WL-TR-91-3021

A CONSISTENT SHEAR DEFORMABLE THEORY FOR LAMINATED PLATES

R. S. Sandhu, W. E. Wolfe, S. J. Hong, and H. S. Chohan
Department of Civil Engineering
The Ohio State University

July 1991

Final Report for Period July 1985 - August 1990

FLIGHT DYNAMICS DIRECTORATE
WRIGHT LABORATORY
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433-6553

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely Government-related procurement, the United States Government incurs no responsibility or any obligation whatsoever. The fact that the Government may have formulated or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication, or otherwise in any manner construed, as licensing the holder, or any other person or corporation; or as conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

GEORGE P. SENDECKYJ, Project Engineer
Fatigue, Fracture and Reliability Group
Structural Integrity Branch

JOHN K. RYDER, Capt, USAF
Technical Manager
Fatigue, Fracture and Reliability Group
Structural Integrity Branch

FOR THE COMMANDER

JAMES L. RUDD, Chief
Structural Integrity Branch
Structures Division

If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization, please notify WL/FIBEC, WPAFB, OH 45433-6553 to help us maintain a current mailing list.

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS: None
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Available for public release; distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): RF Project 764779/717297	5. MONITORING ORGANIZATION REPORT NUMBER(S): WL-TR-91-3021
6a. NAME OF PERFORMING ORGANIZATION: The Ohio State University Research Foundation; 6b. OFFICE SYMBOL (If applicable): OSURF	7a. NAME OF MONITORING ORGANIZATION: Flight Dynamics Directorate (WL/FIBEC) Wright Laboratory
6c. ADDRESS (City, State, and ZIP Code): 1960 Kenny Road Columbus, Ohio 43210	7b. ADDRESS (City, State, and ZIP Code): Wright-Patterson Air Force Base Dayton, Ohio 45433-6553
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Flight Dynamics Directorate; 8b. OFFICE SYMBOL (If applicable): WL/FIBEC	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: Contract No. F33615-85-C-3213
8c. ADDRESS (City, State, and ZIP Code): Wright-Patterson Air Force Base Dayton, Oh 45433-6553	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2302; TASK NO. N1; WORK UNIT ACCESSION NO. 02

11. TITLE (Include Security Classification)
A Consistent Shear Deformable Theory for Laminated Plates

12. PERSONAL AUTHOR(S)
R. S. Sandhu, W. E. Wolfe, S. J. Hong, and H. S. Chohan

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Final	FROM 7/1/85 TO 8/31/90	1991, July, 22	174

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Composite Laminates; Delamination; Discrete Plate Theory; Finite Element Method; Free-edge Delamination

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A consistent theory for the interlayer coupling of transverse shear in laminated plates is presented. Starting from a generalization of Reissner's variational principle for elastic orthotropic plates to laminated plates, it is shown that the shear force in any lamina may depend upon the shear deformation of that lamina as well as others. The coupling depends on the material properties, the stacking sequence and the applied surface tractions. Several existing theories of homogeneous as well as laminated plates are seen to arise as specializations/approximations of the general theory. The nature of the constitutive coupling is illustrated by studying a 12-layer graphite-epoxy laminate. The theory is implemented in a finite element computer program and applied to the determination of stresses in cross-ply as well as angle-ply free-edge delamination specimens, and to study the frequencies of vibration of laminated plates.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: George P. Sendeckyj	22b. TELEPHONE (Include Area Code): 513-255-6104; 22c. OFFICE SYMBOL: WL/FIBEC

DD Form 1473, JUN 86 — Previous editions are obsolete. — SECURITY CLASSIFICATION OF THIS PAGE

FOREWORD

The research reported herein was supported by the Air Force System Command/ASD under Grant No. F33615-85-C-3213 to The Ohio State Universty. Dr. George P. Sendeckyj of the Flight Dynamics Directorate, Wright Laboratory (WL/FIBEC) was the Program Manager. The report covers part of the work done from 7/1/85 to 8/31/90. At The Ohio State University, the research was carried out under the supervision of Professors Ranbir S. Sandhu and William E. Wolfe by Soon J. Hong and Harpal S. Chohan, Graduate Research Associates. Some parts of this report also form part of Dr. Hong's doctoral dissertation at The Ohio State University. The Instruction and Research Computer Center of The Ohio State University and the Ohio Supercomputer Center provided the computational facilities.

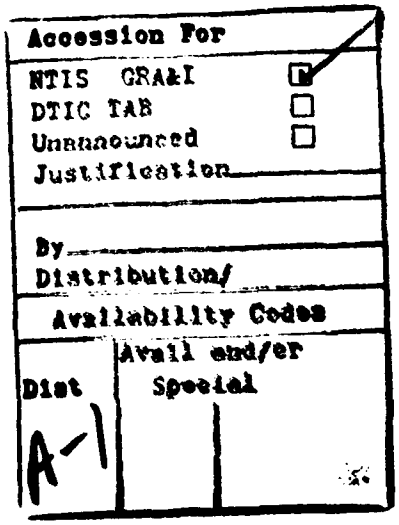


TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Section I

INTRODUCTION

Use of advanced fiber-reinforced composite laminates has been rapidly growing in structural engineering, e.g., in the design of aircraft, space vehicles, automobiles, large-span roof structures, etc. This is due to the high strength/weight ratio and the possibility for optimal design by tailoring the mechanical properties of structural components for a specific application. Increasing use of composite materials in the design of high-performance vehicles has attracted much attention to the dynamic behavior of structural components under service conditions. Experimental procedures can provide information on the real behavior of structures to the designer, but cannot cover all the design possibilities. Therefore, it is important to develop a general, as well as reliable, analysis procedure which can predict the response of composite laminates under a variety of service conditions.

Considerable research effort has been devoted to the development of analytical procedures for the analysis of composite materials. This has resulted in a variety of laminated plate theories and solution methods including, among others, classical thin plate theory [e.g., Reissner 1961, Stavsky 1961], first-order shear deformable theories [e.g., Yang 1966, Whitney 1970], higher-order theories [e.g., Whitney 1973,1974, Nelson 1974, Lo 1977, Reddy 1984a] and discrete laminate theories [e.g., Srinivas 1973, Sun 1973, Pagano 1978,1983, Seide 1980, Green 1982].

Classical thin plate theory (CPT) based on the Kirchhoff hypothesis assumes that the transverse shear deformation is negligible. For the analysis of laminated composites, it is well known [Whitney 1969, Pagano 1969,1970b, Jones 1970, Srinivas 1970] that use of CPT leads to underprediction of the transverse deflection, overprediction of natural frequencies, and higher buckling loads. This is attributed to the fact that the ratio of shear to Young's modulus is lower in most composite materials than in conventional isotropic materials. Also, the error grows with an increase in plate thickness.

This theoretical deficiency of classical thin plate theory was corrected by the shear deformable theory [Yang 1966] in which transverse shear deformation was taken into account, following Mindlin's [1951] work, for the dynamic analysis of laminated plates. Since then, various shear deformable theories have been proposed, including higher-order theories in which the power expansion for displacements contains terms of order higher than one. It has been shown [Whitney 1969, Srinivas 1970] that first order shear deformable theory may be adequate to predict global behavior of laminated plates, e.g., lateral deflection or fundamental natural frequency, but it is not better than CPT in calculating in-plane stresses because it does not include the contributions of higher shear modes. Higher-order theories lead to improved estimates of in-plane stress distributions and of the flexural vibration characteristics.

However, the shear deformable laminate theory, whether it is the first or higher-order theory, has two critical deficiencies. The first is its lack of capability to describe local deformation precisely. Due to this, it is difficult to avoid error in calculating natural frequencies as well as in-plane stresses around laminar interfaces, especially, when shear rigidities of adjacent laminae are quite different [Sun 1973, Lo

1977]. The other deficiency is the violation of equilibrium of the plate because stress continuity at the interface is, in general, not satisfied. The need to eliminate these deficiencies has motivated the development of several discrete laminated plate theories [Srinivas 1973, Sun 1973, Seide 1980] in which variation of anisotropy in the laminate is properly incorporated. The discrete laminate theory not only removes the drawbacks of shear deformable theories noted above, but it also allows different boundary conditions to be specified in each layer. It may be regarded as the most general approach capable of accurately describing the mechanical behavior of any type of laminated plates. Use of discrete laminate theories appeared to give better in-plane stress disribution [Seide 1980] and more accurate natural frequencies [Sun 1973]. However, this theory, in general, involves a large number of field equations, and consequently makes the problems quite complicated.

A basis often used for laminate theories is to assume a pattern of variation of displacements over the thickness of the plate. In such theories, which allow for shear deformation, the constitutive relations of transverse shear are, in general, not satisfied. As a result, it is not possible to avoid some error in evaluating the laminate stiffness. Since the effect of transverse shear deformation is significant in laminated composites, accuracy of analysis can be considerably affected. In particular, its effect becomes more critical in thick laminates or hybrid laminates made of layers with drastically different material properties. Many attempts have been made to treat the shear deformation realistically, but a standard procedure applicable to laminates of arbitrary construction is not available.

Since the boundary value problem of a structure constructed with composite laminates is extremely complex, approximate numerical techniques are often used to

obtain the solution. The most popular tool has been the finite element method which is usually based on a variational formulation. Several different types of element geometries, interpolation schemes and formulation strategies have been introduced, (e.g., Mawenya [1974], Reddy [1980,1984b], Bhashyam [1983], and Putcha [1986]). To provide the basis for different possible formulations, Al-Ghothani [1986] presented complementary variational formulations of the discrete laminate theory of dynamics of laminated plates following Sandhu's [1970,1971,1975,1976] procedure. Various extended and specialized forms of the general variational principle were discussed. However, he failed to derive variational principles for the direct formulation which provides another and often more useful approach for construction of approximate solution procedures.

As part of the current research program, reliable procedures were to be developed for the analysis of stresses and deformations in delamination specimens of composite laminates allowing for the coupling of flexure and extension. This required development of a theoretical model which could realistically describe the mechanical behavior of composite laminates. The discrete laminate theory was selected as quite general. This was extended to include constitutive coupling of force resultants in the lamina. In Section II, the field equations of a discrete laminate theory based on the assumed-displacement field are summarized following Srinivas [1973], Sun [1973] and Seide [1980], and its somewhat ad hoc treatment of transverse shear deformation is discussed. Section III presents a procedure based on a generalization of Reissner's method to incorporate the effect of transverse shear deformation in a consistent manner. A variational formulation of the consistent shear deformable discrete laminate theory of laminated composite plates is proposed in Section IV. Direct as well as complementary formulations are discussed. In Section V a finite element discretization procedure is

introduced. In Section VI, application of the finite element code to evaluation of stresses in some cross-ply and angle-ply free-edge delamination specimens is described along with an application to free vibration analyses.

The development of the coupled shear theory discussed herein is an important step forward in obtaining reliable estimates for stresses and deformations in laminated composites. Clearly, the new theory has certain limitations including its assumptions of vanishing transverse strain. Further refinements on introducing coupled relations for the other force resultants besides shearing forces, and allowing for variation of transverse stress over the thickness of the laminate is apparently necessary for reliable estimation of stresses in a composite laminate.

Section II

FIELD EQUATIONS OF THE DISCRETE LAMINATE THEORY OF COMPOSITE PLATES

2.1 *INTRODUCTION*

In this section, field equations of the discrete laminate theory for dynamics of laminated plates are summarized using the kinematic assumptions proposed by Srinivas [1973], Sun [1973], and Seide [1980]. The domain of definition of all functions is the Cartesian product $\bar{R}x[0,\infty)$, where $\bar{R}$ is the closure of the open, connected spatial region R occupied by the plate and $[0,\infty)$ is the positive time interval.

We consider a laminated plate of uniform thickness h composed of an arbitrary number of thin layers, in which each layer is assumed to be homogeneous, linear elastic with its material axes not necessarily coincident with the geometric coordinate axes (Figure 1.). For the Cartesian reference frame used, the origin is located in the bottom surface of the plate (x_1–x_2 axes) with x_3 axis normal to this plane. Also, in each layer a local coordinate system, $x_i^{(k)}$, is set up in a similar way with the range of $x_3^{(k)}$ limited to the thickness of k^{th} layer.

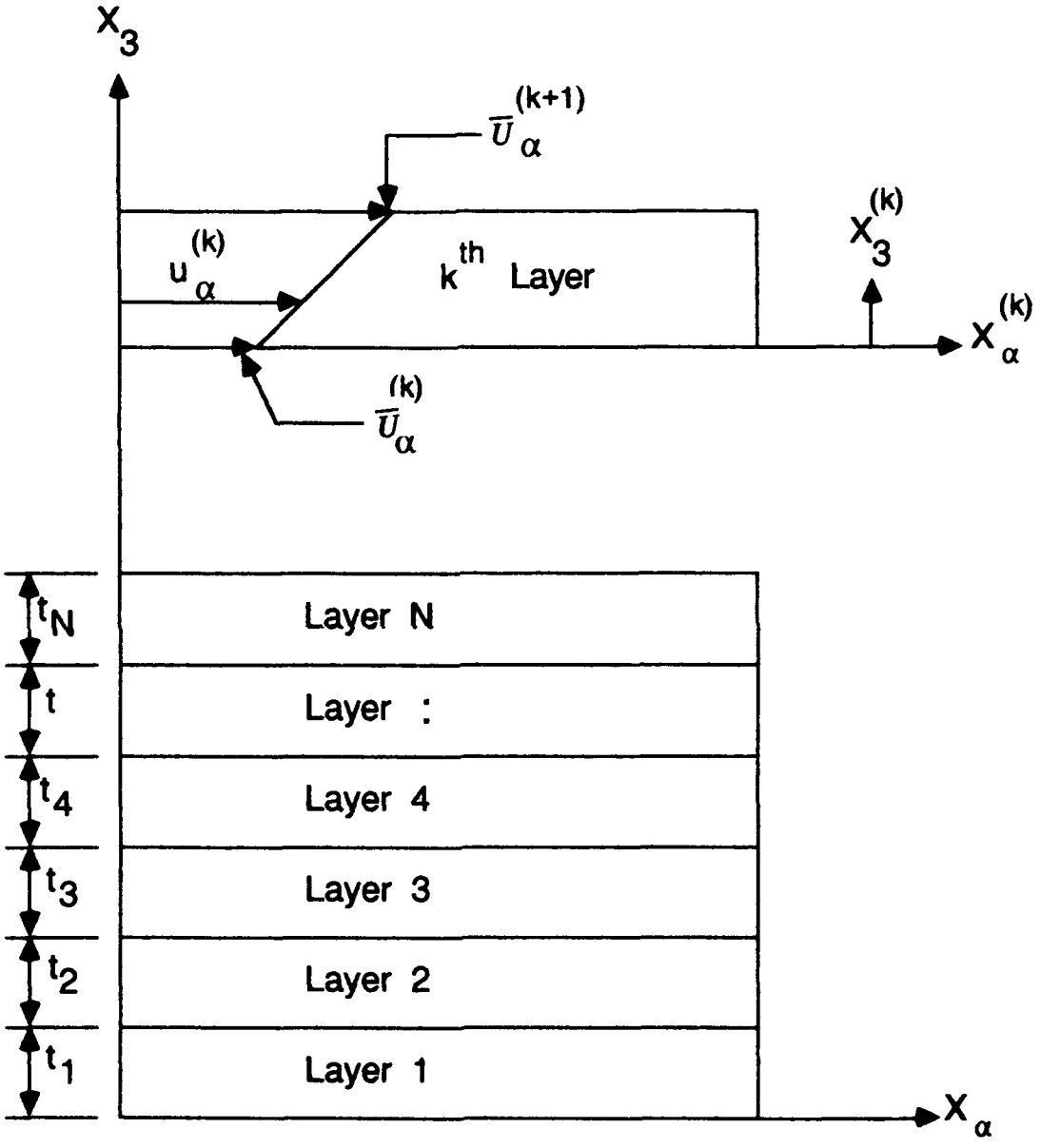


Figure 1: Global and Local Coordinate Systems in a Laminated Plate.

2.2 *FIELD EQUATIONS OF LINEAR ELASTOSTATICS*

Differential equations of equilibrium for linear elastostatics are:

$$\sigma_{ij,j} + f_i = 0 \tag{1}$$

where σ_{ij} and f_i are components of the symmetric Cauchy stress tensor and the body force vector respectively. Here, and in the sequel, we use standard indicial notation. Roman indices take on the range of values 1, 2, 3 and greek indices the values 1, 2. Summation on repeated indices is implied except where indicated otherwise. The superscript (k) denotes the k^{th} layer and is not summed. Parenthesis around a single index indicate "no sum" on that index. Parentheses around a pair of indices denote symmetric part of the tensor defined by the pair. Indices following a subscripted "comma" denote partial differentiation with respect to the spatial co-ordinate defined by the subscript.

For small deformations, the strain-displacement relations are:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

where e_{ij} and u_i are components of the strain tensor and the displacement vector, respectively.

For isothermal elasticity, the constitutive equations are:

$$\sigma_{ij} = E_{ijkl} e_{kl} \tag{3}$$

where, because of the symmetry of e_{ij} and σ_{ij}, the components E_{ijkl} of the elasticity tensor satisfy the symmetry relation

$$E_{ijkl} = E_{jikl} = E_{ijlk} \tag{4}$$

Further, assuming the existence of an energy function implies $E_{ijkl} = E_{klij}$. For a general anisotropic material, the elasticity tensor with components E_{ijkl} has 21 independent elements. If inverse of (3) exists,

$$e_{ij} = C_{ijkl}\sigma_{kl} \tag{5}$$

where C_{ijkl} are components of the compliance tensor.

2.3 *SPECIALIZATION TO A LAMINATED PLATE*

2.3.1 Kinematics

For a laminated plate subject to bending and stretching, in order to reduce the problem to one in two dimensions, the functional dependence of the displacements upon the transverse coordinate x_3 is made explicit. Often, the in-plane displacements are assumed to vary linearly within each layer and the thickness stretch is assumed to be negligible. Mathematically, for the k^{th} layer, this can be expressed as

$$u_\alpha^{(k)}(x_i,t) = \bar{u}_\alpha^{(k)}(x_\beta,t) + x_3^{(k)}\phi_\alpha^{(k)}(x_\beta,t) \tag{6}$$

$$u_3^{(k)}(x_i,t) = w^{(k)}(x_\beta,t) \tag{7}$$

where $\bar{u}_\alpha^{(k)}$, $w^{(k)}$ are the associated displacements at the bottom surface of the k^{th} layer; and $\phi_\alpha^{(k)}$ are the rotations of the cross section of the k^{th} layer. For small deformation, the kinematic relations for the k^{th} layer are (2):

$$e_{ij}^{(k)} \equiv \frac{1}{2}(u_{i,j}^{(k)} + u_{j,i}^{(k)}) \equiv u_{(i,j)}^{(k)} \tag{8}$$

Substituting (6) and (7) into (8), the strain-displacement relations for k^{th} layer become

$$e_{\alpha\beta}^{(k)} = \bar{e}_{\alpha\beta}^{(k)} + x_3^{(k)}\kappa_{\alpha\beta}^{(k)} \tag{9}$$

$$e_{\alpha 3}^{(k)} = \frac{1}{2}(\phi_\alpha^{(k)} + w_{,\alpha}^{(k)}) \tag{10}$$

$$e_{33}^{(k)} = 0 \tag{11}$$

where the kinematic variables are defined as

$$\bar{e}^{(k)}_{\alpha\beta} \equiv \frac{1}{2}(\bar{u}^{(k)}_{\alpha,\beta} + \bar{u}^{(k)}_{\beta,\alpha}) \equiv \bar{u}^{(k)}_{(\alpha,\beta)} \tag{12}$$

$$\kappa^{(k)}_{\alpha\beta} \equiv \frac{1}{2}(\phi^{(k)}_{\alpha,\beta} + \phi^{(k)}_{\beta,\alpha}) \equiv \phi^{(k)}_{(\alpha,\beta)} \tag{13}$$

2.3.2 Equilibrium Equations

The three-dimensional equations of motion of the k^{th} layer are

$$\sigma^{(k)}_{ij,j} + f^{(k)}_i = \rho^{(k)} \ddot{u}^{(k)}_i \tag{14}$$

where $\rho^{(k)}$ is the mass density. Here, superposed dots denote time derivatives of the order denoted by the number of dots. Regarding $\sigma^{(k)}_{ij}$, $f^{(k)}_i$, $u^{(k)}_i$ as functions of x_3, (14) is equivalent to

$$\int_0^{t_k} (\sigma^{(k)}_{ij,j} + f^{(k)}_i - \rho^{(k)} \ddot{u}^{(k)}_i) x_3^n dx_3 = 0$$

for $n = 0, 1, \ldots\ldots \infty$. The integration leads to a countable set of equations involving functions of x_1 and x_2. As an approximation, the values $n = 0, 1$ are generally used. Evidently, higher order equilibrium theory would use higher order of n as well. For the k^{th} layer, integration of (14) and the first moment of two of the equations (viz. $i = 1, 2$) over the thickness of the layer, for the displacement assumptions (6) and (7), gives [Al-Ghothani 1986]

$$N^{(k)}_{\alpha\beta,\beta} + (T^{(k)}_\alpha - T^{(k-1)}_\alpha) + F^{(k)}_\alpha - P^{(k)} \ddot{\bar{u}}^{(k)}_\alpha - R^{(k)} \ddot{\phi}^{(k)}_\alpha = 0 \tag{15}$$

$$M^{(k)}_{\alpha\beta,\beta} - Q^{(k)}_\alpha + G^{(k)}_\alpha + t_k T^{(k)}_\alpha - R^{(k)} \ddot{\bar{u}}^{(k)}_\alpha - I^{(k)} \ddot{\phi}^{(k)}_\alpha = 0 \tag{16}$$

$$Q^{(k)}_{\alpha,\alpha} + (T^{(k)}_3 - T^{(k-1)}_3) + F^{(k)}_3 - P^{(k)}_3 \ddot{w}^{(k)} = 0 \tag{17}$$

where

$$Q_{\alpha}^{(k)} = \int_0^{t_k} \sigma_{\alpha 3}^{(k)} dx_3^{(k)} \tag{18}$$

$$(N_{\alpha\beta}^{(k)}, M_{\alpha\beta}^{(k)}) = \int_0^{t_k} (1, x_3^{(k)})\, \sigma_{\alpha\beta}^{(k)}\, dx_3^{(k)} \tag{19}$$

$$(F_{\alpha}^{(k)}, G_{\alpha}^{(k)}) = \int_0^{t_k} (1, x_3^{(k)})\, f_{\alpha}^{(k)}\, dx_3^{(k)} \tag{20}$$

$$F_3^{(k)} = \int_0^{t_k} f_3^{(k)}\, dx_3^{(k)} \tag{21}$$

$$\{P^{(k)}, R^{(k)}, I^{(k)}\} = \int_0^{t_k} \{1, x_3^{(k)}, (x_3^{(k)})^{(2)}\}\, \rho^{(k)}\, dx_3^{(k)} \tag{22}$$

$$T_i^{(k)} = \sigma_{i3}^{(k)}(x_3^{(k)} = t_k) = \sigma_{i3}^{(k+1)}(x_3^{(k+1)} = 0) \tag{23}$$

$$T_i^{(k-1)} = \sigma_{i3}^{(k)}(x_3^{(k)} = 0) = \sigma_{i3}^{(k-1)}(x_3^{(k-1)} = t_{k-1}) \tag{24}$$

and t_k is the thickness of the k^{th} layer.

2.3.3 Constitutive Equations

For a composite lamina having material symmetry with respect to its middle surface, coupling of the extensional stresses and the shear strains vanishes and (3) reduces to [Al-Ghothani 1986]

$$\sigma_{\alpha\beta}^{(k)} = E_{\alpha\beta\gamma\delta}^{(k)} e_{\gamma\delta}^{(k)} + E_{\alpha\beta 33}^{(k)} e_{33}^{(k)} \tag{25}$$

$$\sigma_{\alpha 3}^{(k)} = 2\, E_{\alpha 3\gamma 3}^{(k)} e_{\gamma 3}^{(k)} \tag{26}$$

$$\sigma_{33}^{(k)} = E_{33\gamma\delta}^{(k)} e_{\gamma\delta}^{(k)} + E_{3333}^{(k)} e_{33}^{(k)} \tag{27}$$

Substituting (25) into (19) and using (11), the constitutive equations of bending and stretching are obtained in terms of plate kinematic variables and force resultants. Namely,

$$\begin{Bmatrix} N_{\alpha\beta}^{(k)} \\ M_{\alpha\beta}^{(k)} \end{Bmatrix} = \begin{bmatrix} A_{\alpha\beta\gamma\delta}^{(k)} & B_{\alpha\beta\gamma\delta}^{(k)} \\ B_{\alpha\beta\gamma\delta}^{(k)} & D_{\alpha\beta\gamma\delta}^{(k)} \end{bmatrix} \begin{Bmatrix} \bar{e}_{\gamma\delta}^{(k)} \\ \kappa_{\gamma\delta}^{(k)} \end{Bmatrix} \tag{28}$$

where

$$(A_{\alpha\beta\gamma\delta}^{(k)}, B_{\alpha\beta\gamma\delta}^{(k)}, D_{\alpha\beta\gamma\delta}^{(k)}) = (t_k, \frac{t_k^{(2)}}{2}, \frac{t_k^3}{3}) E_{\alpha\beta\gamma\delta}^{(k)} \tag{29}$$

It is well known from the exact elasticity solutions [Pagano 1969,1970a] that the transverse shear stress distribution is close to parabolic over the thickness of each layer. In (10), however, the transverse shear strains are constant through the thickness of a layer, which implies the constant shear stresses through (26). Furthermore, if interface continuity requirement of the transverse shear stress is enforced, the shear stress distribution becomes constant over the thickness of the entire laminate, which is far from the real situation. As a result, direct use of (26) for obtaining the plate constitutive equations yields an error in the evaluation of the plate stiffness. The usual measure to avoid this error is to multiply the shear stiffness in (26) by a coefficient K, but a standard method to determine the value of K is not available. Therefore, a procedure to obtain the shear stiffness matrix which takes into account parabolic distribution of shear stress needs to be developed so as to ensure reliability of the theory. This issue is addressed in Section III by developing consistent transverse shear constitutive relations which allow for realistic stress distributions.

2.4 *BOUNDARY CONDITIONS*

For the k^{th} layer, the boundary conditions associated with the field equations are:

$$N_{\alpha\beta}^{(k)}\eta_\beta = \hat{N}_\alpha^{(k)}(x_\beta,t) \qquad on\ S_1^{(k)}x[0,\infty) \tag{30}$$

$$M_{\alpha\beta}^{(k)}\eta_\beta = \hat{M}_\alpha^{(k)}(x_\beta,t) \qquad on\ S_3^{(k)}x[0,\infty) \tag{31}$$

$$Q_\alpha^{(k)} = \hat{Q}_\alpha^{(k)}(x_\beta,t) \qquad on\ S_5^{(k)}x[0,\infty) \tag{32}$$

$$\bar{u}_\alpha^{(k)} = \hat{\bar{u}}_\alpha^{(k)}(x_\beta,t) \qquad on\ S_2^{(k)}x[0,\infty) \tag{33}$$

$$\phi_\alpha^{(k)} = \hat{\phi}_\alpha^{(k)}(x_\beta,t) \qquad on\ S_4^{(k)}x[0,\infty) \tag{34}$$

$$w^{(k)} = \hat{w}^{(k)}(x_\beta,t) \qquad on\ S_6^{(k)}x[0,\infty) \tag{35}$$

where x_β are the coordinates along the edge boundary $S^{(k)}$ of the spatial region R occupied by the plate; a circumflex denotes the value of the prescribed quantity on $S^{(k)}$; and η_β are components of the unit outward normal to $S^{(k)}$. The boundary segments $S_1^{(k)}$, $S_2^{(k)}$ are complementary subsets of $S^{(k)}$, and so are $S_3^{(k)}$, $S_4^{(k)}$. and $S_5^{(k)}$, $S_6^{(k)}$.

2.5 *INITIAL CONDITIONS*

The initial conditions for the problem are

$$\bar{u}_\alpha^{(k)}(x_\beta,0) = \bar{u}_{\alpha o}^{(k)}(x_\beta) \tag{36}$$

$$\phi_\alpha^{(k)}(x_\beta,0) = \phi_{\alpha o}^{(k)}(x_\beta) \tag{37}$$

$$w^{(k)}(x_\beta,0) = w_o^{(k)}(x_\beta) \tag{38}$$

$$\dot{\bar{u}}_\alpha^{(k)}(x_\beta,0) = \dot{\bar{u}}_{\alpha o}^{(k)}(x_\beta) \tag{39}$$

$$\dot{\phi}_\alpha^{(k)}(x_\beta,0) = \dot{\phi}_{\alpha o}^{(k)}(x_\beta) \tag{40}$$

$$\dot{w}^{(k)}(x_\beta,0) = \dot{w}_o^{(k)}(x_\beta) \tag{41}$$

2.6 *INTERLAMINAR CONTINUITY CONDITIONS*

Since it is assumed that all the layers are perfectly bonded, continuity of displacements and tractions along interlaminar surfaces must be satisfied. The displacement continuity conditions are:

$$\bar{u}_\alpha^{(k+1)} = \bar{u}_\alpha^{(k)} + t_k \phi_\alpha^{(k)} \tag{42}$$

$$w^{(k+1)} = w^{(k)} \tag{43}$$

and the traction continuity conditions are:

$$\sigma_{i3}^{(k)}(x_3^{(k)} = t_k) = \sigma_{i3}^{(k+1)}(x_3^{(k+1)} = 0) \tag{44}$$

Through these continuity conditions, all the field equations defined for each layer can be combined to give the governing equations of the laminated plate.

In approximate solution procedures, two distinct situations may arise. In case the interlaminar traction components and the layerwise shear forces are admitted as field variables, continuity can be directly enforced. On the other hand, if a displacement type approach is used, the shearing forces obtained through material contitutive relations can be grossly in error if the simplistic kinematic assumptions (6) and (7) are used. An alternative often employed is to evaluate shearing stresses from consideration of equilibrium, i.e., obtaining $\sigma_{\alpha\beta}^{(k)}$ through the material constitutive relations but $\sigma_{\alpha 3}^{(k)}$ and $\sigma_{33}^{(k)}$ using (14). We discuss this point in Section V where the new theory is applied to free-edge delamination specimens.

Section III

CONSISTENT TREATMENT OF TRANSVERSE SHEAR DEFORMATION

3.1 INTRODUCTION

Laminate theories based on assumed displacements, in general, do not satisfy the constitutive relations of transverse shear. Since the effect of transverse shear deformation is significant in laminated composites, there could be certain loss of accuracy in the analysis due to this error. In particular, its effect could be significant in thick laminates and hybrid laminates composed of layers with drastically different shear rigidities. For this reason, to enhance the reliability of laminate theory, development of a procedure to incorporate transverse shear effect properly is necessary.

In this section, the development of constitutive equations of transverse shear in a consistent manner is described. The assumptions and notation of a discrete laminate theory given in Section II are used. The theoretical basis for development is a generalization of Reissner's mixed variational principle of linear elastic orthotropic plates to laminated composites. Reissner's principle was stated on an ad hoc basis. Herein, it is derived as an extension of the general variational principle for linear elastostatics based upon the general procedures for coupled linear problems introduced by Sandhu and his co-workers [1970, 1971, 1975, 1976]. A summary of these procedures is given in Appendix A. Throughout, it is assumed that all the functions are defined on $\bar{R}$, closure of the open connected spatial region of interest R. A rectangular Cartesian coordinate system is used.

3.2 *COMPLEMENTARY FORM OF FIELD EQUATIONS OF LINEAR ELASTOSTATICS*

The field equations (1), (2) and (5) of elasticity can be written as follows.

$$\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} + f_\alpha = 0 \tag{45}$$

$$\sigma_{3\alpha,\alpha} + \sigma_{33,3} + f_3 = 0 \tag{46}$$

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{47}$$

$$e_{\alpha 3} = \frac{1}{2}(u_{\alpha,3} + u_{3,\alpha}) \tag{48}$$

$$e_{33} = u_{3,3} \tag{49}$$

$$e_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} + 2C_{\alpha\beta\gamma 3}\sigma_{\gamma 3} + C_{\alpha\beta 33}\sigma_{33} \tag{50}$$

$$e_{\alpha 3} = C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 333}\sigma_{33} \tag{51}$$

$$e_{33} = C_{33\gamma\delta}\sigma_{\gamma\delta} + 2C_{33\gamma 3}\sigma_{\gamma 3} + C_{3333}\sigma_{33} \tag{52}$$

Here we have separated the equations involving spatial co-ordinate x_3 from the others.

3.3 *SELF-ADJOINT FORM OF FIELD EQUATIONS*

Coupled field equations of linear elastostatics (45)-(52) can be written in self-adjoint matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial 3}\delta_{\alpha\gamma} & L_2 \\ 0 & 0 & \frac{\partial}{\partial 3} & \frac{\partial}{\partial \gamma} & 0 \\ 0 & -\frac{\partial}{\partial 3} & C_{3333} & 2C_{33\gamma 3} & C_{33\gamma\delta} \\ -\frac{\partial}{\partial 3}\delta_{\alpha\gamma} & -\frac{\partial}{\partial \alpha} & 2C_{\alpha 333} & 4C_{\alpha 3\gamma 3} & 2C_{\alpha 3\gamma\delta} \\ -L_1 & 0 & C_{\alpha\beta 33} & 2C_{\alpha\beta\gamma 3} & C_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{bmatrix} u_\gamma \\ u_3 \\ \sigma_{33} \\ \sigma_{\gamma 3} \\ \sigma_{\gamma\delta} \end{bmatrix} = \begin{bmatrix} -f_\alpha \\ -f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{53}$$

in which $\delta_{\alpha\beta}$ is the identity tensor, $\frac{\partial}{\partial\theta} \equiv \frac{\partial}{\partial x_\theta}$, and

$$L_1 \equiv \frac{1}{2}(\delta_{\alpha\gamma}\frac{\partial}{\partial\beta} + \delta_{\beta\gamma}\frac{\partial}{\partial\alpha})$$

$$L_2 \equiv \frac{1}{2}(\delta_{\alpha\gamma}\frac{\partial}{\partial\delta} + \delta_{\alpha\delta}\frac{\partial}{\partial\gamma})$$

The operators on the diagonal of the matrix in (53) are symmetric tensors. If we define

$$<f, g>_R \equiv \int_R fg\, dR \tag{54}$$

the off-diagonal operators constitute adjoint pairs i.e.,

$$<u_\alpha, \sigma_{\alpha 3,3}>_R = -<\sigma_{\alpha 3}, u_{\alpha,3}>_R + <u_\alpha, \sigma_{\alpha 3}\eta_3>_{\partial R} \tag{55}$$

$$<u_3, \sigma_{33,3}>_R = -<\sigma_{33}, u_{3,3}>_R + <u_3, \sigma_{33}\eta_3>_{\partial R} \tag{56}$$

$$<u_\alpha, \sigma_{\alpha\beta,\beta}>_R = -<\sigma_{\alpha\beta}, u_{\alpha,\beta}>_R + <u_\alpha, \sigma_{\alpha\beta}\eta_\beta>_{\partial R} \tag{57}$$

$$<u_3, \sigma_{\alpha 3,\alpha}>_R = -<\sigma_{\alpha 3}, u_{3,\alpha}>_R + <u_3, \sigma_{\alpha 3}\eta_\alpha>_{\partial R} \tag{58}$$

(55) through (58) are sufficient to ensure self-adjointness of the matrix of operators in (53) in the sense of (A.25). Consistent boundary conditions associated with the field equations (53) are:

$$u_\alpha\eta_\beta = \hat{u}_\alpha\eta_\beta \quad \text{and} \quad u_\alpha\eta_3 = \hat{u}_\alpha\eta_3 \qquad \text{on} \quad S_1 \tag{59}$$

$$u_3\eta_3 = \hat{u}_3\eta_3 \quad \text{and} \quad u_3\eta_\beta = \hat{u}_3\eta_\beta \qquad \text{on} \quad S_1 \tag{60}$$

$$-(\sigma_{\alpha\beta}\eta_\beta + \sigma_{\alpha 3}\eta_3) = -\hat{t}_\alpha \quad \text{on} \quad S_2 \tag{61}$$

$$-(\sigma_{33}\eta_3 + \sigma_{\alpha 3}\eta_\alpha) = -\hat{t}_3 \qquad \text{on} \quad S_2 \tag{62}$$

where a superposed circumflex denotes the prescribed value of the quantity over the boundary surface; $\hat{t}_i$ and η_i are the components of the prescribed traction vector and of outward unit vector normal to ∂R, respectively. In addition, S_1 and S_2 are complementary subsets of ∂R. We note that in a physical problem, each component of

displacement or traction may be specified over different parts of the boundary. However, in the interest of conciseness, we denote the part of the boundary on which displacement is specified as S_1 and the portion on which traction is specified as S_2. This representation is symbolic and in no way indicative of limitations on the theory in this respect.

3.4 A GENERAL VARIATIONAL PRINCIPLE

Using the definition (A.26), the governing function for the field equations (53) and associated consistent boundary conditions (59)-(62) can be written as

$$\begin{aligned}\Omega_1 = {} & \langle u_\alpha, \sigma_{\alpha 3,3} \rangle_R + \langle u_\alpha, \sigma_{\alpha\beta,\beta} \rangle_R + \langle u_3, \sigma_{33,3} \rangle_R + \langle u_3, \sigma_{\alpha 3,\alpha} \rangle_R + 2\langle u_\alpha, f_\alpha \rangle_R \\ & + 2\langle u_3, f_3 \rangle_R - \langle \sigma_{33}, u_{3,3} \rangle_R - \langle \sigma_{\alpha 3}, (u_{\alpha,3} + u_{3,\alpha}) \rangle_R - \langle \sigma_{\alpha\beta}, u_{\alpha,\beta} \rangle \\ & + \langle \sigma_{33}, C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta} \rangle_R \\ & + \langle 2\sigma_{\alpha 3}, C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta} \rangle_R \\ & + \langle \sigma_{\alpha\beta}, C_{\alpha\beta 33}\sigma_{33} + 2C_{\alpha\beta\gamma 3}\sigma_{\gamma 3} + C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} \rangle_R \\ & + \langle \sigma_{\alpha\beta}, u_\alpha \eta_\beta - 2\hat{u}_\alpha \eta_\beta \rangle_{S_1} + \langle \sigma_{33}, u_3 \eta_3 - 2\hat{u}_3 \eta_3 \rangle_{S_1} \\ & + \langle \sigma_{\alpha 3}, (u_\alpha - 2\hat{u}_\alpha)\eta_3 \rangle_{S_1} + \langle \sigma_{\alpha 3}, (u_3 - 2\hat{u}_3)\eta_\alpha \rangle_{S_1} \\ & - \langle u_\alpha, (\sigma_{\alpha\beta}\eta_\beta + \sigma_{\alpha 3}\eta_3) - 2\hat{t}_\alpha \rangle_{S_2} - \langle u_3, \sigma_{33}\eta_3 - 2\hat{t}_3 \rangle_{S_2} \end{aligned} \tag{63}$$

Let $\{\bar{v}\} = \{\bar{u}_\alpha, \bar{u}_3, \bar{\sigma}_{\alpha\beta}, \bar{\sigma}_{\alpha 3}, \bar{\sigma}_{33}\}$ be an admissible state corresponding to the set of field variables $\{v\} = \{u_\alpha, u_3, \sigma_{\alpha\beta}, \sigma_{\alpha 3}, \sigma_{33}\}$. Assuming that $\{v\}+\lambda\{\bar{v}\}$, for λ a scalar, is an admissible state for all λ, i.e. Ω is defined at every point in a neighborhood of v, Gateaux differential of Ω_1 along $\bar{v}$ is

$$\begin{aligned}\Delta_{\bar{v}}\Omega = {} & -2\langle \bar{\sigma}_{\alpha 3}, (u_{\alpha,3} + u_{3,\alpha}) - 2C_{33\alpha 3}\sigma_{33} - 4C_{\alpha 3\gamma 3}\sigma_{\gamma 3} - 2C_{\gamma\delta\alpha 3}\sigma_{\gamma\delta} \rangle_R \\ & -2\langle \bar{\sigma}_{\alpha\beta}, u_{\alpha,\beta} - C_{33\alpha\beta}\sigma_{33} - 2C_{\gamma 3\alpha\beta}\sigma_{\gamma 3} - C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} \rangle_R \end{aligned}$$

$$
\begin{aligned}
&-<\bar{\sigma}_{33}, u_{33} - C_{3333}\sigma_{33} - 2C_{\alpha 333}\sigma_{\alpha 3} - C_{\alpha\beta 33}\sigma_{\alpha\beta}>_R \\
&+2<\bar{u}_\alpha, \sigma_{\alpha 3,3} + \sigma_{\alpha\beta,\beta} + f_\alpha>_R + 2<\bar{u}_3, \sigma_{33,3} + \sigma_{\alpha 3,\alpha} + f_3>_R \\
&+2<\bar{\sigma}_{\alpha\beta}, (u_\alpha - \hat{u}_\alpha)\eta_\beta>_{S_1} + 2<\bar{\sigma}_{33}, (u_3 - \hat{u}_3)\eta_3>_{S_1} \\
&+2<\bar{\sigma}_{\alpha 3}, (u_3 - \hat{u}_3)\eta_\alpha>_{S_1} + 2<\bar{\sigma}_{\alpha 3}, (_\alpha - \hat{u}_\alpha)\eta_3>_{S_1} \\
&-2<\bar{u}_\alpha, (\sigma_{\alpha\beta}\eta_\beta + \sigma_{\alpha 3}\eta_3) - \hat{t}_\alpha>_{S_2} - 2<\bar{u}_3, (\sigma_{33}\eta_3 + \sigma_{\alpha 3}\eta_\alpha) - \hat{t}_3>_{S_2} \qquad (64)
\end{aligned}
$$

Because of the self-adjointness of the operator matrix (53), (51)-(54), and linearity and nondegenarcy of the bilinear mapping, the Gateaux differential (64) vanishes if and only if all the field equations and boundary conditions are satisfied.

3.5 EXTENDED VARIATIONAL PRINCIPLES AND A SPECIALIZATION

Equations (55) through (58) relate pairs of off-diagonal operators in the operator matrix of (53) and may be used to eliminate either of elements in each pair from the governing function Ω_1. Elimination of an operator A_{ij} implies that the state variable u_j need not be in the domain M_{ij} *of* A_{ij}. This may result in relaxing the requirement of differentiability of u_j, thereby extending the space of admissible states.

Through this procedure, numerous extended forms of the function Ω_1 are possible. Using (55)-(58) simultaneously to eliminate $\sigma_{\alpha\beta,\beta}$, $\sigma_{\alpha 3,\alpha}$, $\sigma_{\alpha 3,3}$ and $\sigma_{33,3}$ from Ω_1, the domain of the functional is extended to include nondifferentiable stress state. Explicitly, this functional is

$$
\begin{aligned}
\Omega_2 = &-2<\sigma_{\alpha\beta}, u_{\alpha,\beta}>_R - 2<\sigma_{\alpha 3}, (u_{\alpha,3} + u_{3,\alpha})>_R - 2<\sigma_{33}, u_{3,3}>_R \\
&+2<u_\alpha, f_\alpha>_R + 2<u_3, f_3>_R \\
&+<\sigma_{33}, C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta}>_R
\end{aligned}
$$

$$+ <2\sigma_{\alpha 3}, C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}>_R$$

$$+ <\sigma_{\alpha\beta}, C_{\alpha\beta 33}\sigma_{33} + 2C_{\alpha\beta\gamma 3}\sigma_{\gamma 3} + C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}>_R$$

$$+ 2<\sigma_{\alpha\beta}, (u_\alpha - \hat{u}_\alpha)\eta_\beta>_{S_1} + 2<\sigma_{33}, (u_3 - \hat{u}_3)\eta_3>_{S_1}$$

$$+ 2<\sigma_{\alpha 3}, (u_\alpha - \hat{u}_\alpha)\eta_3>_{S_1} + 2<\sigma_{\alpha 3}, (u_3 - \hat{u}_3)\eta_\alpha>_{S_1}$$

$$+ 2<u_\alpha, \hat{t}_\alpha>_{S_2} + 2<u_3, \hat{t}_3>_{S_2} \tag{65}$$

This is equivalent to the Hellinger-Reissner mixed variational principle. For this functional, certain specializations are possible by constraining the admissible state to satisfy some of the field equations. Assuming that $(53)_5$ is identically satisfied, i.e., the constitutive equation is exactly satisfied for the "inplane" deformations and stresses, Ω_2 reduces to

$$\Omega_3 = -<\sigma_{\alpha\beta}, u_{\alpha,\beta}>_R - 2<\sigma_{\alpha 3}, (u_{\alpha,3} + u_{3,\alpha})>_R - 2<\sigma_{33}, u_{3,3}>_R$$

$$+ 2<u_\alpha, f_\alpha>_R + 2<u_3, f_3>_R$$

$$+ <\sigma_{33}, C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta}>_R$$

$$+ <2\sigma_{\alpha 3}, C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}>_R$$

$$+ 2<\sigma_{\alpha\beta}, (u_\alpha - \hat{u}_\alpha)\eta_\beta>_{S_1} + 2<\sigma_{33}, (u_3 - \hat{u}_3)\eta_3>_{S_1}$$

$$+ 2<\sigma_{\alpha 3}, (u_\alpha - \hat{u}_\alpha)\eta_3>_{S_1} + 2<\sigma_{\alpha 3}, (u_3 - \hat{u}_3)\eta_\alpha>_{S_1}$$

$$+ 2<u_\alpha, \hat{t}_\alpha>_{S_2} + 2<u_3, \hat{t}_3>_{S_2} \tag{66}$$

The only assumption to obtain Ω_3 is that the kinematic and constitutive relations of 1-2 plane are satisfied. In connection with the use of this functional in deriving a plate theory this point is noteworthy because most theories based on the assumed displacement field satisfy this requirement. Reissner [1984] presented a mixed

variational principle equivalent to Ω_3 which was derived using a Lagrange multiplier technique and partial Legendre transformation. For some special types of elastic materials with certain symmetry of material properties, the procedure for obtaining the explicit form of the principle was discussed. However, an explicit expression of the principle for a general anisotropic material was not given. The derivation above shows Reissner's ad hoc formulation to be a special case of the general variational principle of linear elastostatics. Also, Ω_3 in (66) would be more convenient than Reissner's mixed variational principle for the general anisotropic case.

If we assume further that the displacement boundary conditions on S_1 are identically satisfied, Ω_3 reduces to

$$\begin{aligned}\Omega_4 = & -\langle\sigma_{\alpha\beta}, u_{\alpha,\beta}\rangle_R - 2\langle\sigma_{\alpha 3}, (u_{\alpha,3}+u_{3,\alpha})\rangle_R - 2\langle\sigma_{33}, u_{3,3}\rangle_R \\ & + 2\langle u_\alpha, f_\alpha\rangle_R + 2\langle u_3, f_3\rangle_R \\ & + \langle\sigma_{33}, C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta}\rangle_R \\ & + \langle 2\sigma_{\alpha 3}, C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\ & + 2\langle u_\alpha, \hat{t}_\alpha\rangle_{S_2} + 2\langle u_3, \hat{t}_3\rangle_{S_2} \end{aligned} \tag{67}$$

3.6 *A VARIATIONAL PRINCIPLE FOR A LAMINATED PLATE*

The functional (67) written explicitly is:

$$\begin{aligned}\Omega_4 = \int_R \{ & \frac{1}{2}\sigma_{\alpha\beta}u_{\alpha,\beta} + \sigma_{\alpha 3}(u_{\alpha,3}+u_{3,\alpha}) + \sigma_{33}u_{3,3} \\ & - \sigma_{\alpha 3}\bar{e}_{\alpha 3} - \frac{1}{2}\sigma_{33}\bar{e}_{33} - u_i f_i \}\, dR - \int_{S_2} u_i \hat{t}_i\, ds \end{aligned} \tag{68}$$

in which

$$\bar{e}_{\alpha 3} = C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 333}\sigma_{33} \tag{69}$$

$$\bar{e}_{33} = C_{33\gamma\delta}\sigma_{\gamma\delta} + 2C_{33\gamma 3}\sigma_{\gamma 3} + C_{3333}\sigma_{33} \tag{70}$$

Recalling that in the derivation of the above functional the in-plane kinematic and constitutive relation $(53)_5$ was assumed to be identically satisfied, with some algebra, vanishing of the Gateaux differential of $\Omega_4(u_\alpha,\sigma_{\alpha\beta})$ along the path $(v_\alpha,\tau_{\alpha\beta})$, gives

$$0 = \Delta_{(v_\alpha,\tau_{\alpha\beta})}\Omega_4 = \int_R \{\, \sigma_{ij} v_{i,j} + \tau_{\alpha 3}(u_{\alpha,3} + u_{3,\alpha} - 2\bar{e}_{\alpha 3}) + \tau_{33}(u_{3,3} - \bar{e}_{33}) - v_i f_i \,\} dR - \int_{S_2} v_i \hat{t}_i \, ds \tag{71}$$

Using (55)-(58) yields

$$0 = \Delta_{(v_\alpha,\tau_{\alpha\beta})}\Omega_4 = \int_R \{ -(\sigma_{ij,j} + f_i)v_i + \tau_{\alpha 3}(u_{\alpha,3} + u_{3,\alpha} - 2\bar{e}_{\alpha 3}) + \tau_{33}(u_{3,3} - \bar{e}_{33})\} dR - \int_{S_2} v_i(t_i - \hat{t}_i) \, ds \tag{72}$$

Using the notation defined in Figure 1, the variational equation for a laminate composed of N layers is

$$0 = \int_A \{ \sum_{k=1}^{N} \int_0^{t_k} [-(\sigma_{ij,j}^{(k)} + f_i)v_i^{(k)} + \tau_{\alpha 3}^{(k)}(u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)} - 2\bar{e}_{\alpha 3}^{(k)}) + \tau_{33}^{(k)}(u_{3,3} - \bar{e}_{33}^{(k)})] dx_3^{(k)} \} dA - \int_{S_2} \left| \sum_{k=1}^{N} \int_0^{t_k} v^{(k)}(t_i^{(k)} - \hat{t}_i^{(k)}) \, dk_3^{(k)} \right| ds \tag{73}$$

3.7 CONSTITUTIVE EQUATIONS OF TRANSVERSE SHEAR

3.7.1 Assumed Transverse Shear Stresses

In order to use the mixed variational principle, developed in the previous subsection, to set up constitutive equations for the force resultants, following Reisner [1984] we propose a state of stresses in equilibrium. The stresses are stated in terms of the force resultants as follows. Assuming the components $\sigma_{\alpha\beta}^{(k)}$ to be linear in x_3, i.e.

$$\sigma_{\alpha\beta}^{(k)} = \bar{\sigma}_{\alpha\beta}^{(k)} + C_{\alpha\beta}^{(k)} x_3^{(k)} \tag{74}$$

where $\bar{\sigma}_{\alpha\beta}^{(k)}$ and $C_{\alpha\beta}^{(k)}$ are independent of $x_3^{(k)}$ coordinate, and using the definitions of force resultants it is easy to show that

$$\sigma_{\alpha\beta}^{(k)} = \frac{2}{t_k}(2 - 3\frac{x_3^{(k)}}{t_k}) N_{\alpha\beta}^{(k)} + \frac{6}{t_k^2}(2\frac{x_3^{(k)}}{t_k} - 1) M_{\alpha\beta}^{(k)} \tag{75}$$

The equilibrium equations of elasticity are, separating the in-plane equilibrium equations from the transverse ones, and ignoring inertia terms:

$$\sigma_{\alpha\beta,\beta}^{(k)} + \sigma_{\alpha 3,3}^{(k)} + f_\alpha^{(k)} = 0 \tag{76}$$

$$\sigma_{3\alpha,\alpha}^{(k)} + \sigma_{33,3}^{(k)} + f_3^{(k)} = 0 \tag{77}$$

Integrating (76) with respect to $x_3^{(k)}$,

$$\sigma_{\alpha 3}^{(k)} = T_{\alpha}^{(k-1)} - \int_0^{x_3^{(k)}} (\sigma_{\alpha\beta,\beta}^{(k)} + f_{\alpha}^{(k)})\, dx_3^{(k)} \tag{78}$$

Substituting (75) into (78),

$$\sigma_{\alpha 3}^{(k)} = T_{\alpha}^{(k-1)} - [4(\frac{x_3^{(k)}}{t_k}) - 3(\frac{x_3^{(k)}}{t_k})^2] N_{\alpha\beta,\beta}^{(k)} - \frac{6}{t_k}[(\frac{x_3^{(k)}}{t_k})^2 - (\frac{x_3^{(k)}}{t_k})] M_{\alpha\beta,\beta}^{(k)} - F_{\alpha}^{(k)} \tag{79}$$

Substituting (15) and (16) in (79) and again ignoring inertia terms,

$$\sigma_{\alpha 3}^{(k)} = T_{\alpha}^{(k-1)} + [4(\frac{x_3^{(k)}}{t_k}) - 3(\frac{x_3^{(k)}}{t_k})^2] [F_{\alpha}^{(k)} + (T_{\alpha}^{(k)} - T_{\alpha}^{(k-1)})]$$

$$- \frac{6}{t_k}[(\frac{x_3^{(k)}}{t_k})^2 - (\frac{x_3^{(k)}}{t_k})] [Q_{\alpha}^{(k)} - G_{\alpha}^{(k)} - t_k T_{\alpha}^{(k)}] - F_{\alpha}^{(k)} \tag{80}$$

In case of no body force, i.e. $F_{\alpha}^{(k)} = G_{\alpha}^{(k)} = 0$, (80) reduces to

$$\sigma_{\alpha 3}^{(k)} = \zeta_1^{(k)} Q_{\alpha}^{(k)} + \zeta_2^{(k)} T_{\alpha}^{(k-1)} + \zeta_3^{(k)} T_{\alpha}^{(k)} \tag{81}$$

where

$$\zeta_1^{(k)} = \frac{6}{t_k^2}(1 - \frac{x_3^{(k)}}{t_k}) x_3^{(k)}$$

$$\zeta_2^{(k)} = 3(\frac{x_3^{(k)}}{t_k})^2 - 4(\frac{x_3^{(k)}}{t_k}) + 1$$

$$\zeta_3^{(k)} = 3(\frac{x_3^{(k)}}{t_k})^2 - 2(\frac{x_3^{(k)}}{t_k})$$

For a monoclinic material, it is only necessary to describe $\sigma_{\alpha 3}^{(k)}$ in terms of $Q_{\alpha}^{(k)}$ to evaluate $\bar{e}_{\alpha 3}$ in (69). Using the engineering notation for elastic constants $\bar{Q}_{ij}^{(k)}$ and $S_{ij}^{(k)}$,

$$\begin{Bmatrix} 2\bar{e}_{23}^{(k)} \\ 2\bar{e}_{13}^{(k)} \end{Bmatrix} = \frac{1}{D^{(k)}} \begin{bmatrix} \bar{Q}_{55}^{(k)} & -\bar{Q}_{45}^{(k)} \\ -\bar{Q}_{45}^{(k)} & \bar{Q}_{44}^{(k)} \end{bmatrix} \begin{Bmatrix} \sigma_{23}^{(k)} \\ \sigma_{13}^{(k)} \end{Bmatrix} \equiv \begin{bmatrix} S_{44}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \sigma_{23}^{(k)} \\ \sigma_{13}^{(k)} \end{Bmatrix} \tag{82}$$

where

$$D^{(k)} = \bar{Q}_{44}^{(k)} \bar{Q}_{55}^{(k)} - (\bar{Q}_{45}^{(k)})^2$$

3.7.2 Constitutive Equations for Shear Resultants

Neglecting $\bar{e}_{33}^{(k)}$ and noting that $u_{3,3}^{(k)} = 0$ for this formulation, (73) yields the Euler equations for transverse shear

$$0 = \int_A \sum_{k=1}^{N} \left[\int_0^{t_k} \tau_{\alpha 3}^{(k)} (u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)} - 2\bar{e}_{\alpha 3}^{(k)}) \, dx_3^{(k)} \right] dA \tag{83}$$

Using (6), (7), (81) and (82), and denoting the "variation" in any quantity by the prefixed symbol δ, (83) can be rewritten as

$$\begin{aligned} 0 = \int_A \sum_{k=1}^{N} \{ & \delta[H_2^{(k)}]^T \left[\{L^{(k)}\} (\phi_2^{(k)} + w_{,2}^{(k)}) - [N^{(k)}][H_2^{(k)}, H_1^{(k)}] \begin{Bmatrix} S_{44}^{(k)} \\ S_{45}^{(k)} \end{Bmatrix} \right] \\ & + \delta[H_1^{(k)}]^T \left[\{L^{(k)}\} (\phi_1^{(k)} + w_{,1}^{(k)}) - [N^{(k)}][H_2^{(k)}, H_1^{(k)}] \begin{Bmatrix} S_{45}^{(k)} \\ S_{55}^{(k)} \end{Bmatrix} \right] \} \, dA \end{aligned} \tag{84}$$

where

$$[H_\alpha^{(k)}]^T = [Q_\alpha^{(k)}, \; T_\alpha^{(k-1)}, \; T_\alpha^{(k)}] \tag{85}$$

$$\{L^{(k)}\}^T = [L_1^{(k)}, \; L_2^{(k)}, \; L_3^{(k)}] \tag{86}$$

$$[N^{(k)}] = \begin{bmatrix} n_{11}^{(k)} & n_{12}^{(k)} & n_{13}^{(k)} \\ n_{12}^{(k)} & n_{22}^{(k)} & n_{23}^{(k)} \\ n_{13}^{(k)} & n_{23}^{(k)} & n_{33}^{(k)} \end{bmatrix} \tag{87}$$

$$L_j^{(k)} = \int_0^{t_k} \zeta_j^{(k)}\, dx_3^{(k)} \qquad j= 1,2,3 \tag{88}$$

$$n_{ij}^{(k)} = \int_0^{t_k} \zeta_i^{(k)} \zeta_j^{(k)}\, dx_3^{(k)} \qquad i,j= 1,2,3 \tag{89}$$

Explicit evaluation of integrals in (88) and (89) gives

$$L_1^{(k)} = 1, \qquad L_2^{(k)} = L_3^{(k)} = 0$$

$$n_{11}^{(k)} = \frac{6}{5t_k}, \qquad n_{12}^{(k)} = n_{13}^{(k)} = -\frac{1}{10}, \qquad n_{22}^{(k)} = n_{33}^{(k)} = \frac{2}{15} t_k, \qquad n_{23}^{(k)} = -\frac{1}{30} t_k$$

Vanishing of the integral in (84), for arbitrary values of $\delta Q_\alpha^{(k)}$ and $\delta T_\alpha^{(k)}$ gives the following constitutive equations:

$$\begin{bmatrix} \phi_1^{(k)} + w_{,1} \\ \phi_2^{(k)} + w_{,2} \end{bmatrix} = \begin{bmatrix} S_{55}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{44}^{(k)} \end{bmatrix} \left\{ \frac{6}{5t_k} \begin{bmatrix} Q_1^{(k)} \\ Q_2^{(k)} \end{bmatrix} - \frac{1}{10} \begin{bmatrix} T_1^{(k-1)} \\ T_2^{(k-1)} \end{bmatrix} - \frac{1}{10} \begin{bmatrix} T_1^{(k)} \\ T_2^{(k)} \end{bmatrix} \right\} \qquad k= 1, 2, \ldots N \tag{90}$$

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = & -\frac{1}{10} \begin{bmatrix} S_{55}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{44}^{(k)} \end{bmatrix} \begin{bmatrix} Q_1^{(k)} \\ Q_2^{(k)} \end{bmatrix} - \frac{1}{10} \begin{bmatrix} S_{55}^{(k+1)} & S_{45}^{(k+1)} \\ S_{45}^{(k+1)} & S_{44}^{(k+1)} \end{bmatrix} \begin{bmatrix} Q_1^{(k+1)} \\ Q_2^{(k+1)} \end{bmatrix} \\ & - \frac{t_k}{30} \begin{bmatrix} S_{55}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{44}^{(k)} \end{bmatrix} \begin{bmatrix} T_1^{(k-1)} \\ T_2^{(k-1)} \end{bmatrix} - \frac{t_{k+1}}{30} \begin{bmatrix} S_{55}^{(k+1)} & S_{45}^{(k+1)} \\ S_{45}^{(k+1)} & S_{44}^{(k+1)} \end{bmatrix} \begin{bmatrix} T_1^{(k+1)} \\ T_2^{(k+1)} \end{bmatrix} \\ & + \frac{2}{15} \left\{ t_k \begin{bmatrix} S_{55}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{44}^{(k)} \end{bmatrix} + t_{k+1} \begin{bmatrix} S_{55}^{(k+1)} & S_{45}^{(k+1)} \\ S_{45}^{(k+1)} & S_{44}^{(k+1)} \end{bmatrix} \right\} \begin{bmatrix} T_1^{(k)} \\ T_2^{(k)} \end{bmatrix} \qquad k= 1, 2, \ldots N-1 \end{aligned} \tag{91}$$

For a laminate of N layers, (90) and (91) constitute $2(2N-1)$ equations. These equations may be solved for $T_\alpha^{(k)}$ and $Q_\alpha^{(k)}$ in terms of $\phi_\alpha^{(k)} + w_{,\alpha}$. To do this, it is convenient to rewrite these in matrix form as

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ - & - \\ \mathbf{K}_{ab}^{T} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \chi_a \\ - \\ \chi_b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_a \\ - \\ 0 \end{bmatrix} + \begin{bmatrix} \tau_a \\ - \\ \tau_b \end{bmatrix} \tag{92}$$

where

$$\mathbf{K}_{aa} = \frac{6}{5} \begin{bmatrix} \frac{1}{t_1}\mathbf{M}_1 & 0 & 0 & \cdots & 0 \\ & \frac{1}{t_2}\mathbf{M}_2 & 0 & \cdots & 0 \\ & & \frac{1}{t_3}\mathbf{M}_3 & \cdots & 0 \\ & & & & \cdot \\ & \textit{Symm.} & & & \cdot \\ & & & & \frac{1}{t_n}\mathbf{M}_n \end{bmatrix} \tag{93}$$

$$\mathbf{K}_{ab} = -\frac{1}{10} \begin{bmatrix} \mathbf{M}_1 & 0 & 0 & \cdots & \cdot & 0 \\ \mathbf{M}_2 & \mathbf{M}_2 & 0 & \cdots & \cdot & 0 \\ 0 & \mathbf{M}_3 & \mathbf{M}_3 & \cdots & \cdot & 0 \\ & & \mathbf{M}_4 & & \cdot & 0 \\ & & & & \cdot & 0 \\ & & & & \mathbf{M}_{n-1} & \mathbf{M}_{n-1} \\ & & & & & \mathbf{M}_n \end{bmatrix} \tag{94}$$

$$\mathbf{K}_{bb} = \begin{bmatrix} \mathbf{N}_1 & -\frac{t_2}{30}\mathbf{M}_2 & 0 & \cdots & \cdot & \cdot & 0 \\ -\frac{t_2}{30}\mathbf{M}_2 & \mathbf{N}_2 & -\frac{t_3}{30}\mathbf{M}_3 & \cdots & \cdot & \cdot & 0 \\ 0 & -\frac{t_3}{30}\mathbf{M}_3 & \mathbf{N}_3 & \cdots & \cdot & \cdot & 0 \\ & & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \frac{t_{n-2}}{30}\mathbf{M}_{n-2} & 0 \\ & \textit{Symm.} & & & \frac{t_{n-2}}{30}\mathbf{M}_{n-2} & \mathbf{N}_{n-2} & -\frac{t_{n-1}}{30}\mathbf{M}_{n-1} \\ & & & & 0 & -\frac{t_{n-1}}{30}\mathbf{M}_{n-1} & \mathbf{N}_{n-1} \end{bmatrix} \tag{95}$$

$$\chi_a^T = [Q_1^{(1)}, Q_2^{(1)}, Q_1^{(2)}, Q_2^{(2)}, \ldots\ldots\ldots, Q_1^{(n)}, Q_2^{(n)}] \tag{96}$$

$$\chi_b^T = [T_1^{(1)}, T_2^{(1)}, T_1^{(2)}, T_2^{(2)}, \ldots\ldots\ldots, T_1^{(n-1)}, T_2^{(n-1)}] \tag{97}$$

$$\mathbf{R}_a^T = [\gamma_1^{(1)}, \gamma_2^{(1)}, \gamma_1^{(2)}, \gamma_2^{(2)}, \ldots\ldots\ldots, \gamma_1^{(n-1)}, \gamma_2^{(n-1)}, \gamma_1^{(n)}, \gamma_2^{(n)}] \tag{98}$$

$$\tau_a^T = \frac{1}{10}[g_1^{(0)}, g_2^{(0)}, 0, 0, \ldots\ldots, 0, 0, g_1^{(n)}, g_2^{(n)}] \tag{99}$$

$$\tau_b^T = \frac{1}{30}[t_1 g_1^{(0)}, t_1 g_2^{(0)}, 0, 0, \ldots\ldots, 0, 0, t_n g_1^{(n)}, t_n g_2^{(n)}] \tag{100}$$

and

$$\mathbf{M}_k = \begin{bmatrix} S_{55}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{44}^{(k)} \end{bmatrix} \tag{101}$$

$$\mathbf{N}_k = \frac{2}{15}(t_k \mathbf{M}_k + t_{k+1}\mathbf{M}_{k+1}) \qquad k=1,\ 2,\ \ldots\ldots,\ n\text{-}1 \tag{102}$$

$$\begin{bmatrix} g_1^{(k)} \\ g_2^{(k)} \end{bmatrix} = \begin{bmatrix} S_{55}^{(k)} & S_{45}^{(k)} \\ S_{45}^{(k)} & S_{44}^{(k)} \end{bmatrix} \begin{bmatrix} T_1^{(k)} \\ T_2^{(k)} \end{bmatrix} \qquad k=\ 1\ \textit{or}\ n \tag{103}$$

$$\gamma_\alpha^{(k)} = \phi_\alpha^{(k)} + w_{,\alpha} \qquad k=\ 1,\ 2,\ \ldots\ldots,\ n \tag{104}$$

Solving (92) to eliminate χ_b,

$$\bar{\mathbf{K}}\,\chi_a = \bar{\mathbf{R}} \tag{105}$$

where

$$\bar{\mathbf{K}} = \mathbf{K}_{aa} - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\mathbf{K}_{ab}^{T}$$

$$\bar{\mathbf{R}} = \mathbf{R}_a - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\tau_b + \tau_a$$

Inverting (105),

$$\chi_a = \bar{\mathbf{K}}^{-1}\bar{\mathbf{R}} \equiv \Lambda\bar{\mathbf{R}} \tag{106}$$

Equations (105) and (106) represent the relations between the shear forces and the shear strains. Here $\bar{\mathbf{K}}$ is symmetric because of symmetry of $\mathbf{K}_{bb}^{1}$ and, therefore, Λ is also symmetric. In (104) and (105), $\bar{\mathbf{R}}$ depends upon the shear stresses specified on the laminate surfaces so that the constitutive equations of transverse shear include dependence upon these quantities.

In general, $\bar{\mathbf{K}}$ and Λ are full matrices. Thus (105) and (106) may be rewritten, in the absence of surface tractions, as

$$(\phi_\alpha^{(k)} + w_{,\alpha}^{(k)}) = \sum_{j=1}^{n}\mu_{\alpha\beta}^{(kj)}Q_{\beta 3}^{(j)} \tag{107}$$

and

$$Q_\alpha^{(k)} = \sum_{j=1}^{n}\lambda_{\alpha\beta}^{(kj)}(\phi_\beta^{(j)} + w_{,\beta}^{(j)}) \qquad k=1,\ 2,\ \ldots,\ n \tag{108}$$

where $\lambda_{\alpha\beta}^{(kj)}$ and $\mu_{\alpha\beta}^{(kj)}$ are coefficients defined by the material properties, thickness of layers and stacking sequence of a laminate. From these relations, it is seen that the shear force in a layer is a linear combination of the transverse shear strains of all other layers and vice versa. This result is due to continuity of shear stresses in the interfaces and shows that conventional approaches to handle transverse shear are not

appropriate. Also, symmetry of matrices $\bar{\mathbf{K}}$ and Λ implies $\lambda_{\alpha\beta}^{(kj)} = \lambda_{\alpha\beta}^{(jk)}$ and $\mu_{\alpha\beta}^{(kj)} = \mu_{\alpha\beta}^{(jk)}$, which means that the contribution of unit shear strain in the j^{th} layer to the shear force of the k^{th} layer is the same as the shear force in the j^{th} layer caused by the shear strain in the k^{th} layer.

3.7.3 Specializations to the Mindlin-Type Laminate Theory

The procedure described above can be used to obtain the shear constitutive equations of Mindlin-type plate theory. For a homogeneous isotropic plate, (90) may be written as

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{5}{6} h \begin{bmatrix} \bar{Q}_{55} & 0 \\ 0 & \bar{Q}_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 + w_{,1} \\ \phi_2 + w_{,2} \end{Bmatrix} + \frac{1}{12} h \begin{Bmatrix} T_1^+ + T_1^- \\ T_2^+ + T_2^- \end{Bmatrix} \tag{109}$$

where T_α^+, T_α^- (α=1,2) denote the shearing stresses specified on the top and the bottom surface respectively. If the plate surfaces are traction-free, the relation (109) reduces to Reissner's [1947] shear constitutive equations with the shear correction factor k=5/6.

For Mindlin-type laminate theory [Yang 1966, Whitney 1970], rotation of the plate cross-section is constant and the plate shear force resultants are the algebraic sum of shear forces of all layers, i.e.,

$$\phi_\alpha = \phi_\alpha^{(k)} \text{ for all k} \quad \text{and} \quad Q_\alpha = \sum_{k=1}^{n} Q_\alpha^{(k)} \tag{110}$$

In this case, the shear constitutive equations (108) reduce to

$$Q_\alpha = \left(\sum_{k=1}^{n} \sum_{j=1}^{n} \lambda_{\alpha\beta}^{(kj)} \right) (\phi_\beta + w_{,\beta}) \tag{111}$$

3.7.4 Traction-Free Edges

For the case of free-edge delamination specimens, the transverse shear stress σ_{23} at the free-edge is known to be zero. This implies that $T_2^{(k)}$ and $Q_2^{(k)}$ at the free edge are zero. Consequently, the known quantities $T_2^{(k)}$ cannot be condensed out of (92). Explicitly specifying $T_2^{(k)}=0$ and $Q_2^{(k)}=0$, (92) may be rewritten in the form

$$\begin{bmatrix} \mathbf{K}_{(1)aa} & \mathbf{0} & \mathbf{K}_{(1)ab} & \mathbf{0} \\ - & - & - & - \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ - & - & - & - \\ \mathbf{K}^{I}_{(1)ab} & \mathbf{0} & \mathbf{K}_{(1)bb} & \mathbf{0} \\ - & - & - & - \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \chi_{(1)a} \\ - \\ \chi_{(2)a} \\ - \\ \chi_{(1)b} \\ - \\ \chi_{(2)b} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{(1)a} \\ - \\ 0 \\ - \\ 0 \\ - \\ 0 \end{bmatrix} + \begin{bmatrix} \tau_{(1)a} \\ - \\ 0 \\ - \\ \tau_{(1)b} \\ - \\ 0 \end{bmatrix} \tag{112}$$

where

$$\chi^T_{(1)a} = [Q_1^{(1)}, Q_1^{(2)}, \ldots\ldots\ldots, Q_1^{(n)}] \tag{113}$$

$$\chi^T_{(2)a} = [Q_2^{(1)}, Q_2^{(2)}, \ldots\ldots\ldots, Q_2^{(n)}] \tag{114}$$

$$\chi^T_{(1)b} = [T_1^{(1)}, T_1^{(2)}, \ldots\ldots\ldots, T_1^{(n-1)}] \tag{115}$$

$$\chi^T_{(2)b} = [T_2^{(1)}, T_2^{(2)}, \ldots\ldots\ldots, T_2^{(n-1)}] \tag{116}$$

$$\tau^T_{(1)a} = \frac{1}{10}[g_1^{(0)}, 0, 0, \ldots\ldots, 0, 0, g_1^{(n)}] \tag{117}$$

$$\tau^I_{(1)b} = \frac{1}{30}[t_1 g_1^{(0)}, 0, 0, \ldots\ldots, 0, 0, t_n g_1^{(n)}] \tag{118}$$

$$\mathbf{R}^T_{(1)a} = [\gamma_1^{(1)}, \gamma_1^{(2)}, \ldots\ldots\ldots, \gamma_1^{(n-1)}, \gamma_1^{(n)}] \tag{119}$$

and $\mathbf{K}_{(1)aa}$, $\mathbf{K}_{(1)ab}$ and $\mathbf{K}_{(1)bb}$ are obtained by taking the rows and columns corresponding to $T_1^{(k)}$ and $Q_1^{(k)}$ from , $\mathbf{K}_{ab}$ and $\mathbf{K}_{bb}$ respectively.

Eliminating $\chi_{(1)b}$ from (112),

$$\bar{\mathbf{K}}_{(1)}\chi_{(1)a} = \bar{\mathbf{R}}_{(1)} \tag{120}$$

where

$$\bar{\mathbf{K}}_{(1)} = \mathbf{K}_{(1)aa} - \mathbf{K}_{(1)ab}\mathbf{K}^{-1}_{(1)bb}\mathbf{K}^{T}_{(1)ab}$$

$$\bar{\mathbf{R}}_{(1)} = \mathbf{R}_{(1)a} - \mathbf{K}_{(1)ab}\mathbf{K}^{-1}_{(1)bb}\tau_{(1)b} + \tau_{(1)a}$$

In the absence of surface stresses, constitutive relations of the form (107), (108) at the traction free edge are

$$(\phi_1^{(k)} + w_{,1}^{(k)}) = \sum_{j=1}^{n} \bar{\mu}_{11}^{(kj)} Q_1^{(j)} \tag{121}$$

and

$$Q_1^{(k)} = \sum_{j=1}^{n} \bar{\lambda}_{11}^{(kj)} (\phi_1^{(j)} + w_{,1}^{(j)}) \qquad k=1,\ 2,\ \ldots,\ n \tag{122}$$

where $\bar{\lambda}_{11}^{(kj)}$ and $\bar{\mu}_{11}^{(kj)}$ are the constitutive coeeficients at the free-edge.

3.8 AN EXAMPLE OF COUPLED SHEAR CONSTITUTIVE RELATIONS

For a graphite-epoxy laminate, made up of 12 layers, each 0.005 inch thick, let the material properties referred to the material axes be

$$E_{11} = 19.0\times10^{6}\ ,\quad E_{22} = E_{33} = 1.5\times10^{6}\quad (psi)$$

$$G_{12} = G_{13} = 0.8\times10^{6}\ ,\quad G_{23} = 0.528\times10^{6}\quad (psi) \tag{123}$$

$$\nu_{12} = \nu_{13} = 0.3\ ,\quad \nu_{23} = 0.42$$

To study the role of coupling in constitutive relations for shear forces, we consider the stackings $[0_3/90_3]_s$ and $[+45_3/-45_3]_s$.

Table 1 shows the transverse shear stiffness coefficients for the $[0_3/90_3]$ laminate and Table 2 contains those for the $[+45_3/-45_3]$ laminate. Only the coefficients corresponding to the transverse shear stress resultants $Q_1^{(k)}$ have been listed, i.e., the $\lambda_{1\alpha}^{(kj)}$s. For the first laminate, $Q_1^{(k)}$ and $Q_2^{(k)}$ are uncoupled; i.e., $\lambda_{12}^{(kj)} = \lambda_{21}^{(kj)} = 0$. For the second laminate the coefficients for $Q_1^{(k)}$ and $Q_2(k)$ are identical due to the fibre orientation of 45°; i.e., $\lambda_{11}^{(kj)} = \lambda_{22}^{(kj)}$. The diagonal terms, $\lambda_{\alpha\beta}^{kk}$, represent the shearing force in each layer due to unit shear deformation of the same layer. The off-diagonal terms represent the coupling between layers. As is evident, for the cases studied the interlayer coupling is not "strong" i.e., the shearing force in any layer is not significantly influenced by the deformation of the others. Also, the effect is localized i.e., the contribution of deformation of any layer to the shearing force in another decreases sharply with the distance between the layers. Table 3 and Table 4 show the inverse of the stiffness coefficients i.e, the compliance coefficients $\mu_{1\alpha}^{(kj)}$.

It should be noted that in the case where $G_{13} = G_{23}$ there will be no coupling between $Q_1^{(k)}$ and $Q_2^{(k)}$ by virtue of $\overline{Q}_{45}^{(k)}$ in (82) being zero. Moreover, the inter-layer coupling will be independent of the fibre orientation as $\overline{Q}_{44}^{(k)}$ and $\overline{Q}_{55}^{k}$ will no longer be affected by the orientation of the fibres.

Table 1

Transverse Shear Stiffnesses λ_{11}^{kJ} for $[0_3/90_3]$ Laminate

Layer	Stiffness Coefficients					
(k)	λ_{11}^{k1}	λ_{11}^{k2}	λ_{11}^{k3}	λ_{11}^{k4}	λ_{11}^{k5}	λ_{11}^{k6}
1	3447.698	133.869	22.184	3.137	0.538	0.092
2	133.869	3603.755	155.289	21.958	3.767	0.646
3	22.184	155.289	3576.224	128.611	22.066	3.786
4	3.137	21.958	128.611	2404.119	110.513	18.961
5	0.538	3.767	22.066	110.513	2382.900	106.872
6	0.092	0.646	3.786	18.961	106.872	2382.300
7	0.016	0.111	0.649	3.253	18.337	106.770
8	0.003	0.019	0.111	0.559	3.149	18.337
9	*	0.003	0.019	0.099	0.559	3.253
10	*	*	0.004	0.019	0.111	0.649
11	*	*	*	0.003	0.019	0.111
12	*	*	*	*	0.003	0.015

* denotes coefficients smaller than 10^{-3} in magnitude.

$\lambda_{12}^{kJ} = \lambda_{21}^{kJ} = 0.$

Table 2

Transverse Shear Stiffnesses $\lambda_{i\alpha}^{kj}$s for $[+45_3/-45_3]_s$ laminate.

Layer.	Stiffness Coefficients											
(k)	λ_{11}^{k1}	λ_{12}^{k1}	λ_{11}^{k2}	λ_{12}^{k2}	λ_{11}^{k3}	λ_{12}^{k3}	λ_{11}^{k4}	λ_{12}^{k4}	λ_{11}^{k5}	λ_{12}^{k5}	λ_{11}^{k6}	λ_{12}^{k6}
1	2861.599	586.094	111.202	22.666	18.946	3.238	3.136	*	0.538	*	0.092	*
2	111.202	22.666	2991.757	612.002	132.622	22.667	21.957	*	3.767	*	0.646	*
3	18.946	3.238	132.622	22.667	2990.123	586.098	128.609	*	22.066	*	3.785	*
4	3.137	*	21.957	*	128.609	*	2990.234	-586.111	133.294	-22.781	22.870	-3.908
5	0.538	*	3.767	*	22.066	*	133.294	-22.781	2995.707	-612.803	134.230	-27.360
6	0.092	*	0.646	*	3.786	*	22.869	-3.908	134.233	-27.360	2995.900	-613.590
7	0.016	*	0.111	*	0.649	*	3.924	-0.670	23.030	-4.693	134.260	-27.492
8	0.003	*	0.019	*	0.111	*	0.673	-0.114	3.950	-0.801	23.031	-4.693
9	*	*	0.003	*	0.019	*	0.115	-0.015	0.673	-0.114	3.923	-0.670
10	*	*	*	*	0.003	*	0.019	*	0.114	*	0.649	*
11	*	*	*	*	*	*	0.003	*	0.019	*	0.111	*
12	*	*	*	*	*	*	*	*	0.003	*	0.015	*

* denotes coefficients smaller than 10^{-3} in magnitude.

Table 3

Transverse Shear Compliances μ_{11}^{kj} for $[0_3/90_3]_s$ Laminate

Layer	Compliance Coefficients ($\times 10^{-3}$)					
(k)	μ_{11}^{k1}	μ_{11}^{k2}	μ_{11}^{k3}	μ_{11}^{k4}	μ_{11}^{k5}	μ_{11}^{k6}
1	0.2904	-0.0107	-0.0013	-0.0002	*	*
2	-0.0107	0.2784	-0.0119	-0.0018	-0.0002	*
3	-0.0013	-0.0119	0.2807	-0.0148	-0.0018	-0.0002
4	-0.0002	-0.0018	-0.0148	0.4176	-0.0191	-0.0024
5	*	-0.0002	-0.0028	-0.0191	0.4214	-0.0186
6	*	*	-0.0002	-0.0024	-0.0186	0.4214
7	*	*	*	-0.0003	-0.0023	-0.0186
8	*	*	*	*	-0.0003	-0.0024
9	*	*	*	*	*	-0.0003
10	*	*	*	*	*	*
11	*	*	*	*	*	*
12	*	*	*	*	*	*

* denotes coefficients smaller than 10^{-6} in magnitude.

Table 4

Transverse Shear Compliances $\mu_{1\alpha}^{k1}$ for $[+45_3/-45_3]$ Laminate

Layer	Compliance Coefficients ($\times 10^{-3}$)											
(k)	μ_{11}^{k1}	μ_{12}^{k1}	μ_{12}^{k2}	μ_{12}^{k2}	μ_{11}^{k3}	μ_{12}^{k3}	μ_{11}^{k4}	μ_{12}^{k4}	μ_{11}^{k5}	μ_{12}^{k5}	μ_{11}^{k6}	μ_{12}^{k6}
1	0.3653	-0.0748	-0.0135	0.0027	-0.0017	0.0004	-0.0002	*	*	*	*	*
2	-0.0135	0.0027	0.3500	-0.0716	-0.0155	0.0035	-0.0018	*	-0.0002	*	*	*
3	-0.0017	0.0004	-0.0155	0.0035	0.3492	-0.0684	-0.0148	*	-0.0018	*	-0.0002	*
4	-0.0002	*	-0.0018	*	-0.0148	*	0.3492	0.0684	-0.0155	-0.0035	-0.0019	-0.0004
5	*	*	-0.0002	*	-0.0018	*	-0.0155	-0.0035	0.3498	0.0716	-0.0155	-0.0031
6	*	*	*	*	-0.0002	*	-0.0019	-0.0004	-0.0155	-0.0031	0.3498	0.0716
7	*	*	*	*	*	*	-0.0002	*	-0.0019	-0.0004	-0.0155	-0.0031
8	*	*	*	*	*	*	*	*	-0.0002	*	-0.0019	-0.0004
9	*	*	*	*	*	*	*	*	*	*	-0.0002	*
10	*	*	*	*	*	*	*	*	*	*	*	*
11	*	*	*	*	*	*	*	*	*	*	*	*
12	*	*	*	*	*	*	*	*	*	*	*	*

* denotes coefficients smaller than 10^{-6} in magnitude.

3.9 *DISCUSSION*

For determining the constitutive equations for transverse shear in discrete laminated plate theory, a mixed variational principle of linear elastostatics has been derived. The basis for derivation was the method proposed by Sandhu [1970,1971,1975] for the variational formulation of linear coupled problems with multiple field variables. The variational principle is equivalent to Reissner's mixed variational principle [1984], but more convenient for application to a material with general anisotropy. Using this mixed variational principle, a procedure to obtain the constitutive relations for transverse shear has been developed for a discrete laminate theory which is based on the assumptions of linear in-plane displacements and parabolic transverse shear stresses over the thickness of each layer. The procedure allows for the interlaminar continuities of stresses and displacements. Resulting constitutive equations show that the shear force resultants of a layer are coupled with the shear strains of the other layers as well as of different directions (x_1 and x_2). As indicated by earlier investigators, the shear stiffness of x_1-x_2 and x_2-x_3 sections, in general, are different and vary with stacking sequence of a laminate. Also, the consistent shear constitutive relations for the Mindlin-type laminate theory have been derived as a special case. Actual computation of the shear stiffness requires inversion of a square matrix with constant elements. This can be carried out with a high-speed digital computer without much difficulty. The constitutive relation for shear at the freee-edge or surfaces on which shearing stresses are specified is different from the relation when these stresses are not specified. The example of a 12-layer graphite-epoxy laminate was considered using $[0_3/90_3]_s$ and $[+45_3/-45_3]_s$ stackings and the extent of coupling studied.

Section IV

VARIATIONAL FORMULATION OF DISCRETE LAMINATE THEORY

4.1 INTRODUCTION

Procedures for obtaining approximate numerical solutions to boundary value problems are often based on variational formulations. For systematic development of variational principles governing linear and certain nonlinear problems, general procedures have been developed. Mikhlin [1965] set up the problem in an inner product space and stated the basic variational theorem for a self-adjoint boundary value problem with homogeneous boundary conditions. For deriving variational principles governing initial value problems, Gurtin [1963,1964] used convolution product as the nondegenerate bilinear mapping and explicitly included nonhomogeneous initial and boundary conditions in the formulation. Sandhu [1970,1971] extended these ideas to the general linear coupled problem. In the context of application of the finite element method, Prager [1968] included in the variational formulation jump discontinuities which may exist across interelement boundaries. By introducing the concept of boundary operators consistent with the field operators, Sandhu [1975] examined the general case of linear operators with nonhomogeneous boundary conditions and internal jump discontinuities.

For mechanics of the fiber-reinforced composite laminates, little work has been done on variational formulation of the problem. Al-Ghothani [1986], following Sandhu [1970,1971,1975,1976], presented a variational formulation of dynamics of laminated

composite plate. General variational principle was derived based on the complementary form of an extension of Seide's [1980] discrete laminate plate theory to include inertial force, allowing for nonhomogeneous boundary conditions and internal jump discontinuities. Various extended and specialized forms of the general variational principles were discussed. However, he failed to derive direct variational formulation which gives other types of variational principles. Furthermore, the laminate theory used did not treat the effect of transverse shear deformation adequately. This effect is important in studying local deformation and possibly in modelling higher vibration modes.

In this section, a variational formulation of the problem of vibration of a laminated composite plate allowing for nonhomogeneous boundary conditions as well as internal discontinuities is developed. The theory used is the one described in Section II along with the constitutive equations of the transverse shear derived in Section III. Even though the principal concern of the research program is the behavior of free-edge delamination specimens under static loading, inertia effects and arbitrary geometric configuration and loading are included in the variational formulation because of the ease with which such generality could be introduced. Both the direct and the complementary forms of the field equations are considered. Extended variational principles based on self-adjointness of the operator matrix are introduced along with several specializations. One of their specializations formed the basis of the finite element approximation described in Section V.

4.2 *INTEGRAL FORM OF FIELD EQUATIONS*

4.2.1 General Comments

To set up the function governing the motion of laminated plates, it is necessary to write the field equations in a way that the operator matrix is self-adjoint in a certain space. The self-adjointness of operators is not an absolute notion, but rather, it is relative to the choice of an appropriate bilinear mapping. Thus, there are two possible ways to set up variational principles governing the problem; one is to find a bilinear mapping that makes the field operators self-adjoint, and the other is to transform the field equations so that they can be self-adjoint with respect to a familiar form of bilinear mapping. For various intial-boundary value problem, Gurtin's [1963,1964] procedure, which belongs to the latter approach, has been successfully applied [Sandhu 1971,1987], and we follow it for the present problem although other forms can be used. Transformation of the differential form of the field equations to the equivalent integral form is done by applying Laplace Transform and taking inverse after appropriate rearrangement. The procedure removes the time derivatives from the equilibrium equations and includes initial conditions explicitly. For the field equations given in Section II along with the constitutive equations of transverse shear derived in Section III, integral form of the field equations is presented below. Throughout, an asterisk (*) denotes the convolution integral, i.e.

$$u * v = \int_0^t u(\tau)\, v(t-\tau)\, d\tau \tag{124}$$

We note that the convolution satisfies distributive, associative and commutative laws.

4.2.2 Kinematics

Equations (9) and (10) upon taking convolution with the time variable become:

$$t^* e_{\alpha\beta}^{(k)} = t^* \bar{e}_{\alpha\beta}^{(k)} + x_3^{(k)} t^* \kappa_{\alpha\beta}^{(k)} \tag{125}$$

$$t^* e_{\alpha 3}^{(k)} = \frac{1}{2} t^* (\phi_\alpha^{(k)} + w_{,\alpha}^{(k)}) \tag{126}$$

where, by (12) and (13),

$$t^* \bar{e}_{\alpha\beta}^{(k)} = \frac{1}{2} t^* (\bar{u}_{\alpha,\beta}^{(k)} + \bar{u}_{\beta,\alpha}^{(k)}) \tag{127}$$

$$t^* \kappa_{\alpha\beta}^{(k)} = \frac{1}{2} t^* (\phi_{\alpha,\beta}^{(k)} + \phi_{\beta,\alpha}^{(k)}) \tag{128}$$

4.2.3 Equilibrium Equations

Equations (15) through (17) upon convolution with t and appropriate integration to eliminate derivatives with respect to time give;

$$t^* N_{\alpha\beta,\beta}^{(k)} + t^* (T_\alpha^{(k)} - T_\alpha^{(k-1)}) + t^* F_\alpha - P^{(k)} \bar{u}_\alpha^{(k)} - R^{(k)} \phi_\alpha^{(k)} + X_\alpha^{(k)} = 0 \tag{129}$$

$$t^* M_{\alpha\beta,\beta}^{(k)} - t^* Q_\alpha + t^* (t_k T_\alpha^{(k)} + G_\alpha) - R^{(k)} \bar{u}_\alpha^{(k)} - I^{(k)} \phi_\alpha^{(k)} + Y_\alpha^{(k)} = 0 \tag{130}$$

$$t^* Q_{\alpha,\alpha}^{(k)} + t^* (T_3^{(k)} - T_3^{(k-1)}) + t^* F_3^{(k)} - P^{(k)} w^{(k)} + Z^{(k)} = 0 \tag{131}$$

where

$$X_\alpha^{(k)} = P^{(k)} (\bar{u}_{\alpha 0}^{(k)} + t^* \dot{\bar{u}}_{\alpha 0}^{(k)}) + R^{(k)} (\phi_{\alpha 0}^{(k)} + t^* \dot{\phi}_{\alpha 0}^{(k)}) \tag{132}$$

$$Y_\alpha^{(k)} = R^{(k)} (\bar{u}_{\alpha 0}^{(k)} + t^* \dot{\bar{u}}_{\alpha 0}^{(k)}) + I^{(k)} (\phi_{\alpha 0}^{(k)} + t^* \dot{\phi}_{\alpha 0}^{(k)}) \tag{133}$$

$$Z^{(k)} = P^{(k)} (w_0^{(k)} + t^* \dot{w}_0^{(k)}) \tag{134}$$

The initial conditions (36) through (41) appear explicitly in the equilibrium equations above.

4.2.4 Constitutive Equations

Equation (28) upon convolution with t yields;

$$t^* \begin{bmatrix} N^{(k)}_{\alpha\beta} \\ M^{(k)}_{\alpha\beta} \end{bmatrix} = t^* \begin{bmatrix} A^{(k)}_{\alpha\beta\gamma\delta} & B^{(k)}_{\alpha\beta\gamma\delta} \\ B^{(k)}_{\alpha\beta\gamma\delta} & D^{(k)}_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{bmatrix} \bar{e}^{(k)}_{\gamma\delta} \\ \kappa^{(k)}_{\gamma\delta} \end{bmatrix} \tag{135}$$

and (108) upon convolution with t, noting (10) gives,

$$t^* Q^{(k)}_{\alpha} = 2t^* \sum_{j=1}^{n} \lambda^{(kj)}_{\alpha\beta} e^{(j)}_{\beta 3} \tag{136}$$

The inverse relations are

$$t^* \begin{bmatrix} \bar{e}^{(k)}_{\alpha\beta} \\ \kappa^{(k)}_{\alpha\beta} \end{bmatrix} = t^* \begin{bmatrix} \bar{A}^{(k)}_{\alpha\beta\gamma\delta} & \bar{B}^{(k)}_{\alpha\beta\gamma\delta} \\ \bar{B}^{(k)}_{\alpha\beta\gamma\delta} & \bar{D}^{(k)}_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{bmatrix} N^{(k)}_{\gamma\delta} \\ M^{(k)}_{\gamma\delta} \end{bmatrix} \tag{137}$$

and

$$t^* (\phi^{(k)}_{\alpha} + w^{(k)}_{,\alpha}) = t^* \sum_{j=1}^{n} \mu^{(kj)}_{\alpha\beta} Q^{(j)}_{\beta} \tag{138}$$

4.2.5 Boundary Conditions

As with the field equations, the boundary conditions (30) through (35) upon convolution with t give

$$t^* N^{(k)}_{\alpha\beta} \eta_{\beta} = t^* \hat{N}_{\alpha} \qquad on \ S^{(k)}_1 \tag{139}$$

$$t^* M^{(k)}_{\alpha\beta} \eta_{\beta} = t^* \hat{M}_{\alpha} \qquad on \ S^{(k)}_3 \tag{140}$$

$$t^* Q^{(k)}_{\alpha} = t^* \hat{Q}^{(k)}_{\alpha} \qquad on \ S^{(k)}_5 \tag{141}$$

$$t^* \bar{u}^{(k)}_{\alpha} = t^* \hat{\bar{u}}^{(k)}_{\alpha} \qquad on \ S^{(k)}_2 \tag{142}$$

$$t^* \phi^{(k)}_{\alpha} = t^* \hat{\phi}^{(k)}_{\alpha} \qquad on \ S^{(k)}_4 \tag{143}$$

$$t^* w^{(k)} = t^* \hat{w}^{(k)} \qquad on \ S^{(k)}_6 \tag{144}$$

4.2.6 Interlaminar Continuity of Displacements

Equations (42) and (43) upon convolution with t give

$$t^* \bar{u}_\alpha^{(k+1)} = t^* \bar{u}_\alpha^{(k)} + t^* t_k \phi_\alpha^{(k)} \tag{145}$$

$$t^* w^{(k)} = t^* w^{(k+1)} \tag{146}$$

4.3 DIRECT VARIATIONAL FORMULATION

4.3.1 Self-Adjoint Form of the Field Equations

The field equations of fiber-reinforced composite laminated plate expressed in integral form, (125)-(136) and (145)-(146) can be written in the self-adjoint matrix form as

$$\begin{bmatrix}
\mathbf{A}_1 & \mathbf{B}_1 & \mathbf{D}_{1,2} & 0 & \mathbf{D}_{1,3} & 0 & \cdots & \mathbf{D}_{1,n-1} & 0 & \mathbf{D}_{1,n} \\
0 & \mathbf{C}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & \mathbf{A}_2 & \mathbf{B}_2 & \mathbf{D}_{2,3} & 0 & \cdots & \mathbf{D}_{2,n-1} & 0 & \mathbf{D}_{2,n} \\
 & 0 & \mathbf{C}^T & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & & A_3 & \mathbf{B}_3 & \cdots & & \mathbf{D}_{3,n-1} & 0 & \mathbf{D}_{3,n} \\
 & & & 0 & & & 0 & 0 & 0 \\
 & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & \mathbf{A}_{n-1} & \mathbf{B}_{n-1} & \mathbf{D}_{n-1,n} \\
 & & & & & & & 0 & \mathbf{C}^T \\
 & & & & & & & & \mathbf{A}_n
\end{bmatrix}
\begin{Bmatrix}
\mathbf{U}_1 \\ \Xi_1 \\ \mathbf{U}_2 \\ \Xi_2 \\ \mathbf{U}_3 \\ \Xi_3 \\ \cdot \\ \cdot \\ \mathbf{U}_{n-1} \\ \Xi_{n-1} \\ \mathbf{U}_n
\end{Bmatrix}
=
\begin{Bmatrix}
-\mathbf{r}_1 + \Xi_0 \\ 0 \\ -\mathbf{r}_2 \\ 0 \\ -\mathbf{r}_3 \\ 0 \\ \cdot \\ \cdot \\ -\mathbf{r}_{n-1} \\ 0 \\ -\mathbf{r}_n - \Xi_n
\end{Bmatrix} \tag{147}$$

or, symbolically,

$$[X]\{Y\} = \{Z\}$$

Here, only the operators in the upper triangular region have been entered. The below diagonal operators are adjoints of the above diagonal operators, i.e., the operator A_{ij} is adjoint of A_{ji} in the sense of the bilinear mapping used to set up the variational

formulation. To satisfy the self-adjointness condition (A-25) in Appendix A, it is sufficient that the elements X_{ij} of the matrix X be adjoints of elements X_{ji} for $i \neq j$ and the diagonal elements be self-adjoint. Explicitly, the symbolic operators applying in (147) are:

$$\mathbf{A}_k = \begin{vmatrix} -P^{(k)}\delta_{\alpha\gamma} & 0 & t^*\Gamma_1 & -R^{(k)}\delta_{\alpha\gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & -t^* A^{(k)}_{\alpha\beta\gamma\delta} & t^* & 0 & -t^* B^{(k)}_{\alpha\beta\gamma\delta} & 0 & 0 & 0 & 0 \\ -t^*\Gamma_2 & t^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R^{(k)}\delta_{\alpha\gamma} & 0 & 0 & I^{(k)}\delta_{\alpha\gamma} & 0 & t^*\Gamma_1 & 0 & 0 & -t^* \\ 0 & t^* B^{(k)}_{\alpha\beta\gamma\delta} & 0 & 0 & t^* D^{(k)}_{\alpha\beta\gamma\delta} & t^* & 0 & 0 & 0 \\ 0 & 0 & 0 & -t^*\Gamma_2 & t^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -P^{(k)} & 0 & t^*\delta_{\alpha\gamma}\dfrac{\partial}{\partial\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^*\lambda^{kk}_{\gamma\alpha} & t^* \\ 0 & 0 & 0 & -t^* & 0 & 0 & -t^*\delta_{\alpha\gamma}\dfrac{\partial}{\partial\gamma} & t^* & 0 \end{vmatrix} \tag{148}$$

$$\mathbf{B}_k^T = \begin{vmatrix} t^* & 0 & 0 & t^* t_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^* & 0 & 0 \end{vmatrix} \tag{149}$$

$$\mathbf{C}_k^T = \begin{vmatrix} -t^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -t^* & 0 & 0 \end{vmatrix} \tag{150}$$

$$\mathbf{D}_{i,j} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^*\lambda^{ij}_{\gamma\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \qquad i,j = 1, 2, \ldots, n \tag{151}$$

$$\mathbf{U}_k^T = [\bar{u}^{(k)}_\alpha, \bar{e}^{(k)}_{\gamma\delta}, N^{(k)}_{\alpha\beta}, \phi^{(k)}_\alpha, \kappa^{(k)}_{\gamma\delta}, M^{(k)}_{\alpha\beta}, w^{(k)}, 2e^{(k)}_{\alpha 3}, Q^{(k)}_\gamma] \tag{152}$$

$$\Xi_k^T = [T_\alpha^{(k)}, T_3^{(k)}] \tag{153}$$

$$\mathbf{r}_k^T = [t^* F_\gamma^{(k)} + X_\gamma^{(k)}, 0, 0, t^* G_\gamma + Y_\gamma^{(k)}, 0, 0, t^* F_3^{(k)} + Z^{(k)}, 0, 0] \tag{154}$$

$$\Xi_0^T = [t^* T_\alpha^{(0)}, 0, 0, 0, 0, 0, t^* T_3^{(0)}, 0, 0] \tag{155}$$

$$\Xi_n^T = [t^* T_\alpha^{(n)}, 0, 0, t^* t_n T_\alpha^{(n)}, 0, 0, t^* T_3^{(n)}, 0, 0] \tag{156}$$

$$\Gamma_1 = \frac{1}{2}\left(\delta_{\alpha\gamma}\frac{\partial}{\partial\beta} + \delta_{\beta\gamma}\frac{\partial}{\partial\alpha}\right) \tag{157}$$

$$\Gamma_2 = \frac{1}{2}\left(\delta_{\alpha\gamma}\frac{\partial}{\partial\delta} + \delta_{\alpha\delta}\frac{\partial}{\partial\gamma}\right) \tag{158}$$

Here, $T_i^{(0)}$ and $T_i^{(n)}$ are specified shear stress components on the top and bottom surface of the plate and $\delta_{\alpha\beta}$ is Kronecker's delta. Operator matrix in (147) is self-adjoint in the sense of (A-25) if the bilinear mapping is defined as

$$\langle u, v \rangle = \int_R u^* v \, dv \tag{159}$$

which is linear and nondegenerate. Nondegeneracy of this bilinear mapping was shown by Gurtin [1963,1964].

The boundary conditions consistent with the operator equations (147) are

$$t^* N_{\alpha\beta}^{(k)} \eta_\beta = t^* \hat{N}_\alpha \qquad on \; S_1^{(k)}(x_\alpha^{(k)}) \tag{160}$$

$$t^* M_{\alpha\beta}^{(k)} \eta_\beta = t^* \hat{M}_\alpha \qquad on \; S_3^{(k)}(x_\alpha^{(k)}) \tag{161}$$

$$t^* Q_\alpha^{(k)} \eta_\alpha = t^* \hat{Q}_\alpha^{(k)} \eta_\alpha \qquad on \; S_5^{(k)}(x_\alpha^{(k)}) \tag{162}$$

$$t^* u_\alpha^{(k)} \eta_\beta = t^* \hat{u}_\alpha^{(k)} \eta_\beta \qquad on \; S_2^{(k)}(x_\alpha sup(k)) \tag{163}$$

$$t^* \phi_\alpha^{(k)} \eta_\beta = t^* \hat{\phi}_\alpha^{(k)} \eta_\beta \qquad on \; S_4^{(k)}(x_\alpha^{(k)}) \tag{164}$$

$$t^* w^{(k)} \eta_\alpha = t^* \hat{w}^{(k)} \eta_\alpha \qquad on \; S_6^{(k)}(x_\alpha^{(k)}) \tag{165}$$

and the internal jump discontinuity conditions are

$$t^*(N^{(k)}_{\alpha\beta}\eta_\beta)' = t^*(g^{(k)}_1)_\alpha \qquad on\ \ S^{(k)}_{1i}(x^{(k)}_\alpha) \tag{166}$$

$$t^*(M^{(k)}_{\alpha\beta}\eta_\beta)' = t^*(g^{(k)}_3)_\alpha \qquad on\ \ S^{(k)}_{3i}(x^{(k)}_\alpha) \tag{167}$$

$$t^*(Q^{(k)}_\alpha\eta_\alpha)' = t^*(g^{(k)}_{5i})_\alpha\eta_\alpha \qquad on\ \ S^{(k)}_{5i}(x^{(k)}_\alpha) \tag{168}$$

$$t^*(\bar{u}^{(k)}_\alpha\eta_\beta)' = t^*(g^{(k)}_2)_\alpha\eta_\beta \qquad on\ \ S^{(k)}_{2i}(x^{(k)}_\alpha) \tag{169}$$

$$t^*(\phi^{(k)}_\alpha\eta_\beta)' = t^*(g^{(k)}_4)_\alpha\eta_\beta \qquad on\ \ S^{(k)}_{4i}(x^{(k)}_\alpha) \tag{170}$$

$$t^*(w^{(k)}\eta_\alpha)' = t^*(g^{(k)}_6)\eta_\alpha \qquad on\ \ S^{(k)}_{6i}(x^{(k)}_\alpha) \tag{171}$$

The following relations are satisfied by off-diagonal operators.

$$\begin{aligned}<\bar{u}^{(k)}_\alpha, t^* N^{(k)}_{\alpha\beta,\beta}>_{R^{(k)}} &= -<t^*\bar{u}^{(k)}_{\alpha,\beta}, N^{(k)}_{\alpha\beta}>_{R^{(k)}} \\ &\quad - <\bar{u}^{(k)}_\alpha, t^* N^{(k)}_{\alpha\beta}\eta_\beta>_{S^{(k)}_1} + <N^{(k)}_{\alpha\beta}, t^*\bar{u}^{(k)}_\alpha\eta_\beta>_{S^{(k)}_2} \\ &\quad - <\bar{u}^{(k)}_\alpha, t^*(N^{(k)}_{\alpha\beta}\eta_\beta)'>_{S^{(k)}_{1i}} + <N^{(k)}_{\alpha\beta}, t^*(\bar{u}^{(k)}_\alpha\eta_\beta)'>_{S^{(k)}_{2i}}\end{aligned} \tag{172}$$

$$\begin{aligned}<\phi^{(k)}_\alpha, t^* M^{(k)}_{\alpha\beta,\beta}>_{R^{(k)}} &= -<t^*\phi^{(k)}_{\alpha,\beta}, M^{(k)}_{\alpha\beta}>_{R^{(k)}} \\ &\quad - <\phi^{(k)}_\alpha, t^* M^{(k)}_{\alpha\beta}\eta_\beta>_{S^{(k)}_3} + <M^{(k)}_{\alpha\beta}, t^*\phi^{(k)}_\alpha\eta_\beta>_{S^{(k)}_4} \\ &\quad - <\phi^{(k)}_\alpha, t^*(M^{(k)}_{\alpha\beta}\eta_\beta)'>_{S^{(k)}_{3i}} + <M^{(k)}_{\alpha\beta}, t^*(\phi^{(k)}_\alpha\eta_\beta)'>_{S^{(k)}_{4i}}\end{aligned} \tag{173}$$

$$\begin{aligned}<w^{(k)}, t^* Q^{(k)}_{\alpha,\alpha}>_{R^{(k)}} &= -<t^* w^{(k)}_{,\alpha}, Q^{(k)}_\alpha>_{R^{(k)}} \\ &\quad - <w^{(k)}, t^* Q^{(k)}_\alpha\eta_\alpha>_{S^{(k)}_5} + <Q^{(k)}_\alpha, t^* w^{(k)}\eta_\alpha>_{S^{(k)}_6} \\ &\quad - <w^{(k)}, t^*(Q^{(k)}_\alpha\eta_\alpha)'>_{S^{(k)}_{5i}} + <Q^{(k)}_\alpha, t^*(w^{(k)}\eta_\alpha)'>_{S^{(k)}_{6i}}\end{aligned} \tag{174}$$

4.3.2 A General Variational Principle

Using (A-29), the governing function for the operator equations (147) is defined as

$$\Omega = \sum_{k=1}^{n} <\mathbf{U}_k^T, \mathbf{A}_k\mathbf{U}_k>_{R^{(k)}} + \sum_{k=1}^{n-1} <\mathbf{U}_k^T, \mathbf{B}_k\Xi_k>_{R^{(k)}} + \sum_{k=2}^{n} <\mathbf{U}_k^T, \mathbf{C}\Xi_{k-1}>_{R^{(k)}}$$

$$+ \sum_{k=1}^{n}\sum_{j=1}^{n} <\mathbf{U}_k^T, \mathbf{D}_{j,k}\mathbf{U}_j>_{R^{(k)}} - \sum_{k=1}^{n} <\mathbf{U}_k^T, \mathbf{D}_{k,k}\mathbf{U}_k>_{R^{(k)}}$$

$$+ \sum_{k=1}^{n-1} <\Xi_k^T, \mathbf{B}_k^T\mathbf{U}_k>_{R^{(k)}} + \sum_{k=1}^{n-1} <\Xi_k^T, \mathbf{C}^T\mathbf{U}_{k+1}>_{R^{(k)}}$$

$$+ 2\sum_{k=1}^{n} <\mathbf{U}_k^T, \mathbf{r}_k>_{R^{(k)}} - 2<\mathbf{U}_1^T, \Xi_0>_{R^{(1)}} + 2<\mathbf{U}_n^T, \Xi_n>_{R^{(n)}}$$

$$+ \textit{Boundary Terms} + \textit{Internal Jump Terms} \qquad (175)$$

Substituting (148)-(171) into (175), the explicit form of the governing function is obtained.

$$\Omega = \sum_{k=1}^{n} \{ <\bar{u}_\alpha^{(k)}, -P^{(k)}\bar{u}_\alpha^{(k)} + t^* N_{\alpha\beta,\beta}^{(k)} - R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}$$

$$+ <\bar{e}_{\alpha\beta}^{(k)}, -t^* A_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)} + t^* N_{\alpha\beta}^{(k)} - t^* B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, -t^*\bar{u}_{\alpha,\beta}^{(k)} + t^*\bar{e}_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ <\phi_\alpha^{(k)}, -R^{(k)}\bar{u}_\alpha^{(k)} - I^{(k)}\phi_\alpha^{(k)} + t^* M_{\alpha\beta,\beta}^{(k)} - t^* Q_\alpha^{(k)}>_{R^{(k)}}$$

$$+ <\kappa_{\alpha\beta}^{(k)}, -t^* B_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)} - t^* D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)} + t^* M_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, -t^*\phi_{\alpha,\beta}^{(k)} + t^*\kappa_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+ <w^{(k)}, -P^{(k)}w^{(k)} + t^* Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$+ <2e_{\alpha 3}^{(k)}, -2t^*\lambda_{\alpha\beta}^{44}e_{\beta 3}^{(k)} + t^* Q_\alpha^{(k)}>_{R^{(k)}}$$

$$+ \langle Q_\alpha^{(k)},\ -t^*\phi_\alpha^{(k)} - t^* w_{,\alpha}^{(k)} + 2t^* e_{\alpha 3}^{(k)} \rangle_{R^{(k)}} \}$$

$$+ \sum_{k=1}^{n-1} \{ \langle \bar{u}_\alpha^{(k)},\ t^* T_\alpha^{(k)} \rangle_{R^{(k)}} + \langle \phi_\alpha^{(k)},\ t^* t_k T_\alpha^{(k)} \rangle_{R^{(k)}} + \langle w^{(k)},\ t^* T_3^{(k)} \rangle_{R^{(k)}} \}$$

$$+ \sum_{k=2}^{n} \{ \langle \bar{u}_\alpha^{(k)},\ -t^* T_\alpha^{(k-1)} \rangle_{R^{(k)}} + \langle w^{(k)},\ -t^* T_3^{(k-1)} \rangle_{R^{(k)}} \}$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{n} \langle 2e_{\alpha 3}^{(k)},\ -2t^* \lambda_{\alpha\beta}^{(k,j)} e_{\beta 3}^{(j)} \rangle_{R^{(k)}} - \sum_{k=1}^{n} \langle 2e_{\alpha 3}^{(k)},\ -2t^* \lambda_{\alpha\beta}^{kk} e_{\beta 3}^{(k)} \rangle_{R^{(k)}}$$

$$+ \sum_{k=1}^{n-1} \{ \langle T_\alpha^{(k)},\ t^* \bar{u}_\alpha^{(k)} + t^* t_k \phi_\alpha^{(k)} \rangle_{R^{(k)}} + \langle T_3^{(k)},\ t^* w^{(k)} \rangle_{R^{(k)}} \}$$

$$+ \sum_{k=1}^{n-1} \{ \langle T_\alpha^{(k)},\ -t^* \bar{u}_\alpha^{(k+1)} \rangle_{R^{(k)}} + \langle T_3^{(k)},\ -t^* w^{(k+1)} \rangle_{R^{(k)}} \}$$

$$+ 2 \sum_{k=1}^{n} \{ \langle \bar{u}_\alpha^{(k)},\ t^* F_\alpha^{(k)} + X_\alpha^{(k)} \rangle_{R^{(k)}} + \langle \phi_\alpha^{(k)},\ t^* G_\alpha^{(k)} + Y_\alpha^{(k)} \rangle_{R^{(k)}} + \langle w^{(k)},\ t^* F_3^{(k)} + Z^{(k)} \rangle_{R^{(k)}} \}$$

$$- 2\langle \bar{u}_\alpha^{(1)},\ t^* T_\alpha^{(0)} \rangle_{R^{(1)}} - 2\langle w^{(1)},\ t^* T_3^{(0)} \rangle_{R^{(1)}}$$

$$+ 2\langle \bar{u}_\alpha^{(n)},\ t^* T_\alpha^{(n)} \rangle_{R^{(n)}} + 2\langle \phi_\alpha^{(n)},\ t^* t_n T_\alpha^{(n)} \rangle_{R^{(n)}} + 2\langle w^{(n)},\ t^* T_3^{(n)} \rangle_{R^{(n)}}$$

$$+ \sum_{k=1}^{n} \{ \langle \bar{u}_\alpha^{(k)},\ t^* (N_{\alpha\beta}^{(k)} \eta_\beta - 2\hat{N}_\alpha^{(k)}) \rangle_{S_1^{(k)}} + \langle \phi_\alpha^{(k)},\ t^* (M_{\alpha\beta}^{(k)} \eta_\beta^{(k)} - 2\hat{M}_\alpha^{(k)}) \rangle_{S_3^{(k)}}$$

$$+ \langle w^{(k)},\ t^* (Q_\alpha^{(k)} - 2\hat{Q}_\alpha^{(k)}) \eta_\alpha \rangle_{S_5^{(k)}} + \langle N_{\alpha\beta}^{(k)},\ t^* (\bar{u}_\alpha^{(k)} - 2\hat{\bar{u}}_\alpha^{(k)}) \eta_\beta \rangle_{S_2^{(k)}}$$

$$+ \langle M_{\alpha\beta}^{(k)},\ t^* (\phi_\alpha^{(k)} - 2\hat{\phi}_\alpha^{(k)}) \eta_\beta \rangle_{S_4^{(k)}} + \langle Q_\alpha^{(k)},\ t^* (w^{(k)} - 2\hat{w}^{(k)}) \eta_\alpha \rangle_{S_6^{(k)}}$$

$$+ \sum_{k=1}^{n} \{ \langle \bar{u}_\alpha^{(k)},\ t^* ((N_{\alpha\beta}^{(k)} \eta_\beta)' - 2(g_1^{(k)})_\alpha) \rangle_{S_{1t}^{(k)}} + \langle \phi_\alpha^{(k)},\ t^* ((M_{\alpha\beta}^{(k)} \eta_\beta^{(k)})' - 2(g_3^{(k)})_\alpha) \rangle_{S_{3t}^{(k)}}$$

$$+ \langle w^{(k)},\ t^* ((Q_\alpha^{(k)} \eta_\alpha)' - 2(g_5^{(k)})_\alpha \eta_\alpha) \rangle_{S_{5t}^{(k)}} + \langle N_{\alpha\beta}^{(k)},\ t^* ((\bar{u}_\alpha^{(k)} \eta_\beta)' - 2(g_2^{(k)})_\alpha \eta_\beta) \rangle_{S_{2t}^{(k)}}$$

$$+ < M_{\alpha\beta}^{(k)}, t^*((\phi_\alpha^{(k)}\eta_\beta)' - 2(g_4^{(k)})_\alpha \eta_\beta) >_{S_{4i}^{(k)}} + < Q_\alpha^{(k)}, t^*((w^{(k)}\eta_\alpha)' - 2(g_6^{(k)})\eta_\alpha) >_{S_{6i}^{(k)}}$$

(176)

This function is defined over the set

$$\{v^{(k)}\} = \{\bar{u}_\alpha^{(k)}, \phi_\alpha^{(k)}, w^{(k)}, N_{\alpha\beta}^{(k)}, M_{\alpha\beta}, Q_\alpha^{(k)}, \bar{e}_{\alpha\beta}^{(k)}, \kappa_{\alpha\beta}^{(k)}, e_{\alpha 3}^{(k)}, T_\alpha^{(k)}, T_3^{(k)}\}$$

where each of the functions in the set is sufficiently smooth for the governing function to exist. This requires $\bar{u}_\alpha^{(k)}, \phi_\alpha^{(k)}, w_\alpha^{(k)}, N_{\alpha\beta}^{(k)}, M_{\alpha\beta}^{(k)}, Q_\alpha^{(k)}$ to be continuous and such that their derivatives have a finite number of discontinuities. The collection of all possible sets within the domain of Ω is the set of admissible states. Let

$$\{\bar{v}^{(k)}\} = \{\bar{v}_\alpha^{(k)}, \bar{\psi}_\alpha^{(k)}, \bar{w}^{(k)}, \bar{n}_{\alpha\beta}^{(k)}, \bar{m}_{\alpha\beta}, \bar{q}_\alpha^{(k)}, \bar{\epsilon}_{\alpha\beta}^{(k)}, \bar{\kappa}_{\alpha\beta}^{(k)}, \bar{e}_{\alpha 3}^{(k)}, \bar{T}_\alpha^{(k)}, \bar{T}_3^{(k)}\}$$

be an admissible state. Assume $v + \lambda\bar{v} \in$ the set of admissible states for all values of the scalar λ, i.e. the function Ω is defined in a neighborhood of v. The Gateaux differential (A-10) of the function Ω defined by (176) along the path $v^{(k)}$, using (172) through (174) to eliminate $\bar{v}_{\alpha,\beta}^{(k)}, \bar{\psi}_{\alpha,\beta}^{(k)}, \bar{w}_{,alpha}^{(k)}, \bar{n}_{\alpha,\beta}^{(k)}, \bar{m}_{\alpha,\beta}^{(k)}, \bar{q}_{\alpha,\alpha}^{(k)}$ is

$$\Delta_{\bar{v}}\Omega = 2\sum_{k=1}^{n}\{ < \bar{v}_\alpha^{(k)}, t^* N_{\alpha\beta,\beta}^{(k)} - P^{(k)}\bar{u}_\alpha^{(k)} - R^{(k)}\phi_\alpha^{(k)} + t^* F_\alpha^{(k)} + X_\alpha^{(k)} >_{R^{(k)}}$$

$$+2\sum_{k=1}^{n-1} < \bar{v}_\alpha^{(k)}, t^* T_\alpha^{(k)} >_{R^{(k)}} + 2 < \bar{v}_\alpha^{(n)}, t^* T_\alpha^{(n)} >_{R^{(n)}}$$

$$-2\sum_{k=2}^{n} < \bar{v}_\alpha^{(k)}, t^* T_\alpha^{(k-1)} >_{R^{(k)}} - 2 < \bar{v}_\alpha^{(1)}, t^* T_\alpha^{(0)} >_{R^{(n)}}$$

$$+2\sum_{k=1}^{n} < \bar{\epsilon}_{\alpha\beta}^{(k)}, -t^* A_{\alpha\beta\gamma\delta}\bar{e}_{\gamma\delta}^{(k)} + -t^* B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)} + t^* N_{\alpha\beta}^{(k)} >_{R^{(k)}}$$

$$+2\sum_{k=1}^{n} < \bar{n}_{\alpha\beta}^{(k)}, -t^* \bar{u}_{\alpha,\beta}^{(k)} + t^* \bar{e}_{\alpha\beta}^{(k)} >_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\vec{\psi}_{\alpha}^{(k)},\ -R^{(k)}\vec{\bar{u}}_{\alpha}^{(k)}-I^{(k)}\phi_{\alpha}^{(k)}+t^{*}M_{\alpha\beta,\beta}^{(k)}-t^{*}Q_{\alpha}^{(k)}+t^{*}G_{\alpha}^{(k)}+Y_{\alpha}^{(k)}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\vec{\psi}_{\alpha}^{(k)},\ t^{*}t_{k}T_{\alpha}^{(k)}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\tilde{\kappa}_{\alpha\beta}^{(k)},\ -t^{*}B_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)}-t^{*}D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}+t^{*}M_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\tilde{m}_{\alpha\beta}^{(k)},\ -t^{*}\phi_{\alpha,\beta}^{(k)}+t^{*}\kappa_{\alpha\beta}^{(k)}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\tilde{w}^{(k)},\ -P^{(k)}w^{(k)}+t^{*}Q_{\alpha,\beta}^{(k)}+t^{*}F_{3}^{(k)}+Z^{(k)}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n-1}<\tilde{w},\ t^{*}T_{3}^{(k)}>_{R^{(k)}}+2<\tilde{w}^{(n)},\ t^{*}T_{3}^{(n)}>_{R^{(k)}}$$

$$-2\sum_{k=2}^{n}<\tilde{w},\ t^{*}T_{3}^{(k-1)}>_{R^{(k)}}-2<\tilde{w}^{(1)},\ t^{*}T_{3}^{(0)}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<2\tilde{e}_{\alpha 3}^{(k)},\ \sum_{j=1}^{n}t^{*}(Q_{\alpha}^{(k)}-2\lambda_{\alpha\beta}^{(kj)}e_{\beta 3}^{(j)})>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\tilde{q}_{\alpha}^{(k)},\ t^{*}(2e_{\alpha 3}^{(k)}-w_{,\alpha}^{(k)}-\phi_{\alpha}^{(k)})>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\tilde{q}_{\alpha}^{(k)},\ t^{*}(2e_{\alpha 3}^{(k)}-w_{,\alpha}^{(k)}-\phi_{\alpha}^{(k)})>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}<\tilde{t}_{\alpha}^{(k)},\ t^{*}(\vec{\bar{u}}_{\alpha}^{(k)}+t_{k}\phi_{\alpha}^{(k)}-\vec{\bar{u}}_{\alpha}^{(k-1)})_{R^{(k)}}+2\sum_{k=1}^{n-1}<\tilde{t}_{3}^{(k)},\ t^{*}(w^{(k)}-w^{(k+1)})>_{R^{(k)}}$$

$$+\sum_{k=1}^{n}\{\ <\tilde{v}_{\alpha}^{(k)},\ 2t^{*}(N_{\alpha\beta}^{(k)}\eta_{\beta}-\hat{N}_{\alpha}^{(k)})>_{S_{1}^{(k)}}+\ <\tilde{\psi}_{\alpha}^{(k)},\ 2t^{*}(M_{\alpha\beta}^{(k)}\eta_{\beta}^{(k)}-2\hat{M}_{\alpha}^{(k)})>_{S_{3}^{(k)}}$$

$$+ <\tilde{w}^{(k)}, 2t^*(Q_\alpha^{(k)}\eta_\alpha - \hat{Q}_\alpha^{(k)}\eta_\alpha)>_{S_5^{(k)}} + <\tilde{n}_{\alpha\beta}^{(k)}, 2t^*(\bar{u}_\alpha^{(k)}\eta_\beta - \hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+ <\tilde{m}_{\alpha\beta}^{(k)}, 2t^*(\phi_\alpha^{(k)}\eta_\beta - \hat{\phi}_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}} + <\tilde{q}_\alpha^{(k)}, 2t^*(w^{(k)}\eta_\alpha - \hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{ <\tilde{v}_\alpha^{(k)}, 2t^*((N_{\alpha\beta}^{(k)}\eta_\beta)' - (g_1^{(k)})_\alpha)>_{S_{1i}^{(k)}} + <\tilde{\psi}_\alpha^{(k)}, 2t^*((M_{\alpha\beta}^{(k)}\eta_\beta^{(k)})' - 2(g_3^{(k)})_\alpha)>_{S_{3i}^{(k)}}$$

$$+ <\tilde{w}^{(k)}, 2t^*((Q_\alpha^{(k)}\eta_\alpha)' - (g_5^{(k)})_\alpha\eta_\alpha)>_{S_{5i}^{(k)}} + <\tilde{n}_{\alpha\beta}^{(k)}, 2t^*((\bar{u}_\alpha^{(k)}\eta_\beta)' - (g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$+ <\tilde{m}_{\alpha\beta}^{(k)}, 2t^*((\phi_\alpha^{(k)}\eta_\beta)' - (g_4^{(k)})_\alpha\eta_\beta)>_{S_{4i}^{(k)}} + <\tilde{q}_\alpha^{(k)}, 2t^*((w^{(k)}\eta_\alpha)' - (g_6)_\alpha)>_{S_{6i}^{(k)}}$$

(177)

The Gateaux differential vanishes if and only if all the field equations along with the boundary conditions and the internal discontinuity conditions are satisfied because of linearity and nondegeneracy of the bilinear mapping.

Rearranging the terms, the governing function (176) may be written as

$$\Omega = \sum_{k=1}^{n}\{ -<\bar{\bar{u}}_\alpha^{(k)}, P^{(k)}\bar{\bar{u}}_\alpha^{(k)}>_{R^{(k)}} - <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - <w^{(k)}, P^{(k)}w^{(k)}>_{R^{(k)}}$$

$$-<\bar{\bar{e}}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)}\bar{\bar{e}}_{\gamma\delta}^{(k)}>_{R^{(k)}} - <\kappa_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+<\bar{\bar{u}}_\alpha^{(k)}, t^* N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}} - <N_{\alpha\beta}^{(k)}, t^*\bar{\bar{u}}_{\alpha,\beta}^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* M_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}$$

$$-<M_{\alpha\beta}^{(k)}, t^*\phi_{\alpha,\beta}^{(k)}>_{R^{(k)}} + <w^{(k)}, t^* Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}} - <Q_\alpha^{(k)}, t^* w_{,\alpha}^{(k)}>_{R^{(k)}}$$

$$-2<\bar{\bar{u}}_\alpha^{(k)}, R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - 2<\phi_\alpha^{(k)}, t^* Q_\alpha^{(k)}>_{R^{(k)}} - 2<\bar{\bar{e}}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+2<\bar{\bar{e}}_{\alpha\beta}^{(k)}, t^* N_{\alpha\beta}^{(k)}>_{R^{(k)}} + 2<M_{\alpha\beta}^{(k)}, t^*\kappa_{\alpha\beta}^{(k)}>_{R^{(k)}} + 2<2e_{\alpha 3}^{(k)}, t^* Q_\alpha^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)}, -2t^*\lambda_{\alpha\beta}^{(kj)}e_{\beta 3}^{j}>_{R^{(k)}}$$

$$\sum_{k=1}^{n-1} <T_\alpha^{(k)}, t^*(\bar{u}_\alpha^{(k)} + t_k\phi_\alpha^{(k)} - \bar{u}_\alpha^{(k+1)})>_{R^{(k)}} + <T_3^{(k)}, t^*(w^{(k)} - w^{(k+1)})>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)}, t^* F_3^{(k)} + Z^{(k)}>_{R^{(k)}} \}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w^{(1)}, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<\phi_\alpha^{(n)}, t^* t_n T_\alpha^{(n)}>_{R^{(n)}} + 2<w^{(n)}, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^*(N_{\alpha\beta}^{(k)}\eta_\beta - 2\hat{N}_\alpha^{(k)})>_{S_1^{(k)}} + <\phi_\alpha^{(k)}, t^*(M_{\alpha\beta}^{(k)}\eta_\beta^{(k)} - 2\hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$+ <w^{(k)}, t^*(Q_\alpha^{(k)}\eta_\alpha - 2\hat{Q}_\alpha^{(k)}\eta_\alpha)>_{S_5^{(k)}} + <N_{\alpha\beta}^{(k)}, t^*(\bar{u}_\alpha^{(k)}\eta_\beta - 2\hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, t^*(\phi_\alpha^{(k)}\eta_\beta - 2\phi_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}} + <Q_\alpha^{(k)}, t^*(w^{(k)}\eta_\alpha - 2\hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^*((N_{\alpha\beta}^{(k)}\eta_\beta)' - 2(g_1^{(k)})_\alpha)>_{S_{1i}^{(k)}} + <\phi_\alpha^{(k)}, t^*((M_{\alpha\beta}^{(k)}\eta_\beta^{(k)})' - 2(g_3^{(k)})_\alpha)>_{S_{3i}^{(k)}}$$

$$+ <w^{(k)}, t^*((Q_\alpha^{(k)}\eta_\alpha)' - 2(g_5^{(k)})_\alpha\eta_\alpha)>_{S_{5i}^{(k)}} + <N_{\alpha\beta}^{(k)}, t^*((\bar{u}_\alpha^{(k)}\eta_\beta)' - 2(g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, t^*((\phi_\alpha^{(k)}\eta_\beta)' - 2(g_4^{(k)})_\alpha\eta_\beta)>_{S_{4i}^{(k)}} + <Q_\alpha^{(k)}, t^*((w^{(k)}\eta_\alpha)' - 2(g_6^{(k)})\eta_\alpha)>_{S_{6i}^{(k)}}$$

(178)

4.3.3 Extended Variational Principles

Equations (172)-(174) relate pairs of operators in the operator matrix of (147). These relations may be used to eliminate either of $N_{\alpha\beta,\beta}^{(k)}$ or $\bar{u}_{\alpha,\beta}^{(k)}$, either of $M_{\alpha\beta,\beta}^{(k)}$ or $\phi_{\alpha,\beta}^{(k)}$ and either of $Q_{\alpha,\alpha}^{(k)}$ or $w_{,\alpha}^{(k)}$ from the governing function Ω in (176), leading to numerous different forms of variational formulations. Elimination of an

operator A_{ij} implies that state of variable u_j need not be in the domain M_{ij} of A_{ij}. Where A_{ij} are differential operators, this results in relaxing the requirement of differentiability in u_j, thereby extending the space of admissible states. In the context of the finite element method, it is clear that the extension of the admissible space provides greater freedom in selection of approximation functions. To llustrate the procedure, we present six extended variational principles.

Using (172) to eliminate $N^{(k)}_{\alpha\beta,\beta}$, (178) gives

$$\begin{aligned}
\Omega_1 = \sum_{k=1}^{n} \{ & -\langle \bar{u}_\alpha^{(k)}, P^{(k)} \bar{u}_\alpha^{(k)} \rangle_{R^{(k)}} - \langle \phi_\alpha^{(k)}, I^{(k)} \phi_\alpha^{(k)} \rangle_{R^{(k)}} - \langle w^{(k)}, P^{(k)} w^{(k)} \rangle_{R^{(k)}} \\
& - \langle \bar{e}_{\alpha\beta}^{(k)}, t^* A^{(k)}_{\alpha\beta\gamma\delta} \bar{e}^{(k)}_{\gamma\delta} \rangle_{R^{(k)}} - \langle \kappa_{\alpha\beta}^{(k)}, t^* D^{(k)}_{\alpha\beta\gamma\delta} \kappa^{(k)}_{\gamma\delta} \rangle_{R^{(k)}} \\
& + 2 \langle N^{(k)}_{\alpha\beta}, t^* (\bar{e}^{(k)}_{\alpha\beta} - \bar{u}^{(k)}_{\alpha,\beta}) \rangle_{R^{(k)}} + \langle \phi^{(k)}_\alpha, t^* M^{(k)}_{\alpha\beta,\beta} \rangle_{R^{(k)}} \\
& - \langle M^{(k)}_{\alpha\beta}, t^* \phi^{(k)}_{\alpha,\beta} \rangle_{R^{(k)}} + \langle w^{(k)}, t^* Q^{(k)}_{\alpha,\alpha} \rangle_{R^{(k)}} - \langle Q^{(k)}_\alpha, t^* w^{(k)}_{,\alpha} \rangle_{R^{(k)}} \\
& - 2 \langle \bar{u}_\alpha^{(k)}, R^{(k)} \phi_\alpha^{(k)} \rangle_{R^{(k)}} - 2 \langle \phi_\alpha^{(k)}, t^* Q_\alpha^{(k)} \rangle_{R^{(k)}} - 2 \langle \bar{e}^{(k)}_{\alpha\beta}, t^* B^{(k)}_{\alpha\beta\gamma\delta} \kappa^{(k)}_{\gamma\delta} \rangle_{R^{(k)}} \\
& + 2 \langle M^{(k)}_{\alpha\beta}, t^* \kappa^{(k)}_{\alpha\beta} \rangle_{R^{(k)}} + 2 \langle 2e^{(k)}_{\alpha 3}, t^* Q^{(k)}_\alpha \rangle_{R^{(k)}} \}
\end{aligned}$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{n} \langle 2e^{(k)}_{\alpha 3}, -2t^* \lambda^{(kj)}_{\alpha\beta} e^j_{\beta 3} \rangle_{R^{(k)}}$$

$$+ \sum_{k=1}^{n-1} \{ \langle T^{(k)}_\alpha, t^* (\bar{u}^{(k)}_\alpha + t_k \phi^{(k)}_\alpha - \bar{u}^{(k+1)}_\alpha) \rangle_{R^{(k)}} + \langle T^{(k)}_3, t^* (w^{(k)} - w^{(k+1)}) \rangle_{R^{(k)}} \}$$

$$+ 2 \sum_{k=1}^{n} \{ \langle \bar{u}^{(k)}_\alpha, t^* F^{(k)}_\alpha + X^{(k)}_\alpha \rangle_{R^{(k)}} + \langle \phi^{(k)}_\alpha, t^* G^{(k)}_\alpha + Y^{(k)}_\alpha \rangle_{R^{(k)}} + \langle w^{(k)}, t^* F^{(k)}_3 + Z^{(k)} \rangle_{R^{(k)}} \}$$

$$- 2 \langle \bar{u}^{(1)}_\alpha, t^* T^{(0)}_\alpha \rangle_{R^{(1)}} - 2 \langle w^{(1)}, t^* T^{(0)}_3 \rangle_{R^{(1)}}$$

$$+ 2 \langle \bar{u}^{(n)}_\alpha, t^* T^{(n)}_\alpha \rangle_{R^{(n)}} + 2 \langle \phi^{(n)}_\alpha, t^* t_n T^{(n)}_\alpha \rangle_{R^{(n)}} + 2 \langle w^{(n)}, t^* T^{(n)}_3 \rangle_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_\alpha^{(k)}, t^*\hat{N}_\alpha^{(k)}>_{S_1^{(k)}} + <\phi_\alpha^{(k)}, t^*(M_{\alpha\beta}^{(k)}\eta_\beta^{(k)} - 2\hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$+<w^{(k)}, t^*(Q_\alpha^{(k)}\eta_\alpha - 2\hat{Q}_\alpha^{(k)}\eta_\alpha)>_{S_5^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*(\bar{u}_\alpha^{(k)}\eta_\beta - \hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+<M_{\alpha\beta}^{(k)}, t^*(\phi_\alpha^{(k)}\eta_\beta - 2\phi_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}} + <Q_\alpha^{(k)}, t^*(w^{(k)}\eta_\alpha - 2\hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_\alpha^{(k)}, t^*(g_1^{(k)})_\alpha>_{S_{1i}^{(k)}} + <\phi_\alpha^{(k)}, t^*((M_{\alpha\beta}^{(k)}\eta_\beta^{(k)})' - 2(g_3^{(k)})_\alpha)>_{S_{3i}^{(k)}}$$

$$+<w^{(k)}, t^*((Q_\alpha^{(k)}\eta_\alpha)' - 2(g_5^{(k)})_\alpha\eta_\alpha)>_{S_{5i}^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*((\bar{u}_\alpha^{(k)}\eta_\beta)' - (g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$+<M_{\alpha\beta}^{(k)}, t^*((\phi_\alpha^{(k)}\eta_\beta)' - 2(g_4^{(k)})_\alpha\eta_\beta)>_{S_{4i}^{(k)}} + <Q_\alpha^{(k)}, t^*((w^{(k)}\eta_\alpha)' - 2(g_6^{(k)})\eta_\alpha)>_{S_{6i}^{(k)}}$$

(179)

where $N_{\alpha\beta}^{(k)}$ need not be differentiable. In addition, eliminating $M_{\alpha\beta,\beta}^{(k)}$ from (179) Ω_1 reduces to

$$\Omega_2 = \sum_{k=1}^{n}\{-<\bar{u}_\alpha^{(k)}, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} - <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - <w^{(k)}, P^{(k)}w^{(k)}>_{R^{(k)}}$$

$$-<\bar{e}_{\alpha\beta}^{(k)}, t^*A_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)}>_{R^{(k)}} - <\kappa_{\alpha\beta}^{(k)}, t^*D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+2<N_{\alpha\beta}^{(k)}, t^*(\bar{e}_{\alpha\beta}^{(k)} - \bar{u}_{\alpha,\beta}^{(k)})>_{R^{(k)}} + 2<M_{\alpha\beta}^{(k)}, t^*(\kappa_{\alpha\beta}^{(k)} - \phi_{\alpha,\beta}^{(k)})>_{R^{(k)}}$$

$$+<w^{(k)}, t^*Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}} - <Q_\alpha^{(k)}, t^*w_{,\alpha}^{(k)}>_{R^{(k)}} + 2<2e_{\alpha 3}^{(k)}, t^*Q_\alpha^{(k)}>_{R^{(k)}}$$

$$-2<\bar{u}_\alpha^{(k)}, R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - 2<\phi_\alpha^{(k)}, t^*Q_\alpha^{(k)}>_{R^{(k)}} - 2<\bar{e}_{\alpha\beta}^{(k)}, t^*B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n}<2e_{\alpha 3}^{(k)}, -2t^*\lambda_{\alpha\beta}^{(kj)}e_{\beta 3}^{j}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1}\{<T_\alpha^{(k)}, t^*(\bar{u}_\alpha^{(k)}+t_k\phi_\alpha^{(k)}-\bar{u}_\alpha^{(k+1)})>_{R^{(k)}} + <T_3^{(k)}, t^*(w^{(k)}-w^{(k+1)})>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* G_\alpha^{(k)}+Y_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)}, t^* F_3^{(k)}+Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w^{(1)}, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<\phi_\alpha^{(n)}, t^* t_n T_\alpha^{(n)}>_{R^{(n)}} + 2<w^{(n)}, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{ -2<\bar{u}_\alpha^{(k)}, t^* \hat{N}_\alpha^{(k)}>_{S_1^{(k)}} - 2<\phi_\alpha^{(k)}, t^* \hat{M}_\alpha^{(k)}>_{S_3^{(k)}}$$

$$+<w^{(k)}, t^*(Q_\alpha^{(k)}\eta_\alpha - 2\hat{Q}_\alpha^{(k)}\eta_\alpha)>_{S_5^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*(\bar{u}_\alpha^{(k)}\eta_\beta - \hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)}, t^*(\phi_\alpha^{(k)}\eta_\beta - \hat{\phi}_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}} + <Q_\alpha^{(k)}, t^*(w^{(k)}\eta_\alpha - 2\hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{ -2<\bar{u}_\alpha^{(k)}, t^*(g_1^{(k)})_\alpha>_{S_{1i}^{(k)}} - 2<\phi_\alpha^{(k)}, t^*(g_3^{(k)})_\alpha>_{S_{3i}^{(k)}}$$

$$+<w^{(k)}, t^*((Q_\alpha^{(k)}\eta_\alpha)' - 2(g_5^{(k)})_\alpha\eta_\alpha)>_{S_{5i}^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*((\bar{u}_\alpha^{(k)}\eta_\beta)' - (g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)}, t^*((\phi_\alpha^{(k)}\eta_\beta)' - (g_4^{(k)})_\alpha\eta_\beta)>_{S_{4i}^{(k)}} + <Q_\alpha^{(k)}, t^*((w^{(k)}\eta_\alpha)' - 2(g_6^{(k)})\eta_\alpha)>_{S_{6i}^{(k)}}$$

(180)

Also, $Q_{\alpha,\alpha}^{(k)}$ can be eliminated using (174) from (180) to give

$$\Omega_3 = \sum_{k=1}^{n}\{ -<\bar{u}_\alpha^{(k)}, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} - <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - <w^{(k)}, P^{(k)}w^{(k)}>_{R^{(k)}}$$

$$-<\bar{e}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)}>_{R^{(k)}} - <\kappa_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}} - 2<\bar{u}_\alpha^{(k)}, R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}$$

$$+2<N_{\alpha\beta}^{(k)}, t^*(\phi_\alpha^{(k)} - \bar{u}_{\alpha,\beta}^{(k)})>_{R^{(k)}} + 2<M_{\alpha\beta}^{(k)}, t^*(\kappa_{\alpha\beta}^{(k)} - \phi_{\alpha,\beta}^{(k)})>_{R^{(k)}}$$

$$-2<Q_\alpha^{(k)}, t^*(2e_{\alpha 3}^{(k)}-\phi_\alpha^{(k)}-w_{,\alpha})>_{R^{(k)}} - 2<\bar{e}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)} \overset{(k)}{N}_{\gamma\delta}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)}, -2t^* \lambda_{\alpha\beta}^{(kj)} e_{\beta 3}^{j}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1}\{<T_\alpha^{(k)}, t^*(\bar{u}_\alpha^{(k)}+t_k\phi_\alpha^{(k)}-\bar{u}_\alpha^{(k+1)})>_{R^{(k)}} + <T_3^{(k)}, t^*(w^{(k)}-w^{(k+1)})>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* G_\alpha^{(k)}+Y_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)}, t^* F_3^{(k)}+Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w^{(1)}, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<\phi_\alpha^{(n)}, t^* t_n T_\alpha^{(n)}>_{R^{(k)}} + 2<w^{(n)}, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{ -2<\bar{u}_\alpha^{(k)}, t^* \hat{N}_\alpha^{(k)}>_{S_1^{(k)}} - 2<\phi_\alpha^{(k)}, t^* \hat{M}_\alpha^{(k)}>_{S_3^{(k)}}$$

$$-2<w^{(k)}, t^* \hat{Q}_\alpha^{(k)}\eta_\alpha>_{S_5^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*(\bar{u}_\alpha^{(k)}\eta_\beta-\hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)}, t^*(\phi_\alpha^{(k)}\eta_\beta-\phi_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}} + 2<Q_\alpha^{(k)}, t^*(w^{(k)}\eta_\alpha-\hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{ -2<\bar{u}_\alpha^{(k)}, t^*(g_1^{(k)})_\alpha>_{S_{1t}^{(k)}} - 2<\phi_\alpha^{(k)}, t^*(g_3^{(k)})_\alpha>_{S_{3t}^{(k)}}$$

$$-2<w^{(k)}, t^*(g_5^{(k)})_\alpha\eta_\alpha>_{S_{5t}^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*((\bar{u}_\alpha^{(k)}\eta_\beta)'-(g_2^{(k)})_\alpha\eta_\beta)>_{S_{2t}^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)}, t^*((\phi_\alpha^{(k)}\eta_\beta)'-(g_4^{(k)})_\alpha\eta_\beta)>_{S_{4t}^{(k)}} + 2<Q_\alpha^{(k)}, t^*((w^{(k)}\eta_\alpha)'-(g_6^{(k)})\eta_\alpha)>_{S_{6t}^{(k)}}$$

(181)

where none of the stress resultants $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$ and $Q_\alpha^{(k)}$ need be differentiable.

Alternatively, extended formulations which do not have derivatives of the kinematic variables $\bar{u}_\alpha^{(k)}$, $\phi_\alpha^{(k)}$ and $w^{(k)}$ may be derived. Elimination of $\bar{u}_{\alpha,\beta}^{(k)}$ from Ω using (172) results in

$$\Omega_4 = \sum_{k=1}^{n}\{ -<\bar{u}_\alpha^{(k)}, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} - <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - <w^{(k)}, P^{(k)}w^{(k)}>_{R^{(k)}}$$

$$-<\bar{e}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)}>_{R^{(k)}} - <\kappa_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+2<\bar{u}_\alpha^{(k)}, t^* N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* M_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}$$

$$-<M_{\alpha\beta}^{(k)}, t^*\phi_{\alpha,\beta}>_{R^{(k)}} + <w^{(k)}, t^* Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}} - <Q_\alpha^{(k)}, t^* w_{,\alpha}^{(k)}>_{R^{(k)}}$$

$$-2<\bar{u}_\alpha^{(k)}, R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - 2<\phi_\alpha^{(k)}, t^* Q_\alpha^{(k)}>_{R^{(k)}} - 2<\bar{e}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+2<\bar{e}_{\alpha\beta}^{(k)}, t^* N_{\alpha\beta}^{(k)}>_{R^{(k)}} + 2<M_{\alpha\beta}^{(k)}, t^*\kappa_{\alpha\beta}^{(k)}>_{R^{(k)}} + 2<2e_{\alpha 3}^{(k)}, t^* Q_\alpha^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)}, -2t^*\lambda_{\alpha\beta}^{(kj)}e_{\beta 3}^{j}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1}\{<T_\alpha^{(k)}, t^*(\bar{u}_\alpha^{(k)} + t_k\phi_\alpha^{(k)} - \bar{u}_\alpha^{(k+1)})>_{R^{(k)}} + <T_3^{(k)}, t^*(w^{(k)} - w^{(k+1)})>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)}, t^* F_3^{(k)} + Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w^{(1)}, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<\phi_\alpha^{(k)}, t^* t_n T_\alpha^{(n)}>_{R^{(n)}} + 2<w^{(n)}, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{ 2<\bar{u}_\alpha^{(k)}, t^*(N_{\alpha\beta}^{(k)}\eta_\beta - \hat{N}_\alpha^{(k)})>_{S_1^{(k)}} + <\phi_\alpha^{(k)}, t^*(M_{\alpha\beta}^{(k)}\eta_\beta^{(k)} - 2\hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$+ <w^{(k)}, t^*(Q_\alpha^{(k)} - 2\hat{Q}_\alpha^{(k)})\eta_\alpha>_{S_3^{(k)}} - 2<N_{\alpha\beta}^{(k)}, t^*\hat{\bar{u}}_\alpha^{(k)}\eta_\beta>_{S_2^{(k)}}$$

$$+ < M^{(k)}_{\alpha\beta}, t^*(\phi^{(k)}_\alpha - 2\hat{\phi}^{(k)}_\alpha)\eta_\beta >_{S^{(k)}_4} + < Q^{(k)}_\alpha, t^*(w^{(k)} - 2\hat{w}^{(k)})\eta_\alpha >_{S^{(k)}_6}$$

$$+ \sum_{k=1}^{n} \{ 2 < \bar{u}^{(k)}_\alpha, t^*((N^{(k)}_{\alpha\beta}\eta_\beta)' - (g^{(k)}_1)_\alpha) >_{S^{(k)}_{1t}} + < \phi^{(k)}_\alpha, t^*((M^{(k)}_{\alpha\beta}\eta^{(k)}_\beta)' - 2(g^{(k)}_3)_\alpha) >_{S^{(k)}_{3t}}$$

$$+ < w^{(k)}, t^*((Q^{(k)}_\alpha \eta_\alpha)' - 2(g^{(k)}_5)_\alpha \eta_\alpha) >_{S^{(k)}_{5t}} - 2 < N^{(k)}_{\alpha\beta}, t^*(g^{(k)}_2)_\alpha \eta_\beta >_{S^{(k)}_{2t}}$$

$$+ < M^{(k)}_{\alpha\beta}, t^*((\phi^{(k)}_\alpha \eta_\beta)' - 2(g^{(k)}_4)_\alpha \eta_\beta) >_{S^{(k)}_{4t}} + < Q^{(k)}_\alpha, t^*((w^{(k)}\eta_\alpha)' - 2(g^{(k)}_6)\eta_\alpha) >_{S^{(k)}_{6t}}$$

(182)

Using (173) to eliminate $\phi^{(k)}_{\alpha,\beta}$ from Ω_4,

$$\Omega_5 = \sum_{k=1}^{n} \{ - < \bar{u}^{(k)}_\alpha, P^{(k)} \bar{u}^{(k)}_\alpha >_{R^{(k)}} - < \phi^{(k)}_\alpha, I^{(k)} \phi^{(k)}_\alpha >_{R^{(k)}} - < w^{(k)}, P^{(k)} w^{(k)} >_{R^{(k)}}$$

$$- < \bar{e}^{(k)}_{\alpha\beta}, t^* A^{(k)}_{\alpha\beta\gamma\delta} \bar{e}^{(k)}_{\gamma\delta} >_{R^{(k)}} - < \kappa^{(k)}_{\alpha\beta}, t^* D^{(k)}_{\alpha\beta\gamma\delta} \kappa^{(k)}_{\gamma\delta} >_{R^{(k)}}$$

$$+ 2 < \bar{u}^{(k)}_\alpha, t^* N^{(k)}_{\alpha\beta,\beta} >_{R^{(k)}} + 2 < \phi^{(k)}_\alpha, t^* M^{(k)}_{\alpha\beta,\beta} >_{R^{(k)}}$$

$$+ < w^{(k)}, t^* Q^{(k)}_{\alpha,\alpha} >_{R^{(k)}} - < Q^{(k)}_\alpha, t^* w^{(k)}_{,\alpha} >_{R^{(k)}}$$

$$- 2 < \bar{u}^{(k)}_\alpha, R^{(k)} \phi^{(k)}_\alpha >_{R^{(k)}} - 2 < \phi^{(k)}_\alpha, t^* Q^{(k)}_\alpha >_{R^{(k)}} - 2 < \bar{e}^{(k)}_{\alpha\beta}, t^* B^{(k)}_{\alpha\beta\gamma\delta} \kappa^{(k)}_{\gamma\delta} >_{R^{(k)}}$$

$$+ 2 < \bar{e}^{(k)}_{\alpha\beta}, t^* N^{(k)}_{\alpha\beta} >_{R^{(k)}} + 2 < M^{(k)}_{\alpha\beta}, t^* \kappa^{(k)}_{\alpha\beta} >_{R^{(k)}} + 2 < 2e^{(k)}_{\alpha 3}, t^* Q^{(k)}_\alpha >_{R^{(k)}} \}$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{n} < 2e^{(k)}_{\alpha 3}, -2t^* \lambda^{(kj)}_{\alpha\beta} e^{j}_{\beta 3} >_{R^{(k)}}$$

$$+ \sum_{k=1}^{n-1} \{ < T^{(k)}_\alpha, t^*(\bar{u}^{(k)}_\alpha + t_k \phi^{(k)}_\alpha - \bar{u}^{(k+1)}_\alpha) >_{R^{(k)}} + < T^{(k)}_3, t^*(w^{(k)} - w^{(k+1)}) >_{R^{(k)}} \}$$

$$+ 2 \sum_{k=1}^{n} \{ < \bar{u}^{(k)}_\alpha, t^* F^{(k)}_\alpha + X^{(k)}_\alpha >_{R^{(k)}} + < \phi^{(k)}_\alpha, t^* G^{(k)}_\alpha + Y^{(k)}_\alpha >_{R^{(k)}} + < w^{(k)}, t^* F^{(k)}_3 + Z^{(k)} >_{R^{(k)}} \}$$

$$
\begin{aligned}
&-2\langle \bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)} \rangle_{R^{(1)}} - 2\langle w^{(1)}, t^* T_3^{(0)} \rangle_{R^{(1)}} \\
&+2\langle \bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)} \rangle_{R^{(n)}} + 2\langle \phi_\alpha^{(n)}, t^* t_n T_\alpha^{(n)} \rangle_{R^{(k)}} + 2\langle w^{(n)}, t^* T_3^{(n)} \rangle_{R^{(n)}} \\
&+\sum_{k=1}^{n} \{ 2\langle \bar{u}_\alpha^{(k)}, t^* (N_{\alpha\beta}^{(k)} \eta_\beta - \hat{N}_\alpha^{(k)}) \rangle_{S_1^{(k)}} + 2\langle \phi_\alpha^{(k)}, t^* (M_{\alpha\beta}^{(k)} \eta_\beta^{(k)} - \hat{M}_\alpha^{(k)}) \rangle_{S_3^{(k)}} \\
&\quad + \langle w^{(k)}, t^* (Q_\alpha^{(k)} - 2\hat{Q}_\alpha^{(k)}) \eta_\alpha \rangle_{S_5^{(k)}} - 2\langle N_{\alpha\beta}^{(k)}, t^* \hat{\bar{u}}_\alpha^{(k)} \eta_\beta \rangle_{S_2^{(k)}} \\
&\quad -2\langle M_{\alpha\beta}^{(k)}, t^* \hat{\phi}_\alpha^{(k)} \eta_\beta \rangle_{S_4^{(k)}} + \langle Q_\alpha^{(k)}, t^* (w^{(k)} - 2\hat{w}^{(k)}) \eta_\alpha \rangle_{S_6^{(k)}} \\
&+\sum_{k=1}^{n} \{ 2\langle \bar{u}_\alpha^{(k)}, t^* ((N_{\alpha\beta}^{(k)} \eta_\beta)' - (g_1^{(k)})_\alpha) \rangle_{S_{1i}^{(k)}} + 2\langle \phi_\alpha^{(k)}, t^* ((M_{\alpha\beta}^{(k)} \eta_\beta^{(k)})' - (g_3^{(k)})_\alpha) \rangle_{S_{3i}^{(k)}} \\
&\quad + \langle w^{(k)}, t^* ((Q_\alpha^{(k)} \eta_\alpha)' - 2(g_5^{(k)})_\alpha \eta_\alpha) \rangle_{S_{5i}^{(k)}} - 2\langle N_{\alpha\beta}^{(k)}, t^* (g_2^{(k)})_\alpha \eta_\beta \rangle_{S_{2i}^{(k)}} \\
&\quad -2\langle M_{\alpha\beta}^{(k)}, t^* (g_4^{(k)})_\alpha \eta_\beta \rangle_{S_{4i}^{(k)}} + \langle Q_\alpha^{(k)}, t^* ((w^{(k)} \eta_\alpha)' - 2(g_6^{(k)}) \eta_\alpha) \rangle_{S_{6i}^{(k)}} \qquad (183)
\end{aligned}
$$

where $\bar{u}_\alpha^{(k)}$ and $\phi_\alpha^{(k)}$ need not be differentiable. If we eliminate $w_{,\alpha}^{(k)}$ from (183), the extended formulation which does not require continuous differentiability in any of the kinematic variables is realized.

$$
\begin{aligned}
\Omega_6 = \sum_{k=1}^{n} \{ &-\langle \bar{u}_\alpha^{(k)}, P^{(k)} \bar{u}_\alpha^{(k)} \rangle_{R^{(k)}} - \langle \phi_\alpha^{(k)}, I^{(k)} \phi_\alpha^{(k)} \rangle_{R^{(k)}} - \langle w^{(k)}, P^{(k)} w^{(k)} \rangle_{R^{(k)}} \\
&-\langle \bar{e}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)} \bar{e}_{\gamma\delta}^{(k)} \rangle_{R^{(k)}} - \langle \kappa_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)} \rangle_{R^{(k)}} \\
&+2\langle \bar{u}_\alpha^{(k)}, t^* N_{\alpha\beta,\beta}^{(k)} \rangle_{R^{(k)}} + 2\langle \phi_\alpha^{(k)}, t^* M_{\alpha\beta,\beta}^{(k)} \rangle_{R^{(k)}} + 2\langle w^{(k)}, t^* Q_{\alpha,\alpha}^{(k)} \rangle_{R^{(k)}} \\
&-2\langle \bar{u}_\alpha^{(k)}, R^{(k)} \phi_\alpha^{(k)} \rangle_{R^{(k)}} - 2\langle \phi_\alpha^{(k)}, t^* Q_\alpha^{(k)} \rangle_{R^{(k)}} - 2\langle \bar{e}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)} \rangle_{R^{(k)}} \\
&+2\langle \bar{e}_{\alpha\beta}^{(k)}, t^* N_{\alpha\beta}^{(k)} \rangle_{R^{(k)}} + 2\langle M_{\alpha\beta}^{(k)}, t^* \kappa_{\alpha\beta}^{(k)} \rangle_{R^{(k)}} + 2\langle 2e_{\alpha 3}^{(k)}, t^* Q_\alpha^{(k)} \rangle_{R^{(k)}} \}
\end{aligned}
$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)}, -2t^{*}\lambda_{\alpha\beta}^{(k,j)}e_{\beta 3}^{j}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1}\{<T_{\alpha}^{(k)}, t^{*}(\bar{u}_{\alpha}^{(k)}+t_{k}\phi_{\alpha}^{(k)}-\bar{u}_{\alpha}^{(k+1)})>_{R^{(k)}} + <T_{3}^{(k)}, t^{*}(w^{(k)}-w^{(k+1)})>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_{\alpha}^{(k)}, t^{*}F_{\alpha}^{(k)}+X_{\alpha}^{(k)}>_{R^{(k)}} + <\phi_{\alpha}^{(k)}, t^{*}G_{\alpha}^{(k)}+Y_{\alpha}^{(k)}>_{R^{(k)}} + <w^{(k)}, t^{*}F_{3}^{(k)}+Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_{\alpha}^{(1)}, t^{*}T_{\alpha}^{(0)}>_{R^{(1)}} - 2<w^{(1)}, t^{*}T_{3}^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_{\alpha}^{(n)}, t^{*}T_{\alpha}^{(n)}>_{R^{(n)}} + 2>\phi_{\alpha}^{(k)}, t^{*}t_{n}T_{\alpha}^{(n)}>_{R^{(k)}} + 2<w^{(n)}, t^{*}T_{3}^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{ 2<\bar{u}_{\alpha}^{(k)}, t^{*}(N_{\alpha\beta}^{(k)}\eta_{\beta}-\hat{N}_{\alpha}^{(k)})>_{S_{1}^{(k)}} + 2<\phi_{\alpha}^{(k)}, t^{*}(M_{\alpha\beta}^{(k)}\eta_{\beta}^{(k)}-\hat{M}_{\alpha}^{(k)})>_{S_{3}^{(k)}}$$

$$+ 2<w^{(k)}, t^{*}(Q_{\alpha}^{(k)}-\hat{Q}_{\alpha}^{(k)})\eta_{\alpha}>_{S_{5}^{(k)}} - 2<N_{\alpha\beta}^{(k)}, t^{*}\hat{\bar{u}}_{\alpha}^{(k)}\eta_{\beta}>_{S_{2}^{(k)}}$$

$$-2<M_{\alpha\beta}^{(k)}, t^{*}\phi_{\alpha}^{(k)}\eta_{\beta}>_{S_{4}^{(k)}} - 2<Q_{\alpha}^{(k)}, t^{*}\hat{w}^{(k)}\eta_{\alpha})>_{S_{6}^{(k)}}$$

$$+\sum_{k=1}^{n}\{ 2<\bar{u}_{\alpha}^{(k)}, t^{*}((N_{\alpha\beta}^{(k)}\eta_{\beta})'-(g_{1}^{(k)})_{\alpha})>_{S_{1i}^{(k)}} + 2<\phi_{\alpha}^{(k)}, t^{*}((M_{\alpha\beta}^{(k)}\eta_{\beta}^{(k)})'-(g_{3}^{(k)})_{\alpha})>_{S_{3i}^{(k)}}$$

$$+2<w^{(k)}, t^{*}((Q_{\alpha}^{(k)}\eta_{\alpha})'-(g_{5}^{(k)})_{\alpha}\eta_{\alpha})>_{S_{5i}^{(k)}} - 2<N_{\alpha\beta}^{(k)}, t^{*}(g_{2}^{(k)})_{\alpha}\eta_{\beta}>_{S_{2i}^{(k)}}$$

$$-2<M_{\alpha\beta}^{(k)}, t^{*}(g_{4}^{(k)})_{\alpha}\eta_{\beta}>_{S_{4i}^{(k)}} - 2<Q_{\alpha}^{(k)}, t^{*}(g_{6}^{(k)})\eta_{\alpha}>_{S_{6i}^{(k)}} \qquad (184)$$

In the context of use in the finite element procedure, Ω_3 leads to displacement formulations while Ω_6 leads to stress formulations. Evidently, other extensions of the general variational principle than the ones presented here are possible. For example, elimination of derivatives of certain force resultants from Ω_1 and Ω_2 or derivatives of kinematic variables from Ω_4 and Ω_5 results in so-called mixed formulations.

4.3.4 Some Specializations

If the admissible state is constrained to satisfy some field equations and/or boundary conditions, certain specialized forms of the variational principle are realized. This procedure is used to reduce the number of free variables in the governing function. Also, certain assumptions in the spatial or temporal variation of some variables can lead to approximate theories. Some specializations of the extended variational principles stated in the previous section are presented below.

Specialization of the function Ω_3, to the case where (146) and (169)-(171) are identically satisfied i.e. if displacement w is constant through the thickness of plate and the jump discontinuities in 'displacement' components over internal surfaces are identically satisfied leads to

$$\begin{aligned}
\Omega_7 = \sum_{k=1}^{n} \{ & -<\bar{u}_\alpha^{(k)}, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} - <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - <w, P^{(k)}w>_{R^{(k)}} \\
& -<\bar{e}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)} \bar{e}_{\gamma\delta}^{(k)}>_{R^{(k)}} - <\kappa_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)}>_{R^{(k)}} \\
& +2<t^* N_{\alpha\beta}^{(k)}, (\bar{e}_{\alpha\beta}^{(k)} - \bar{u}_{\alpha,\beta}^{(k)})>_{R^{(k)}} + 2<t^* M_{\alpha\beta}^{(k)}, (\kappa_{\alpha\beta}^{(k)} - \phi_{\alpha,\beta})>_{R^{(k)}} \\
& +2<t^* Q_\alpha^{(k)}, (2e_{\alpha 3}^{(k)} - w_{,\alpha} - \phi_\alpha^{(k)})>_{R^{(k)}} - 2<\bar{e}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)}>_{R^{(k)}} \\
& -2<\bar{u}_\alpha^{(k)}, R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} \}
\end{aligned}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)}, -2t^* \lambda_{\alpha\beta}^{(k,j)} e_{\beta 3}^{(j)}>_{R^{(k)}} + \sum_{k=1}^{n-1} <T_\alpha^{(k)}, t^*(\bar{u}_\alpha^{(k)} + t_k \phi_\alpha^{(k)} - \bar{u}_\alpha^{(k+1)})>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n} \{ <\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R^{(k)}} + <w, t^* F_3^{(k)} + Z^{(k)}>_{R^{(k)}} \}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<\phi_\alpha^{(k)}, t^* t_n T_\alpha^{(n)}>_{R^{(k)}} + 2<w, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_\alpha^{(k)}, t^* \hat{N}_\alpha^{(k)}>_{S_1^{(k)}} - 2<\phi_\alpha^{(k)}, t^* \hat{M}_\alpha^{(k)}>_{S_3^{(k)}}$$

$$-2<w, t^* \hat{Q}_\alpha^{(k)}\eta_\alpha>_{S_5^{(k)}} + 2<N_{\alpha\beta}^{(k)}, t^*(\bar{u}_\alpha^{(k)} - \hat{\bar{u}}_\alpha^{(k)})\eta_\beta>_{S_2^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)}, t^*(\phi_\alpha^{(k)} - \hat{\phi}_\alpha^{(k)})\eta_\beta>_{S_4^{(k)}} + 2<Q_\alpha^{(k)}, t^*(w - \hat{w})\eta_\alpha>_{S_6^{(k)}} \qquad (185)$$

Since $w^{(k)}$ was assumed to be constant over the thickness of layer, its continuity in the interface implies that the lateral displacement is constant through the thickness of plate. If we further specialize Ω_7 to identically satisfy the kinematic relations (125)-(128),

$$\Omega_8 = \sum_{k=1}^{n}\{-<\bar{u}_\alpha^{(k)}, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} - <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} - <w, P^{(k)}w>_{R^{(k)}}$$

$$-<\bar{e}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(k)}>_{R^{(k)}} - <\kappa_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$-2<\bar{e}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}} - 2<\bar{u}_\alpha^{(k)}, R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n}<2e_{\alpha 3}^{(k)}, -2t^* \lambda_{\alpha\beta}^{(kj)} e_{\beta 3}^{j}>_{R^{(k)}} + \sum_{k=1}^{n-1}\{<T_\alpha^{(k)}, t^*(\bar{u}_\alpha^{(k)} + t_k\phi_\alpha^{(k)} - \bar{u}_\alpha^{(k+1)})>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}\{<\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R^{(k)}} + <w, t^* F_3^{(k)} + Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<\phi_\alpha^{(k)}, t^* t_n T_\alpha^{(n)}>_{R^{(k)}} + 2<w, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_\alpha^{(k)}, t^* \hat{N}_\alpha^{(k)}>_{S_1^{(k)}} - 2<\phi_\alpha^{(k)}, t^* \hat{M}_\alpha^{(k)}>_{S_3^{(k)}}$$

$$-2<w,\ t^*\hat{Q}_\alpha^{(k)}\eta_\alpha>_{S_5^{(k)}}+2<N_{\alpha\beta}^{(k)},\ t^*(\bar{u}_\alpha^{(k)}-\hat{\bar{u}}_\alpha^{(k)})\eta_\beta>_{S_7^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)},\ t^*(\phi_\alpha^{(k)}-\hat{\phi}_\alpha^{(k)})\eta_\beta>_{S_4^{(k)}}+2<Q_\alpha^{(k)},\ t^*(w-\hat{w})\eta_\alpha>_{S_6^{(k)}} \tag{186}$$

Furtherfore, if the interface continuity condition of the in-plane displacements (145) and the displacement boundary conditions (163)-(165) are identically satisfied, Ω_8 in (186) reduces to the potential energy type variational principle. Here, two different forms of expression are possible, depending on which of the variables is eliminated. When $\bar{u}_\alpha^{(k)}$ is eliminated, it becomes

$$\Omega_9=-\sum_{k=1}^{n}\{<\bar{u}_\alpha^{(1)},\ P^{(k)}\bar{u}_\alpha^{(1)}>_{R^{(k)}}+2<\bar{u}_\alpha^{(1)},\ P^{(k)}\sum_{i=1}^{k-1}t_i\phi_\alpha^i>_{R^{(k)}}+<\sum_{i=1}^{k-1}t_i\phi_\alpha^i,\ P^{(k)}\sum_{j=1}^{k-1}t_j\phi_\alpha^{(j)}>_{R^{(k)}}$$

$$+<\phi_\alpha^{(k)},\ I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}+<w,\ P^{(k)}w>_{R^{(k)}}+2<\bar{u}_\alpha^{(1)},\ R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}$$

$$+2<\sum_{i=1}^{k-1}t_i\phi_\alpha^i,\ R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}+<\bar{e}_{\alpha\beta}^{(1)},\ t^*A_{\alpha\beta\gamma\delta}^{(k)}\bar{e}_{\gamma\delta}^{(1)}>_{R^{(k)}}$$

$$+2<\bar{e}_{\alpha\beta}^{(1)},\ t^*A_{\alpha\beta\gamma\delta}^{(k)}\sum_{i=1}^{k-1}t_i\kappa_{\gamma\delta}^i>_{R^{(k)}}+<\sum_{i=1}^{k-1}t_i\kappa_{\alpha\beta}^{(k)},\ t^*A_{\alpha\beta\gamma\delta}^{(k)}\sum_{j=1}^{k-1}t_j\kappa_{\gamma\delta}^{(j)}>_{R^{(k)}}$$

$$+<\kappa_{\alpha\beta}^{(k)},\ t^*D_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}+2<\bar{e}_{\alpha\beta}^{(1)},\ t^*B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>_{R^{(k)}}$$

$$+2<\sum_{i=1}^{k-1}t_i\kappa_{\alpha\beta}^i,\ t^*B_{\alpha\beta\gamma\delta}^{(k)}\kappa_{\gamma\delta}^{(k)}>\}$$

$$-\sum_{k=1}^{n}\sum_{j=1}^{n}<2e_{\alpha3}^{(k)},\ +2t^*\lambda_{\alpha\beta}^{(kj)}e_{\beta3}^j>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}\{<\bar{u}_\alpha^{(1)},\ t^*F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}}+<\sum_{i=1}^{k-1}t_i\phi_\alpha^i,\ t^*F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}}$$

$$+<\phi_\alpha^{(k)},\ t^*G_\alpha^{(k)}+Y_\alpha^{(k)}>_{R^{(k)}}+<w,\ t^*F_3^{(k)}+Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w, t^* T_3^{(0)}>_{R^{(1)}} + 2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(n)}>_{R^{(n)}}$$

$$+2<\sum_{i=1}^{n} t_i \phi_\alpha^i, t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<w, t^* T_3^{(n)}>_{R^{(n)}}$$

$$-2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(1)}, t^* \hat{N}_\alpha^{(k)}>_{S_1^{(k)}} + <\sum_{i=1}^{k-1} t_i \phi_\alpha^i, t^* \hat{N}_\alpha^{(k)}>_{S_1^{(k)}} + <\phi_\alpha^{(k)}, t^* \hat{M}_\alpha^{(k)}>_{S_3^{(k)}}$$

$$+<w, t^* \hat{Q}_\alpha^{(k)} \eta_\alpha>_{S_5^{(k)}}\} \tag{187}$$

which is the variational principle for Sun's [1973] theory. On the other hand, eliminating $\phi_\alpha^{(k)}$, we have

$$\Omega_{10} = \sum_{k=1}^{n}\{-<\bar{u}_\alpha^{(k)}, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} - \frac{1}{t_k^{(2)}}<(\bar{u}_\alpha^{(k+1)} - \bar{u}_\alpha^{(k)}), I^{(k)}(\bar{u}_\alpha^{(k+1)} - \bar{u}_\alpha^{(k)})>_{R^{(k)}} - <w, P^{(k)} w>_{R^{(k)}}$$

$$-<\bar{e}_{\alpha\beta}^{(k)}, t^* A_{\alpha\beta\gamma\delta}^{(k)} \bar{e}_{\gamma\delta}^{(k)}>_{R^{(k)}} - \frac{1}{t_k^{(2)}}<(\bar{e}_{\alpha\beta}^{(k+1)} - \bar{e}_{\alpha\beta}^{(k)}, t^* D_{\alpha\beta\gamma\delta}^{(k)}(\bar{e}_{\gamma\delta}^{(k+1)} - \bar{e}_{\alpha\beta}^{(k)})>_{R^{(k)}}$$

$$-\frac{2}{t_k}<\bar{e}_{\alpha\beta}^{(k)}, t^* B_{\alpha\beta\gamma\delta}^{(k)}(\bar{e}_{\gamma\delta}^{(k+1)} - \bar{e}_{\gamma\delta}^{(k)})>_{R^{(k)}} - \frac{2}{t_k}<\bar{u}_\alpha^{(k)}, R^{(k)}(\bar{u}_\alpha^{(k+1)} - \bar{u}_\alpha^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)}, -2t^* \lambda_{\alpha\beta}^{(kj)} e_{\beta 3}^{j}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)}, t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R^{(k)}} + \frac{1}{t_k}<(\bar{u}_\alpha^{(k+1)}, t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R^{(k)}}$$

$$+ <w, t^* F_3^{(k)} + Z^{(k)}>_{R^{(k)}}\} - 2<\bar{u}_\alpha^{(1)}, t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w, t^* T_3^{(0)}>_{R^{(1)}}$$

$$+2<\bar{u}_\alpha^{(n)}, t^* T_\alpha^{(n)}>_{R^{(n)}} + \frac{2}{t_k}<\bar{u}_\alpha^{(k+1)} - \bar{u}_\alpha^{(k)}, t^* t_n T_\alpha^{(k)}>_{R^{(k)}} + 2<w, t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{ -2<\bar{u}_\alpha^{(k)}, t^* \hat{N}_\alpha^{(k)})>_{S_1^{(k)}} - \frac{2}{t_k}<(\bar{u}_\alpha^{(k+1)} - \bar{u}_\alpha^{(k)}), t^* \hat{M}_\alpha^{(k)})>_{S_3^{(k)}} - 2<w, t^* \hat{Q}_\alpha^{(k)} \eta_\alpha)>_{S_5^{(k)}}$$

which is the variational principle for Srinivas' [1973] and Seide's [1980] theories. In connection with finite element formulation, it is worth noting that use of Ω_{10} is more convenient if in-plane streching of individual layer needs to be specified.

Clearly, a large number of other specializations from other extended variational principles are possible. We desist from an attempt to make a comprehensive catalogue of such possibilities.

4.4 COMPLEMENTARY VARIATIONAL PRINCIPLES

4.4.1 General

An alternative procedure to set up variational principles governing the problem is to write the operator equations in complementary form instead of the direct formulation (147). In this formulation, it is assumed the kinematic relations are satisfied. Al-Ghothani [1986] presented the complementary formulation of laminated composite plate for the dynamic case using a discrete laminate theory and discussed various specializations of the extended variational principles. In this section, we present the complementary form of the field equations given in the previous section. Except the constitutive equations for transverse shear, the formulation is the same as the one given by Al-Ghothani [1986]. Since an extensive discussion on the extended principles and various specializations, some of which led to the variational principles of various approximate theories, has been given in [Al-Ghothani 1986], those investigations will not be repeated here. However, some extensions of the general complementary variational principle and specializations which are not included in [Al-Ghothani 1986], but are interesting are presented.

4.4.2 Complementary Form of the Field Equations

Assuming the kinematic relations (125)-(128) are satisfied, the field equations (129)-(131), (137)-(139) and (145),(146) may be written in the self-adjoint matrix form as

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & \mathbf{E}_{1,2} & \mathbf{0} & \mathbf{E}_{1,3} & \mathbf{0} & \cdots & \mathbf{E}_{1,n-1} & \mathbf{0} & \mathbf{E}_{1,n} \\ \mathbf{0} & \mathbf{C}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ & \mathbf{A}_2 & \mathbf{B}_2 & \mathbf{E}_{2,3} & \mathbf{0} & \cdots & \mathbf{E}_{2,n-1} & \mathbf{0} & \mathbf{E}_{2,n} & \\ & \mathbf{0} & \mathbf{C}^T & \mathbf{0} & \cdots & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ & & \mathbf{A}_3 & \mathbf{B}_3 & \cdots & & \mathbf{E}_{3,n-1} & \mathbf{0} & \mathbf{E}_{3,n} & \\ & & & \mathbf{0} & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ & & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \mathbf{A}_{n-1} & \mathbf{B}_{n-1} & \mathbf{E}_{n-1,n} & \\ & & & & & & & \mathbf{0} & \mathbf{C}^T & \\ & & & & & & & & \mathbf{A}_n & \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \Xi_1 \\ \mathbf{U}_2 \\ \Xi_2 \\ \mathbf{U}_3 \\ \Xi_3 \\ \cdot \\ \cdot \\ \mathbf{U}_{n-1} \\ \Xi_{n-1} \\ \mathbf{U}_n \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_1 + \Xi_0 \\ \mathbf{0} \\ -\mathbf{r}_2 \\ \mathbf{0} \\ -\mathbf{r}_3 \\ \mathbf{0} \\ \cdot \\ \cdot \\ -\mathbf{r}_{n-1} \\ \mathbf{0} \\ -\mathbf{r}_n - \Xi_n \end{bmatrix} \tag{189}$$

where we redefine the operators

$$\mathbf{A}_k = \begin{bmatrix} -P^{(k)}\delta_{\alpha\gamma} & -R^{(k)}\delta_{\alpha\gamma} & 0 & \frac{1}{2}t^*\Pi_1 & 0 & 0 \\ -R^{(k)}\delta_{\alpha\gamma} & -I^{(k)}\delta_{\alpha\gamma} & 0 & 0 & \frac{1}{2}t^*\Pi_1 & -t^* \\ 0 & 0 & -P^{(k)} & 0 & 0 & t^*\delta_{\alpha\gamma}\dfrac{\partial}{\partial\alpha} \\ -\frac{1}{2}t^*\Pi_2 & 0 & 0 & t^*\overline{A}^{(k)}_{\alpha\beta\gamma\delta} & t^*\overline{B}_{\alpha\beta\gamma\delta} & 0 \\ 0 & -\frac{1}{2}t^*\Pi_2 & 0 & t^*\overline{B}_{\alpha\beta\gamma\delta} & t^*\overline{D}_{\alpha\beta\gamma\delta} & 0 \\ 0 & -t^* & -t^*\delta_{\alpha\gamma}\dfrac{\partial}{\partial\alpha} & 0 & 0 & t^*\mu^{kk}_{\gamma\alpha} \end{bmatrix} \tag{190}$$

$$\mathbf{B}_k^T = \begin{bmatrix} t^* & t^*t_k & 0 & 0 & 0 & 0 \\ 0 & 0 & t^* & 0 & 0 & 0 \end{bmatrix} \tag{191}$$

$$\mathbf{C}_k^T = \begin{bmatrix} -t^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t^* & 0 & 0 & 0 \end{bmatrix} \tag{192}$$

$$\mathbf{E}_{i,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{ij}_{\gamma\alpha} \end{pmatrix} \qquad i,j = 1, 2, \ldots, n \tag{193}$$

$$\mathbf{U}_k^T = [\bar{u}_\gamma^{(k)}, \phi_\gamma^{(k)}, w^k, N_{\gamma\delta}^{(k)}, M_{\gamma\delta}^{(k)}, Q_\alpha^{(k)}], \qquad k = 1, 2, \ldots, n \tag{194}$$

$$\Xi_k^T = [T_\alpha^{(k)}, T_3^{(k)}] \qquad k = 1, 2, \ldots, n-1 \tag{195}$$

$$\mathbf{r}_k^T = [t^* F_\alpha^{(k)} + X_\alpha^{(k)}, t^* G_\alpha + Y_\alpha^{(k)}, t^* F_3^{(k)} + Z^{(k)}, 0, 0, 0] \tag{196}$$

$$\Xi_0^T = [t^* T_\alpha^{(0)}, 0, t^* T_3^{(0)}, 0, 0, 0] \tag{197}$$

$$\Xi_n^T = [t^* T_\alpha^{(n)}, t^* t_n T_\alpha^{(n)}, t^* T_3^{(n)}, 0, 0, 0] \tag{198}$$

$$\Pi_1 = \delta_{\alpha\delta} \frac{\partial}{\partial\gamma} + \delta_{\alpha\gamma} \frac{\partial}{\partial\delta} \tag{199}$$

$$\Pi_2 = \delta_{\alpha\gamma} \frac{\partial}{\partial\beta} + \delta_{\beta\gamma} \frac{\partial}{\partial\alpha} \tag{200}$$

The operator matrix in (189) is self-adjoint in the sense of (A-25) if the bilinear mapping defined in (159) is used. This self-adjoint form of the field equations (189) is different from Al-Ghothani's [1986] in that it includes matrices $\mathbf{E}_{i,j}$ representing the coupling of transverse shear constitutive relations between layers based on the consistent shear theory developed in Section III.

4.4.3 General Complementary Variational Principle

For the operator matrix equation (189), the governing function is defined, following (A-29), as

$$J = \sum_{k=1}^{n} \langle \mathbf{U}_k^T, \mathbf{A}_k \mathbf{U}_k \rangle_{F^{(k)}} + \sum_{k=1}^{n-1} \langle \mathbf{U}_k^T, \mathbf{B}_k \Xi_k \rangle_{F^{(k)}} + \sum_{k=2}^{n} \langle \mathbf{U}_k^T, \mathbf{C} \Xi_{k-1} \rangle_{F^{(k)}}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} < \mathbf{U}_k^T, \mathbf{E}_{j,k}\mathbf{U}_j >_{R^{(k)}} - \sum_{k=1}^{n} < \mathbf{U}_k^T, \mathbf{E}_{k,k}\mathbf{U}_k >_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1} <\Xi_k^T, \mathbf{B}_k^T\mathbf{U}_k>_{R^{(k)}} + \sum_{k=1}^{n-1} <\Xi_k^T, \mathbf{C}^T\mathbf{U}_{k+1}>_{R^{(k)}}$$

$$+2\sum_{k=1}^{n} <\mathbf{U}_k^T, \mathbf{r}_k>_{R^{(k)}} - 2<\mathbf{U}_1^T, \Xi_0>_{R^{(1)}} + 2<\mathbf{U}_n^T, \Xi_n>_{R^{(n)}}$$

$$+ \textit{Boundary Terms} + \textit{Internal Jump Terms} \qquad (201)$$

Substituting (190)-(200) into (201), the explicit form of the governing function is obtained.

$$J = \sum_{k=1}^{n} \{ <\bar{u}_\alpha^{(k)}, -P^{(k)}\bar{u}_\alpha^{(k)} + t^* N_{\alpha\beta,\beta}^{(k)} - R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}$$

$$+ <\phi_\alpha^{(k)}, -R^{(k)}\bar{u}_\alpha^{(k)} - I^{(k)}\phi_\alpha^{(k)} + t^* M_{\alpha\beta,\beta}^{(k)} - t^* Q_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)}, -P^{(k)}w^{(k)} + t^* Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$+ <N_{\alpha\beta}^{(k)}, -t^*\bar{u}_{\alpha,\beta}^{(k)} + t^*(\bar{A}_{\alpha\beta\gamma\delta}^{(k)}N_{\gamma\delta}^{(k)} + \bar{B}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}^{(k)})>_{R^{(k)}}$$

$$+ <M_{\alpha\beta}^{(k)}, -t^*\phi_{\alpha,\beta}^{(k)} + t^*(\bar{B}_{\alpha\beta\gamma\delta}^{(k)}N_{\gamma\delta}^{(k)} + \bar{D}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}^{(k)})>_{R^{(k)}}$$

$$+ <Q_\alpha^{(k)}, -t^*\phi_\alpha^{(k)} - t^* w_{,\alpha}^{(k)} + t^*\mu_{\alpha\beta}^{kk}Q_\beta^{(k)}>_{R^{(k)}} \}$$

$$+\sum_{k=1}^{n-1} \{ <\bar{u}_\alpha^{(k)}, t^* T_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)}, t^* t_f T_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)}, t^* T_3^{(k)}>_{R^{(k)}} \}$$

$$+\sum_{k=2}^{n} \{ <\bar{u}_\alpha^{(k)}, -t^* T_\alpha^{(k-1)}>_{R^{(k)}} + <w^{(k)}, -t^* T_3^{(k-1)}>_{R^{(k)}} \}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <Q_\alpha^{(k)}, t^*\mu_{\alpha\beta}^{(kj)}Q_\beta^{(j)}>_{R^{(k)}} - <Q_\alpha^{(k)}, t^*\mu_{\alpha\beta}^{kk}Q_\beta^{(k)}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1} \{ <T_\alpha^{(k)}, t^*\bar{u}_\alpha^{(k)} + t^* t_f \phi_\alpha^{(k)}>_{R^{(k)}} + <T_3^{(k)}, t^* w^{(k)}>_{R^{(k)}} \}$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ -t^*\bar{u}_\alpha^{(k+1)}>_{R^{(k)}}\ +\ <T_3^{(k)},\ -t^*w^{(k+1)}>_{R^{(k)}}\ \}$$

$$+2\sum_{k=1}^{n}\{\ <\bar{u}_\alpha^{(k)},\ t^*F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}}\ +\ <\phi_\alpha^{(k)},\ t^*G_\alpha^{(k)}+Y_\alpha^{(k)}>_{R^{(k)}}\ +\ <w^{(k)},\ t^*F_3^{(k)}+Z^{(k)}>_{R^{(k)}}\ \}$$

$$-2<\bar{u}_\alpha^{(1)},\ t^*T_\alpha^{(0)}>_{R^{(1)}}-2<w^{(1)},\ t^*T_3^{(0)}>_{R^{(1)}}+2<(\bar{u}_\alpha^{(n)}+t_n\phi_\alpha^{(n)}),\ t^*T_\alpha^{(n)}>_{R^{(n)}}+2<w^{(n)},\ t^*T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{\ -<\bar{u}_\alpha^{(k)},\ t^*(N_{\alpha\beta}^{(k)}\eta_\beta-2\hat{N}_\alpha^{(k)})>_{S_1^{(k)}}\ -\ <\phi_\alpha^{(k)},\ t^*(M_{\alpha\beta}^{(k)}\eta_\beta^{(k)}-2\hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$-<w^{(k)},\ t^*(Q_\alpha^{(k)}\eta_\alpha-2\hat{Q}_\alpha^{(k)}\eta_\alpha)>_{S_5^{(k)}}+\ <N_{\alpha\beta}^{(k)},\ t^*(\bar{u}_\alpha^{(k)}\eta_\beta-2\hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ t^*(\phi_\alpha^{(k)}\eta_\beta-2\hat{\phi}_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}}+\ <Q_\alpha^{(k)},\ t^*(w^{(k)}\eta_\alpha-2\hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}\}$$

$$+\sum_{k=1}^{n}\{\ -<\bar{u}_\alpha^{(k)},\ t^*((N_{\alpha\beta}^{(k)}\eta_\beta)'-2(g_1^{(k)})_\alpha)>_{S_{1i}^{(k)}}\ -\ <\phi_\alpha^{(k)},\ t^*((M_{\alpha\beta}^{(k)}\eta_\beta^{(k)})'-2(g_3^{(k)})_\alpha)>_{S_{3i}^{(k)}}$$

$$-<w^{(k)},\ t^*((Q_\alpha^{(k)}\eta_\alpha)'-2(g_{5i}^{(k)}))>_{S_{5i}^{(k)}}+\ <N_{\alpha\beta}^{(k)},\ t^*((\bar{u}_\alpha^{(k)}\eta_\beta)'-2(g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ t^*((\phi_\alpha^{(k)}\eta_\beta)'-2(g_4^{(k)})_\alpha\eta_\beta)>_{S_{4i}^{(k)}}+\ <Q_\alpha^{(k)},\ t^*((w^{(k)}\eta_\alpha)'-2(g_6^{(k)})\eta_\alpha)>_{S_{6i}^{(k)}}\}$$

$$=\sum_{k=1}^{n}\{\ -<\bar{u}_\alpha^{(k)},\ P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}}-\ <\phi_\alpha^{(k)},\ I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}-\ <w^{(k)},\ P^{(k)}w^{(k)}>_{R^{(k)}}$$

$$-2<\bar{u}_\alpha^{(k)},\ R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}-2<\phi_\alpha^{(k)},\ t^*Q_\alpha^{(k)}>_{R^{(k)}}$$

$$+<\bar{u}_\alpha^{(k)},\ t^*N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}+\ <\phi_\alpha^{(k)},\ t^*M_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}}+\ <w^{(k)},\ t^*Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$-<N_{\alpha\beta}^{(k)},\ t^*\bar{u}_{\alpha,\beta}^{(k)}>_{R^{(k)}}-\ <M_{\alpha\beta}^{(k)},\ t^*\phi_{\alpha,\beta}^{(k)}>_{R^{(k)}}-\ <Q_\alpha^{(k)},\ t^*w_{,\alpha}^{(k)}>_{R^{(k)}}$$

$$+<N_{\alpha\beta}^{(k)},\ t^*\bar{A}_{\alpha\beta\gamma\delta}^{(k)}N_{\gamma\delta}^{(k)}>_{R^{(k)}}+\ <M_{\alpha\beta}^{(k)},\ t^*\bar{D}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}>_{R^{(k)}}$$

$$+2<N_{\alpha\beta}^{(k)},\ t^*\bar{B}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n-1}\{\ <\bar{u}_\alpha^{(k)},\ t^*T_\alpha^{(k)}>_{R^{(k)}}+\ <\phi_\alpha^{(k)},\ t^*t_kT_\alpha^{(k)}>_{R^{(k)}}+\ <w^{(k)},\ t^*T_3^{(k)}>_{R^{(k)}}\ \}$$

$$+\sum_{k=2}^{n}\{\ <\bar{u}_\alpha^{(k)},\ -t^*T_\alpha^{(k-1)}>_{R^{(k)}}+\ <w^{(k)},\ -t^*T_3^{(k-1)}>_{R^{(k)}}\ \}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n}\ <Q_\alpha^{(k)},\ t^*\mu_{\alpha\beta}^{(kj)}Q_\beta^{(j)}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ t^*\bar{u}_\alpha^{(k)}+t^*t_k\phi_\alpha^{(k)}>_{R^{(k)}}+\ <T_3^{(k)},\ t^*w^{(k)}>_{R^{(k)}}\ \}$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ -t^*\bar{u}_\alpha^{(k+1)}>_{R^{(k)}}+\ <T_3^{(k)},\ -t^*w^{(k+1)}>_{R^{(k)}}\ \}$$

$$+2\sum_{k=1}^{n}\{\ <\bar{u}_\alpha^{(k)},\ t^*F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}}+\ <\phi_\alpha^{(k)},\ t^*G_\alpha^{(k)}+Y_\alpha^{(k)}>_{R^{(k)}}+\ <w^{(k)},\ t^*F_3^{(k)}+Z^{(k)}>_{R^{(k)}}\ \}$$

$$-2<\bar{u}_\alpha^{(1)},\ t^*T_\alpha^{(0)}>_{R^{(1)}}-2<w^{(1)},\ t^*T_3^{(0)}>_{R^{(1)}}+2<(\bar{u}_\alpha^{(n)}+t_n\phi_\alpha^{(n)}),\ t^*T_\alpha^{(n)}>_{R^{(n)}}+2<w^{(n)},\ t^*T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{\ -<\bar{u}_\alpha^{(k)},\ t^*(N_{\alpha\beta}^{(k)}\eta_\beta-2\hat{N}_\alpha^{(k)})>_{S_1^{(k)}}-\ <\phi_\alpha^{(k)},\ t^*(M_{\alpha\beta}^{(k)}\eta_\beta^{(k)}-2\hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$-\ <w^{(k)},\ t^*(Q_\alpha^{(k)}\eta_\alpha-2\hat{Q}_\alpha^{(k)}\eta_\alpha)>_{S_5^{(k)}}+\ <N_{\alpha\beta}^{(k)},\ t^*(\bar{u}_\alpha^{(k)}\eta_\beta-2\hat{\bar{u}}_\alpha^{(k)}\eta_\beta)>_{S_2^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ t^*(\phi_\alpha^{(k)}\eta_\beta-2\hat{\phi}_\alpha^{(k)}\eta_\beta)>_{S_4^{(k)}}+\ <Q_\alpha^{(k)},\ t^*(w^{(k)}\eta_\alpha-2\hat{w}^{(k)}\eta_\alpha)>_{S_6^{(k)}}\}$$

$$+\sum_{k=1}^{n}\{\ -<\bar{u}_\alpha^{(k)},\ t^*((N_{\alpha\beta}^{(k)}\eta_\beta)'-2(g_1^{(k)})_\alpha)>_{S_{1i}^{(k)}}-\ <\phi_\alpha^{(k)},\ t^*((M_{\alpha\beta}^{(k)}\eta_\beta^{(k)})'-2(g_3^{(k)})_\alpha)>_{S_{3i}^{(k)}}$$

$$-\ <w^{(k)},\ t^*((Q_\alpha^{(k)}\eta_\alpha)'-2(g_{5i}^{(k)}))>_{S_{5i}^{(k)}}+\ <N_{\alpha\beta}^{(k)},\ t^*((\bar{u}_\alpha^{(k)}\eta_\beta)'-2(g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$+\ <M_{\alpha\beta}^{(k)},\ t^*((\phi_\alpha^{(k)}\eta_\beta)'-2(g_4^{(k)})_\alpha\eta_\beta)>_{S_{4i}^{(k)}}+\ <Q_\alpha^{(k)},\ t^*((w^{(k)}\eta_\alpha)'-2(g_6^{(k)})\eta_\alpha)>_{S_{6i}^{(k)}}\}$$

As in direct formulation, it can be shown [Al-Ghothani 1986] that the Gateaux differential of J vanishes if and only if the field equations (189) along with the boundary conditions and the internal jump discontinuity conditions are satisfied.

4.4.4 Extended Complementary Variational Principles

Following the principles and methodology presented previously, it is possible to develop extended variational principles for the complementary form of the field equations (189) as well. Relations (172)-(174) can be used to eliminate some of the operators from (202). Simultaneous use of those relations to eliminate $N^{(k)}_{\alpha\beta,\beta}$, $M^{(k)}_{\alpha\beta,\beta}$ and $Q^{(k)}_{\alpha,\alpha}$ results in

$$
\begin{aligned}
J_1 = \sum_{k=1}^{n} \{ & -\langle \bar{u}^{(k)}_{\alpha}, P^{(k)} \bar{u}^{(k)}_{\alpha} \rangle_{R^{(k)}} - \langle \phi^{(k)}_{\alpha}, I^{(k)} \phi^{(k)}_{\alpha} \rangle_{R^{(k)}} - \langle w^{(k)}, P^{(k)} w^{(k)} \rangle_{R^{(k)}} \\
& -2\langle \bar{u}^{(k)}_{\alpha}, R^{(k)} \phi^{(k)}_{\alpha} \rangle_{R^{(k)}} - 2\langle \phi^{(k)}_{\alpha}, t^* Q^{(k)}_{\alpha} \rangle_{R^{(k)}} \\
& -2\langle N^{(k)}_{\alpha\beta}, t^* \bar{u}^{(k)}_{\alpha,\beta} \rangle_{R^{(k)}} - 2\langle M^{(k)}_{\alpha\beta}, t^* \phi^{(k)}_{\alpha,\beta} \rangle_{R^{(k)}} - 2\langle Q^{(k)}_{\alpha}, t^* w^{(k)}_{,\alpha} \rangle_{R^{(k)}} \\
& + \langle N^{(k)}_{\alpha\beta}, t^* \bar{A}^{(k)}_{\alpha\beta\gamma\delta} N^{(k)}_{\gamma\delta} \rangle_{R^{(k)}} + \langle M^{(k)}_{\alpha\beta}, t^* \bar{D}^{(k)}_{\alpha\beta\gamma\delta} M^{(k)}_{\gamma\delta} \rangle_{R^{(k)}} \\
& + 2\langle N^{(k)}_{\alpha\beta}, t^* \bar{B}^{(k)}_{\alpha\beta\gamma\delta} M^{(k)}_{\gamma\delta} \rangle_{R^{(k)}} \}
\end{aligned}
$$

$$
+ \sum_{k=1}^{n-1} \{ \langle \bar{u}^{(k)}_{\alpha}, t^* T^{(k)}_{\alpha} \rangle_{R^{(k)}} + \langle \phi^{(k)}_{\alpha}, t^* t_k T^{(k)}_{\alpha} \rangle_{R^{(k)}} + \langle w^{(k)}, t^* T^{(k)}_{3} \rangle_{R^{(k)}} \}
$$

$$
+ \sum_{k=2}^{n} \{ \langle \bar{u}^{(k)}_{\alpha}, -t^* T^{(k-1)}_{\alpha} \rangle_{R^{(k)}} + \langle w^{(k)}, -t^* T^{(k-1)}_{3} \rangle_{R^{(k)}} \}
$$

$$
+ \sum_{k=1}^{n} \sum_{j=1}^{n} \langle Q^{(k)}_{\alpha}, t^* \mu^{(kj)}_{\alpha\beta} Q^{(j)}_{\beta} \rangle_{R^{(k)}}
$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ t^*\bar{u}_\alpha^{(k)}+t^*t_k\phi_\alpha^{(k)}>_{R^{(k)}}+\ <T_3^{(k)},\ t^*w^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ -t^*\bar{u}_\alpha^{(k+1)}>_{R^{(k)}}+\ <T_3^{(k)},\ -t^*w^{(k+1)}>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{\ <\bar{u}_\alpha^{(k)},\ t^*F_\alpha^{(k)}+X_\alpha^{(k)}>_{R^{(k)}}+\ <\phi_\alpha^{(k)},\ t^*G_\alpha^{(k)}+Y_\alpha^{(k)}>_{R^{(k)}}+\ <w^{(k)},\ t^*F_3^{(k)}+Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)},\ t^*T_\alpha^{(0)}>_{R^{(1)}}-2<w^{(1)},\ t^*T_3^{(0)}>_{R^{(1)}}+2<(\bar{u}_\alpha^{(n)}+t_n\phi_\alpha^{(n)}),\ t^*T_\alpha^{(n)}>_{R^{(n)}}+2<w^{(n)},\ t^*T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{\ 2<\bar{u}_\alpha^{(k)},\ t^*\hat{N}_\alpha^{(k)}>_{S_1^{(k)}}+2<\phi_\alpha^{(k)},\ t^*\hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$+2<w^{(k)},\ t^*\hat{Q}_\alpha^{(k)}\eta_\alpha>_{S_5^{(k)}}+2<N_{\alpha\beta}^{(k)},\ t^*(\bar{u}_\alpha^{(k)}-\hat{\bar{u}}_\alpha^{(k)})\eta_\beta>_{S_2^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)},\ t^*(\phi_\alpha^{(k)}-\hat{\phi}_\alpha^{(k)})\eta_\beta>_{S_4^{(k)}}+2<Q_\alpha^{(k)},\ t^*(w^{(k)}-\hat{w}^{(k)})\eta_\alpha>_{S_6^{(k)}}\}$$

$$+\sum_{k=1}^{n}\{\ 2<\bar{u}_\alpha^{(k)},\ t^*(g_1^{(k)})_\alpha>_{S_{1l}^{(k)}}+2<\phi_\alpha^{(k)},\ t^*(g_3^{(k)})_\alpha>_{S_{3l}^{(k)}}$$

$$+2<w^{(k)},\ t^*(g_{5l}^{(k)})>_{S_5^{(k)}}+2<N_{\alpha\beta}^{(k)},\ t^*((\bar{u}_\alpha^{(k)}\eta_\beta)'-(g_2^{(k)})_\alpha\eta_\beta)>_{S_{2l}^{(k)}}$$

$$+2<M_{\alpha\beta}^{(k)},\ t^*((\phi_\alpha^{(k)}\eta_\beta)'-(g_4^{(k)})_\alpha\eta_\beta)>_{S_{4l}^{(k)}}+2<Q_\alpha^{(k)},\ t^*((w^{(k)}\eta_\alpha)'-(g_6^{(k)})\eta_\alpha)>_{S_{6l}^{(k)}}\}$$

(203)

Here, the force resultants need not be differentiable. Alternatively, eliminating $\bar{u}_{\alpha,3}^{(k)}$, $\phi_{\alpha,\beta}^{(k)}$ and $w_{,\alpha}^{(k)}$ from (202),

$$J_2=\sum_{k=1}^{n}\{\ -<\bar{u}_\alpha^{(k)},\ P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}}-\ <\phi_\alpha^{(k)},\ I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}-\ <w^{(k)},\ P^{(k)}w^{(k)}>_{R^{(k)}}$$

$$-2<\bar{u}_\alpha^{(k)},\ R^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}-2<\phi_\alpha^{(k)},\ t^*Q_\alpha^{(k)}>_{R^{(k)}}$$

$$+2<\bar{u}_\alpha^{(k)},\ t^* N_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}} + 2<\phi_\alpha^{(k)},\ t^* M_{\alpha\beta,\beta}^{(k)}>_{R^{(k)}} + 2<w^{(k)},\ t^* Q_{\alpha,\alpha}^{(k)}>_{R^{(k)}}$$

$$+<N_{\alpha\beta}^{(k)},\ t^* \bar{A}_{\alpha\beta\gamma\delta}^{(k)} N_{\gamma\delta}^{(k)}>_{R^{(k)}} + <M_{\alpha\beta}^{(k)},\ t^* \bar{D}_{\alpha\beta\gamma\delta}^{(k)} M_{\gamma\delta}>_{R^{(k)}}$$

$$+2<N_{\alpha\beta}^{(k)},\ t^* \bar{B}_{\alpha\beta\gamma\delta}^{(k)} M_{\gamma\delta}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n-1}\{\ <\bar{u}_\alpha^{(k)},\ t^* T_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)},\ t^* t_k T_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)},\ t^* T_3^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=2}^{n}\{\ <\bar{u}_\alpha^{(k)},\ -t^* T_\alpha^{(k-1)}>_{R^{(k)}} + <w^{(k)},\ -t^* T_3^{(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n} <Q_\alpha^{(k)},\ t^* \mu_{\alpha\beta}^{(k,j)} Q_\beta^{(j)}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ t^* \bar{u}_\alpha^{(k)} + t^* t_k \phi_\alpha^{(k)}>_{R^{(k)}} + <T_3^{(k)},\ t^* w^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n-1}\{\ <T_\alpha^{(k)},\ -t^* \bar{u}_\alpha^{(k+1)}>_{R^{(k)}} + <T_3^{(k)},\ -t^* w^{(k+1)}>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{n}\{\ <\bar{u}_\alpha^{(k)},\ t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)},\ t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)},\ t^* F_3^{(k)} + Z^{(k)}>_{R^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)},\ t^* T_\alpha^{(0)}>_{R^{(1)}} - 2<w^{(1)},\ t^* T_3^{(0)}>_{R^{(1)}} + 2<(\bar{u}_\alpha^{(n)} + t_n \phi_\alpha^{(n)}),\ t^* T_\alpha^{(n)}>_{R^{(n)}} + 2<w^{(n)},\ t^* T_3^{(n)}>_{R^{(n)}}$$

$$+\sum_{k=1}^{n}\{\ -2<\bar{u}_\alpha^{(k)},\ t^*(N_{\alpha\beta}^{(k)}\eta_\beta - \hat{N}_\alpha^{(k)})>_{S_1^{(k)}} - 2<\phi_\alpha^{(k)},\ t^*(M_{\alpha\beta}^{(k)}\eta_\beta - \hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$-2<w^{(k)},\ t^*(Q_\alpha^{(k)} - \hat{Q}_\alpha^{(k)})\eta_\alpha>_{S_5^{(k)}} - 2<N_{\alpha\beta}^{(k)},\ t^* \hat{\bar{u}}_\alpha^{(k)}\eta_\beta>_{S_2^{(k)}}$$

$$-2<M_{\alpha\beta}^{(k)},\ t^* \hat{\phi}_\alpha^{(k)}\eta_\beta>_{S_4^{(k)}} - 2<Q_\alpha^{(k)},\ t^* \hat{w}^{(k)}\eta_\alpha>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{\ -2<\bar{u}_\alpha^{(k)},\ t^*((N_{\alpha\beta}^{(k)}\eta_\beta)' - (g_1^{(k)})_\alpha)>_{S_{1t}^{(k)}} - 2<\phi_\alpha^{(k)},\ t^*((M_{\alpha\beta}^{(k)}\eta_\beta^{(k)})' - (g_3^{(k)})_\alpha)>_{S_{3t}^{(k)}}$$

$$-2<w^{(k)},\, t^*((Q_\alpha^{(k)}\eta_\alpha)'-(g_{5i}^{(k)}))>_{S_{5i}^{(k)}} - 2<N_{\alpha\beta}^{(k)},\, t^*(g_2^{(k)})_\alpha\eta_\beta)>_{S_{2i}^{(k)}}$$

$$-2<M_{\alpha\beta}^{(k)},\, t^*(g_4^{(k)})_\alpha\eta_\beta>_{S_{4i}^{(k)}} - 2<Q_\alpha^{(k)},\, t^*(g_6^{(k)})\eta_\alpha>_{S_{6i}^{(k)}}\} \qquad (204)$$

In addition to extensions illustrated above, it is obvious that numerous other extended principles are possible [Al-Ghothani 1986].

4.4.5 Some Specializations

As in the direct formulation, some specializations of the complementary extended variational principles are possible by requiring that certain field equations and/or boundary conditions be identically satisfied.

If we assume that the equilibrium equations (129)-(131) are identically satisfied, J_2 results in

$$J_3 = \sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(k)},\, P^{(k)}\bar{u}_\alpha^{(k)}>_{R^{(k)}} + <\phi_\alpha^{(k)},\, I^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}} + <w^{(k)},\, P^{(k)}w^{(k)}>_{R^{(k)}} + 2<\bar{u}_\alpha^{(k)},\, K^{(k)}\phi_\alpha^{(k)}>_{R^{(k)}}$$

$$+<N_{\alpha\beta}^{(k)},\, t^*\bar{A}_{\alpha\beta\gamma\delta}^{(k)}N_{\gamma\delta}^{(k)}>_{R^{(k)}} + <M_{\alpha\beta}^{(k)},\, t^*\bar{D}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}>_{R^{(k)}}$$

$$+2<N_{\alpha\beta}^{(k)},\, t^*\bar{B}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} <Q_\alpha^{(i)},\, t^*\mu_{\alpha\beta}^{(i,j)}Q_\beta^{(j)}>_{R^{(i)}}$$

$$+\sum_{k=1}^{n}\{ -2<\bar{u}_\alpha^{(k)},\, t^*(N_{\alpha\beta}^{(k)}\eta_\beta - \hat{N}_\alpha^{(k)})>_{S_1^{(k)}} - 2<\phi_\alpha^{(k)},\, t^*(M_{\alpha\beta}^{(k)}\eta_\beta^{(k)} - \hat{M}_\alpha^{(k)})>_{S_3^{(k)}}$$

$$-2<w^{(k)},\, t^*(Q_\alpha^{(k)} - \hat{Q}_\alpha^{(k)})\eta_\alpha>_{S_5^{(k)}} - 2<N_{\alpha\beta}^{(k)},\, t^*\hat{\bar{u}}_\alpha^{(k)}\eta_\beta>_{S_2^{(k)}}$$

$$-2<M_{\alpha\beta}^{(k)},\, t^*\hat{\phi}_\alpha^{(k)}\eta_\beta>_{S_4^{(k)}} - 2<Q_\alpha^{(k)},\, t^*\hat{w}^{(k)}\eta_\alpha>_{S_6^{(k)}}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_{\alpha}^{(k)},\ t^{*}((N_{\alpha\beta}^{(k)}\eta_{\beta})'-(g_{1}^{(k)})_{\alpha})>_{S_{1i}^{(k)}}-2<\phi_{\alpha}^{(k)},\ t^{*}((M_{\alpha\beta}^{(k)}\eta_{\beta}^{(k)})'-(g_{3}^{(k)})_{\alpha})>_{S_{3i}^{(k)}}$$

$$-2<w^{(k)},\ t^{*}((Q_{\alpha}^{(k)}\eta_{\alpha})'-(g_{5i}^{(k)}))>k_{S_{5i}^{(k)}}-2<N_{\alpha\beta}^{(k)},\ t^{*}(g_{2}^{(k)})_{\alpha}\eta_{\beta})>_{S_{2i}^{(k)}}$$

$$-2<M_{\alpha\beta}^{(k)},\ t^{*}(g_{4}^{(k)})_{\alpha}\eta_{\beta}>_{S_{4i}^{(k)}}-2<Q_{\alpha}^{(k)},\ t^{*}(g_{6}^{(k)})\eta_{\alpha}>_{S_{6i}^{(k)}}\} \tag{205}$$

If we further specialize (205) to satisfy the stress boundary conditions on $S_1^{(k)}$, $S_3^{(k)}$ and $S_5^{(k)}$, and the internal jump discontinuity conditions

$$J_{4}=\sum_{k=1}^{n}\{<\bar{u}_{\alpha}^{(k)},\ P^{(k)}\bar{u}_{\alpha}^{(k)}>_{R^{(k)}}+<\phi_{\alpha}^{(k)},\ I^{(k)}\phi_{\alpha}^{(k)}>_{R^{(k)}}+<w^{(k)},\ P^{(k)}w^{(k)}>_{R^{(k)}}+2<\bar{u}_{\alpha}^{(k)},\ R^{(k)}\phi_{\alpha}^{(k)}>_{R^{(k)}}$$

$$+<N_{\alpha\beta}^{(k)},\ t^{*}\bar{A}_{\alpha\beta\gamma\delta}^{(k)}N_{\gamma\delta}^{(k)}>_{R^{(k)}}+<M_{\alpha\beta}^{(k)},\ t^{*}\bar{D}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}>_{R^{(k)}}$$

$$+2<N_{\alpha\beta}^{(k)},\ t^{*}\bar{B}_{\alpha\beta\gamma\delta}^{(k)}M_{\gamma\delta}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n}<Q_{\alpha}^{(k)},\ t^{*}\mu_{\alpha\beta}^{(kj)}Q_{\beta}^{(j)}>_{R^{(k)}}$$

$$+\sum_{k=1}^{n}\{-2<N_{\alpha\beta}^{(k)},\ t^{*}\hat{\bar{u}}_{\alpha}^{(k)}\eta_{\beta}>_{S_{2}^{(k)}}-2<M_{\alpha\beta}^{(k)},\ t^{*}\hat{\phi}_{\alpha}^{(k)}\eta_{\beta}>_{S_{4}^{(k)}}-2<Q_{\alpha}^{(k)},\ t^{*}\hat{w}^{(k)}\eta_{\alpha}>_{S_{6}^{(k)}} \tag{206}$$

4.5 *DISCUSSION*

Based on the discrete laminated plate theory, which accounts for the effect of transverse shear deformation in a consistent manner, a systematic development of variational principles for dynamics of linear elastic composite laminated plate has been presented. The direct as well as complementary formulation are considered. Complementary self-adjoint form of the field equations is the same as the one presented by Al-Ghothani [1986], except for the coupling terms of transverse shear constitutive equations between layers which have been introduced in the consistent

shear deformable theory presented in Section III. Nonhomogeneous boundary conditions and internal jump discontinuities have been explicitly included in general variational principles. Allowance of jump discontinuity terms in variational formulation is necessary in the context of direct approximation in finite element spaces since the space of approximants may not be sufficiently smooth. Also, extensions of the general variational principles through elimination of certain field operators and specializations by restricting some of the field equations and/or boundary conditions to be identically satisfied have been proposed. Figure 2 and Figure 3 diagrammatically depict possible extensions of the general variational principles based on the direct and the complementary formulation, respectively. In either case, the specializations listed in this section are shown. These formulations can provide a basis for development of alternative approaches to approximate solution of the problem and also for development of approximation theories. Evidently, other extended forms could be used as starting points for specialization.

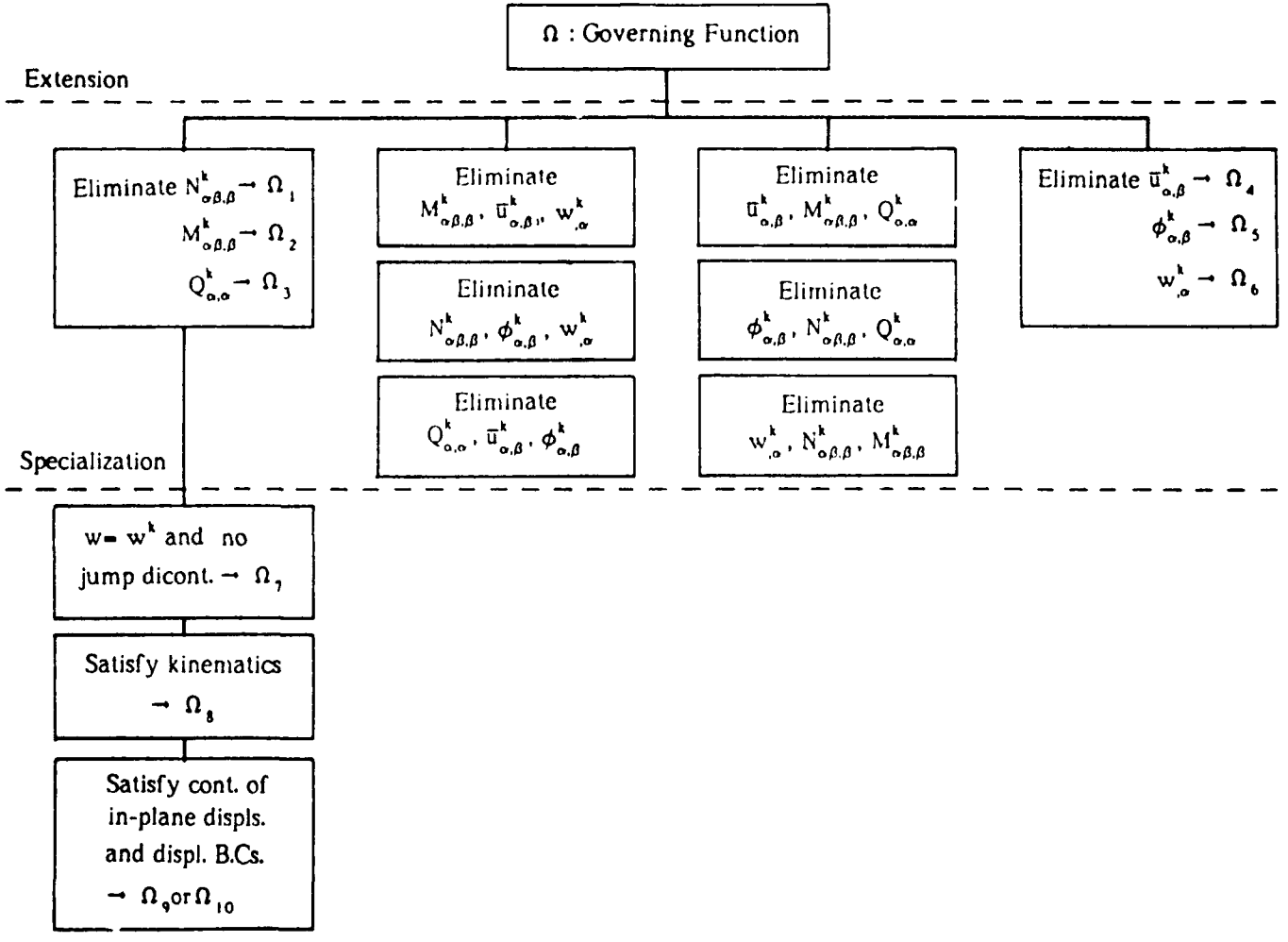


Figure 2: Family of Variational Principles by Direct Formulation

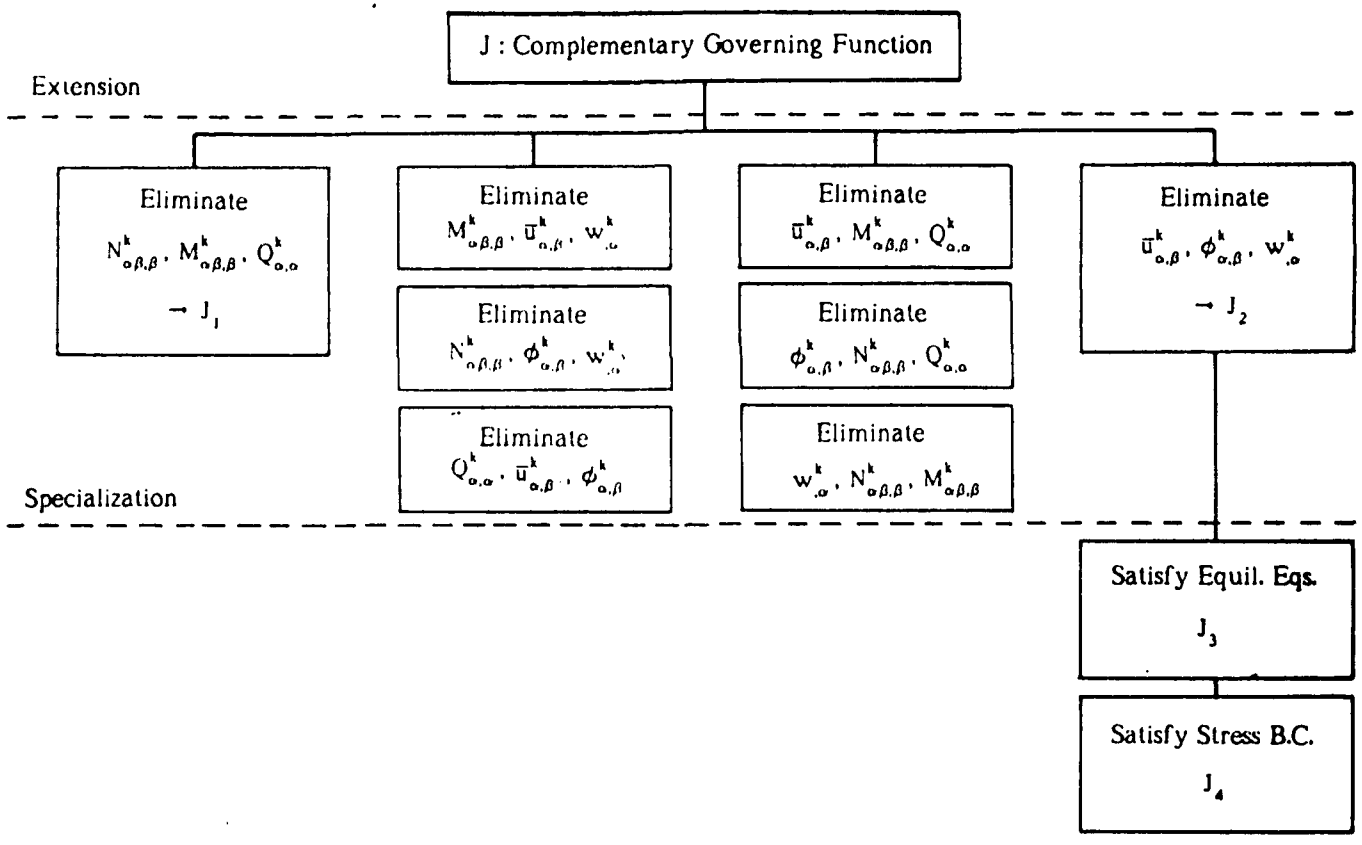


Figure 3: Family of Complementary Variational Principles

Section V

FINITE ELEMENT FORMULATION OF A SPECIAL DISCRETE LAMINATE THEORY OF PLATES

5.1 *INTRODUCTION*

In the finite element procedure, the region under consideration is subdivided into a finite number of disjoint subregions (elements), and the field variables of the problem are approximated by functions which are continuous along the boundary of elements, but have limited smoothness. Consider the open connected region R in an Euclidean space discretized by a finite number of elements R_e (R_e, e=1, . . ., m) such that

$$\overline{R} = \lim_{m\to\infty} \bigcup_{e=1}^{m} \overline{R}_e \tag{207}$$

in which the elements satisfy the property

$$R_e \bigcap R_f = \phi \qquad if \ e \neq f \tag{208}$$

and are connected at a finite number of nodal points. Here, $\overline{R}$ and $\overline{R}_e$ denote the closures of R and R_e, respectively.

Since the field variables of the problem are represented by functions which may not be sufficiently smooth over the region R, the variational principles derived in the previous section may not be valid over the region R. However, if approximate functions have adequate smoothness over each element, and internal discontinuities across element boundaries are explicitly included, they are valid. For such case, we define the governing function over the region R as

$$\Omega = \sum_{e=1}^{m} \Omega_e \tag{209}$$

where Ω_e (e=1,2,....,m) is the set of functions governing the problem over indicated element R_e. Sandhu [1976] showed that Gateaux differential of Ω in (209) vanishes if and only if the field equations, the boundary conditions and the continuity conditions across the interelement boundaries are satisfied. If there are discontinities across the interelement boundaries, the actual jump quantities need to be explicitly included in Ω_e [Sandhu 1976].

In the following, we present a finite element formulation based on the variational principle using the governing function Ω_9 which is defined in terms of displacement field variables $\bar{u}_\alpha^{(1)}$, $\phi_\alpha^{(k)}$ and w, and is a specialization of the extension of the general variational principle to cases where $N_{\alpha\beta}$, $M_{\alpha\beta}$, Q_α may not be continuously differentiable. The specialization is to the case where $w^{(k)} = w$ for all k, the strain-displacement equations are identically satisfied, the in-plane displacements u_α are continuous and the specified displacement boundary conditions are identically satisfied.

5.2 *SPATIAL DISCRETIZATION OF GOVERNING FUNCTION*

The governing function Ω_9, satisfying the kinematic relations and displacement continuities in the laminar interfaces, can be rewritten as the sum of functions Ω_e (e=1,2....,m) defined on each element of the finite element representation as

$$\Omega_e = -\sum_{k=1}^{n} \{ <\bar{u}_\alpha^{(1)}, P^{(k)}\bar{u}_\alpha^{(1)}>_{R_e^{(k)}} + 2<\bar{u}_\alpha^{(1)}, P^{(k)}\sum_{i=1}^{k-1} t_i \phi^i>_{R_e^{(k)}} + <\sum_{i=1}^{k-1} t_i \phi_\alpha^i, P^{(k)}\sum_{j=1}^{k-1} t_j \phi_\alpha^{(j)}>_{R_e^{(k)}}$$

$$+ <\phi_\alpha^{(k)}, I^{(k)}\phi_\alpha^{(k)}>_{R_e^{(k)}} + <w, P^{(k)}w>_{R_e^{(k)}} + 2<\bar{u}_\alpha^{(1)}, R^{(k)}\phi_\alpha^{(k)}>_{R_e^{(k)}}$$

$$+2<\sum_{i=1}^{k-1} t_i \phi_\alpha^i,\ R^{(k)}\phi_\alpha^{(k)}>_{R_e^{(k)}} + <\bar{e}_{\alpha\beta}^{(1)},\ t^* A_{\alpha\beta\gamma\delta}^{(k)} \bar{e}_{\gamma\delta}^{(1)}>_{R_e^{(k)}}$$

$$+2<\bar{e}_{\alpha\beta}^{(1)},\ t^* A_{\alpha\beta\gamma\delta}^{(k)} \sum_{i=1}^{k-1} t_i \kappa_{\gamma\delta}^i>_{R_e^{(k)}} + <\sum_{i=1}^{k-1} t_i \kappa_{\alpha\beta}^{(k)},\ t^* A_{\alpha\beta\gamma\delta}^{(k)} \sum_{j=1}^{k-1} t_j \kappa_{\gamma\delta}^{(j)}>_{R_e^{(k)}}$$

$$+<\kappa_{\alpha\beta}^{(k)},\ t^* D_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)}>_{R_e^{(k)}} + 2<\bar{e}_{\alpha\beta}^{(1)},\ t^* B_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)}>_{R_e^{(k)}}$$

$$+2<\sum_{i=1}^{k-1} t_i \kappa_{\alpha\beta}^i,\ t^* B_{\alpha\beta\gamma\delta}^{(k)} \kappa_{\gamma\delta}^{(k)}>_{R_e^{(k)}}\}$$

$$-\sum_{k=1}^{n}\sum_{j=1}^{n} <2e_{\alpha 3}^{(k)},\ 2t^* \lambda_{\alpha\beta}^{(k,j)} e_{\beta 3}^j>_{R_e^{(k)}}$$

$$+2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(1)},\ t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R_e^{(k)}} + <\sum_{i=1}^{k-1} t_i \phi_\alpha^i,\ t^* F_\alpha^{(k)} + X_\alpha^{(k)}>_{R_e^{(k)}}$$

$$+<\phi_\alpha^{(k)},\ t^* G_\alpha^{(k)} + Y_\alpha^{(k)}>_{R_e^{(k)}} + <w,\ t^* F_3^{(k)} + Z^{(k)}>_{R_e^{(k)}}\}$$

$$-2<\bar{u}_\alpha^{(1)},\ t^* T_\alpha^{(0)}>_{R_e^{(1)}} - 2<w,\ t^* T_3^{(0)}>_{R_e^{(1)}} + 2<\bar{u}_\alpha^{(1)},\ t^* T_\alpha^{(n)}>_{R_e^{(n)}}$$

$$+2<\sum_{i=1}^{n} t_i \phi_\alpha^i,\ t^* T_\alpha^{(n)}>_{R_e^{(n)}} + 2<w,\ t^* T_3^{(n)}>_{R_e^{(n)}}$$

$$-2\sum_{k=1}^{n}\{ <\bar{u}_\alpha^{(1)},\ t^* \hat{N}_\alpha^{(k)}>_{S_{1e}^{(k)}} + <\sum_{i=1}^{k-1} t_i \phi_\alpha^i,\ t^* \hat{N}_\alpha^{(k)}>_{S_{1e}^{(k)}} + <\phi_\alpha^{(k)},\ t^* \hat{M}_\alpha^{(k)}>_{S_{3e}^{(k)}}$$

$$+<w,\ t^* \hat{Q}_\alpha^{(k)} \eta_\alpha>_{S_{5e}^{(k)}}\} \tag{210}$$

where $S_{1e}^{(k)}$, $S_{2e}^{(k)}$ and $S_{5e}^{(k)}$ are the intersections of the boundary surface of an element with $S_1^{(k)}$, $S_2^{(k)}$ and $S_5^{(k)}$, respectively. For spatial discretization of this function, it is assumed that the field variables are interpolated over an element domain as

$$\bar{\mathbf{u}}^{(1)}(\mathbf{x}, t) = \mathbf{H}_u^T(\mathbf{x})\mathbf{U}(t) \tag{211}$$

$$\phi^{(k)}(\mathbf{x},t) = \mathbf{H}_{\phi}^{T}(\mathbf{x})\Phi^{(k)}(t) \tag{212}$$

$$\mathbf{w}(\mathbf{x},t) = \mathbf{H}_{w}^{T}(\mathbf{x})\mathbf{W}(t) \tag{213}$$

where

$$\bar{\mathbf{u}}^{(k)} = [\bar{u}_1^{(k)}, \bar{u}_2^{(k)}]^T$$

$$\phi^{(k)} = [\phi_1^{(k)}, \phi_2^{(k)}]^T$$

and $\mathbf{U}(t)$, $\Phi^{(k)}(t)$, $\mathbf{W}(t)$ are the vector functions of time defined at the nodal points and $\mathbf{H}_u$, $\mathbf{H}_\phi$, $\mathbf{H}_w$ are respectively, the matrices of spatial interpolation functions for the field variables indicated by subscripts. Also, the generalized strains may be expressed as

$$\mathbf{e}^{(k)} = \mathbf{T}_u^T \mathbf{U}(t) \tag{214}$$

$$\kappa^{(k)} = \mathbf{T}_\phi^T \Phi^{(k)}(t) \tag{215}$$

$$\gamma^{(k)} = \mathbf{T}_s^T \mathbf{W}(t) + \mathbf{H}_\phi^T \Phi^{(k)}(t) \tag{216}$$

where

$$\mathbf{e}^{(k)} = [\bar{e}_{11}^{(k)}, \bar{e}_{22}^{(k)}, 2\bar{e}_{12}^{(k)}]^T$$

$$\kappa^{(k)} = [\kappa_{11}^{(k)}, \kappa_{22}^{(k)}, 2\kappa_{12}^{(k)}]^T$$

$$\gamma^{(k)} = 2[e_{23}^{(k)}, e_{13}^{(k)}]^T$$

and $\mathbf{T}_u$, $\mathbf{T}_\phi$ and $\mathbf{T}_s$ are, respectively, the transformation matrices derived from the displacement interpolation functions $\mathbf{H}_u$,$\mathbf{H}_\phi$ and $\mathbf{H}_w$ by suitable differentiation and reorganization of terms.

Substituting (211)-(216) into (210), the spatially discretized governing function is obtained.

$$\Omega_{\cdot} = -\sum_{k=1}^{n}\{\mathbf{U}^T\mathbf{M}_{uu}^{(k)}*\mathbf{U} + 2\mathbf{U}^T\mathbf{M}_{u\Phi I}^{(k)}*\sum_{i=1}^{k-1}t_i\Phi^i + \sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{M}_{\Phi\Phi I}^{(k)}*\sum_{j=1}^{k-1}t_j\Phi^{(j)}$$

$$+(\Phi^{(k)})^T\mathbf{M}_{\Phi\Phi I}^{(k)}*\Phi^{(k)} + \mathbf{W}^T\mathbf{M}_{ww}^{(k)}*\mathbf{W} + 2\mathbf{U}^T\mathbf{M}_{u\Phi R}^{(k)}*\Phi^{(k)}$$

$$+2\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{M}_{\Phi\Phi R}^{(k)}*\Phi^{(k)} + t^*\mathbf{U}^T\mathbf{K}_{uu,A}^{(k)}*\mathbf{U} + 2t^*\mathbf{U}^T\mathbf{K}_{u\Phi,A}^{(k)}*\sum_{i=1}^{k-1}t_i\Phi^i$$

$$+t^*\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{K}_{\Phi\Phi,A}^{(k)}*\sum_{j=1}^{k-1}t_j\Phi^{(j)} + t^*(\Phi^{(k)})^T\mathbf{K}_{\Phi\Phi I}^{(k)}*\Phi^{(k)} + 2t^*\mathbf{U}^T\mathbf{K}_{u\Phi B}^{(k)}*\Phi^{(k)}$$

$$+2t^*\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{K}_{\Phi\Phi B}^{(k)}*\Phi^{(k)}\}$$

$$-t^*\sum_{k=1}^{n}\{\mathbf{W}^T\sum_{j=1}^{n}\mathbf{K}_{sww}^{(k,j)}*\mathbf{W} + 2\mathbf{W}^T\sum_{j=1}^{n}\mathbf{K}_{sw\Phi}^{(k,j)}*\Phi^{(j)} + (\Phi^{(k)})^T\sum_{j=1}^{n}\mathbf{K}_{s\Phi\Phi}^{(k,j)}*\Phi^{(j)}\}$$

$$+2\sum_{k=1}^{n}\{t^*\mathbf{U}^T*\mathbf{f}^{(k)} + \mathbf{U}^T*\mathbf{b}_x^{(k)} + t^*\sum_{i=1}^{k-1}t_i(\Phi^i)^T*\mathbf{h}^{(k)} + \sum_{i=1}^{k-1}t_i(\Phi^i)^T*\mathbf{r}_x^{(k)}$$

$$+t^*(\Phi^{(k)})^T*\mathbf{g}^{(k)} + (\Phi^{(k)})^T*\mathbf{b}_y^{(k)} + t^*\mathbf{W}^T*\mathbf{f}_3^{(k)} + \mathbf{W}^T*\mathbf{b}_z^{(k)}\}$$

$$+2t^*\{-\mathbf{U}^T*\tau^o + \mathbf{W}^T*(\mathbf{p}^{(n)}-\mathbf{p}^o) + \mathbf{U}^T*\tau^{(n)} + \sum_{i=1}^{n}t_i(\Phi^i)^T*\mathbf{t}^{(n)}\}$$

$$-2t^*\sum_{k=1}^{n}\{\mathbf{U}^T*\mathbf{R}_{n1}^{(k)} + \sum_{i=1}^{k-1}t_i(\Phi^i)^T*\mathbf{R}_{n2}^{(k)} + (\Phi^{(k)})^T*\mathbf{R}_m^{(k)} + \mathbf{W}^T*\mathbf{R}_q^{(k)}\} \tag{217}$$

where

$$\mathbf{M}_{uu}^{(k)} = \int_{R_e^{(k)}}\mathbf{H}_u P^{(k)}\mathbf{H}_u^T\,dR \tag{218}$$

$$\mathbf{M}_{u\Phi I}^{(k)} = \int_{R_e^{(k)}}\mathbf{H}_u P^{(k)}\mathbf{H}_\Phi^T\,dR \tag{219}$$

$$\mathbf{M}_{u\Phi\Phi}^{(k)} = \int_{R_e^{(k)}}\mathbf{H}_\Phi P^{(k)}\mathbf{H}_\Phi^T\,dR \tag{220}$$

$$\mathbf{M}^{(k)}_{u\phi R} = \int_{R_e^{(k)}} \mathbf{H}_u R^{(k)} \mathbf{H}_\phi^T \, dR \tag{221}$$

$$\mathbf{M}^{(k)}_{\phi\phi I} = \int_{R_e^{(k)}} \mathbf{H}_\phi I^{(k)} \mathbf{H}_\phi^T \, dR \tag{222}$$

$$\mathbf{M}^{(k)}_{\phi\phi R} = \int_{R_e^{(k)}} \mathbf{H}_\phi R^{(k)} \mathbf{H}_\phi^T \, dR \tag{223}$$

$$\mathbf{M}^{(k)}_{ww} = \int_{R_e^{(k)}} \mathbf{H}_w P^{(k)} \mathbf{H}_w^T \, dR \tag{224}$$

$$\mathbf{K}^{(k)}_{uuA} = \int_{R_e^{(k)}} \mathbf{T}_u \mathbf{A}^{(k)} \mathbf{T}_u^T \, dR \tag{225}$$

$$\mathbf{K}^{(k)}_{u\phi A} = \int_{R_e^{(k)}} \mathbf{T}_u \mathbf{A}^{(k)} \mathbf{T}_\phi^T \, dR \tag{226}$$

$$\mathbf{K}^{(k)}_{u\phi B} = \int_{R_e^{(k)}} \mathbf{T}_u \mathbf{B}^{(k)} \mathbf{T}_\phi^T \, dR \tag{227}$$

$$\mathbf{K}^{(k)}_{\phi\phi A} = \int_{R_e^{(k)}} \mathbf{T}_\phi \mathbf{A}^{(k)} \mathbf{T}_\phi^T \, dR \tag{228}$$

$$\mathbf{K}^{(k)}_{\phi\phi B} = \int_{R_e^{(k)}} \mathbf{T}_\phi \mathbf{B}^{(k)} \mathbf{T}_\phi^T \, dR \tag{229}$$

$$\mathbf{K}^{(k)}_{\phi\phi D} = \int_{R_e^{(k)}} \mathbf{T}_\phi \mathbf{D}^{(k)} \mathbf{T}_\phi^T \, dR \tag{230}$$

$$\mathbf{K}^{(kj)}_{sww} = \int_{R_e^{(k)}} \mathbf{T}_s \Lambda^{(kj)} \mathbf{T}_s^T \, dR \tag{231}$$

$$\mathbf{K}^{(kj)}_{sw\phi} = \int_{R_e^{(k)}} \mathbf{T}_s \Lambda^{(kj)} \mathbf{H}_\phi^T \, dR \tag{232}$$

$$\mathbf{K}^{(k,l)}_{\phi\Phi\Phi} = \int_{R_e^{(k)}} \mathbf{H}_\phi \Lambda^{(k,l)} \mathbf{H}_\phi^T \, dR \tag{233}$$

$$\mathbf{f}^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_u \mathbf{F}^{(k)} \, dR \tag{234}$$

$$\mathbf{f}_3^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_w F_3^{(k)} \, dR \tag{235}$$

$$\mathbf{g}^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_\phi \mathbf{G} \, dR \tag{236}$$

$$\mathbf{h}^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_\phi \mathbf{F}^{(k)} \, dR \tag{237}$$

$$\mathbf{r}_x^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_\phi \mathbf{X}^{(k)} \, dR \tag{238}$$

$$\mathbf{b}_x^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_u \mathbf{X}^{(k)} \, dR \tag{239}$$

$$\mathbf{b}_y^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_\phi \mathbf{Y}^{(k)} \, dR \tag{240}$$

$$\mathbf{b}_z^{(k)} = \int_{R_e^{(k)}} \mathbf{H}_w Z^{(k)} \, dR \tag{241}$$

$$(\tau^o, \tau^{(n)}) = \int_{R_e^{(k)}} \mathbf{H}_u (\mathbf{T}^o, \mathbf{T}^{(n)}) \, dR \tag{242}$$

$$\mathbf{t}^{(n)} = \int_{R_e^{(k)}} \mathbf{H}_\phi \mathbf{T}^{(n)} \, dR \tag{243}$$

$$(\mathbf{p}^o, \mathbf{p}^{(n)}) = \int_{R_e^{(k)}} \mathbf{H}_w (T_3^o, T_3^{(n)}) \, dR \tag{244}$$

$$\mathbf{R}_{n1}^{(k)} = \int_{S_{1e}^{(k)}} \mathbf{H}_u \mathbf{n}^{(k)} \, ds \tag{245}$$

$$\mathbf{R}_{n2}^{(k)} = \int_{S_{1e}^{(k)}} \mathbf{H}_{\phi} \mathbf{n}^{(k)} \, ds \tag{246}$$

$$\mathbf{R}_{m}^{(k)} = \int_{S_{2e}^{(k)}} \mathbf{H}_{\phi} \mathbf{m}^{(k)} \, ds \tag{247}$$

$$\mathbf{R}_{q}^{(k)} = \int_{S_{2e}^{(k)}} \mathbf{H}_{w} \mathbf{q}^{(k)} \, ds \tag{248}$$

and

$$\mathbf{X}^{(k)} = [X_1^{(k)}, X_2^{(k)}]^T \tag{249}$$

$$\mathbf{Y}^{(k)} = [Y_1^{(k)}, Y_2^{(k)}]^T \tag{250}$$

$$\mathbf{F}^{(k)} = [F_1^{(k)}, F_2^{(k)}]^T \tag{251}$$

$$\mathbf{G}^{(k)} = [G_1^{(k)}, G_2^{(k)}]^T \tag{252}$$

$$\mathbf{T}^{o} = [T_1^{o}, T_2^{o}]^T \tag{253}$$

$$\mathbf{T}^{(n)} = [T_1^{(n)}, T_2^{(n)}]^T \tag{254}$$

$$\mathbf{n}^{(k)} = [\hat{N}_1^{(k)}, \hat{N}_2^{(k)}] \tag{255}$$

$$\mathbf{m}^{(k)} = [\hat{M}_1^{(k)}, \hat{M}_2^{(k)}] \tag{256}$$

$$\mathbf{q}^{(k)} = [\hat{Q}_1^{(k)}, \hat{Q}_2^{(k)}] \tag{257}$$

In the above expressions, $\mathbf{A}^{(k)}$, $\mathbf{B}^{(k)}$ and $\mathbf{D}^{(k)}$ denote, respectively, the matrices of the stiffness $A_{\alpha\beta\gamma\delta}^{(k)}$, $B_{\alpha\beta\gamma\delta}^{(k)}$ and $D_{\alpha\beta\gamma\delta}^{(k)}$. For convenience, we rewrite the quantities involving summations in (217) as

$$\sum_{k=1}^{n} \{ 2\mathbf{U}^T \mathbf{M}_{u\phi p}^{(k)} * \sum_{i=1}^{k-1} t_i \Phi^i \} = 2\mathbf{U}^T \sum_{k=1}^{n-1} (\sum_{i=k+1}^{n} \mathbf{M}_{u\phi p}^{i}) * t_k \Phi^{(k)} \tag{258}$$

$$\sum_{k=1}^{n}\{\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{M}_{\phi\phi p}^{(k)}*\sum_{j=1}^{k-1}t_j\Phi^{(j)}\}=\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T\sum_{i=k+1}^{n}\left[\mathbf{M}_{\phi\phi p}^{i}*(\sum_{j=1}^{i-1}t_j\Phi^{(j)})+(\sum_{j=i+1}^{n}\mathbf{M}_{\phi\phi p}^{(j)})*t_i\Phi^i\right] \tag{259}$$

$$\sum_{k=1}^{n}\{2\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{M}_{\phi\phi R}^{(k)}*\Phi^{(k)}\}=2\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T*\{\sum_{i=k+1}^{n}\mathbf{M}_{\phi\phi R}^{i}\Phi^i\} \tag{260}$$

$$\sum_{k=1}^{n}\{2\mathbf{U}^T\mathbf{K}_{u\phi A}^{(k)}*\sum_{i=1}^{k-1}t_i\Phi^i\}=2\mathbf{U}^T\sum_{k=1}^{n-1}(\sum_{i=k+1}^{n}\mathbf{K}_{u\phi A}^{i})*t_k\Phi^{(k)} \tag{261}$$

$$\sum_{k=1}^{n}\{\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{K}_{\phi\phi A}^{(k)}*\sum_{j=1}^{k-1}t_j\Phi^{(j)}\}=\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T\sum_{i=k+1}^{n}\left[\mathbf{K}_{\phi\phi A}^{i}*(\sum_{j=1}^{i-1}t_j\Phi^{(j)})+(\sum_{j=i+1}^{n}\mathbf{K}_{\phi\phi A}^{(j)})*t_i\Phi^i\right] \tag{262}$$

$$\sum_{k=1}^{n}\{2\sum_{i=1}^{k-1}t_i(\Phi^i)^T\mathbf{K}_{\phi\phi B}^{(k)}*\Phi^{(k)}\}=2\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T*\{\sum_{i=k+1}^{n}\mathbf{K}_{\phi\phi B}^{i}\Phi^i\} \tag{263}$$

Using above equalities, the discretized function (217) becomes

$$\begin{aligned}\Omega_e=-\{&\sum_{k=1}^{n}\mathbf{U}^T\mathbf{M}_{uu}^{(k)}*\mathbf{U}+2\mathbf{U}^T\sum_{k=1}^{n-1}(\sum_{i=k+1}^{n}\mathbf{M}_{u\phi p}^{i})*t_k\Phi^{(k)}+\sum_{k=1}^{n}(\Phi^{(k)})^T\mathbf{M}_{\phi\phi I}^{(k)}*\Phi^{(k)}\\&+\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T\sum_{i=k+1}^{n}\left[\mathbf{M}_{\phi\phi p}^{i}*(\sum_{j=1}^{i-1}t_j\Phi^{(j)})+(\sum_{j=i+1}^{n}\mathbf{M}_{\phi\phi p}^{(j)})*t_i\Phi^i\right]+\mathbf{W}^T\sum_{k=1}^{n}\mathbf{M}_{ww}^{(k)}*\mathbf{W}\\&+2\mathbf{U}^T\sum_{k=1}^{n}\{\mathbf{M}_{u\phi R}^{(k)}*\Phi^{(k)}\}+2\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T*\{\sum_{i=k+1}^{n}\mathbf{M}_{\phi\phi R}^{i}\Phi^i\}\\&+t^*\mathbf{U}^T\sum_{k=1}^{n}\mathbf{K}_{uuA}^{(k)}*\mathbf{U}+2t^*\mathbf{U}^T\sum_{k=1}^{n-1}(\sum_{i=k+1}^{n}\mathbf{K}_{u\phi A}^{i})*t_k\Phi^{(k)}\\&+\sum_{k=1}^{n-1}t_k(\Phi^{(k)})^T\sum_{i=k+1}^{n}\left[\mathbf{K}_{\phi\phi A}^{i}*(\sum_{j=1}^{i-1}t_j\Phi^{(j)})+(\sum_{j=i+1}^{n}\mathbf{K}_{\phi\phi A}^{(j)})*t_i\Phi^i\right]\end{aligned}$$

$$+t^{*}\sum_{k=1}^{n}(\Phi^{(k)})^{T}\mathbf{K}_{\phi\phi D}^{(k)}*\Phi^{(k)}+2t^{*}\sum_{k=1}^{n}\mathbf{U}^{T}\mathbf{K}_{u\phi B}^{(k)}*\Phi^{(k)}+\sum_{k=1}^{n-1}t_{k}(\Phi^{(k)})^{T}*\{\sum_{i=k+1}^{n}\mathbf{K}_{\phi\phi B}^{i}\Phi^{i}\}\}$$

$$-t^{*}\sum_{k=1}^{n}\{\mathbf{W}^{T}\sum_{j=1}^{n}\mathbf{K}_{sww}^{(kj)}*\mathbf{W}+2\mathbf{W}^{T}\sum_{j=1}^{n}\mathbf{K}_{sw\phi}^{(kj)}*\Phi^{(j)}+(\Phi^{(k)})^{T}\sum_{j=1}^{n}\mathbf{K}_{s\phi\phi}^{(kj)}*\Phi^{(j)}\}$$

$$+2\sum_{k=1}^{n}\{t^{*}\mathbf{U}^{T}*\mathbf{f}^{(k)}+\mathbf{U}^{T}*\mathbf{b}_{x}^{(k)}+t^{*}\sum_{i=1}^{k-1}t_{i}(\Phi^{i})^{T}*\mathbf{h}^{(k)}+t^{*}\sum_{i=1}^{k-1}t_{i}(\Phi^{i})^{T}*\mathbf{r}_{x}^{(k)}$$

$$+t^{*}(\Phi^{(k)})^{T}*\mathbf{g}^{(k)}+(\Phi^{(k)})^{T}*\mathbf{b}_{y}^{(k)}+t^{*}\mathbf{W}^{T}*\mathbf{f}_{3}^{(k)}+\mathbf{W}^{T}*\mathbf{b}_{z}^{(k)}\}$$

$$+2t^{*}\{-\mathbf{U}^{T}*\tau^{o}+\mathbf{W}^{T}*(\mathbf{p}^{(n)}-\mathbf{p}^{o})+\mathbf{U}^{T}*\tau^{(n)}+\sum_{i=1}^{n}t_{i}(\Phi^{i})^{T}*\mathbf{t}^{(n)}\}$$

$$-2t^{*}\sum_{k=1}^{n}\{\mathbf{U}^{T}*\mathbf{R}_{n1}^{(k)}+\sum_{i=1}^{k-1}t_{i}(\Phi^{i})^{T}*\mathbf{R}_{n2}^{(k)}+(\Phi^{(k)})^{T}*\mathbf{R}_{m}^{(k)}+\mathbf{W}^{T}*\mathbf{R}_{q}^{(k)}\} \tag{264}$$

5.3 SEMIDISCRETE EQUATIONS OF MOTION

To obtain the semidiscrete equations of motion of laminate plate, it is convenient to rewrite the spatially discretized variational principle, (264) in matrix form as

$$\Omega_{e}=-\mathrm{X}_{e}^{T}\mathbf{S}_{e}*\mathrm{X}_{e}+2\,\mathrm{X}_{e}^{T}*\mathbf{R}_{e} \tag{265}$$

where

$$\mathbf{X}_e = \begin{pmatrix} \mathbf{U} \\ - \\ \Phi^{(1)} \\ \Phi^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{(n)} \\ - \\ \mathbf{W} \end{pmatrix} \qquad \mathbf{R}_e = \begin{pmatrix} P_u \\ - \\ P_{\phi^{(1)}} \\ P_{\phi^{(2)}} \\ \cdot \\ \cdot \\ \cdot \\ P_{\phi^{(n)}} \\ - \\ P_w \end{pmatrix} \qquad \mathbf{S}_e = \begin{pmatrix} S_{11} & | & S_{12} & | & 0 \\ - & | & - & | & - \\ S_{12}^T & | & S_{22} & | & S_{23} \\ - & | & - & | & - \\ 0 & | & S_{23}^T & | & S_{33} \end{pmatrix} \tag{266}$$

Here, elements of the submatrices of $\mathbf{S}_e$ and the load vector $\mathbf{R}_e$ are, explicitly,

$$S_{11} = \sum_{k=1}^{n} (\mathbf{M}_{uu}^{(k)} + t^* \mathbf{K}_{uuA}^{(k)})$$

$$(S_{12})_{1j} = t_j \sum_{k=j+1}^{n} (\mathbf{M}_{u\phi p}^{(k)} + t^* \mathbf{K}_{u\phi A}^{(k)}) + \mathbf{M}_{u\phi R}^{(j)} + t^* \mathbf{K}_{u\phi B}^{(j)} \quad , \; j = 1, 2, \ldots, n\text{-}1$$

$$(S_{12})_{1n} = \mathbf{M}_{u\phi R}^{(n)} + t^* \mathbf{K}_{u\phi B}^{(n)}$$

$$(S_{22})_{ii} = t_i^{(2)} \sum_{k=i+1}^{n} (\mathbf{M}_{\phi\phi p}^{(k)} + t^* \mathbf{K}_{\phi\phi A}^{(k)}) + \mathbf{M}_{\phi\phi I}^{i} + t^* (\mathbf{K}_{\phi\phi D}^{i} + \mathbf{K}_{s\phi\phi}^{ii}) \quad , \quad i = 1, 2, \ldots, n\text{-}1$$

$$(S_{22})_{nn} = \mathbf{M}_{\phi\phi I}^{(n)} + t^* (\mathbf{K}_{\phi\phi D}^{(n)} + \mathbf{K}_{s\phi\phi}^{nn})$$

$$(S_{22})_{ij} = t_i t_j \sum_{k=j+1}^{n} (\mathbf{M}_{\phi\phi p}^{(k)} + t^* \mathbf{K}_{\phi\phi A}^{(k)}) + t_i \mathbf{M}_{\phi\phi R}^{(j)} + t^* (t_i \mathbf{K}_{\phi\phi B}^{(j)} + \mathbf{K}_{s\phi\phi}^{ij})$$

$$i = 1, 2, \ldots, n\text{-}1 \text{ and } j = i+1, i+2, \ldots, n\text{-}1$$

$$(S_{22})_{in} = t_i \mathbf{M}_{\phi\phi R}^{(n)} + t^* (t_i \mathbf{K}_{\phi\phi B}^{(n)} + \mathbf{K}_{s\phi\phi}^{in}), \qquad i = 1, 2, \ldots, n\text{-}1$$

$$(S_{23})_{i1} = t^* \sum_{k=1}^{n} \mathbf{K}_{sw\phi}^{ki}, \qquad i = 1, 2, \ldots, n\text{-}1$$

$$S_{33} = \sum_{k=1}^{n}\left[\mathbf{M}^{(k)}_{www} + t^* \sum_{j=1}^{n} \mathbf{K}^{(kj)}_{sww}\right]$$

$$P_u = \sum_{k=1}^{n}\left(t^*(\mathbf{f}^{(k)} - \mathbf{R}^{(k)}_{n1}) + \mathbf{b}^{(k)}_x\right) + t^*(\tau^{(n)} - \tau^o)$$

$$P_{\phi^{(k)}} = \sum_{i=1+k}^{n}\left(t^*(t_k \mathbf{h}^i - \mathbf{R}^i_{n2}) + t_k \mathbf{r}^i_x\right) + t^*(\mathbf{g}^{(k)} + t_k \tau^{(n)} - \mathbf{R}^{(k)}_m) + \mathbf{b}^{(k)}_y \,, \qquad k = 1, 2, \ldots, n\text{-}1$$

$$\mathbf{P}_{\phi^{(n)}} = t^*(\mathbf{g}^{(n)} + t_n \tau^{(n)} - \mathbf{R}^{(n)}_m) + \mathbf{b}^{(n)}_y$$

$$\mathbf{P}_w = \sum_{k=1}^{n}\left(t^*(\mathbf{f}^{(k)}_3 - \mathbf{R}^{(k)}_q) + \mathbf{b}^{(k)}_z - \mathbf{p}^o + \mathbf{p}^{(n)}\right)$$

The spatially discretized governing function for the global system is given by

$$\Omega = \sum_{e=1}^{m} \Omega_e = -\mathrm{X}^T \mathbf{S}^* \mathrm{X} + 2\,\mathrm{X}^T * \mathbf{R} \tag{267}$$

where X is the vector of values of field variables at the system nodal points, **R** is the set of corresponding 'forcing' quantities and **S** is the system matrix corresponding to $\mathbf{S}_e$ for an element. Here, summation in (267) indicates matrix assembly following usual (direct stiffness) procedure. Vanishing of the differential of Ω in (267) with respect to X gives the set of equations:

$$\mathbf{S}\mathrm{X} = \mathbf{M}\ddot{\mathrm{X}} + t^*\mathbf{K}\mathrm{X} = \mathbf{R} \tag{268}$$

where

$$
\mathbf{M} = \begin{vmatrix}
\sum_{k=1}^{n} \mathbf{M}_{uu}^{(k)} & t_1 \sum_{k=2}^{n} \mathbf{M}_{u\phi p}^{(k)} + \mathbf{M}_{u\phi R}^{(1)} & t_2 \sum_{k=3}^{n} \mathbf{M}_{u\phi p}^{(k)} + \mathbf{M}_{u\phi R}^{(2)} & t_3 \sum_{k=4}^{n} \mathbf{M}_{u\phi p}^{(k)} + \mathbf{M}_{u\phi R}^{3} & \cdots \\
 & t_1^{(2)} \sum_{k=2}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + \mathbf{M}_{\phi\phi I}^{(1)} & t_1 t_2 \sum_{k=3}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_1 \mathbf{M}_{\phi\phi R}^{(2)} & t_1 t_3 \sum_{k=4}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_1 \mathbf{M}_{\phi\phi R}^{3} & \cdots \\
 & & t_2^{(2)} \sum_{k=3}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + \mathbf{M}_{\phi\phi I}^{(2)} & t_2 t_3 \sum_{k=4}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_2 \mathbf{M}_{\phi\phi R}^{3} & \cdots \\
 & & & t_3^{(2)} \sum_{k=4}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + \mathbf{M}_{\phi\phi I}^{3} & \cdots
\end{vmatrix}
$$

$$
\begin{vmatrix}
\cdots & t_{n-2} \sum_{k=n-1}^{n} \mathbf{M}_{u\phi p}^{(k)} + \mathbf{M}_{u\phi R}^{n-2} & t_{n-1} \sum_{k=n}^{n} \mathbf{M}_{u\phi p}^{(k)} + \mathbf{M}_{u\phi R}^{(n-1)} & \mathbf{M}_{u\phi R}^{(n)} & 0 \\
\cdots & t_1 t_{n-2} \sum_{k=n-1}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_1 \mathbf{M}_{\phi\phi R}^{n-2} & t_1 t_{n-1} \sum_{k=n}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_1 \mathbf{M}_{\phi\phi R}^{(n-1)} & t_1 \mathbf{M}_{\phi\phi R}^{(n)} & 0 \\
\cdots & t_2 t_{n-2} \sum_{k=n-1}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_2 \mathbf{M}_{\phi\phi R}^{n-2} & t_2 t_{n-1} \sum_{k=n}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_2 \mathbf{M}_{\phi\phi R}^{(n-1)} & t_2 \mathbf{M}_{\phi\phi R}^{(n)} & 0 \\
\cdots & t_3 t_{n-2} \sum_{k=n-1}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_3 \mathbf{M}_{\phi\phi R}^{n-2} & t_3 t_{n-1} \sum_{k=n}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_2 \mathbf{M}_{\phi\phi R}^{n-2} & t_3 \mathbf{M}_{\phi\phi R}^{(n)} & 0 \\
 & \vdots & \vdots & \vdots & \vdots \\
 & t_{n-2}^{(2)} \sum_{k=n-1}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + \mathbf{M}_{\phi\phi I}^{n-2} & t_{n-2} t_{n-1} \sum_{k=n}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + t_{n-2} \mathbf{M}_{\phi\phi R}^{(n-1)} & t_{n-2} \mathbf{M}_{\phi\phi R}^{(n)} & 0 \\
 & & t_{n-1}^{(2)} \sum_{k=n}^{n} \mathbf{M}_{\phi\phi p}^{(k)} + \mathbf{M}_{\phi\phi I}^{(n-1)} & t_{n-1} \mathbf{M}_{\phi\phi R}^{(n)} & 0 \\
 & & & \mathbf{M}_{\phi\phi I}^{(n)} & 0 \\
 & & & & \sum_{k=1}^{n} \mathbf{M}_{\cdots}^{(k)}
\end{vmatrix}
$$

$$
\mathbf{K} = \begin{vmatrix}
\sum_{k=1}^{n} \mathbf{K}_{uuA}^{(k)} & t_1 \sum_{k=2}^{n} \mathbf{K}_{u\phi A}^{(k)} + \mathbf{K}_{u\phi B}^{(1)} & t_2 \sum_{k=3}^{n} \mathbf{K}_{u\phi A}^{(k)} + \mathbf{K}_{u\phi B}^{(2)} & \cdots \\
 & t_1^{(2)} \sum_{k=2}^{n} \mathbf{K}_{\phi\phi A}^{(k)} + \mathbf{K}_{\phi\phi D}^{(1)} + \mathbf{K}_{s\phi\phi}^{11} & t_1 t_2 \sum_{k=3}^{n} \mathbf{K}_{\phi\phi A}^{(k)} + t_1 \mathbf{K}_{\phi\phi B}^{(2)} + \mathbf{K}_{s\phi\phi}^{12} & \cdots \\
 & & t_2^{(2)} \sum_{k=3}^{n} \mathbf{K}_{\phi\phi A}^{(k)} + \mathbf{K}_{\phi\phi D}^{(2)} + \mathbf{K}_{s\phi\phi}^{22} & \cdots \\
 & & & \cdots \\
 & & & \cdots \\
 & & & \cdots
\end{vmatrix}
$$

$$
\begin{vmatrix}
\cdots & t_{n-1} \sum_{k=n}^{n} \mathbf{K}_{u\phi A}^{(k)} + \mathbf{K}_{u\phi B}^{(n-1)} & \mathbf{K}_{u\phi B}^{(n)} & 0 \\
\cdots & t_1 t_{n-1} \sum_{k=n}^{n} \mathbf{K}_{\phi\phi A}^{(k)} + t_1 \mathbf{K}_{\phi\phi B}^{(n-1)} + \mathbf{K}_{s\phi\phi}^{1,n-1} & t_1 \mathbf{K}_{\phi\phi B}^{(n)} + \mathbf{K}_{s\phi\phi}^{1n} & \sum_{k=1}^{n} \mathbf{K}_{sw\phi}^{k1} \\
\cdots & t_2 t_{n-1} \sum_{k=n}^{n} \mathbf{K}_{\phi\phi A}^{(k)} + t_2 \mathbf{K}_{\phi\phi B}^{(n-1)} + \mathbf{K}_{s\phi\phi}^{2,n-1} & t_2 \mathbf{K}_{\phi\phi B}^{(n)} + \mathbf{K}_{s\phi\phi}^{2n} & \sum_{k=1}^{n} \mathbf{K}_{sw\phi}^{k2} \\
\cdots & \vdots & \vdots & \vdots \\
\cdots & \vdots & \vdots & \vdots \\
\cdots & \vdots & \vdots & \vdots \\
 & \vdots & \vdots & \vdots \\
 & t_{n-1}^{(2)} \sum_{k=n}^{n} \mathbf{K}_{\phi\phi A}^{(k)} + \mathbf{K}_{\phi\phi D}^{(n-1)} + \mathbf{K}_{s\phi\phi}^{n-1,n-1} & t_{n-1} \mathbf{K}_{\phi\phi B}^{(n)} + \mathbf{K}_{s\phi\phi}^{n-1,n} & \sum_{k=1}^{n} \mathbf{K}_{sw\phi}^{k,n-1} \\
 & & \mathbf{K}_{\phi\phi D}^{(n)} + \mathbf{K}_{s\phi\phi}^{nn} & \sum_{k=1}^{n} \mathbf{K}_{sw\phi}^{kn} \\
 & & & \sum_{k=1}^{n} \sum_{j=1}^{n} \mathbf{K}_{sww}^{(kj)}
\end{vmatrix}
$$

The semidiscretized equations of motion for the entire system can be obtained by differenting (268) twice with respect to time.

$$\mathbf{M}\ddot{X} + \mathbf{K}X = \mathbf{R} \tag{269}$$

where mass matrix **M** and stiffness matrix **K** are symmetric. Furthermore, **M** and **K** are positive definite and semi-positive definite, respectively.

5.4 FREE VIBRATION ANALYSIS

For free vibration analysis, the load vector **R** is set to zero and (269) becomes

$$\mathbf{M}\ddot{X} + \mathbf{K}X = \mathbf{0} \tag{270}$$

Assuming harmonic motion of the system, i.e., assuming the solution by $X = \psi e^{i\omega t}$, where ψ is the amplitude vector, ω is the natural frequency and $i=\sqrt{-1}$, we obtain generalized eigenvalue problem

$$(\mathbf{K}-\omega^{(2)}\mathbf{M})\psi = \mathbf{0} \tag{271}$$

Before applying the boundary constraints, this eigenvalue problem has three zero eigenvalues corresponding to the rigid body motions. If equations corresponding to the constrained nodes are removed before solving the eigenvalue problem, matrix **K** becomes positive-definite and consequently all eigenvalues are positive real and corresponding eigenvectors representing vibration modes are orthogonal with respect to the mass matrix **M**.

5.5 *SPECIALIZATION TO THE STATIC CASE*

A specialization to the static problem can be made by dropping the inertial terms from the set of equations (269). The resulting system of linear algebraic equations is

$$\mathbf{K}X = \mathbf{R} \tag{272}$$

Equation (272) can be modified to account for known boundary constraints and subsequently solved for the unknown vector X comprising kinematic quantities.

5.6 *FREE-EDGE DELAMINATION SPECIMENS*

For the case of a free-edge delamination specimen, the stresses σ_{2i} at the free-edge are known to be zero. As the field variables in the present formulation are the nodal values of displacements and rotations, these stress-free constraints cannot be directly applied to the system (272). Since in this formulation stresses are secondary variables to be computed from the obtained displacements, appropriate refinement of the mesh near the edge boundary would be needed to realize σ_{2i} as close to zero as possible. However, as discussed in Section 3.7.4, different constitutive equations (122) apply at the free edge for the case of transverse shear. This leads to two sets of constitutive relations, one applicable in the interior of the laminate and the other at the free edge. Hence, in evaluating the integrals in (231) through (233), (108) were used for elements lying entirely within the interior of the laminate. For elements having one edge coinciding with the free edge, values of $\Lambda^{(kj)}$ at each of its nodes were chosen from either (108) or (122) depending on whether the node was on the free edge or in the interior. Then the value of $\Lambda^{(kj)}$ at the gauss points could be determined by interpolating the nodal values using some interpolating functions. For the examples of application given in this report, the same interpolation was used as adopted for the displacements.

5.7 *CALCULATION OF STRESSES*

After obtaining the displacement solution, secondary computations to obtain the stress components can be carried out in the following manner.

5.7.1 In-Plane Stress Components

Using the known nodal displacements in conjunction with the interpolating functions, strains can be obtained at any desired location over an element. Subsequently using (25), and noting that $e_{33}^{(k)}=0$ in this formulation, the components of in-plane stress can be calculated. Alternatively, (28) may be used to obtain the stress resultants $N_{\alpha\beta}$, $M_{\alpha\beta}$ and (75) used to obtain $\sigma_{\alpha\beta}$

5.7.2 Transverse Shear Stresses

Having the solution for the nodal displacements and rotations, the resultants $Q_\alpha^{(k)}$ can be determined from the constitutive relations (108). Subsequently, use of (92) will yeild χ_b and hence $T_\alpha^{(k)}$. The transverse shear stresses $\sigma_{\alpha 3}$ can then be evaluated from (81).

For the case of the free-edge delamination specimen, interpolation over the edge elements as described in Section 5.6 was done to obtain $Q_\alpha^{(k)}$ and $T_\alpha^{(k)}$ at the desired locations over the element.

5.7.3 Transverse Normal Stress

To compute the normal stress σ_{33}, the equilibrium equation (77) is integrated w.r.t x_3^k to obtain

$$\sigma_{33}^{(k)} = T_3^{(k-1)} - \int_0^{x_3^{(k)}} \sigma_{3\alpha,\alpha}\, dx_3^{(k)} \tag{273}$$

using (81),

$$\sigma_{33}^{(k)} = T_3^{(k-1)} - \int_0^{x_3^k} [\zeta_1^{(k)} Q_{\alpha,\alpha}^{(k)} + \zeta_2^{(k)} T_{\alpha,\alpha}^{(k-1)} + \zeta_3^{(k)} T_{\alpha,\alpha}^{(k)}] dx_3^{(k)} \tag{274}$$

Substituting for $\zeta_1^{(k)}$, $\zeta_2^{(k)}$, and ζ_3^k from (81) and evaluating the integrals leads to

$$\begin{aligned}\sigma_{33}^{(k)} = T_3^{(k-1)} &- (\frac{x_3^k}{t_k})^2 \left[3 - 2(\frac{x_3^{(k)}}{t_k})\right] Q_{\alpha,\alpha} \\ &- x_3^{(k)} \left[(\frac{x_3^k}{t_k})^{(2)} - 2(\frac{x_3^k}{t_k}) + 1\right] T_{\alpha,\alpha}^{(k-1)} \\ &- x_3^{(k)} \left[(\frac{x_3^k}{t_k})^{(2)} - (\frac{x_3^{(k)}}{t_k})\right] T_{\alpha,\alpha}^{(k)}\end{aligned} \tag{275}$$

Section VI

APPLICATIONS

6.1 INTRODUCTION

The formulation presented in the preceding section was incorporated in a computer program. A nine-noded Heterosis element, shown in Figure B.1, was used for finite element analysis. For verification of the code, the program was used to solve the problem of a three-layer, simply supported, square sandwich plate with isotropic/orthotropic outer skin layers. The results were compared with an exact series solution. This example is the same as used by Mawenya [1974]. The formulation was then applied to the solution of a Free Edge Delamination (FED) problem. A four-layer coupon under uniform axial strain whose solution has been presented by Pagano[1978] was considered. Displacements and stresses along the midsection were computed and compared with Pagano's results.

Though the constitutive relations for transverse shear have been derived using the equations of static equilibrium, the problem of free vibration of a simply supported laminated plate was also solved by including inertial terms in the finite element formulation. Developing constitutive equations for the coupled theory, allowing for the inertia terms, Schoeppner [1990] has shown that the effect of inertia on constitutive relations decreases with decrease in layer thickness. Thus, for sufficiently small layer thickness (any lamina can be arbitrarily replaced by a suitable number of sublayers) the constitutive relations developed in Section IV would be applicable. The natural

frequencies of free vibration of a sandwich plate were computed and the fundamental frequency compared with an exact elasticity solution by Srinivas [1970].

6.2 *PLATE ANALYSIS*

A three-layer, simply supported, square plate uniformly loaded in the transverse direction was considered for analysis. Two separate cases in which the outer layers were respectively isotropic and orthotropic were considered. The plate dimensions and material properties were the same as in a similar example considered by Mawenya [1974], and were

Plate Dimensions

Length of each side $=10$ *in.*

Thickness of outer layers $t^{(1)} = t^{(3)} = 0.028$ *in.*

Thickness of core $t^{(2)} = 0.75$ *in.*

Material Properties

a). Isotropic Outer Layers

$$E^{(1)} = E^{(3)} = 10^7 \ lb/in.^2$$

$$\nu^{(1)} = \nu^{(3)} = 0.3$$

$$G^{(2)} = 3 \times 10^4 \ lb/in.^2$$

b). Orthotropic Outer Layers

$$E_{11}^{(1)} = E_{11}^{(3)} = 10^7 \ lb/in.^2$$

$$E_{22}^{(1)} = E_{22}^{(3)} = 4 \times 10^6 \ lb/in.^2$$

$$G_{12}^{(1)} = G_{12}^{(3)} = 1.875 \times 10^6 \ lb/in.^2$$

$$\nu_{12}^{(1)} = \nu_{12}^{(3)} = 0.3$$

$$G_{13}^{(2)} = 3 \times 10^4 \ lb/in.^2$$

$$G_{23}^{(2)} = 1.2 \times 10^4 \ lb/in.^2$$

Lateral Loading

$$q = 1.0 \ lb/in.^2$$

Due to the symmetry of the problem, only one quadrant of the plate needed to be considered for finite element analyses under static conditions. A typical discretized quarter along with associated boundary conditions is shown in Figure 4. Discretization was done using 1x1, 2x2, 4x4 and 8x8 meshes.

The results for maximum lateral deflection at the plate midpoint are shown in Table 5 and Table 6 along with the exact series solutions [Mawenya 1974] and a comparison with results obtained by Moazzami et al. [1991] using the conventional Discrete Layer theory. The CPU time used on a Cray Y-MP8/864 is also listed. It should be noted here that the Heterosis element used is nine-noded except for the lateral displacement degrees of freedom, for which it is eight-noded. Moazzami's results, on the other hand, were obtained using a fully nine-noded element.

These results were obtained using a generalized plane stress assumption for the purpose of matching them with the series solution. Solving (27) for $e_{33}^{(k)}$ and substituting into (25) and (27),

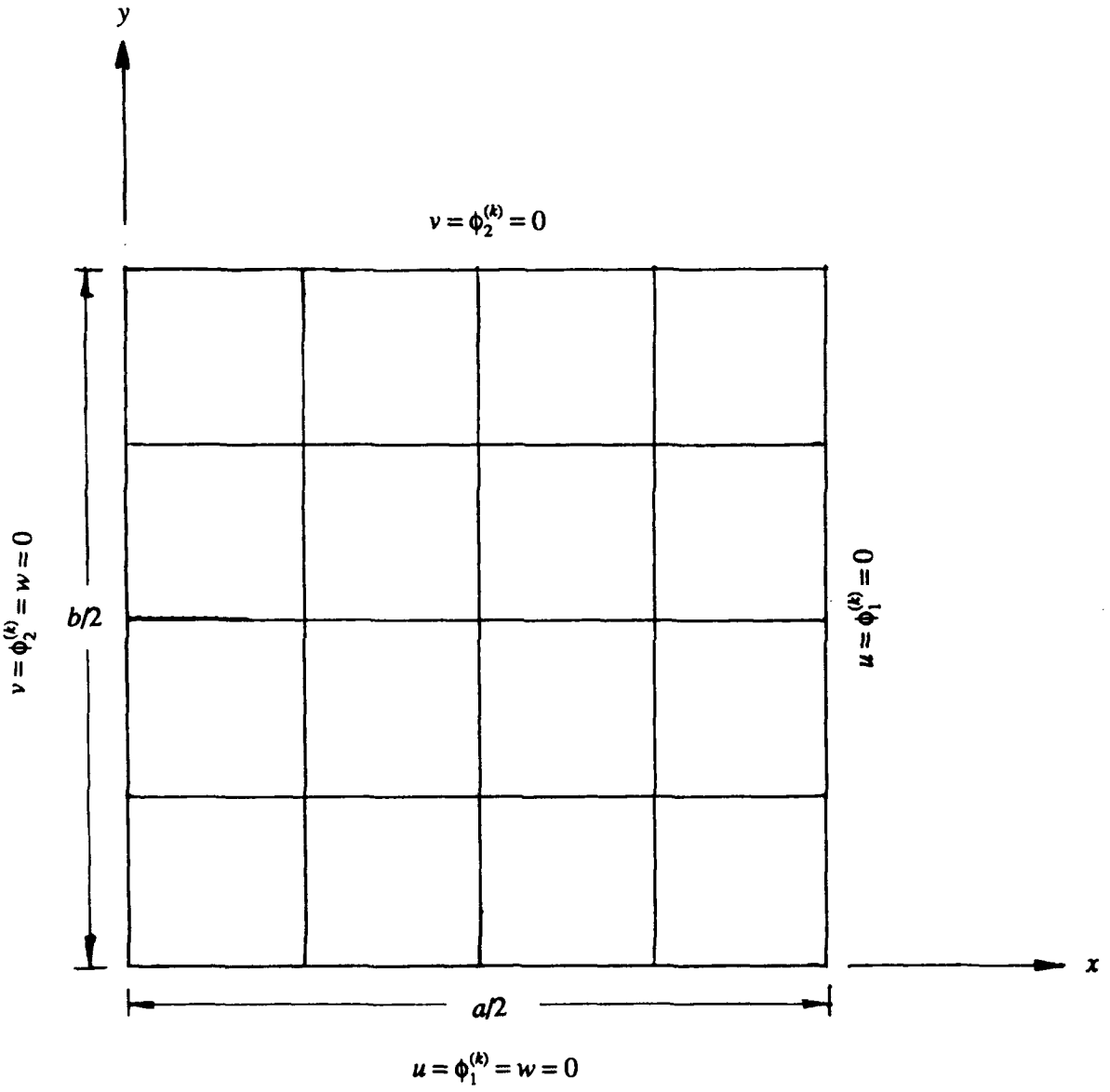


Figure 4: Discretized Quadrant of a Simply Supported Plate

Table 5

Maximum Lateral Deflections for the Isotropic Case

No. of Elements	CPU (Secs.)	Maxm. Defl. from Code	Max. Defl. Moazzami et al.
1x1	0.113	0.00069822	0.00073539
2x2	0.411	0.00073456	0.00074087
4x4	1.733	0.00073537	0.00074008
8x8	9.912	0.00073538	0.00074006
Series Soln. [Mawenya 1974]		0.00074000	

Table 6

Maximum Lateral Deflections for the Orthotropic Case

No. of Elements	CPU (Secs.)	Maxm. Defl. from Code	Max. Defl. Moazzami et al.
1x1	0.114	0.0011452	0.0012140
2x2	0.414	0.0012063	0.0012224
4x4	1.783	0.0012074	0.0012216
8x8	9.909	0.0012075	0.0012216
Series Soln. [Mawenya 1974]		0.0012300	

$$\sigma_{\alpha\beta}^{(k)} = \bar{E}_{\alpha\beta\gamma\delta}^{(k)} e_{\gamma\delta}^{(k)} + \bar{E}_{\alpha\beta 33}^{(k)} \sigma_{33}^{(k)} \tag{276}$$

$$\sigma_{\alpha 3}^{(k)} = 2\bar{E}_{\alpha 3\gamma 3}^{(k)} e_{\gamma 3}^{(k)} \tag{277}$$

where

$$\bar{E}_{\alpha\beta\gamma\delta}^{(k)} = E_{\alpha\beta\gamma\delta}^{(k)} - \frac{E_{\alpha\beta 33}^{(k)}}{E_{3333}^{(k)}} E_{33\gamma\delta}^{(k)}$$

$$\bar{E}_{\alpha\beta 33}^{(k)} = \frac{E_{\alpha\beta 33}^{(k)}}{E_{3333}^{(k)}} \tag{278}$$

$$\bar{E}_{\alpha 3\gamma 3}^{(k)} = E_{\alpha 3\gamma 3}^{(k)}$$

Assuming generalized plane stress state in a lamina, i.e.,

$$\int_0^{t_k} \sigma_{33}^{(k)} \, dx_3^{(k)} = 0 \tag{279}$$

the constitutive equations of bending and stretching are obtained as in (28), where now

$$(A_{\alpha\beta\gamma\delta}^{(k)}, B_{\alpha\beta\gamma\delta}^{(k)}, D_{\alpha\beta\gamma\delta}^{(k)}) = (t_k, \frac{t_k^2}{2}, \frac{t_k^3}{3}) \bar{E}_{\alpha\beta\gamma\delta}^{(k)} \tag{280}$$

We note here that the assumption of $u_3^{(k)}$ constant over the thickness of each layer implies $e_{33}^{(k)} = 0$. We also note that (25) and (27) would not contain $e_{33}^{(k)}$ and hence there would be no need for its elimination. The in-plane constitutive equations can be obtained as was done in (28) without any assumptions on $\sigma_{33}^{(k)}$. However, for comparison with the series solution based on Kirchoff's theory, (280) were used. The choice of the two constitutive relations is easily incorporated into the program with the inclusion of a few lines of code.

The assumption of generalized plane stress state defined by (279) was first used by Mindlin [1951] for homogeneous plate theory and later by Whitney [1970] for

laminated plate theory. They, however, assumed the integral of $\sigma_{33}^{(k)}$ over the entire thickness of the laminate to vanish as opposed to over each layer indicated by (279). Hong's [1988] assumption (279) is clearly invalid for a multi-layered laminate. However, for a single layer plate or the sandwich plate with thin outer layers under consideration, it is acceptable.

6.3 *APPLICATION TO FREE EDGE DELAMINATION (FED) SPECIMEN*

Pagano[1978] presented solutions for the free-edge problem of a coupon subjected to uniform axial strain ϵ. A four-layer laminate as shown in Figure 5 was considered. The material properties were:

$$E_{11} = 20.0 \times 10^6 \ , \quad E_{22} = E_{33} = 2.1 \times 10^6 \quad (psi)$$

$$G_{12} = G_{13} = G_{23} = 0.85 \times 10^6 \quad (psi) \tag{281}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.21$$

To match the ratios of the spatial dimensions with those used by Pagano, the laminate length : width : layer thickness (i.e., L : 2b : h) was selected as 11 : 3 : 0.1875. Two stacking sequences viz., $[+45/-45]_s$ and $[0/90]_s$ were considered.

6.3.1 Angle Ply Specimen

In the x direction the specimen was discretized using 11 equal sized elements. In the y direction a sequence of mesh refinement was done by starting with three equal elements and succesively subdividing the elements along the two traction-free edges further into three equal elements. Using this pattern, 3x11, 7x11, 11x11 and 15x11 element meshes were used. This scheme is illustrated in Figure 6 where refinement from the 3x11 to the 7x11 mesh is depicted. A nine-noded Heterosis element shown

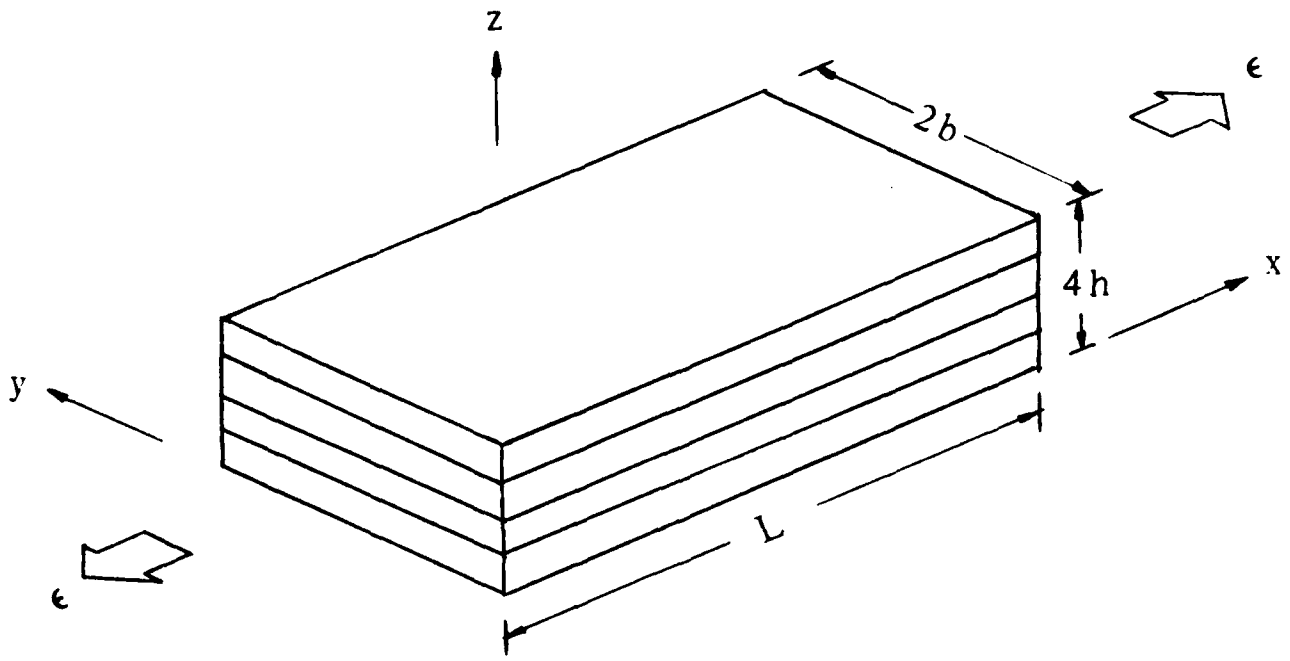


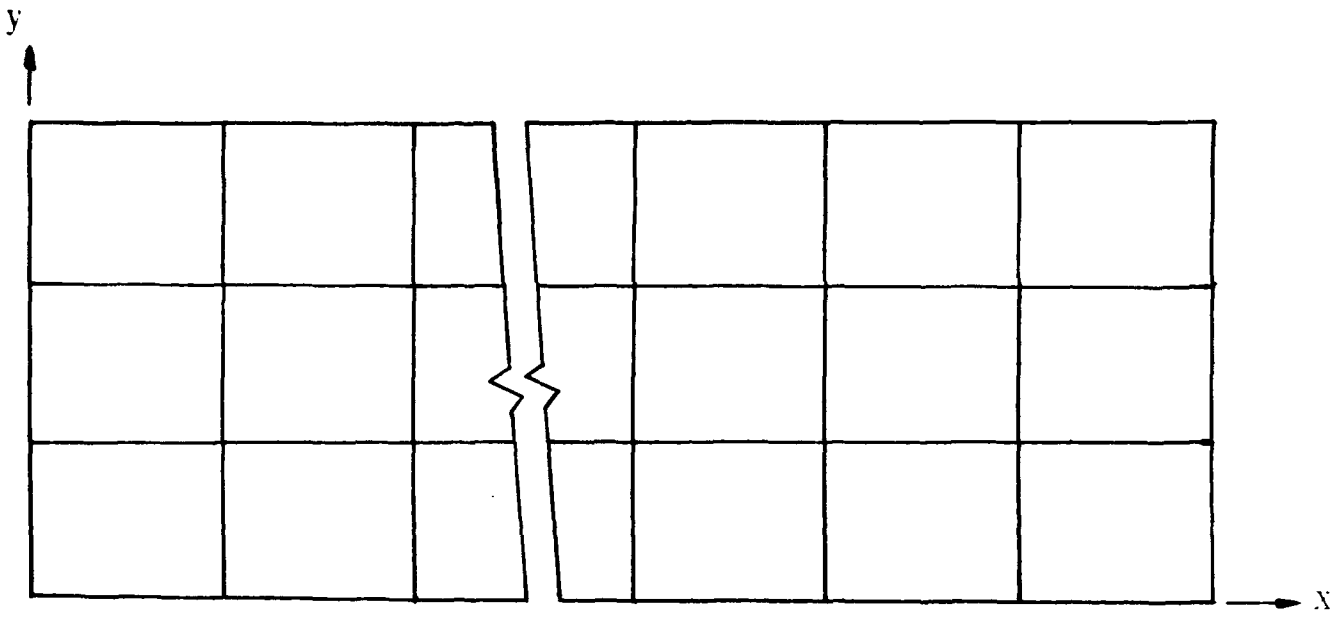
Figure 5: Four-Ply Coupon Subjected to Axial Strain

in Figure B.1 was used for the finite element analysis.

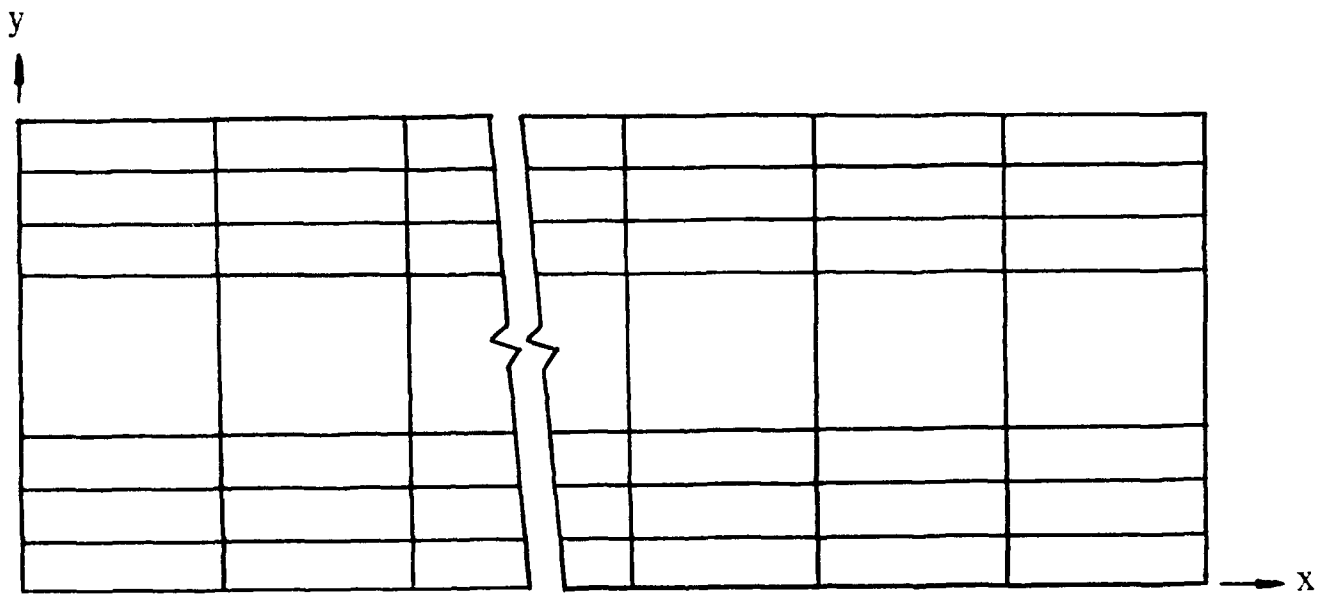
The results for the inplane stresses σ_{11} *and* σ_{21} for the various meshes are compared with Pagano's solution in Figure 7 and Figure 8. These stresses are plotted across the middle surface of the top layer for comparison with Pagano's solution. The stresses are shown over half the width of the coupon, where $(y\text{-}b)/b = 0$, 1, respectively represent the center and the free edge of the laminate.

The result for the 3x11 mesh shows a jump across the element boundary. It should be noted that as only half the laminate width is represented in the plots, the 3x11 mesh is represented by $1\frac{1}{2}$ elements. For the chosen element, in-plane strains, and consequently the stresses, are linear over the element. The edge element in the 3x11 mesh attempts to model both the sharply varying stresses near the free edge and the relatively flatter ones near the center by a single linear fit. Hence the stresses near the inner boundary of the edge element tend to be overestimated. This explains the kink in the 3x11 solution across the element boundary. With mesh refinement the kink was observed to reduce for the 7x11 mesh and was not noticeable in the solution for the 11x11 element mesh. Moreover, it was observed that further refinement to a 15x11 mesh did not show "substantial" improvement in the solution. A comparison of the total CPU time for the different meshes using a CRAY X-MP/28 is given in Table 7.

The results in Figure 7 and Figure 8 show that the finite element results overestimate Pagano's solution for σ_{11} and σ_{12} by about 4 percent at the center and 12 percent at the free edge. The same amount of error is observed at the free-edge in the longitudinal displacements plotted in Figure 9 for the "optimum" 11x11 mesh. The transverse shear stress σ_{13} computed using (81) is shown in Figure 10. Comparison with Pagano's exact solution shows that the numerical results grossly



3x11 MESH



7X11 MESH

Figure 6: Sequential Refinement Scheme for the Finite Element Mesh

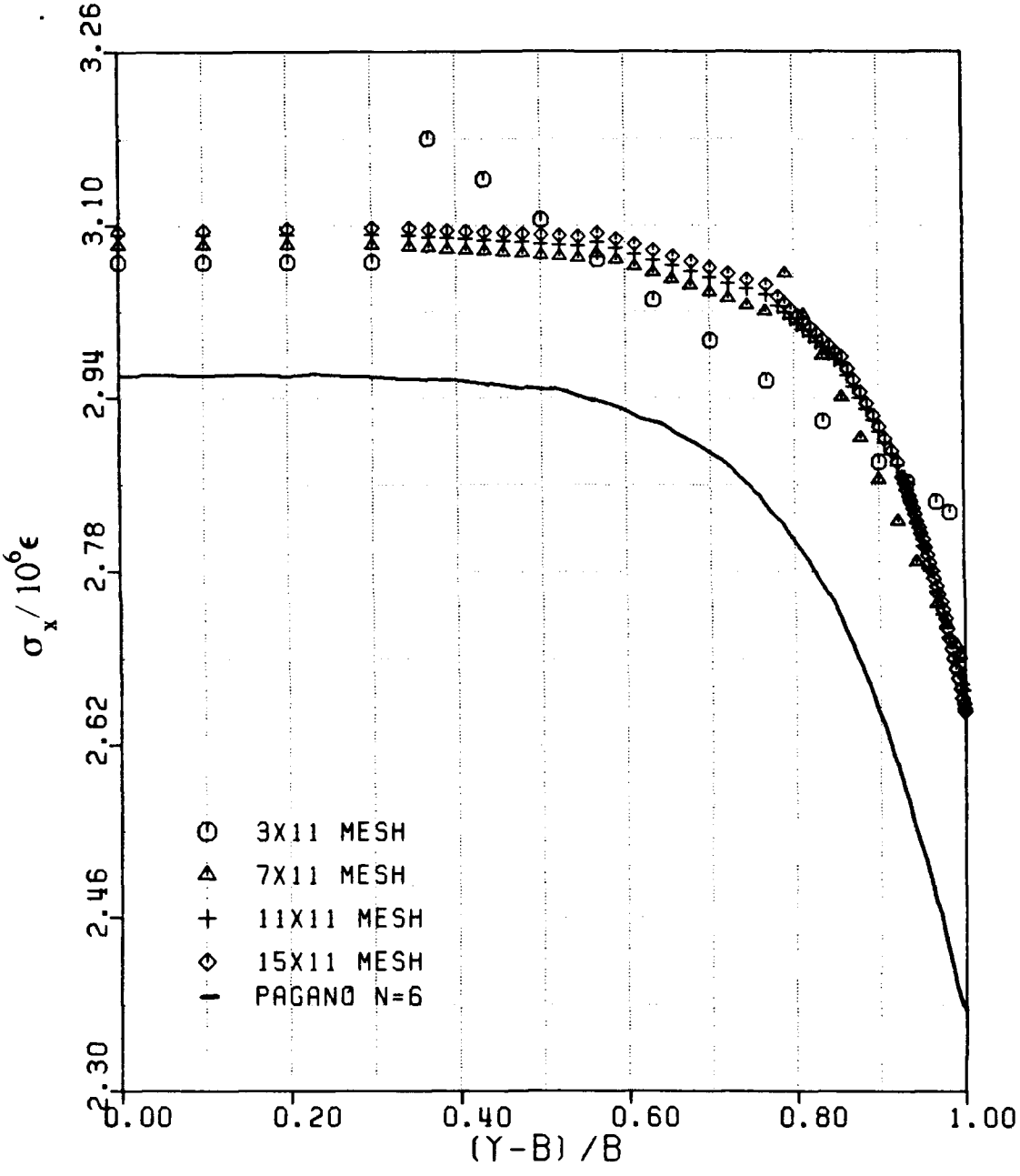


Figure 7: Distribution of X-stress along Center of Top Layer (Angle-Ply).

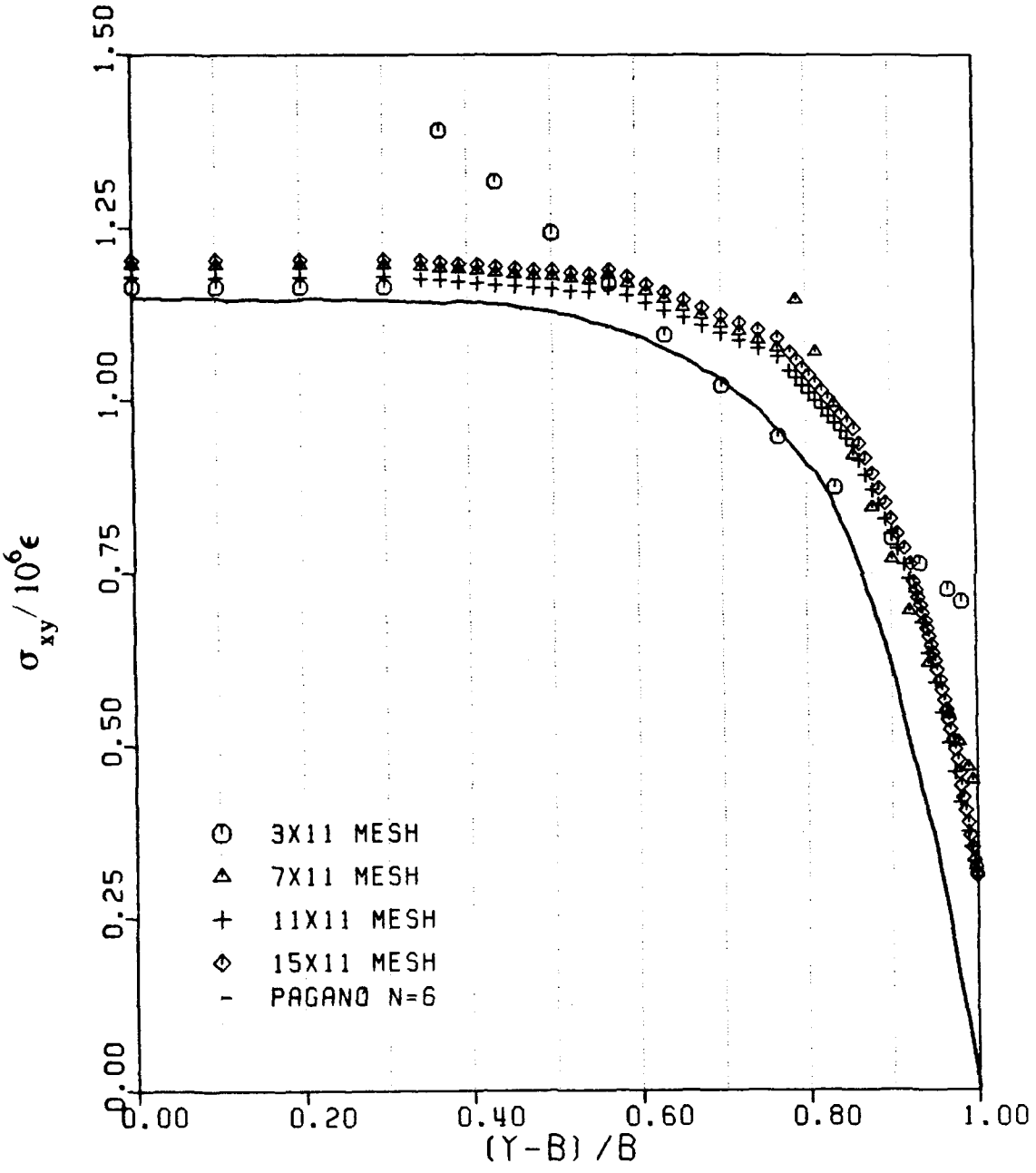


Figure 8: Distribution of XY-stress along Center of Top Layer (Angle-Ply).

Table 7

Comparison of CPU time for different Finite Element Meshes.

F.E.M Mesh	Total CPU time on a CRAY X-MP/28 (Seconds.)
3x11	7.798
7x11	23.424
11x11	42.664
15x11	68.921

underestimate the shearing stress.

To further enhance the accuracy of the solution, refinement in the thickness direction, i.e. the division of each layer into sublayers, was attempted. Each layer was divided into three sublayers of thickness $h/3$. Further subdivision into five layers was done by dividing the sublayer of interest (i.e., the one containing the location at which stresses were desired) into three equal layers of thickness $h/9$. To save on computational effort, the symmetry of the problem was exploited by considering only the top two plies and specifying the transverse displacements, w, at the middle surface of the laminate equal to zero. Hence, solutions for the cases of $N = 2$, N=6 and N=10 were obtained, where N represents the total number of resulting layers; N=2 being the case with no sublayers. The scheme for sublayer division is depicted in Figure 11.

The results for inplane stresses for different N using a 11x11 element mesh are compared with Pagano's results in Figure 12 to Figure 15. A comparison of CPU time for various number of sublayers is given in Table 8. Table 7 and Table 8 show that for the problem using the 11x11 mesh, exploiting the symmetry by considering only two plies reduced the CPU time from 42.664 to 15.852 seconds.

Figure 12 and Figure 13 show significant improvement in the results for inplane stresses at the free edge with increasing number of sublayers, though the stresses at the center of the laminate remain essentially unchanged. The results for axial displacement are shown in Figure 14 and show that the displacements obtained using N=6 and N=10 agree closely with Pagano's solution.

The transverse shear stress σ_{13} at the $\pm 45°$ interface was computed using (81) and is shown in Figure 15. With increasing N, the computed free-edge stress steadily increased in magnitude. Pagano [1978] observed that σ_{13} at the edge grows with

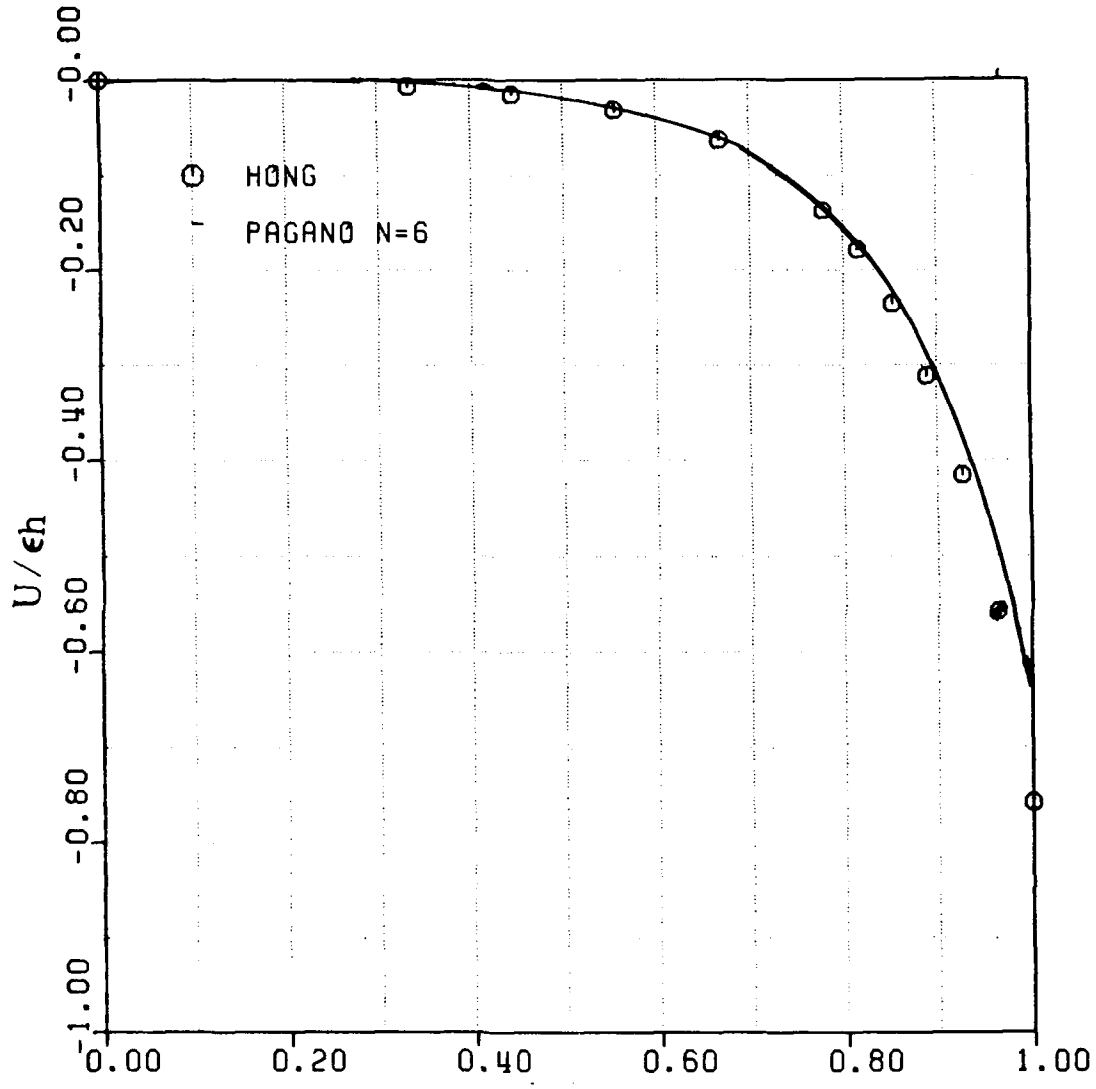


Figure 9: Axial Displacement Across Top Surface (Angle-Ply)

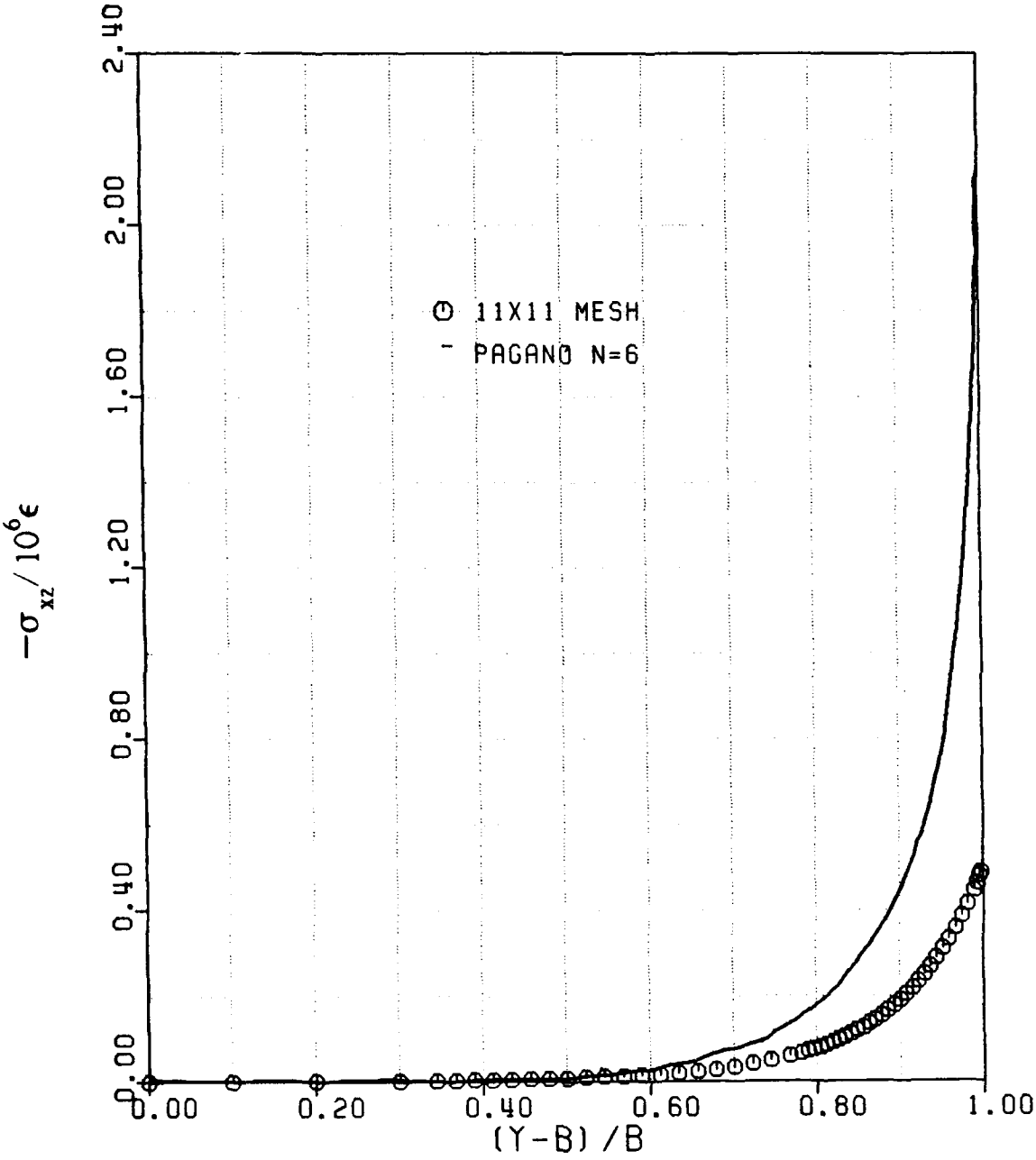


Figure 10: Distribution of XZ-stress along 45/-45 interface (Angle-Ply).

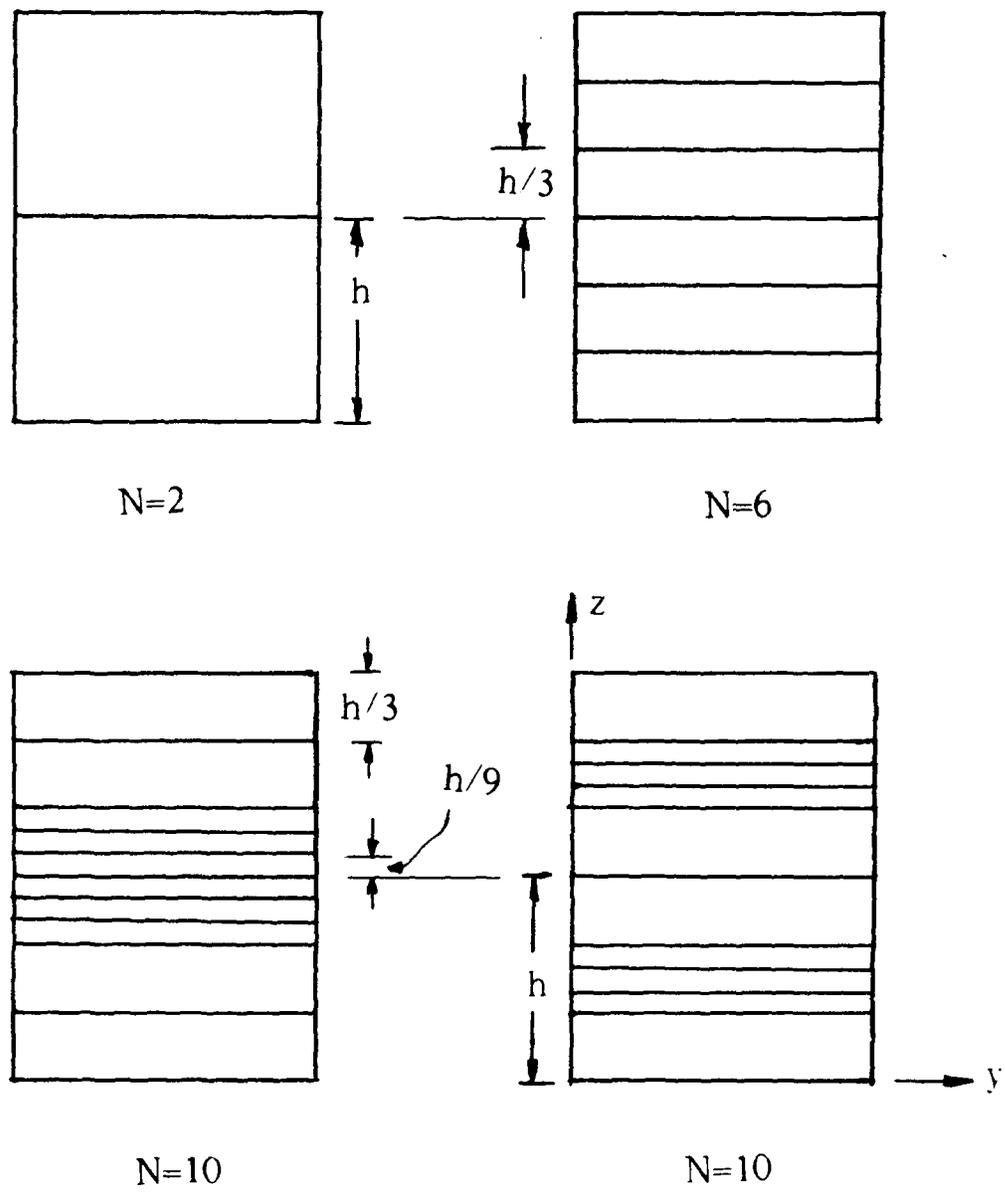


Figure 11: Division of Individual Plies into Sublayers

Table 8

Comparison of CPU Time for 11x11 Mesh with Different Number of Sublayers.

N	Total CPU time on a CRAY X-MP/28 (Seconds.)
N=2	15.852
N=6	79.333
N=10	197.084

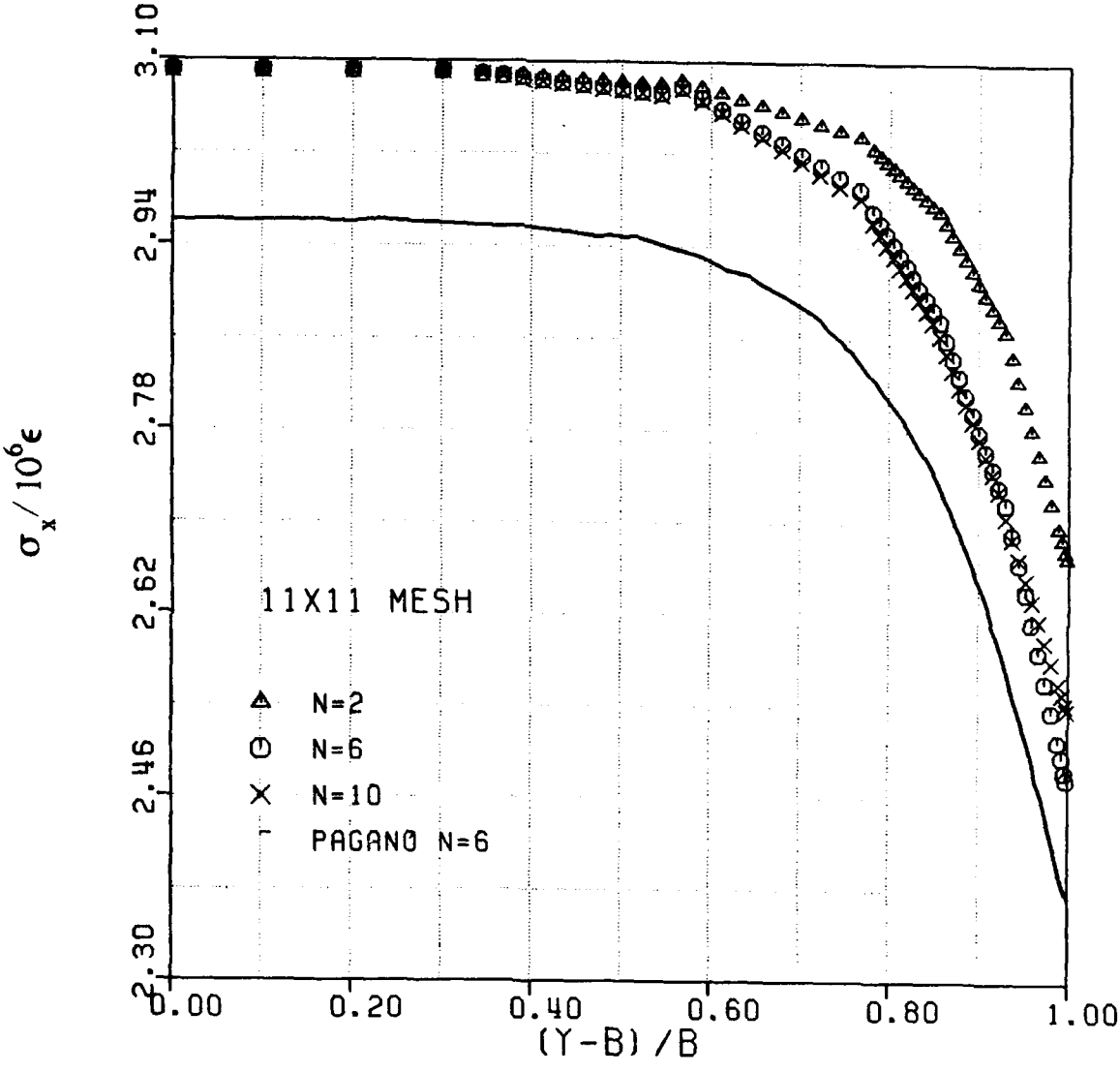


Figure 12: Distribution of X-stress Along Centre of Top Layer for Various N (Angle Ply).

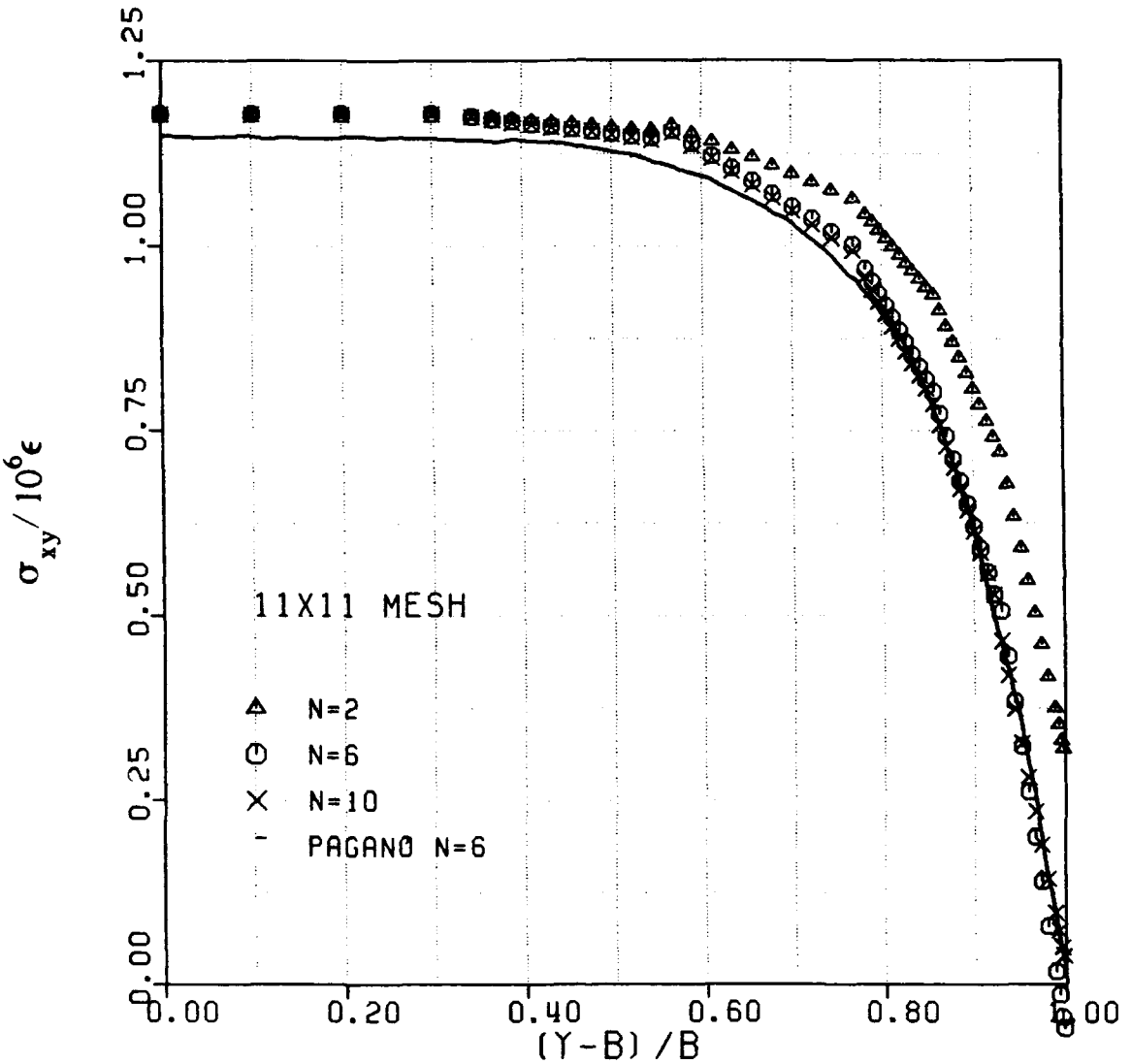


Figure 13: Distribution of XY-stress Along Center of Top Layer for Various N (Angle-Ply).

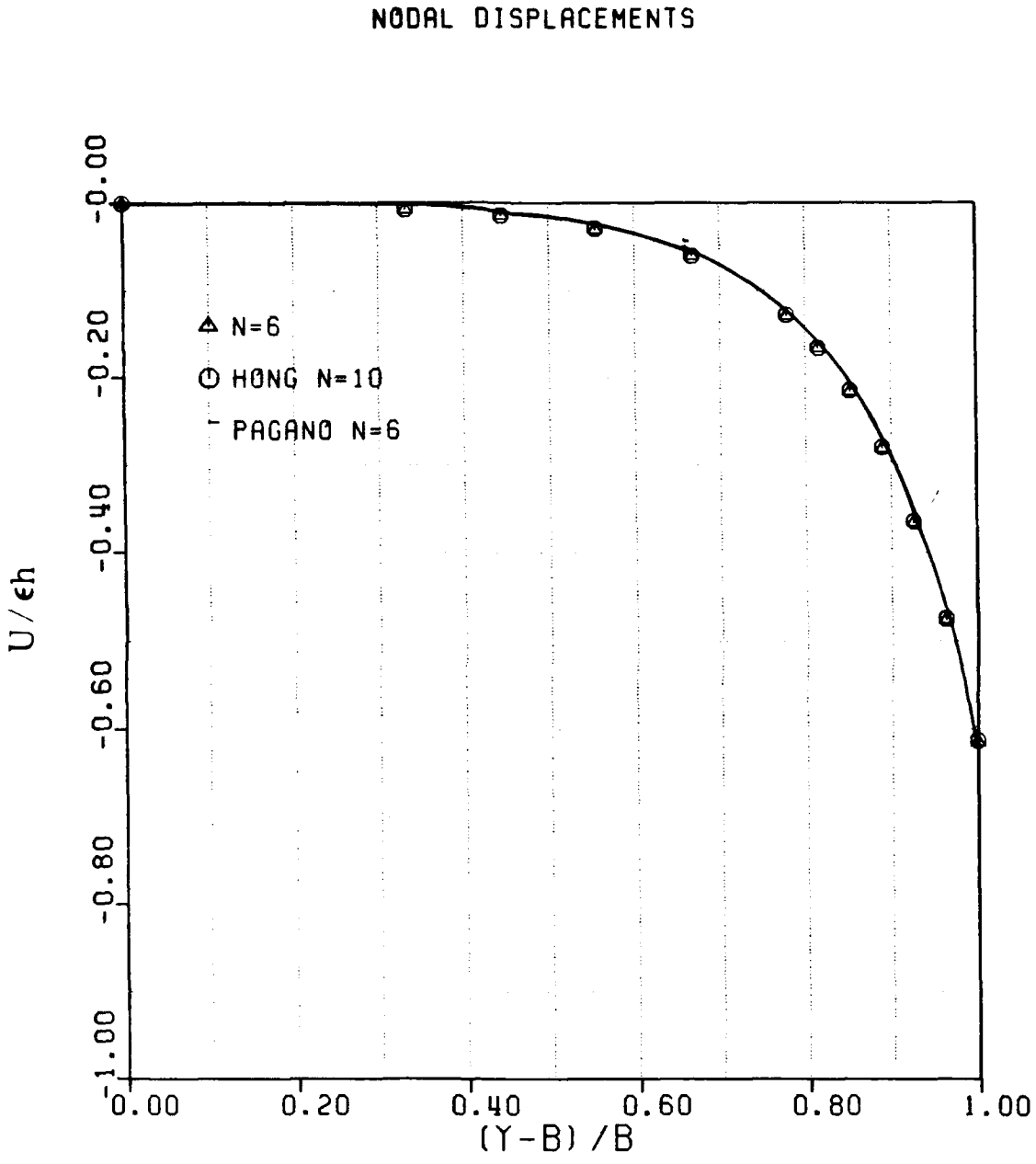


Figure 14: Axial Displacement Across Top Surface with N=6, 10. (Angle-Ply).

increasing N and could not determine whether a finite limit was approached for large N. The values of the transverse stress close to the free edge for different number of sublayers are presented in Table 9.

To observe the effect of the consistent shear treatment, the results were compared with those obtained by Moazzami et al. [1991] using the discrete laminate theory along with a constant shear correction factor. The stresses and displacements at element centers for a 11x11 mesh with N=2 are compared in Table 10 to Table 13.

Table 10 shows that the results for σ_{11} do not show any "substantial" difference. The results for σ_{12}, σ_{13}, and u_1 in Table 11 to Table 13, however, indicate that those obtained from the present theory are slightly better than those obtained using the existing discrete laminate theory. As observed in Section 6.2, only marginal improvement was expected by the introduction of consistent shear coupling.

In summary, it was observed that the results improved to a certain extent only with spatial refinement of the finite element mesh. Convergence after the 11x11 mesh was slow. Further enhancement in the solution was obtained by dividing each ply into sublayers. As the inplane displacements are assumed linear over the thickness of each layer, division into sublayers resulted in piecewise linear displacements over each ply contributing to increased accuracy.

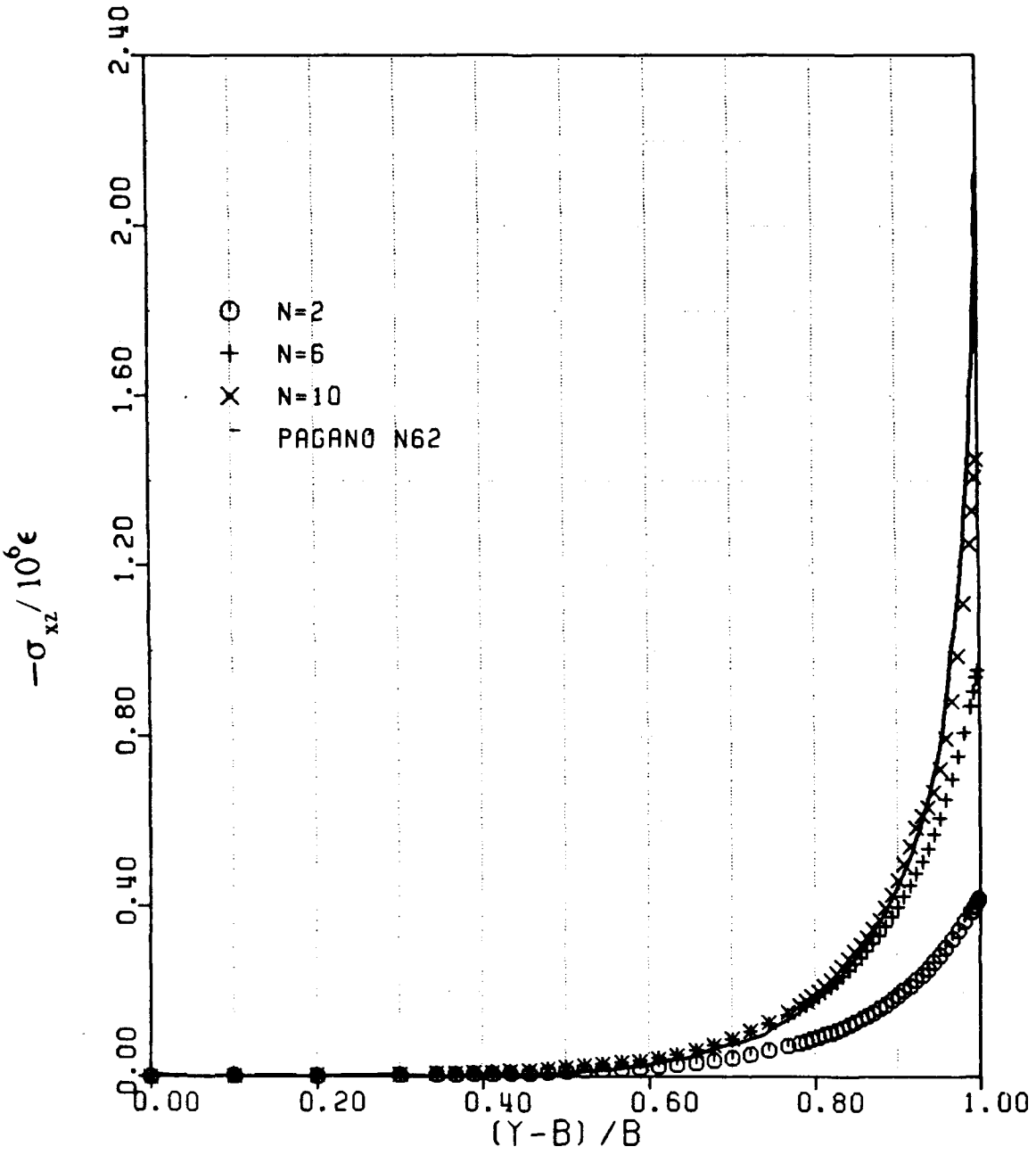


Figure 15: Distribution of XZ-stress Along 45/-45 Interface for Various N (Angle-Ply).

Table 9

Growth of XZ-stress at Free Edge with Increasing N

N	The Present Study	Pagano
2	0.416	1.664
6	0.955	2.213
10	1.450	-

Table 10

Comparison of X-stresses with those Obtained by Moazzami et al.

(y-b)/b	Present Study	Moazzami	Pagano (N=6)*
0.000	3.0912	3.0882	2.902
0.444	3.0853	3.0785	2.90
0.666	3.0593	3.0584	2.89
0.815	2 9999	3.0067	2.78
0.888	2.9229	2.9317	2.67
0.962	2.7712	2.7658	2.25

* Note: These numbers are approximated from the plots in Pagano [197

Table 11

Comparison of XY-stresses with those Obtained by Moazzami et al.

(y-b)/b	Present Study	Moazzami	Pagano (N=6)*
0.000	1.1793	1.1995	1.15
0.444	1.1668	1.1921	1.14
0.666	1.1137	1.1526	1.05
0.815	0.9923	1.0504	0.88
0.888	0.8348	0.9023	0.64
0.962	0.5248	0.6409	0.25

* Note: These numbers are approximated from the plots in Pagano [197

Table 12

Comparison of XZ-stresses with those Obtained by Moazzami et al.

(y-b)/b	Present Study	Moazzami	Pagano (N=6)*
0.000	-0.0030	-0	-0.00
0.444	-0.0143	-0.0072	-0.019
0.666	-0.0613	-0.0387	-0.09
0.815	-0.1752	-0.1268	-0.26
0.888	-0.3046	-0.2385	-0.38
0.962	-0.5465	-0.4662	-0.939

* Note: These numbers are approximated from the plots in Pagano [197

Table 13

Comparison of X-displacements with those Obtained by Moazzami et al.

(y-b)/b	Present Study	Moazzami	Pagano (N=6)*
0.000	~0	~0	0.00
0.444	-0.0143	-0.0071	-0.011
0.666	-0.0615	-0.0387	-0.055
0.815	-0.1765	-0.1244	-0.16
0.888	-0.3069	-0.2338	-0.26
0.962	-0.5507	-0.4571	-0.47

* Note: These numbers are approximated from the plots in Pagano [197

6.3.2 Cross-Ply Specimen.

The second example of the FED specimen solved was a [0/90]$_s$ laminate. The material properties and laminate geometery were the same as before. Results for σ_{23} and σ_{33} were computed for comparison with Pagano's solution.

Figure 16 compares the results obtained for σ_{23} with those given by Pagano [1978]. The results for N=6 matched well with those of Pagano's except over the edge element. Notice that a singularity exists at the edge due to the different constitutive relations at the free edge. As the $\Lambda^{(kj)}s$ were interpolated over the edge element, the edge element should be kept relatively small to better approximate the singularity. Hence further spatial refinement was done by using 15x11 and 19x11 element meshes. These results are shown in Figure 17 and Figure 18. These figures show that with mesh refinement near the edge of the laminate, a sharp peak was observed in the edge element. The magnitude of the peak increased with decreasing width of the edge element. It appears that if the edge element is sufficiently small and is ignored, the stresses in the penultimate element can be considered to realistically represent the conditions very close to the free edge.

Equation (275) was used to compute the normal stress. The derivatives were computed at each location using a central difference method. The size of the interval used was 0.05 of the element width, reducing to 0.02 of the element width near the free edge.

The results for σ_{33} for the different element meshes are shown in Figure 19. The hump noticeable in Pagano's solution was not observed. It was observed that the edge values of the stresses approached Pagano's solution as the mesh near the laminate edge was refined.

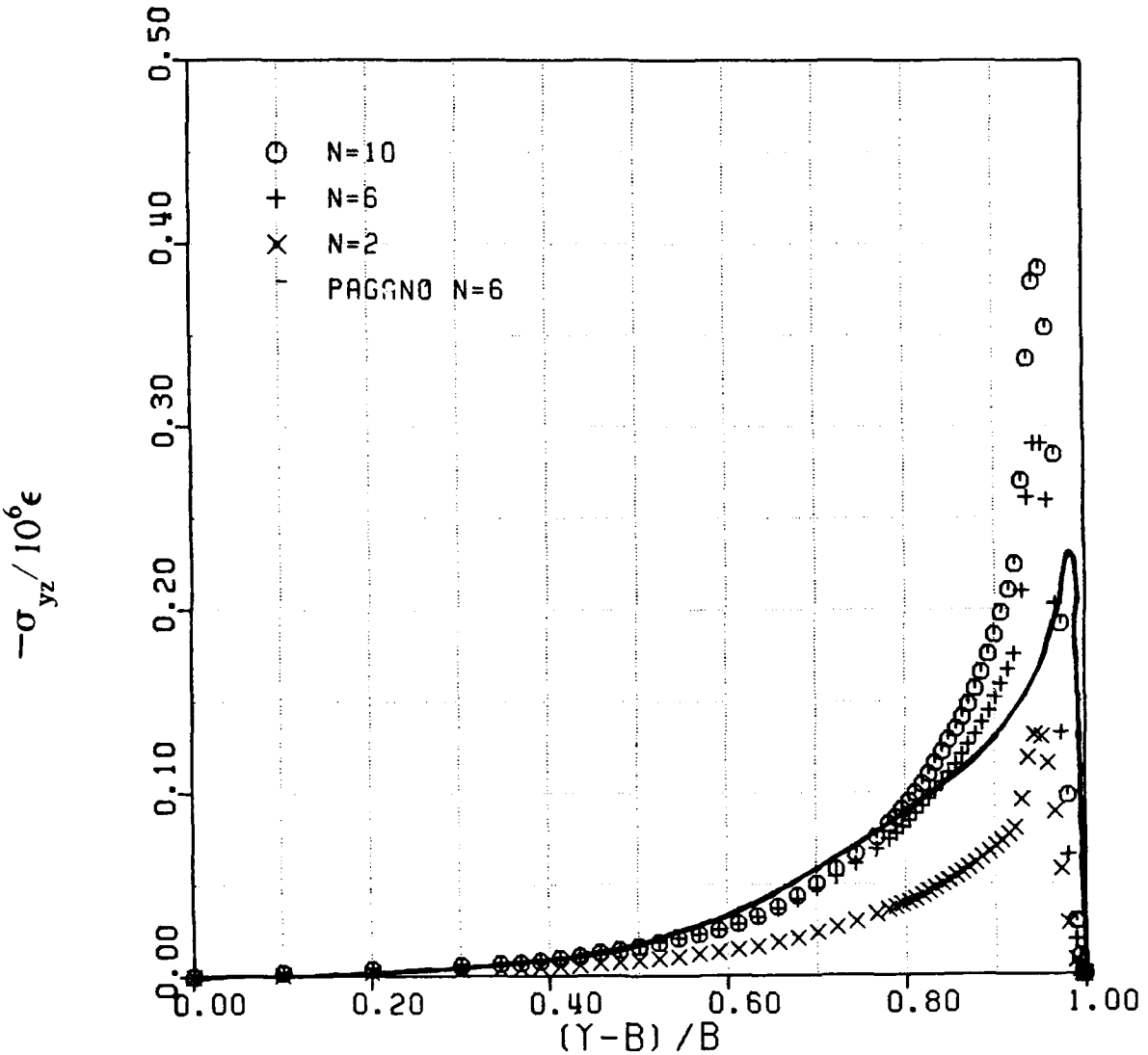


Figure 16: Distribution of YZ-stress Along 0/90 Interface (11x11 Mesh).

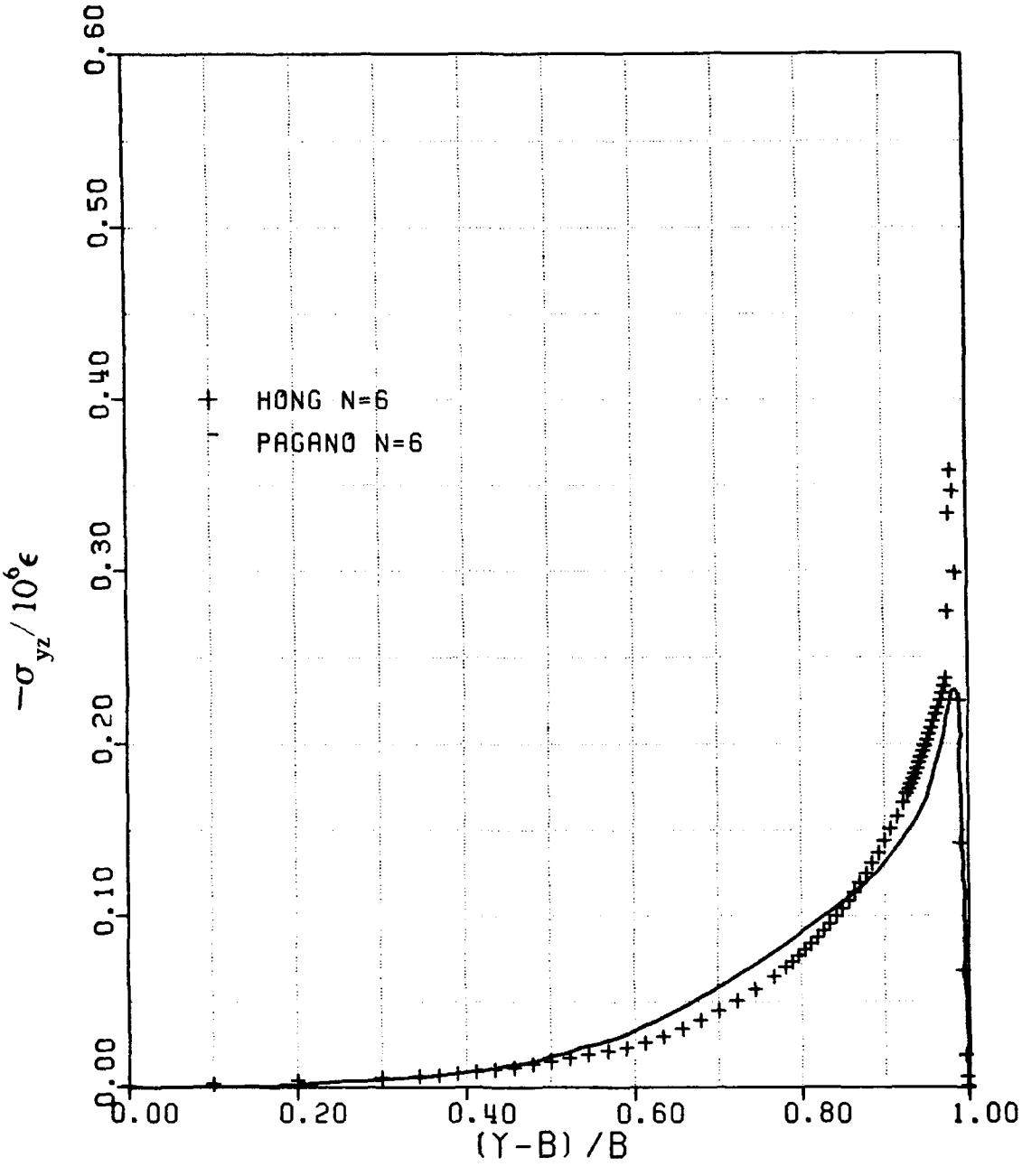


Figure 17: Distribution of YZ-stress Along 0/90 Interface (15x11 Mesh).

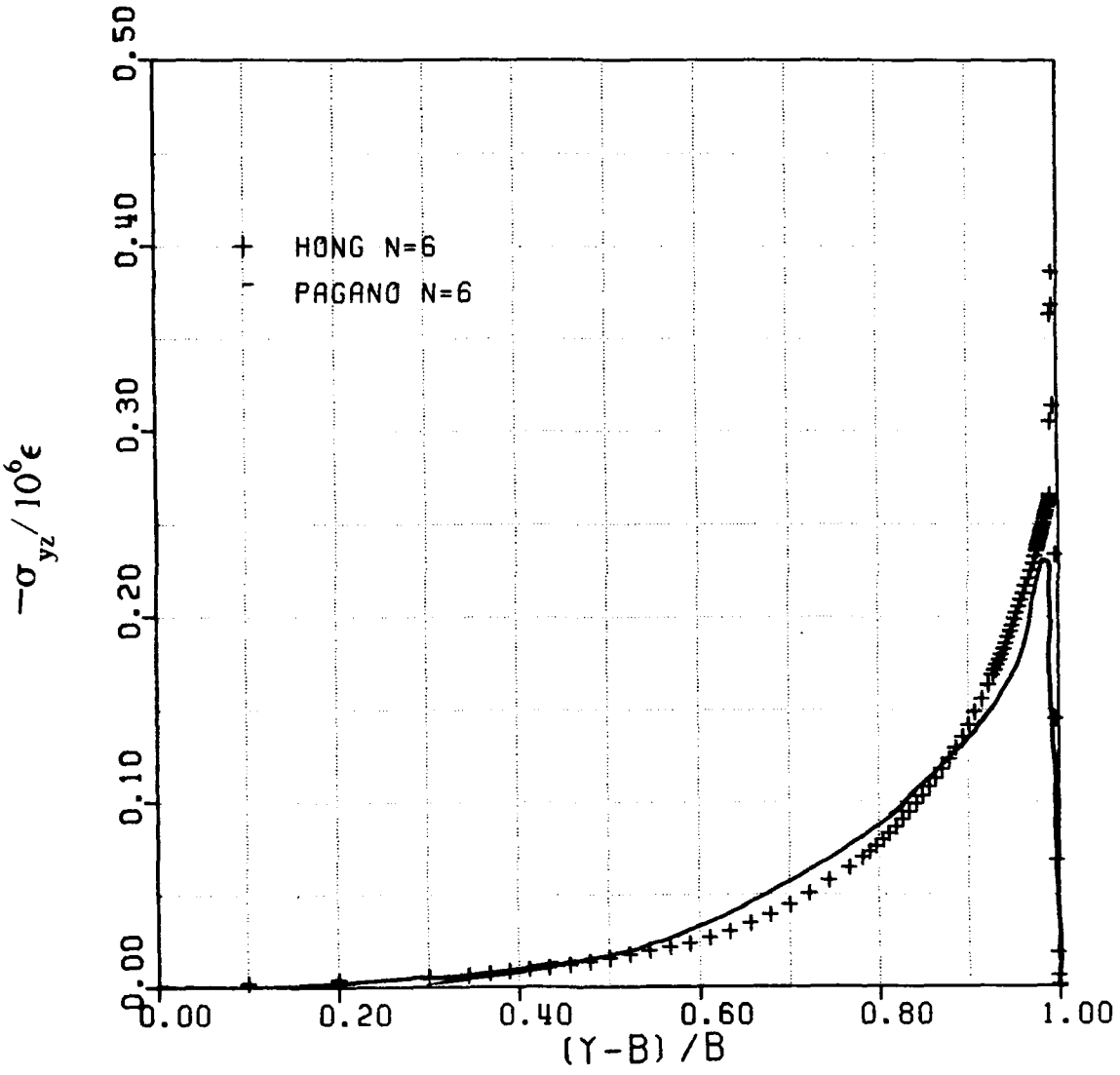


Figure 18: Distribution of YZ-stress Along 0/90 Interface (19x11 Mesh).

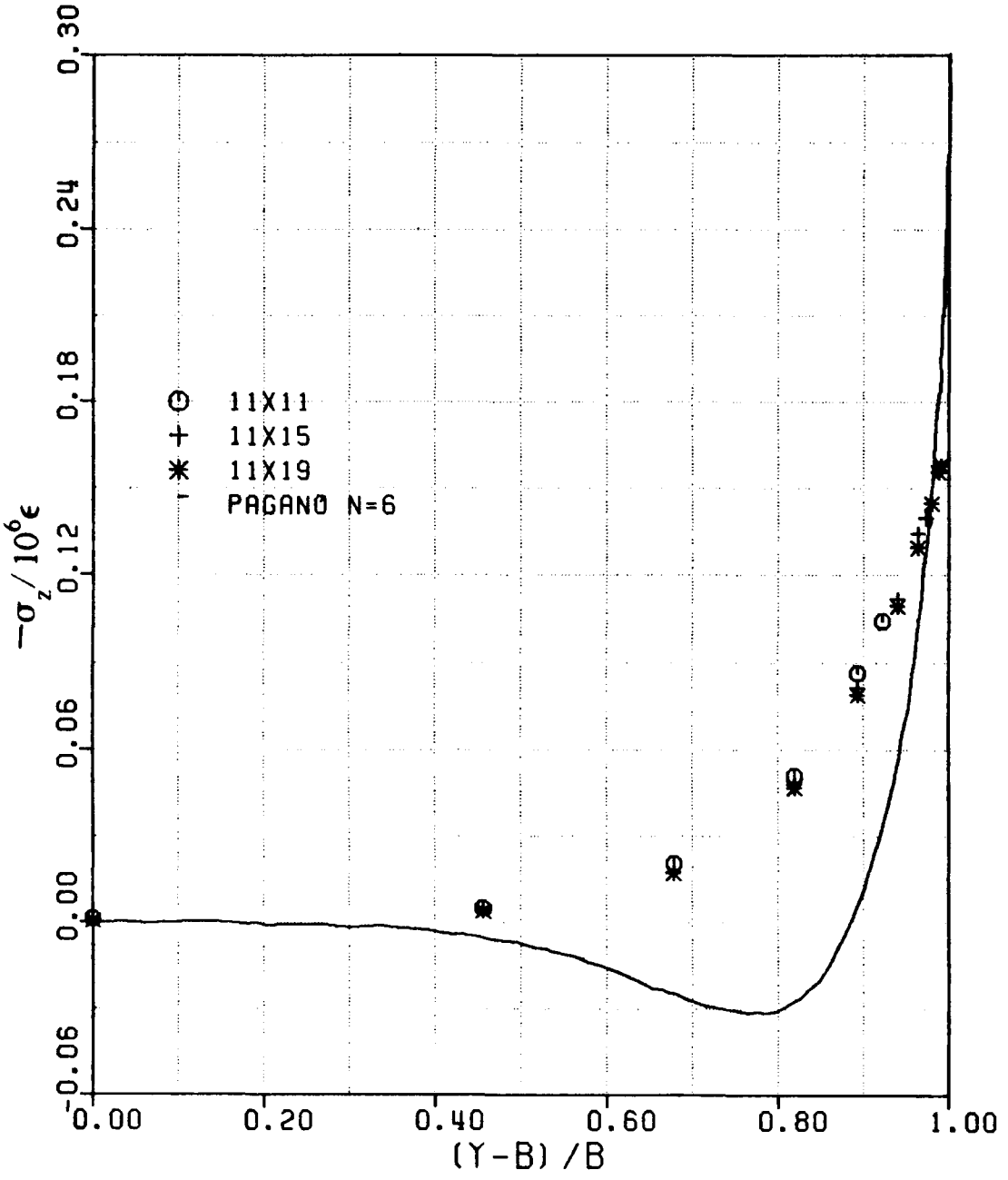


Figure 19: Distribution of Z-stress Along 0/90 Interface.

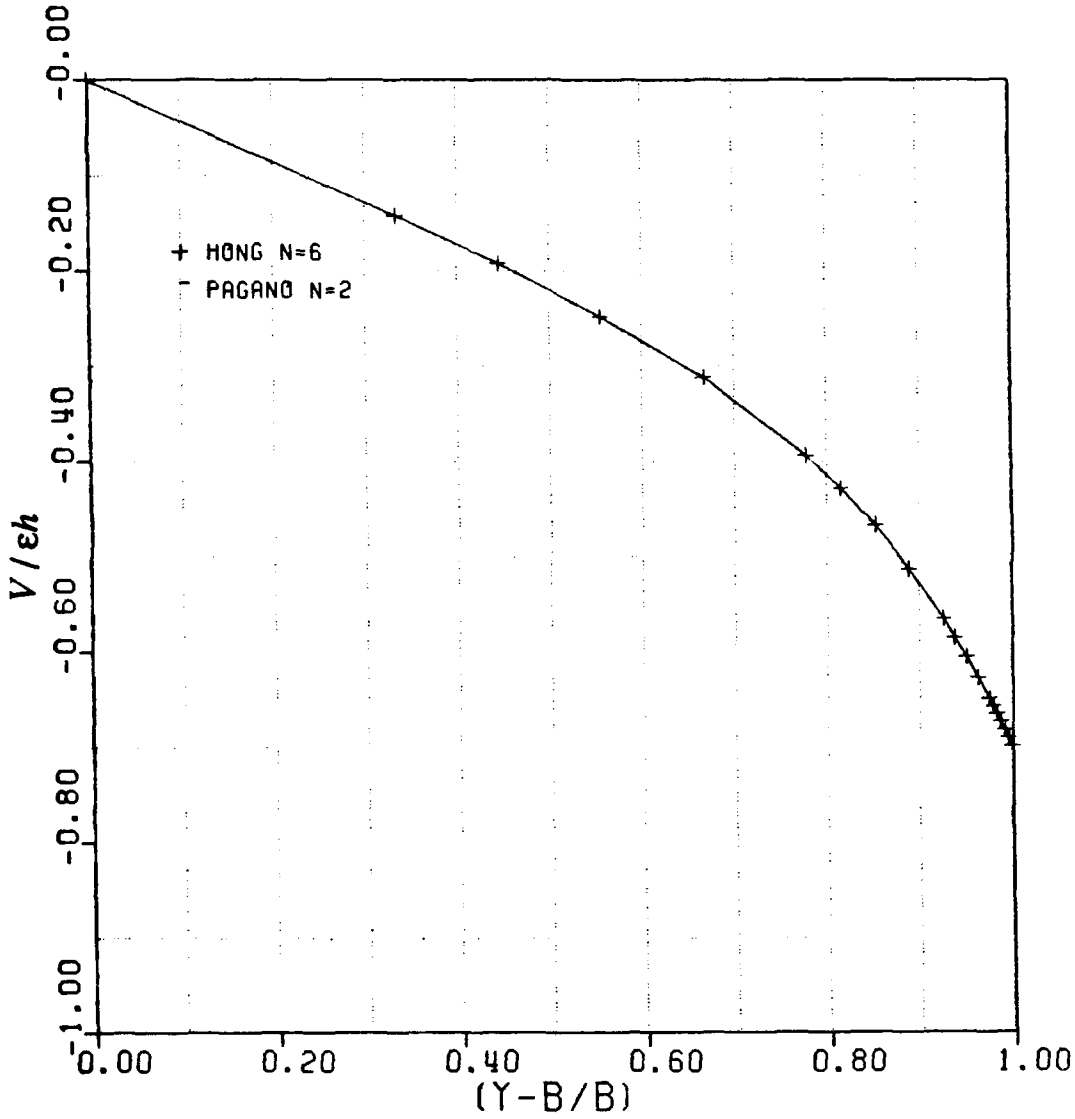


Figure 20: Distribution of Y-displacement along Top Surface (19x11 Mesh).

The distribution of the tranverse inplane displacement is shown in Figure 20 and agrees well with Pagano's result.

6.4 *NATURAL FREQUENCIES OF FREE VIBRATION OF A SANDWICH PLATE.*

Though the constitutive relations for transverse shear have been obtained from the equations of static equilibrium, inertial terms were included in the finite element formulation to compute the natural frequencies of free vibration of a simply supported plate. A sandwich plate whose 3-D elasticity solution for the fundamental frequency was presented by Srinivas [1970] was considered. The orthotropic laminated plate consisted of three layers as shown in Figure 21. The top and bottom layers had the same thickness and material properties while the thickness and material properties of the middle layer were different. Square geometry with thickness/side ratio of 0.1 (h/a=h/b=0.1) was used and three different cases of material properties given in Table 14 were considered.

For a simply supported laminated plate, the boundary conditions are [Srinivas 1970,1973]:

$$u_1 = w = 0 \qquad at\ x_2 = 0 \text{ and } b \tag{282}$$

$$u_2 = w = 0 \qquad at\ x_1 = 0 \text{ and } a \tag{283}$$

With the present formulation, these boundary conditions were restated as

$$u_1^{(1)} = \phi_1^{(k)} = w = 0 \qquad at\ x_2 = 0 \text{ and } b \tag{284}$$

$$u_2^{(1)} = \phi_2^{(k)} = w = 0 \qquad at\ x_1 = 0 \text{ and } a \tag{285}$$

Figure 22 shows the finite element mesh along with the boundary conditions.

Table 14

Lamination Data for Sandwich Plate

Case	t_1/h	t_2/h	t_3/h	$\rho^{(1)}/\rho^{(2)}$	$\rho^{(3)}/\rho^{(2)}$	$\bar{Q}_{66}^{(1)}/\bar{Q}_{66}^{(2)}$	$\bar{Q}_{66}^{(3)}/\bar{Q}_{66}^{(2)}$
I	0.1	0.8	0.1	1.0	1.0	1	1
II	0.1	0.8	0.1	1.0	1.0	10	10
III	0.1	0.8	0.1	1.0	1.0	50	50

* For all layers, ratios of orthotropic elastic constants were:

$$\bar{Q}_{11} : \bar{Q}_{12} : \bar{Q}_{22} : \bar{Q}_{44} : \bar{Q}_{55} : \bar{Q}_{66} = 3.802 : 0.879 : 1.996 : 1.015 : 0.608 : 1.0$$

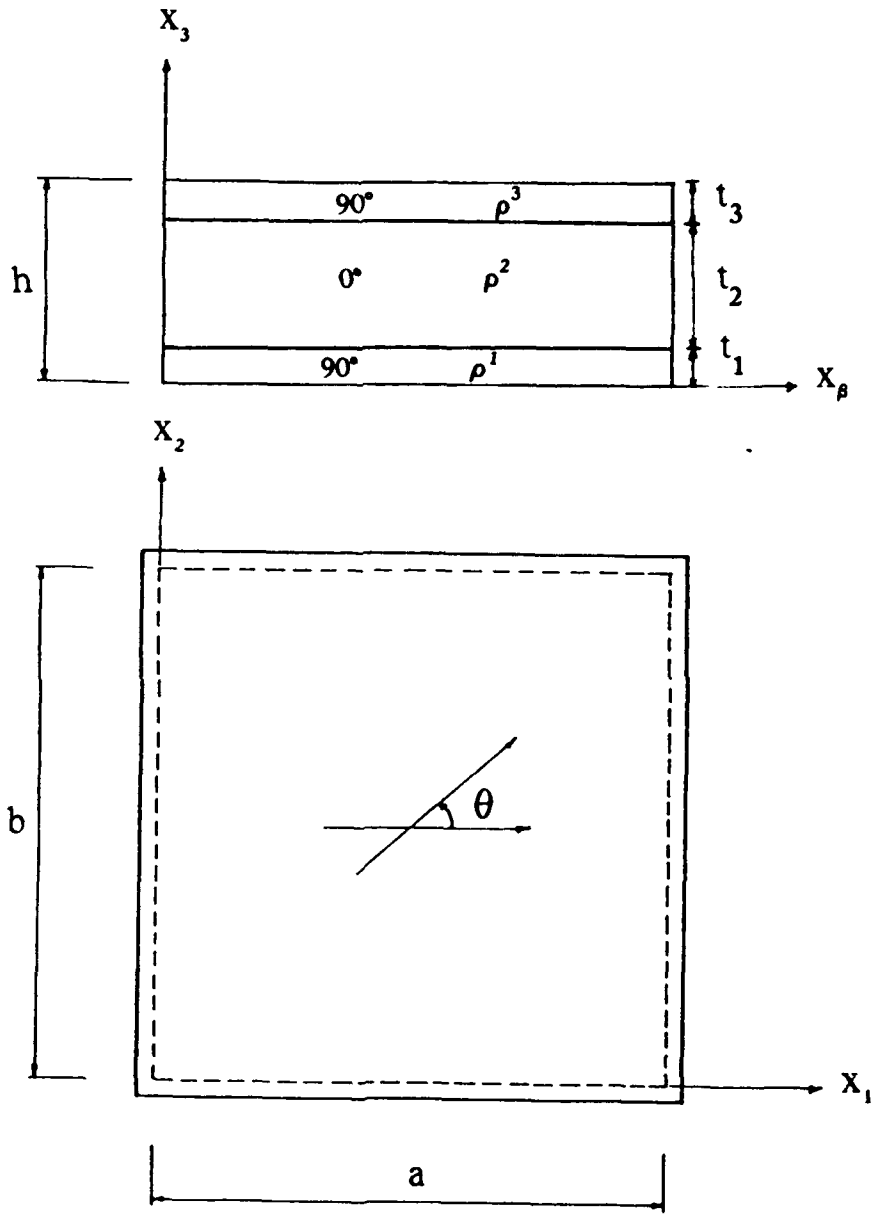


Figure 21: Planform and Cross Section of Laminated Plate of the Test Problem

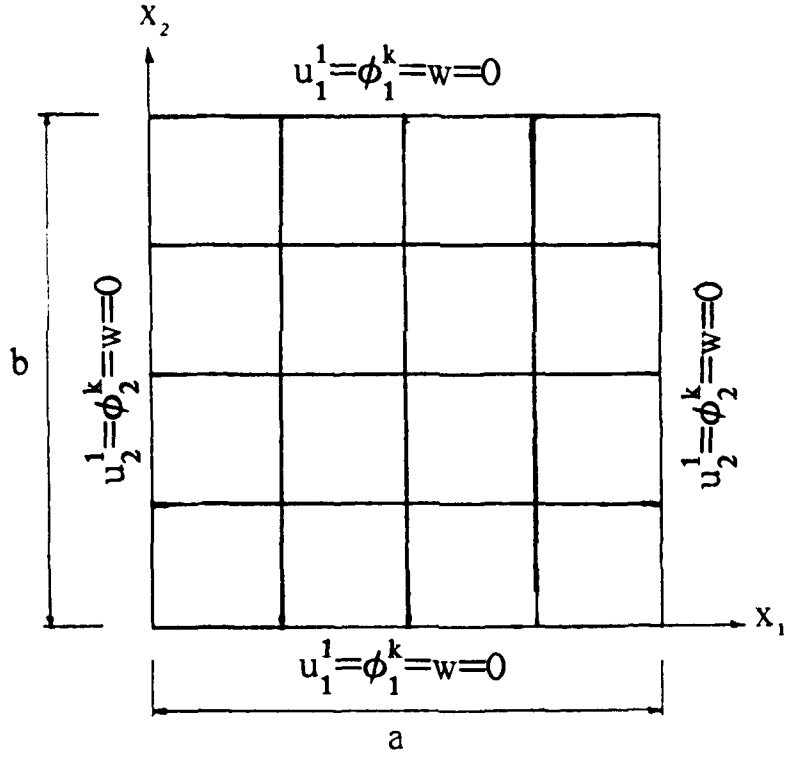


Figure 22: Finite Element Mesh and Boundary Conditions

Table 15 shows the nondimensionalized fundamental natural frequency obtained using 4, 16 and 36 element mesh for cases I, II, III for which the ratio of elastic constants of the middle layer to those of the outer layers was 1, 10 and 50, respectively. It was seen that the accuracy for Case I was excellent, but the error became larger as the ratio of the elastic constants of the outer layers to the middle layer increased. Figure 23 illustrates the convergence of the numerical solutions with spatial mesh refinement. It was seen that convergence after refinement beyond 16 elements was slower than from 4 to 16 elements.

To examine the effect of consistent treatment of transverse shear deformation, the same example problem was solved using the code with a simple shear correction factor k=5/6, assuming the constitutive equations of shear in each layer to be uncoupled. Table 16 shows the non-dimensionalized fundamental natural frequency obtained based on this approach for the three cases. The quantity in parentheses is percentage error.

The results obtained with k=5/6 were compared with the result obtained with the consistent theory. These comparisons for cases I, II and III are illustrated, respectively, in Figure 24, Figure 25 and Figure 26. It was observed that the uncoupled approach overpredicts the natural frequency. The difference increases as the difference in stiffnesses of the outer and the inner layer grows. Further, Figure 24 through Figure 26 show that the best accuracy was obtained with the 4x4 element mesh, but the solution did not show monotonic convergence.

To further study the effect of the proposed consistent shear constitutive relations, in addition to comparing the fundamental frequency the higher frequencies obtained using the proposed theory and a shear correction factor of 5/6 were also compared. Figure 27 to Figure 29 compare the frequencies of a 537 degree-of-freedom system

Table 15

Non-Dimensionalized Fundamental Frequency by FEM based on the Consistent Shear Deformable Theory

Mesh	Case I	Case II	Case III
4	0.094697 (2.4%)	0.19682 (2.9%)	0.31097 (3.8%)
16	0.092952 (0.5%)	0.19472 (1.8%)	0.30919 (3.2%)
36	0.092900 (0.4%)	0.19463 (1.7%)	0.30911 (3.1%)
Exact 3-D (Srinivas)	0.09248	0.19132	0.29954

* $\lambda = \omega h \sqrt{\frac{\rho_2}{\overline{Q}_{66}^{(2)}}}$ *where* ω *is natural frequency.*

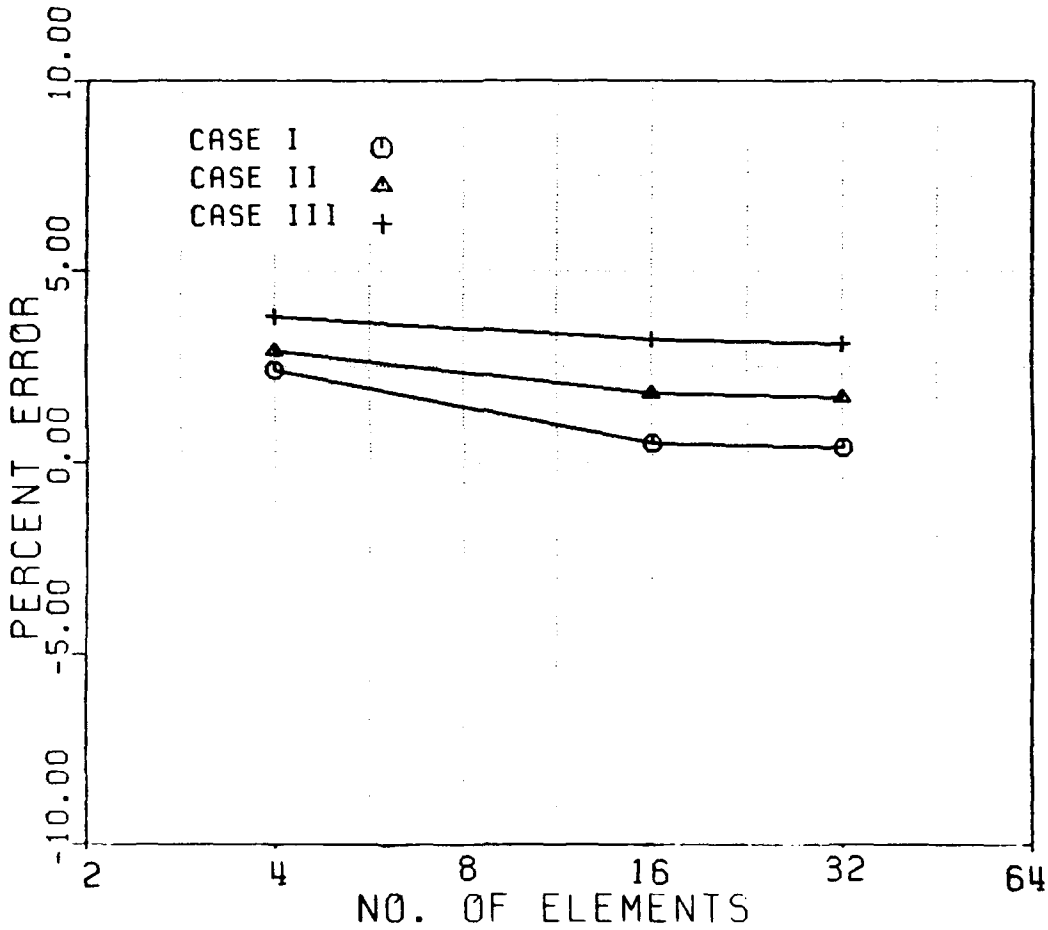


Figure 23: Convergence Rate of the Solution with Mesh Refinement

Table 16

Non-Dimensionalized Fundamental Frequency by FEM using shear correction factor k=5/6

Mesh	Case I	Case II	Case III
4	0.09498 (2.7%)	0.19710 (3.0%)	0.31489 (5.1%)
16	0.09317 (0.7%)	0.19584 (2.3%)	0.31305 (4.5%)
36	0.09341 (0.7%)	0.19805 (3.5%)	0.32232 (7.6%)
Exact (Srinivas)	0.09248	0.19132	0.29954

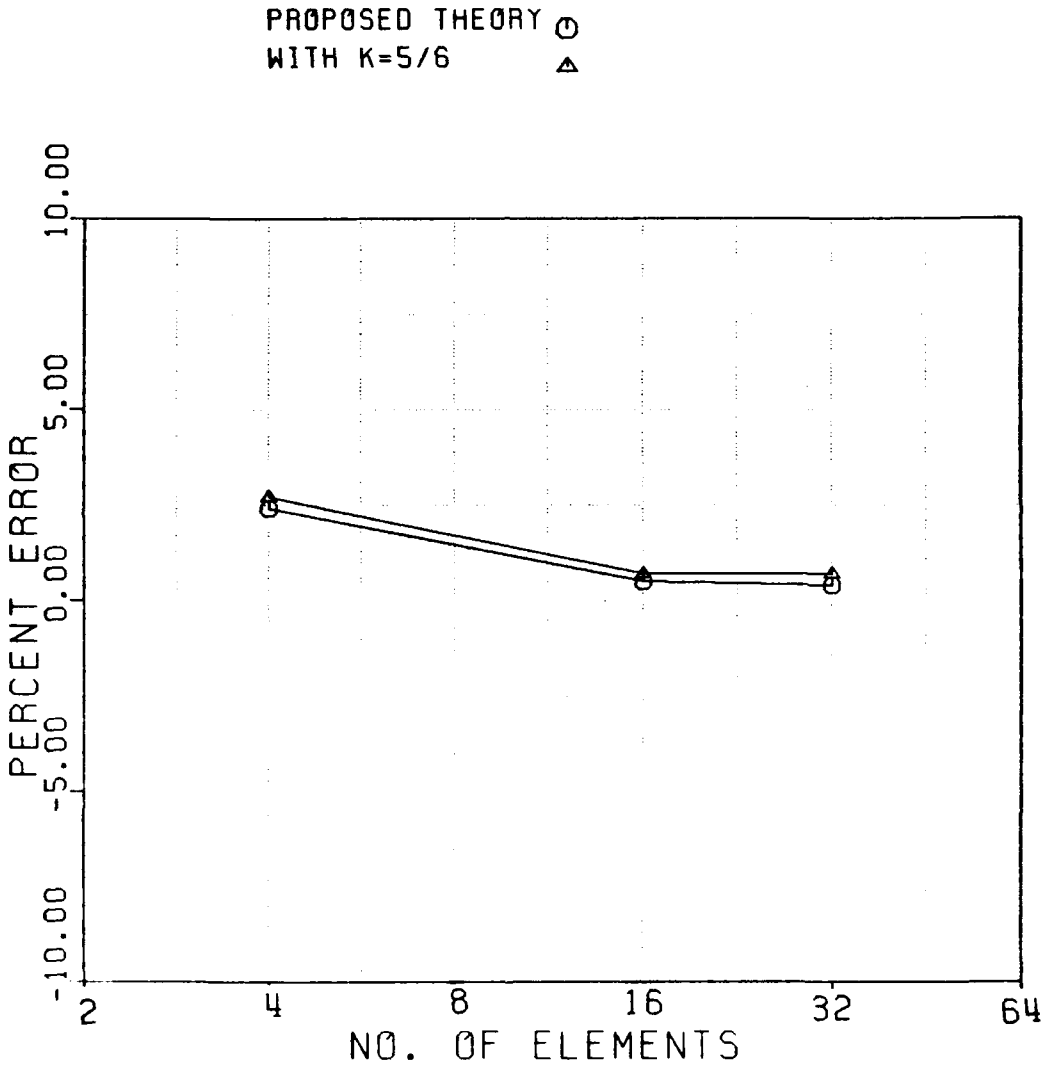


Figure 24: Effect of Consistent Shear Correction for Case I

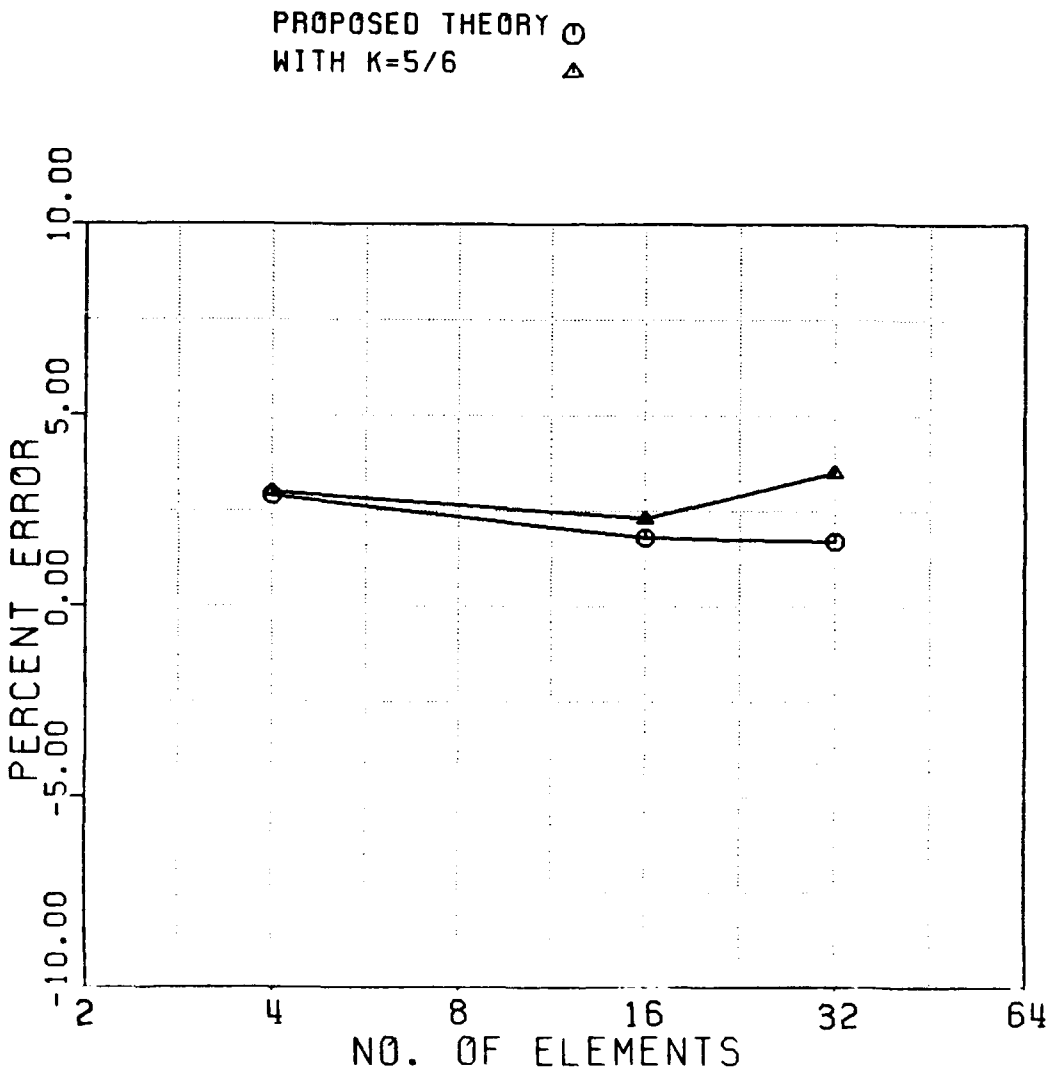


Figure 25: Effect of Consistent Shear Correction for Case II

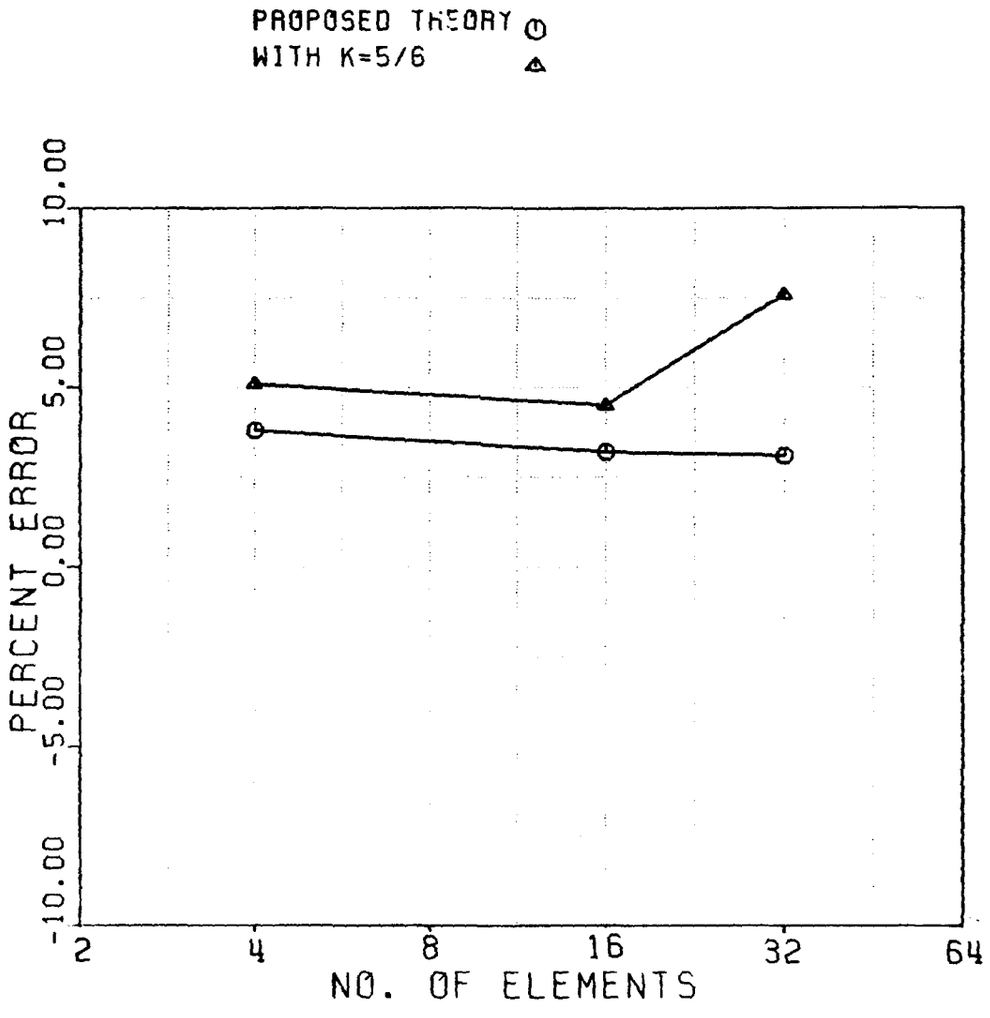


Figure 26: Effect of Consistent Shear Correction for Case III

(4x4) mesh for the three cases of the sandwich plate studied. The values of the lower frequencies computed using the consistent shear approach differ only slightly from those by the discrete laminate using a shear correction factor of 5/6. However, the higher frequencies are distinctly different.

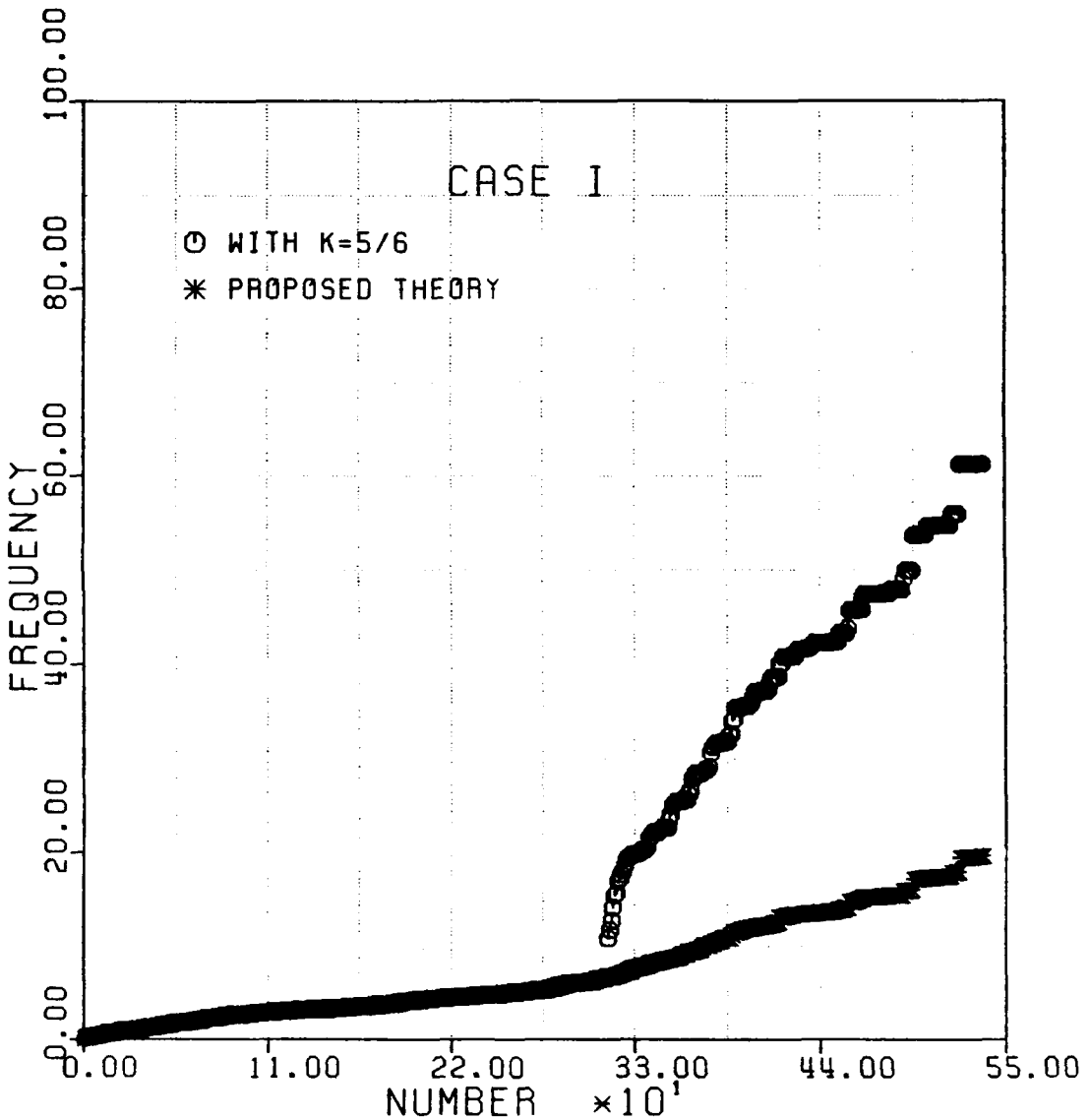


Figure 27: Natural Frequencies for Case I.

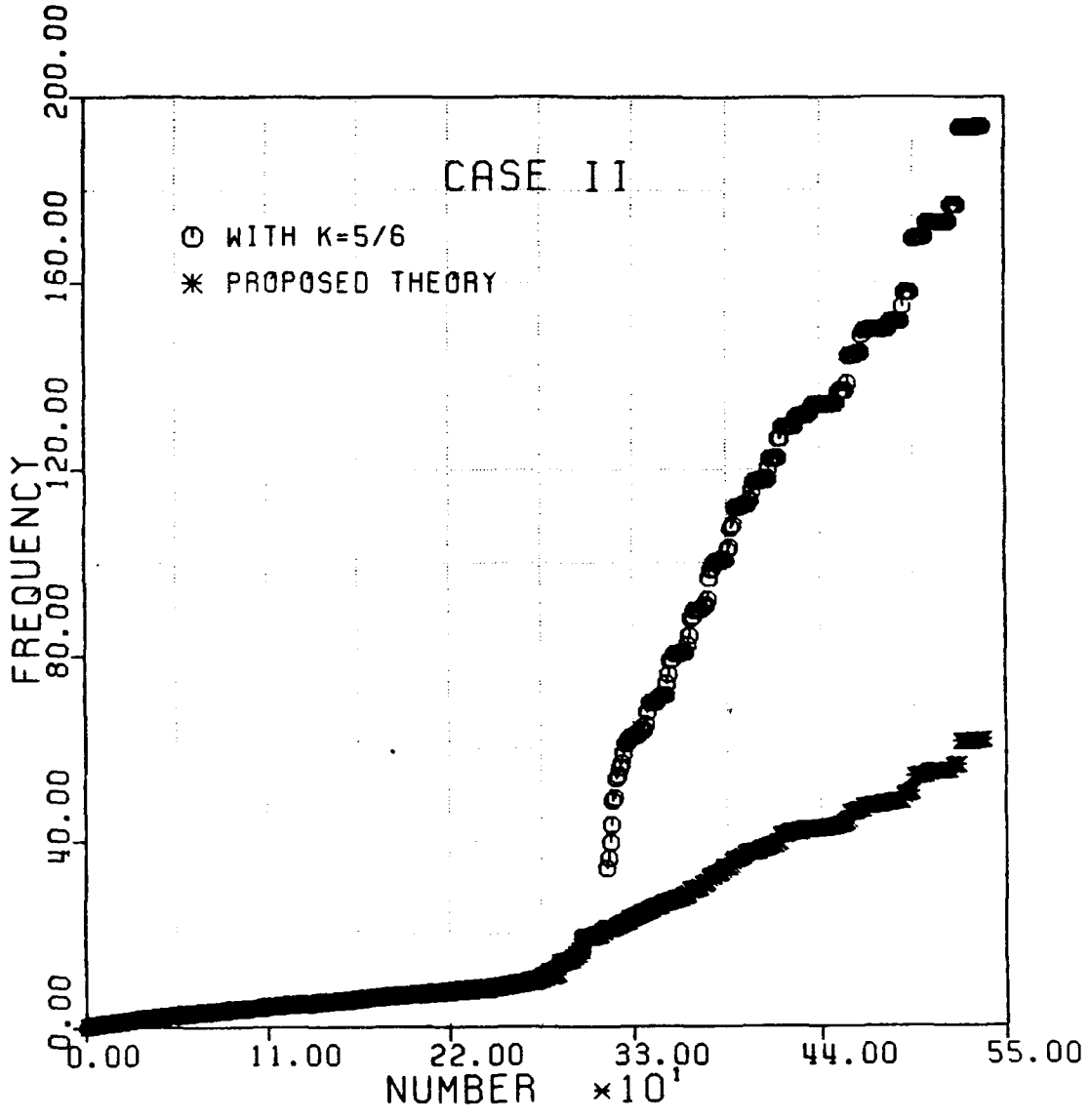


Figure 28: Natural Frequencies for Case II.

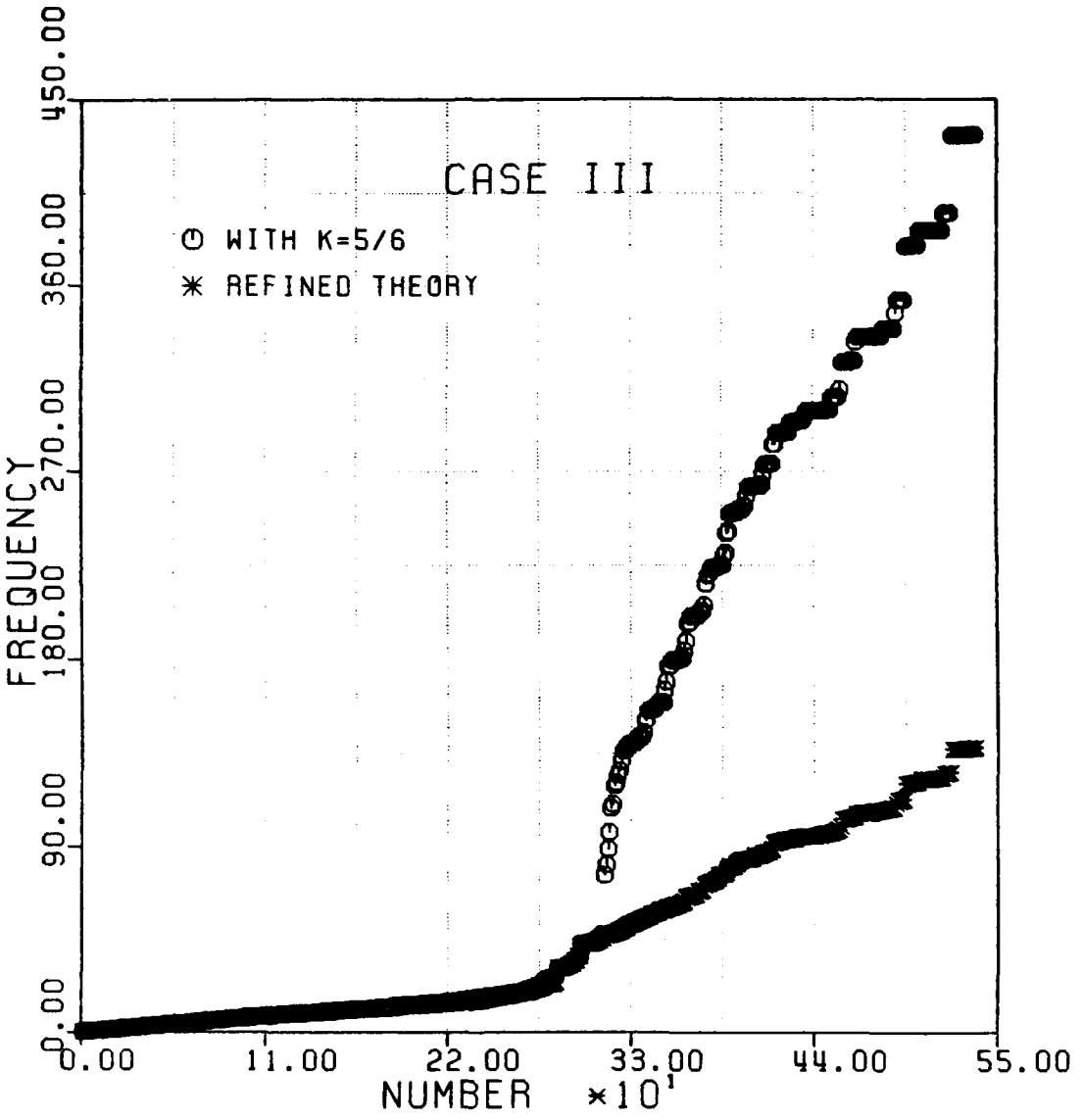


Figure 29: Natural Frequencies for Case III.

Section VII

DISCUSSION

Due to the inherent complexity associated with material anisotropy and inhomogeneity of composites, a laminated plate often shows quite different mechanical characteristics from the homogeneous isotropic counterpart. Therefore, it is essential that any simplified theory satisfy equilibrium, kinematic and constitutive relations as closely as possible to ensure reliable results.

The investigation reported here, aimed at development of stress and deformation analysis of laminated composites resulted in the following accomplishments:

a. Development of a systematic and general theory to consistently incorporate the transverse shear-deformation effect in composite laminates.

b. Derivation of variational principles governing the refined theory to provide a basis for development of efficient and reliable Ritz type as well as finite element approximation procedures.

c. Implementation of the theory into a finite element computer program including code verification.

d. Application of the theory to free-edge delamination specimens.

A discrete laminated plate theory based on the assumption of a layerwise linear variation of in-plane displacements has been further refined by incorporating the effect of transverse shear deformation in a consistent manner, viz. allowing for the coupling of shear deformation effects between layers. This development was facilitated by a

mixed variational principle of linear elasticity derived using Sandhu's generalized procedure for variational formulation of linear coupled boundary value problems. This mixed variational principle is more useful for the application to a general anisotropic material than Reissner's [1984] approach. A parabolic distribution of shear stresses over the thickness of each layer was assumed. Continuity of stresses and displacements in the layer interfaces were allowed for. Distinctive features of the resulting constitutive relations for transverse shear are:

1. Shear force resultants for each layer are a linear combination of the shear strains of all the layers. Directional coupling of the constitutive relations disappears for orthotropically constructed laminates.
2. Coefficients in a linear combination of shear strains are determined by parameters related to lamination schemes such as material properties, thickness of layers and stacking sequence of layers.
3. The shear constitutive relations also depend upon tangential stresses specified on the laminate surfaces.

The fact that the shear force over a layer is coupled with the shear strains of other layers in a linear fashion seems to be a striking result because such coupling has not been anticipated in earlier work. However, this result can be attributed to the consideration of continuity of shear stresses in the interface of layers which has not been taken into account in previous studies. Coupling of shear constitutive relations of all layers could result in a better understanding of how a laminate composed of many layers might react to applied forces.

The nature and extent of the shear coupling was studied by looking at the constitutive matrix for a 12-layer graphite epoxy laminate. It was observed that the

shearing force in a layer was not significantly influenced by the strains in other layers. Also, the effect was local, i.e, contribution from other layers decreased sharply with increasing distance between layers.

Using the refined laminate theory, a systematic development of variational principles for static as well as dynamic analysis of laminated composite plates was carried out. Direct as well as complementary formulations were developed. The complementary self-adjoint form of the field equations obtained is an advancement over the one presented by Al-Ghothani [1986], insomuch as the present work contains coupling terms of the transverse shear constitutive equations between layers. Nonhomogeneous boundary conditions and internal jump discontinuities have been explicitly included in the general variational principles. Allowance for jump discontinuity terms in the variational formulation is meaningful in the context of direct approximation in finite element spaces since the space of approximants may not be sufficiently smooth. Extensions of the general variational principles through elimination of certain field operators and specializations by restricting some of the field equations and/or boundary conditions to be identically satisfied have been proposed.

Based on a special variational principle, a finite element procedure which uses $u_\alpha^{(1)}$, $\phi_\alpha^{(k)}$ and w as the field variables has been formulated and a finite element code has been developed. The computer program was used to study the effect of the constitutive coupling by solving the static problem of a laminated coupon under axial extension. The results were compared with the solution provided by Pagano [1978]. The displacement solution was seen to agree well. In addition to refinement of the Finite Element mesh, increased accuracy was seen to result from the division of each layer into sublayers. The results for inplane stresses compared well though those for

transverse shear and normal stresses did not. It was observed that refinement of the finite element mesh along the length of the free-edge delamination specimen did not significantly contribute to improvement in accuracy. However, refinement in the lateral direction (y-direction or the x_2 axis) near the free-edge gave considerable improvement. Also, refinement of layers into sublayers improved the results. This points to the desirability of using a higher order variation over the x_3 co-ordinate.

Though the constitutive relations have been derived from the equations of static equilibrium, the problem of free vibration of a sandwich plate was also studied by including inertial terms in the variation formulation. The fundamental frequency was compared with that obtained from an elastodynamic solution. It was observed that

1. In certain cases, a consistent shear correction improves the accuracy considerably in predicting natural frequencies over the shear correction by a simple factor which has been widely used in previous theories.
2. Improvement of accuracy depends upon the material properties of each layer. With increase in the difference of material properties in the individual layers, the significance of a consistent shear correction was more pronounced.

The higher frequencies were also studied and it was observed that the higher modes differed sharply from those obtained using a single shear correction factor.

These limited numerical tests illustrate the validity of a consistent shear correction procedure. Considering numerous design possibilities of composite laminates, use of a procedure to treat transverse shear deformation in a consistent manner rather than shear correction on an ad-hoc basis seems to be desirable.

REFERENCES

1. Al-Ghothani, A.M., *A UNIFIED APPROACH TO THE DYNAMICS OF BENDING AND EXTENSION OF MODERATELY THICK LAMINATED COMPOSITE PLATES*, Ph.D Dissertation, The Ohio State University, Columbus, Ohio, 1986.

2. Bhashyam, G.R. and Gallagher, R.H., *A Triangular Shear Flexible Finite Element for Moderately Thick Laminated Composite Plates*, Comp. Meth. Appl. Mech. Eng., Vol. 40, 309-320, 1983.

3. Green, A.E. and Naghdi, P.M., *A Theory of Laminated Composite Plates*, IMA J. Appl. Math., Vol. 29, 1-23, 1982.

4. Gurtin, S.G., *Variational Principles in the Linear Theory of Viscoelasticity*, Arch. Rat. Mech. Anal., Vol. 13, 179-191, 1963.

5. Gurtin, S.G., *Variational Principles in the Linear Elastodynamics*, Arch. Rat Mech. Anal., Vol. 16, 234-254, 1964.

6. Hong, S.J., *A CONSISTENT SHEAR DEFORMABLE THEORY FOR THE VIBRATION OF LAMINATED PLATES*, PhD. Dissertation, The Ohio State University, 1988.

7. Hughes, T.J.R. and Cohen, M., *The 'Heterosis' Finite Element for Plate Bending*, Computers & Structures, Vol. 9, 445-450, 1978.

8. Jones, A.T., *Exact Natural Frequencies for Cross-Ply Laminates*, J. Comp. Matls., Vol. 4, 476-491, 1970.

9. Lo, K.H., Christensen, R.M. and Wu, E.M., *A High-Order Theory of Plate Deformation, Part 2: Laminated Plates*, J. Appl. Mech., Vol. 44, 669-676, 1977.

10. Mawenya, A.S. and Davies, J.D., *Finite Element Bending Analysis of Multilayer Plates*, Int. J. Num. Methods Eng., Vol. 8, 215-225, 1974.

11. Mikhlin, S. G., *THE PROBLEM OF THE MINIMUM OF A QUADRATIC FUNCTIONAL*, Holden-Day, San Francisco, 1965.

12. Mindlin, R.D., *Influence of Rotatory Inertia and Shear Deformation on Flexural Motions of Isotropic, Elastic Plates*, J. Appl. Mech., Vol. 18, 31-38, 1951.

13. Nelson, R.B. and Lorch, D.R., *A Refined Theory for Laminated Orthotropic Plates*, J. Appl. Mech., Vol. 41, 177-183, 1974.

14. Pagano, N.J., *Exact solutions for Composite Laminates in Cylindrical Bending*, J. Comp. Matls., Vol. 3, 348-411, 1969.

15. Pagano, N.J., *Exact Solution for Rectangular Bidirectional Composites and Sandwitch Plates*, J. Comp. Matls., Vol. 4, 20-34, 1970a.

16. Pagano, N.J., *Influence of Shear Coupling in Cylindrical Bending of Anisotropic Laminates*, J. Comp. Matls., Vol. 4, 330-343, 1970b.

17. Pagano, N.J., *Stress Fields in Composite Laminates*, Int. J. Solids Struct., Vol. 14, 385-400, 1978.

18. Pagano, N.J. and Soni, S.R., *Global-Local Laminate Variational Model*, Int. J. Solids Struct., Vol. 19, 207-228, 1983.

19. Prager, W., *Variational Principles for Elastic Plates with Relaxed Continuity Requirements*, Int. J. Solids Struct., Vol. 4, 837-844, 1968.

20. Putcha, N.S. and Reddy, J.N., *Stability and Natural Vibration Analysis of Laminated Plates by using a Mixed Element based on a Refined Plate Theory*, J. Sound and Vib., Vol. 104, 285-300, 1986.

21. Reddy, J.N., *A Penalty Plate-Bending Element for the Analysis of Laminated Anistropic Composite Plates*, Int. J. Numer. Methods Engrg., Vol. 15, 1187-1206, 1980.

22. Reddy, J.N., *A Simple Higher-Order Theory for Laminated Composite Plates*, J. Appl. Mech., Vol. 51, 745-752, 1984.

23. Reddy, J.N. and Kuppusamy, T., *Natural Vibration of Laminated Anisotropic Plates*, J. Sound and Vib., Vol. 94, 63-69, 1984.

24. Reissner, E., *On Bending of Elastic Plates*, Quart. Appl. Math., Vol. 5, 55-68, 1947.

25. Reissner, E. and Stavsky, Y., *Bending and Stretching of Certain Types of Heterogeneous Aeolotropic Elastic Plates*, J. Appl. Mech., Vol. 28, 402-408, 1961.

26. Reissner, E., *On a Certain Mixed Variational Theorem and a Proposed Application*, Int. J. Numer. Methods Engrg., Vol. 20, 1366-1368, 1984.

27. Moazzami, M., Sandhu, R.S., and W.E. Wolfe, *A FINITE ELEMENT PROCEDURE FOR ANALYSIS OF LAMINATED PLATES* Technical Report WL-TR-91-3023, Flight Dynamics Directorate, Wright Laboratory, Wright-Patterson Air Force Base, Ohio, 1991.

28. Sandhu, R. S. and Pister, K. S., *A Variational Principle for Linear Coupled Field Problems*, Int. J. Engng. Sci., Vol. 8, 989-999, 1970.

29. Sandhu, R. S. and Pister, K. S., *Variational Principles for Boundary Value and Initial Boundary Value Problems*, Int. J. Solids. Struct., Vol. 7, 639-654, 1971.

30. Sandhu, R. S. and Salaam, U., *Variational Formulation of Linear Problems with Nonhomogeneous Boundary Conditions and Internal Discontinuities*, Comp. Meth. Appl. Mech. Eng., Vol. 7, 75-91, 1975.

31. Sandhu, R. S., *Variational Principles for Finite Element Approximations*, in FINITE ELEMENTS IN WATER RESOURCES ENGINEERING, G.F. Pinder and C.A. Brebbia (Eds.), Pentech Press, 1976.

32. Sandhu, R.S. and Hong, S.J., *Dynamics of Fluid-Saturated Soils- Variational Formulation*, Int. J. Numer. Anal. Meth. Eng., Vol. 11, 241-255, 1987.

33. Schoeppner, G., *Personal Communication*, 1990.

34. Seide, P., *An Approximate Theory for the Bending of Laminated Plates*, in MECAHNICS TODAY, N. Nasser (ed.), Vol. 5, 451-466, 1980.

35. Srinivas, S. and Rao, A.K., *Bending, Vibration and Buckling of Simply Supported Thick Orthotropic Rectangular Plates and Laminates*, Int. J. Solids Structs., Vol. 6, 1463-1482, 1970.

36. Srinivas, S., *Refined Analysis of Composite Laminates*, J. Sound and Vib., Vol. 30, 495-507, 1973.

37. Stavsky, Y., *Bending and Stretching of Laminated Aeolotropic Plates*, J. Eng. Mech. Div., ASCE, Vol. 97, 31-55, 1961.

38. Sun, C.T. and Whitney, J.M., *Theories for the Dynamic Response of Laminated Plates*, AIAA J., Vol. 11, 178-183, 1973.

39. Whitney, J.M., *The Effect of Transverse Shear Deformation on the Bending of Laminated Plates*, J. Appl. Mech., Vol. 36, 534-547, 1969.

40. Whitney, J.M. and Pagano, N.J., *Shear Deformation in Heterogeneous Anisotropic Plates*, J. Appl. Mech., Vol. 37, 1031-1036, 1970.

41. Whitney, J.M. and Sun, C.T., *A Higher Order Theory for Extensional Motion of Laminated Composites*, J. Sound and Vib., Vol. 30, 85-97, 1973.

42. Whitney, J.M. and Sun, C.T., *A Refined Theory for Laminated Anisotropic, Cylindrical Shells*, J. Appl. Mech., Vol. 41, 177-183, 1974.

43. Yang, P. C., Norris, C. H., and Stavsky, Y., *Elastic Wave Propagation in Heterogeneous Plates*, Int. J. Solids Struct., Vol. 2, 665-684, 1966.

Appendix A

VARIATIONAL FORMULATION OF LINEAR COUPLED PROBLEMS

The procedure for obtaining variational principles governing linear coupled boundary value problem is summarized following Sandhu [1970,1971,1975,1976]. The procedure can be considered as an extension of Mikhlin's [1965] basic variational theorem to coupled linear boundary value problems, including nonhomogeneous boundary conditions and internal discontinuities which may exist in the physical problem or arise in connection with numerical approximation procedures.

A.1 MATHEMATICAL PRELIMINARIES

A.1.1 Boundary Value Problem

Consider the boundary value problem

$$A(u) = f \qquad on \ \ R \tag{A.1}$$

$$C(u) = g \qquad on \ \ \partial R \tag{A.2}$$

where A and C are the linear, bounded operators, u is the field variable, R is an open connected region in an Euclidean space and ∂R is its boundary. Let V_R and $V_{\partial R}$ be linear vector spaces defined on the regions indicated by subscripts such that

$$f \in V_R \quad and \quad g \in V_{\partial R}$$

Then, the operators can be regarded as transformations defined over sets W_R, $W_{\partial R}$ such that

$$A: W_R \rightarrow V_R \tag{A.3}$$

$$C: W_{\partial R} \rightarrow V_{\partial R} \tag{A.4}$$

For A, C differential operators, W_R, $W_{\partial R}$ are, in general, dense subsets in V_R, $V_{\partial R}$, respectively.

A.1.2 Bilinear Mapping

Let V and S be linear vector spaces. A bilinear mapping $B(w,v)$: $V \times V \rightarrow S$ assigns an element in S to an ordered pair of elements $w,v \in V$ while preserving linearity. For convenience, we shall use the notation

$$B(w,v) \equiv \langle w,v \rangle \tag{A.5}$$

Bilinear mapping B is said to be nondegenerate if

$$\langle w, v \rangle = 0 \quad \textit{for all } w \in V \text{ if and } \textit{only} \text{ if } v = 0 \tag{A.6}$$

and symmetric if

$$\langle w, v \rangle = \langle v, w \rangle \quad \textit{for all } v,w \in V \tag{A.7}$$

A.1.3 Self-Adjoint Operator

Let $A: V \rightarrow V$ be an operator on the linear vector space V. Operator $A^*: V \rightarrow V$ is said to be adjoint of A with respect to a bilinear mapping $\langle \ , \ \rangle_R: V \times V \rightarrow S$, a linear vector space, if

$$\langle w, Av \rangle_R = \langle v, A^* w \rangle_R + D_{\partial R}(v,w) \tag{A.8}$$

for all $v,w \in V$. We assume here that the domain of A, A^*, a dense subset in V can be extended to V. Here, the subscript R of bilinear mapping indicates that V is defined over spatial region R and $D_{\partial R}(v,w)$ represents quantities associated with the boundary

∂R *of* R. If $A = A^*$, A is said to be self-adjoint. In particular, an operator A is said to be symmetric with respect to bilinear mapping if

$$<w, Av>_R = <v, Aw>_R \tag{A.9}$$

A.1.4 Gateaux Differential

Consider a continuous function $\Omega : V \to S$. Gateaux differential of Ω is defined as

$$\Delta_v \Omega(u) = \lim_{\lambda \to 0} \frac{1}{\lambda} [\Omega(u + \lambda v) - \Omega(u)] \tag{A.10}$$

provided the limit exists. Here, v is referred to as the 'path' and λ is a scalar. We note that for $u, v \in V$, $u + \lambda v \in V$. If the Gateaux differential exists at every point in a neighborhood of $v = u_0$

$$\Delta_v \Omega(u) = \frac{d}{d\lambda} \Omega(u + \lambda v)|_{\lambda = 0}$$

A.2 *BASIC VARIATIONAL PRINCIPLE*

For the boundary value problem (A-1) with homogeneous boundary condition, Mikhlin [1965] used the inner product as the nondegenerate symmetric bilinear mapping on the linear vector space of square integrable functions and showed that the unique solution u_o minimizes the functional

$$\Omega(u) = <Au, u>_R - 2<u, f>_R \tag{A.11}$$

if the linear operator A is positive definite and self-adjoint. Conversely, u_o which minimizes the functional (A-11) is the solution of the problem (A-1).

Taking Gateaux differential of the function (A-11),

$$\Delta_v \Omega(u) = \lim_{\lambda \to 0} \frac{1}{\lambda} [<A(u + \lambda v>, u + \lambda v> - 2 <u + \lambda v> \quad <Au, u> + 2 <u, f>]$$

$$= <Au, v> + <Av, u> - 2 <v, f>$$

$$= 2\langle v,\ Au - f \rangle \qquad \text{(A.12)}$$

The Gateaux differential evidently vanishes at the solution $u = u_o$ such that $Au_o - f = 0$. Also, since the bilinear mapping is nondegenerate, vanishing of the Gateaux differential for all v implies $Au_o - f = 0$. To prove the minimization property, the bilinear mapping has to be into the real line and the operator must be positive. However, in general, it is only necessary to use vanishing of Gateaux differential as equivalent to (A-1) being satisfied.

A.3 VARIATIONAL FORMULATION OF THE COUPLED PROBLEM

Through generalization of Mikhlin's theorem, Sandhu [1970,1971,1975,1976] constructed a framework to handle the inverse problem of variational calculus for the linear coupled problem with multiple field variables. To include the nonhomogeneous boundary conditions and internal discontinuity conditions explicitly in the formulation, the concept of consistent boundary operators was introduced.

Consider the coupled boundary value problem with multiple field variables

$$\mathbf{A}(\mathbf{u}) = \mathbf{f} \qquad \text{on } R \qquad \text{(A.13)}$$

$$\mathbf{C}(\mathbf{u}) = \mathbf{g} \qquad \text{on } \partial R \qquad \text{(A.14)}$$

in which $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{u}$ are, respectively, the field operator matrix, the boundary operator matrix and the vector of field variables, and $\mathbf{f}$, $\mathbf{g}$ are the vectors of known forcing functions. Explicitly,

$$\sum_{j=1}^{n} A_{ij} u_j = f_i \qquad \text{on } R \qquad \text{(A.15)}$$

$$\sum_{j=1}^{n} C_{ij} u_j = g_i \qquad \text{on } \partial R_i, \qquad i = 1, 2, \ldots, n \qquad \text{(A.16)}$$

in which ∂R_i denote segments of ∂R such that

$$\partial R = \bigcup_{i=1}^{n} \partial R_i \tag{A.17}$$

and n is the number of independent field variables. Operators A_{ij} are regarded as the transformations

$$A_{ij} : M_{ij} \rightarrow P_{ij} \tag{A.18}$$

where

$$u_i \in W_i = \bigcap_{j=1}^{n} M_{ij} \tag{A.19}$$

$$f_i \in V_i = \bigcup_{j=1}^{n} P_{ij} \tag{A.20}$$

Thus, the range of A_{ij} constitutes a product space

$$V = V_1 x V_2 x . \; . \; . \; . \; . \; V_n \tag{A.21}$$

Let V be a vector space defined as the direct sum

$$V = V_1 + V_2 + . \; . \; . \; . \; . \; . \; . + V_n \tag{A.22}$$

and an element $u \in V$ be the ordered set

$$u = \{u_1, u_2, . \; . \; . \; . \; . \; , u_n\} \tag{A.23}$$

such that $u_i \in V_i$. Then, a bilinear mapping on V_i may be defined as

$$< u, v >_R = < u_1, v_1 >_R + . \; . \; . \; . \; . \; . \; . + < u_n, v_n >_R \tag{A.24}$$

The set of operators A_{ij} is said to be self-adjoint with respect to this bilinear mapping if [Sandhu, 1976]

$$\sum_{j=1}^{n} < u_j, A_{ji} u_i >_R = < u_i, \sum_{j=1}^{n} A_{ij} u_j >_R + D_{\partial R}(u_i, u_j) \tag{A.25}$$

where $D_{\partial R}(u_i, u_j)$ denote quantities associated with ∂R. If the set of operators A_{ij} is self adjoint, as a generalization of Mikhlin's theorem, the function governing the problem (A-13) and (A-14) was defined as

$$\Omega = \sum_{i=1}^{n}\sum_{j=1}^{n}\{<u_i, A_{ij}\mu_j - 2f_i>_R + <u_i, C_{ij}\mu_j - 2g_i>_{\partial R}\} \tag{A.26}$$

For vanishing of the Gateaux differential of this function to imply (A-13) and (A-14), the boundary operators C_{ij} must be consistent with the field operators A_{ij}. Sandhu [1976] stated the consistency condition of the boundary operators as

$$D_{\partial R}(u_i, u_j) = <u_i, \sum_{j=1}^{n} C_{ij}u_j>_{\partial R} - \sum_{j=1}^{n} <u_j, C_{ji}u_i>_{\partial R} \tag{A.27}$$

In other words, for (A-26) to be the governing function in variational formulation of the problem given by (A-13) and (A-14), the boundary operators must satisfy (A-27). Sandhu [1975] showed that appropriate boundary terms should be included in the formulation even if the boundary conditions are homogeneous.

In applying the finite element method to obtain an approximate solution, the approximation space may have limited smoothness, e.g., when nonconforming elements are used. Prager [1968] was the first to discuss this aspect in connection with variational formulation. To allow for this, Sandhu [1975] wrote the internal discontinuity conditions in the form

$$(C_{ij}\mu_j)' = g'_i \qquad on \ \partial R_i \tag{A.28}$$

where a prime denotes the internal jump discontinuity along internal surfaces ∂R_i embedded in the region R. Since (A-28) has the same form as the boundary conditions, it is apparent that this condition can be included in the governing function in the same way. The governing function allowing for (A-28) is

$$\Omega = \sum_{i=1}^{n'} \sum_{j=1}^{n} \{ <u_i, A_{ij}\mu_j - 2f_i>_R + <u_i, C_{ij}\mu_j - 2g_i>_{\partial R} + <u_i, (C_{ij}\mu_j)' - 2g'_i>_{\partial R'_i} \} \quad (A.29)$$

This is the general form of the governing function in the variational formulation of linear coupled boundary value problem with multiple field variables. The essential step in setting up a variational formulation of a boundary value problem is to write the field equations in a form that the matrix of field operators is self-adjoint with respect to certain bilinear mapping and the boundary conditions are consistent with the field operators. The procedure is also applicable to initial-boundary value problems using appropriate bilinear mappings.

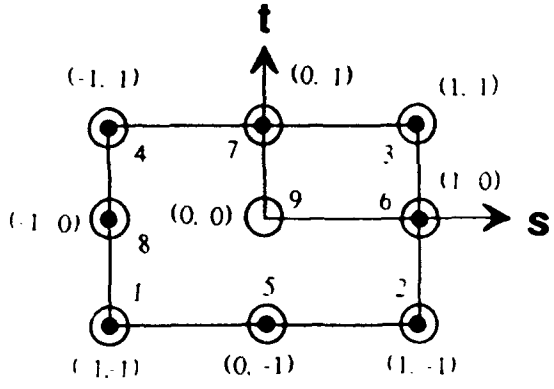
Appendix B

EVALUATION OF ELEMENT MATRICES

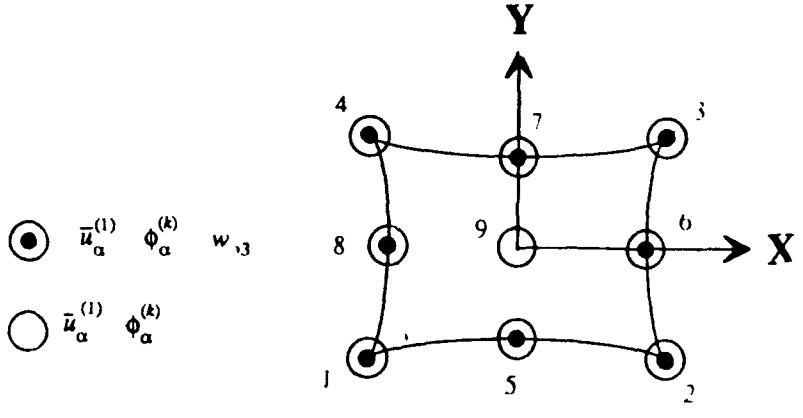
B.1 INTERPOLATION FUNCTIONS OF THE 'HETEROSIS' ELEMENT

The computer program developed incorporated the 'Heterosis' plate bending element [Hughes 1978] along with reduced/selective integration technique. The element matrices can be formed following the usual procedure of isoparametric element formulation. However, the 'Heterosis' element differs from other isoparametric elements in using different interpolation scheme for lateral displacement on the one hand and inplane displacement and rotations of the cross-section on the other. In-plane displacements $u_\alpha^{(1)}$ and rotations of cross-section $\phi_\alpha^{(k)}$ are approximated by qaudratic functions for 8-node isoparametric element while the lateral displacement w is approximated by 9-node Lagrange interpolation functions. Consequently, the number of degrees of freedom at the center node is less than that at other nodes by one. Interpolation scheme of the 'Heterosis' element is shown in Figure B.1. Using such interpolation scheme, Hughes [1978] was able to avoid spurious zero energy mode of stiffness matrix which can be caused by use of reduced integration.

Interpolations functions of 8-node isoparametric element and 9-node Lagrange element in terms of natural coordinates (s,t) and their derivatives with respect to s and t are given below.



(a) Local



(b) Global

Figure B.1: Global and Local Coordinate Systems of 'Heterosis' Element

$$\mathbf{N} = \frac{1}{4}\begin{Bmatrix} (1-s)(1-t)(-1-s-t) \\ (1+s)(1-t)(-1+s-t) \\ (1+s)(1+t)(-1+s+t) \\ (1-s)(1+t)(-1-s+t) \\ 2(1-s^2)(1-t) \\ 2(1+s)(1-t^2) \\ 2(1-s^2)(1+t) \\ 2(1-s)(1-t^2) \end{Bmatrix} \quad \frac{\partial \mathbf{N}}{\partial s} = \frac{1}{4}\begin{Bmatrix} (1-t)(2s+t) \\ (1-t)(2s-t) \\ (1+t)(2s+t) \\ (1+t)(2s-t) \\ -4s(1-t) \\ 2(1-t^2) \\ -4s(1+t) \\ -2(1-t^2) \end{Bmatrix} \quad \frac{\partial \mathbf{N}}{\partial t} = \frac{1}{4}\begin{Bmatrix} (1-s)(2t+s) \\ (1+s)(2t-s) \\ (1+s)(2t+s) \\ (1-s)(2t-s) \\ -2(1-s^2) \\ -4t(1+s) \\ 2(1-s^2) \\ -4t(1-s) \end{Bmatrix} \tag{B.1}$$

$$\mathbf{L} = \frac{1}{4}\begin{Bmatrix} st(1-s)(1-t) \\ st(1+s)(t-1) \\ st(1+s)(1+t) \\ st(s-1)(t+1) \\ 2t(1-s^2)(t-1) \\ 2s(s+1)(1-t^2) \\ 2t(1-s^2)(t+1) \\ 2s(s-1)(1-t^2) \\ 4(1-s^2)(1-t^2) \end{Bmatrix} \quad \frac{\partial \mathbf{L}}{\partial s} = \frac{1}{4}\begin{Bmatrix} t(2s-1)(t-1) \\ t(2s+1)(t-1) \\ t(2s+1)(t+1) \\ t(2s-1)(t+1) \\ 4st(1-t) \\ 2(2s+1)(1-t^2) \\ -4st(t+1) \\ 2(2s-1)(1-t^2) \\ 8s(t^2-1) \end{Bmatrix} \quad \frac{\partial \mathbf{L}}{\partial t} = \frac{1}{4}\begin{Bmatrix} s(2t-1)(s-1) \\ s(2t-1)(s+1) \\ s(2t+1)(s+1) \\ s(2t+1)(s-1) \\ 2(2t+1)(1-s^2) \\ -4st(s+1) \\ 2(2t+1)(1-s^2) \\ 4st(1-s) \\ 8t(s^2-1) \end{Bmatrix} \tag{B.2}$$

Here, **N** and **L** denote interpolation functions for 8-node isoparametric and 9-node Lagrange element, respectively.

B.2 EVALUATION OF STIFFNESS AND MASS MATRICES

Since the field variables are interpolated over an element in natural coordinates (s,t), it is necessray to set up the relation of the global coordinates and natural (local) coordinates for evaluation of the element matrices defined in Section VI. We consider a mapping of global coordinate system (x_1, x_2) to local coordinate system (s,t). We assume that this mapping is one-to-one and onto. By chain rule, the derivative in each coordinate system is related by

$$\begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \qquad or \qquad \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} \tag{B.3}$$

where Jacobian matrix **J** and its inverse is defined as

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \qquad and \qquad \mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{bmatrix} \tag{B.4}$$

Here, $|\mathbf{J}|$ is the determinant of Jacobian matrix. Using (B-3) and (B-4), one can obtain the expressions of the matrices $\mathbf{T}_u$, $\mathbf{T}_\phi$ and $\mathbf{T}_s$ defined in (214)-(216) in natural coordinates. Following the concept of isoparametric formulation, global coordinates are interpolated over an element as

$$x = \Psi^T \underline{x} \tag{B.5}$$

where Ψ is the vector of interpolation functions used for field variable. $\underline{x}$ is the vector of global coordinate values at nodal points. Since in the 'Heterosis' element different interpolations functions **N** and **L** are used for $\phi_\alpha^{(k)}$ and w, Ψ in (B-5) must be **L** for evaluating $\mathbf{T}_u$ and $\mathbf{T}_\phi$ while be **N** for $\mathbf{T}_s$.

Using (B-1)-(B-5) and the interpolation functions defined in (211)-(216),

$$\mathbf{H}_u = \mathbf{H}_\phi \begin{bmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \tag{B.6}$$

$$\mathbf{H}_s = \mathbf{N} \tag{B.7}$$

$$\mathbf{T}_u^T = \mathbf{T}_\phi^T = \frac{1}{|\mathbf{J}_L|} \begin{bmatrix} \underline{y}^T & 0 \\ 0 & -\underline{x}^T \\ -\underline{x}^T & \underline{y}^T \end{bmatrix} \mathbf{P} \tag{B.8}$$

$$\mathbf{T}_s^T = \frac{1}{|\mathbf{J}_N|} \begin{bmatrix} \underline{y}^T \\ -\underline{x}^T \end{bmatrix} \mathbf{R} \tag{B.9}$$

where

$$|\mathbf{J}_L| = -\underline{x}^T \mathbf{P}\, \underline{y}$$

$$|\mathbf{J}_N| = -\underline{x}^T \mathbf{R}\, \underline{y}$$

$$\mathbf{P} = \mathbf{L}_{,t}\mathbf{L}_{,s}^T - \mathbf{L}_{,s}\mathbf{L}_{,t}^T$$

$$\mathbf{R} = \mathbf{N}_{,t}\mathbf{N}_{,s}^T - \mathbf{N}_{,s}\mathbf{N}_{,t}^T$$

Here, a subscripted comma denotes partial differentiation with respect to the variables following the comma.

In element matrices given in (218)-(233), the integrands are functions of natural coordinates (s-t). Therefore, the surface integration extends over the natural coordinate surface. Since, in general,

$$dR = |\mathbf{J}|\, ds\, dt \tag{B.10}$$

integration in each coordinate system is related by

$$\int\int_A F(x_\alpha)\, dx\, dy = \int_{-1}^{1}\int_{-1}^{1} F(s,t)\, |\mathbf{J}| ds\, dt \tag{B.11}$$

Using Gaussian quadrature

$$\int_R F(x,y)\, dR = \sum_{i=1}^{m}\sum_{j=1}^{m} F(s_{ij},t_{ij})\, |\mathbf{J}_{ij}|\, |W_{ij}| \tag{B.12}$$

where m is the number of Gaussian quadrature points and W_{ij} are weighting values. Here, it should be mentioned that the in the 'Heterosis' element numerical integration was performed by selective/reduced integration technique, viz., two-point Gaussian quadrature for evaluation of transverse strain energy term while three-point quadrature is used for other quantities. Therefore, (231)-(233) were evaluated by two-point quadrature and the remaining quantities were evaluated by three-point quadrature.

☆ U.S. Government Printing Office 1992—648-127/62351