an H-function. In particular,

$$F(x) = \begin{cases} \dfrac{k}{c}\, H^{m\ \ n+1}_{p+1\ q+1}\left[\,cx : (1,1),\ \left\{\left(a_i + A_i, A_i\right)\right\}\ ;\ \left\{\left(b_j + B_j, B_j\right)\right\},\ (0,1)\,\right], & I = 0 \\[2ex] \dfrac{-k}{c}\, H^{m+1\ \ n}_{p+1\ q+1}\left[\,cx : \left\{\left(a_i + A_i, A_i\right)\right\},\ (1,1)\ ;\ (0,1),\ \left\{\left(b_j + B_j, B_j\right)\right\}\,\right], & I > 0 \end{cases} \tag{3.9}$$

where I is as given in Eq (2.62).

It is possible to introduce an arbitrary, positive generalizing constant, u, to Eq (3.9). The new generalized formula for the H-function representation of the c.d.f. of an H-function variate is

$$
* \quad F(x) = \begin{cases} \frac{uk}{c} H_{p+1\ q+1}^{m\ \ n+1}\left[cx : (1,u), \left\{\left(a_i+A_i, A_i\right)\right\} ; \left\{\left(b_j+B_j, B_j\right)\right\}, (0,u) \right], & I=0 \\ \frac{-uk}{c} H_{p+1\ q+1}^{m+1\ \ n}\left[cx : \left\{\left(a_i+A_i, A_i\right)\right\}, (1,u) ; (0,u), \left\{\left(b_j+B_j, B_j\right)\right\} \right], & I>0 \end{cases} \tag{3.10}
$$

where I is as given in Eq (2.62) and u>0 is arbitrary.

3.5. SPECIAL CASES - STATISTICAL DISTRIBUTIONS

With the parameters given below, the H-function distribution can represent all of the following probability density functions as a special case [Carter, 1972; Carter and Springer, 1977, pp. 547-549; Mathai and Saxena, 1978, pp. 10-12; Eldred, 1979, pp. 104-108; Springer, 1979, pp. 202-207; Cook, 1981, pp. 85-87; Cook and Barnes, 1981, p. 300; Bodenschatz and Boedigheimer, 1983, pp. 17-22; Bodenschatz et al, 1990]. The H-function representations of the cumulative distribution functions and a formula for the moments about the origin are also presented. In some of the representations, v denotes an arbitrary positive constant. In the c.d.f. representations, u denotes an arbitrary positive constant.

Gamma distribution

$$f(x|r,\lambda) = \frac{\lambda^{r}}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

$$= \frac{\lambda}{\Gamma(r)} H_{0\ 1}^{1\ 0}[\ \lambda x : ; (r-1,1)\ ] \qquad x > 0$$

$$r,\lambda > 0$$

$$F(x|r,\lambda) = \frac{1}{\Gamma(r)} H_{1\ 2}^{1\ 1}[\ \lambda x : (1,1) ; (r,1), (0,1)\ ]$$

* $$= \frac{u}{\Gamma(r)} H_{1\ 2}^{1\ 1}[\ \lambda x : (1,u) ; (r,1), (0,u)\ ]$$

$$x > 0$$

$$\mu_{r^*} = \frac{1}{\lambda^{r^*}} \frac{\Gamma\left(r+r^*\right)}{\Gamma(r)}$$

Exponential distribution (Gamma distribution with r=1)

(Weibull distribution with β=1)

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

$$= \lambda H_{0\ 1}^{1\ 0}[\ \lambda x : ; (0,1)\ ] \qquad x > 0$$

$$\lambda > 0$$

* $$F(x|\lambda) = H_{1\ 2}^{1\ 1}[\ \lambda x : (1,1) ; (1,1), (0,1)\ ]$$

* $$= u H_{1\ 2}^{1\ 1}[\ \lambda x : (1,u) ; (1,1), (0,u)\ ] \qquad x > 0$$

$$\mu_r = \frac{1}{\lambda^r}\,\Gamma(1+r)$$

Chi-Square distribution

(Gamma distribution with $r = \frac{\nu}{2}$ and $\lambda = \frac{1}{2}$)

$$f(x|\nu) = \frac{1}{2^{\left(\frac{\nu}{2}\right)}\,\Gamma\left(\frac{\nu}{2}\right)}\; x^{\left(\frac{\nu}{2} - 1\right)}\; e^{-\left(\frac{x}{2}\right)}$$

$$= \frac{1}{2\;\Gamma\left(\frac{\nu}{2}\right)}\, H^{1\,0}_{0\,1}\left[\frac{x}{2} : ; \left(\frac{\nu}{2} - 1, 1\right)\right]$$

$x > 0$

$\nu > 0$

* $$F(x|\nu) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)}\, H^{1\,1}_{1\,2}\left[\frac{x}{2} : (1,1) \; ; \left(\frac{\nu}{2}, 1\right), (0,1)\right]$$

* $$= \frac{u}{\Gamma\left(\frac{\nu}{2}\right)}\, H^{1\,1}_{1\,2}\left[\frac{x}{2} : (1,u) \; ; \left(\frac{\nu}{2}, 1\right), (0,u)\right]$$

$x > 0$

$$\mu_r = \frac{2^r}{\Gamma\left(\frac{\nu}{2}\right)}\,\Gamma\left(\frac{\nu}{2} + r\right)$$

Weibull distribution

$$f(x|\beta,\lambda) = \beta\,\lambda\,x^{(\beta-1)}\,e^{-\lambda x^{\beta}}$$

$$= \lambda^{\left(\frac{1}{\beta}\right)}\,H^{1\,0}_{0\,1}\left[\lambda^{\left(\frac{1}{\beta}\right)} x : ; \left(1-\frac{1}{\beta},\frac{1}{\beta}\right)\right]$$

$x > 0$

$\beta,\lambda > 0$

* $$F(x|\beta,\lambda) = H^{1\,1}_{1\,2}\left[\lambda^{\left(\frac{1}{\beta}\right)} x : (1,1) ; \left(1,\frac{1}{\beta}\right), (0,1)\right]$$

* $$= u\,H^{1\,1}_{1\,2}\left[\lambda^{\left(\frac{1}{\beta}\right)} x : (1,u) ; \left(1,\frac{1}{\beta}\right), (0,u)\right]$$

$x > 0$

$$\mu_r = \frac{1}{\lambda^{\frac{r}{\beta}}}\,\Gamma\left(1+\frac{r}{\beta}\right)$$

Rayleigh distribution (Weibull distribution with β=2)

$$f(x|\lambda) = 2\lambda\,x\,e^{-\lambda x^{2}}$$

$$= \sqrt{\lambda}\,H^{1\,0}_{0\,1}\left[\sqrt{\lambda}\,x : ; \left(\frac{1}{2},\frac{1}{2}\right)\right] \qquad x > 0$$

$\lambda > 0$

$$* \quad F(x|\lambda) = H_{1\ 2}^{1\ 1}\left[\sqrt{\lambda}\, x : (1,1) \ ; \left(1,\frac{1}{2}\right), (0,1) \right]$$

$$* \quad = u\, H_{1\ 2}^{1\ 1}\left[\sqrt{\lambda}\, x : (1,u) \ ; \left(1,\frac{1}{2}\right), (0,u) \right] \qquad x > 0$$

$$\mu_r = \frac{1}{\sqrt{\lambda}^{\,r}} \Gamma\left(1+ \frac{r}{2}\right)$$

Maxwell distribution

$$f(x|\theta) = \frac{4}{\theta^3 \sqrt{\pi}}\ x^2\ e^{-\left(\frac{x^2}{\theta^2}\right)}$$

$$= \frac{2}{\theta \sqrt{\pi}} H_{0\ 1}^{1\ 0}\left[\frac{x}{\theta} : \ ; \left(1,\frac{1}{2}\right) \right] \qquad x > 0$$

$$\theta > 0$$

$$* \quad F(x|\theta) = \frac{2}{\sqrt{\pi}} H_{1\ 2}^{1\ 1}\left[\frac{x}{\theta} : (1,1) \ ; \left(\frac{3}{2},\frac{1}{2}\right), (0,1) \right]$$

$$* \quad = \frac{2u}{\sqrt{\pi}} H_{1\ 2}^{1\ 1}\left[\frac{x}{\theta} : (1,u) \ ; \left(\frac{3}{2},\frac{1}{2}\right), (0,u) \right]$$

$$x > 0$$

$$\mu_r = \frac{2\theta^r}{\sqrt{\pi}} \Gamma\left(\frac{3}{2} + \frac{r}{2}\right)$$

Half-Normal distribution

$$f(x|\sigma) = \frac{2}{\sqrt{2\pi}\,\sigma}\; e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\, H^{1\,0}_{0\,1}\left[\frac{x}{\sqrt{2}\,\sigma} : \;;\; \left(0,\frac{1}{2}\right)\right] \qquad x > 0$$

$$\sigma > 0$$

$$F(x|\sigma) = \frac{1}{\sqrt{\pi}\,\sigma}\, H^{1\,1}_{1\,2}\left[\frac{x}{\sqrt{2}\,\sigma} : (1,1)\;;\; \left(\frac{1}{2},\frac{1}{2}\right),\ (0,1)\right]$$

* $$= \frac{u}{\sqrt{\pi}\,\sigma}\, H^{1\,1}_{1\,2}\left[\frac{x}{\sqrt{2}\,\sigma} : (1,u)\;;\; \left(\frac{1}{2},\frac{1}{2}\right),\ (0,u)\right]$$

$$x > 0$$

$$\mu_r = \frac{\left(\sqrt{2}\,\sigma\right)^r}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2} + \frac{r}{2}\right)$$

Beta distribution of the first kind

$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\; x^{(\alpha-1)}\;(1-x)^{(\beta-1)} \qquad 0 < x < 1$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\, H^{1\,0}_{1\,1}[\,x : (\alpha+\beta-1,1)\;;\;(\alpha-1,1)\,]$$

$$\alpha,\beta > 0$$

$$F(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\, H^{1\,1}_{2\,2}[\,x : (1,1),\ (\alpha+\beta,1)\;;$$

$$(\alpha,1),\ (0,1)\,]$$

$$* \quad = u \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} H_{2\ 2}^{1\ 1}[\ x : (1,u),\ (\alpha+\beta,1)\ ;\ (\alpha,1),\ (0,u)\ ] \qquad x > 0$$

$$\mu_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+r)}$$

Power Function distribution (Beta distribution with β=1)

$$f(x|\alpha) = \alpha\, x^{(\alpha-1)} \qquad 0 < x < 1$$

$$= \alpha\, H_{1\ 1}^{1\ 0}[\ x : (\alpha,1)\ ;\ (\alpha-1,1)\ ]$$

$$= v\alpha\, H_{1\ 1}^{1\ 0}[\ x : (v(\alpha-1)+1,v)\ ;\ (v(\alpha-1),v)\ ] \qquad \alpha > 0$$

$$* \quad F(x|\alpha) = \alpha\, H_{2\ 2}^{1\ 1}[\ x : (1,1),\ (\alpha+1,1)\ ;\ (\alpha,1),\ (0,1)\ ]$$

$$* \quad = uv\alpha\, H_{2\ 2}^{1\ 1}[\ x : (1,u),\ (v\alpha+1,v)\ ;\ (v\alpha,v),\ (0,u)\ ] \qquad x > 0$$

$$\mu_r = \frac{\alpha}{\alpha+r}$$

Uniform distribution (Beta distribution with $\alpha=\beta$=1)

(Power Function distribution with α=1)

$$f(x) = 1 \qquad 0 < x < 1$$

$$= H_{1\ 1}^{1\ 0}[\ x : (1,1)\ ;\ (0,1)\ ]$$

$$= v\, H^{1\,0}_{1\,1}[\ x : (1,v)\ ;\ (0,v)\ ]$$

* $$F(x) = H^{1\,1}_{2\,2}[\ x : (1,1),\ (2,1)\ ;\ (1,1),\ (0,1)\ ]$$

* $$= uv\, H^{1\,1}_{2\,2}[\ x : (1,u),\ (v+1,v)\ ;\ (v,v),\ (0,u)\ ]$$

$x > 0$

$$\mu_r = \frac{1}{1+r}$$

Pareto distribution

$$f(x|\alpha) = \alpha\, x^{-(\alpha+1)} \qquad x > 1$$

$$= \alpha\, H^{0\,1}_{1\,1}[\ x : (-\alpha,1)\ ;\ (-\alpha-1,1)\ ]$$

$$= v\alpha\, H^{0\,1}_{1\,1}[\ x : (1-v(\alpha+1),v)\ ;\ (-v(\alpha+1),v)\ ]$$

$\alpha > 0$

* $$F(x|\alpha) = \alpha\, H^{0\,2}_{2\,2}[\ x : (1,1),\ (1-\alpha,1)\ ;\ (-\alpha,1),\ (0,1)\ ]$$

* $$= uv\alpha\, H^{0\,2}_{2\,2}[\ x : (1,u),\ (1-v\alpha,v)\ ;\ (-v\alpha,v),\ (0,u)]$$

$x > 0$

$$\mu_r = \frac{\alpha}{\alpha-r} \qquad \text{for } \alpha>r$$

Half-Cauchy distribution

$$f(x|\theta) = \frac{2\theta}{\pi\left(x^2 + \theta^2\right)}$$

$$= \frac{1}{\theta\pi} H_{1\ 1}^{1\ 1}\left[\frac{x}{\theta} : \left(0,\frac{1}{2}\right) ; \left(0,\frac{1}{2}\right) \right] \qquad x > 0$$

$$\theta > 0$$

$$* \qquad F(x|\theta) = \frac{1}{\pi} H_{2\ 2}^{1\ 2}\left[\frac{x}{\theta} : (1,1), \left(\frac{1}{2},\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,1) \right]$$

$$* \qquad = \frac{u}{\pi} H_{2\ 2}^{1\ 2}\left[\frac{x}{\theta} : (1,u), \left(\frac{1}{2},\frac{1}{2}\right) ; \left(\frac{1}{2},\frac{1}{2}\right), (0,u) \right]$$

$$x > 0$$

$$\mu_r = \begin{cases} \dfrac{\theta^r}{\pi} \Gamma\left(\dfrac{1}{2} + \dfrac{r}{2}\right) \Gamma\left(\dfrac{1}{2} - \dfrac{r}{2}\right) & \text{if r is even} \\ \text{do not exist} & \text{if r is odd} \end{cases}$$

Half-Student distribution

$$f(x|\nu) = \frac{2\ \Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\ \Gamma\left(\frac{\nu}{2}\right) \left(1 + \frac{x^2}{\nu}\right)^{\left(\frac{\nu+1}{2}\right)}}$$

$$= \frac{1}{\sqrt{\nu\pi}\ \Gamma\left(\frac{\nu}{2}\right)} H_{1\,1}^{1\,1}\left[\frac{x}{\sqrt{\nu}} : \left(\frac{1-\nu}{2},\frac{1}{2}\right) ;\ \left(0,\frac{1}{2}\right)\right]$$

$x > 0$

$\nu > 0$

$$* \qquad F(x|\nu) = \frac{1}{\sqrt{\pi}\ \Gamma\left(\frac{\nu}{2}\right)} H_{2\,2}^{1\,2}\left[\frac{x}{\sqrt{\nu}} : (1,1),\ \left(1-\frac{\nu}{2},\frac{1}{2}\right) ;\ \left(\frac{1}{2},\frac{1}{2}\right),\ (0,1)\right]$$

$$* \qquad = \frac{u}{\sqrt{\pi}\ \Gamma\left(\frac{\nu}{2}\right)} H_{2\,2}^{1\,2}\left[\frac{x}{\sqrt{\nu}} : (1,u),\ \left(1-\frac{\nu}{2},\frac{1}{2}\right) ;\ \left(\frac{1}{2},\frac{1}{2}\right),\ (0,u)\right]$$

$x > 0$

$$\mu_r = \begin{cases} \dfrac{\left(\sqrt{\nu}\right)^r}{\sqrt{\pi}\ \Gamma\left(\frac{\nu}{2}\right)} \Gamma\left(\frac{1}{2}+\frac{r}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{r}{2}\right) & \text{if } \nu-r\neq-2J \text{ for } J=0,1,2,\ldots. \\ \text{do not exist} & \text{if } \nu-r=-2J \text{ for } J=0,1,2,\ldots. \end{cases}$$

F distribution

$$f(x|\nu,\omega) = \frac{\Gamma\left(\frac{\nu+\omega}{2}\right) \; \nu^{\left(\frac{\nu}{2}\right)} \; \omega^{\left(\frac{\omega}{2}\right)} \; x^{\left(\frac{\nu}{2} - 1\right)}}{\Gamma\left(\frac{\nu}{2}\right) \; \Gamma\left(\frac{\omega}{2}\right) \; (\nu x+\omega)^{\left(\frac{\nu+\omega}{2}\right)}}$$

$$= \frac{\nu}{\omega \, \Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\omega}{2}\right)} H_{1\,1}^{1\,1}\left[\frac{\nu x}{\omega} : \left(-\frac{\omega}{2},1\right) ; \left(\frac{\nu}{2} -1,1\right) \right]$$

$x > 0$

$\nu,\omega > 0$

* $$F(x|\nu,\omega) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\omega}{2}\right)} H_{2\,2}^{1\,2}\left[\frac{\nu x}{\omega} : (1,1), \left(1- \frac{\omega}{2},1\right) ; \left(\frac{\nu}{2},1\right), (0,1) \right]$$

* $$= \frac{u}{\Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\omega}{2}\right)} H_{2\,2}^{1\,2}\left[\frac{\nu x}{\omega} : (1,u), \left(1- \frac{\omega}{2},1\right) ; \left(\frac{\nu}{2},1\right), (0,u) \right]$$

$x > 0$

$$\mu_r = \begin{cases} \dfrac{\omega^r}{\nu^r\,\Gamma\left(\frac{\nu}{2}\right)\,\Gamma\left(\frac{\omega}{2}\right)}\,\Gamma\left(\frac{\nu}{2}+r\right)\,\Gamma\left(\frac{\omega}{2}-r\right) & \\ \qquad\qquad \text{if } \frac{\omega}{2}-r \neq -J \quad \text{for } J=0,1,2,\ldots \\ \text{do not exist} \qquad \text{if } \frac{\omega}{2}-r = -J \quad \text{for } J=0,1,2,\ldots \end{cases}$$

Beta distribution of the second kind

$$f(x|\alpha,\beta) = \left(\frac{\beta}{\alpha}\right)^{\alpha} \frac{\Gamma(\alpha+\beta)\quad x^{(\alpha-1)}}{\Gamma(\alpha)\,\Gamma(\beta)\,\left(1+\frac{\beta x}{\alpha}\right)^{(\alpha+\beta)}}$$

$$= \frac{\beta}{\alpha\,\Gamma(\alpha)\,\Gamma(\beta)}\, H_{1\ 1}^{1\ 1}\left[\frac{\beta x}{\alpha} : (-\beta,1)\ ;\ (\alpha-1,1)\right]$$

$x > 0$

$\alpha,\beta > 0$

* $$F(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\, H_{2\ 2}^{1\ 2}\left[\frac{\beta x}{\alpha} : (1,1),\ (1-\beta,1)\ ;\ (\alpha,1),\ (0,1)\right]$$

* $$= \frac{u}{\Gamma(\alpha)\,\Gamma(\beta)}\, H_{2\ 2}^{1\ 2}\left[\frac{\beta x}{\alpha} : (1,u),\ (1-\beta,1)\ ;\ (\alpha,1),\ (0,u)\right]$$

$x > 0$

$$\mu_r = \begin{cases} \dfrac{\alpha^r}{\beta^r\,\Gamma(\alpha)\,\Gamma(\beta)}\,\Gamma(\alpha+r)\,\Gamma(\beta-r) \\ \qquad\qquad\qquad \text{if } \beta-r\neq -J \quad \text{for } J=0,1,2,\ldots. \\ \text{do not exist} \qquad \text{if } \beta-r=-J \quad \text{for } J=0,1,2,\ldots. \end{cases}$$

General Hypergeometric distribution

$$f(x|a,b,c,d,r) = \frac{d\;a^{\frac{c}{d}}\;\Gamma(b)\;\Gamma\left(\frac{r-c}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\;\Gamma(r)\;\Gamma\left(\frac{b-c}{d}\right)}\;x^{c-1}\;M\left(b,r,-ax^{d}\right)$$

$$= \frac{d\;a^{\frac{c}{d}}\;\Gamma(b)\;\Gamma\left(\frac{r-c}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\;\Gamma(r)\;\Gamma\left(\frac{b-c}{d}\right)}\;x^{c-1}\;{}_1F_1\left(b,r,-ax^{d}\right)$$

$$= \frac{a^{\frac{1}{d}}\;\Gamma\left(\frac{r-c}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\;\Gamma\left(\frac{b-c}{d}\right)}\;H^{1\;1}_{1\;2}\left[\,a^{\frac{1}{d}}x : \left(1-b+\frac{c-1}{d},\frac{1}{d}\right) ; \left(\frac{c-1}{d},\frac{1}{d}\right),\left(1-r+\frac{c-1}{d},\frac{1}{d}\right)\right]$$

$x > 0$

$$* \qquad F(x|a,b,c,d,r) = \frac{\Gamma\left(\frac{r-c}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\ \Gamma\left(\frac{b-c}{d}\right)}\ H^{1\ 2}_{2\ 3}\left[\ a^{\frac{1}{d}} x :\ (1,1),\ \left(1-b+\frac{c}{d},\frac{1}{d}\right)\ ;\ \left(\frac{c}{d},\frac{1}{d}\right),\ \left(1-r+\frac{c}{d},\frac{1}{d}\right),\ (0,1)\ \right]$$

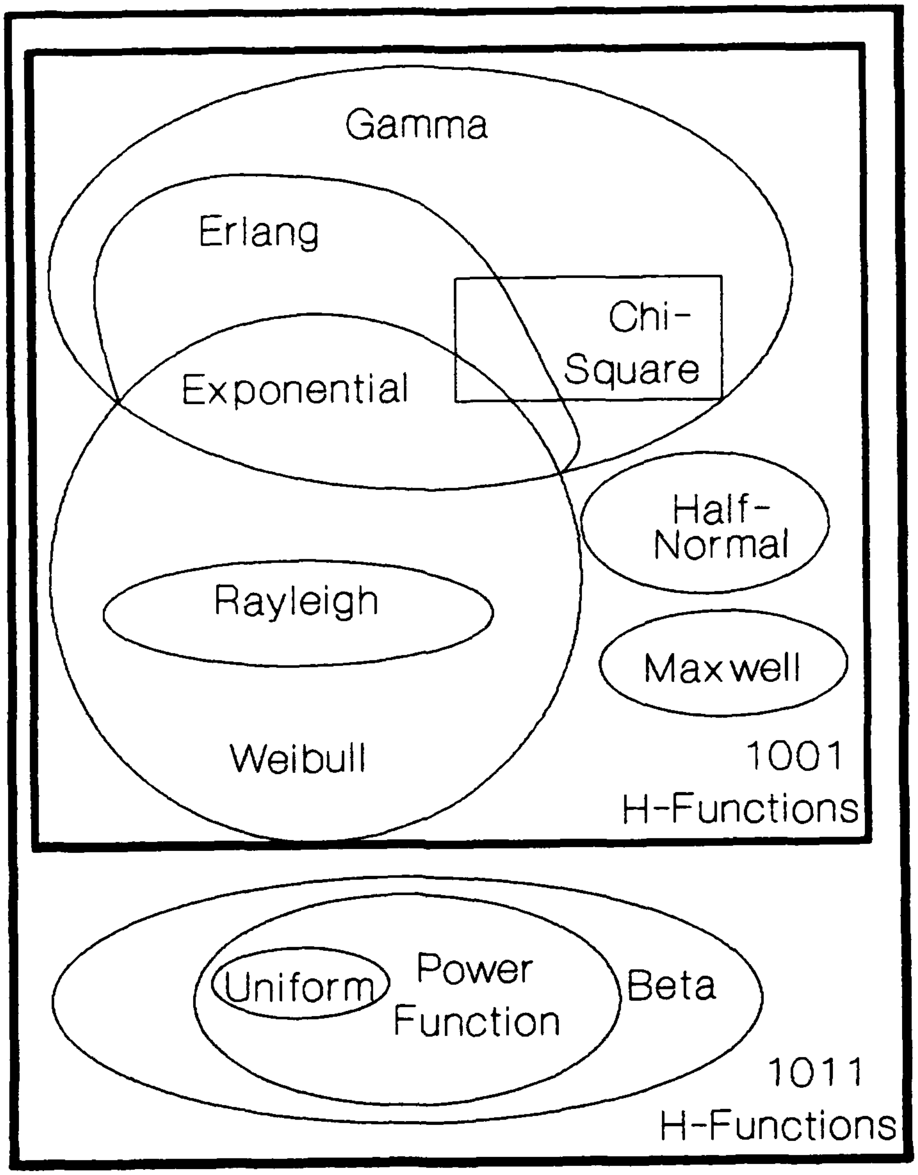
$$* \qquad = \frac{u\,\Gamma\left(\frac{r-c}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\ \Gamma\left(\frac{b-c}{d}\right)}\ H^{1\ 2}_{2\ 3}\left[\ a^{\frac{1}{d}} x :\ (1,u),\ \left(1-b+\frac{c}{d},\frac{1}{d}\right)\ ;\ \left(\frac{c}{d},\frac{1}{d}\right),\ \left(1-r+\frac{c}{d},\frac{1}{d}\right),\ (0,u)\ \right]$$

$$x > 0$$

$$\mu_{r^*} = \begin{cases} \dfrac{\Gamma\left(\frac{r-c}{d}\right)\ \Gamma\left(\frac{c}{d}+\frac{r^*}{d}\right)\ \Gamma\left(b-\frac{c}{d}-\frac{r^*}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\ \Gamma\left(\frac{b-c}{d}\right)\ \Gamma\left(r-\frac{c}{d}-\frac{r^*}{d}\right)\ a^{\frac{r^*}{d}}} & \text{if } b-\frac{c}{d}-\frac{r^*}{d} \neq -J \quad \text{for } J=0,1,2,\ldots \\ \text{do not exist} & \text{if } b-\frac{c}{d}-\frac{r^*}{d} = -J \quad \text{for } J=0,1,2,\ldots \end{cases}$$

It should be noted that since the H-function distribution is only defined over positive values, symmetric and doubly infinite distributions like the normal and Student's t must be manipulated in their folded forms. While the normal, Student's t, and Cauchy distributions are not special cases of the H-function, their folded forms, the half-normal, half-Student's t, and half-Cauchy, are representable as H-function distributions.

The venn diagram in Figure 1 shows the relationship between many common first and second order H-function distributions. The Erlang distribution is simply a gamma distribution with an integer shape parameter r. The exponential distribution is a proper subset of both the Weibull distribution and the Erlang distribution, which is a proper subset of the gamma distribution. One exponential distribution (with $\lambda=\frac{1}{2}$) is a special case of the Chi-Square distribution, which is also a proper subset of the gamma distribution. If a Chi-Square distribution has an even number of degrees of freedom, ν, it is also an Erlang distribution with $\lambda=\frac{1}{2}$. The Rayleigh distribution is a proper subset of the Weibull distribution. All of these named distributions, plus the Half-Normal and Maxwell distributions, are special cases of the first order $H_{0\ 1}^{1\ 0}$ class of H-function distributions.



* **Figure 1. Venn Diagram of Certain Common Statistical Distributions as $H_{0\ 1}^{1\ 0}$ and $H_{1\ 1}^{1\ 0}$ H-Function Distributions**

It is well known that the Uniform distribution is a special case of the Power Function distribution, which is a special case of the Beta distribution. All of these distributions are represented as $H_{1\ 1}^{1\ 0}$ H-function distributions. Further, it was shown in Section 2.4.6 that the $H_{0\ 1}^{1\ 0}$ class of H-functions is a proper subset of the $H_{1\ 1}^{1\ 0}$ class of H-functions. Therefore, all of the named distributions in Figure 1 can be exactly represented as H-function distributions in the $H_{1\ 1}^{1\ 0}$ class.

An H-function representation has not been given for the p.d.f. of the Lognormal or Logistic distributions. Conversely, no one has proven that these distributions cannot be represented as H-functions. This is an area for future research.

* 3.6. ARBITRARY RANGES FOR TYPE VI H-FUNCTION VARIATES

The H-functions in some of the p.d.f. representations in Section 3.5 are Type VI convergent according to Table 1 in Section 2.3. Consequently, they represent the desired p.d.f. over a certain range of the variable and, since they lack poles in the opposite half-plane, are zero for other values of the variable.

Specifically, the H-function representations of the Beta (first kind), Power Function, and Uniform distributions above are all $H_{1\,1}^{1\,0}$ Type VI convergent H-functions with no RHP poles. As listed above, they represent the desired p.d.f. over $0 < x < 1$ and are zero for $x > 1$.

Similarly, the H-function representation of the Pareto distribution above is an $H_{1\,1}^{0\,1}$ Type VI convergent H-function with no LHP poles. As listed, it represents the Pareto p.d.f. for $x > 1$ and is zero over $0 < x < 1$.

This limitation (as discussed in Section 2.6) of the H-function to represent a general power function or beta-type function for all $x > 0$ is actually an advantage when representing the statistical distributions. Using the H-function representation eliminates the need to specify the range of the variable for which the density is nonzero. The H-function representation gives the desired p.d.f. for the appropriate range of the variable and is automatically zero otherwise.

The newly discovered technique of scaling these Type VI convergent H-functions discussed in Section 2.6 is also applicable here. By changing the value of c in the definition of the H-function, it is possible to alter the point at which the H-function changes from representing the special case to taking the value zero. For the statistical distributions mentioned earlier, this allows more flexibility in representing the same functional form over a more general range. Given below are the new H-function representations of the densities for the Three-Parameter Beta distribution, the Power Function distribution over (0,M), the Uniform distribution over (0,M), and the Pareto distribution over (ε,∞). In some of the representations, v denotes an arbitrary positive constant.

Three-Parameter Beta distribution of the first kind

$$f(x|\alpha,\beta,M) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \frac{1}{M^{\alpha+\beta-1}}\ x^{(\alpha-1)}\ (M-x)^{(\beta-1)}$$

$$0 < x < M$$

* $$= \frac{\Gamma(\alpha+\beta)}{M\ \Gamma(\alpha)} H_{1\ 1}^{1\ 0}\left[\frac{x}{M} : (\alpha+\beta-1,1)\ ;\ (\alpha-1,1)\right]$$

$$\alpha,\beta,M > 0$$

Power Function distribution over (0,M)

(Three-Parameter Beta distribution with β=1)

$$f(x|\alpha,M) = \alpha \frac{1}{M^{\alpha}}\ x^{(\alpha-1)} \qquad 0 < x < M$$

* $$= \frac{\alpha}{M} H_{1\ 1}^{1\ 0}\left[\frac{x}{M} : (\alpha,1)\ ;\ (\alpha-1,1)\right]$$

* $$= \frac{v\alpha}{M} H_{1\ 1}^{1\ 0}\left[\frac{x}{M} : (v(\alpha-1)+1,v)\ ;\ (v(\alpha-1),v)\right]$$

$$\alpha,M > 0$$

Uniform distribution over (0,M)

(Three-Parameter Beta distribution with $\alpha=\beta=1$)

(Power Function distribution over (0,M) with $\alpha=1$)

$$f(x|M) = \frac{1}{M} \qquad 0 < x < M$$

$$* \qquad = \frac{1}{M} H_{1\ 1}^{1\ 0}\left[\frac{x}{M} : (1,1) \ ; \ (0,1) \right]$$

$$* \qquad = \frac{v}{M} H_{1\ 1}^{1\ 0}\left[\frac{x}{M} : (1,v) \ ; \ (0,v) \right] \qquad M > 0$$

Pareto distribution over (ε,∞)

$$f(x|\alpha,\varepsilon) = \alpha\ \varepsilon^{\alpha}\ x^{-(\alpha+1)} \qquad x > \varepsilon$$

$$* \qquad = \frac{\alpha}{\varepsilon} H_{1\ 1}^{0\ 1}\left[\frac{x}{\varepsilon} : (-\alpha,1) \ ; \ (-\alpha-1,1) \right]$$

$$* \qquad = \frac{v\alpha}{\varepsilon} H_{1\ 1}^{0\ 1}\left[\frac{x}{\varepsilon} : (1-v(\alpha+1),v) \ ; \ (-v(\alpha+1),v) \right]$$

$$\alpha,\varepsilon > 0$$

3.7. TRANSFORMATIONS OF INDEPENDENT H-FUNCTION VARIATES

A significant advantage of using the H-function representations of statistical distributions is that they make finding the distribution of an algebraic combination of independent random variables much easier. Carter [1972; Carter and Springer, 1977, pp. 549-557; Springer, 1979, pp. 207-219] showed that the product, quotient, or power of independent H-function variates themselves had an H-function distribution.

In fact, the resulting density of the new random variable can immediately be written (as an H-function) using only the parameters of the H-functions in the algebraic combination.

Springer [1979, pp. 217-219] described a method for finding the H-function distribution of an algebraic combination (including constants, products, quotients, and powers) of independent H-function variates. Cook [1981, p. 92] combined the three theorems of Carter [1972; Carter and Springer, 1977, pp. 549-557; Springer, 1979, pp. 207-217] into one very complicated theorem. Using Cook's theorem, the H-function distribution of $Y = \prod_{j=1}^{V} X_j^{P_j}$ is immediately available, where X_j, $j=1,\ldots,V$ are mutually independent H-function variates and P_j, $j=1,\ldots,V$ may be positive or negative. Carter's separate results are given below, not Cook's combined results.

3.7.1. <u>DISTRIBUTION OF A PRODUCT</u>

If X_1, $X_2,\ldots,$ X_N are mutually independent H-function variates with densities $f_1(x_1)$, $f_2(x_2),\ldots,$ $f_N(x_N)$, respectively, where

$$f_i(x_i) = \begin{cases} k_i\, H^{m_i\, n_i}_{p_i\, q_i}\left[c_i x_i : \left\{\left(a_{ij}, A_{ij}\right)\right\}_{j=1,\ldots,p_i} ; \left\{\left(b_{ij}, B_{ij}\right)\right\}_{j=1,\ldots,q_i} \right] & x_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

for $i=1,2,\ldots,N$, then the p.d.f. of the random variable $Y = \prod_{i=1}^{N} X_i$ is given by

$$f_Y(y) = \begin{cases} \left(\prod_{i=1}^{N} k_i\right) H^{\sum_{i=1}^{N} m_i \;\; \sum_{i=1}^{N} n_i}_{\sum_{i=1}^{N} p_i \;\; \sum_{i=1}^{N} q_i}\left[\left(\prod_{i=1}^{N} c_i\right) y : \left\{\left(a_{ij}, A_{ij}\right)\right\} ; \left\{\left(b_{ij}, B_{ij}\right)\right\} \right] & y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

where the sequence of parameters $\left\{\left(a_{ij}, A_{ij}\right)\right\}$ is

$$\left\{ \begin{array}{ll} j=1,2,\ldots,n_i & \text{for } i=1,2,\ldots,N \\ \text{followed by} & \\ j=n_i+1, n_i+2, \ldots, p_i & \text{for } i=1,2,\ldots,N \end{array} \right\}$$

and the sequence of parameters $\left\{\left(b_{ij},B_{ij}\right)\right\}$ is

$$\left\{\begin{array}{ll} j=1,2,\ldots,m_i & \text{for } i=1,2,\ldots,N \\ \text{followed by} & \\ j=m_i+1,m_i+2,\ldots,q_i & \text{for } i=1,2,\ldots,N \end{array}\right\}$$

In effect, the formula retains the gamma terms in their previous place (numerator or denominator) in the integrand of the H-function.

3.7.2. <u>DISTRIBUTION OF A QUOTIENT</u>

If X_1 and X_2 are independent H-function variates with densities $f_1(x_1)$ and $f_2(x_2)$, respectively, where

$$f_i(x_i) = \begin{cases} k_i\, H_{p_i\ q_i}^{m_i\ n_i}\left[c_i x_i : \left\{\left(a_{ij},A_{ij}\right)\right\}_{j=1,\ldots,p_i} ; \left\{\left(b_{ij},B_{ij}\right)\right\}_{j=1,\ldots,q_i} \right] & x_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

for i=1,2 then the p.d.f. of the random variable $Y = \dfrac{X_1}{X_2}$ is given by

$$f_Y(y) = \begin{cases} \dfrac{k_1 k_2}{\left(c_2\right)^2} \; H^{m_1+n_2 \;\; n_1+m_2}_{p_1+q_2 \;\; q_1+p_2} \left[\dfrac{c_1}{c_2}\, y : \begin{matrix} \left\{\left(d,D\right)\right\} ; \\ \left\{\left(e,E\right)\right\} \end{matrix} \right] & y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

where the sequence of parameters $\left\{\left(d,D\right)\right\}$ is

$$\left\{ \left(a_{11}, A_{11}\right), \ldots, \left(a_{1n_1}, A_{1n_1}\right), \right.$$

$$\left(1-b_{21} - 2B_{21}, B_{21}\right), \ldots, \left(1-b_{2m_2} - 2B_{2m_2}, B_{2m_2}\right),$$

$$\left(a_{1n_1+1}, A_{1n_1+1}\right), \ldots, \left(a_{1p_1}, A_{1p_1}\right),$$

$$\left. \left(1- b_{2m_2+1} - 2B_{2m_2+1}, B_{2m_2+1}\right), \ldots, \left(1- b_{2q_2} - 2B_{2q_2}, B_{2q_2}\right) \right\}$$

and the sequence of parameters $\left\{\left(e,E\right)\right\}$ is

$$\Big\{ \left(b_{11},\ B_{11}\right),\ldots,\left(b_{1m_1},\ B_{1m_1}\right),$$

$$\left(1-a_{21}\ -2A_{21},\ A_{21}\right),\ldots,\left(1-a_{2n_2}\ -2A_{2n_2},\ A_{2n_2}\right),$$

$$\left(b_{1m_1+1},\ B_{1m_1+1}\right),\ldots,\left(b_{1q_1},\ B_{1q_1}\right),$$

$$\left(1-\ a_{2n_2+1}\ -2A_{2n_2+1},\ A_{2n_2+1}\right),\ldots,\left(1-\ a_{2p_2}\ -2A_{2p_2},\ A_{2p_2}\right)\Big\}$$

3.7.3. DISTRIBUTION OF A VARIATE TO A POWER

If X is an H-function variate with density f(x) where

$$f(x) = \begin{cases} k\, H_{p\ q}^{m\ n}\left[cx : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,q} \right] & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

then the p.d.f. of the random variable $Y = X^P$ is given by

$$f_Y(y) = \begin{cases} k\,c^{P-1}\, H_{p\ q}^{m\ n}\left[c^P y : \begin{array}{c}\left\{\left(a_i - A_i P + A_i, A_i P\right)\right\}_{i=1,\dots,p} ; \\ \left\{\left(b_j - B_j P + B_j, B_j P\right)\right\}_{j=1,\dots,q}\end{array}\right] & y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

when $P > 0$ and

$$f_Y(y) = \begin{cases} k\,c^{P-1}\, H_{q\ p}^{n\ m}\left[c^P y : \begin{array}{c}\left\{\left(1 - b_j + B_j P - B_j, -B_j P\right)\right\}_{j=1,\dots,q} ; \\ \left\{\left(1 - a_i + A_i P - A_i, -A_i P\right)\right\}_{i=1,\dots,p}\end{array}\right] & y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

when $P < 0$.

3.7.4. USE OF JACOBS' (B,b) PLOT IN FINDING POWERS OF FIRST ORDER H-FUNCTION VARIATES

Even first order H-functions (possessing only one gamma term) can represent a wide variety of distributional forms. For example, an H-function distribution can have a shape which is not quite that of either a Weibull density or a gamma density. The plot given in Figure 2 of the (B,b) parameter space for first order H-function distributions by Jacobs [1986, p. 53; Jacobs et al, 1987, p. 134] shows the ability of first order H-function variates to model an infinite number of other, unnamed distributional forms.

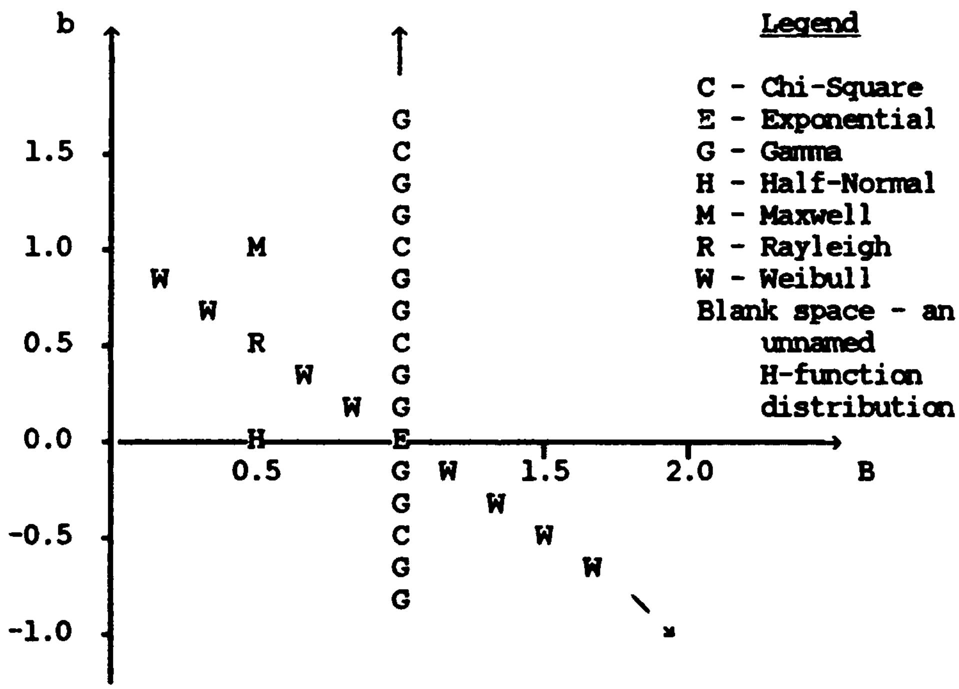


Figure 2. Classical Statistical Distributions as First Order H-Functions in (B,b) Space

Jacobs [1986, p. 55; Jacobs et al, 1987, p. 144] showed how the (B,b) plane can be used to determine the H-function distribution of positive powers of H-function variates. Since Eq (3.16) in Section 3.7.3 for P>0 does not change the values of m, n, p, or q, positive powers of first order H-function distributions are also first order H-function distributions. Further, Jacobs noted that all positive powers of the H-function variate X with parameters (b*,B*) lie on the line in the (B,b) plane parallel to the line representing the Weibull family of probability density functions and through the point (B*,b*).

Jacobs [1986, p. 55; Jacobs et al, 1987, p. 144] gave three examples of the use of the (B,b) plane in finding the distribution of the power of H-function variates. He made a slight error when he stated that the square of a half-normal random variable has a Chi-Square distribution. Instead, the square of a standard (i.e. σ=1) half-normal random variable has a Chi-Square distribution with ν=1. He correctly stated that the square of a Rayleigh random variable has an exponential distribution and that any positive power of a Weibull random variable has another, different Weibull distribution.

* In particular, if X has a Rayleigh distribution with parameter λ, then $Y=X^2$ has an exponential distribution with parameter λ. If X has a Weibull distribution with parameters β and λ, then $Y=X^P$ where P>0 has a Weibull distribution with

parameters $\frac{\beta}{P}$ and λ. I do not believe these relationships were commonly known.

* There are other new relationships between statistical distributions available from using the (B,b) plane of first order H-function distributions. If X has a half-normal distribution with parameter σ, then $Y=X^2$ has a gamma distribution with parameters $r = \frac{1}{2}$ and $\lambda = \frac{1}{2\sigma^2}$. If X has a Maxwell distribution with parameter θ, then $Y=X^2$ has a gamma distribution with parameters $r = \frac{3}{2}$ and $\lambda = \frac{1}{\theta^2}$. If X has a Maxwell distribution with parameter $\theta = \sqrt{2}$, then $Y=X^2$ has a Chi-Square distribution with parameter $\nu=3$. If X has a Rayleigh distribution with parameter $\lambda = \frac{1}{2}$, then $Y=X^2$ has a Chi-Square distribution with parameter $\nu=2$. I do not believe these relationships are commonly known, either.

* The (B,b) plane is useful for finding the parameters for powers of first order H-function variates. If $P>1$, the parameters for $Y=X^P$ can be found by traveling down an imaginary line toward the lower right of the graph and parallel to the line representing the Weibull family of densities. If $0<P<1$, the parameters for $Y=X^P$ can be found by traveling up an imaginary line toward the upper left of the graph and parallel to the line representing the Weibull family of densities. In

either case, the parameter B for Y is simply the product of P and the parameter B for X.

3.7.5. DISTRIBUTION OF A SUM

Although the products, quotients, and powers of independent H-function variates have H-function distributions, this characteristic does not, in general, extend to sums. Analogs to Carter's results in Sections 3.7.1, 3.7.2, and 3.7.3 do not currently exist for sums of independent H-function variates.

Because the Laplace transform of the p.d.f. of an H-function variate is readily obtained as another H-function, it was hoped the distribution of a general sum of H-function variates had an H-function distribution. The Laplace transform of a density which is nonzero only over positive values can be used like the moment generating function or characteristic function. The Laplace transform of the p.d.f. of the sum of two independent random variables is the product of the Laplace transforms of the individual densities. If this product were available as another H-function, it could be inverted analytically, yielding the desired density expressed as an H-function. The problem reduced to determining whether the product of two H-functions was, in general, another H-function.

Because the H-function can exactly represent nearly every common mathematical function and statistical density, there was ample reason to suspect that the product of two H-functions was, in general, another H-function. Indeed, there are many cases where two individual functions and their product are all special cases of the H-function. Also, the sum of certain independent H-function variates does have an H-function distribution. Examples of this are independent exponentially or gamma distributed variates with a common value of λ.

Unfortunately, the product of two H-functions might not, in general, be another H-function. Recently, Prudnikov et al [1990, p. 354] gave the following result

$$\int_0^\infty x^{\alpha-1} \; H_{u\,v}^{s\,t}\left[\sigma x : \underset{i=1,\ldots,u}{\left\{\left(c_i, C_i\right)\right\}} \; ; \; \underset{j=1,\ldots,v}{\left\{\left(d_j, D_j\right)\right\}} \right]$$

$$H_{p\,q}^{m\,n}\left[\omega x^r : \underset{i=1,\ldots,p}{\left\{\left(a_i, A_i\right)\right\}} \; ; \; \underset{j=1,\ldots,q}{\left\{\left(b_j, B_j\right)\right\}} \right] \; dx$$

$$= \frac{1}{\sigma^{\alpha}} H^{m+t\ n+s}_{p+v\ q+u}\left[\frac{\omega}{\sigma^{\alpha}} : \left\{\left(a_i, A_i\right)\right\}_{i=1,\ldots,n}, \left\{\left(1-d_j-\alpha D_j, rD_j\right)\right\}_{j=1,\ldots,v},\right.$$

$$\left\{\left(a_i, A_i\right)\right\}_{i=n+1,\ldots,p} ; \left\{\left(b_j, B_j\right)\right\}_{j=1,\ldots,m}, \left\{\left(1-c_i-\alpha C_i, rC_i\right)\right\}_{i=1,\ldots,u},$$

$$\left.\left\{\left(b_j, B_j\right)\right\}_{j=m+1,\ldots,q}\right] \tag{3.18}$$

under certain general conditions. Comparing the left side of Eq (3.18) to Eq (1.7), the Mellin transform of the product of two H-functions is another H-function. Using a result from Erdélyi [1954, p. 308, No. 13], it was possible to verify the result in Eq (3.18). Still, Eq (3.18) is not entirely satisfying. The H-function on the right side of Eq (3.18) may not even be a valid H-function or Mellin-Barnes integral. The variable is α, the transform variable, which does not appear in the argument location, but in the parameters of the H-function. The variable α will appear inside certain products and quotients of gamma functions in the integrand of the H-function.

It has not been shown that the Mellin transform of an H-function is another H-function. However, by Eq (2.68), the Mellin transform of an H-function is basically products and quotients of gamma functions. Considering Eq (3.18), if the product of the two H-functions inside the integral produced an H-function, then the Mellin transform integral would yield its products and quotients of gamma functions. Therefore, the H-function on the right of Eq (3.18) would have to represent those products and quotients of gamma functions.

An H-function representation has not been given for a single gamma function. An H-function representation of products and quotients of gamma functions seems even less likely.

Although this discussion is not a rigorous mathematical proof, it leads to the conjecture that the product of two H-functions is not, in general, another H-function. Further research is needed to prove or disprove this conjecture.

Because the product of two H-functions was not available as another H-function, a pure analytical solution for the density of the sum of independent H-function variates as another H-function was not possible. Analogs to Carter's results for the sum of independent H-function variates do not currently exist. Therefore, development of a practical technique to find the H-function distribution for the sum (or a close approximation to it) was necessary. A description of this technique is presented in the next chapter.

CHAPTER 4

FINDING AN H-FUNCTION DISTRIBUTION FOR THE SUM

Because the product of two H-functions was not available as another H-function, it was necessary to develop a method to find a close approximation for the density of the sum of two or more independent H-function variates. Ideally, this approximation would also have an H-function representation. This would allow the inclusion of the sum of random variables in more complicated algebraic combinations with other independent random variables.

For example, suppose the density of the random variable Z is desired, where

$$Z = \frac{\left(Y_1\right)^3}{Y_2\,Y_3} \tag{4.1}$$

$$Y_1 = X_1+X_2+X_3 \tag{4.2}$$

$$Y_2 = X_4+X_5 \tag{4.3}$$

$$Y_3 = X_6+X_7+X_8+X_9 \tag{4.4}$$

and the X_i are mutually independent H-function variates for

$i=1,\ldots,9$. If the Y_i $(i=1,2,3)$ were available as H-function variates, Carter's results in Sections 3.7.1 through 3.7.3 could be used. Eq (3.16) for a variate to a power in Section 3.7.3 could be used to find the H-function distribution of $\left(Y_1\right)^3$. Similarly, Eq (3.12) for a product of H-function variates in Section 3.7.1 could be used to find the H-function distribution of $Y_2\ Y_3$. Finally, Eq (3.14) for a quotient of H-function variates in Section 3.7.2 could be used to find the H-function distribution for the density of Z. Even a complicated algebraic combination of independent random variables such as this becomes almost trivial if the Y_i are available as H-function variates.

Cook [1981; Cook and Barnes, 1981] developed a method for finding the p.d.f. and c.d.f. (in tabular form) of an algebraic combination involving products, quotients, powers, and sums of independent H-function variates. Cook also developed a FORTRAN computer program which implements the technique.

Cook's method first uses Carter's results for products, quotients, and powers so that a sum of independent H-function variates remains. The Laplace transform of each term in the sum is obtained, then evaluated and multiplied at corresponding values of the transform variable. This yields a tabular representation for the Laplace transform of the sum. This is then numerically inverted from transform space using Crump's method, yielding a tabular representation of the p.d.f. for the algebraic combination.

Since only a tabular representation of the p.d.f. of the sum is available with Cook's method, it is not possible to use Carter's results at the next more general level as in finding the density of Z in Eq (4.1). Thus, Cook's method will work for certain algebraic combinations involving products, quotients, powers, and sums of independent H-function variates, but not others.

Springer [1979, pp. 250-268] describes an approach due to Carter [1972] to approximate the p.d.f. for the sum or difference of independent variates based on the moments of the sum or difference. Carter [1972] also wrote a FORTRAN computer program to calculate the moments of an algebraic combination of independent H-function variates and approximate the p.d.f. and c.d.f. from these moments. The approximation procedure was developed by Hill [1969] and, if possible, uses either a Gram-Charlier type A series (Hermite polynomial) or a Laguerre polynomial series. If a series approximation is not possible, the first four moments are used to fit a probability distribution from the Pearson family. As Carter [1972] himself notes "... there were many situations in which the methods did not work or in which the approximations were totally unsatisfactory."

Since, at best, only a series approximation of the p.d.f. of the sum or difference is available with this method, it is still not possible to use Carter's results at the next more general level as in finding the density of Z in Eq (4.1). Thus, this method will also work for certain algebraic combinations involving products, quotients, powers, sums, and differences of independent H-function variates, but not others.

4.1. MOMENTS OF THE SUM

A practical method to approximate the density of the sum of independent H-function variates with another H-function distribution is based on the moments of the sum. The moments of the sum can be computed using the moments of the individual variates. Suppose $X_1, X_2, \ldots, X_n$ are mutually independent random variables and

$$Y = \sum_{i=1}^{n} X_i \tag{4.5}$$

The r^{th} moment about the origin of Y is

$$\mu_r = E\left[\, Y^r \,\right] = E\left[\, \left(X_1 + X_2 + \cdots + X_n\right)^r \,\right] \tag{4.6}$$

For the case n=2, the moments of Y can be found in terms of the moments of X_1 and X_2 by using the binomial formula.

$$\mu_r = E\left[\, Y^r \,\right] = E\left[\, \left(X_1+X_2\right)^r \,\right]$$

$$= E\left[\, \sum_{i=0}^{r} \binom{r}{i} \; X_1^{\,r-i} \; X_2^{\,i} \,\right]$$

$$= \sum_{i=0}^{r} \binom{r}{i} E\left[\, X_1^{\,r-i} \; X_2^{\,i} \,\right]$$

$$= \sum_{i=0}^{r} \binom{r}{i} E\left[\, X_1^{\,r-i} \,\right] E\left[\, X_2^{\,i} \,\right] \tag{4.7}$$

since mathematical expectation is a linear operator and X_1 and X_2 are independent. Here, $\binom{r}{i}$ is the binomial coefficient defined as

$$\binom{r}{i} = \frac{r!}{i!\,(r-i)!} \tag{4.8}$$

Eq (4.7) gives the moments of Y in terms of the moments of X_1 and X_2. There are r+1 terms in the sum in Eq (4.7).

For the case n≥2, the moments of Y can be found in terms of the moments of X_1 to X_n by using a generalization of the binomial formula.

$$\mu_r = E\left[\, Y^r \,\right] = E\left[\, \left(X_1+X_2+\cdots+X_n\right)^r \,\right]$$

$$= E\left[\, \sum \frac{r!}{i_1!\, i_2! \cdots i_n!} \; X_1^{\,i_1} \; X_2^{\,i_2} \cdots X_n^{\,i_n} \,\right]$$

$$= \sum \frac{r!}{i_1!\, i_2! \cdots i_n!} E\left[X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n} \right]$$

$$= \sum \frac{r!}{i_1!\, i_2! \cdots i_n!} \prod_{j=1}^{n} E\left[X_j^{i_j} \right] \qquad (4.9)$$

where the sum is over all $i_1, i_2, \ldots, i_n$ such that

$$\sum_{j=1}^{n} i_j = r \qquad (4.10)$$

Here, the multinomial coefficient replaces the binomial coefficient. Eq (4.9) gives the moments of Y in terms of the moments of X_1 through X_n. There are $\binom{r+n-1}{r}$ terms in the sum in Eq (4.9).

4.2. H-FUNCTION PARAMETER ESTIMATES

It is possible to use the moments of an H-function to estimate its parameters. Bodenschatz and Boedigheimer [1983; Boedigheimer et al, 1984] developed and verified an effective and reliable method to estimate the H-function parameters using the method of moments. The technique can be used to curve-fit a mathematical function or to estimate the density of a particular probability distribution. Their FORTRAN computer program will accept known moments, univariate data, ordered pair data from a relative frequency, or ordered pair data directly from the function. Output from the program are the parameters of the H-function whose moments most closely match the given moments.

This technique to estimate the parameters of the H-function from the exact moments of an unknown distribution was also generally described by Jacobs et al [1987]. A more complete derivation is given below, which also shows how, through algebraic manipulation, it is possible to reduce the number of equations by two.

4.2.1. METHOD OF MOMENTS

The method of moments equates an appropriate number of analytic moments of the H-function with corresponding known moments or moments calculated from data. The method uses 2(p+q)+2 moments because there are this many parameters in an H-function distribution. This produces a system of nonlinear equations in the parameters of the H-function.

Eq (4.9) gives the exact moments of a sum of independent variates in terms of the moments of the individual variates. If each random variable in the sum follows an H-function distribution, the exact moments of the individual variates are available by Eq (3.5). Using these moments in Eq (4.9) yields the exact moments of the sum of independent H-function variates.

There are $2(p+q)+2$ unknown parameters of the H-function distribution, specifically (a_i, A_i), $i=1,\ldots,p$, (b_j, B_j), $j=1,\ldots,q$, k, and c. It is necessary, therefore, to create $2(p+q)+2$ equations. If consecutive moments of the sum are used starting with the zeroth moment, the equations are

$$\mu_r = E\left[\, Y^r \,\right] = \frac{k}{c^{r+1}} \quad I(r+1) \tag{4.11}$$

for $r=0,1,\ldots,2(p+q)+1$ where

$$I(r+1) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j+B_j+B_j r\right) \prod_{i=1}^{n} \Gamma\left(1-a_i-A_i-A_i r\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i+A_i+A_i r\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_j-B_j-B_j r\right)} \tag{4.12}$$

It is obvious the equations are nonlinear since they involve gamma functions. Given the exact moments of the sum, μ_r, the system of nonlinear equations as in Eq (4.11) needs to be solved for the H-function parameters.

4.2.2. <u>REDUCING THE SYSTEM OF NONLINEAR EQUATIONS</u>

It is possible to eliminate the parameters k and c from the system of equations through algebraic manipulation. This procedure reduces by two the number of simultaneous nonlinear equations which need to be solved to give the H-function parameter estimates. Each equation of Eq (4.11) can be solved for k, producing

$$k = \frac{\mu_r \ c^{r+1}}{I(r+1)} \tag{4.13}$$

for $r=0,\ldots,2(p+q)+1$, where $I(r+1)$ is given by Eq (4.12). Since all the equations equal k,

$$k = \frac{\mu_0 \ c}{I(1)} = \frac{\mu_1 \ c^2}{I(2)} = \frac{\mu_2 \ c^3}{I(3)} = \cdots$$

$$= \frac{\mu_{2(p+q)+1} \ c^{2(p+q)+2}}{I\left(2(p+q)+2\right)} \tag{4.14}$$

where $I(r+1)$ is given by Eq (4.12). The adjacent equations in Eq (4.14) can be solved for c to give

$$c = \frac{\mu_r \ I(r+2)}{\mu_{r+1} \ I(r+1)} \tag{4.15}$$

for $r=0,\ldots,2(p+q)$, where $I(r+1)$ is given by Eq (4.12). Since all the equations equal c,

$$c = \frac{\mu_0 \; I(2)}{\mu_1 \; I(1)} = \frac{\mu_1 \; I(3)}{\mu_2 \; I(2)} = \cdots$$

$$= \frac{\mu_{2(p+q)} \quad I\big(2(p+q)+2\big)}{\mu_{2(p+q)+1} \quad I\big(2(p+q)+1\big)} \tag{4.16}$$

where I(r+1) is given by Eq (4.12). The adjacent equations in Eq (4.16) can be solved to give the following homogeneous equations:

$$\frac{\mu_i \quad \mu_{i+2} \quad \big(I(i+2)\big)^2}{\big(\mu_{i+1}\big)^2 \quad I(i+1) \quad I(i+3)} - 1 = 0 \tag{4.17}$$

for $i=0,\ldots,2(p+q)-1$, where I(r+1) is given by Eq (4.12). This reduced system of equations involves only the H-function parameters (a_i, A_i), $i=1,\ldots,p$, and (b_j, B_j), $j=1,\ldots,q$.

Once the nonlinear system of equations in Eq (4.17) is solved, c and k can be found by backsubstitution. Using Eq (4.16),

$$c = \frac{\mu_0 \; I(2)}{\mu_1 \; I(1)} \tag{4.18}$$

where I(r+1) is given by Eq (4.12). Using this estimate of c in Eq (4.14) produces

$$k = \frac{\mu_0 \, c}{I(1)} \tag{4.19}$$

where I(r+1) is given by Eq (4.12).

Note that although this algebraic manipulation has reduced the number of equations from 2(p+q)+2 to 2(p+q), all 2(p+q)+2 moments of the sum are still necessary.

4.2.3. SOLVING THE SYSTEM OF NONLINEAR EQUATIONS

Even after reducing the system of nonlinear equations by two, there will always be at least two simultaneous equations involving gamma functions. To complicate matters further, the unknowns (the H-function parameters) appear in the argument of every gamma function. A general analytic solution seems unlikely, if not impossible.

Bodenschatz and Boedigheimer [1983] conducted a literature review of various numerical methods applicable to the problem. Most methods are based on Newton's method, which uses the Jacobian matrix of first partial derivatives to move toward the solution.

Powell developed a quasi-Newton hybrid algorithm which includes the beneficial features of the Levenberg-Marquardt method and implements the calculation-saving strategy of Broyden's procedure. Powell's method was available in an

International Mathematical and Statistical Library (IMSL) routine named ZSPOW. In comparison to available software designed to solve systems of nonlinear equations, ZSPOW had outstanding performance and initial estimates of the parameters had little effect on the algorithm's super linear convergence [Hiebert, 1980].

Bodenschatz and Boedigheimer [1983] used ZSPOW to solve many systems of nonlinear equations as in Eq (4.17) with great success. In their FORTRAN computer program, they allow the user to make initial estimates of the H-function parameters and also provide default guesses which nearly always led to convergence to the correct parameters.

When trying to fit a third- or higher-order H-function, however, the numerical solution of the system of nonlinear equations with ZSPOW is sometimes numerically unstable. Several other numerical methods were tried, but none worked as well as ZSPOW in solving the type of nonlinear equations generated using the method of moments with the H-function. An area of further research is to develop a better way to solve the system of nonlinear equations.

Jacobs et al [1987] discussed using equations as in Eq (4.11) to estimate the parameters of an H-function if the exact moments, μ_r, of the unknown distribution were available.

They also gave analytic solutions to these equations for certain first order H-function distributions with B=1 or $B=\frac{1}{2}$. However, restrictions such as these may not give an H-function with as good a fit to the moments as an unrestricted solution to the equations might allow.

* 4.3. SPECIAL CONSIDERATIONS FOR TYPE VI H-FUNCTION VARIATES

Sums of certain H-function distributions have special properties which allow an analytic solution to the system of nonlinear equations. For example, if $Y = \sum_{i=1}^{n} X_i$ where the X_i are mutually independent random variables which have beta distributions over (0,1), then the p.d.f. of Y will be nonzero only over (0,n). Similarly, if $Y = \sum_{i=1}^{n} X_i$ where the X_i are mutually independent random variables which have Pareto distributions over $(1,\infty)$, then the p.d.f. of Y will be nonzero only over (n,∞). A new way to take advantage of these ranges was discovered which eliminates the need for a numerical solution to the system of nonlinear equations.

Certain simplifications occur if each H-function distribution in the sum is a Type VI convergent H-function according to Table 1 in Section 2.3. Because the sums described above will also have restricted ranges, they must also be Type VI convergent H-functions. Sections 2.6 and 3.6

gave the new H-function representations of these types of functions over more general ranges.

$H_{1\,1}^{1\,0}$ Type VI convergent H-functions have a "B" gamma term in the numerator and an "A" gamma term in the denominator of the definition of the H-function, with A=B. Many $H_{1\,1}^{1\,0}$ Type VI convergent H-functions have A=B=1. Scaling over a range different from (0,1) is achieved through the H-function parameter c. If the range of the sum, Y, is only over (0,n), then in the H-function representation of the p.d.f. of Y, $c=\frac{1}{n}$. Exploiting the unique nature of $H_{1\,1}^{1\,0}$ Type VI convergent H-functions has reduced the number of parameters to estimate from six to three. The three nonlinear equations in the unknown parameters k, b, and a are

$$\mu_0 = \frac{k}{c}\,\frac{\Gamma(b+1)}{\Gamma(a+1)} \tag{4.20}$$

$$\mu_1 = \frac{k}{c^2}\,\frac{\Gamma(b+2)}{\Gamma(a+2)} \tag{4.21}$$

$$\mu_2 = \frac{k}{c^3}\,\frac{\Gamma(b+3)}{\Gamma(a+3)} \tag{4.22}$$

where $c=\frac{1}{n}$, the range of the sum is over (0,n), and μ_0 through μ_2 are the exact moments of the sum from Eq 4.9. These three nonlinear equations allow an analytic solution. The solution is

$$* \quad a = \frac{\mu_0\,\mu_1}{c\left(\mu_0\mu_2 - \left(\mu_1\right)^2\right)} - \frac{\mu_0\,\mu_2}{\mu_0\mu_2 - \left(\mu_1\right)^2} - 1 \tag{4.23}$$

$$* \quad b = c\,\frac{\mu_1}{\mu_0}\,(a+1) \;\; - 1 \tag{4.24}$$

$$* \quad k = \mu_0\; c\,\frac{\Gamma(a+1)}{\Gamma(b+1)} \tag{4.25}$$

where c and μ_0 through μ_2 are as given above. A possible H-function representation for the density of the sum of $H_{1\,1}^{1\,0}$ Type VI convergent H-functions is

$$* \quad f(y) = k\, H_{1\,1}^{1\,0}\left[\, \frac{1}{n}\, y \;:\; (a,1)\;;\;(b,1) \,\right] \tag{4.26}$$

where the range of the sum is over (0,n) and a, b, and k are given by Eq (4.23), Eq (4.24), and Eq (4.25), respectively.

A similar approach applies to sums of $H_{1\,1}^{0\,1}$ Type VI convergent H-functions which have an "A" gamma term in the numerator and a "B" gamma term in the denominator of the definition of the H-function, with A=B. Again, let A=B=1. Scaling over a range different from $(1,\infty)$ is achieved through the H-function parameter c. If the range of the sum, Y, is over (n,∞), then in the H-function representation of the p.d.f. of Y, $c = \frac{1}{n}$. Exploiting the unique nature of $H_{1\,1}^{0\,1}$ Type VI convergent H-functions, the three nonlinear equations in the unknown parameters k, b, and a are

$$\mu_0 = \frac{k}{c} \frac{\Gamma(-a)}{\Gamma(-b)} \tag{4.27}$$

$$\mu_1 = \frac{k}{c^2} \frac{\Gamma(-a-1)}{\Gamma(-b-1)} \tag{4.28}$$

$$\mu_2 = \frac{k}{c^3} \frac{\Gamma(-a-2)}{\Gamma(-b-2)} \tag{4.29}$$

where $c= \frac{1}{n}$, the range of the sum is over (n,∞), and μ_0 through μ_2 are the exact moments of the sum from Eq 4.9. These three nonlinear equations again allow an analytic solution. The solution for the unknown parameters a, b, and k is identical to the ones given by Eq (4.23), Eq (4.24), and Eq (4.25), respectively. A possible H-function representation for the density of the sum of $H_{1\,1}^{0\,1}$ Type VI convergent H-functions is

$$* \qquad f(y) = k\, H_{1\,1}^{0\,1}\left[\frac{1}{n} y : (a,1)\ ;\ (b,1) \right] \tag{4.30}$$

where the range of the sum is over (n,∞) and a, b, and k are given by Eq (4.23), Eq (4.24), and Eq (4.25), respectively.

* 4.4. DEMONSTRATION OF THE TECHNIQUE

A FORTRAN computer program was developed to implement the new technique of finding an H-function distribution which approximates the distribution of the sum of two or more independent H-function variates. The computer program was designed to run interactively, asking questions of the user and expecting a response.

Input to the program can be done interactively or through an input file. A user will input the number of terms (independent random variables) in the sum and the type of each variate. As currently configured, the program will accept up to five terms in the sum. The type of each variate may be one of the named special cases or a general H-function distribution. If the type is a special case, the program will convert the parameters of the special case into the correct H-function parameters for the H-function representation. Alternatively, a user may input the H-function parameters directly.

The program will then compute, using Eq (3.5), the correct moments of each H-function variate in the sum. The program will verify a zeroth moment of unity and query the user if the H-function parameters do not give a valid density. If desired, the program will recompute the constant k to produce a valid H-function distribution. The moments of each term in the sum are used as in Eq (4.9) to compute the exact moments of the sum.

If the sum involves only $H_{1\,1}^{1\,0}$ Type VI H-function distributions or $H_{1\,1}^{0\,1}$ Type VI H-function distributions, the analytic solutions for the H-function parameters derived in Section 4.3 are used. Otherwise, an adapted version of the program of Bodenschatz and Boedigheimer [1983] is called to create and solve, using ZSPOW, the system of nonlinear equations as in Eq (4.17). This produces parameters of the H-function distribution whose moments will closely match the exact moments of the sum.

If there are only two terms in the sum, an attempt can be made to fit an H-function distribution with up to five gamma terms (fifth-order). If there are more than two terms in the sum, the program currently computes only the zeroth through the fifth moments about the origin of the sum. This limits the choice of an H-function distribution for the sum to a first- or second-order H-function.

This is not a serious restriction for two reasons. First- and second-order H-function distributions can represent a wide variety of distributional forms, including nearly every named special case. Secondly, numerical and computational limitations exist in finding the moments of the sum and solving large systems of nonlinear equations with ZSPOW. Until these limitations are addressed, an attempt to fit a third- or higher-order H-function would probably not be successful.

The moments of the new H-function distribution are computed for comparison to the exact moments of the sum. The program will then, if desired, create an input file for another computer program [Cook, 1981; Cook and Barnes, 1981] which will evaluate and plot the resulting p.d.f. and c.d.f.

An output file is always created, giving the H-function parameters and moments of each term in the sum, the exact moments of the sum, the estimated H-function parameters, and the moments of this new H-function distribution.

The FORTRAN source code of the computer program described above is not provided in an appendix. Instead, it is available from Dr. J. Wesley Barnes in the Department of Mechanical Engineering at the University of Texas at Austin. If requested, Dr. Barnes will transfer the program by electronic mail or by floppy disk.

The new FORTRAN computer program which implements the technique currently operates on the instructional VAX cluster at the University of Texas at Austin. This cluster links two VAX 6420 computers and several VAX 11/780 computers and uses the VMS operating system. The VAX FORTRAN compiler was used and the program linked to IMSL for the call to ZSPOW. The program should successfully run on other computers which support ZSPOW in IMSL and which have ANSI FORTRAN capabilities.

The new technique and computer program are demonstrated in the following examples. In all but the final example, it is possible to find the exact distribution of these sums of variates by other methods. Thus, we can measure the effectiveness of the new technique by comparing the resulting H-function distribution to the exact distribution. Of course, the new technique also works with sums for which the solution by other methods is very difficult (or impossible for all practical purposes).

4.4.1. EXAMPLE 1 - SUM OF THREE INDEPENDENT, IDENTICALLY DISTRIBUTED GAMMA VARIATES

Suppose we want to use this technique to find the H-function distribution of the sum of three independent, identically distributed gamma variates with parameters r=2 and λ=2. Using the computer program, we input "3" as the number of terms in the sum, "gamma" as the type of each variate, and the parameters "2" for r and "2" for λ.

The moments of each variate in the sum are computed and combined to give the exact moments of the sum. These are

$\mu_0 =$ 1.0000000 $\qquad$ $\mu_1 =$ 3.0000000

$\mu_2 =$ 10.5000000 $\qquad$ $\mu_3 =$ 42.0000000

$\mu_4 =$ 189.0000000 $\qquad$ $\mu_5 =$ 945.0000000

The program asks whether we want to fit an H-function distribution to the moments of the sum and, if so, the order of

H-function to fit. Since all three variates in the sum are first order H-functions, we try to fit another first order H-function. We enter "1 0 0 1" for m, n, p, and q, respectively. The program uses the zeroth through third moments of the sum to create and solve the system of nonlinear equations, producing the H-function parameter estimates

$b = 4.99999988853$

$B = 0.99999999256$

$k = 0.01666666667$

$c = 2.00000000000$

Allowing for roundoff error, we recognize these as the parameters of the H-function representation of the gamma distribution with $r=6$ and $\lambda=2$. The procedure found the "correct" distribution since the gamma distribution with a common λ has the reproductive property. The program automatically uses these H-function parameters to compute the moments of the new H-function distribution.

$\mu_0 = 1.0000000$ $\mu_1 = 3.0000000$

$\mu_2 = 10.5000000$ $\mu_3 = 42.0000000$

$\mu_4 = 189.0000000$ $\mu_5 = 945.0000000$

The program then asks whether we want to create an input file for another computer program [Cook, 1981; Cook and Barnes, 1981] which will evaluate and plot the resulting p.d.f. and c.d.f. If so, it asks the range and interval of the variable where we want these functions evaluated. We try the range 0.1 to 7.0 with an interval of 0.1. The complete output file is provided in Appendix A.

4.4.2. EXAMPLE 2 - SUM OF TWO INDEPENDENT ERLANG VARIATES WITH DIFFERENT λ

Suppose we want to find an H-function distribution for $Y = X_1+X_2$ where X_1 has an Erlang (gamma) distribution with parameters $r=2$ and $\lambda=4$ and X_2 has an Erlang (gamma) distribution with parameters $r=1$ and $\lambda=2$. Note that X_2 also has an exponential distribution with $\lambda=2$. Using the computer program, we input "2" as the number of terms in the sum, "gamma" as the type of each variate, the parameters r="2" and λ="4" for X_1, and the parameters r="1" and λ="2" for X_2. Of course, we could have input X_2 as type "exponential" with λ="2" or input both random variables as general (unnamed) H-function variates by giving their H-function parameter representations.

The moments of each variate in the sum are computed and combined to give the exact moments of the sum. These are

$\mu_0 = 1.0000000$	$\mu_1 = 1.0000000$
$\mu_2 = 1.3750000$	$\mu_3 = 2.4375000$
$\mu_4 = 5.3437500$	$\mu_5 = 14.0625000$
$\mu_6 = 43.4179688$	$\mu_7 = 154.4238281$

Since both variates in the sum are first order H-functions, we try to fit another first order H-function to the moments of the sum. We enter "1 0 0 1" for m, n, p, and q, respectively. The program uses the zeroth through third moments of the sum to create and solve the system of nonlinear equations, producing the H-function parameter estimates

b = 3.02485587519

B = 1.26389176312

k = 0.74895896608

c = 6.53330240202

The program automatically uses these H-function parameters to compute the moments of the new H-function distribution.

μ_0 = 1.0000000	μ_1 = 1.0000000
μ_2 = 1.3750000	μ_3 = 2.4375000
μ_4 = 5.3382458	μ_5 = 14.0112602
μ_6 = 43.0835662	μ_7 = 152.4885870

We choose to evaluate and plot the resulting p.d.f. and c.d.f. over the range 0.05 to 5.50 with an interval of 0.05. The complete output file is provided in Appendix A.

4.4.3. EXAMPLE 3 - SUM OF TWO INDEPENDENT STANDARD UNIFORM VARIATES

Suppose we want to find an H-function distribution for $Y = X_1+X_2$ where X_1 and X_2 each have a uniform distribution over (0,1). Using the computer program, we input "2" as the number of terms in the sum and "uniform over (0,1)" as the type of each variate.

The moments of each variate in the sum are computed and combined to give the exact moments of the sum. These are

$\mu_0 = 1.0000000$ $\mu_1 = 1.0000000$

$\mu_2 = 1.1666667$ $\mu_3 = 1.5000000$

$\mu_4 = 2.0666667$ $\mu_5 = 3.0000000$

$\mu_6 = 4.5357143$ $\mu_7 = 7.0833333$

$\mu_8 = 11.3555556$ $\mu_9 = 18.6000000$

$\mu_{10} = 31.0151515$

Since both variates in the sum are *Type VI convergent* $H_{1\ 1}^{1\ 0}$ H-functions over (0,1), the program uses the zeroth through second moments of the sum to compute the analytic solution for the H-function parameters. We get the H-function parameter estimates

b = 1.50000000000

B = 1.00000000000

a = 4.00000000000

A = 1.00000000000

k = 9.02703333685

c = 0.50000000000

The program automatically uses these H-function parameters to compute the moments of the new H-function distribution.

$\mu_0 = 1.0000000$	$\mu_1 = 1.0000000$
$\mu_2 = 1.1666667$	$\mu_3 = 1.5000000$
$\mu_4 = 2.0625000$	$\mu_5 = 2.9791667$
$\mu_6 = 4.4687500$	$\mu_7 = 6.9062500$
$\mu_8 = 10.9348958$	$\mu_9 = 17.6640625$
$\mu_{10} = 29.0195312$	

We choose to evaluate and plot the resulting p.d.f. and c.d.f. over the range 0.05 to 2.00 with an interval of 0.05. The complete output file is provided in Appendix A.

4.4.4. EXAMPLE 4 - SUM OF TWO INDEPENDENT, IDENTICALLY DISTRIBUTED BETA VARIATES

Suppose we want to find an H-function distribution for $Y = X_1+X_2$ where X_1 and X_2 are independent variates with beta distributions over (0,1) with parameters $\alpha=1$ and $\beta=2$. Using the computer program, we input "2" as the number of terms in the sum, "beta over (0,1)" as the type of each variate, "1" for α, and "2" for β.

The moments of each variate in the sum are computed and combined to give the exact moments of the sum. These are

$\mu_0 = 1.0000000$	$\mu_1 = 0.6666667$
$\mu_2 = 0.5555556$	$\mu_3 = 0.5333333$
$\mu_4 = 0.5666667$	$\mu_5 = 0.6507937$

$\mu_6 =$ 0.7952381 $\mu_7 =$ 1.0222222

$\mu_8 =$ 1.3703704 $\mu_9 =$ 1.9030303

$\mu_{10} =$ 2.7229437

Since both variates in the sum are Type VI convergent $H_{1\ 1}^{1\ 0}$ H-functions over (0,1), the program uses the zeroth through second moments of the sum to compute the analytic solution for the H-function parameters. We get the H-function parameter estimates

b = 1.33333333333

B = 1.00000000000

a = 6.00000000000

A = 1.00000000000

k = 302.35856086513

c = 0.50000000000

The program automatically uses these H-function parameters to compute the moments of the new H-function distribution.

$\mu_0 =$ 1.0000000 $\mu_1 =$ 0.6666667

$\mu_2 =$ 0.5555556 $\mu_3 =$ 0.5349794

$\mu_4 =$ 0.5706447 $\mu_5 =$ 0.6571060

$\mu_6 =$ 0.8031296 $\mu_7 =$ 1.0296533

$\mu_8 =$ 1.3728711 $\mu_9 =$ 1.8915113

$\mu_{10} =$ 2.6796410

We choose to evaluate and plot the resulting p.d.f. and c.d.f. over the range 0.05 to 2.00 with an interval of 0.05. The complete output file is provided in Appendix A.

4.4.5. EXAMPLE 5 - SUM OF TWO INDEPENDENT VARIATES WITH WEIBULL AND RAYLEIGH DISTRIBUTIONS

A final example demonstrates the technique and computer program when the exact distribution of the sum is very difficult or impossible to obtain.

Suppose we want to find an H-function distribution for $Y = X_1+X_2$ where X_1 has a Weibull distribution with parameters $\beta=5$ and $\lambda=4$ and X_2 has a Rayleigh distribution with parameter $\lambda=3$. Using the computer program, we input "2" as the number of terms in the sum, "Weibull" as the type of variate X_1 with parameters β="5" and λ="4", and "Rayleigh" as the type of variate X_2 with parameter λ="3". Of course, we could have input both random variables as general H-function variates by giving their H-function parameter representations.

The moments of each variate in the sum are computed and combined to give the exact moments of the sum. These are

μ_0 = 1.0000000	μ_1 = 1.2075051
μ_2 = 1.5550061	μ_3 = 2.1228284
μ_4 = 3.0567312	μ_5 = 4.6225066
μ_6 = 7.3134235	μ_7 = 12.0647384
μ_8 = 20.6900000	μ_9 = 36.7858797

μ_{10} = 67.6453356

Since both variates in the sum are first order H-functions, we try to fit another first order H-function to the moments of the sum. We enter "1 0 0 1" for m, n, p, and q, respectively. The program uses the zeroth through third moments of the sum to create and solve the system of nonlinear equations, producing the H-function parameter estimates

b = 5.29104742893

B = 0.62358286635

k = 0.02369785777

c = 2.45993399245

The program automatically uses these H-function parameters to compute the moments of the new H-function distribution.

μ_0 = 1.0000000	μ_1 = 1.2075051
μ_2 = 1.5550061	μ_3 = 2.1228284
μ_4 = 3.0568212	μ_5 = 4.6235079
μ_6 = 7.3191453	μ_7 = 12.0889706
μ_8 = 20.7770136	μ_9 = 37.0686930
μ_{10} = 68.5084603	

We choose to evaluate and plot the resulting p.d.f. and c.d.f. over the range 0.05 to 2.40 with an interval of 0.05. The complete output file is provided in Appendix A.

CHAPTER 5

COMPARING THE ESTIMATED H-FUNCTION TO THE EXACT DISTRIBUTION OF THE SUM

It is natural to want to measure the effectiveness of the newly developed technique which finds an H-function distribution whose moments closely approximate the exact moments of the sum of independent H-function variates. Ideally, the approximate H-function distribution should be compared to the exact distribution for the sum. Unfortunately, the exact distribution of the sum is often difficult to obtain. Because of this, analysts usually resort to computer simulation to analyze the resulting distribution. Even this approach yields only information about the resulting distribution, not the exact density itself.

Four examples included in this thesis were chosen as representative of sums of independent H-function variates where the exact distribution of the sum can be obtained. A FORTRAN computer program was written to compare the approximate H-function distribution to the exact distribution for the sum, using several different measures of merit. The newly developed technique was also effective in finding an H-function distribution whose moments closely approximate the exact moments of the sum even when the exact distribution is very difficult or impossible to obtain.

5.1. FINDING THE EXACT DISTRIBUTION OF THE SUM

As mentioned above, it is often difficult to obtain the exact distribution for the sum of independent random variables. While the moment generating function, Laplace transform, or characteristic function (Fourier transform) of specific densities are sometimes available from tables, the product of these transform functions will often produce a functional form that is not easily inverted.

Suppose $X_1, X_2, \ldots, X_n$ are mutually independent random variables with respective densities $f_1(x_1)$, $f_2(x_2)$, ..., $f_n(x_n)$, each nonzero only for positive values of the variable. Suppose we want the density of $Y = \sum_{i=1}^{n} X_i$. If the Laplace transforms of the densities are $\mathcal{L}_s\left\{f_1(x_1)\right\}$, $\mathcal{L}_s\left\{f_2(x_2)\right\}$, ..., $\mathcal{L}_s\left\{f_n(x_n)\right\}$, respectively, then the density of Y is available as the inversion integral

$$f_Y(y) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{st} \prod_{i=1}^{n} \left[\mathcal{L}_s\left\{ f_i(x_i) \right\} \right] ds \qquad (5.1)$$

The contour integral of a complex variable in Eq (5.1) is usually difficult to evaluate. The approach is similar (and as difficult) if the moment generating function or characteristic function (Fourier transform) is used instead of the Laplace transform.

In certain situations, there are easier ways to find the exact distribution of the sum of independent random variables. These are discussed in the following sections.

5.1.1. CONVOLUTION INTEGRAL

If the product of two transform functions can not be recognized as the transform of a specific function, the inversion may be done with the convolution integral. If X_1 and X_2 are independent random variables with respective densities $f_1(x_1)$ and $f_2(x_2)$, each nonzero only for positive values of the variable, then the density of $Y=X_1+X_2$ is

$$f_Y(y) = \int_0^y f_1(y-w)\ f_2(w)\ dw \qquad (5.2)$$

The convolution integral in Eq (5.2) is often easier to evaluate than Eq (5.1).

The convolution integral was used to find the exact distribution of the sum of selected random variables with uniform, power function, or beta distributions. Only 2 of the 15 exact distributions given below for the sum of two independent variates are used as examples in this thesis. The others are only included for the reader's benefit, perhaps by saving the reader the work of deriving them. Consider the mutually independent random variables $X_1, X_2, \ldots, X_{10}$ with densities

$$f_1(x_1) = \begin{cases} 1 & 0 < x_1 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

$$f_2(x_2) = \begin{cases} 2\,x_2 & 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

$$f_3(x_3) = \begin{cases} 2 - 2x_3 & 0 < x_3 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

$$f_4(x_4) = \begin{cases} 3\left(x_4\right)^2 & 0 < x_4 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

$$f_5(x_5) = \begin{cases} 3\left(1-x_5\right)^2 & 0 < x_5 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

$$f_6(x_6) = \begin{cases} 1 & 0 < x_6 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

$$f_7(x_7) = \begin{cases} 2\,x_7 & 0 < x_7 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

$$f_8(x_8) = \begin{cases} 2 - 2x_8 & 0 < x_8 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.10}$$

$$f_9(x_9) = \begin{cases} 3\left(x_9\right)^2 & 0 < x_9 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.11}$$

$$f_{10}(x_{10}) = \begin{cases} 3\left(1-x_{10}\right)^2 & 0 < x_{10} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.12}$$

If $Y=X_1+X_6$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} y & 0 < y \le 1 \\ 2-y & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.13}$$

If $Y=X_1+X_2$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} y^2 & 0 < y \le 1 \\ 2y-y^2 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

If $Y=X_1+X_3$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} 2y-y^2 & 0 < y \le 1 \\ (y-2)^2 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.15}$$

If $Y=X_2+X_7$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{2}{3} y^3 & 0 < y \le 1 \\ \frac{-2}{3} y^3 + 4y - \frac{8}{3} & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.16}$$

If $Y=X_3+X_8$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{2}{3} y^3 - 4y^2 + 4y & 0 < y \le 1 \\ \frac{-2}{3} (y-2)^3 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

If $Y=X_2+X_3$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{-2}{3} y^3 + 2y^2 & 0 < y \le 1 \\ \frac{2}{3} y^3 - 2y^2 + \frac{8}{3} & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.18}$$

If $Y=X_1+X_4$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} y^3 & 0 < y \le 1 \\ 1 - (y-1)^3 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.19}$$

If $Y=X_1+X_5$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} 1 + (y-1)^3 & 0 < y \le 1 \\ -(y-2)^3 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.20}$$

If $Y=X_2+X_4$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{1}{2} y^4 & 0 < y \le 1 \\ \frac{-1}{2} y^4 + 3y^2 - 2y & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.21}$$

If $Y=X_2+X_5$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{1}{2}y^4 - 2y^3 + 3y^2 & 0 < y \le 1 \\ \frac{-1}{2}y^4 + 2y^3 - 8y + 8 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \qquad (5.22)$$

If $Y=X_3+X_4$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{-1}{2} y^4 + 2y^3 & 0 < y \le 1 \\ \frac{1}{2} y^4 - 2y^3 + 3y^2 - 4y + 4 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \qquad (5.23)$$

If $Y=X_3+X_5$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{-1}{2} y^4 + 4y^3 - 9y^2 + 6y & 0 < y \le 1 \\ \frac{1}{2} (y-2)^4 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \qquad (5.24)$$

If $Y=X_4+X_9$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{3}{10} y^5 & 0 < y \le 1 \\ \frac{-3}{10} y^5 + 6y^2 - 9y + \frac{18}{5} & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \qquad (5.25)$$

If $Y=X_5+X_{10}$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{3}{10} y^5 - 3y^4 + 12y^3 - 18y^2 + 9y & 0 < y \le 1 \\ \frac{-3}{10} (y-2)^5 & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.26}$$

If $Y=X_4+X_5$, then the exact p.d.f. of Y is

$$f(y) = \begin{cases} \frac{3}{10} y^5 - \frac{3}{2} y^4 + 3y^3 & 0 < y \le 1 \\ \frac{-3}{10} y^5 + \frac{3}{2} y^4 - 3y^3 + 6y^2 - 12y + \frac{48}{5} & 1 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.27}$$

These exact distributions could be compared to the corresponding approximate H-function distributions. The exact distribution in Eq (5.13) is compared to the approximate H-function distribution in Section 5.3.3 below. The exact distribution in Eq (5.17) is compared to the approximate H-function distribution in Section 5.3.4 below.

Appendix B contains graphical depictions of the 15 exact distributions in Eq (5.13) through (5.27) above. The graph of the p.d.f. of each random variable in the sum is followed by the graph of the p.d.f. of the sum. Each series of three graphs is referenced to the corresponding equation number.

5.1.2. REPRODUCTIVE DISTRIBUTIONS

As discussed in Section 1.4.1.3, certain distributions have the reproductive property, which considerably simplifies finding the exact distribution of the sum of independent random variables with distributions of these forms. The gamma distribution is an H-function distribution with the reproductive property, provided λ is common among all distributions in the sum.

Suppose we want the density of $Y = \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots, X_n$ are mutually independent gamma distributed random variables with respective parameters (r_1, λ), (r_2, λ), ..., (r_n, λ), and λ is common to all the distributions. It is well known that Y also has a gamma distribution with parameters $r = \sum_{i=1}^{n} r_i$ and λ. Since the exponential distribution is a special case of the gamma distribution with r=1, the sum of n independent, identically distributed exponential random variables has a gamma distribution with parameters r=n and λ. This result is also widely known.

The reproductive property of the gamma distribution was used to find the exact distribution of the sum of three independent, identically distributed gamma variates. This exact distribution is compared to the approximate H-function distribution in Section 5.3.1 below.

* 5.1.3. ERLANG DISTRIBUTIONS WITH DIFFERENT λ

Another new finding permits finding the exact distribution of the sum of independent Erlang distributed random variables, even when λ is different. An Erlang distribution is simply a gamma distribution with an integer shape parameter r.

Suppose we want the density of $Y=X_1+X_2$ where X_1 and X_2 are independent Erlang distributed random variables with respective parameters (r_1,λ_1) and (r_2,λ_2), with $\lambda_1 \neq \lambda_2$. The Laplace transform of the density of Y is the product of the Laplace transforms of the Erlang densities.

$$\mathcal{L}_s\left\{f_Y(y)\right\} = \left[\frac{\lambda_1}{s+\lambda_1}\right]^{r_1} \left[\frac{\lambda_2}{s+\lambda_2}\right]^{r_2}$$

$$= (\lambda_1)^{r_1} (\lambda_2)^{r_2} \left[\frac{1}{s+\lambda_1}\right]^{r_1} \left[\frac{1}{s+\lambda_2}\right]^{r_2} \tag{5.28}$$

We proceed by decomposing the final two terms by partial fractions. The approach presented below is essentially the same as that described by Kleinrock [1975] and is routinely given in texts covering Laplace transforms including Churchill [1972], Widder [1941], Thompson [1960], Smith [1966], Doetsch [1971], Davies [1978], or LePage [1961]. Since r_1 and r_2 are positive integers in the Erlang distribution, we have λ_1 and λ_2 as (possibly repeated) linear factors of the denominator.

$$\left(\frac{1}{s+\lambda_1}\right)^{r_1}\left(\frac{1}{s+\lambda_2}\right)^{r_2} = \sum_{i=1}^{r_1}\frac{A_{r_1-i}}{(s+\lambda_1)^i} + \sum_{i=1}^{r_2}\frac{B_{r_2-i}}{(s+\lambda_2)^i} \tag{5.29}$$

To solve for the constants A_j, we first rewrite Eq (5.29) as

$$\left(\frac{1}{s+\lambda_2}\right)^{r_2} = \sum_{i=1}^{r_1} A_{r_1-i}\,(s+\lambda_1)^{r_1-i} + (s+\lambda_1)^{r_1}\sum_{i=1}^{r_2}\frac{B_{r_2-i}}{(s+\lambda_2)^i} \tag{5.30}$$

We can immediately solve for A_0 by setting $s=-\lambda_1$ in Eq (5.30). Hence,

$$A_0 = \frac{1}{(\lambda_2-\lambda_1)^{r_2}} \tag{5.31}$$

Then by taking successive derivatives of Eq (5.30) with respect to s and evaluating both sides at $s=-\lambda_1$, we have

$$A_j = \frac{1}{j!}\,\frac{d^{(j)}}{ds^{(j)}}\left[\left(\frac{1}{s+\lambda_2}\right)^{r_2}\right]_{s=-\lambda_1} \tag{5.32}$$

$$= \frac{(-1)^j}{j!} \quad \frac{(r_2)(r_2+1)\ldots.(r_2+j-1)}{\left(\lambda_2-\lambda_1\right)^{r_2+j}}$$

$$= (-1)^j \begin{pmatrix} r_2+j-1 \\ j \end{pmatrix} \frac{1}{\left(\lambda_2-\lambda_1\right)^{r_2+j}} \tag{5.33}$$

for $j=1,\ldots,r_1-1$. Note that in solving for each of the A_i, all terms on the right side of Eq (5.30) vanish except for one.

We can take similar steps to solve for the B_j. We get

$$B_0 = \frac{1}{\left(\lambda_1-\lambda_2\right)^{r_1}} \tag{5.34}$$

$$B_j = \frac{1}{j!} \quad \frac{d^{(j)}}{ds^{(j)}} \left[\left(\frac{1}{s+\lambda_1}\right)^{r_1} \right]_{s=-\lambda_2} \tag{5.35}$$

$$= \frac{(-1)^j}{j!} \quad \frac{(r_1)(r_1+1)\ldots.(r_1+j-1)}{\left(\lambda_1-\lambda_2\right)^{r_1+j}}$$

$$= (-1)^j \begin{pmatrix} r_1+j-1 \\ j \end{pmatrix} \frac{1}{\left(\lambda_1-\lambda_2\right)^{r_1+j}} \tag{5.36}$$

for $j=1,\ldots,r_2-1$.

Hence, we can write the Laplace transform of the p.d.f. of Y as

$$\mathcal{L}_s\left\{f_Y(y)\right\} = (\lambda_1)^{r_1} (\lambda_2)^{r_2} \left(\frac{1}{s+\lambda_1}\right)^{r_1} \left(\frac{1}{s+\lambda_2}\right)^{r_2} \tag{5.37}$$

$$= (\lambda_1)^{r_1}(\lambda_2)^{r_2} \left[\sum_{i=1}^{r_1} \frac{A_{r_1-i}}{(s+\lambda_1)^i} + \sum_{i=1}^{r_2} \frac{B_{r_2-i}}{(s+\lambda_2)^i} \right] \tag{5.38}$$

$$= (\lambda_1)^{r_1}(\lambda_2)^{r_2} \left[\sum_{j=0}^{r_1-1} \frac{(-1)^j \binom{r_2+j-1}{j}}{\left(\lambda_2-\lambda_1\right)^{r_2+j} \left(s+\lambda_1\right)^{r_1-j}} + \sum_{j=0}^{r_2-1} \frac{(-1)^j \binom{r_1+j-1}{j}}{\left(\lambda_1-\lambda_2\right)^{r_1+j} \left(s+\lambda_2\right)^{r_2-j}} \right] \tag{5.39}$$

We now use the linearity property of the inverse Laplace transform to invert this transform term by term. We obtain the p.d.f. of Y, $f_Y(y)$.

$$* \; f_Y(y) = (\lambda_1)^{r_1}(\lambda_2)^{r_2} \left[\sum_{j=0}^{r_1-1} \frac{(-1)^j \binom{r_2+j-1}{j}}{(\lambda_2-\lambda_1)^{r_2+j} (r_1-j-1)!} \cdot y^{r_1-j-1} e^{-\lambda_1 y} \right.$$

$$\left. + \sum_{j=0}^{r_2-1} \frac{(-1)^j \binom{r_1+j-1}{j}}{(\lambda_1-\lambda_2)^{r_1+j} (r_2-j-1)!} y^{r_2-j-1} e^{-\lambda_2 y} \right]$$

(5.40)

$$* \qquad = \frac{(\lambda_1)^{r_1}(\lambda_2)^{r_2} e^{-\lambda_1 y}}{(\lambda_2-\lambda_1)^{r_2}} \cdot$$

$$\sum_{j=0}^{r_1-1} \frac{(-1)^j \binom{r_2+j-1}{j}}{(\lambda_2-\lambda_1)^j (r_1-j-1)!} y^{r_1-j-1}$$

$$+\frac{(\lambda_1)^{r_1}(\lambda_2)^{r_2}\ e^{-\lambda_2 y}}{(\lambda_1-\lambda_2)^{r_1}}\cdot$$

$$\sum_{j=0}^{r_2-1}\frac{(-1)^j\binom{r_1+j-1}{j}}{(\lambda_1-\lambda_2)^j\ (r_2-j-1)!}\ y^{r_2-j-1}$$

$$y > 0 \tag{5.41}$$

The result is simply a sum of exponential and gamma-type terms. There are r_1+r_2 terms in the sum defining the p.d.f. of $Y=X_1+X_2$.

The practical method described above can be extended to find the exact distribution of the sum of an arbitrary number of independent Erlang random variables. Suppose we want the distribution of $Y=\sum_{i=1}^{n} X_i$ where $X_1,\ldots,X_n$ are independent Erlang distributed random variables with parameters r_i and λ_i, $i=1,\ldots,n$. Also assume $\lambda_i \neq \lambda_j$ for $i \neq j$. If any scale parameters, λ, were common among the random variables in the sum, they could be immediately combined into another Erlang random variable using the reproductive property.

We again decompose the product of Laplace transforms of the individual p.d.f.'s by partial fractions.

$$\mathcal{L}_s\left\{f_Y(y)\right\} = \prod_{i=1}^{n}(\lambda_i)^{r_i} \prod_{i=1}^{n}\left(\frac{1}{s+\lambda_i}\right)^{r_i} \tag{5.42}$$

$$= \prod_{i=1}^{n}(\lambda_i)^{r_i} \left[\sum_{i=1}^{r_1} \frac{A_{1,r_1-i}}{(s+\lambda_1)^i} + \sum_{i=1}^{r_2} \frac{A_{2,r_2-i}}{(s+\lambda_2)^i} + \cdots + \sum_{i=1}^{r_n} \frac{A_{n,r_n-i}}{(s+\lambda_n)^i} \right] \tag{5.43}$$

$$= \prod_{i=1}^{n}(\lambda_i)^{r_i} \sum_{k=1}^{n} \left[\sum_{i=1}^{r_k} \frac{A_{k,r_k-i}}{(s+\lambda_k)^i} \right] \tag{5.44}$$

$$= \prod_{i=1}^{n}(\lambda_i)^{r_i} \sum_{k=1}^{n} \left[\sum_{m=0}^{r_k-1} \frac{A_{k,r_k-m-1}}{(s+\lambda_k)^{m+1}} \right] \tag{5.45}$$

The constants $A_{k,j}$ can be found as

$$A_{k,j} = \frac{1}{j!} \frac{d^{(j)}}{ds^{(j)}} \left[(s+\lambda_k)^{r_k} \prod_{i=1}^{n}\left(\frac{1}{s+\lambda_i}\right)^{r_i} \right]_{s=-\lambda_k} \tag{5.46}$$

for $j=0,1,\ldots,r_k-1$ and $k=1,2,\ldots,n$.

We again use the linearity property of the inverse Laplace transform to invert this transform term by term. We obtain the p.d.f. of $Y=\sum_{i=1}^{n} X_i$, $f_Y(y)$.

$$* \quad f_Y(y) = \prod_{i=1}^{n} (\lambda_i)^{r_i} \sum_{k=1}^{n} \left[\sum_{m=0}^{r_k-1} A_{k,r_k-m-1} \frac{y^m}{m!} e^{-\lambda_k y} \right] \qquad y > 0 \tag{5.47}$$

where $A_{k,j}$ are as given in Eq (5.46) for $j=0,1,\ldots,r_k-1$ and $k=1,2,\ldots,n$.

Eq (5.47) gives the exact p.d.f. for the sum of an arbitrary number of independent Erlang random variables with different scale parameters λ_i. Note that there are $\sum_{k=1}^{n} r_k$ terms in the sum and each term has either an exponential or gamma-type form.

This new result was used to find the exact distribution of the sum of two independent Erlang variates. This exact distribution is compared to the approximate H-function distribution in Section 5.3.2 below.

5.2. MEASURES OF MERIT

There is no universally accepted measure to determine how "close" an approximation is to the exact distribution. It is likely that any standard statistical test for goodness of fit (e.g. Chi-Square or Kolomogorov-Smirnov) would fail to reject the null hypothesis that the approximate H-function distribution and the exact distribution were equal. However, this is because these tests are not very powerful or discriminatory, not necessarily because the distributions are nearly identical.

One way to compare the approximate H-function distribution to the exact distribution involves comparing the corresponding moments. Assuming ZSPOW successfully "solved" the system of nonlinear equations, there should at least be a perfect match of the moments used. For sums of variates where an analytic solution for the H-function parameters was possible, at least three moments should match perfectly. Higher order moments of the approximate H-function distribution may be in error to some degree.

Although there is no standard measure of the "closeness" of two distributions, there are several measures that are commonly used. These are described below and computed for the examples listed. The measures were included in a FORTRAN computer program which compares the approximate H-function distribution to the exact distribution of the sum.

These measures, when considered collectively, give some idea of how well the approximate H-function distribution matches the exact distribution of the sum of independent H-function variates. Of course, the measures can only be computed when it is possible to find the exact distribution of the sum. In many cases, the exact distribution of the sum of independent H-function variates is very difficult to obtain.

In the descriptions that follow, let the density of the exact distribution of the sum of independent variates be represented by f(y), let the density of the approximate H-function distribution be H(y), and let each density be evaluated at n equally spaced values of the variable. The common interval between consecutive y values is Δy. The densities should be evaluated over the whole range of values the variable is likely to assume.

5.2.1. ESTIMATED SUMS OF SQUARES OF ERROR

The estimated sums of squares of error (SSE) is obtained by adding the squared difference between f(y) and H(y) for all n values of the variable where the functions were evaluated.

$$\text{Estimated SSE} = \sum_{i=1}^{n} \left[f\left(y_i\right) - H\left(y_i\right) \right]^2 \qquad (5.48)$$

5.2.2. ESTIMATED MEAN SQUARED ERROR

Unless f(y) and H(y) are identical, the estimated SSE will increase if n is increased. This limitation of the estimated SSE is corrected by dividing by n, yielding the estimated mean squared error (MSE).

$$\text{Estimated MSE} = \frac{\text{Estimated SSE}}{n} \qquad (5.49)$$

5.2.3. MAXIMUM ABSOLUTE DIFFERENCE

The maximum absolute difference (MAD) measures the maximum vertical distance of the two densities over the evaluated y values.

$$\text{MAD} = \max_{i=1,\ldots,n} \left| f\left(y_i\right) - H\left(y_i\right) \right| \qquad (5.50)$$

5.2.4. INTEGRATED ABSOLUTE DENSITY DIFFERENCE

The integrated absolute density difference (IADD) is a measure of the positive area between the two densities. If $f(y)=0$ for all $y \le 0$,

$$\text{IADD} = \int_0^{\infty} \left| f(y) - H(y) \right| \, dy \qquad (5.51)$$

Since the area under a valid p.d.f. is unity, $0 < \text{IADD} < 2$.

IADD can be estimated by summing the positive area of rectangles (or trapezoids) with width Δy and height $\left| f(y_i) - H(y_i) \right|$. The computer program uses the larger of the vertical distances of the densities at the left and right endpoints of each interval as the height of each rectangle.

$$\text{Estimated IADD} = \sum_{i=1}^{n} \left[\Delta y \ \ \max(A,B) \right] \tag{5.52}$$

where

$$A = \left| f(y_{i-1}) - H(y_{i-1}) \right| \tag{5.53}$$

$$B = \left| f(y_i) - H(y_i) \right| \tag{5.54}$$

and $\left| f(y_0) - H(y_0) \right| = 0$. Therefore, the estimated IADD will be an upper bound for the IADD.

* 5.3. DEMONSTRATED RESULTS

As shown below, the new technique was successful in finding an H-function distribution which closely approximates the exact distribution of the sum of independent H-function variates.

5.3.1. EXAMPLE 1 - SUM OF THREE INDEPENDENT, IDENTICALLY DISTRIBUTED GAMMA VARIATES

In this example, all moments of the approximate H-function distribution were identical to those of the exact distribution of the sum. This occurred whenever the reproductive property was applicable. In these cases, the new technique always found the H-function distribution which represented the exact distribution. The measures of merit were

Estimated SSE = 0.0000000004

Estimated MSE = 0.0000000000

MAD = 0.0000133056

Estimated IADD = 0.0000050304

As expected, all measures of merit show a very close fit. Even the small amount of error is probably due to computer roundoff error in the evaluation of the H-function distribution by the sum of residues. If the approximate H-function distribution were graphically compared to the exact distribution, the two graphs would be indistinguishable.

5.3.2. EXAMPLE 2 - SUM OF TWO INDEPENDENT ERLANG VARIATES WITH DIFFERENT λ

Since the parameter λ was different for the Erlang variates in this example, the reproductive property of the gamma or Erlang distribution did not apply. Only the zeroth through third moments were identical between the approximate H-function distribution and the exact distribution. The measures of merit in this example were

Estimated SSE = 0.0006853443

Estimated MSE = 0.0000062304

MAD = 0.0100130852

Estimated IADD = 0.0065133019

All measures of merit show a close fit of the approximate H-function distribution to the exact distribution. The approximate H-function distribution deviated the most from the exact distribution near y=0.0, achieving its maximum absolute deviation of 0.01 at y=0.1. In the more critical right tail, all absolute residual values were less than 0.0006.

The two distributions are compared graphically in Figure 3 below. The densities are nearly identical.

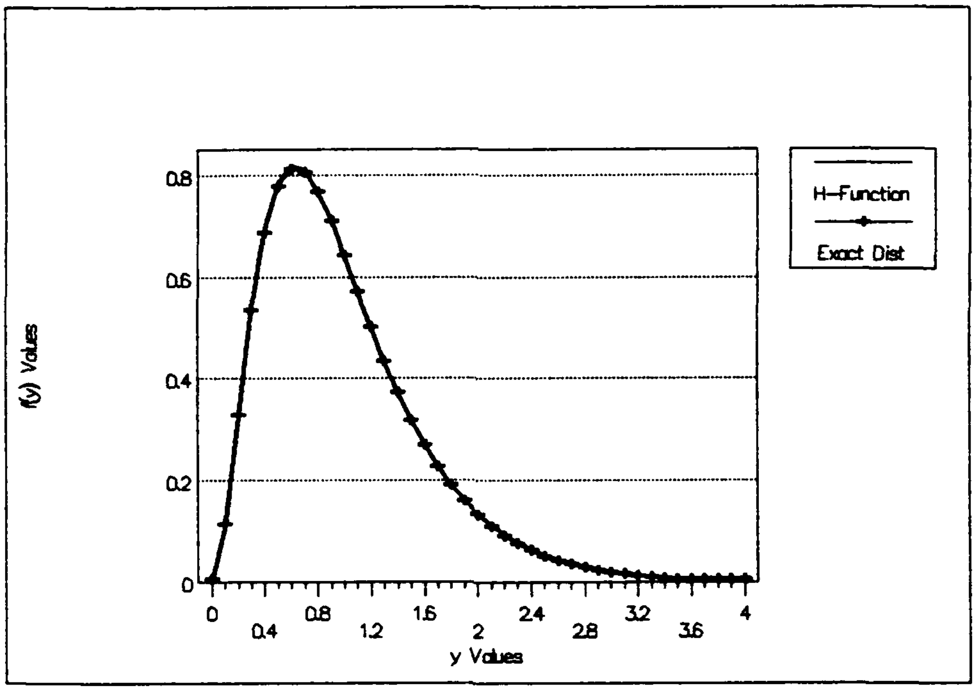


Figure 3. Graphical Comparison of the H-Function and the Exact Distribution of Example 2

5.3.3. EXAMPLE 3 - SUM OF TWO INDEPENDENT STANDARD UNIFORM VARIATES

This is one common example where the new H-function technique did not work very well. It is well known that the sum of two independent standard uniform variates has the triangular distribution as in Eq (5.13). There are two distinct functional forms, one for $y \in (0,1]$ and one for $y \in [1,2)$. The derivative of the exact distribution is not continuous at y=1. In general, the new H-function technique was less effective when the derivative of the exact distribution was not continuous.

Of course, either linear function in Eq (5.13) could be exactly represented as an H-function. With a shift in the argument of one H-function, it is even possible to perfectly represent the exact distribution of the sum as a sum of H-functions. But in the new technique, we try to fit a single H-function to the moments of the sum. This produces an H-function variate which (correctly) cannot assume values less than zero or greater than two. Even the moments of the approximate H-function distribution were *reasonably close* to those of the exact distribution. Still, the measures of merit were

Estimated SSE = 0.0920133337

Estimated MSE = 0.0023003333

MAD = 0.1511936368

Estimated IADD = 0.0752034552

The approximate H-function distribution was not as effective in representing the exact distribution as in the earlier examples. The approximate H-function distribution achieved its maximum absolute deviation of 0.15 at y=1.0. It more closely approximated the exact distribution in both tails. The two distributions are compared graphically in Figure 4 below.

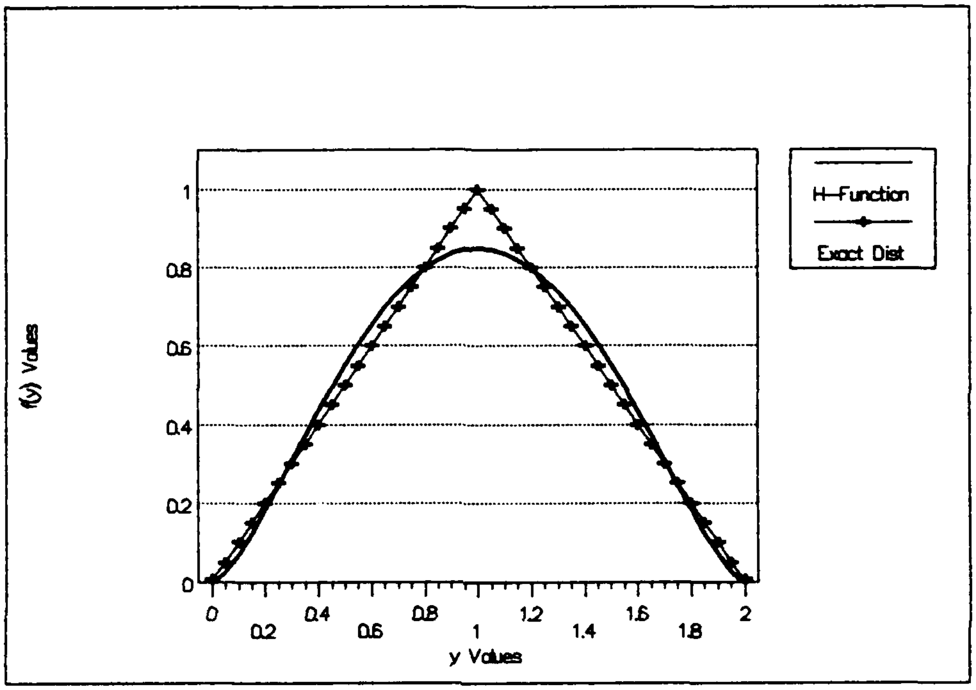


Figure 4. Graphical Comparison of the H-Function and the Exact Distribution of Example 3

Attempts to fit a higher-order H-function to the moments of the sum were generally unsuccessful. Numerical instability when ZSPOW tried to solve the system of nonlinear equations led to execution errors, either floating overflow or division by zero.

It is interesting, though, that when a first-order H-function is fit to the moments of the sum, a slightly better fit is obtained, at least over the range of the variable from zero to two. The measures of merit were

Estimated SSE = 0.0620352924

Estimated MSE = 0.0015508823

MAD = 0.1107462928

Estimated IADD = 0.0581953790

The drawback of this slightly better fit is that this H-function distribution is positive for all $y>0$. At $y=2.0$, the first-order H-function distribution has $f(2)=.0348$ and $F(2)=.9966$. Therefore, the probability of achieving a value greater than two with the first-order H-function distribution is .0034. Of course, this event is *impossible* and the probability should be zero. Most of the measures given above would increase (worsen) if they were computed over a larger range of the variable since the exact distribution has $f(y)=0$ for $y>2$ but the first-order H-function distribution has $f(y)>0$ for $y>2$.

5.3.4. EXAMPLE 4 - SUM OF TWO INDEPENDENT, IDENTICALLY DISTRIBUTED BETA VARIATES

This example demonstrates the technique for the sum of two Type VI convergent $H_{1\ 1}^{1\ 0}$ variates over (0,1) where the first derivative of the exact distribution is continuous at $y=1$. The exact distribution is given by Eq (5.17). Again, the moments of the approximate H-function distribution were reasonably close to those of the exact distribution. The measures of merit were

Estimated SSE = 0.0358601689

Estimated MSE = 0.0008965042

MAD = 0.0579816425

Estimated IADD = 0.0442126253

The approximate H-function distribution more closely matched the exact distribution than in the previous example, but not as close as in the gamma and Erlang examples. While the two functions which represent the exact distribution have a continuous first derivative at $y=1$, the second derivative is not continuous there. The approximate H-function distribution has continuous derivatives of all orders for $y \in (0,2)$. The two distributions are compared graphically in Figure 5 below.

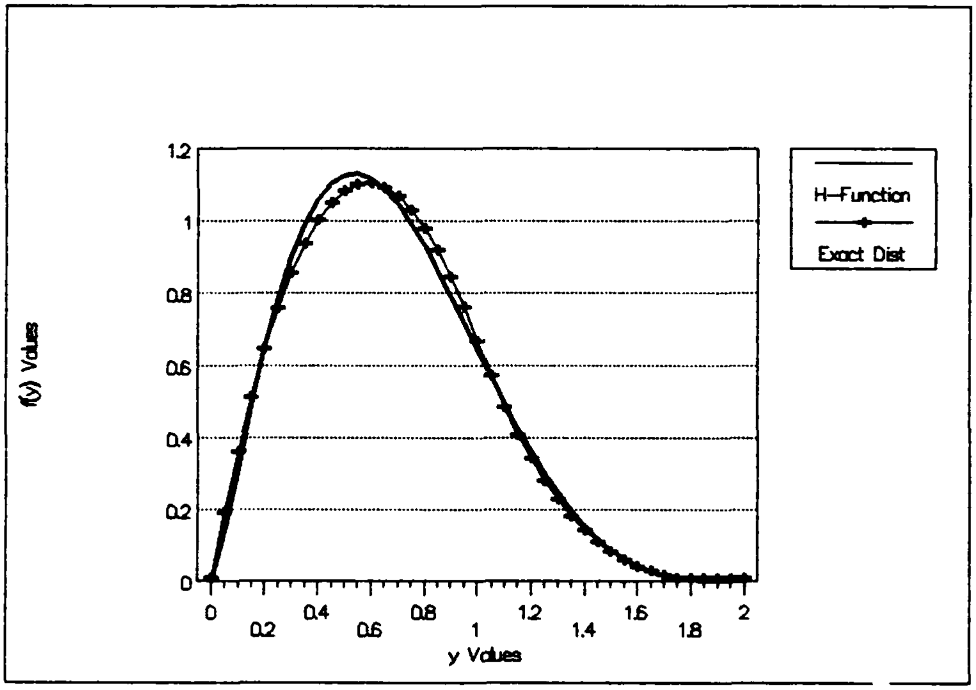


Figure 5. Graphical Comparison of the H-Function and the Exact Distribution of Example 4

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

This thesis has presented a practical method to find an H-function distribution for the sum of two or more independent H-function variates. The method finds the moments about the origin of each variate in the sum and uses these to find the moments of the sum. The parameters of an H-function are then estimated using the method of moments. This produces an H-function distribution whose moments closely approximate those of the sum of independent H-function variates.

The new technique is especially useful because the H-function can exactly represent nearly every named statistical distribution of positive random variables. By rewriting the distributions of the random variables in the sum in their H-function representations, an analyst can find the sum of two or more independent variates with practically any distributional form.

Further, simple formulas exist which immediately give the distribution, as an H-function, of the random variable defined as the product, quotient, or power of independent H-function variates. The new technique for the sum allows an analyst to exploit these powerful H-function results to find the H-function distribution of more complicated algebraic combinations of independent random variables.

When the exact distribution of an algebraic combination of independent random variables is difficult or impossible to obtain, an analyst may resort to computer simulation to analyze its distributional properties. The method presented here, when combined with the other H-function results, give an H-function distribution for the algebraic combination, possibly precluding the need to rely on computer simulation.

A FORTRAN computer program which implements the new technique is demonstrated through five examples. Of course, the program was also tested on many other examples. When it was possible to find the exact distribution for the sum of independent random variables, the approximate H-function distribution was compared to the exact distribution.

In cases where the exact distribution of the sum is widely known (e.g. reproductive distributions), the method always found the correct H-function representation for the sum. In other cases, only the moments used to estimate the H-function parameters agreed with those of the sum. Other, higher-order moments were in error to some degree. The method was less effective in finding an H-function distribution which closely approximated the exact distribution of the sum of independent variates over a restricted range. These sums produce an exact distribution with two functional forms over distinct ranges of the variable and do not have continuous derivatives of all orders at $y=1$.

The exact distribution of the general sum of independent Erlang variates with different scale parameters, λ, was derived in Section 5.1.3. An Erlang variate is simply a gamma variate with an integer shape parameter r. The derivation used partial fractions to decompose the product of Laplace transforms of the individual densities. This produced a sum of terms, each of which could easily be inverted from transform space, yielding the desired density of the sum of independent variates.

Throughout the thesis, a number of other new results relating to the H-function were given. A previously unstated restriction on the variable in the H-function representations of power functions and beta-type functions was highlighted. Several ways of overcoming this limitation when representing mathematical functions were presented. These include scaling the variable, allowing a slightly different function to be represented for a range of the variable, or the use of complex parameters.

The restriction, however, is an advantage when representing certain statistical distributions. For random variables with a restricted range of the variable, the H-function representation eliminates the need to specify the range for which the density is nonzero. The H-function distribution exactly represents the desired density over the appropriate range and is zero elsewhere.

Through scaling the variable of these H-functions with the parameter c, the value where the H-function changes from representing the special case to taking the value zero can be changed. This allowed the H-function to represent the power functions and beta-type functions over a more general range and also gave the H-function representation of the Unit Step Function and its complement. The same technique was used to give the H-function representation of the Three-Parameter Beta p.d.f., the Power Function p.d.f. over (0,M), the Uniform p.d.f. over (0,M), and the Pareto p.d.f. over (ε,∞).

Analytic solutions were derived for the system of nonlinear equations generated by the method of moments to estimate the H-function parameters for certain distributions. These distributions have restricted ranges of the variable and must be represented by Type VI convergent H-functions.

Many new H-function representations of other mathematical functions were also given, including those of the complementary error function, erfc x, the complementary incomplete gamma function, $\Gamma(\alpha,x)$, the complementary incomplete beta function, $\overline{B_x(\alpha,\beta)}$, and the inverse hyperbolic cotangent function, arccoth z.

Four new theorems show when and how a generalizing constant may be present in an H-function representation. Many generalized H-function representations are given, including those of every cumulative distribution function of an H-function variate. The generalizing constant was also possible in the H-function representations of power functions, the error function and its complement, the incomplete gamma function and its complement, the incomplete beta function and its complement, many inverse trigonometric and hyperbolic functions, and the logarithmic functions.

Sufficient convergence conditions were developed for the alternate definition of the H-function. These show where the H-function may be evaluated by the sum of LHP residues or the negative sum of RHP residues, without first changing the form of the alternate definition of the H-function to that of the primary definition.

The hierarchical structure among classes of H-functions was given through seven new theorems. Every class of H-functions is wholly contained in many higher-order classes of H-functions through the application of the duplication, triplication, and multiplication formulas for the gamma function. In the other direction, the hierarchical structure gives new reduction properties that are less obvious than the known reduction properties.

Several new results were obtained for powers of first-order H-function variates. The square of a random variable with a Rayleigh distribution with parameter λ has an exponential distribution with parameter λ. If X has a Weibull distribution with parameters β and λ, then $Y=X^P$ where $P>0$ has a Weibull distribution with parameters $\frac{\beta}{P}$ and λ. The square of a random variable with a half-normal distribution with parameter σ has a gamma distribution with parameters $r = \frac{1}{2}$ and $\lambda = \frac{1}{2\sigma^2}$. The square of a random variable with a Maxwell distribution with parameter θ has a gamma distribution with parameters $r = \frac{3}{2}$ and $\lambda = \frac{1}{\theta^2}$. The square of a random variable with a Maxwell distribution with parameter $\theta = \sqrt{2}$ has a Chi-Square distribution with parameter $\nu=3$. The square of a random variable with a Rayleigh distribution with parameter $\lambda = \frac{1}{2}$ has a Chi-Square distribution with parameter $\nu=2$.

Many areas of further research proposed by Eldred [1979, pp. 258-9], Cook [1981, pp. 145-147], Kellogg [1984, pp.170-171], and Jacobs [1986, pp. 149-152] remain unfinished. Readers interested in H-function research should consider those recommendations and some additional areas that follow.

In Bayesian statistics, the parameters of a distribution are considered to be random variables with a particular distribution. After sampling from the distribution, the information gained about the parameters is reflected by updating the prior distribution and creating a posterior distribution. To my knowledge, a Bayesian approach has not been attempted with the H-function.

Numerical methods exist for certain commonly-used transforms such as the Laplace and Fourier transforms. If these methods are sufficiently accurate and will work in both directions (finding a transform and inverting a transform), it would be possible to find the distribution (in tabular form) of sums or differences of independent random variables. If a numerical method to invert a Mellin transform exists, it could be used to evaluate the H-function.

The computer program by Cook [1981, Cook and Barnes, 1981] will evaluate an H-function by the sum of residues for certain values of the real variable, x, when given the parameters of the H-function. The program's capabilities should be extended to allow complex parameters of the H-function and to evaluate the H-function at values of the complex variable, z.

The numerical solution of the system of nonlinear equations with ZSPOW is sometimes numerically unstable. Several other numerical methods were tried, but none worked as well as ZSPOW in solving the type of nonlinear equations generated using the method of moments with the H-function. Further research might develop a better way to solve the system of nonlinear equations.

While the H-function has been shown to exactly represent many mathematical functions and statistical distributions, there are a few functions and distributions which have not yet been shown to be H-functions. Conversely, no one has proven that these functions cannot be represented as H-functions. Examples include the gamma function, certain trigonometric and hyperbolic functions, and the Lognormal and Logistic densities. A related issue is to prove or disprove that the product of two H-functions is another H-function.

To my knowledge, the H-function has not been applied to the study of complex-valued random variables. The H-function seems the appropriate tool to analyze random variables which take values in the complex plane.

The relationship between the H-function and the infinite summability of integrals described in Section 2.7 needs to be more extensively studied. Is there a particular summability scheme the H-function always follows to obtain the moments of the functions representable as H-functions?

Theorems similar to those in Section 2.4.6 could be developed to show upgrade and reduction results for H-functions by applying the triplication and multiplication formulas for the gamma function. Then, the search could begin for a class of H-functions which includes most of the known special cases. For example, Figure 1 showed the $H_{1\,1}^{1\,0}$ class includes many common statistical distributions, including all those in the $H_{0\,1}^{1\,0}$ class. The half-Cauchy, half-Student, and F distributions, however, are in the $H_{1\,1}^{1\,1}$ class of H-function distributions. Using combinations of theorems based on the duplication, triplication, or multiplication formulas, is there a class of H-functions which includes both the $H_{1\,1}^{1\,0}$ and $H_{1\,1}^{1\,1}$ classes of H-functions?

The H-function continues to be an extremely rewarding area of research for me. I am convinced that we have much to discover about its ability to simplify many difficult problems in mathematics, probability, and statistics. Those who pursue H-function research should find it both challenging and rewarding.

APPENDIX A

OUTPUT FROM COMPUTER PROGRAMS

Listed below are the complete output files from the FORTRAN computer programs for each of the five examples in Section 4.4. The output from the program which finds a H-function distribution for the sum of independent H-function variates is followed by the output from a different program [Cook, 1981; Cook and Barnes, 1981] which evaluates the p.d.f. and c.d.f. of the resulting H-function distribution. For the first four examples for which it was possible to find the exact distribution, the output from the program which compares the approximate H-function distribution to the exact distribution is also presented.

EXAMPLE 1 - SUM OF THREE INDEPENDENT, IDENTICALLY DISTRIBUTED GAMMA VARIATES

```
PROGRAM SUMVAR RUN IN DOUBLE PRECISION
FOR THE SUM OF 3 INDEPENDENT H-FUNCTION VARIATES
INPUT PARAMETERS FOR VARIABLE NUMBER 1 OF THE SUM:
   M, N, P, Q  =   1  0  0  1
   K, C  =     2.0000000   2.0000000
   b, B  =       1.00000      1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.0000000
```

```
      THE SECOND MOMENT IS          1.5000000
      THE THIRD  MOMENT IS          3.0000000
      THE FOURTH MOMENT IS          7.5000000
      THE FIFTH  MOMENT IS         22.5000000

INPUT PARAMETERS FOR VARIABLE NUMBER 2 OF THE SUM:
   M, N, P, Q =   1  0  0  1
   K, C  =    2.0000000   2.0000000
   b, B  =      1.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.0000000
      THE SECOND MOMENT IS          1.5000000
      THE THIRD  MOMENT IS          3.0000000
      THE FOURTH MOMENT IS          7.5000000
      THE FIFTH  MOMENT IS         22.5000000

INPUT PARAMETERS FOR VARIABLE NUMBER 3 OF THE SUM:
   M, N, P, Q =   1  0  0  1
   K, C  =    2.0000000   2.0000000
   b, B  =      1.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.0000000
      THE SECOND MOMENT IS          1.5000000
      THE THIRD  MOMENT IS          3.0000000
      THE FOURTH MOMENT IS          7.5000000
      THE FIFTH  MOMENT IS         22.5000000

FOR THE RANDOM VARIABLE GIVEN AS THE SUM
OF THE ABOVE INDEPENDENT H-FUNCTION
VARIATES, THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          3.0000000
      THE SECOND MOMENT IS         10.5000000
      THE THIRD  MOMENT IS         42.0000000
      THE FOURTH MOMENT IS        189.0000000
      THE FIFTH  MOMENT IS        945.0000000

PROGRAM H_FIT RUN WITH DOUBLE PRECISION
INPUT WAS OF TYPE0
DEFAULT INITIAL GUESS WAS USED
THE ZEROTH MOMENT WAS USED IN THE FIT
RESULTS OF ZSPOW -
NUMERATOR:
  SMALLB(1)=     4.99999988533181114
```

```
  BIGB(1)=       0.99999999256084261 3
 DENOMINATOR:
 VALUES OF K & C ARE:
   K=      0.016666666666666667
   C=      2.000000000000000000
   FNORM=      0.000000000000000000

 ESTIMATED H-FUNCTION PARAMETERS
 FOR THE SUM OF THE INDEPENDENT
 H-FUNCTION VARIATES ARE:
    M, N, P, Q  =    1  0  0  1
    K, C  =      0.0166667    2.0000000
    b, B  =       5.00000      1.00000
 THE MOMENTS ABOUT THE ORIGIN OF THIS
 H-FUNCTION ARE:
       THE ZEROTH MOMENT IS          1.0000000
       THE FIRST  MOMENT IS          3.0000000
       THE SECOND MOMENT IS         10.5000000
       THE THIRD  MOMENT IS         42.0000000
       THE FOURTH MOMENT IS        189.0000000
       THE FIFTH  MOMENT IS        945.0000000

DETERMINE P.D.F.(Z) AND C.D.F.(Z)
FOR VALUES OF Z FROM   0.1000 TO   7.0000
WITH STEP SIZE   0.1000
FOR THE SUM OF  1 TERMS, WHERE
THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.
CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,
PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =   1.3653
FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):
Z = Y1
VARIATE X 1 IS TYPE NUMBER  4
INPUT PARAMETERS ARE THETA =    0.01667, PHI =   2.00000
AND POWER =   1.00000
THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:
              1  0
 0.01667  H          (   2.00000 X):
              0  1                       (   5.000,   1.000)
THE P.D.F. FOR TERM  1 OF THE SUM IS GIVEN BY:
              1  0
 0.01667  H          (   2.00000 Z), WHERE
              0  1
(BA(I),GBA(I)): (   5.000,   1.000) (
 CONVERGENCE TYPE = 1
     D =  1.00     E = -1.00     L =  4.50     R = 1.0000
```

Z	PDF(Z)	CDF(Z)
0.1000	0.000004	0.000000
0.2000	0.000114	0.000004
0.3000	0.000711	0.000039
0.4000	0.002454	0.000184
0.5000	0.006131	0.000594
0.6000	0.012491	0.001500
0.7000	0.022104	0.003201
0.8000	0.035284	0.006040
0.9000	0.052057	0.010378
1.0000	0.072179	0.016564
1.1000	0.095173	0.024910
1.2000	0.120392	0.035673
1.3000	0.147079	0.049037
1.4000	0.174427	0.065110
1.5000	0.201638	0.083918
1.6000	0.227959	0.105408
1.7000	0.252721	0.129458
1.8000	0.275360	0.155881
1.9000	0.295425	0.184444
2.0000	0.312587	0.214870
2.1000	0.326632	0.246857
2.2000	0.337455	0.280088
2.3000	0.345051	0.314240
2.4000	0.349495	0.348994
2.5000	0.350935	0.384039
2.6000	0.349570	0.419087
2.7000	0.345643	0.453868
2.8000	0.339422	0.488139
2.9000	0.331193	0.521685
3.0000	0.321246	0.554320
3.1000	0.309871	0.585887
3.2000	0.297347	0.616256
3.3000	0.283939	0.645327
3.4000	0.269892	0.673023
3.5000	0.255433	0.699292
3.6000	0.240764	0.724102
3.7000	0.226062	0.747443
3.8000	0.211484	0.769319
3.9000	0.197163	0.789749
4.0000	0.183207	0.808764
4.1000	0.169708	0.826405
4.2000	0.156737	0.842723
4.3000	0.144347	0.857772
4.4000	0.132578	0.871613
4.5000	0.121454	0.884309

4.6000	0.110989	0.895926
4.7000	0.101186	0.906529
4.8000	0.092040	0.916185
4.9000	0.083540	0.924959
5.0000	0.075667	0.932914
5.1000	0.068398	0.940112
5.2000	0.061710	0.946613
5.3000	0.055572	0.952472
5.4000	0.049956	0.957745
5.5000	0.044830	0.962480
5.6000	0.040164	0.966726
5.7000	0.035926	0.970527
5.8000	0.032086	0.973925
5.9000	0.028614	0.976957
6.0000	0.025481	0.979659
6.1000	0.022659	0.982063
6.2000	0.020123	0.984200
6.3000	0.017847	0.986096
6.4000	0.015808	0.987777
6.5000	0.013985	0.989265
6.6000	0.012357	0.990581
6.7000	0.010905	0.991742
6.8000	0.009612	0.992767
6.9000	0.008462	0.993669
7.0000	0.007440	0.994464

NUMBER OF POLES EVALUATED = 83

PROGRAM COMPAR RUN IN DOUBLE PRECISION

X	Y (H-FCN)	Y (EVAL)	RESIDUAL
0.1000	0.0000043666	0.0000043666	0.0000000000
0.2000	0.0001144013	0.0001144013	0.0000000000
0.3000	0.0007112599	0.0007112599	0.0000000000
0.4000	0.0024539353	0.0024539352	0.0000000001
0.5000	0.0061313241	0.0061313240	0.0000000001
0.6000	0.0124911263	0.0124911264	-0.0000000001
0.7000	0.0221042938	0.0221042943	-0.0000000005
0.8000	0.0352839727	0.0352839739	-0.0000000012
0.9000	0.0520572460	0.0520572483	-0.0000000023
1.0000	0.0721788138	0.0721788177	-0.0000000039
1.1000	0.0951731110	0.0951731171	-0.0000000061
1.2000	0.1203921499	0.1203921587	-0.0000000088
1.3000	0.1470787062	0.1470787183	-0.0000000121
1.4000	0.1744272435	0.1744272593	-0.0000000158
1.5000	0.2016376070	0.2016376269	-0.0000000199
1.6000	0.2279587426	0.2279587669	-0.0000000243
1.7000	0.2527214162	0.2527214452	-0.0000000290

1.8000	0.2753601348	0.2753601684	-0.0000000336
1.9000	0.2954252728	0.2954253111	-0.0000000383
2.0000	0.3125868609	0.3125869037	-0.0000000428
2.1000	0.3266316878	0.3266317349	-0.0000000471
2.2000	0.3374553719	0.3374554229	-0.0000000510
2.3000	0.3450509461	0.3450510007	-0.0000000546
2.4000	0.3494953096	0.3494953673	-0.0000000577
2.5000	0.3509346792	0.3509347395	-0.0000000603
2.6000	0.3495699434	0.3495700059	-0.0000000625
2.7000	0.3456426020	0.3456426662	-0.0000000642
2.8000	0.3394217767	0.3394218420	-0.0000000653
2.9000	0.3311926087	0.3311926746	-0.0000000659
3.0000	0.3212462161	0.3212462821	-0.0000000660
3.1000	0.3098712736	0.3098713394	-0.0000000658
3.2000	0.2973471912	0.2973472562	-0.0000000650
3.3000	0.2839388042	0.2839388681	-0.0000000639
3.4000	0.2698924499	0.2698925124	-0.0000000625
3.5000	0.2554332757	0.2554333366	-0.0000000609
3.6000	0.2407636152	0.2407636741	-0.0000000589
3.7000	0.2260622642	0.2260623210	-0.0000000568
3.8000	0.2114844971	0.2114845515	-0.0000000544
3.9000	0.1971626696	0.1971627215	-0.0000000519
4.0000	0.1832072739	0.1832073232	-0.0000000493
4.1000	0.1697083224	0.1697083691	-0.0000000467
4.2000	0.1567369566	0.1567370007	-0.0000000441
4.3000	0.1443471907	0.1443472324	-0.0000000417
4.4000	0.1325777187	0.1325777577	-0.0000000390
4.5000	0.1214537217	0.1214537587	-0.0000000370
4.6000	0.1109886370	0.1109886714	-0.0000000344
4.7000	0.1011858411	0.1011858741	-0.0000000330
4.8000	0.0920402391	0.0920402710	-0.0000000319
4.9000	0.0835397256	0.0835397566	-0.0000000310
5.0000	0.0756665181	0.0756665496	-0.0000000315
5.1000	0.0683983578	0.0683983922	-0.0000000344
5.2000	0.0617095716	0.0617096146	-0.0000000430
5.3000	0.0555720188	0.0555720692	-0.0000000504
5.4000	0.0499558733	0.0499559385	-0.0000000652
5.5000	0.0448303394	0.0448304269	-0.0000000875
5.6000	0.0401642223	0.0401643429	-0.0000001206
5.7000	0.0359264174	0.0359265830	-0.0000001656
5.8000	0.0320862857	0.0320865265	-0.0000002408
5.9000	0.0286140192	0.0286143510	-0.0000003318
6.0000	0.0254807806	0.0254812775	-0.0000004969
6.1000	0.0226590473	0.0226597557	-0.0000007084
6.2000	0.0201226143	0.0201235965	-0.0000009822
6.3000	0.0178466675	0.0178480594	-0.0000013919

6.4000	0.0158078894	0.0158099039	-0.0000020145
6.5000	0.0139846566	0.0139874081	-0.0000027515
6.6000	0.0123566022	0.0123603635	-0.0000037613
6.7000	0.0109048623	0.0109100486	-0.0000051863
6.8000	0.0096119998	0.0096191863	-0.0000071865
6.9000	0.0084623041	0.0084718905	-0.0000095864
7.0000	0.0074402961	0.0074536017	-0.0000133056

```
ESTIMATED SSE IS          0.0000000004
ESTIMATED MSE IS          0.0000000000
THE MAXIMUM ABSOLUTE DIFFERENCE BETWEEN THE
H-FUNCTION AND THE ACTUAL FUNCTION IS              0.0000133056
THE INTEGRATED ABSOLUTE DENSITY DIFFERENCE
(AN ESTIMATE OF THE AREA BETWEEN THE
H-FUNCTION AND THE ACTUAL DENSITY) IS              0.0000050304
```

<u>EXAMPLE 2 - SUM OF TWO INDEPENDENT ERLANG VARIATES</u>

<u>WITH DIFFERENT λ</u>

```
PROGRAM SUMVAR RUN IN DOUBLE PRECISION
FOR THE SUM OF 2 INDEPENDENT H-FUNCTION VARIATES
INPUT PARAMETERS FOR VARIABLE NUMBER 1 OF THE SUM:
   M, N, P, Q =   1  0  0  1
   K, C  =    4.0000000   4.0000000
   b, B  =      1.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          0.5000000
      THE SECOND MOMENT IS          0.3750000
      THE THIRD  MOMENT IS          0.3750000
      THE FOURTH MOMENT IS          0.4687500
      THE FIFTH  MOMENT IS          0.7031250
      THE  NEXT  MOMENT IS          1.2304688
      THE  NEXT  MOMENT IS          2.4609375

INPUT PARAMETERS FOR VARIABLE NUMBER 2 OF THE SUM:
   M, N, P, Q =   1  0  0  1
   K, C  =    2.0000000   2.0000000
   b, B  =      0.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          0.5000000
      THE SECOND MOMENT IS          0.5000000
      THE THIRD  MOMENT IS          0.7500000
```

```
      THE FOURTH MOMENT IS          1.5000000
      THE FIFTH  MOMENT IS          3.7500000
      THE  NEXT  MOMENT IS         11.2500000
      THE  NEXT  MOMENT IS         39.3750000

FOR THE RANDOM VARIABLE GIVEN AS THE SUM
OF THE ABOVE INDEPENDENT H-FUNCTION
VARIATES, THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.0000000
      THE SECOND MOMENT IS          1.3750000
      THE THIRD  MOMENT IS          2.4375000
      THE FOURTH MOMENT IS          5.3437500
      THE FIFTH  MOMENT IS         14.0625000
      THE  NEXT  MOMENT IS         43.4179688
      THE  NEXT  MOMENT IS        154.4238281

PROGRAM H_FIT RUN WITH DOUBLE PRECISION
INPUT WAS OF TYPE0
DEFAULT INITIAL GUESS WAS USED
THE ZEROTH MOMENT WAS USED IN THE FIT
RESULTS OF ZSPOW -
NUMERATOR:
  SMALLB(1)=     3.024855875185841148
  BIGB(1)=       1.263891763119380202
DENOMINATOR:
VALUES OF K & C ARE:
  K=      0.748958966084013356
  C=      6.533302402015585986
  FNORM=      0.000000000000000000

ESTIMATED H-FUNCTION PARAMETERS
FOR THE SUM OF THE INDEPENDENT
H-FUNCTION VARIATES ARE:
   M, N, P, Q  =   1  0  0  1
   K, C  =    0.7489590   6.5333024
   b, B  =       3.02486     1.26389
THE MOMENTS ABOUT THE ORIGIN OF THIS
H-FUNCTION ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.0000000
      THE SECOND MOMENT IS          1.3750000
      THE THIRD  MOMENT IS          2.4375000
      THE FOURTH MOMENT IS          5.3382458
      THE FIFTH  MOMENT IS         14.0112602
      THE  NEXT  MOMENT IS         43.0835662
```

```
      THE  NEXT  MOMENT IS       152.4885870

DETERMINE P.D.F.(Z) AND C.D.F.(Z)
FOR VALUES OF Z FROM   0.0500 TO   5.5000
WITH STEP SIZE   0.0500
FOR THE SUM OF  1 TERMS, WHERE
THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.
CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,
PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =   1.7376
FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):
Z = Y1
VARIATE X 1 IS TYPE NUMBER  4
INPUT PARAMETERS ARE THETA =    0.74896, PHI =   6.53330
AND POWER =   1.00000
THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:
              1  0
 0.74896  H          (   6.53330 X):
              0  1                      (   3.025,    1.264)
THE P.D.F. FOR TERM  1 OF THE SUM IS GIVEN BY:
              1  0
 0.74896  H          (   6.53330 Z), WHERE
              0  1
(BA(I),GBA(I)): (   3.025,    1.264) (
 CONVERGENCE TYPE = 1
     D =  1.26     E = -1.26     L =  2.52     R = 0.7438
```

Z	PDF(Z)	CDF(Z)
0.0500	0.026956	0.000430
0.1000	0.104760	0.003557
0.1500	0.211027	0.011377
0.2000	0.326668	0.024812
0.2500	0.438949	0.043985
0.3000	0.540207	0.068521
0.3500	0.626386	0.097754
0.4000	0.695859	0.130880
0.4500	0.748580	0.167060
0.5000	0.785467	0.205474
0.5500	0.807989	0.245367
0.6000	0.817884	0.286063
0.6500	0.816969	0.326975
0.7000	0.807023	0.367609
0.7500	0.789713	0.407555
0.8000	0.766559	0.446483
0.8500	0.738910	0.484136
0.9000	0.707945	0.520319
0.9500	0.674675	0.554892
1.0000	0.639955	0.587762

1.0500	0.604498	0.618875
1.1000	0.568889	0.648210
1.1500	0.533601	0.675770
1.2000	0.499009	0.701581
1.2500	0.465403	0.725687
1.3000	0.433003	0.748142
1.3500	0.401967	0.769010
1.4000	0.372404	0.788363
1.4500	0.344381	0.806276
1.5000	0.317931	0.822827
1.5500	0.293060	0.838096
1.6000	0.269752	0.852159
1.6500	0.247973	0.865096
1.7000	0.227679	0.876981
1.7500	0.208814	0.887888
1.8000	0.191315	0.897886
1.8500	0.175116	0.907041
1.9000	0.160147	0.915418
1.9500	0.146338	0.923075
2.0000	0.133618	0.930070
2.0500	0.121917	0.936454
2.1000	0.111168	0.942277
2.1500	0.101305	0.947585
2.2000	0.092263	0.952421
2.2500	0.083984	0.956824
2.3000	0.076410	0.960831
2.3500	0.069486	0.964476
2.4000	0.063163	0.967790
2.4500	0.057391	0.970802
2.5000	0.052127	0.973538
2.5500	0.047330	0.976022
2.6000	0.042959	0.978278
2.6500	0.038980	0.980325
2.7000	0.035359	0.982182
2.7500	0.032066	0.983866
2.8000	0.029072	0.985393
2.8500	0.026351	0.986778
2.9000	0.023880	0.988033
2.9500	0.021636	0.989170
3.0000	0.019599	0.990200
3.0500	0.017751	0.991133
3.1000	0.016074	0.991978
3.1500	0.014553	0.992743
3.2000	0.013174	0.993435
3.2500	0.011924	0.994062
3.3000	0.010792	0.994630

3.3500	0.009766	0.995143
3.4000	0.008836	0.995608
3.4500	0.007994	0.996028
3.5000	0.007232	0.996409
3.5500	0.006541	0.996753
3.6000	0.005917	0.997064
3.6500	0.005351	0.997345
3.7000	0.004839	0.997600
3.7500	0.004376	0.997830
3.8000	0.003957	0.998038
3.8500	0.003578	0.998226
3.9000	0.003235	0.998397
3.9500	0.002925	0.998550
4.0000	0.002644	0.998690
4.0500	0.002391	0.998815
4.1000	0.002161	0.998929
4.1500	0.001954	0.999032
4.2000	0.001766	0.999125
4.2500	0.001597	0.999209
4.3000	0.001444	0.999285
4.3500	0.001305	0.999353
4.4000	0.001180	0.999415
4.4500	0.001067	0.999472
4.5000	0.000964	0.999522
4.5500	0.000872	0.999568
4.6000	0.000789	0.999610
4.6500	0.000713	0.999647
4.7000	0.000645	0.999681
4.7500	0.000584	0.999712
4.8000	0.000528	0.999740
4.8500	0.000478	0.999765
4.9000	0.000433	0.999788
4.9500	0.000392	0.999808
5.0000	0.000356	0.999827
5.0500	0.000323	0.999844
5.1000	0.000292	0.999859
5.1500	0.000266	0.999873
5.2000	0.000242	0.999886
5.2500	0.000221	0.999897
5.3000	0.000200	0.999908
5.3500	0.000185	0.999918
5.4000	0.000170	0.999927
5.4500	0.000157	0.999935
5.5000	0.000145	0.999942

NUMBER OF POLES EVALUATED = 89

PROGRAM COMPAR RUN IN DOUBLE PRECISION

X	Y (H-FCN)	Y (EVAL)	RESIDUAL
0.0500	0.0269560422	0.0338687172	-0.0069126750
0.1000	0.1047604975	0.1147735827	-0.0100130852
0.1500	0.2110272898	0.2189047501	-0.0078774603
0.2000	0.3266679624	0.3300759702	-0.0034080078
0.2500	0.4389488040	0.4376919836	0.0012568204
0.3000	0.5402072867	0.5352071763	0.0050001104
0.3500	0.6263855876	0.6189637207	0.0074218669
0.4000	0.6958594647	0.6873218538	0.0085376109
0.4500	0.7485799015	0.7400141770	0.0085657245
0.5000	0.7854665809	0.7776709976	0.0077955833
0.5500	0.8079891806	0.8014756091	0.0065135715
0.6000	0.8178843945	0.8129177174	0.0049666771
0.6500	0.8169693788	0.8136205051	0.0033488737
0.7000	0.8070228765	0.8052225091	0.0018003674
0.7500	0.7897133043	0.7892999138	0.0004133905
0.8000	0.7665590050	0.7673183012	-0.0007592962
0.8500	0.7389101583	0.7406055613	-0.0016954030
0.9000	0.7079449386	0.7103397230	-0.0023947844
0.9500	0.6746747268	0.6775470467	-0.0028723199
1.0000	0.6399547674	0.6431069326	-0.0031521652
1.0500	0.6044977968	0.6077611209	-0.0032633241
1.1000	0.5688889715	0.5721253654	-0.0032363939
1.1500	0.5336009937	0.5367022861	-0.0031012924
1.2000	0.4990087333	0.5018945066	-0.0028857733
1.2500	0.4654029237	0.4680174730	-0.0026145493
1.3000	0.4330027002	0.4353115704	-0.0023088702
1.3500	0.4019668816	0.4039533060	-0.0019864244
1.4000	0.3724039823	0.3740654440	-0.0016614617
1.4500	0.3443809949	0.3457260524	-0.0013450575
1.5000	0.3179310207	0.3189764773	-0.0010454566
1.5500	0.2930598403	0.2938282943	-0.0007684540
1.6000	0.2697515271	0.2702693074	-0.0005177803
1.6500	0.2479732076	0.2482686787	-0.0002954711
1.7000	0.2276790667	0.2277812745	-0.0001022078
1.7500	0.2088136949	0.2087513166	0.0000623783
1.8000	0.1913148623	0.1911154218	0.0001994405
1.8500	0.1751157994	0.1748051079	0.0003106915
1.9000	0.1601470554	0.1597488398	0.0003982156
1.9500	0.1463379965	0.1458736804	0.0004643161
2.0000	0.1336179992	0.1331066060	0.0005113932
2.0500	0.1219173875	0.1213755376	0.0005418499
2.1000	0.1111681551	0.1106101339	0.0005580212
2.1500	0.1013045089	0.1007423864	0.0005621225
2.2000	0.0922632656	0.0917070504	0.0005562152

2.2500	0.0839841259	0.0834419409	0.0005421850
2.3000	0.0764098507	0.0758881208	0.0005217299
2.3500	0.0694863578	0.0689899994	0.0004963584
2.4000	0.0631627543	0.0626953630	0.0004673913
2.4500	0.0573913195	0.0569553491	0.0004359704
2.5000	0.0521274483	0.0517243794	0.0004030689
2.5500	0.0473295640	0.0469600610	0.0003695030
2.6000	0.0429590129	0.0426230642	0.0003359487
2.6500	0.0389799364	0.0386769844	0.0003029520
2.7000	0.0353591407	0.0350881930	0.0002709477
2.7500	0.0320659524	0.0318256831	0.0002402693
2.8000	0.0290720785	0.0288609122	0.0002111663
2.8500	0.0263514586	0.0261676457	0.0001838129
2.9000	0.0238801228	0.0237218028	0.0001583200
2.9500	0.0216360554	0.0215013070	0.0001347484
3.0000	0.0195990566	0.0194859415	0.0001131151
3.0500	0.0177506130	0.0176572119	0.0000934011
3.1000	0.0160737745	0.0159982144	0.0000755601
3.1500	0.0145530391	0.0144935125	0.0000595266
3.2000	0.0131742345	0.0131290204	0.0000452141
3.2500	0.0119244243	0.0118918938	0.0000325305
3.3000	0.0107918007	0.0107704277	0.0000213730
3.3500	0.0097656028	0.0097539623	0.0000116405
3.4000	0.0088360115	0.0088327943	0.0000032172
3.4500	0.0079940810	0.0079980955	-0.0000040145
3.5000	0.0072316960	0.0072418379	-0.0000101419
3.5500	0.0065414505	0.0065567237	-0.0000152732
3.6000	0.0059166029	0.0059361217	-0.0000195188
3.6500	0.0053510603	0.0053740084	-0.0000229481
3.7000	0.0048392420	0.0048649142	-0.0000256722
3.7500	0.0043761116	0.0044038736	-0.0000277620
3.8000	0.0039570862	0.0039863804	-0.0000292942
3.8500	0.0035779964	0.0036083458	-0.0000303494
3.9000	0.0032350765	0.0032660609	-0.0000309844
3.9500	0.0029249240	0.0029561618	-0.0000312378
4.0000	0.0026444213	0.0026755985	-0.0000311772
4.0500	0.0023907019	0.0024216056	-0.0000309037
4.1000	0.0021613239	0.0021916766	-0.0000303527
4.1500	0.0019539329	0.0019835396	-0.0000296067
4.2000	0.0017664002	0.0017951361	-0.0000287359
4.2500	0.0015968516	0.0016246006	-0.0000277490
4.3000	0.0014436486	0.0014702432	-0.0000265946
4.3500	0.0013051134	0.0013305330	-0.0000254196
4.4000	0.0011798299	0.0012040833	-0.0000242534
4.4500	0.0010667957	0.0010896381	-0.0000228424
4.5000	0.0009643209	0.0009860600	-0.0000217391

4.5500	0.0008721016	0.0008923190	-0.0000202174
4.6000	0.0007885237	0.0008074822	-0.0000189585
4.6500	0.0007132715	0.0007307051	-0.0000174336
4.7000	0.0006450051	0.0006612232	-0.0000162181
4.7500	0.0005836256	0.0005983440	-0.0000147184
4.8000	0.0005283933	0.0005414409	-0.0000130476
4.8500	0.0004781645	0.0004899465	-0.0000117820
4.9000	0.0004330106	0.0004433471	-0.0000103365
4.9500	0.0003921517	0.0004011779	-0.0000090262
5.0000	0.0003557943	0.0003630181	-0.0000072238
5.0500	0.0003226795	0.0003284866	-0.0000058071
5.1000	0.0002919937	0.0002972388	-0.0000052451
5.1500	0.0002664966	0.0002689625	-0.0000024659
5.2000	0.0002422554	0.0002433754	-0.0000011200
5.2500	0.0002211998	0.0002202218	0.0000009780
5.3000	0.0002000859	0.0001992705	0.0000008154
5.3500	0.0001845803	0.0001803119	0.0000042684
5.4000	0.0001702609	0.0001631567	0.0000071042
5.4500	0.0001565074	0.0001476334	0.0000088740
5.5000	0.0001445091	0.0001335868	0.0000109223

```
ESTIMATED SSE IS            0.0006853443
ESTIMATED MSE IS            0.0000062304
THE MAXIMUM ABSOLUTE DIFFERENCE BETWEEN THE
H-FUNCTION AND THE ACTUAL FUNCTION IS             0.0100130852
THE INTEGRATED ABSOLUTE DENSITY DIFFERENCE
(AN ESTIMATE OF THE AREA BETWEEN THE
H-FUNCTION AND THE ACTUAL DENSITY) IS             0.0065133019
```

EXAMPLE 3 - SUM OF TWO INDEPENDENT STANDARD UNIFORM VARIATES

```
PROGRAM SUMVAR RUN IN DOUBLE PRECISION
FOR THE SUM OF 2 INDEPENDENT H-FUNCTION VARIATES
INPUT PARAMETERS FOR VARIABLE NUMBER 1 OF THE SUM:
   M, N, P, Q =   1  0  1  1
   K, C  =    1.0000000   1.0000000
   a, A  =      1.00000     1.00000
   b, B  =      0.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          0.5000000
      THE SECOND MOMENT IS          0.3333333
      THE THIRD  MOMENT IS          0.2500000
```

```
        THE FOURTH MOMENT IS          0.2000000
        THE FIFTH  MOMENT IS          0.1666667
        THE  NEXT  MOMENT IS          0.1428571
        THE  NEXT  MOMENT IS          0.1250000
        THE  NEXT  MOMENT IS          0.1111111
        THE  NEXT  MOMENT IS          0.1000000
        THE  NEXT  MOMENT IS          0.0909091

INPUT PARAMETERS FOR VARIABLE NUMBER 2 OF THE SUM:
   M, N, P, Q  =    1  0  1  1
   K, C  =    1.0000000   1.0000000
   a, A  =      1.00000     1.00000
   b, B  =      0.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
        THE ZEROTH MOMENT IS          1.0000000
        THE FIRST  MOMENT IS          0.5000000
        THE SECOND MOMENT IS          0.3333333
        THE THIRD  MOMENT IS          0.2500000
        THE FOURTH MOMENT IS          0.2000000
        THE FIFTH  MOMENT IS          0.1666667
        THE  NEXT  MOMENT IS          0.1428571
        THE  NEXT  MOMENT IS          0.1250000
        THE  NEXT  MOMENT IS          0.1111111
        THE  NEXT  MOMENT IS          0.1000000
        THE  NEXT  MOMENT IS          0.0909091

FOR THE RANDOM VARIABLE GIVEN AS THE SUM
OF THE ABOVE INDEPENDENT H-FUNCTION
VARIATES, THE MOMENTS ABOUT THE ORIGIN ARE:
        THE ZEROTH MOMENT IS          1.0000000
        THE FIRST  MOMENT IS          1.0000000
        THE SECOND MOMENT IS          1.1666667
        THE THIRD  MOMENT IS          1.5000000
        THE FOURTH MOMENT IS          2.0666667
        THE FIFTH  MOMENT IS          3.0000000
        THE  NEXT  MOMENT IS          4.5357143
        THE  NEXT  MOMENT IS          7.0833333
        THE  NEXT  MOMENT IS         11.3555556
        THE  NEXT  MOMENT IS         18.6000000
        THE  NEXT  MOMENT IS         31.0151515

PROGRAM H_FIT RUN WITH DOUBLE PRECISION
INPUT WAS OF TYPE0
THE ZEROTH MOMENT WAS USED IN THE FIT
RESULTS OF ANALYTIC SOLUTION -
NUMERATOR:
```

```
  SMALLB(1)=       1.499999999999999667
  BIGB(1)=         1.000000000000000000
DENOMINATOR:
  SMALLA(1)=       3.999999999999999334
  BIGA(1)=         1.000000000000000000
VALUES OF K & C ARE:
  K=      9.027033336853110823
  C=      0.500000000000000000

ESTIMATED H-FUNCTION PARAMETERS
FOR THE SUM OF THE INDEPENDENT
H-FUNCTION VARIATES ARE:
   M, N, P, Q  =    1  0  1  1
   K, C  =      9.0270333   0.5000000
   a, A  =        4.00000     1.00000
   b, B  =        1.50000     1.00000
THE MOMENTS ABOUT THE ORIGIN OF THIS
H-FUNCTION ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.0000000
      THE SECOND MOMENT IS          1.1666667
      THE THIRD  MOMENT IS          1.5000000
      THE FOURTH MOMENT IS          2.0625000
      THE FIFTH  MOMENT IS          2.9791667
      THE  NEXT  MOMENT IS          4.4687500
      THE  NEXT  MOMENT IS          6.9062500
      THE  NEXT  MOMENT IS         10.9348958
      THE  NEXT  MOMENT IS         17.6640625
      THE  NEXT  MOMENT IS         29.0195312

DETERMINE P.D.F.(Z) AND C.D.F.(Z)
FOR VALUES OF Z FROM   0.0500 TO   2.0000
WITH STEP SIZE   0.0500
FOR THE SUM OF  1 TERMS, WHERE
THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.
CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,
PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =   4.7785
FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):
Z = Y1
VARIATE X 1 IS TYPE NUMBER  4
INPUT PARAMETERS ARE THETA =    9.02703, PHI =   0.50000
AND POWER =   1.00000
THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:
              1  0                   (    4.000,    1.000)
 9.02703  H         (    0.50000 X):
              1  1                   (    1.500,    1.000)
```

THE P.D.F. FOR TERM 1 OF THE SUM IS GIVEN BY:

```
                    1 0
 9.02703  H             (    0.50000 Z), WHERE
                    1 1
(BA(I),GBA(I)): (   1.500,   1.000) (
(CD(I),GCD(I)): (   4.000,   1.000) (
```

CONVERGENCE TYPE = 7

D = 0.00 E = 0.00 L = -2.50 R = 1.0000

Z	PDF(Z)	CDF(Z)
0.0500	0.025842	0.000523
0.1000	0.070299	0.002876
0.1500	0.124083	0.007706
0.2000	0.183346	0.015375
0.2500	0.245633	0.026091
0.3000	0.309153	0.039958
0.3500	0.372518	0.057003
0.4000	0.434599	0.077189
0.4500	0.494465	0.100426
0.5000	0.551329	0.126585
0.5500	0.604526	0.155498
0.6000	0.653488	0.186967
0.6500	0.697733	0.220768
0.7000	0.736853	0.256655
0.7500	0.770506	0.294362
0.8000	0.798410	0.333610
0.8500	0.820340	0.374104
0.9000	0.836126	0.415541
0.9500	0.845645	0.457612
1.0000	0.848826	0.500000
1.0500	0.845645	0.542388
1.1000	0.836126	0.584459
1.1500	0.820340	0.625896
1.2000	0.798410	0.666390
1.2500	0.770506	0.705638
1.3000	0.736853	0.743345
1.3500	0.697733	0.779232
1.4000	0.653488	0.813033
1.4500	0.604526	0.844502
1.5000	0.551329	0.873415
1.5500	0.494465	0.899574
1.6000	0.434599	0.922811
1.6500	0.372518	0.942997
1.7000	0.309153	0.960042
1.7500	0.245633	0.973909
1.8000	0.183346	0.984625
1.8500	0.124083	0.992294

1.9000	0.070297	0.997124
1.9500	0.025793	0.999477
2.0000	-0.001952	0.999977

NUMBER OF POLES EVALUATED = 100

PROGRAM COMPAR RUN IN DOUBLE PRECISION

X	Y (H-FCN)	Y (EVAL)	RESIDUAL
0.0500	0.0258419798	0.0500000000	-0.0241580202
0.1000	0.0702990184	0.1000000000	-0.0297009816
0.1500	0.1240833016	0.1500000000	-0.0259166984
0.2000	0.1833464944	0.2000000000	-0.0166535056
0.2500	0.2456325663	0.2500000000	-0.0043674337
0.3000	0.3091534688	0.3000000000	0.0091534688
0.3500	0.3725176396	0.3500000000	0.0225176396
0.4000	0.4345990979	0.4000000000	0.0345990979
0.4500	0.4944645689	0.4500000000	0.0444645689
0.5000	0.5513288954	0.5000000000	0.0513288954
0.5500	0.6045258790	0.5500000000	0.0545258790
0.6000	0.6534882583	0.6000000000	0.0534882583
0.6500	0.6977334553	0.6500000000	0.0477334553
0.7000	0.7368531561	0.7000000000	0.0368531561
0.7500	0.7705055551	0.7500000000	0.0205055551
0.8000	0.7984095245	0.8000000000	-0.0015904755
0.8500	0.8203402272	0.8500000000	-0.0296597728
0.9000	0.8361258520	0.9000000000	-0.0638741480
0.9500	0.8456452546	0.9500000000	-0.1043547454
1.0000	0.8488263632	1.0000000000	-0.1511736368
1.0500	0.8456452546	0.9500000000	-0.1043547454
1.1000	0.8361258520	0.9000000000	-0.0638741480
1.1500	0.8203402272	0.8500000000	-0.0296597728
1.2000	0.7984095245	0.8000000000	-0.0015904755
1.2500	0.7705055551	0.7500000000	0.0205055551
1.3000	0.7368531561	0.7000000000	0.0368531561
1.3500	0.6977334553	0.6500000000	0.0477334553
1.4000	0.6534882583	0.6000000000	0.0534882583
1.4500	0.6045258790	0.5500000000	0.0545258790
1.5000	0.5513288954	0.5000000000	0.0513288954
1.5500	0.4944645689	0.4500000000	0.0444645689
1.6000	0.4345990979	0.4000000000	0.0345990979
1.6500	0.3725176396	0.3500000000	0.0225176396
1.7000	0.3091534687	0.3000000000	0.0091534687
1.7500	0.2456325660	0.2500000000	-0.0043674340
1.8000	0.1833464889	0.2000000000	-0.0166535111
1.8500	0.1240831901	0.1500000000	-0.0259168099
1.9000	0.0702967579	0.1000000000	-0.0297032421
1.9500	0.0257925005	0.0500000000	-0.0242074995

```
 2.0000      -0.0019520985       0.0000000000       -0.0019520985
ESTIMATED SSE IS             0.0920133337
ESTIMATED MSE IS             0.0023003333
THE MAXIMUM ABSOLUTE DIFFERENCE BETWEEN THE
H-FUNCTION AND THE ACTUAL FUNCTION IS              0.1511736368
THE INTEGRATED ABSOLUTE DENSITY DIFFERENCE
(AN ESTIMATE OF THE AREA BETWEEN THE
H-FUNCTION AND THE ACTUAL DENSITY) IS              0.0752034552
```

EXAMPLE 4 - SUM OF TWO INDEPENDENT, IDENTICALLY

DISTRIBUTED BETA VARIATES

```
PROGRAM SUMVAR RUN IN DOUBLE PRECISION
FOR THE SUM OF 2 INDEPENDENT H-FUNCTION VARIATES
INPUT PARAMETERS FOR VARIABLE NUMBER 1 OF THE SUM:
   M, N, P, Q  =    1  0  1  1
   K, C  =     2.0000000   1.0000000
   a, A  =       2.00000     1.00000
   b, B  =       0.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          0.3333333
      THE SECOND MOMENT IS          0.1666667
      THE THIRD  MOMENT IS          0.1000000
      THE FOURTH MOMENT IS          0.0666667
      THE FIFTH  MOMENT IS          0.0476190
      THE  NEXT  MOMENT IS          0.0357143
      THE  NEXT  MOMENT IS          0.0277778
      THE  NEXT  MOMENT IS          0.0222222
      THE  NEXT  MOMENT IS          0.0181818
      THE  NEXT  MOMENT IS          0.0151515

INPUT PARAMETERS FOR VARIABLE NUMBER 2 OF THE SUM:
   M, N, P, Q  =    1  0  1  1
   K, C  =     2.0000000   1.0000000
   a, A  =       2.00000     1.00000
   b, B  =       0.00000     1.00000
   THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          0.3333333
      THE SECOND MOMENT IS          0.1666667
      THE THIRD  MOMENT IS          0.1000000
      THE FOURTH MOMENT IS          0.0666667
```

```
      THE FIFTH  MOMENT IS          0.0476190
      THE  NEXT  MOMENT IS          0.0357143
      THE  NEXT  MOMENT IS          0.0277778
      THE  NEXT  MOMENT IS          0.0222222
      THE  NEXT  MOMENT IS          0.0181818
      THE  NEXT  MOMENT IS          0.0151515

FOR THE RANDOM VARIABLE GIVEN AS THE SUM
OF THE ABOVE INDEPENDENT H-FUNCTION
VARIATES, THE MOMENTS ABOUT THE ORIGIN ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          0.6666667
      THE SECOND MOMENT IS          0.5555556
      THE THIRD  MOMENT IS          0.5333333
      THE FOURTH MOMENT IS          0.5666667
      THE FIFTH  MOMENT IS          0.6507937
      THE  NEXT  MOMENT IS          0.7952381
      THE  NEXT  MOMENT IS          1.0222222
      THE  NEXT  MOMENT IS          1.3703704
      THE  NEXT  MOMENT IS          1.9030303
      THE  NEXT  MOMENT IS          2.7229437

PROGRAM H_FIT RUN WITH DOUBLE PRECISION
INPUT WAS OF TYPE0
THE ZEROTH MOMENT WAS USED IN THE FIT
RESULTS OF ANALYTIC SOLUTION -
NUMERATOR:
  SMALLB(1)=     1.333333333333333204
  BIGB(1)=       1.000000000000000000
DENOMINATOR:
  SMALLA(1)=     5.999999999999999556
  BIGA(1)=       1.000000000000000000
VALUES OF K & C ARE:
  K=   302.358560865130122863
  C=     0.500000000000000000

ESTIMATED H-FUNCTION PARAMETERS
FOR THE SUM OF THE INDEPENDENT
H-FUNCTION VARIATES ARE:
   M, N, P, Q =    1  0  1  1
   K, C  =  302.3585609   0.5000000
   a, A  =      6.00000      1.00000
   b, B  =      1.33333      1.00000
THE MOMENTS ABOUT THE ORIGIN OF THIS
H-FUNCTION ARE:
      THE ZEROTH MOMENT IS          1.0000000
```

```
THE FIRST  MOMENT IS        0.6666667
THE SECOND MOMENT IS        0.5555556
THE THIRD  MOMENT IS        0.5349794
THE FOURTH MOMENT IS        0.5706447
THE FIFTH  MOMENT IS        0.6571060
THE  NEXT  MOMENT IS        0.8031296
THE  NEXT  MOMENT IS        1.0296533
THE  NEXT  MOMENT IS        1.3728711
THE  NEXT  MOMENT IS        1.8915113
THE  NEXT  MOMENT IS        2.6796410

 DETERMINE P.D.F.(Z) AND C.D.F.(Z)
 FOR VALUES OF Z FROM   0.0500 TO   2.0000
 WITH STEP SIZE   0.0500
 FOR THE SUM OF  1 TERMS, WHERE
 THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.
 CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,
 PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =    4.7785
 FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):
 Z = Y1
 VARIATE X 1 IS TYPE NUMBER  4
 INPUT PARAMETERS ARE THETA =  302.35856, PHI =    0.50000
 AND POWER =    1.00000
 THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:
              1 0                  (    6.000,    1.000)
302.35856  H          (    0.50000 X):
              1 1                  (    1.333,    1.000)
 THE P.D.F. FOR TERM  1 OF THE SUM IS GIVEN BY:
              1 0
302.35856  H          (    0.50000 Z), WHERE
              1 1
(BA(I),GBA(I)): (    1.333,    1.000) (
(CD(I),GCD(I)): (    6.000,    1.000) (
 CONVERGENCE TYPE = 7
      D =  0.00     E =  0.00     L = -4.67     R = 1.0000
     Z          PDF(Z)      CDF(Z)
   0.0500    0.136922    0.003018
   0.1000    0.313676    0.014247
   0.1500    0.488431    0.034343
   0.2000    0.648272    0.062838
   0.2500    0.787247    0.098821
   0.3000    0.902663    0.141169
   0.3500    0.993688    0.188680
   0.4000    1.060648    0.240137
   0.4500    1.104632    0.294361
   0.5000    1.127231    0.350243
```

0.5500	1.130362	0.406760
0.6000	1.116150	0.462990
0.6500	1.086825	0.518123
0.7000	1.044659	0.571459
0.7500	0.991905	0.622412
0.8000	0.930756	0.670510
0.8500	0.863314	0.715383
0.9000	0.791555	0.756769
0.9500	0.717316	0.794498
1.0000	0.642271	0.828487
1.0500	0.567924	0.858736
1.1000	0.495600	0.885314
1.1500	0.426434	0.908349
1.2000	0.361374	0.928025
1.2500	0.301178	0.944568
1.3000	0.246416	0.958234
1.3500	0.197475	0.969306
1.4000	0.154565	0.978082
1.4500	0.117726	0.984864
1.5000	0.086842	0.989954
1.5500	0.061651	0.993643
1.6000	0.041760	0.996207
1.6500	0.026665	0.997899
1.7000	0.015767	0.998944
1.7500	0.008399	0.999535
1.8000	0.003848	0.999831
1.8500	0.001390	0.999954
1.9000	0.000326	0.999993
1.9500	0.000026	1.000000
2.0000	-0.000001	1.000000

NUMBER OF POLES EVALUATED = 100

PROGRAM COMPAR RUN IN DOUBLE PRECISION

X	Y (H-FCN)	Y (EVAL)	RESIDUAL
0.0500	0.1369215248	0.1900833333	-0.0531618085
0.1000	0.3136760423	0.3606666667	-0.0469906244
0.1500	0.4884305603	0.5122500001	-0.0238194398
0.2000	0.6482720923	0.6453333335	0.0029387588
0.2500	0.7872469795	0.7604166670	0.0268303125
0.3000	0.9026634638	0.8580000005	0.0446634633
0.3500	0.9936878985	0.9385833342	0.0551045643
0.4000	1.0606483104	1.0026666679	0.0579816425
0.4500	1.1046322658	1.0507500018	0.0538822640
0.5000	1.1272305972	1.0833333358	0.0438972614
0.5500	1.1303624979	1.1009166700	0.0294458279
0.6000	1.1161501217	1.1040000043	0.0121501174

0.6500	1.0868253832	1.0930833388	-0.0062579556
0.7000	1.0446588587	1.0686666735	-0.0240078148
0.7500	0.9919045410	1.0312500084	-0.0393454674
0.8000	0.9307563999	0.9813333435	-0.0505769436
0.8500	0.8633140228	0.9194166789	-0.0561026561
0.9000	0.7915554397	0.8460000145	-0.0544445748
0.9500	0.7173157774	0.7615833504	-0.0442675730
1.0000	0.6422707533	0.6666666865	-0.0243959332
1.0500	0.5679242735	0.5715833504	-0.0036590769
1.1000	0.4955995779	0.4860000145	0.0095995634
1.1500	0.4264335126	0.4094166789	0.0170168337
1.2000	0.3613736037	0.3413333435	0.0200402602
1.2500	0.3011776887	0.2812500084	0.0199276803
1.3000	0.2464159201	0.2286666735	0.0177492466
1.3500	0.1974750104	0.1830833388	0.0143916716
1.4000	0.1545646276	0.1440000043	0.0105646233
1.4500	0.1177258977	0.1109166700	0.0068092277
1.5000	0.0868420048	0.0833333358	0.0035086690
1.5500	0.0616509293	0.0607500018	0.0009009275
1.6000	0.0417604104	0.0426666679	-0.0009062575
1.6500	0.0266652853	0.0285833342	-0.0019180489
1.7000	0.0157674444	0.0180000005	-0.0022325561
1.7500	0.0083987678	0.0104166670	-0.0020178992
1.8000	0.0038476138	0.0053333335	-0.0014857197
1.8500	0.0013897874	0.0022500001	-0.0008602127
1.9000	0.0003256263	0.0006666667	-0.0003410404
1.9500	0.0000264951	0.0000833333	-0.0000568382
2.0000	-0.0000011508	0.0000000000	-0.0000011508

ESTIMATED SSE IS 0.0358601689
ESTIMATED MSE IS 0.0008965042
THE MAXIMUM ABSOLUTE DIFFERENCE BETWEEN THE
H-FUNCTION AND THE ACTUAL FUNCTION IS 0.0579816425
THE INTEGRATED ABSOLUTE DENSITY DIFFERENCE
(AN ESTIMATE OF THE AREA BETWEEN THE
H-FUNCTION AND THE ACTUAL DENSITY) IS 0.0442126253

EXAMPLE 5 - SUM OF TWO INDEPENDENT VARIATES WITH WEIBULL AND RAYLEIGH DISTRIBUTIONS

PROGRAM SUMVAR RUN IN DOUBLE PRECISION
FOR THE SUM OF 2 INDEPENDENT H-FUNCTION VARIATES
INPUT PARAMETERS FOR VARIABLE NUMBER 1 OF THE SUM:
M, N, P, Q = 1 0 0 1

```
  K, C  =     1.3195079   1.3195079
  b, B  =       0.80000     0.20000
  THE MOMENTS ABOUT THE ORIGIN ARE:
     THE ZEROTH MOMENT IS         1.0000000
     THE FIRST  MOMENT IS         0.6958418
     THE SECOND MOMENT IS         0.5095992
     THE THIRD  MOMENT IS         0.3889251
     THE FOURTH MOMENT IS         0.3072421
     THE FIFTH  MOMENT IS         0.2500000
     THE  NEXT  MOMENT IS         0.2087525
     THE  NEXT  MOMENT IS         0.1783597
     THE  NEXT  MOMENT IS         0.1555701
     THE  NEXT  MOMENT IS         0.1382589
     THE  NEXT  MOMENT IS         0.1250000

INPUT PARAMETERS FOR VARIABLE NUMBER 2 OF THE SUM:
  M, N, P, Q  =   1  0  0  1
  K, C  =     1.7320508   1.7320508
  b, B  =       0.50000     0.50000
  THE MOMENTS ABOUT THE ORIGIN ARE:
     THE ZEROTH MOMENT IS         1.0000000
     THE FIRST  MOMENT IS         0.5116634
     THE SECOND MOMENT IS         0.3333333
     THE THIRD  MOMENT IS         0.2558317
     THE FOURTH MOMENT IS         0.2222222
     THE FIFTH  MOMENT IS         0.2131931
     THE  NEXT  MOMENT IS         0.2222222
     THE  NEXT  MOMENT IS         0.2487252
     THE  NEXT  MOMENT IS         0.2962963
     THE  NEXT  MOMENT IS         0.3730879
     THE  NEXT  MOMENT IS         0.4938272

FOR THE RANDOM VARIABLE GIVEN AS THE SUM
OF THE ABOVE INDEPENDENT H-FUNCTION
VARIATES, THE MOMENTS ABOUT THE ORIGIN ARE:
     THE ZEROTH MOMENT IS         1.0000000
     THE FIRST  MOMENT IS         1.2075051
     THE SECOND MOMENT IS         1.5550061
     THE THIRD  MOMENT IS         2.1228284
     THE FOURTH MOMENT IS         3.0567312
     THE FIFTH  MOMENT IS         4.6225066
     THE  NEXT  MOMENT IS         7.3134235
     THE  NEXT  MOMENT IS        12.0647384
     THE  NEXT  MOMENT IS        20.6900000
     THE  NEXT  MOMENT IS        36.7858797
     THE  NEXT  MOMENT IS        67.6453356
```

```
PROGRAM H_FIT RUN WITH DOUBLE PRECISION
INPUT WAS OF TYPE0
DEFAULT INITIAL GUESS WAS USED
THE ZEROTH MOMENT WAS USED IN THE FIT
RESULTS OF ZSPOW -
NUMERATOR:
  SMALLB(1)=      5.291047428929997909
  BIGB(1)=        0.623582866349619888
DENOMINATOR:
VALUES OF K & C ARE:
  K=      0.023697857773422697
  C=      2.459933992450636764
  FNORM=      0.000000000000000000

ESTIMATED H-FUNCTION PARAMETERS
FOR THE SUM OF THE INDEPENDENT
H-FUNCTION VARIATES ARE:
   M, N, P, Q  =   1  0  0  1
   K, C  =     0.0236979   2.4599340
   b, B  =       5.29105     0.62358
THE MOMENTS ABOUT THE ORIGIN OF THIS
H-FUNCTION ARE:
      THE ZEROTH MOMENT IS          1.0000000
      THE FIRST  MOMENT IS          1.2075051
      THE SECOND MOMENT IS          1.5550061
      THE THIRD  MOMENT IS          2.1228284
      THE FOURTH MOMENT IS          3.0568212
      THE FIFTH  MOMENT IS          4.6235079
      THE  NEXT  MOMENT IS          7.3191453
      THE  NEXT  MOMENT IS         12.0889706
      THE  NEXT  MOMENT IS         20.7770136
      THE  NEXT  MOMENT IS         37.0686930
      THE  NEXT  MOMENT IS         68.5084603

DETERMINE P.D.F.(Z) AND C.D.F.(Z)
FOR VALUES OF Z FROM   0.0500 TO    2.4000
WITH STEP SIZE   0.0500
FOR THE SUM OF  1 TERMS, WHERE
THE MAXIMUM NUMBER OF POLES TO BE EVALUATED IS  100.
CRUMP PARAMETERS: NUMBER OF COMPLEX VALUES = 1001,
PERCENT OF HIGHEST Z VALUE = 1.00, AXIS POINT A =    3.9820
FORM FOR OVERALL PROBLEM (WHERE YJ = XJ**PJ):
Z = Y1
VARIATE X 1 IS TYPE NUMBER  4
INPUT PARAMETERS ARE THETA =    0.02370, PHI =   2.45993
```

```
AND POWER =    1.00000
THE P.D.F. FOR VARIATE X 1 IS GIVEN BY:
                 1  0
 0.02370  H           (    2.45993 X):
                 0  1                     (    5.291,    0.624)
THE P.D.F. FOR TERM  1 OF THE SUM IS GIVEN BY:
                 1  0
 0.02370  H           (    2.45993 Z), WHERE
                 0  1
(BA(I),GBA(I)): (    5.291,    0.624) (
 CONVERGENCE TYPE = 1
     D =  0.62      E = -0.62      L =   4.79      R = 1.3425
```

Z	PDF(Z)	CDF(Z)
0.0500	0.000000	0.000000
0.1000	0.000000	0.000000
0.1500	0.000007	0.000000
0.2000	0.000067	0.000001
0.2500	0.000388	0.000011
0.3000	0.001561	0.000054
0.3500	0.004860	0.000202
0.4000	0.012505	0.000612
0.4500	0.027742	0.001579
0.5000	0.054613	0.003580
0.5500	0.097399	0.007305
0.6000	0.159805	0.013648
0.6500	0.244059	0.023651
0.7000	0.350144	0.038419
0.7500	0.475360	0.058986
0.8000	0.614318	0.086185
0.8500	0.759413	0.120521
0.9000	0.901657	0.162079
0.9500	1.031744	0.210484
1.0000	1.141135	0.264908
1.0500	1.223000	0.324137
1.1000	1.272868	0.386673
1.1500	1.288939	0.450859
1.2000	1.272029	0.515016
1.2500	1.225228	0.577563
1.3000	1.153351	0.637120
1.3500	1.062279	0.692578
1.4000	0.958308	0.743133
1.4500	0.847563	0.788296
1.5000	0.735552	0.827869
1.5500	0.626854	0.861907
1.6000	0.524979	0.890669
1.6500	0.432336	0.914559

1.7000	0.350321	0.934080
1.7500	0.279456	0.949778
1.8000	0.219576	0.962209
1.8500	0.170015	0.971907
1.9000	0.129779	0.979366
1.9500	0.097705	0.985021
2.0000	0.072574	0.989252
2.0500	0.053205	0.992375
2.1000	0.038504	0.994650
2.1500	0.027513	0.996287
2.2000	0.019386	0.997448
2.2500	0.013441	0.998262
2.3000	0.009088	0.998819
2.3500	0.005791	0.999188
2.4000	0.002836	0.999396

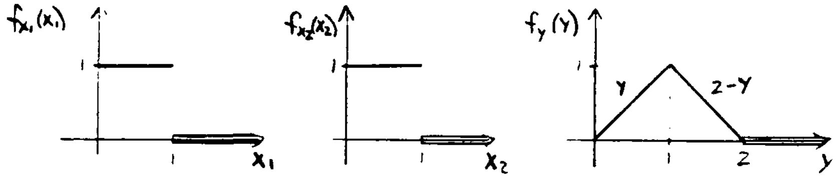
NUMBER OF POLES EVALUATED = 94

APPENDIX B

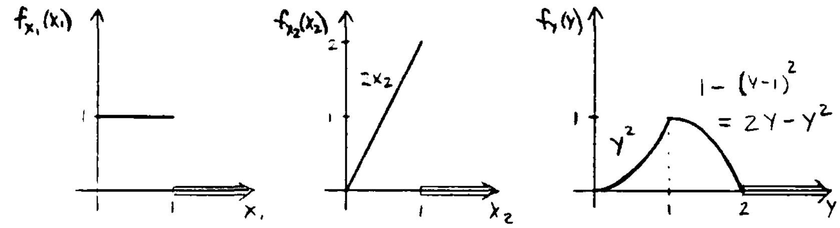
GRAPHICAL DEPICTIONS OF SUMS OF SELECTED INDEPENDENT UNIFORM, POWER FUNCTION, AND BETA VARIATES

Listed below are graphical depictions of the examples in Section 5.1.1 of analytic convolutions performed to find the exact distribution of the sum of selected independent random variables. The exact distributions of sums of two independent variates with certain uniform, power function, and beta distributions were found using the convolution integral. These graphs may provide some insight to the sum of independent variates with a restricted range. These sums produce a distribution with two functional forms over distinct ranges of the variable and do not have continuous derivatives of all orders at x=1.

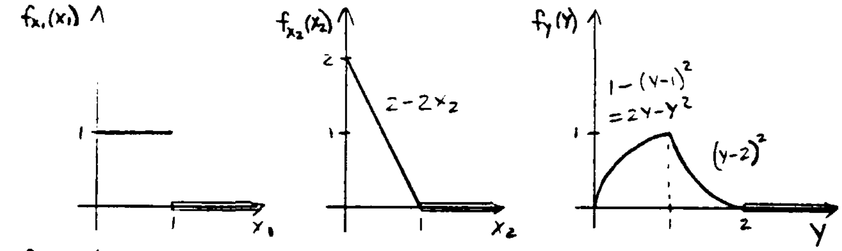
The two leftmost graphs in each row show the densities of each random variable in the sum, $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$. The rightmost graph in each row shows the density of $Y=X_1+X_2$, $f_Y(y)$, when X_1 and X_2 are independent. Each series of graphs is also referenced to the corresponding equation number from Section 5.1.1.



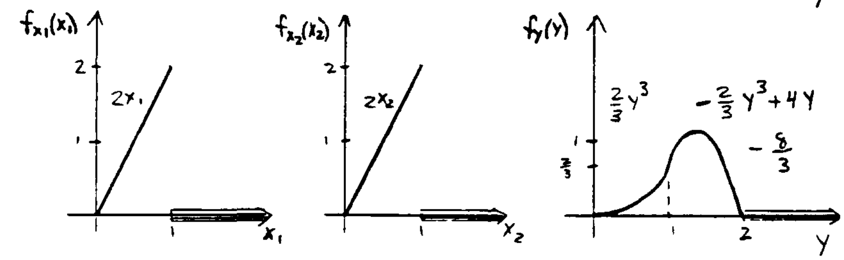
(5.13)



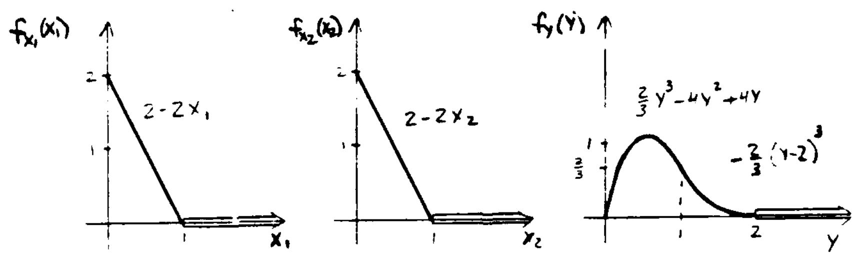
(5.14)



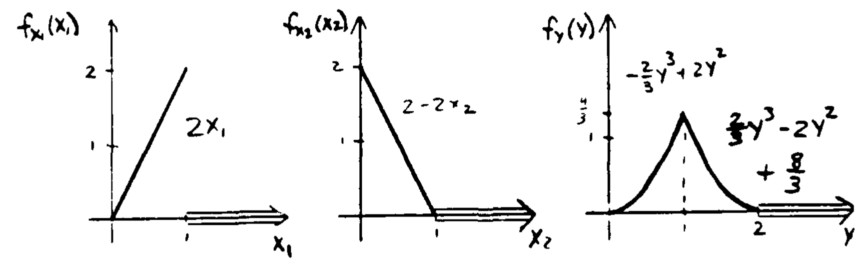
(5.15)



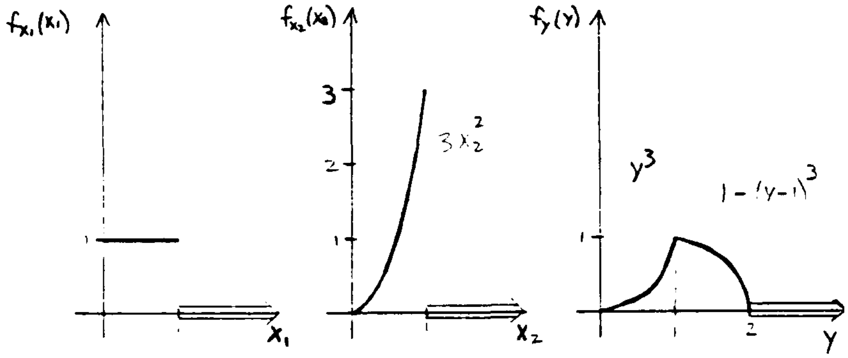
(5.16)



(5.17)

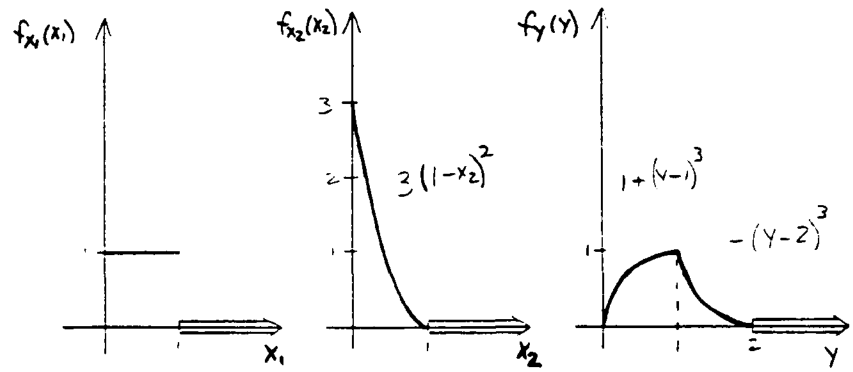


(5.18)



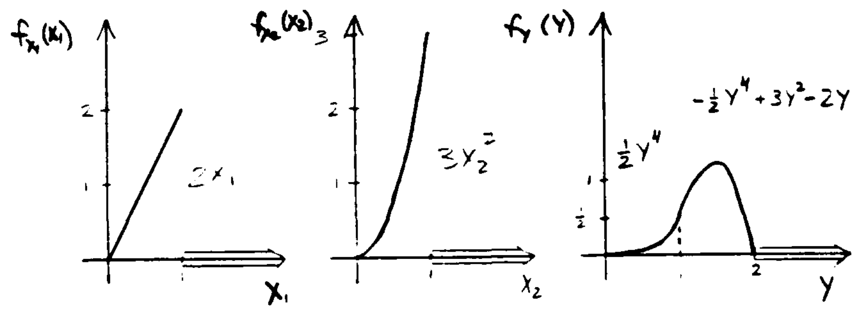
$f_{X_1}(x_1)$
X_1
$f_{X_2}(x_2)$
3
2
1
$3x_2^2$
X_2
$f_Y(y)$
y^3
$1-(y-1)^3$
1
2
Y

(5.19)



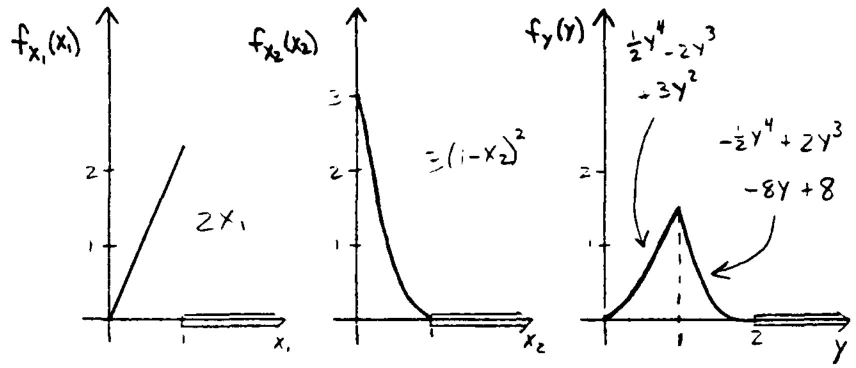
$f_{X_1}(x_1)$
X_1
$f_{X_2}(x_2)$
3
2
1
$3(1-x_2)^2$
X_2
$f_Y(y)$
$1+(y-1)^3$
$-(y-2)^3$
1
2
Y

(5.20)



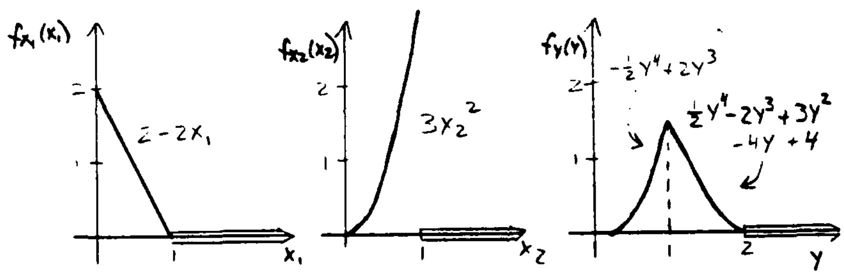
$f_{X_1}(x_1)$
2
1
$2x_1$
X_1
$f_{X_2}(x_2)$
3
2
1
$3x_2^2$
X_2
$f_Y(y)$
$-\frac{1}{2}y^4+3y^2-2y$
$\frac{1}{2}y^4$
1
$\frac{1}{2}$
2
Y

(5.21)



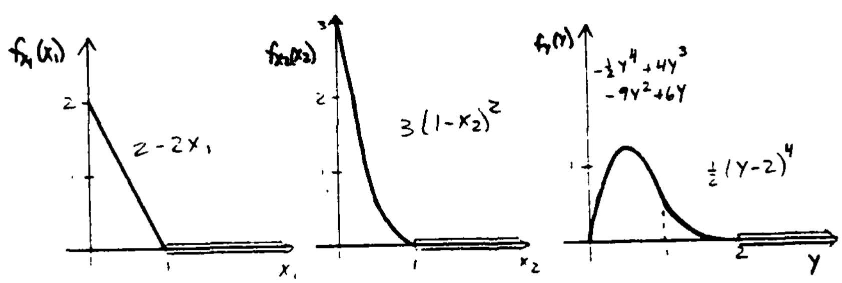
$f_{X_1}(x_1)$
$2x_1$
x_1
$f_{X_2}(x_2)$
$3(1-x_2)^2$
x_2
$f_Y(y)$
$\frac{1}{2}y^4 - 2y^3 + 3y^2$
$-\frac{1}{2}y^4 + 2y^3 - 8y + 8$
y

(5.22)



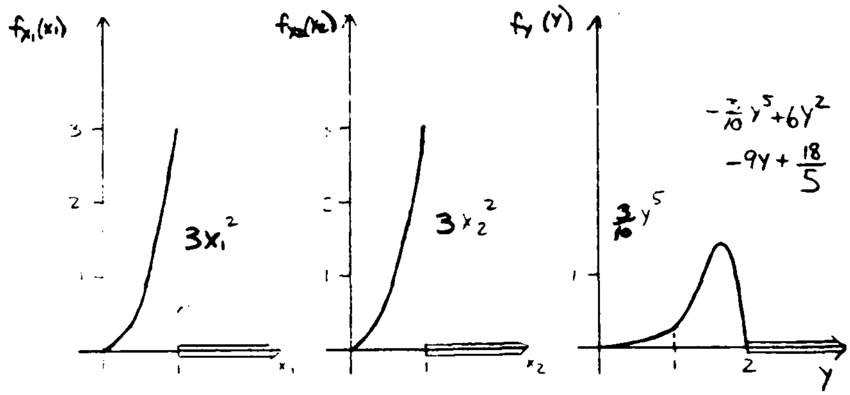
$f_{X_1}(x_1)$
$2-2x_1$
x_1
$f_{X_2}(x_2)$
$3x_2^2$
x_2
$f_Y(y)$
$-\frac{1}{2}y^4 + 2y^3$
$\frac{1}{2}y^4 - 2y^3 + 3y^2 - 4y + 4$
y

(5.23)



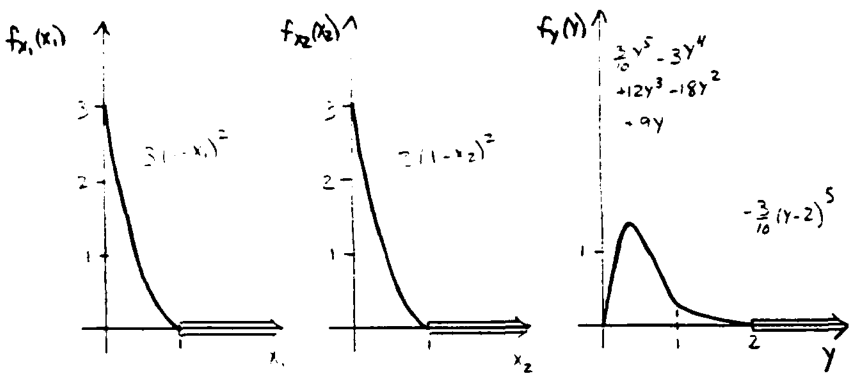
$f_{X_1}(x_1)$
$2-2x_1$
x_1
$f_{X_2}(x_2)$
$3(1-x_2)^2$
x_2
$f_Y(y)$
$-\frac{1}{2}y^4 + 4y^3 - 9y^2 + 6y$
$\frac{1}{2}(y-2)^4$
y

(5.24)



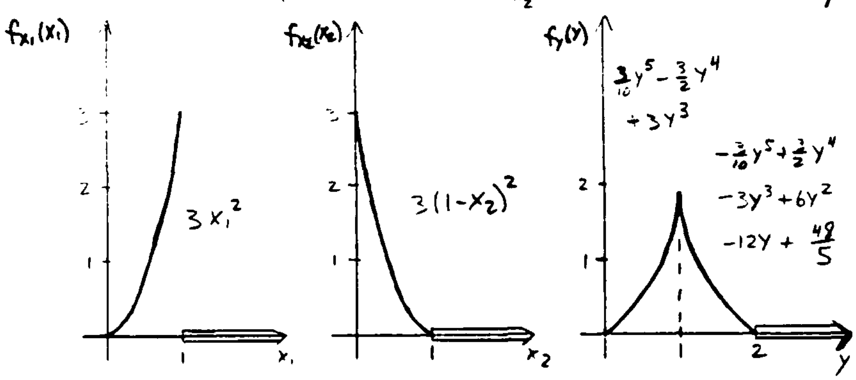
$f_{X_1}(x_1)$
$3x_1^2$
$f_{X_2}(x_2)$
$3x_2^2$
$f_Y(y)$
$\frac{3}{10}y^5$
$-\frac{3}{10}y^5+6y^2$
$-9y+\frac{18}{5}$
y

(5.25)



$f_{X_1}(x_1)$
$3(1-x_1)^2$
$f_{X_2}(x_2)$
$3(1-x_2)^2$
$f_Y(y)$
$\frac{3}{10}y^5-3y^4$
$+12y^3-18y^2$
$+9y$
$-\frac{3}{10}(y-2)^5$
y

(5.26)



$f_{X_1}(x_1)$
$3x_1^2$
$f_{X_2}(x_2)$
$3(1-x_2)^2$
$f_Y(y)$
$\frac{3}{10}y^5-\frac{3}{2}y^4$
$+3y^3$
$-\frac{3}{10}y^5+\frac{3}{2}y^4$
$-3y^3+6y^2$
$-12y+\frac{49}{5}$
y

(5.27)

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Captain Carl Dinsmore Bodenschatz was born in Champaign-Urbana, Illinois on June 16, 1957, the son of Mary Joan Dinsmore Bodenschatz and Carl Alvin Bodenschatz. In 1975, he graduated from Hempfield Area High School in Greensburg, Pennsylvania. After working as a Quality Control Lab Technician at Kennametal, Inc. in Latrobe, Pennsylvania for 14 months, he entered West Virginia University in Morgantown, West Virginia. In college, he majored in Mathematics with a minor in Statistics. In May 1980, he received the degree of Bachelor of Arts (magna cum laude) from West Virginia University. Upon graduation, he received a commission in the U.S. Air Force through the four-year ROTC program. In December 1983, he received the degree of Master of Science in Operations Research from the Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio. His other Air Force assignments have included working as a Training Systems Analyst and Operational Test and Evaluation Analyst at Eglin Air Force Base, Florida and as an Instructor in the Department of Mathematical Sciences at the U.S. Air Force Academy, Colorado. He is the principal author of "New and Generalized H-function Distributions", published in 1990 by the American Journal of Mathematical and Management Sciences. He entered the Graduate School of The University of Texas in August 1989. He married Deborah Murdock of Greensburg, Pennsylvania on May 31, 1980. They have three children, Luke Murdock born in 1981 and twins John Oakwood and Paul Oveson born in 1984.

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