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Henry E. Kyburg, Jr.

Abstract

Four main results are arrived at in this paper. (1) Closed convex sets of classical probability functions provide a representation of belief that includes the representations provided by Shafer probability mass functions as a special case. (2) The impact of "uncertain evidence" can be (formally) represented by Dempster conditioning, in Shafer's framework. (3) The impact of "uncertain evidence" can be (formally) represented in the framework of convex sets of classical probabilities by classical conditionalization. (4) The probability intervals that result from Dempster/Shafer updating on uncertain evidence are included in (and may be properly included in) the intervals that result from Bayesian updating on uncertain evidence.

Bayesian and Non-Bayesian Evidential Updating*

1. Recent work in both vision systems (Wesley) and in knowledge representation (Lowrance, Barnett, Quinlan, Dillard) has employed an alternative, often referred to as Dempster/Shafer updating, to classical Bayesian updating of uncertain knowledge. Various other investigators have gone beyond classical Bayesian conditionalization (MYCIN, EMYCIN, DENDRAL,...) but in a less systematic manner. It is appropriate to examine the formal relations between various Bayesian and non-Bayesian approaches to what has come to be called evidence theory, in order to explore the question of whether the new techniques are really more powerful than the old, and the question of whether, if they are, this increment of power is bought at too high a price.

2. Orthodox probability theory supposes (1) that we commence with known statistical distributions, (2) that these distributions are such as to give rise to real-valued probabilities, and (3) that these probabilities can be modified by using Bayes' theorem to conditionalize on evidence that is taken to be certain. There are thus three ways to modify the classical theory.

We may dispense with the supposition that we are dealing with known statistical distributions. The best known advocate of this gambit was L. J. Savage, who argued that probabilities represent personal, subjective, opinions, and not objective distributions of quantities in the world. This approach has

given rise to Bayesian statistics, based on that fact that the opinions of most people are such that, faced with frequency data, they will converge reasonably rapidly. Furthermore, in practice, it is common to recognize that some opinions are better than others, and to use as prior distributions in statistical inference distributions representing the opinions of knowledgeable experts. This approach has been incorporated in some expert systems, for example, PROSPECTOR. It has both virtues and limitations. A purely pragmatic virtue is that it allows us to get on with our business even when we don't have the knowledge of prior distributions we would like to have. It has the practical virtue that the considered opinions of genuinely knowledgeable experts are formed in response to, and reflect with some degree of accuracy, relative frequencies in nature. But it has two drawbacks: it does not incorporate any indication of whether the opinion is a wild guess, or a considered judgement based on long experience; and it calls for expert opinions even in the face of total, acknowledged ignorance.

This suggests the second departure from the classical picture; abandoning the assumption that our probabilities are point-valued. This has recently been hailed as a novel departure (Lowrance, 1982, p. 21; Garvey, et. al., 1981, p. 319; Dillard, 1982, p. 1; Lowrance and Garvey, 1982, p. 7; Wesley and Hanson, 1982, p. 16; Quinlan, 1982, p. 9). The idea of representing probabilities by intervals is not new (cf. Kyburg, Good, Levi, Smith), and the notion of probabilities that constitute a field richer than that of the real numbers goes back even further

(Keynes, 1921, pp. 38-40, offers a formal philosophical treatment of such entities; B. O. Koopman, 1941, 1942, offers a mathematical characterization). Even the standard subjectivistic or personalist view of probability can be construed in this way; while each person has a set of real-valued probabilities defined over a given field, a group of people will reflect a set of probability functions defined over the field. We may quite reasonably focus our attention on the maximum and minimum of these functions evaluated at a member of the field.[1]

In general the representation in terms of intervals seems superior to the representation in terms of point values. Even in the ideal case, in which all of our measures are based on statistical inference from suitably massive quantities of data, it is most natural to construe these measures as being constrained by intervals. In confidence interval estimation, for example, what we get from our statistics is a high confidence that a given parameter is contained in a certain interval. This translates neatly and conveniently into an interval constraint. The results of statistical inference should reflect indeterminacy or vagueness. What we can properly claim to know is not that a parameter has certain value, but (with probability or high confidence) that it lies within certain limits. This limitation of human knowledge should surely be mirrored in computer based systems.

The third departure from the classical scheme is to consider alternatives to Bayes' theorem as a way of updating probabilities in the light of new evidence. This departure is recent, and was first stated in Dempster, 1967. Dempster's novel rule of

combination, subsequently adopted by Shafer (1976), is often referred to as a "generalization" of Bayesian inference (Shafer, 1981, p. 337: "The theory of belief functions ... is a thoroughgoing generalization of the Bayesian theory ..."; Lowrance and Garvey, 1982, p. 9: "Dempster's rule can be viewed as a direct generalization of Bayes' rule ..."; Dillard, 1982, p. 1; Garvey, et. al., p. 319; Lowrance 1982, p. 21). This suggests, on the one hand, that Bayes' rule can be regarded as a special or limiting case of Dempster's rule, which is true, and on the other hand that Dempster's rule can be applied where Bayes' rule cannot, which is false. Dempster himself recognizes (1967, 1968) that his rule results from the imposition of additional constraints on the Bayesian analysis (see note 5).

One criticism of the usual Bayesian approach to evidential updating is the quantity of information that may be required to specify the probability function covering the field of propositions with which we are concerned. This may be <u>empirical</u> information (if the underlying probabilities are thought of as being based on statistical knowledge), <u>psychological</u> information (if a personalistic interpretation of probability is adopted), or <u>logical</u> information (if we interpret probability as degree of confirmation, <u>a la</u> Carnap 1951). Suppose we consider a field of propositions based on the logically independent propositions $\underline{p_1 \cdots p_n}$; the set of what Carnap called "state descriptions:" induced by this basis consists of $\underline{2^n}$ atoms, each of which is the conjunction of the $\underline{n}$ (negated or unnegated) $\underline{p_i}$. It is obvious that for reasonably large $\underline{n}$ this assignment of probabilities

presents great difficulties. But once we have those $2^{\underline{n}}$ numbers, we're done - we can calculate all conditional probabilities as well as the probability of any proposition in the field based on $\underline{P}_1 \cdots \underline{P}_n$.

Is there a saving in effort if we go to a Dempster/Shafer System? Using the handy representation in Shafer (1976), we take θ, the universal set, to be the set of all $2^{\underline{n}}$ possibilities represented by the state descriptions, and assign a mass to each subset of θ. This requires 2 exp $2^{\underline{n}}$ assignments! As far as the number of parameters to be taken account of is concerned, we are exponentially worse off. But if we construe probabilities as intervals, or represent them by sets of simple probability functions, we are just as badly off. (For an example relating mass assignments to interval assignments, see table I in the appendix. For the general equivalence, see theorem 1 below.) Dillard (p. 4) refers to "computational limitations" and Lowrance and Garvey (1982) mention that with large θ, maintaining the model is "computationally infeasible".

In either case, we need to find some systematic and computationally feasible procedure for obtaining the masses or probabilities we need. Bayesian and non-Bayesian approaches are in essentially the same difficult situation in this respect, although there are often plausible ways of systematizing the parameter assignments on either view.

3. Whether the representation of our initial knowledge state is given by an assignment of masses to subsets of θ or by a set of classical probability distributions over the atoms of θ, it

is important that these masses or probabilities be justifiable. As already suggested, a straightforward way of obtaining them is through statistical inference, which (when possible) yields interval valued estimates of relative frequencies. But there may also be other ways to obtain masses or intervals of probability. If so, then the deep and difficult problem arises of how to combine both statistical and non-statistical sources of information.[2]

It has been suggested that Dempster/Shafer updating relieves us of the necessity of making assumptions about the joint probabilities of the objects we are concerned about. Thus, Quinlan claims that INFERNO "makes no assumptions whatever about the joint probability distributions of pieces of knowledge ..." (Quinlan 1982). Other writers have made similar claims — e.g., Wesley and Hanson, 1982, p. 15. (To make independence assumptions is exactly to make assumptions about joint probability distributions.)

It is clear that the assignment of masses to subsets of θ invloves just as much in the way of "assumptions" as the assignment of a priori probabilities to the corresponding propositions. In view of the reducibility of the Dempster/Shafer formalism to the formalism provided by convex sets of classical probability functions (to be shown below), moreover, we may recapture the assumptions about joint probability distributions from the convex Bayesian representation.

4. One important novelty claimed for the Dempster/Schafer

system is its ability to handle uncertain evidence. But even this is not in itself anti-Bayesian. there are also Bayesian methods for handling uncertain evidence. One of these, used in PROSPECTOR and mentioned by Lowrance (1982, p. 17) is known in the philosophical world as Jeffrey's rule. (It is presented and discussed in Jeffrey, 1965.) It follows from Bayes' theorem that

$$P(A) = P(A/B)P(B) + P(A/-B)P(-B).$$

If you adopt a new (coherent) probability function P', there are essentially no constraints on $P'(A)$. But one often confronts situations where if a shift in probability originates in the assignment of a new probability to B, that should not affect the conditional probability of A given B: $P(A/B) = P'(A/B)$. We have learned something new about B, but we haven't learned anything new about the bearing of the truth of B on the truth of A.

Given such a situation the response of a shift in the probability of B from $P(B)$ to $P'(B)$, resulting from new evidence, should propagate itself according to:

$$P(A) = P(A/B)P'(B) + P(A/-B)P'(-B)$$

When new evidence leads us to shift our credence in B from $P(B)$ to $P'(B)$, a corresponding shift in probability is induced for every other proposition in the field: the new probability of a proposition A is the weighted average of the probability of A, given B, and the probability of A given not-B, weighted by the new probabilities of B and not-B.

Lowrance (1982) worries about the problem of iterating this move. Having made it, should we then update the probability of B in the light of the new probability P'(A)? Wesley and Hanson (1982, p. 15) worry about a potential "violation of Bayes' law". But what is offered is not a relaxation method; it is a method of evaluating the impact of evidence which warrants a shift in the support for B. It makes no sense to consider updating P'(B) in the light of the new value of P(A); P'(B) is the source of the updating. No contradiction lurks here. But there is a difficulty for mechanical updating - the notion of a source is clear to us, but may not even be represented in an artificial system.

Other Bayesian updating procedures are possible (cf. Hartry Field, 1978; Diaconis and Zabell, 1982), but it is hard to think of one so simple and often so natural. This is particularly true in the epistemological framework considered by Shafer; the weights of the subsets of θ assigned masses reflect our a priori intuitions; there is no way in which the values of these masses, given our observations, can be changed without changing the model entirely. What impact given evidence has should not also change according to the evidence we happen to have. Shafer himself has explored the relation between Jeffrey's rule and his own updating recommendations in (Shafer 1981).

5. In order to investigate more closely the relations between the Bayesian and Dempster/Shafer strategies for updating, it will be helpful to have several formal results. In the present section we establish the partial equivalence between the assignment of masses to subsets of θ (the space of possibilities)

and the assignment of a convex set of simple classical probability functions defined over the atoms of Θ. The equivalence is only partial, since some plausible situations do not have a natural representation in terms of mass functions.

Shafer's belief functions are defined relative to a frame of discernment Θ, and are given by either a belief function or a mass function defined over the subsets of Θ. The atoms of Θ are the most specific states of affairs that concern us in a given context. The belief function $\underline{Bel}$ and the mass function $\underline{m}$ are related by:

$$\underline{Bel}(\underline{X}) = \sum_{\underline{A} \subset \underline{X}} \underline{m}(\underline{A})$$

Throughout, "$\emptyset$" is to be understood as allowing improper inclusion. Proofs have been relegated to Appendix A.)

Our first observation is that to every belief function defined over a frame of discernment, there corresponds a closed set of classical probability functions $\underline{S_p}$ defined over the atoms of Θ such that for any $\underline{X} \subset \Theta$,

$$\underline{Bel}(\underline{X}) = \min_{\underline{P} \in \underline{S_p}} \underline{P}(\underline{X}).$$

This result is stated as Theorem 1 in appendix A, and proved there. The proof gives a way of constructing members of the set of classical probability functions, but the intuitive idea is simply this: Consider a set $\underline{X}$, to which is assigned mass $\underline{m}(\underline{X})$. That mass may be construed as probability mass that may be assigned in any way (subject to other constraints) to the atoms

of $\underline{X}$. We obtain the set of classical probability functions that corresponds to the mass function $\underline{m}$ by considering all the ways in which the mass that is not assigned to atoms by $\underline{m}$ can be assigned to atoms while maintaining the constraints imposed by the assignment of mass to sets of atoms. Tables I and II in the appendix show both the general and a specific computation for a simple four-atom frame of discernment.

An example that shows the converse does not hold is the following[3]: Consider a compound experiment consisting of either (1) tossing a fair coin twice, or (2) drawing a coin from a bag containing 40% two headed and 60% two tailed coins and tossing it twice. The two parts (1) and (2) are performed in some unknown ratio $\underline{P}$, so that, for example the probability that the first toss lands heads is $\underline{p}*1/2 + (1-\underline{p})*0.4$, $0 < \underline{p} < 1$. Let $\underline{A}$ be the event that the first toss lands heads, and $\underline{B}$ the event that the second toss lands tails. The representation by a convex set of probability functions is straight-forward, but where $\underline{P}_*$ is $\min_{\underline{P} \in \underline{S}_{\underline{p}}}$,

$$\underline{P}_*(\underline{A} \cup \underline{B}) = 0.75 \quad < 0.9 = \underline{P}_*(\underline{A}) + \underline{P}_*(\underline{B}) - \underline{P}_*(\underline{A} \cap \underline{B}) = .4 + .5 - 0$$

By theorem 2.1 of Shafer 1976, $\underline{Bel}(\underline{A} \cup \underline{B}) \geq \underline{Bel}(\underline{A}) + \underline{Bel}(\underline{B}) - \underline{Bel}(\underline{A} \cap \underline{B})$, $\underline{P}_*$ is therefore not a belief function. It is possible to compute a mass function, but the masses assigned to the union of any three atoms must be negative.

Subject to the condition, however, that $\underline{P}_*(\underline{A} \cup \underline{B}) \geq \underline{P}_*(\underline{A}) + \underline{P}_*(\underline{B}) - \underline{P}_*(\underline{A} \cap \underline{B})$, we can represent any closed convex set of classical probability functions by a Shafer mass function. This

is theorem 2 in the appendix.

These two theorems show that the representation of uncertain knowledge provided by Shafer's probability mass functions is equivalent to a representation provided by a convex set of classical probability functions, and that the representation of uncertain knowledge by a convex set of classical probability functions is equivalent to a representation provided by a probability mass function so long as the convex set of probability functions satisfies the general relation $\underline{P}_*(\underline{A} \cup \underline{B}) \geq \underline{P}_*(\underline{A}) + \underline{P}_*(\underline{B}) - \underline{P}_*(\underline{A} \cap \underline{B})$.

6. Of more interest that the mere representation of belief is the possibility of representing the way that beliefs should change in response to new evidence. Thus what we propose to look at in this section is the relation between Dempster/Shafer updating, and convex Bayesian updating. We shall first look at the relation in the case of evidence that is "certain"; and then we shall look at it in the case of "uncertain evidence".

Suppose that our beliefs can be represented either by a closed convex set of classical probability functions $S_{\underline{P}}$, or by a Shafer mass function. Let $\underline{B}$ be evidence assigned probability 1, or support 1. Shafer defines upper and lower conditional support functions thus:

$$\underline{\text{Bel}}(\underline{A} \,|\, \underline{B}) = (\underline{\text{Bel}}(\underline{A} \cup \overline{\underline{B}}) - \underline{\text{Bel}}(\overline{\underline{B}}))/(1 - \underline{\text{Bel}}(\overline{\underline{B}}))$$

$$\underline{P}^*(\underline{A} \,|\, \underline{B}) = \underline{P}^*(\underline{A} \cap \underline{B})/\underline{P}^*(\underline{B}),$$

where $\underline{P}^*(\underline{X}) = 1 - \underline{Bel}(\overline{\underline{X}})$ is called the plausibility of $\underline{X}$.

Theorem 3 in the appendix shows that the following inequalities hold:

$$\underline{\min}_{P \in S_p} \underline{P}(\underline{A}|\underline{B}) \leq \underline{Bel}(\underline{A}|\underline{B}) \leq \underline{P}^*(\underline{A}|\underline{B}) \leq \underline{\max}_{P \in S_p} \underline{P}(\underline{A}|\underline{B})$$

For the case of a frame of discernment with four atoms, illustrated in table 1 of the appendix, we have the following, where $\underline{X}_i$ is the mass assigned to the set $\underline{i}$ in θ, $\underline{X}_{ij}$ is the mass assigned to the union of sets $\underline{i}$ abd $\underline{j}$, etc.

$$\underline{\min}\ \underline{P}(\underline{A}|\underline{B}) \quad \frac{\underline{X}_1}{(\underline{X}_1+\underline{X}_3)+(\underline{X}_{13}+\underline{X}_{23}+\underline{X}_{34})+(\underline{X}_{123}+\underline{X}_{134}+\underline{X}_{234})+\underline{X}_\theta}$$

$$\underline{Bel}(\underline{A}|\underline{B}) = \frac{\underline{X}_1+[\underline{X}_{12}+\underline{X}_{14}+\underline{X}_{124}]}{(\underline{X}_1+\underline{X}_3)+(\underline{X}_{13}+\underline{X}_{23}+\underline{X}_{34})+(\underline{X}_{123}+\underline{X}_{134}+\underline{X}_{234})+\underline{X}_\theta+[\underline{X}_{12}+\underline{X}_{14}+\underline{X}_{124}]}$$

$$\underline{P}^*(\underline{A}|\underline{B}) = \frac{\underline{X}_1+(\underline{X}_{12}+\underline{X}_{13}+\underline{X}_{14})+(\underline{X}_{123}+\underline{X}_{124}+\underline{X}_{134})+\underline{X}_\theta}{(\underline{X}_1+\underline{X}_3)+(\underline{X}_{12}+\underline{X}_{13}+\underline{X}_{14})+(\underline{X}_{123}+\underline{X}_{124}+\underline{X}_{134})+\underline{X}_\theta+[\underline{X}_{23}+\underline{X}_{34}+\underline{X}_{234}]}$$

$$\underline{\max}\ \underline{P}(\underline{A}|\underline{B}) = \frac{X_1+(X_{12}+X_{13}+X_{14})+(X_{123}+X_{124}+X_{134})+\underline{X}_\theta}{(X_1+X_3)+(X_{12}+X_{13}+X_{14})+(X_{123}+X_{124}+X_{134})+\underline{X}_\theta}$$

We observe that:

(1) $\underline{\min}\ \underline{P}(\underline{A}|\underline{B}) = \underline{Bel}(\underline{A}|\underline{B})$ iff $\underline{X}_{12}+\underline{X}_{14}+\underline{X}_{124} = 0$

(2) $\underline{Bel}(\underline{A}|\underline{B}) = \underline{P}^*(\underline{A}|\underline{B})$ iff $\underline{X}_{13}+\underline{X}_{123}\underline{X}_{134}+\underline{X}_\theta = 0$

(3) $\underline{\max}\ \underline{P}(\underline{A}|\underline{B}) = \underline{P}^*(\underline{A}|\underline{B})$ iff $\underline{X}_{23}+\underline{X}_{34}+\underline{X}_{234} = 0$

Before turning to a discussion of the inequalities of theorem 3, we show that they hold in general, and are not restricted to the case of "certain" evidence. Given two lemmas, the proof of the general result (theorem 4 of the appendix) is trivial. The two lemmas themselves may not be without interest.

7. The first lemma (lemma 1 of the appendix) states that by expanding the frame of discernment θ, we can represent the impact of uncertain evidence as the impact of "certain" evidence. This is not to say that we need to specify that evidence; it is that there is an algorithm by means of which the impact of the uncertain evidence can be represented as the impact of other "certain" evidence.

The general idea of the argument is this. Suppose that θ is the frame of discernment, and that our initial belief function is $\underline{Bel}_1$. The impact of uncertain evidence can be represented by a simple support function $\underline{Bel}_C$ whose single focus is $C \in 2^\theta$, to which $\underline{Bel}_C$ attributes mass s (and therefore mass $1-s$ to θ). To give a representation by "certain" evidence, we split every atom of θ into two new atoms to obtain θ'. We define a new belief function on θ', $\underline{Bel}_1'$, which is such that

(a) if $X \subset \theta$, $\underline{Bel}_1'(X) = \underline{Bel}_1(X)$

(b) if $X \subset \theta$, $(\underline{Bel}_1 \oplus \underline{Bel}_C)(X) = \underline{Bel}_1'(X \mid E)$, where E is a subset of θ' such that the evidence partially supporting C provides total support for E.

Two remarks on this construction are in order. First, we

have given no rule for finding the "possibility" E. But in general that should be no problem. Suppose C is the proposition that there is a squirrel on the roof of the barn. The light is bad, so Bel_C assigns a mass of only .8 to C, and assigns the remaining mass to θ. We take E in θ' to be the proposition that it seems (.8) to be the case that there is a squirrel on the roof, for which the evidence is conclusive. The index 0.8 indicates the force of the seeming, and is reflected in our assignment of masses in θ'. In many situations it seems quite natural to replace "uncertain evidence" by the "certain" data on which it is based.

Even the case discussed by Daiconis and Zabell (1982) does not seem too difficult. The case is one in which we have one degree of belief that a Shakespearean actor to be heard on a record is Gielgud (say a half), but after hearing his voice for a while, we come to have a degree of belief of .8 that it is Gielgud. It is quite true that we would be hard put to it to describe in language the acoustic characteristics we come to assign to that voice with probability 1 that in turn provide evidence that it is the voice of Gielgud. But we can always refer to those characteristics as "the characteristics I have been (consciously or unconsciously) reacting to".

Second, however, whether or not we can always do this is unimportant for the comparison of Bayesian and Dempster conditioning. We can regard the introduction of E to be merely a computational device that helps us to compare the distribution of masses in θ according to the function $Bel_1 \oplus Bel_C$ to the corresponding set of Bayesian conditional distributions.

Lemma 2 of the appendix proves a corresponding fact about Jeffrey's rule for uncertain evicence.[4] It, too, may be represented as the effect of (possibly artificial) "certain" evidence. The argument is similar. Suppose our original degrees of belief are defined over a certain field of propositions. We introduce a new elementary statement into that field, thereby dividing each atom of the original field into two new atoms. The new elementary statement stands for that statement that, if it were "certain", would have just the effect that our "uncertain" evidence does. We then show that the resulting new probabilities obtained by conditionalizing on our new statement are exactly those yielded by applying Jeffrey's rule to the shift in probability of the "uncertain" evidence.

With these two results, and our previous theorem that shows the relation of Dempster/Shafer and convex Bayes updating to the case of "certain" evidence, it follows immediately that the inequalities of theorem 3 hold whether or not the updating is done on the basis of "certain" evidence. In any case, the intervals resulting from Dempster/Shafer updating will be subintervals, and may be proper subintervals, of the intervals resulting from the application of conditionalization to sets of classical probability functions.[5]

8. Dempster/Shafer evidential updating, we have seen, leads to more tightly constrained representations of rational belief than does convex Bayesian updating.[6] It might be thought that this is a virtue. But whether or not this is a Good Thing

is open to question.

Suppose that $\underline{D} = \underline{D}_1,\ldots,\underline{D}_n$ are alternative decisions open to you, and that you have a utility function defined over the cross product of $\underline{D}$ and the set θ of possible states. You begin with a belief function, and you obtain some evidence. If you combine this evidence with your initial belief function according to convex Bayesian conditionalization, your new beliefs will be characterized by a set of probability functions $\underline{P}_B$. If you perform the combination of evidence according to non-Bayesian procedures, your new beliefs will be characterized by a set of probability functions $\underline{P}_N$ that is (in general) a proper subset of $\underline{P}_B$.[7]

Given any probability function $\underline{P}$ in either $\underline{P}_B$ or $\underline{P}_N$, you can calculate the expected value of each decision: $\underline{E}(\underline{D}_i, \underline{P})$. Let us say that $\underline{D}_i$ is admissible relative to a set of probability functions just in case there is some probability function in the set according to which the expected value of $\underline{D}_i$ is at least as great as the expected value of any other decision.[8] Since $\underline{P}_N$ is included in $\underline{P}_B$, the admissible decisions we obtain if we update in a non-Bayesian way are included among those we obtain if we update in a Bayesian way.

There are three cases to consider. (1) We obtain the same set of admissible decisions by either updating procedure. In this case we have gained nothing by using the stronger procedure. (2) If $\underline{P}_N$ leads to a set of admissible decisions containing more than one member, then so does $\underline{P}_B$, and we must in either case invoke additional constraints in order to generate a unique decision. (3) If $\underline{P}_N$ leads to a unique admissible decision and $\underline{P}_B$

does not, we appear to have accomplished something useful by means of non-Bayesian updating.

But it is open to question whether the added power should be built into the evidential updating rule, or whether it should appear as part of a decision procedure that takes us beyond the evidence. Many people feel that principles of evidence and principles of decision should be kept distinct.

Consider an urn filled with black and white iron balls, some of which are magnetized and some of which are not. It is easy to imagine that by extensive sampling, or by word of the manufacturer, our statistical knowledge about the contents of the urn may be as represented in table II of the appendix, where the set of black balls is represented by A, and the set of magnetized balls is represented by B. Given that this is our initial state, we may ask what our attitude should be toward the proposition that a ball selected from the urn is magnetic, given that it is white.

Dempster conditioning yields the degenerate interval [0.8, 0.8].

Bayesian conditionalization yields the interval [0.5, 0.8]. Suppose you are offered a ticket for $.75 that returns a dollar if the ball is magnetic. On the view identified with Dempster and Shafer, it is not only permissible, but, given the usual utility function, mandatory to buy it. On the convex Bayesian view either accepting or rejecting the offer would be admissible. It is true that, for all you know, the true expectation is positive; but it is also true, for all you know, the true

expectation is negative. If everything you know is true, the expected loss may still be $-.25.

On the other hand, there are cases where Dempster's rule of combination leads to intuitively appealing results, but the convex Bayes approach does not.[9] Suppose you know that 70% of the soft berries in a certain area are good to eat, and that 60% of the red berries are good to eat. What are the chances that a soft red berry is good to eat? Dempster's rule yields .42/.54 = .78, which has intuitive appeal. But the set of distributions compatible with the conditions of the problem as they have been stated leaves the probability of a soft red berry being good to eat completely undetermined: it is the entire interval $[0,1]$! It is possible that 100% of the soft red berries are good, and it is possible that 0% of the soft red berries are good.

It is clear that in applying the rule of combination, we are implicitly constraining the set of (joint) distributions we regard as possible. This is suggested by Shafer's requirement that the items of evidence to be combined be "distinct" or "independent". The most natural sufficient condition that leads to the same result as Dempster's rule of combination is that all the probability functions in our convex set satisfy the three conditions:

(i) $\underline{P}(G) = 1/2$

(ii) $\underline{P}(S/G\&R) = \underline{P}(S/G)$

(iii) $\underline{P}(S/\bar{G}\&R) = \underline{P}(S/\bar{G})$

Condition (i), of course, is our old friend, the principle of

indifference. Conditions (ii) and (iii) represent conditional independence, and it is not hard to imagine that we have warrant for supposing they are satisfied.

The exact necessary and sufficient conditions for agreement between the two methods are that our set of probability functions satisfy one of the two conditions

(iv) $\underline{P}(\bar{G}\&R\&S)/\underline{P}(G\&R\&S) = \underline{P}(\bar{G}\&R)*\underline{P}(\bar{G}\&S)/\underline{P}(G\&R)*\underline{P}(G\&S)$

or (v) $\underline{P}(S/\bar{G}\&R)/\underline{R}(S/\bar{G}) = \underline{P}(\bar{G})/\underline{P}(G) * \underline{P}(S/G\&R)/\underline{P}(S/G)$

If our evidence is statistical in character, it clearly behooves us to unpack the statistical assumptions underlying our employment of non-Bayesian updating procedures. But what if our evidence is not statistical in character?

One plausible response is that Dempster's rule of combination is not designed for all cases in which you have statistical data to serve as input. Sometimes the masses in the belief function are determined by frequencies, and sometimes they are not; only when they are not determined by frequencies should we apply non-Bayesian updating. It is difficult to make a case against this response except by making a case for the claim that all responsible and useful probabilities, even very vague ones, are based on statistical knowledge. But even granting the claim, we must face the problem of how to treat evidence which is mixed — which contains both statistical components and intuitive components. While it is a theorem that Dempster combination is both commutative and associative, it is obviously not the case

that a mixture of Dempster and Bayesian methods need be commutative and associative.

It should be strongly emphasized that the present arguments are not intended as arguments in favor of the general applicability of convex Bayesian conditionalization. Rather, what I have shown is (1) that the representation of belief states by distributions of masses over subsets of a set Θ of possibilities is a special case of the convex Bayesian representation in terms of simple classical probabilities over the atoms of Θ, (2) that the treatments of uncertain evidence in both Bayesian and non-Bayesian updating are reducible to the corresponding treatments of certain evidence, and (3) that non-Bayesian updating yields more determinate belief states as outcomes, but that the benefits afforded by non-Bayesian updating are limited and questionable.

APPENDIX

Theorem 1:

Let m be a probability mass function defined over a frame of discernment θ. Let Bel be the corresponding belief function, $Bel(X) = \sum_{A \subseteq X} m(A)$. Then there is a closed, convex set of classical probability functions S_P defined over the atoms of θ such that for every subset X of θ, $Bel(X) = \min_{P \in S_P} P(X)$.

Proof: Let S_P be the set of classical probability functions P defined on the atoms of θ such that for every $X \subset \theta$, $Bel(X) \leq P(X) \leq 1-Bel(\bar{X})$. S_P is closed, since $P(X) = Bel(X)$, $P(\bar{X}) = 1-Bel(X)$ is a classical probability function. S_P is convex, since for $0 < a < 1$, $aP_1(X) + (1-a)P_2(X)$ lies between $Bel(X)$ and $1-Bel(X)$ whenever $P_1(X)$ and $P_2(X)$ do. Since for any given there is a $P \in S_P$ such that $P(X) = Bel(X)$, $Bel(X) \geq \min_{P \in S_P} P(X)$. And $\min_{P \in S_P} P(X) \geq Bel(X)$ since this inequality holds for every $P \in S_P$.

To show that S_P is non-empty, it suffices to show that there is a $P \in S_P$ such that for every $X \subset \theta$, $Bel(X) \leq P(X)$, since if this is so, then $Bel(\bar{X}) \leq P(\bar{X})$ and $1 - Bel(\bar{X}) \geq 1 - P(\bar{X}) = P(X)$.

Suppose the atoms of θ are ordered lexicographically. For every set X, $X \subset \theta$, add the mass assigned to X, $m(X)$, to the mass assigned to $\{a_i\}$, where $\{a_i\}$ is the lexicographically earliest atom in X. Let the new mass function be m'. Define $P(X) = \sum_{a \in X} m'(\{a\})$. $P(\emptyset)=0$; $P(\theta)=1$, since all the original mass ends up on the atoms, and $P(X) \geq Bel(X)$, since the mass assigned to any subset of X ends up on the atoms of X.

□

Theorem 2

If $\underline{S_P}$ is a closed convex set of classical probability functions defined over the atoms of θ, and for every $\underline{A}$, $B \subset \theta$, $\underline{\min}\ \underline{P}(\underline{A} \cup \underline{B}) \geq \underline{\min}\ \underline{P}(\underline{A}) + \underline{\min}\ \underline{P}(\underline{B}) - \underline{\min}\ \underline{P}(\underline{A} \cap \underline{B})$, then there is a mass function $\underline{m}$ defined over the subsets of θ such that for every $\underline{X}$ in θ, the corresponding $\underline{Bel}$ function satisfies

$$\underline{Bel}\ (\underline{X}) = \underline{\min_{P \in S_P} P}(\underline{X}) .$$

Proof: Since $\underline{S_P}$ is closed and convex, for every $\underline{X} \subset \theta$ there is a $\underline{P} \in \underline{S_P}$ such that $\underline{P}(\underline{X}) = \underline{\min_{P \in S_P}}\ \underline{P}(\underline{X})$. For every $\underline{X} \subset \theta$, define $\underline{P_*}(X)$ to be $\underline{\min_{P \in S_P} P}(\underline{X})$.

By Shafer's Theorem 2.1, if 0 is a frame of discernment function $\underline{Bel}\ 2^{\theta} \rightarrow [0,1]$ is a belief function if and only if

(1) $\underline{Bel}\ (\emptyset) = 0 \qquad \underline{P_*}(\emptyset) = 0$

(2) $\underline{Bel}\ (\theta) = 1 \qquad \underline{P_*}(\theta) = 1$

(3) For every positive integer $\underline{n}$ and every collection $\underline{A_1}, \ldots, \underline{A_n}$ of subsets of θ,

$$\underline{Bel}\ (\underline{A_1} \cup \ldots \cup \underline{A_n}) \geq \sum_{\underline{I} \{1,\ldots,\underline{n}\}} (-1)^{|\underline{I}|+1}\ \underline{Bel}\ (\bigcap_{i \in \underline{I}} \underline{A_i})$$

Since Shafer's theorem 2.2 gives an algorithm to recapture the mass function from the belief function, we need merely establish (3) for our function $\underline{P*}$.

$$(3')\quad P_*(A_1 \cup \ldots \cup A_n) \geq \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i\Big)$$

Suppose, on the contrary, (3') fails. Then there is a specific collection $A_1,\ldots,A_n$, of smallest cardinality n, for which (3') is false, i.e.,

$$P_*(A_1 \cup \ldots \cup A_n) < \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i\Big)$$

But $P_*(A_1 \cup \ldots \cup A_n) \geq P_*(A_n) + P_*(A_1 \cup \ldots \cup A_{n-1}) - P_*((A_1 \cup \ldots \cup A_{n-1}) \cap A_n)$,

by the hypothesis of the theorem. Now

$$P_*((A_1 \cup \ldots \cup A_{n-1}) \cap A_n) = P_*((A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n))$$

and by hypothesis, (3') holds for collections of cardinality of $(n-1)$. Thus

$$(4)\quad P_*((A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)) \geq \sum_{I \subset \{1,\ldots,n-1\}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i \cap A_n\Big)$$

and $$(5)\quad P_*(A_1 \cup \ldots \cup A_{n-1}) \geq \sum_{I \subset \{1,\ldots,n-1\}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i\Big)$$

Let us compute

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i\Big):$$

We evaluate the sum by cases: $|I| = 1$, $|I| > 1$ and $n \notin I$, and $|I| > 1$ and $n \in I$, in each case writing the result in general terms for ease of collection.

$$|I| = 1: \quad P_*(A_n) + \sum_{\substack{I \subset \{1,\ldots,n-1\} \\ |I|=1}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i\Big)$$

$$= \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=1}} (-1)^{|I|+1} P_*\Big(\bigcap_{i \in I} A_i\Big) = P_*(A_n) + \sum_{i<n} P_*(A_i)$$

$|I| > 1$, $I \subset \{1,\ldots,n-1\}$: $\sum_{I\subset\{1,\ldots,n-1\}} (-1)^{|I|+1} P_*(\bigcap_{i\in I} A_i)$

$|I| > 1$, $I \subset \{1,\ldots,n-1\}$: $I = I' \cup \{n\}$, $I' \subset \{1,\ldots,n-1\}$

$$\sum_{I'\subset\{1,\ldots,n-1\}} (-1)^{|I'|+2} P_*(\bigcap_{i\in I'} A_i \cap A_n)$$

$$= \sum_{\substack{I\subset\{1,\ldots,n\} \\ n\in I,\ |I|>1}} (-1)^{|I|+1} P_*(\bigcap_{i\in I} A_i)$$

Combining the three terms, we have,

$$P_*(A_1 \cup \ldots \cup A_n) \geq \sum_{\substack{I\subset\{1,\ldots,n\} \\ |I|=1}} (-1)^{|I|+1} P_*(\bigcap_{i\in I} A_i) + \sum_{\substack{I\subset\{1,\ldots,n\} \\ n\notin I \\ |I|>1}} (-1)^{|I|+1} P_*(\bigcap_{i\in I} A_i)$$

$$+ \sum_{\substack{I\subset\{1,\ldots,n\} \\ n\in I \\ |I|>1}} (-1)^{|I|+1} P_*(\bigcap_{i\in I} A_i)$$

$$= \sum_{I\subset\{1,\ldots,n\}} (-1)^{|I|+1} P_*(\bigcap_{i\in I} A_i)$$

contradicting our assumption that there was an n for which (3') was false.

□

Theorem 3:

Let Θ be a frame of discernment, Bel a belief function, and S_P the corresponding set of Bayesian probability functions. Let B be evidence assigned probability 1, or support 1, and suppose $P(B) > 0$ for every $P \in S_P$. Then for every $A \subset \Theta$,

$$\min_{P \in S_P} P(A|B) \leq Bel(A|B) \leq P^*(A|B) \leq \max_{P \in S_P} P(A|B)$$

where $P^*(A|B) = 1 - Bel(\bar{A}|B)$ is Shafer's plausibility function.

Proof: (All maxima and minima are taken over $P \in S_P$.)

For $X \subset \Theta$, $Bel(X) = \min P(X)$, and $P^*(X) = \max P(X)$.

$$Bel(A|B) = \frac{Bel(A \cup \bar{B}) - Bel(\bar{B})}{1 - Bel(\bar{B})} = \frac{\min P(A \cup \bar{B}) - \min(\bar{B})}{1 - \min(\bar{B})}$$

$$\min P(A|B) = \min \left[\frac{P(A \cap B)}{P(B)} \right] = \min \left[\frac{P(A \cup \bar{B}) - P(\bar{B})}{1 - P(B)} \right]$$

Let $Q \in S_P$ be such that $Q(A \cup \bar{B}) = \min P(A \cup \bar{B})$.

Then

$$\min P(A|B) \leq \frac{Q(A \cup \bar{B}) - Q(\bar{B})}{1 - Q(\bar{B})} \leq \frac{Q(A \cup \bar{B}) - \min P(B)}{1 - \min P(B)} = Bel(A|B)$$

$\underline{\max}\ \underline{P}(\underline{A}|\underline{B}) = \underline{\max}\left[\frac{\underline{P}(\underline{A}\cap\underline{B})}{\underline{P}(\underline{B})}\right]$. Let $\underline{R}\ \underline{S}_{\underline{\epsilon}}$ be such that $\underline{R}(\underline{A}\cap\underline{B}) = \underline{\max}\ \underline{P}(\underline{A}\cap\underline{B})$. Then

$$\underline{\max}\ \underline{P}(\underline{A}|\underline{B}) \geq \frac{\underline{R}(\underline{A}\cap\underline{B})}{\underline{R}(\underline{B})} \geq \frac{\underline{\max}\ \underline{P}(\underline{A}\cap\underline{B})}{\underline{\max}\ \underline{P}(\underline{B})} = \underline{P}^*(\underline{A}|\underline{B}).$$

□

Lemma 1: Let θ be a frame of discernment. Let our initial belief function be $\underline{Bel}_1$. We obtain new evidence whose impact on the frame of discernment θ can be represented by a simple support function (Shafer 1976, p. 7) $\underline{Bel}_C$ whose single focus is $\underline{C} \subset 2^{\theta}$. $\underline{Bel}_C$ attributes mass $\underline{s}$ to $\underline{C}$ and mass $(1-\underline{s})$ to θ.

Let the foci of $\underline{Bel}_1$ — the subsets $\underline{A}$ of θ receiving mass $\underline{m}_1(\underline{A})>0$ — be $\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_n$. We can construct a new frame of discernment θ' and a new belief function $\underline{Bel}_1'$, such that

(a) For every $\underline{X}\subset\theta$, $\underline{Bel}_1'(\underline{X}) = \underline{Bel}_1(\underline{X})$

(b) For every $\underline{X}\subset\theta$, $(\underline{Bel}_1 \oplus \underline{Bel}_C)(\underline{X}) = \underline{Bel}_1'(\underline{X}|\underline{E})$, where $\underline{E}\in 2^{\theta'}$, and the evidence partially supporting $\underline{C}$ provides <u>total</u> support for $\underline{E}$. "$\oplus$" represents the application of Dempster's rule of combination to $\underline{Bel}_1$ and $\underline{Bel}_C$; $\underline{Bel}_1'(\underline{X}|\underline{E})$ represents Dempster's rule of conditioning on $\underline{E}$ —the analog of Bayesian conditionalization (Shafer 1976 p. 67).

Proof: Let e be new to θ, and for every $p \in \theta$ generate two new "possibilities" pe and $p\bar{e}$. Let $\theta' = \{p': \exists p \in \theta(p' = pe \vee p' = p\bar{e})\}$. Let $E = \{p': \exists p \in \theta(p' = pe)\}$. Since the evidence that supports C is to render E certain, we have $C' \subset E$ i.e. $C' = \{p': \exists p \in C(p' = pe)\}$.

We define Bel_1' on the basis of m_1' as follows:

Bel_1' has n foci of the form A_i, each with mass $(1-s)m_1(A_i)$, where m_1 is the mass function associated with Bel_1.

For every i such that $A_i \cap C' = \emptyset$, $A_i \cap \bar{E}$ is to be a focus with mass $s \cdot m_1(A_i)$. For convenience we take the first p of the A_i to be those for which $A_i \cap C' = \emptyset$. Note that p may be 0, but cannot be n, else $Bel_1 \oplus Bel_C$ would be undefined.

The remaining i give rise to the remaining foci. These are of the form $(A_i \cap C') \cup (A_i \cap \bar{E})$, and receive the remaining mass. Since $(A_i \cap C') \cup (A_i \cap E) = (A_i \cap C') \cup (A_i \cap \bar{E})$ is a possibility for $i \neq j$, we write

$$m_1'((A_i \cap C') \cup (A_i \cap \bar{E})) = \sum_{\{j:\ (A_j \cap C') \cup (A_j \cap \bar{E}) = (A_i \cap C') \cup (A_i \cap \bar{E})\}} m_1(A_j) \cdot s$$

Note that $\sum m_1'((A_i \cap C') \cup (A_i \cap \bar{E})) = \sum_{i=p+1}^{n} m_1(A_i) \cdot s$, since

these sets have positive mass only if $A_i \cap C' \neq \emptyset$.

We first show that $\underline{Bel}_1'$ is a belief function. Obviously its mass function $\underline{m}'$ is non-negative for every $\underline{A} \subset \theta'$, so we need only show that $\sum_{\underline{A} \subset \theta'} \underline{m}'(\underline{A}) = 1$. Summing over the three kinds of foci, we have:

$$\sum_{\underline{A} \subset \theta'} \underline{m}'(\underline{A}) = \sum_{\underline{i}=1}^{\underline{n}} (1-\underline{s})\underline{m}_1(\underline{A}_{\underline{i}}) + \sum_{\underline{i}=1}^{\underline{p}} \underline{s} \cdot \underline{m}_1(\underline{A}_{\underline{i}}) + \sum_{\underline{i}=\underline{p}+1}^{\underline{n}} \underline{s} \cdot \underline{m}_1(\underline{A}_{\underline{i}}) = 1.$$

We next show that $\underline{Bel}_1'$ is equivalent to $\underline{Bel}_1$ - i.e. that for any $\underline{X} \subset \theta$, $\underline{Bel}_1'(\underline{X}) = \underline{Bel}_1(\underline{X})$.

$$\underline{Bel}_1'(\underline{X}) = \sum_{\underline{A} \subset \underline{X}} \underline{m}'(\underline{A}) = \sum_{\underline{A}_{\underline{i}} \subset \underline{X}} \underline{m}'(\underline{A}_{\underline{i}}) + \sum_{\underline{A}_{\underline{i}} \cap \underline{E} \subset \underline{X}} \underline{m}'(\underline{A}_{\underline{i}} \cap \underline{E}) + \sum_{(\underline{A}_{\underline{i}} \cap \underline{C}) \cup (\underline{A}_{\underline{i}} \cap \overline{\underline{E}}) \subset \underline{X}} \underline{m}'((\underline{A}_{\underline{i}} \cap \underline{C}) \cup (\underline{A}_{\underline{i}} \cap \overline{\underline{E}}))$$

The first term yields $\sum_{\underline{A}_{\underline{i}} \subset \underline{X}} (1-\underline{s})\underline{m}_1(\underline{A}_{\underline{i}}) = (1-\underline{s}) \sum_{\underline{A}_{\underline{i}} \subset \underline{X}} \underline{m}_1(\underline{A}_{\underline{i}})$

Since $\underline{X} = (\underline{X} \cap \underline{E}) \cup (\underline{X} \cap \overline{\underline{E}})$, $\underline{A}_{\underline{i}} \cap \overline{\underline{E}} \subset \underline{X} \cap \overline{\underline{E}}$ if and only if $\underline{A}_{\underline{i}} \subset \underline{X}$, in view of the fact that $\underline{p}\overline{\underline{e}} \in \underline{A}_{\underline{i}} \cap \overline{\underline{E}}$ if and only if $\underline{p} \in \underline{A}_{\underline{i}}$, and the same holds for $\underline{X}$. Thus the second term yields $\sum_{\substack{\underline{A}_{\underline{i}} \subset \underline{X} \\ 1 \le \underline{i} \le \underline{p}}} \underline{s} \cdot \underline{m}_1(\underline{A}_{\underline{i}})$

To evaluate the third term, we claim that $(A_i \cap C) \cup (A_i \cap \bar{E}) \subset X$ if and only if $A_i \subset X$. If $A_i \subset X$, then $A_i \cap C \subset X$ and $A_i \cap \bar{E} \subset X$ and so $(A_i \cap C) \cup (A_i \cap \bar{E}) \subset X$. Suppose $(A_i \cap C) \cup (A_i \cap \bar{E}) \subset X$. Then $A_i \cap \bar{E} \subset X$, $A_i \cap \bar{E} \subset X \cap \bar{E}$, and by the preceding argument $A_i \subset X$. Thus the third term yields $\sum_{\substack{A_i \subset X \\ p < i < n}} s \cdot m_1(A_i)$

Putting the three parts together, we have $Bel_1'(X) = Bel_1(X)$.

We now show that conditioning on E in the frame of discernment θ' is equivalent to combining uncertain evidence C with Bel_1 in the frame of discernment θ according to Dempster's rule of combination:

For every $X \subset \theta$, $(Bel_1 \oplus Bel_C)(X) = Bel_1'(X|E)$

$$(1) \quad (Bel_1 \oplus Bel_C)(X) = \frac{\sum_{A_i \cap C \neq \emptyset,\ A_i \cap C \subset X} m_1(A_i) \cdot s \ + \sum_{A_i \subset X} m_1(A_i)(1-s)}{1 - \sum_{A_i \cap C = \emptyset} m_1(A_i) \cdot s}$$

(The numerator comprises two sums, since Bel_C has two foci: C and θ with masses s and $(1-s)$ respectively.)

$$(2) \quad Bel_1'(X|E) = \frac{\sum_{A \subset X \cup \bar{E}} m_1'(A) - \sum_{A \subset \bar{E}} m_1'(A)}{1 - \sum_{A \subset \bar{E}} m_1'(A)}$$

$\sum_{A \subset \bar{E}} m_1'(A) = \sum_{i=1}^{p} s \cdot m_1(A_i)$, since only the foci of the form $A_i \cap \bar{E}$ are included in $\bar{E}$: $A_i = (A_i \cap E) \cup (A_i \cap \bar{E})$ is not included in $\bar{E}$, and since $C \subset E$, $(A_i \cap C') \cup (A_i \cap \bar{E})$ is included in $\bar{E}$ only if $A_i \cap C' = \emptyset$, in which case it has no mass.

$$\sum_{A_i \cap C' = \emptyset} m_1(A_i) \cdot s = \sum_{i=1}^{p} s \cdot m_1(A_i).$$

Hence the denominators of (1) and (2) are the same.

It remains to evaluate $\sum_{A \subset X \cap \bar{E}} m_1'(A)$. Consider foci of the form A_i. $A_i \subset X \cup \bar{E}$ if and only if $A_i \subset X$, so these foci yield mass

$$\sum_{A_i \subset X} m'(A_i) = \sum_{A_i \subset X} (1-s) m_1(A_i)$$

corresponding to the right hand term in the numerator of (1).

Consider foci of the form $A_i \cap \bar{E}$. All of these are included in $X \cup \bar{E}$; they yield

$$\sum_{i=1}^{p} m'(A_i) = \sum_{i=1}^{p} s \cdot m_1(A_i) = \sum_{A \subset \bar{E}} m_1'(A),$$

so they drop out of the numerator of (2).

Finally, consider foci of the form $(\underline{A}_i \cap \underline{C}') \cup (\underline{A}_i \cap \bar{\underline{E}})$. We first show that $(\underline{A}_i \cap \underline{C}') \cup (\underline{A}_i \cap \bar{\underline{E}}) \subset \underline{X} \cup \bar{\underline{E}}$ if and only if $\underline{A}_i \cap \underline{C}' \subset \underline{X}$. Suppose $(\underline{A}_i \cap \underline{C}') \cup (\underline{A}_i \cap \bar{\underline{E}}) \subset \underline{X} \cup \bar{\underline{E}}$. Then $\underline{A}_i \cap \underline{C}' \subset (\underline{X} \cup \bar{\underline{E}})$. But $\underline{C} \subset \underline{E}$, so $\underline{A}_i \cap \underline{C}' = \underline{A}_i \cap \underline{C}' \cap \underline{E} \subset \underline{X} \cup \bar{\underline{E}}$ only if $\underline{A}_i \cap \underline{C}' \subset \underline{X}$. Suppose $\underline{A}_i \cap \underline{C}' \subset \underline{X}$. Then since $\underline{A}_i \cap \bar{\underline{E}} \subset \bar{\underline{E}} \subset \underline{X} \cup \bar{\underline{E}}$, $(\underline{A}_i \cap \underline{C}') \cup (\underline{A}_i \cap \bar{\underline{E}}) \subset \underline{X} \cup \bar{\underline{E}}$.

We compute the mass in the numerator of (2) due to foci of this sort. They have mass only when $\underline{A}_i \cap \underline{C}' \neq 0$. And then they have mass

$$\sum_{\{j:\ (\underline{A}_j \cap \underline{C}') \cup (\underline{A}_j \cap \bar{\underline{E}}) = (\underline{A}_i \cap \underline{C}') \cup (\underline{A}_i \cap \bar{\underline{E}})\}} \underline{s} \cdot \underline{m}_1(\underline{A}_j) \quad ;$$

each $\underline{A}_i$ such that $\underline{A}_i \cap \underline{C}' \subset \underline{X}$ contributes $\underline{s} \cdot \underline{m}_1(\underline{A}_i)$. Their total mass is therefore

$$\sum_{\substack{\underline{A}_i \cap \underline{C}' \subset \underline{X} \\ \underline{A}_i \cap \underline{C}' \neq 0}} \underline{s} \cdot \underline{m}_1(\underline{A}_i) \, ,$$

corresponding to the first term of the numerator of (1).

We have therefore shown that $(\underline{Bel}_1 \oplus \underline{Bel}_{\underline{C}})(\underline{X}) = \underline{Bel}_1'(\underline{X} \mid \underline{E})$.

□

Lemma 2

Suppose that $\underline{P}_0$ is an assignment of probabilities to the field of propositions whose basis is $\underline{a}_1, \underline{a}_2, \underline{a}_3, \ldots, \underline{a}_n$. Let $\underline{P}_1$ be generated by a shift in the probability assigned to $\underline{A}$; this shift is the source of our new probability $\underline{P}_1$. By Jeffrey's rule, for all $\underline{X}$,

$$\underline{P}_1(\underline{X}) = \underline{P}_0(\underline{X}|\underline{A})\cdot\underline{P}_1(\underline{A}) + \underline{P}_0(\underline{X}|\underline{\bar{A}})\cdot\underline{P}_1(\underline{\bar{A}}).$$

Then there exists a new field of propositions $\underline{F}'$, and a proposition $\underline{E}$, and a new probability function $\underline{P}_0'$ defined on $\underline{F}'$ such that for every proposition $\underline{X}$ in the old field F,

(a) $\underline{P}_0'(\underline{X}) = \underline{P}_0(\underline{X})$

(b) $\underline{P}_0'(\underline{X}|\underline{E}) = \underline{P}_1(\underline{X})$

Proof:

Add a new atomic proposition $\underline{e}$ to the basis of $\underline{F}$ to obtain the field $\underline{F}'$, and represent it by $\underline{E}$. We impose the constraint $\underline{P}_0'(\underline{A}|\underline{E}) = \underline{P}_1(\underline{A})$; $\underline{P}_0'(\underline{E})$ may have any value that strikes our fancy.

We extend $\underline{P}_0'$ so that for any $\underline{X} \in \underline{F}$, $\underline{P}_0'(\underline{X}) = \underline{P}_0(\underline{X})$; $\underline{P}_0'$ is fully equivalent to $\underline{P}_0$, so far as $\underline{F}$ is concerned, before we obtain information about $\underline{A}$. Specifically, set

$$\underline{k} = \frac{\underline{P}_1(\underline{A})}{\underline{P}_0(\underline{A})}; \quad \underline{k}' = \frac{\underline{P}_1(\underline{\bar{A}})}{\underline{P}_0(\underline{\bar{A}})} = \frac{1-\underline{P}_1(\underline{A})}{1-\underline{P}_0(\underline{A})} = \frac{1-\underline{k}\,\underline{P}_0(\underline{A})}{1-\underline{P}_0(\underline{A})}$$

For $\underline{X} \in \underline{F}'$, set

$$\underline{P}_0'(\underline{X}\wedge\underline{E}) = \underline{P}_0'(\underline{E}) \cdot [\underline{k}\,\underline{P}_0(\underline{X}\wedge\underline{A}) + \underline{k}'\,\underline{P}_0(\underline{X}\wedge\underline{\bar{A}})]$$

$$\underline{P}_0'(\underline{X}\wedge\underline{\bar{E}}) = \underline{P}_0(\underline{X}) - \underline{P}_0'(\underline{X}\wedge\underline{E})$$

Clearly, for $X \subseteq E$,

$$P_0'(X) = P_0'(X \wedge E) + P_0'(X \wedge \bar{E}) = P_0(X)$$

We now show that for $X \in F$, probabilities conditional on E are equal to the probabilities given by Jeffrey's rule: $P_1(X) = P_0'(X|E)$.

For $X \subseteq E$, $P_0'(X|E) = \dfrac{P_0(X \wedge E)}{P_0'(E)}$

$$= \frac{P_0'(E) \cdot [k\, P_0(X \wedge A) + k'\, P_0(X \wedge \bar{A})]}{P_0'(E)}$$

$$= \frac{P_0(X \wedge A)}{P_0(A)} P_1(A) + \frac{P_0(X \wedge \bar{A})}{P_0(\bar{A})} P_1(\bar{A}) = P_1(X)$$

□

Theorem 4

Let a distribution of beliefs be given both by the function Bel_1 and by the prior set of probability distributions S_P. Suppose new evidence is obtained whose impact is given by a simple support function Bel_A assigning positive mass to A and θ, or, alternatively, by a shift in the probability of A on each of the distributions in S_{P_1}; let S_{P_2} be the result of propagating this shift by Jeffrey's rule, and let Bel_2 be the result of applying Dempster's rule of combination. Then

$$\min_{P \in S_{P_2}} P(X) \leq Bel_2(X) \leq 1 - Bel_2(\bar{X}) \leq \max_{P \in S_{P_2}} P(X)$$

for all subsets X of θ.

Proof:

Immediate from lemma 1, lemma 2, and Theorem 3.

Table 1

		Mass	Lower Measure	Upper Measure
①	$A\bar{B}$	X_1	X_1	$1-X_2-X_3-X_4-X_{23}-X_{24}-X_{34}-X_{234}$
②	$\bar{A}B$	X_2	X_2	$1-X_1-X_3-X_4-X_{13}-X_{14}-X_{34}-X_{134}$
③	$\bar{A}\bar{B}$	X_3	X_3	$1-X_1-X_2-X_4-X_{12}-X_{14}-X_{24}-X_{124}$
④	AB	X_4	X_4	$1-X_1-X_2-X_3-X_{12}-X_{13}-X_{23}-X_{123}$
	① ∪ ②	X_{12}	$X_1+X_2+X_{12}$	$1-X_3-X_4-X_{34}$
	① ∪ ③	X_{13}	$X_1+X_3+X_{13}$	$1-X_2-X_4-X_{24}$
	① ∪ ④	X_{14}	$X_1+X_4+X_{14}$	$1-X_2-X_3-X_{23}$
	② ∪ ③	X_{23}	$X_2+X_3+X_{23}$	$1-X_1-X_4-X_{14}$
	② ∪ ④	X_{24}	$X_2+X_4+X_{24}$	$1-X_3-X_1-X_{13}$
	③ ∪ ④	X_{34}	$X_3+X_4+X_{34}$	$1-X_1-X_2-X_{12}$
	① ∪ ② ∪ ③	X_{123}	$X_1+X_2+X_3+X_{12}+X_{13}+X_{23}+X_{123}$	$1-X_4$
	① ∪ ② ∪ ④	X_{124}	$X_1+X_2+X_4+X_{12}+X_{14}+X_{24}+X_{124}$	$1-X_3$
	① ∪ ③ ∪ ④	X_{134}	$X_1+X_3+X_4+X_{13}+X_{14}+X_{34}+X_{134}$	$1-X_2$
	② ∪ ③ ∪ ④	X_{234}	$X_2+X_3+X_4+X_{23}+X_{24}+X_{34}+X_{234}$	$1-X_1$
	θ	X_θ	1	1

$$X_\theta = 1-\Sigma X_i$$

Table II

A: white B: magnetic

	Set	mass	Bel	Frequency
$\underline{AB}$	X_1	0.2		[0.2,0.4]
$A\bar{B}$	X_2	0.2		[0.2,0.4]
$\underline{\bar{A}B}$	X_3	0.1		[0.1,0.1]
$\underline{\bar{A}\bar{B}}$	X_4	0.2		[0.2,0.4]
	X_{12}	0.1		[0.5,0.7]
	X_{13}	0.0		[0.3,0.5]
	X_{14}	0.1		[0.5,0.7]
	X_{23}	0.0		[0.3,0.5]
	X_{24}	0.1		[0.5,0.7]
	X_{34}	0.0		[0.3,0.5]
	X_{123}	0.0		[0.6,0.8]
	X_{124}	0.0		[0.9,0.9]
	X_{134}	0.0		[0.6,0.8]
	X_{234}	0.0		[0.6,0.8]
	θ	0.0		[1.0,1.0]

NOTES

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1. This approach is similar to that of Smith (1961). It is also similar to the approach of Levi (1974, 1981), Good (1962), and Kyburg (1974), but as Levi points out in (1981) there are important differences. Levi represents a credal state by a set of conditional probability functions, Q(x,y). For every y consistent with background knowledge, the set of functions Q(x,y) is convex. Since distinct convex sets of conditional probability functions give rise to the same convex sets of absolute probability functions, the two representations are not equivalent. Smith and Kyburg represent a credal state by the convex closure of all probabilities consistent with a set of probability intervals. Shafer, as will be seen, implicitly offers the same characterization. Dempster (1968) offers a more restricted characterization: the convex set representing the credal state is the largest that both satisfies the interval constraints, and can be obtained from a space of "simple joint

propositions" in a certain way. Levi has shown (1981, pp. 338-392) that these additional restrictions are incompatible with certain natural forms of direct inference of probabilities from known statistics.

2. In another place I shall argue that we can found all our probabilities on direct or indirect statistical inference, or on set-theoretical truths. No other source is needed.

3. This example was suggested in conversation by Teddy Seidenfeld.

4. This result was stated informally by Levi (1967), and is reflected in Diaconis and Zabell, 1982, Theorem 2.1.

5. Dempster (1967, 1968) was well aware that his rule of combination led to results stronger than those that would be given by a mere generalization of orthodox Bayesian inference. His reasons for preferring the rule at which he arrives are essentially philosophical: in an orthodox Bayesian framework, unless you restrict the family of priors, you don't get useful results starting with zero information. But in expert systems, we have no desire or need to start with zero information.

6. Quinlan's (1982) subtitle suggests the opposite: "A cautious approach to uncertain inference."

7. It is not clear that Shafer's belief functions were intended

to be used in a decision-theoretic context. Even if they were, there would be serious difficulties standing in the way of such employment. (See Levi (1978, 1980, 1983), and Seidenfeld (1978)). For present purposes, these difficulties need not concern us.

8. This corresponds to Levi's notion (1981) of E-admissibility.

9. This elegant and simple example was proposed by Jerry Feldman.

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