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A constrained two-armed bandit problem was solved.

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SYSTEM RELIABILITY BY SIMULATION: RANDOM HAZARDS VERSUS IMPORTANCE SAMPLING

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Abstract: Two approaches for simulating the reliability function are considered - one using the total hazard estimator and the other using importance sampling. It is shown both for the Wheatstone Bridge system and also for a triangular system that the total hazard estimator has significantly smaller variance when compared both to the standard importance sampling estimator and also to an improved version of it.

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1. INTRODUCTION

Consider an n component system in which each of the n components is either working or is failed. Let, for i=1,...,n

$$X_i = \begin{cases} 1 & \text{if component } i \text{ is failed} \\ 0 & \text{otherwise} \end{cases}$$

The vector $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is called the state vector. Suppose also that the system is itself either working or failed, and that there exists a nondecreasing function ϕ such that

$$\phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if the system is failed under } \underset{\sim}{X} \\ 0 & \text{otherwise.} \end{cases}$$

Assuming that the X_i are independent random variables with

$$P\{X_i = 1\} = q_i$$

the problem of interest will be to use simulation to estimate

$$p_f = P\{\phi(\underset{\sim}{X}) = 1\}$$

We will be interested in the above in the case where p_f is very small and so the usual raw simulation estimator will need an extremely large number of simulations to obtain an estimator having a small error (relative to p_f). We will consider 2 variance reducing approaches - namely using the random hazard estimator developed by Ross [3] and using importance sampling. The final section compares the efficiency of these approaches for 2 different systems.

2. THE TOTAL HAZARD ESTIMATOR

Suppose that the minimal cut sets - these are minimal sets of components whose failure ensures the system's failure - can be determined for the system under study. The hazard approach suggested in [3] is to simulate all of the components of one of these minimal cut sets - call it C_1 - and let the first random hazard - call it h_1 - be the probability that all of these components are failed; that is

$$h_1 = \prod_{i \epsilon C_1} q_i$$

If all of these simulated components are not failed then determine a minimal cut set for the system conditional on the results of this first simulation. (In other words, if you had an initial list of all the minimal cut sets then eliminate any of these minimal cut sets which contain a working component of C_1 and remove from the other minimal cut sets any failed component of C_1. Now eliminate any of these sets which contain any other one as a subset.) Choose one of these (conditional) minimal cut sets - call it C_2 - and simulate its components. The second random hazard is

$$h_2 = \prod_{i \epsilon C_2} q_i$$

Continue in this manner until either all of the components of a (conditional) minimal cut set are failed or until there are no more minimal cut sets. If there were a total of r minimal cut sets that were simulated then the total hazard estimator of p_f is given by

$$H = \sum_{i=1}^{r} h_i$$

It was shown in [3] that H is an unbiased estimator of p_f.

The above leaves open the question of which minimal cut set to simulate. Sometimes the minimal cut sets will not all be known and additional computation is necessary to determine them. In this case it is probably best to find any minimal cut set - by whatever algorithm is most convenient - and then simulate that one. If all the minimal cut sets are available then, as a rule of thumb, we recommend simulating the one which has the largest probability of having all of its components failed. A partial motivation for this rule of thumb is provided by the following example.

Example 2a: Consider a 2 component system which is failed if either of its components fail - this is called a series system. For this system each component is a minimal cut set. The total hazard estimator which first simulates component 1 can be expressed as

$$H = q_1 + q_2 I$$

where

$$I = \begin{cases} 1 & \text{with probability } 1-q_1 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\mathrm{Var}(H) = q_2^2 q_1 (1-q_1)$$

By symmetry the variance of the hazard estimator when component 2 is first simulated is $q_1^2 q_2 (1-q_2)$; and it is easy to see that Var(H) is thus *minimized when we simulate the component having largest* q_i *first.*

3. THE IMPORTANCE SAMPLING APPROACH

The importance sampling approach is to estimate p_f by simulating random variables X_i which are 1 or 0 not with probabilities q_i but with some other probabilities β_i , i=1,...,n. The estimator

$$\mathrm{Imp} = \phi(\underline{X}) \frac{\prod_{i=1}^{n} q_i^{X_i} (1-q_i)^{1-X_i}}{\prod_{i=1}^{n} \beta_i^{X_i} (1-\beta_i)^{1-X_i}} ,$$

which is called the importance sampling estimator, is also an unbiased estimator of p_f (see [4]). The β_i are usually chosen so that there is a reasonably large probability (usually around 0.5) that $\phi(\underline{X}) = 1$.

To get an idea as to the choice of the β_i that will result in the importance sampling estimator having a small variance, suppose that all of the q_i are equal to q which is very small. Suppose also that the simulations will be done with all of the β_i equal to some value, call it β. One way of choosing β is to choose it so that the maximal possible value of the estimator Imp is as small as possible. Now the estimator Imp is given by

$$\mathrm{Imp} = \phi(\underline{X})\ (q/\beta)^{\Sigma X_i}\ ([1-q]/[1-\beta])^{n-\Sigma X_i}$$

Since q will be much smaller than β and since $\phi(\underline{X})$ will equal 0 when the set of failed components does not contain a minimal cut set it follows that the largest possible value of Imp will occur when ΣX_i is equal to the number of components in the smallest minimal cut set. That is, Imp will be maximal when all of the components in the smallest sized minimal cut set are failed and all of the other components are not failed. Hence, if we let m denote the size of the smallest minimal cut set then

$$\mathrm{Imp} \leq (q/\beta)^m([1-q]/[1-\beta])^{n-m}$$

The choice of β minimizing the right hand side of the above inequality is given by $\beta = m/n$. Such a choice of β will result in a small variance of the importance sampling estimator.

<u>Example 3a:</u> Consider a system in which $m=n/2$. In this case we see that the importance sampling estimator which simulates the components using the value $\beta=1/2$ is such that

$$\mathrm{Imp} \leq (2q)^{n/2}(2-2q)^{n/2} \approx (4q)^{n/2} \quad \text{for q small}$$

As we will show below, this implies that

$$\mathrm{Var(Imp)} \leq (4q)^n/4$$

To see how impressive this is suppose, for instance that q=.01 and n=10. Then since the smallest minimal cut set is of size 5 it follows that

$$p_f \geq (.01)^5$$

and so the variance of the raw simulation estimator, which is equal to $p_f(1-p_f)$, is such that

$$\mathrm{Var(Raw\ estimator)} \geq 10^{-10}$$

On the other hand,

$$\mathrm{Var(Imp)} \leq (.04)^{10}/4 = 2.62 \times 10^{-15}$$

The above example made use of the following result which, though possibly well-known, we have not found in the literature.

Proposition 1: If X is a random variable such that $0 \leq X \leq a$ then

$$\mathrm{Var}(X) \leq E[X]\{a - E[X]\} \leq a^2/4$$

Proof: Let Y be a random variable such that

$$Y = \begin{cases} a & \text{with probability } E[X]/a \\ 0 & \text{otherwise} \end{cases}$$

Now

$$E[Y^2] = E[aX] \geq E[X^2]$$

and since $E[Y] = E[X]$ we thus have

$$\mathrm{Var}(Y) \geq \mathrm{Var}(X)$$

The result now follows since, with p=E[X]/a,

$$\mathrm{Var}(Y) = a^2p(1-p) \leq a^2/4$$

The importance sampling estimator can be improved (in the sense of having its variance reduced) by not initially simulating all of the components but rather only those in a (conditional on the results up to that point) minimal cut set (as in the case of the hazard estimator). If the sets $C_1,\ldots,C_r$ are simulated and the system is failed then the improved importance sampling estimator would be given by

$$\text{Imp Imp} = \phi(\underline{X}) \frac{\Pi(q_i^{X_i}(1-q_i)^{1-X_i})}{\Pi(\beta_i^{X_i}(1-\beta_i)^{1-X_i})}$$

where the products are not over all components but only those whose values were actually simulated. As the usual importance sampling estimator is the product of the improved importance sampling estimator and an independent random variable having mean 1 it will necessarily have a larger variance than the improved version (which, of course, does come with a computational cost since we must determine, at each stage, a minimal cut set).

Remark: The use of the importance sampling estimator is not new to this paper, see for instance [1] and [2] and the references quoted therein. However, the approach for choosing β and the improved version appear to be new.

4. THE COMPARISON

We will compare the variance of the hazard estimator and that of the importance sampling estimator for 2 systems. For both systems we will take $q_i=q$ for all i. The first system, often referred to as the Wheatstone Bridge System, can be pictorially represented as in Figure 1.

FIGURE 1: THE BRIDGE SYSTEM

Its minimal cut sets are {1,2}, {1,3,5}, {2,3,4}, and {4,5}. Table 1 presents the variance of the various estimators for this system. These variances were obtained analytically.

TABLE 1. VARIANCES OF ESTIMATORS FOR THE BRIDGE SYSTEM

	q					
	0.001	0.01	0.05	0.1	0.2	0.5
p_f	$2.002\text{x}10^{-6}$	$2.020\text{x}10^{-4}$	.0052	.0215	.0886	.5
Var(Imp)	$5.351\text{x}10^{-11}$	$5.044\text{x}10^{-7}$	$2.431\text{x}10^{-4}$	$2.84\text{x}10^{-3}$	$2.61\text{x}10^{-2}$	.559
Var(Imp Imp)	$6.510\text{x}10^{-12}$	$6.226\text{x}10^{-8}$	$3.244\text{x}10^{-5}$	$4.32\text{x}10^{-4}$	$5.47\text{x}10^{-3}$	.162
Var(Hazard)	$1.996\text{x}10^{-12}$	$1.960\text{x}10^{-8}$	$1.125\text{x}10^{-5}$	$1.61\text{x}10^{-4}$	$2.04\text{x}10^{-3}$	.047

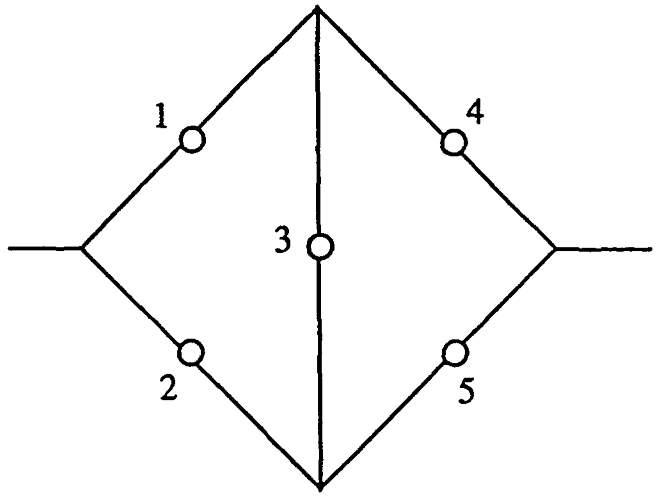

Figure 1: The Bridge System

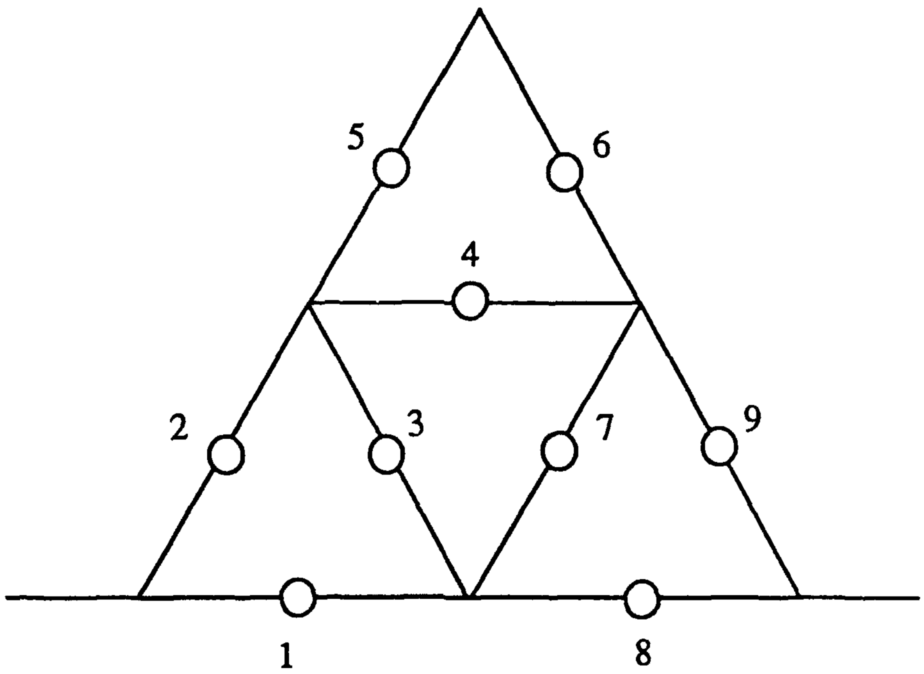

Figure 2: The Triangle System

<u>Notes</u>: The importance sampling estimator used $\beta = 2/5$. Imp Imp is the improved importance sampling, and for this estimator β=.56 (which was seen by computation to result in the smallest variance for this estimator) was used.

<u>Conclusion:</u> The hazard estimator had the smallest variance. It ranged from roughly 27 times better (for small q) to 12 times better (for large q) than the importance sampling estimator, and was roughly 3 times better than the improved importance sampling estimator. All of these estimators were far better than the raw simulation estimator whose variance is $p_f(1-p_f)$.

The second system we will consider is represented in Figure 2.

FIGURE 2: THE TRIANGLE SYSTEM

The variances of the estimators, obtained by a simulation of 100,000 replications, are given in Table 2.

TABLE 2: VARIANCES OF ESTIMATORS FOR THE TRIANGLE SYSTEM

	q 0.001	0.01	0.1	0.2
p_f	2×10^{-6}	2×10^{-4}	2.03×10^{-2}	8.3×10^{-2}
Var(Imp Imp)	4.97×10^{-11}	4.85×10^{-7}	4.02×10^{-3}	6.04×10^{-2}
Var(Hazard)	1.82×10^{-21}	3.22×10^{-12}	1.06×10^{-5}	3.54×10^{-4}

Notes: Imp Imp used β=2/9

Conclusion: The total hazard estimator was far superior. Indeed its variance was smaller than that of the raw simulation estimator - whose variance is $p_f(1-p_f)$ - by a factor of approximately 10^{15}, and smaller than that of the Imp Imp estimator by a factor of approximately 2.7×10^{10}.

Remark: For both systems the smallest sized cut set rule was used to decide which minimal cut set to simulate.

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