THE MULTI-PRODUCT INVENTORY SYSTEM UNDER CONSTRAINT

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Doctor of Philosophy

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by

BILLY M. MALONEY

Dr. Cerry M. Klein Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate Faculty, have examined a dissertation entitled

THE MULTI-PRODUCT INVENTORY SYSTEM
UNDER CONSTRAINT

presented by BILLY M. MALONEY

a candidate for the degree of Doctor of Philosophy

and hereby certify that in their opinion it is worthy of acceptance.

Statement A per telecon Jim Cvlighton
Tape-OPB-D
200 Stovall Ave
Alexandria, VA 22332-0411
NWW 4/15/92

Accession For
Availability Codes
Avail and/or
Dist Special
A-1

Dr. Cerry M. Klein

Dr. Larry G. David

Dr. Hahn-Kyou Rhee

Dr. Lori S. Franz

Dr. Luis G. Occena

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ABSTRACT

This Dissertation is comprised of the following research efforts:

1) Implicit Foundation: Functional relationships between the Lagrangian multipliers and multiple system parameters are identified and used to establish improved bounds on the optimal multiplier in closed form. A recursive process which rapidly converges to the optimal Lagrangian multiplier value is also presented.

2) Horizon Extension: Given an existing inventory system and its related optimal Lagrangian multiplier, an ability to project the multiplier needed to optimize an inventory defined as various shifts occur in the given system is developed. A recursive process which identifies the series of Lagrangian multipliers needed to optimize a inventory horizon in which constrained conditions extend over several periods is also developed.

3) Dual Constraint System: The ability to determine that portion of the constraint set which will be binding at the optimal solution is developed. A routine is also developed which effectively estimates both Lagrangian multipliers needed to optimize the system when both a budget and a storage space constraint remains binding.

4) Real World Application: Potential benefit gained from implementing the algorithms developed within this study is demonstrated within a small hardware company's large volume inventory.

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LIST OF SYMBOLS

TC	Total cost of operating an inventory
C_i	Carrying cost of item i within an inventory
R_i	Reorder cost of item i within an inventory
D_i	Uniform demand rate for item i
S_i	C_i/W_i ratios
Φ	Lagrangian Multiplier (Space Constraint)
Θ	Lagrangian Multiplier (Budget Constraint)
.*	Optimal Lagrangian Multiplier
Φ_L^t	Improved lower bound on Φ^*
Φ_U^t	Improved upper bound on Φ^*
Q_i	ith unconstrained economic reorder quantity
Q_i^*	ith constrained economic reorder quantity
Q_i^h	linear reduction of Q_i
C	Carrying cost characteristic of an inventory
R	Reorder cost characteristic of an inventory
D	Uniform demand rate characteristic
U	General constraint level
CC	Correlation Coefficient
N	Number of items in Inventory
KKT	Karush-Kuhn-Tucker
W_i	Space consumed by the ith inventory item
B_i	Budget consumed by the ith inventory item
B	Budget constraint level
W	Space constraint level
$\delta(.)$	Differential
$f'(.)$	First derivative of function f
ODI	Optimal Displacement Interval
IAW	In Accordance With
WLOG	Without Loss of Generalization
$\in$	Element of
$\notin$	not an element of
Σ	Summation
$\exists$	There exists

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Chapter I

Introduction

1.1 Introduction

"Inventory control is the science-based art of controlling the amount of stocks held within a business to meet economically the demands placed upon that business." [25, pp 1]

As early as 1929, Wilson proposed his 'economic order quantity' (EOQ) formulation for calculating replenishment order sizes for the single item inventory system [25, pp 57]. This initial attempt to provide a scientific basis for the process of determining inventory stockage levels was based on two assumptions. The first was that the total cost, TC, of operating an inventory was comprised of only inventory carrying costs and a cost of placing replenishment orders. In defining these costs, Wilson assumed further that the cost of placing replenishment orders, R, was independent of the size of the order. He also recognized the cost of carrying an item in inventory, C, to encompass all related cost including but not limited to employee costs, overhead cost, direct labor and storage costs as well as costs stemming from losses such as stock deterioration. Wilson's second assumption was that the demand, D, for an item of inventory could be considered uniformly distributed over the period.

It follows from these assumptions that TC for a single item inventory can be expressed as:

$$TC = CQ/2 + RD/Q \tag{1-1}$$

where Q is the reorder quantity. An expression for Q is obtained by differentiating TC with respect to Q and then solving the resulting differential.

$$Q = [2RD/C]^{1/2} \tag{1-2}$$

Hidden within each of these expressions is the assumption that no restriction be placed on the inventory system. Any number of constraints could, however, be imposed on such systems. Storage space constraints, budget constraints, set-up time constraints or even a management imposed constraint stemming from some external pressure are such possible limitations. The equations presented by Wilson work well for single item inventories under a single or multiple constraints since Q simply must be reduced to the level of the most active constraint to be considered optimal. This, however, is not the case when multiple items are carried in inventory. Consideration must then be given to the competition among these items for those limited resources.

1.2 Overview

Using Wilson's fundamental 'Economic Order Quantity' equation as a basis of study, the research outlined in this dissertation presents the results of both empirical and theoretical efforts to improve the methods now available to solve the multiple item inventory system under a single as well as multiple constraints. This effort builds on the work of Maloney [26] and Klein, Ventura and Maloney [23], who focused on the two-item inventory system under a single budgetary or space constraint, as well as on Ventura and Klein [39] who provided an initial extension to the N-item inventory system. In addition to improving the existing algorithm used to solve an N-item inventory, this research expands the scope of study to include cases where multiple constraints are active. An effort to effectively apply these results within a small business setting is also made.

Specifically, the efforts described in [23], [26], and [39] will be re-examined and extended with four primary goals in mind:

1) Improve the efficiency of the Ventura/Klein algorithm by considering multiple system parameters which are allowed to shift concurrently. The current recursive algorithm is only effective for the two item inventory system where the amount of constrained resource consumed by each item of inventory is held at one. There exist, consequently, the need to fully consider the N-item inventory system.

2) Extend the results obtained in achieving (1) to a more realistic multi-period horizon inventory system. The goal will be to develop a capability for finding the Φ^* values needed to optimize a series of inventory systems created by shifting parameters within a known system over a given horizon.

3) Extend the results of (1) and (2) to the multi-constrained inventory system with an initial focus on systems with two linear restriction. With no means available to predict when none, one or both of these constraints will be binding at the optimal solution a trial and error based methodology must currently be used. The goal of this extension will be to develop an effective procedure for determining that portion of a system's constraint set which remains active and then effectively solve for or estimate the needed Lagrangian multipliers.

4) Demonstrate the potential effectiveness of the results obtained from (1), (2) and (3) by implementing the resulting algorithms within a real-world multi-product, multi-constraint inventory system.

An overview of the material to be presented follows. Chapter two presents a statement of the specific inventory problems considered within this study. A literature review tracing three branches of research related to these problems is also presented.

Chapter three describes the empirical efforts made to gain a deeper understanding of how a system's lagrangian multiplier, Φ^*, relates to simultaneous changes in multiple system parameters of the single constraint inventory problem.

Chapter four outlines the analysis conducted on a set of functions related to equations (1-1) and (1-2). This analysis provides the foundation for the algorithms presented in chapter five by providing the shape and character of these functions.

Chapter five presents an algorithm which provides not only easily computed initial bounds around Φ^* for the N-item single constraint inventory problem, but also introduces an algorithm which improves these initial upper and lower bounds. A comparative analysis is presented which demonstrates the effectiveness of this bounding algorithm. Finally, a recursive algorithm is given which converges to Φ^* without using the normally required trial and error based processes.

Chapter six extends the material presented in chapter five, building on the empirical work shown in chapter three to establish the algorithms necessary to identify a new Φ^* from a known system within which multiple parameter perturbations have occurred. These algorithms allow for periodic updating of a constrained inventory system's lagrangian multiplier without repeating the detailed analysis outlined in chapter five.

Chapter seven returns to an empirical study in order to establish the relationships which exist between lagrangian multipliers Φ and Θ from a dual constrained inventory system and shifts which may occur in the respective constraint levels imposed on that system. Building on the results of this empirical study, an algorithm is presented which determines whether both Φ^* and Θ^* must be greater than zero at the optimum solution. When this occurs, the algorithm provides effective estimates for both Φ^* and Θ^* otherwise the optimum solution is presented.

Chapter eight illustrates the potential effectiveness characteristic of the algorithms presented in chapters five, six and seven by implementing them within a real-world, multi-product, multi-constraint inventory system. Inventory data collected from Tyree Parts & Hardware of Columbia Missouri will be used facilitate this demonstration.

Chapter nine provides a summary of the work accomplished in this dissertation and presents several topics suitable for future research.

Chapter II

Problem Statement and Literature Review

2.1 Statement of Problem

The total cost of operating a multi-item single period inventory system can be written as:

$$TC = \sum_{i=1}^{N} (1/2\ C_i Q_i + R_i D_i / Q_i) \qquad (2\text{-}1)$$

where: C_i = cost of carrying the ith item in inventory,
R_i = reorder cost for the ith item of inventory,
D_i = a uniform demand rate for the i^{th} item of inventory
Q_i = the unconstrained order quantity for the ith item of inventory and
N = the number of different items carried in inventory.

The initial focus of this research effort will be to minimize equation (2-1) when only a single limitation on resource, U, is imposed. Building on the resulting foundation our focus will then shift to the multi-constrained system.

2.2 Single Constraint Problem Formulation

A complete formulation for a single storage space constrained inventory problem or equivalently for a capital investment constrained inventory problem is presented below.

Within this formulation, W_i equals the storage space required per unit of inventory or the cost associated with holding each item in inventory while U represents the available resource level.

$$\text{(P)} \qquad \text{Min } TC = \sum_{i=1}^{N} (1/2\ C_iQ_i + R_iD_i/Q_i) \qquad (2\text{-}2)$$

Subject To:

$$\sum_{i=1}^{N} W_iQ_i \leq U \qquad (2\text{-}3)$$

$$-Q_i \leq 0 \quad i=1,2,....N \qquad (2\text{-}4)$$

Utilizing the Lagrange multiplier theory presented by Hildebrand [17] and first applied to constrained optimization problems by Everett [6], Problem (P) can be rewritten in its Lagrangian form. Letting Φ be a nonnegative Lagrange multiplier for equation (2-3) and δ_i the same for each equation (2-4), Problem (P) becomes:

Max $G(\Phi,\delta_i)$

Subject to: $\Phi,\delta_i \geq 0$

where $G(\Phi,\delta_i)=$

$$\text{Min } L= \sum_{i=1}^{N}(1/2C_iQ_i + R_iD_i/Q_i) + \Phi\Big(\sum_{i=1}^{N}W_iQ_i - U\Big) - \sum_{i=1}^{N}\delta_iQ_i \qquad (2\text{-}5)$$

This expression can be simplified since when equation (2-5) is at its optimal value $Q_i > 0$ and $\delta_i = 0$. A restatement of the Lagrangian function associated with Problem (P) therefore becomes:

$$G(\Phi) =$$

$$(P\Phi) \qquad \text{Min } L = \sum_{i=1}^{N} (1/2\, C_iQ_i + R_iD_i/Q_i) + \Phi\Big(\sum_{i=1}^{N} W_iQ_i - U \Big).$$

The Karush-Kuhn-Tucker (KKT) conditions related to problem ($P\Phi$) are:

$$\delta L/\delta Q_i = 0 \qquad i = 1,2,..... \qquad (2\text{-}6)$$

$$\Sigma W_iQ_i - U \leq 0 \qquad i = 1,2..... \qquad (2\text{-}7)$$

$$\Phi\ (\ \Sigma W_iQ_i - U\ ) = 0 \qquad i = 1,2..... \qquad (2\text{-}8)$$

$$\Phi \geq 0 \qquad (2\text{-}9)$$

$$-Q_i^* \leq 0 \qquad i = 1,2...... \qquad (2\text{-}10)$$

Two cases must be considered in attempting to solve Problem ($P\Phi$). The first of these occurs when the sum of the unconstrained reorder quantities is less than the imposed limitation. Under such conditions the constraint is inactive and the unconstrained reorder quantities are optimal. In this case Φ must be equal to 0 in order to satisfy KKT condition (2-8). The second possibility occurs when the sum of Q_i is greater than the imposed constraint level. Here Φ must be greater than zero and each Q_i must be reduced to Q_i^* so that equation (2-8) again is satisfied. From KKT condition (2-6), the optimal reorder quantity for each Q_i^* is given by:

$$Q_i^* = ((2R_iD_i)/(C_i + 2W_i\Phi))^{1/2} \quad i = 1,2...... \tag{2-11}$$

From KKT condition (2-8) and conditions of feasibility:

$$\sum_{i=1}^{N} W_iQ_i^* = U. \tag{2-12}$$

In examining expressions (2-11) and (2-12) the optimum solution to (PΦ) is obtainable once Φ^* is found such that (2-12) is satisfied. The efficient selection of this optimum value, however, has proven to be extremely difficult.

2.3 Multi-Constraint Problem Formulation

Formulation of the multi-constraint system can be illustrated by considering both a space and a budget constraint simultaneously. In such cases, the constraint set can be stated as follows:

$$\sum_{i=1}^{N} W_iQ_i \leq W \tag{2-13}$$

$$\sum_{i=1}^{N} B_iQ_i \leq B \tag{2-14}$$

$$-Q_i \leq 0 \quad i=1,2,....N \tag{2-15}$$

where W_i defines the storage space required per unit of inventory and where B_i defines the cost associated with holding each item in inventory. Here W and B set limits on maximum inventory space and inventory budget respectively. Letting Φ and Θ be a nonnegative Lagrange multipliers for equation (2-13) and (2-14) respectively, the

Lagrangian function associated with Problem (P) can be restated where $G(\Phi,\Theta)=$

$$\text{Min } L = \sum_{i=1}^{N}(1/2\ C_iQ_i + R_iD_i/Q_i) + \Phi\left(\sum_{i=1}^{N}W_iQ_i - W\right) + \Theta\left(\sum_{i=1}^{N}B_iQ_i - B\right). \qquad (P(\Phi,\Theta))$$

The Karush-Kuhn-Tucker (KKT) conditions related to problem $(P(\Phi,\Theta))$ are:

$$\delta L/\delta Q_i = 0 \qquad i = 1,2,....N \qquad (2\text{-}16)$$

$$\Sigma\ W_iQ_i - W \leq 0 \qquad i = 1,2,....N \qquad (2\text{-}17)$$

$$\Sigma\ B_iQ_i - B \leq 0 \qquad i = 1,2,....N \qquad (2\text{-}18)$$

$$\Phi\ (\ \Sigma\ W_iQ_i - W\ ) = 0 \qquad i = 1,2,....N \qquad (2\text{-}19)$$

$$\Theta\ (\ \Sigma\ B_iQ_i - B\ ) = 0 \qquad i = 1,2,....N \qquad (2\text{-}20)$$

$$\Phi \geq 0 \qquad (2\text{-}21)$$

$$\Theta \geq 0 \qquad (2\text{-}22)$$

$$-Q_i^* \leq 0 \qquad i = 1,2,......N \qquad (2\text{-}23)$$

From KKT condition (2-16), the optimal reorder quantity for each Q_i^* is given by:

$$Q_i^* =((2R_iD_i)\ /(C_i + 2W_i\Phi + 2B_i\Theta))^{1/2} \qquad i = 1,2... \qquad (2\text{-}24)$$

As in the case of the single constraint problem, the efficient selection of the lagrangian multipliers Φ^* and Θ^* necessary to minimize problem (P) has also proven to be formidable. The determination of Φ^* as well as Θ^* will be the major concern during the second phase of this research.

2.4 Literature Review

2.4.1 Foundation Review

Early literature pertaining to constrained inventory control, as reflected even in current textbooks, is surprisingly sparse. Holt [19] expanded on the initial foundation of Lagrangian relaxation laid by Everett [6], by recognizing that, when the carrying cost for each item of inventory is equivalent, Φ^* can be computed directly. Lewis [25], later echoed this same observation while clarifying the mathematics involved:

$$\Phi^* = 1/2\{[(2R_1D_1)^{1/2}+(2R_2D_2)^{1/2}]/U\}^2-C/2 \tag{2-25}$$

Holt also presented the first of several techniques developed for obtaining approximate Φ^* values. By using a difference equation approach he established an approximate linear expression for Φ^*. Differentiating a total cost equation by parts and then simplifying, Holt obtained an expression for the differential of Φ:

$$\delta\Phi = \sum_{i=1}^{N} 1/(Qi\ /8R_iD_i)[\delta IQ - \sum_{i=1}^{N} Q_i/4D_i\delta D_i]. \tag{2-26}$$

By replacing each differential ($\delta\Phi$,δI and δD) by finite differences, for example $\delta\Phi = \Phi - \Phi'$, a linear expression for Φ^* was obtained. This complicated process provided an approximate value for Φ' if a related Φ and its associated system were defined and the actual difference was relatively small.

Approximation techniques appear quit often in literature. Hadley and Whitin [13] and Johnson and Montgomery [21] presented marginal cost solutions to Problem ($P\Phi$) which called for the addition to inventory of items which had the largest marginal cost. This process was continued until no addition could be made without violating the imposed constraint. The most current of these approximation approaches are known as "approximate formulation". Britran and Matsuo [3] added to the approximate formulation techniques presented by Gavish [7]. Gavish sought to identify constrained optimization problems which could be solved using known solution processes while Britran sought to modify (P) into a form which could easily be solved.

Johnson and Montgomery [21] used the differentiability of ($P\Phi$) as a basis for the solution they presented. This solution technique recognized that the average cost of an N-item inventory system could be written as a function of the N decision variables, Q_i. The optimal set of Q_i could, in turn, be determined by solving the N+1 equations in N+1 unknowns given by:

$$\delta L/\delta\Phi = 0 \quad \text{and} \quad \delta L/\delta Q_i = 0 \quad i = 1,2,\ldots..$$

Hadley and Whitin [13] along with Buchan and Koeningsberg [4] shifted from these more direct, computationally difficult, solution methods to an iterative approach for obtaining the desired Φ^* value. Their computational procedure follows:

Step 1: Select some $\Phi > 0$.

Step 2: Compute each Q_i using equation (2-11).

Step 3: Compute Smax = Σ Q_i i=1 to n.

Step 4: If Smax > U select a larger Φ and go to Step 2.

Step 5: If Smax < U select a smaller Φ and go to Step 2.

Step 6: When Smax = U stop, the current Φ and Q_i values are optimal.

This process has become the classical solution method for the single constraint inventory problem and is presented universally as the "suggested" or "preferred" solution technique [4],[13],[16],[18],[19],[21],[25],[28],[29],[40].

Hadley and Whitin [13] incorporates this single constraint solution technique into what appears to be the classical approach to handling the inventory problem on which multiple constraints are imposed. This procedure, echoed by Buchan and Koeningsberg [4] as well as Tersine [38], consist of the following steps:

Step 1: Attempt, using the KKT conditions, to solve one of the constraint lagrangian multipliers in terms of the others. If this is possible, solve for each multiplier in-turn using the classical single constraint solution procedure to obtain the needed initial multiplier.

Hadley and Whitin [13, pp 57-58] effectively demonstrated this step when the maximum dollar investment and the number of inventory orders placed in a year are constrained.

Step 2: If step 1 is not possible, as in the case of two linear constraints, solve the inventory problem ignoring both constraints. If solution satisfies both constraints then that solution is optimal, otherwise go to step 3.

Step 3: Include only the first constraint into the problem and solve using the classical single constraint procedure. If this solution satisfies the second constraint then the current solution is optimal, otherwise go to step 4.

Step 4: Repeat step 3 including only second constraint. If this solution also satisfies the first constraint then the current solution optimal, otherwise go to step 5.

Step 5: If both step 3 and step 4 fail then both constraints are active and both multipliers must be greater than zero. Solve using a enumeration computer program.

2.4.2 Recent Research Efforts

A continued review of current publications reveals three active branches in the search for improved solution techniques for Problem (P). As described by Rosenblatt and Rothblum [33], these three branches of research differ primarily in the way each handles the various re-order cycle times for item's comprising an N-item inventory system. Here an inventory item's re-order cycle time is defined as that period of time between successive re-order events.

One of these research branches is based on the idea that, regardless of what individual re-order cycle times exist, a joint cycle can be determine. Within this joint cycle, orders of individual items can then be phased so as not to violate any imposed constraints. A second branch has as its basis the idea that a 'base re-order cycle time' can be identified of which each individual re-order cycle time is either an integer multiple or a power of two multiples. The goal of this second effort is to first group the individual re-order cycle times so that the stated condition holds and then phase the inventory orders over the 'base re-order cycle' so as to again not violate any active constraint. The third research branch is based on the idea that re-order cycle times are independent for each item type carried in inventory. Since all of the item types carried in an inventory system will eventually peak at the same time, it is the focus of this branch of inventory analysis to insure that the imposed constraints are not violated at each of these critical junctures.

The branch of research dealing with the joint re-order cycle, Krone [24], Maxwell [27], Parsons [32], Homer [20], Goyal [10], Paul and Page [30], Silver [36], Zoller [42], Hartley and Thomas [14,15], etc., appears in literature as early as 1964. The efforts outlined by these authors attempt to move away from the classical lagrangian approach by using an order time staggering process found to be effective in solving Problem (P) when the number of items carried in inventory is less than three or when the joint re-order cycle is extremely long. Paul and Page [30] provided a clear illustration of this branch of research using their "equal order interval" constraint. This time interval (t) was defined as follows:

$$t = \left(2 \sum_{i=1}^{N} R_i / \sum_{i=1}^{N} C_i D_i\right)^{1/2} \tag{2-27}$$

The assumptions made for this process were that each item of inventory had a periodic order cycle and that staggering of the resulting order times was always possible. The policy under this scheme was to restore the aggregate inventory to a predetermined level (M) where this level has been defined by

$$M = \tfrac{1}{2}t\{ \sum_{i=1}^{N} W_i D_i + \sum_{i=1}^{N} W_i 2D_i^2 / \sum_{i=1}^{N} W_i D_i \} \tag{2-28}$$

at the end of each order interval. W_i and D_i are defined as before. If the level determined by equation (2-28) is less than the available inventory space then M is set equal to U to determine the replenishment level necessary to satisfy the active constraint.

An overall effect of such staggering is a reduction in the maximum level of aggregate inventory held at any given time while maintaining the EOQ recommended stockage for each individual item. When the number of items carried in inventory is large, however, it is not likely that "Equal Order Intervals" exist. To account for this likelihood and to reduce to total cost of operating such an inventory system, Paul and Page introduced an algorithm which groups N items with unequal order intervals into K equal intervals, where $K \leq N$. With careful selection of the grouping intervals and when the number of groups is small, this approach to the single space constraint inventory problem was shown to often result in a lower total inventory cost than does the classical lagrangian approach. The Paul/Page method, however, is only valid when the resource restriction is on the maximum space consumption. Where the restriction is on the average inventory resource consumption the lagrangian multiplier is optimal [5,37].

Both Goyal [10] and Zoller [42] added to the Page/Paul algorithm. Goyal demonstrated an effective use of the grouping approach described by Page/Paul. Dealing with a three item inventory system, he utilized an analogue approach to determine how best to group the three items into two groups. Zoller addresses the issue of aggregate stock levels more generally demonstrating that within a given time horizon that orders can be phased so that it will never be necessary to have the maximum quantity of each item on hand at the same time. The cost effectiveness of such staggering, however, must quickly be questioned as the number of items carried in inventory grows. The attempt in these papers to withdraw from the classical approach, therefore, was not completely successful since in each case it is likely that a return to that proven approach will at times be necessary.

In a series of publications, Hartley and Thomas [14] and [15] recognized both the limitations of the staggering process and the computational inaccuracy possible with the Lagrangian multiplier approach which was demonstrated by White [40]. They sought to combine the two approaches for an improved algorithm. By examining a two item version of Problem (P) and assuming that an order for product 1 had just been placed, Hartley and Thomas showed that the maximum volume of stock will be minimized by selecting an Optimal Displacement Interval (ODI). The ODI, which simply defines the optimal time lag between ordering item one and item two, is computed as follows:

$$ODI = W_2D_2T/[n_1n_2(W_1D_1 + W_2D_2) \qquad (2\text{-}29)$$

while the maximum stockage level is given by:

$$Smax = W_1Q_1 + W_2Q_2 - [W_1W_2\, D_1D_2T/(W_1D_1 + W_2D_2)n_1n_2$$

where $T = n_1Q_1/D_1 = n_2Q_2/D_2$ and

n_1 and n_2 are positive integers. (2-30)

The optimal order quantities, under this approach, can be obtained by solving the following mixed integer problem:

$$\text{Min } (R_1n_1 + R_2n_2)D_1/Q_1n_1 + Q_1(C_1D_1n_2 + C_2D_2n_1)/2D_1n_1$$

Subject to:

$$Q_1[W_1D_1n_1 + W_2D_2n_2 - W_1W_2\ D_1D_2/(W_1D_1 + W_2D_2)] \leq UD_1n_2 \qquad \Omega(n_1,n_2)$$

To solve Problem $\Omega(n_1,n_2)$, Hartley and Thomas [15] defined an algorithm which first required calculation of the Φ^* value needed to optimize the Lagrangian formulation of Problem (P). This value was then used in establishing bounds on the function $\Omega(n_1,n_2)$. In improving the original Φ^* value, Hartley and Thomas utilized the functional shape of $\Omega(n_1,n_2)$ and an iterative process to facilitate a solution process.

Literature which focuses on that branch of inventory research dealing with the 'base re-order cycles' dates back to the early 1970's. Goyal [9,10], Silver [36], Goyal and Belton [12] as well as Kaspi and Rosenblatt [22] appear to have provided the primary thrust into this research arena. In general this approach provides more flexibility than seen in the joint cycle approach since by definition the base re-order cycle is greater than or equal to the joint cycle. However, as noted by Rosenblatt and Rothblum [33], the computational effort required is more extensive and implementation more difficult.

The third branch of research, identified in current literature focuses on the use of lagrangian multipliers in attempting to solve inventory problem (P). Two recent efforts have been identified which attempt to improve the classical solution process by establishing effective bounds on the desired lagrangian multiplier Φ^*. The first of these was presented by Ziegler [41] who provided an improvement to the Buchan/Koeningsberg iterative process noting from the Karush-Kahn-Tucker conditions (2-6) and (2-8) for problem (PΦ) that:

$$\Phi^* = D_iR_i/W_i(Q_i^*)^2 - C_i/(2W_i) \quad i = 1,2,... \tag{2-31}$$

and $$Q_i^* = [2D_iR_i/(C_i + 2\Phi^*W_i)]^{1/2} \quad i = 1,2,... \tag{2-32}$$

Recognizing that an estimate (Q_i^h) for Q_i^* could always be found by using a linear reduction process, Ziegler utilized the unconstrained Q_i defined by equation (1-2) to define those estimates.

$$Q_i^h = UQ_i \,/\, \sum_{i=1}^{N} W_i(Q_i)^2. \tag{2-34}$$

Incorporating these Q_i^h estimates into equation (2-31) provided i estimates of Φ^*. The largest and smallest of these Φ^* estimates were shown to be initial upper and lower bounds respectively, for the desired Φ^* value. Applying these i estimates to equation (2-3) produces both feasible and infeasible solutions. Ziegler demonstrated that the maximum estimate of Φ^* for which equation (2-3) is greater than U and the minimum estimate of Φ^* for with equation (2-3) is less than U reflect improved lower and upper bounds, respectively.

The second and most recent effort concerning effective bounding of the optimal lagrangian multiplier for problem (P) focuses on the functional relationships which exist between the lagrangian multiplier, Φ^*, and inventory system parameters: carrying costs, reorder costs and demand rates. This research effort is reflected in Maloney [26], Klein, Ventura, and Maloney [23], and Ventura and Klein [39]. In an initial effort, Maloney [26], identified both linear and near linear functional relationships between the single lagrangian multiplier associated with problem (PΦ) and each of the system parameters noted earlier. The lagrangian multiplier needed to optimize such randomly selected inventory systems were identified as each system parameter was to shift by discrete steps away from its initial value. Focusing separately on each parameter and running statistical tests on resulting parameter/multiplier pairs produced the indications of linearity shown in Table 2-1.

Characteristic	F-test	C C
Carrying		
C1	110360	.9977
C2	4744	.9950
Both	3885657	1.000
Reorder		
R1	59706	.9995
R2	68048	.9996
Both	524610	.9999
Demand		
D1	55214	.9994
D2	120841	.9997
Both	429619	.9999

Table 2-1 (Hint of Linearity)

When only the first of two system carrying costs was shifted, this analysis indicated a F-test value of 110360 and a correlation coefficient of .9977. When both carrying cost parameters were shifted simultaneously F-test and Correlation Coefficient values of 3885657 and 1.000 were obtained respectively. Table 2-2 displays the resulting correlation coefficients obtained when similar analysis was conducted on twenty random systems. Graphic presentations of these functional

Characteristic	Correlation Coefficients	
	smallest	largest
C	.9956	1.0000
R	.9910	.9999
D	.9902	.9999

Table 2-2 (Summery of Correlation Data)

relationships can be viewed at appendix 1. Klein, Ventura, and Maloney [23] established the linear relationships indicated in Tables 2-1 and 2-2 above.

Based on the indicated near linear relationships between Φ^* and each of the single variable situations Maloney [26] defined the following functional forms:

$$\Phi = f_i(C_i) \qquad i = 1 \text{ or } 2 \tag{2-34}$$

$$\Phi = q_i(R_i) \qquad i = 1 \text{ or } 2 \tag{2-35}$$

$$\Phi = p_i(D_i) \qquad i = 1 \text{ or } 2 \tag{2-36}$$

where Φ was defined empirically via equation (2-12) as:

$$[(2R_1D_1)/(C_1 + 2\Phi)]^{1/2} + [(2R_2D_2)/(C_2 + 2\Phi)]^{1/2} = U \tag{2-37}$$

for the two-item inventory system. Detailed analysis of these relationships utilizing inverse functional analysis where, for example,

$$C_1 = g(\Phi) = \{4UR_1D_1[2R_2D_2]^{1/2}(C_2 + 2\Phi)\}$$
$$/[U(C_2 + 2\Phi)^{1/2} -(2R_2D_2)^{1/2}]^2 -2\Phi \qquad (2\text{-}38)$$

provided the following conclusions:

FUNCTIONAL DESCRIPTION		
Function	Convexity	Monotonic Character
$f_i(C_i)$	convex	decreasing
$f_i(R_i)$	concave	increasing
$f_i(D_i)$	concave	increasing

Table 2-3 (Summary of Functional Analysis)

Armed with the above information Venture and Klein [39] formalized an effective algorithm, first presented by Maloney [26], which can be used to establish tight bound on the desired optimal multiplier value and which when applied recursively converges rapidly to the optimal value.

Defining the gradient of $\Phi = f_i(C_i)$ as:

$$[\delta g(\Phi)]^{-1} = \{-5W_j 2R_j D_j \sum_{i\neq 1}^{N} W_i[(2R_iD_i)^{1/2}/(C_i+2\delta)^{3/2}]$$

$$/[U - \sum_{i\neq 1}^{N} W_i[2R_iD_i/(C_i+2\delta)]^{1/2}]^3 - 2W_i\}^{-1} \qquad (2\text{-}39)$$

the following three step recursive process was suggested:

Step 0) Compute initial bounds or Φ^* ($\Phi_L{}^0$ and $\Phi_U{}^0$ using equation (3-1). $\Phi_L{}^0$ is computed using C = largest carrying cost in system while $\Phi_U{}^0$ is derived utilizing the smallest cost. Set t = 0.

Step 1) Compute

$$\Phi_L{}^{t+1} = \Phi_L{}^t + \delta[f_L(C_L)](C_L\text{-}C)$$

where $C = g(\Phi^t)$ from equation (3-15).

$$\Phi_U{}^{t+1} = \Phi_L{}^{t+1} + \frac{(\Phi_U{}^t - \Phi_L{}^{t+1})[g(\Phi_L{}^{t+1})\text{-}C_L]}{[g(\Phi_L{}^{t+1}) - g(\Phi_U{}^t)}$$

Step 2) If $(\Phi_U{}^{t+1} - \Phi_L{}^{t+1})/\ \Phi_L{}^{t+1} \leq \varepsilon$ (ε is a small tolerance), set $\Phi = (\Phi_U{}^{t+1} + \Phi_L{}^{t+1})/2$, and stop. Otherwise, set $C = g(\Phi_L{}^{t+1})$, let t = t+1 and go to step 1.

The results of a comparative analysis of this algorithm with that presented by Ziegler demonstrated that the Ventura/ Klein algorithm provided a tighter lower bound 78 percent of the time, a tighter upper bound 100 percent of the time, and a better bounding interval 85 percent of the time. It should be noted, however, that this comparative analysis focused solely on two item inventory systems in which the system parameter Wi (the amount of the budget or space consumed by each item of inventory) was held at one. Application of this algorithm to the N-item inventory system as well as to systems in which the weighing parameter is free to assume any value has yet to be examined. It will be at this point that current efforts begin.

Chapter III

Extension of Original Empirical Study

3.1 Introduction

The purpose of this empirical study was to identify, if possible, consistent relationships between the lagrangian multiplier Φ needed to solve Problem (PΦ) and various inventory system parameters not identified by the empirical study described in [26]. Examined closely, the N-item single constrained inventory system has five interactive parameters which combine to establish a required Φ^* value for a selected inventory system. These independent parameters include: carrying costs, Ci, re-order costs, Ri, uniform demand rates, Di, and a resource consumption rate, W_i for each item of inventory as well as the aggregate inventory constraint level, U. The empirical effort, reported in [26], first identified several near-linear relationships between the system's lagrangian multiplier and single parameter shifts and then focused on those cases where each pair of parameters (ie. both carrying cost) in a two item system shifted simultaneously. The current study, on the other hand, seeks to examine fully the effects of multiple parameter shifts within a N-item inventory system. This chapter will describe the sensitivity analysis used to identify existing relationships, outline the results obtained and draw several general conclusions concerning the inventory system under study.

3.2 Methodology

In order to gain an initial understanding of how Φ^* was related to shifts in system parameters while minimizing the magnitude of that study, the empirical work recorded in [26] considered as its primary focus single parameter shifts. In order to expand that understanding, a full examination of multiple shifts was required. To accomplish this goal a GW Basic program was developed and run on an IBM compatible PC to generate systematic shifts in selected system parameters and then identify the needed Lagrangian multiplier to optimize Problem (P).

For each randomly generated inventory system examined, the same parameter shifting scheme was utilized. Consider the following inventory system parameter matrix:

Carrying Cost	Re-order Cost	Demand Rate
A	B	C
D	E	F
G	H	I

where A - I represent randomly generated system parameters for a three item inventory system. To generate each set of data points, the selected inventory system underwent twenty systematic perturbations. For example, to generate an initial set of empirical data, the carrying cost parameter, A, for the first item in inventory was repeatedly modified by adding 1 to its previous value. Adherents to this shifting scheme generated the desired twenty inventory systems in which parameter A assumed the values indicated below and in which all remaining parameters remained constant.

A ---> A+1 ---> A+2 ---> ----> A+20

Continuing this shifting scheme, the next set of empirical data was generated by altering parameters A and B simultaneously so that:

A ---> A+1 ---> A+2 ---> ----> A+20

B ---> B+1 ---> B+2 ---> ----> B+20

While the next data set stemmed from the simultaneous shifting of parameters A, B and C.

This gradual explosion of the initial inventory system continued until every possible combination of parameter shifts had been examined. For each set of data the slope of the line formed via a simple linear regression and the corresponding correlation coefficient were computed.

3.3 Analysis Results

	Carrying Cost	Re-order Cost	Demand Rate
Item #1	89	41	87
Item #2	5	92	95
Item #3	5	29	52
Item #5	30	7	79

To illustrate this data generation procedure consider the above four item inventory system. The analysis of an initial set of empirical data, generated by shifting the carrying cost for Item #1 from 89 through 109 using unit steps, identifies a regression line which displayed a -.0151 slope and a correlation coefficient (CC) of .9939. These results parallel those obtained by Maloney [26] for similar single parameter perturbations.

A summary of this analysis focusing on selected parameter shifts within the above inventory system follows:

Shifting Parameters	Slope	Correlation Coefficient	Initial Φ^*	Final Φ^*
C1	-.0151	.9939	9.687	9.007
C1,C2	-.1535	.9397	8.205	1.555
C1,C2,C3	-.4575	.9999	7.550	0.580
C1,C2,C3,C4	-.5000	1.0000	7.300	2.291
R1	.0506	.9996	10.071	12.353
R1,R2	.1530	.9996	10.500	17.000
R1,R2,R3	.2565	1.0000	11.055	22.600
R1,R2,R3,R4	.5518	1.0000	12.000	31.880
D1	.0258	.9958	10.000	11.053
D1,D2	.1079	.9999	10.330	15.200
D1,D2,D3	.1729	1.0000	10.655	18.520
D1,D2,D3,D4	.1957	.9999	10.733	19.535
C1,R1	.0221	.9955	10.000	11.000
R1,D1	.1000	.9981	10.216	15.760
C1,R1,D1	.0592	1.0000	10.100	12.770
ALL	.5163	.9715	10.615	33.810

Table 3-1 (Selected Data/Analysis For A Four-Item Inventory)

It should be noted that the notation "R1,R2,R3", for example, under the column headed "Shifting Parameter" indicates those parameters undergoing change. All other system parameters were held constant.

Although Table 3-1 exhibits only a small portion of the data generated from the above inventory system (see Appendix 3), several general attributes emerge. First, note that the range exhibited by the computed CC values lie consistently close to 1.0000, deviating by at most by .0061 in those cases where parameter shifts are all within the same system characteristic and by .0285 when all system parameters were modified. This observation suggest a near linear relationship between Φ^* and multiple shifts in a variety of system parameters.

A second attribute is that as additional parameters from the same cost category (ie. carrying cost parameters 1,2,...) are folded into the analysis the resulting slope of the regression line gradually grows. For example when perturbations are applied to only a single carrying cost the regression slope is -.0151, however, as two, three and then all four carrying cost are simultaneously altered, the resulting regression slope converges to -.5000. This phenomena was observed during the earlier two-item inventory empirical study [26] and appears to hold for the N-item case as well.

A third attribute is that when only carrying cost parameters are modified, the final Lagrangian multiplier is always greater than the initial value obtained. This attribute is mirrored when either the Re-order costs or the Demand rates become the focus of the analysis. In such cases, however, the final Lagrangian multiplier is always less then the initial value. An explanation of these phenomena can be seen by examining the expression given in Chapter I for computing the unconstrained re-order quantity along with those given in Chapter II for computing the constrained re-order quantity and in defining the KKT condition of feasibility. These equation are restated below:

$$Q = [2RD/C]^{1/2} \qquad (1\text{-}2)$$

$$Q_i^* = ((2R_iD_i) / (C_i + 2W_i\Phi))^{1/2} \quad i = 1,2...... \qquad (2\text{-}11)$$

$$\Phi (\Sigma W_iQ_i - U) = 0 \quad i = 1,2..... \qquad (2\text{-}8)$$

It is clear from equation (1-2) that when the carrying cost increases, the unconstrained re-order quantity decreases. Similarly, when either the re-order cost or the demand rate increases, the unconstrained re-order quantity increases. With these patterns in mind, the combination of equations (2-8) and (2-11) implies that if a constraint is binding then at the optimal solution the expression $\Sigma W_iQ^*_i$ must equal U in order to maintain feasibility and Φ must be greater than or equal to zero. These factors suggest that, when a constraint is active, any increase in Q_i caused by a decreased C_i, an increased R_i or an increased D_i must be offset by a corresponding increase in the related Lagrangian multiplier. Similarly, any decrease in Q_i cause by an increased C_i, a decreased R_i or a decreased D_i dictates a corresponding decrease in the resulting Lagrangian multiplier. The truth of these last statements arises from the position Φ holds in the constrained expression for the re-order quantity, equation (2-11), and the necessity to maintain the condition of feasibility shown in equation (2-8).

Since firm conclusions cannot be drawn solely from a single example, a total of six randomly generated inventory systems were subjected to the analysis just described. Table 3-2, which summarizes of the results obtained from examining two 2-item, one 3-item, one 4-item and two 5 item inventory systems, suggest that the above attributes hold in general. The arrows in the last column indicates whether Φ increases or decreases as the selected set of parameters increase.

Shifting Parameters	Min Slope	Max Slope	Min CC	Max CC	Φ
C1	-.0155	-.3878	.9783	.9999	↓
C1,C2	-.1325	-.5000	.9379	1.0000	↓
C1,C2,C3,	-.2557	-.5000	.9935	1.0000	↓
C1,C2,C3,C4	-.4680	-.5000	1.0000	1.0000	↓
C1,C2,C3,C4,C5	-.5000	-.5000	1.0000	1.0000	↓
R1	.0506	1.6057	.9995	.9999	↑
R1,R2	.1530	6.5093	.9996	.9999	↑
R1,R2,R3	.2565	.5858	1.0000	1.0000	↑
R1,R2,R3,R4	.5518	.6369	1.0000	1.0000	↑
R1,R2,R3,R4,R5	.7696	.7696	1.0000	1.0000	↑
D1	.0258	1.3913	.9958	.9999	↑
D1,D2	.1079	1.7565	.9996	1.0000	↑
D1,D2,D3	.1729	.7856	1.0000	1.0000	↑
D1,D2,D3,D4	.1957	1.2057	.9999	.9999	↑
D1,D2,D3,D4,D5	1.5811	1.5811	1.0000	1.0000	↑
C1,R1	.0179	1.1910	.9913	.9998	↑
R1,D1	.1008	5.1628	.9953	.9990	↑
C1,R1,D1	.0592	3.7326	.9877	1.0000	↑
ALL	.5163	12.3557	.9715	.9958	↑

Table 3-2 (Selected Summary of All Data Runs)

In comparing the analysis results presented in Table 3-2 with those shown in Table 3-1 it should be noted that the correlation coefficient values remain close to one even over a wide range of inventory systems. A review of all the data generated within this study indicates that correlation coefficient values range between 1.000 and .7106. However, it should be observed that of the 356 data sets analyzed, 89 % of the correlation coefficient values fall above .99 while more than 95 % fall above .98. In short, the same near linear relationships which were found to exist between single parameter shifts within a two-item system also exist when multiple parameter shifts occur in the N-item inventory.

Before moving on to the functional analysis to be presented in Chapter IV, a graphic examination of the empirical data provides a clear indication of the nature of the relationships suggested by the above CC analysis. Figures 3-1 through 3-7 were constructed using data collected from a randomly generated two-item inventory. Figures 3-1 through 3-3 depict cases where all possible parameters from a single system characteristic undergo perturbation. Figures 3-4 through 3-6 reflect cases where two different system characteristics are paired. Finally Figure 3-7 illustrates those cases when all system characteristics of a single inventory item are shifted. For proofs of several of the indicated linear relations see Appendix 3.

3.4 Conclusion

This chapter has described the sensitivity analysis used to identify existing relationships between the optimal Lagrangian multiplier and multiple shifts in a single constraint inventory system defined as Problem (P) in chapter II. The gradual explosion of each randomly generated system provided a detailed look at the impact of every possible parameter combination on the resulting Lagrangian multiplier.

The correlation coefficient and regression line slope analysis conducted on each set of data identified three attributes which appear to be characteristic of all such inventory systems. These attributes: a near-linear relationship between Φ^* and any

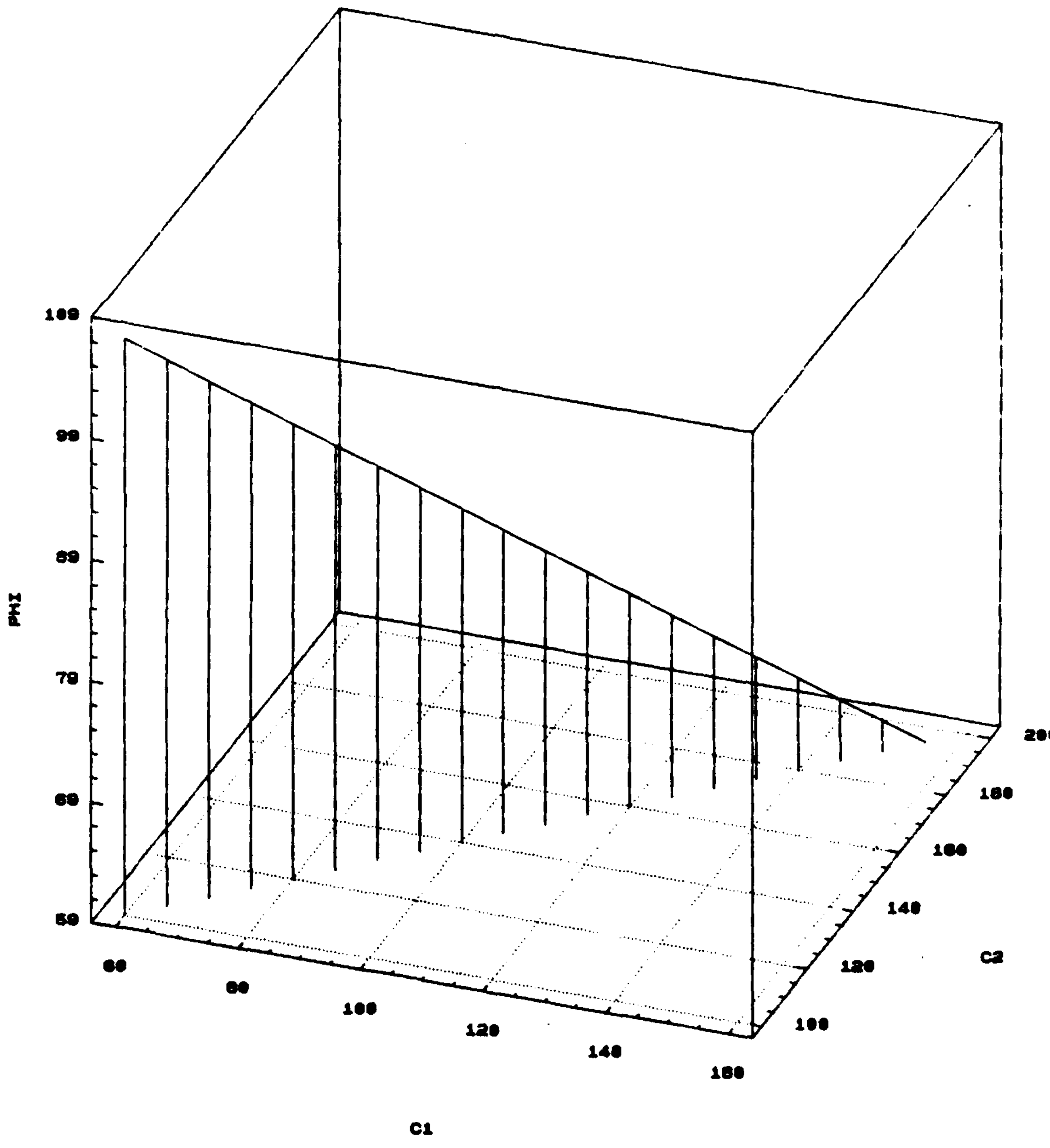

Figure 3-1 (Lagrangian Multiplier vs Carrying Cost 1 and 2)

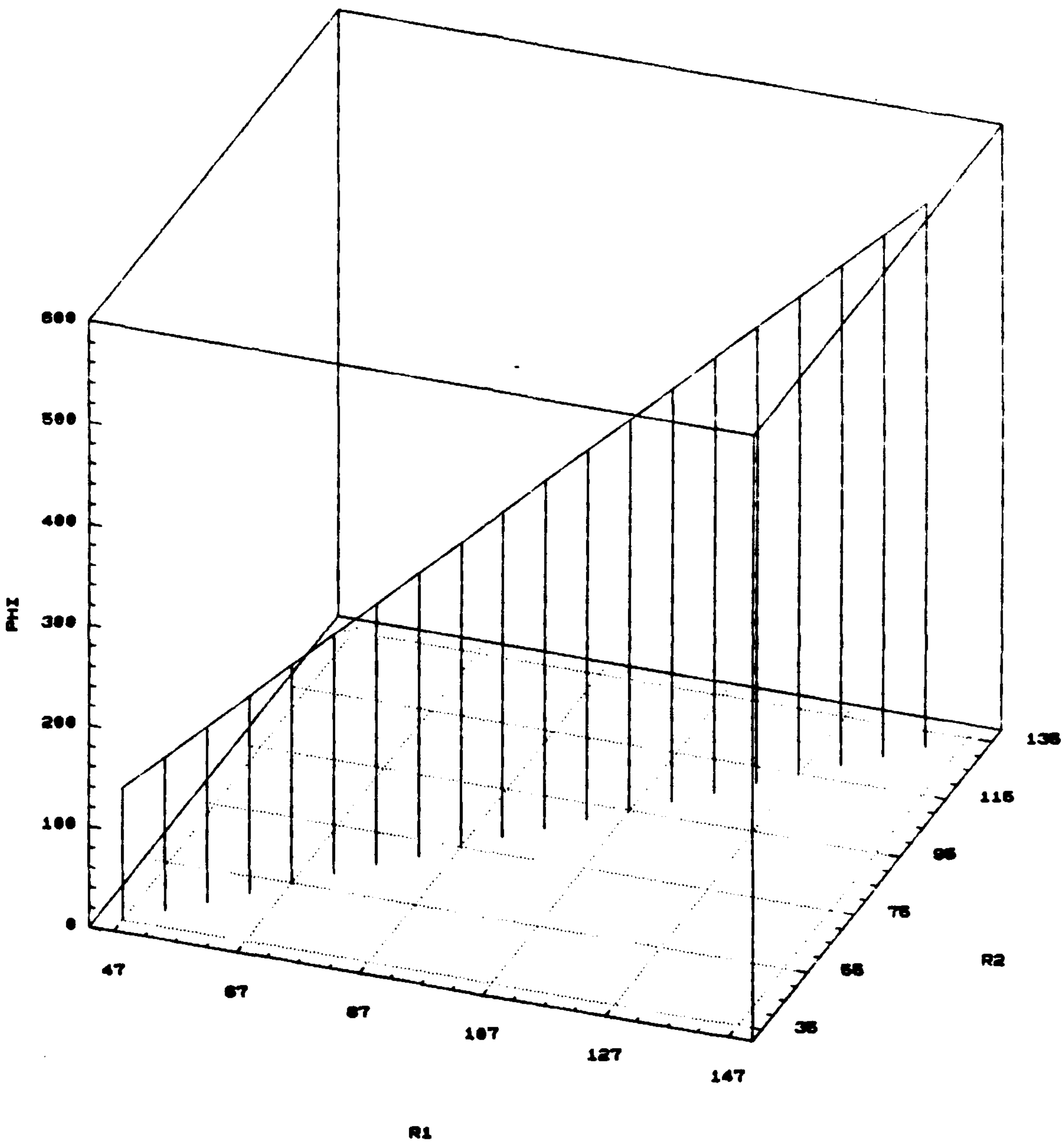

Figure 3-2 (Lagrangian Multiplier vs Re-order Cost 1 and 2)

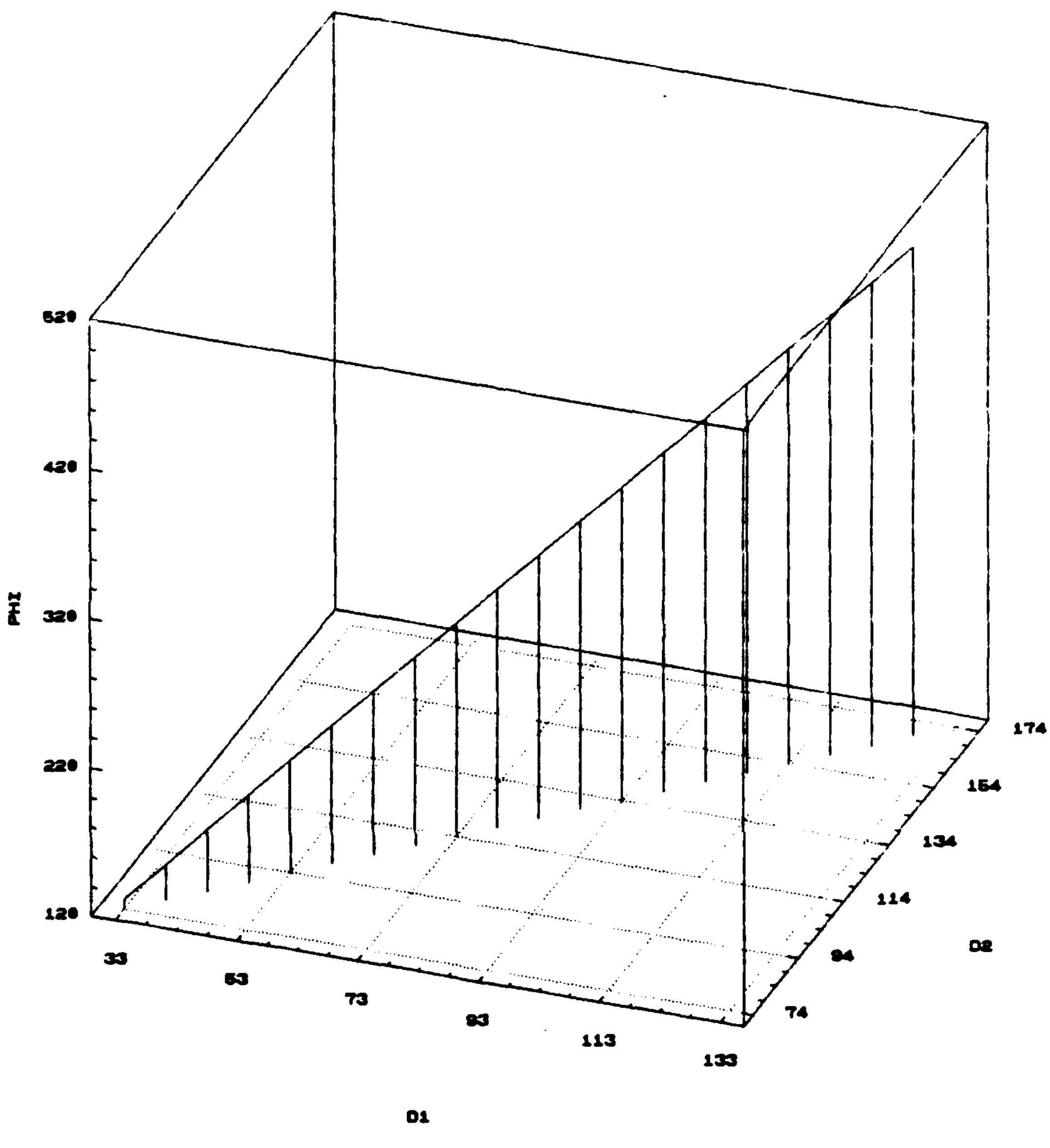

Figure 3-3 (Lagrangian Multiplier vs Demand Rates 1 and 2)

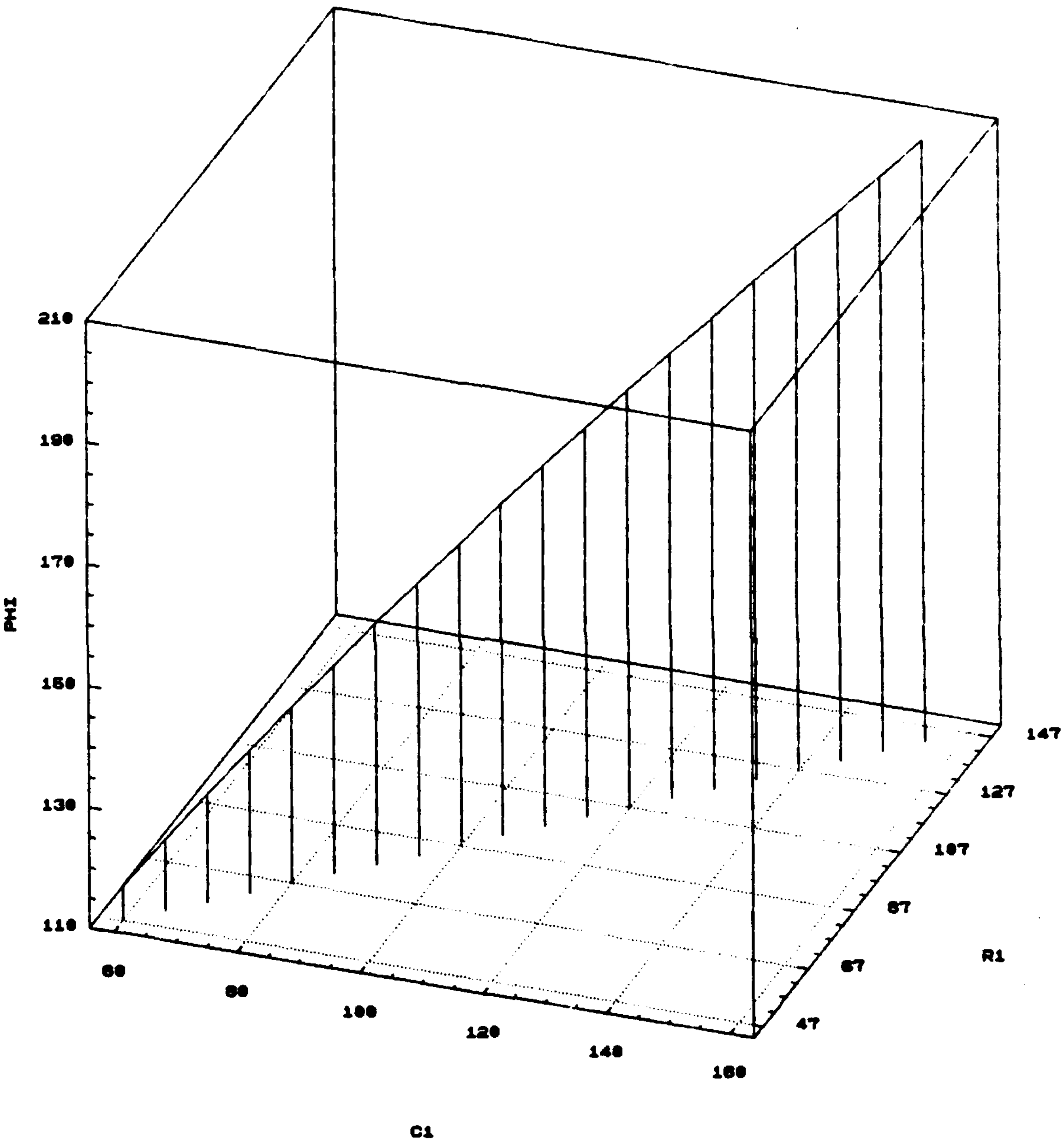

Figure 3-4 (Lagrangian Multiplier vs Carrying Cost 1
and Demand Rate 1)

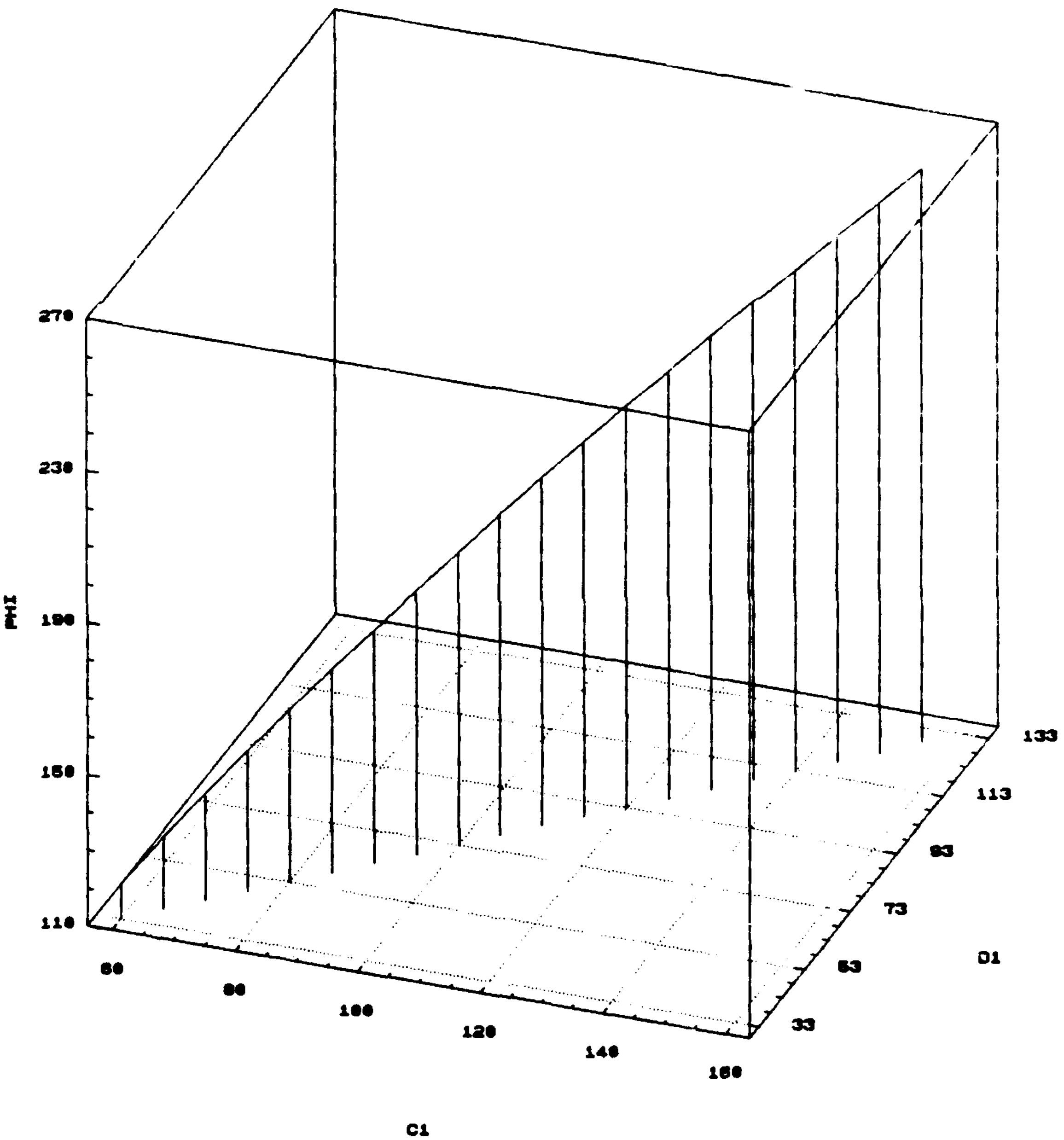

Figure 3-5 (Lagrangian Multiplier vs Carrying Cost 1
and Re-order Cost 1)

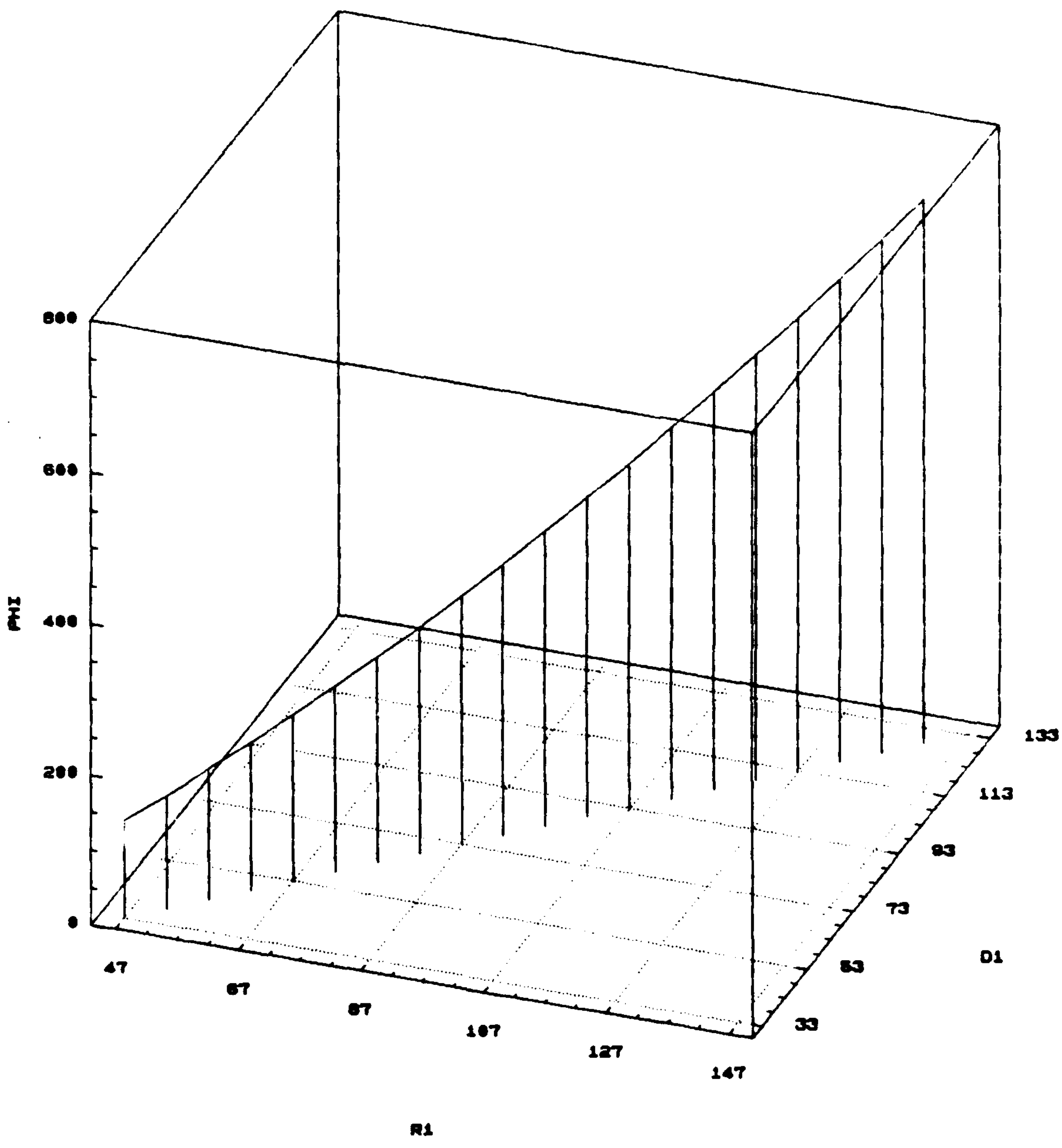

Figure 3-6 (Lagrangian Multiplier vs Re-order Cost 1
and Demand Rate 1)

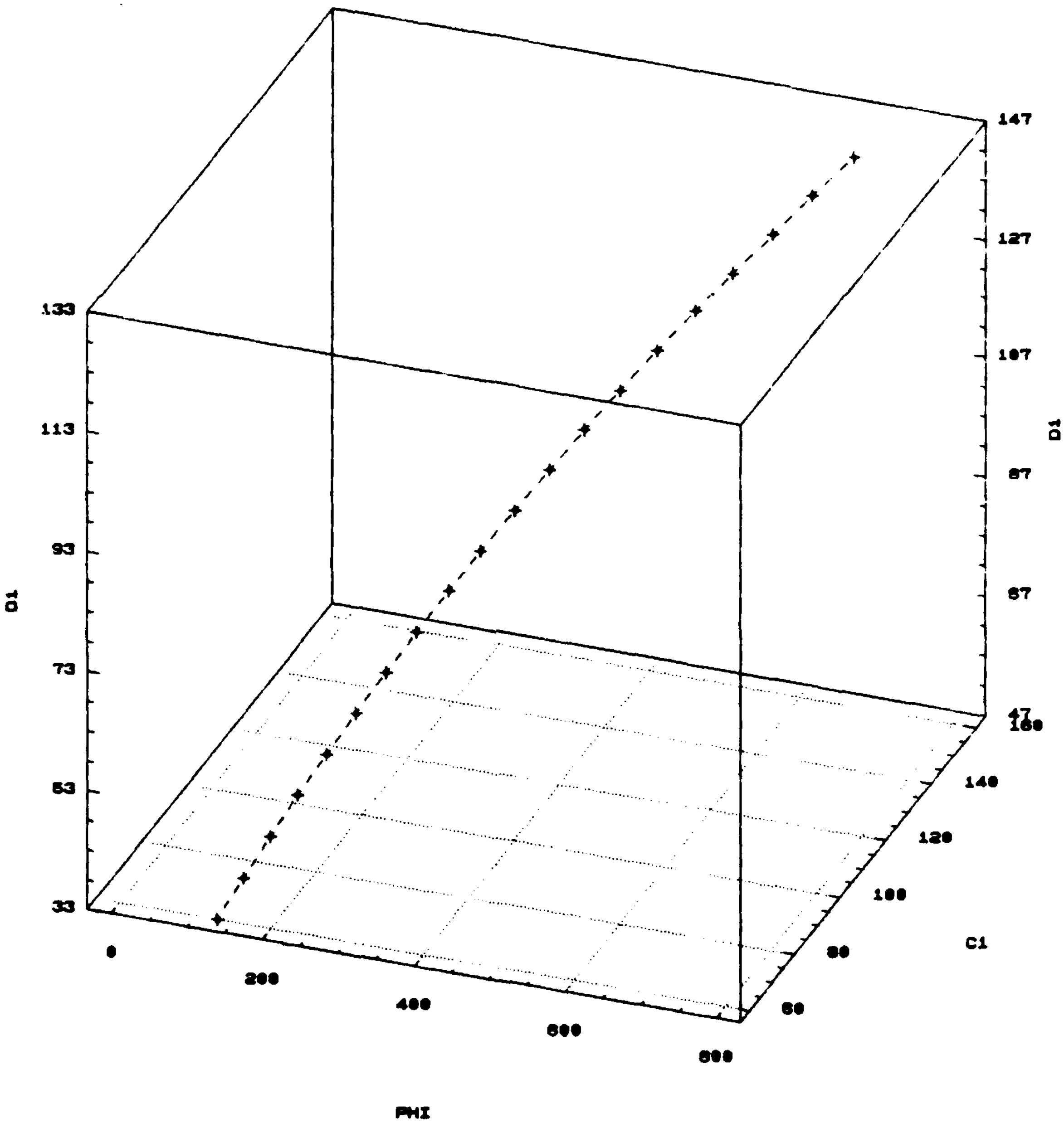

Figure 3-7 (Lagrangian Multiplier vs Carrying Cost 1, Re-order Cost 1 & Demand Rate 1)

combination of system parameter shifts, an increasing regression line slope as additional parameters from the same category are added to the analysis and the monotonic nature of these functions were first identified while examining a single four-item inventory system but were reinforced after examining the 356 different data sets obtained by exploding six random inventory systems.

Finally near-linear relationships, found in the analysis described above, were examined utilizing the graphics package STATGRAPHICS in order to gain a clear picture of the actual nature of selected shifting patterns. The similarity of these plots to those found in Appendix 1 strongly suggest possible improvements to the Ventura/Klein algorithm [39].

Chapter IV

Single Constrained Inventory System Functional Analysis
(Multiple Parameter Perturbations)

4.1 Introduction

The basis for attempting to extend the research documented by [23], [26], and [39] by considering multiple parameter shifts emanates from the difficulty encountered during those efforts to isolate selected system parameters for analysis. Additionally, the results of the empirical study involving multiple parameter perturbations, documented in Chapter III, strongly suggest such a strategy. The approach utilized during previous research efforts, as has been noted, was based on the condition of feasibility, equation (2-8), and resulted in a capability to consider only single parameter shifts within the inventory system. Equations (4-1) and (4-2), in turn, illustrate the condition of feasibility and a resulting single carrying cost parameter function for the N-item inventory system.

$$\sum_{i=1}^{N} W_i[(2R_iD_i)/(C_i + 2W_i\Phi)]^{1/2} = U \tag{4-1}$$

$$C_j = G_j(\Phi) = 2R_jD_j/\{U-\Sigma[(2R_iD_i)/(C_i + 2W_i\Phi)]^{1/2}]^2 - 2W_i\Phi \quad \text{where } j \neq i \tag{4-2}$$

Utilizing the inverse function of equation (4-2), Maloney [26] identified selected functional relationships which were then used to construct the optimizing recursive algorithm presented in [39] for the two item inventory system. To implement this algorithm for a system in which multiple parameters vary, however, the entire algorithm must be applied separately to each shifting parameter. Clearly, as the number of such parameters increases, the algorithm quickly becomes ineffective. In order to over come this difficulty a method of handling multiple parameter perturbations had to be developed. The functional analysis described in this chapter lays the foundation needed for such an advancement.

This chapter discusses the reformulation of equation 4-1 that allows the weighting factors, W_i, to assume any value. This was a major shortfall in the algorithm presented in Ventura and Klein [39]. This reformulation is followed by a description of the analysis performed on that modified equation from which both slope and convexity characteristics for selected multi-parameter shifts were established. Finally, the results of this analysis will be summarized in preparation for the algorithm presented in chapter V.

4.2 Reformulation

$$\sum_{i=1}^{N} [(2R_iD_iW_i)/(C_i/W_i + 2\Phi)]^{1/2} = U \qquad (4\text{-}3)$$

By dividing both the numerator and the denominator of equation 4-1 by W_i, Φ is freed from the influence W_i previously held. A careful examination of equation 4-1 reveals that, except when all W_i's are held at one, the true Φ value is masked by these variables. This masking effect causes the algorithm presented in Ventura and Klein [39] to fail whenever W_i deviates from one.

Based on this observation, two approaches can be distinguished. First, by removing the W_i linkage to Φ, these weighting factors can be considered constant system parameters and carried through the suggested analysis without dynamic impact. Second, by keying on the ratio C_i/W_i as one of the system parameters subject to perturbation, the results of the current empirical study continue to be valid.

4.3 An Implicit Approach

Fundamental to this analysis is the realization that equation (4-3) is a function which cannot be solved explicitly for Φ but from which Φ can be determined as an implicit function of the other system parameters [8]. This can be seen from the fact that the function f(Ci,Ri,Di,Φ), as defined by equation (4-3), is a monotonic function and is differentiable with respect to each system parameter as well as with respect to Φ [26]. By approaching the problem as implicit rather than as explicit, a solution technique can be developed which allows the desired multiple parameter movement.

The potential success of approaching the study of Problem (PΦ) implicitly can be demonstrated considering a two-item inventory system along with the results of the empirical study documented by Maloney [26]. During that study, it was determined that when both carrying cost parameters were allowed to shift concurrently, the resulting slope of the linear regression line was always equal to - ½. The current empirical study also suggest that as each ratio contained in a N-item inventory is in turn added to the analysis, the resulting slope converges to - ½.

Using the principle of implicit functions given by Goodman [8], the proof of this observation becomes trivial. Considering the general case where the amount of space consumed by each item carried in inventory, W_i, can assume any positive value the follow theorem holds.

Theorem 4-1

Let A be an N-item inventory system consisting of carrying costs (C_i), reorder costs (R_i), demand rates (D_i) and resource consumption rates (W_i). When each of the (C_i/W_i) ratio shifts concurrently and uniformly away from their initial values, the slope (M) of the function $f(C_i/W_i,\Phi)$ is -½.

Proof: Let $f(C_i/W_i,\Phi)$ =

$$\sum_{i=1}^{N} [(2R_iD_iW_i)/(C_i/W_i + 2\Phi)]^{1/2} = U \tag{4-5}$$

By definition [8], the derivative of Φ with respect to each C_i/W_i is:

$$\frac{\delta\Phi}{\delta(C_i/W_i)} = \frac{-\delta f/\delta(C_i/W_i)}{\delta f/\delta\Phi\ .} \tag{4-6}$$

It follows that

$$-\delta f/\delta(C_i/W_i) = ½[2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2} \tag{4-7}$$

for each C_i/W_i ratio and that

$$\delta f/\delta\Phi = \sum_{i=1}^{N} - [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}\ . \tag{4-8}$$

Therefore, when each of the (C_i/W_i) ratios shift concurrently and uniformly away from their initial values the slope becomes:

$$\sum_{i=1}^{N} \frac{\delta\Phi}{\delta(C_i/W_i)} = \frac{\sum_{i=1}^{N} \frac{1}{2}[2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}}{\sum_{i=1}^{N} -[2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}} \qquad (4\text{-}9)$$

$$= -\ \frac{1}{2} \qquad \blacksquare$$

It is clear from the above proof that any number and any combination of parameters comprising $f(C_i,R_i,D_i,\Phi)$ can be analyzed to determine slope and functional convexity information. Close examination of equation 4-9 indicates that Theorem 4-1 is valid for any combination of C_i and or W_i shifts which produce the desired uniform change in the C_i/W_i ratios. It follows therefore that the theorem holds whether the uniform movements among the ratios are generated by holding each C_i constant and shifting the W_i's or by holding each W_i constant while shifting C_i's.

4.4 Implicit Function Analysis

Based on the near linear relationships between Φ^* and the multiple variable situations characterized in chapter III, the following functions are defined:

$f_1(\Phi,C_i/W_i)$ $\quad$ $i = 1,2,...$and/or N $\quad$ (4-10)

$f_2(\Phi,R_i)$ $\quad$ $i = 1,2,...$and/or N $\quad$ (4-11)

$f_3(\Phi,D_i)$ $\quad$ $i = 1,2,...$and/or N $\quad$ (4-12)

$f_4(\Phi,R_i,D_i)$ $\quad$ $i = 1,2,...$and/or N $\quad$ (4-13)

$$f_5(\Phi, C_i/W_i, R_i, D_i) \qquad i = 1,2,\ldots \quad \text{or } N \tag{4-14}$$

For these functions, Φ is implicitly defined by the combined effects exerted by each of the other parameters which comprise the system. Notationally, the variables C_i/W_i, R_i and D_i, depict that set of parameters undergoing change. All remaining parameters remain unchanged. For example, equation (4-10) describes the case where some combination of the system's C_i/W_i ratios undergo perturbation while all other parameters remain unchanged.

4.4.1 First Order Analysis

Building on the definition of the partial derivative of Φ with respect to selected parameter shifts utilized in the proof of Theorem 4-1, an examination of the first order derivative for each of these functions was conducted to determine the nature of the corresponding functional slopes. The results of this analysis are contained in the following theorems.

Theorem 4-2

The function defined implicitly between Φ and the ratio C_i/W_i by $f_1(\Phi, C_i/W_i)$, i = 1,2,...., or N, is a strictly monotonically decreasing function.

Proof: Let $f_1(\Phi, C_i/W_i)$ =

$$\sum_{i=1}^{N} [(2R_iD_iW_i)/(C_i/W_i + 2\Phi)]^{1/2} - U \tag{4-15}$$

Theorem 1 [8, pp 165] states that if the first derivative of a function is always less than zero then that function is a strictly monotonically decreasing function. Using the definition for partial derivative [8] and WLOG letting i=1, the gradient of Φ with respect to C_1/W_1 becomes:

$$\frac{\delta\Phi}{\delta(C_1/W_1)} = \frac{-\tfrac{1}{2}[2R_1D_1W_1/(C_1/W_1 + 2\Phi)^3]^{1/2}}{\sum_{i=1}^{N} [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}} \qquad (4\text{-}16)$$

Clearly, equation (4-16) is always less than zero since each of its terms are positive.

∎

Following a similar approach Theorems 4-3 and 4-4 show that, when either a single re-order cost or a single demand rate becomes the focus of analysis, the resulting function is strictly monotonically increasing.

Theorem 4-3

The function defined implicitly between Φ and R_i by $f_2(\Phi,R_i)$, i = 1,2,...., or N, is a strictly monotonically increasing function.

Proof: Let $f_2(\Phi,R_i)$ =

$$\sum_{i=1}^{N} [(2R_iD_iW_i)/(C_i/W_i + 2\Phi)]^{1/2} - U \qquad (4\text{-}17)$$

Theorem 1 [8,pp 165] states that if the first derivative of a function is always greater than zero then that function is a strictly monotonic increasing function. Again, WLOG, letting i=1 and following the same technique utilized in proving Theorem 4-2 yields:

$$\frac{\delta\Phi}{\delta R_1} = \frac{\tfrac{1}{2} R_1^{-\frac{1}{2}} \left[2D_1W_1/(C_1/W_1 + 2\Phi)\right]^{1/2}}{\sum_{i=1}^{N} \left[2R_iD_iW_i/(C_i/W_i + 2\Phi)^3\right]^{1/2}} \tag{4-18}$$

Since each element of equation (4-18) is positive the resulting differential is always greater than zero and therefore any single re-order parameter shift produces a strictly monotonically increasing function. $\blacksquare$

Theorem 4-4

The function defined implicitly between Φ and D_i by $f_3(\Phi, D_i)$, i= 1,2,...,or N, is a strictly monotonically increasing function.

Proof: Follows from proof of Theorem 4-3. $\blacksquare$

To extend each of these theorems to those cases in which multiple parameter shifts are allowed, the contribution made by each shifting parameter must be considered. This collective effect can be measured by summing each of the individual effects [8, pp 190]. In an N-item inventory where J represents the set of parameters from a single characteristic category (ie. re-order cost or demand rate) which undergo change, the following partial derivatives result:

$$\sum_{i\in J} \frac{\delta\Phi}{\delta(C_i/W_i)} = \frac{-\frac{1}{2} \sum_{i\in J} [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}}{\sum_{i=1}^{N} [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}} \tag{4-19}$$

$$\sum_{i\in J} \frac{\delta\Phi}{\delta R_i} = \frac{\frac{1}{2} \sum_{i\in J} R_i^{-\frac{1}{2}} [2D_iW_i/(C_i/W_i + 2\Phi)]^{1/2}}{\sum_{i=1}^{N} [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}} \tag{4-20}$$

$$\sum_{i\in J} \frac{\delta\Phi}{\delta D_i} = \frac{\frac{1}{2} \sum_{i\in J} D_i^{-\frac{1}{2}} [2R_iW_i/(C_i/W_i + 2\Phi)]^{1/2}}{\sum_{i=1}^{N} [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}} \tag{4-21}$$

Careful inspection of equations (4-19), (4-20) and (4-21) shows that even when multiple parameters undergo changes, the results proven in Theorems 4-2, 4-3 and 4-4 hold.

Building on the results of Theorem 4-3 and Theorem 4-4 the following theorem establishes the monotonic nature of the function where Φ is implicitly defined by any combination of re-order and demand shifts.

Theorem 4-5

The function defined implicitly between Φ and both D_i and R_i by $f_4(\Phi,R_i,D_i)$, i= 1,2,...,or/and N, is a strictly monotonically increasing function.

Proof: The first derivative of $f_4(\Phi,R_i,D_i)$ equals the sum of the first partial derivatives

[8, pp 190]. Therefore, from Theorem 4-3 and Theorem 4-4, $f_4(\Phi,R_i,D_i)$, i= 1,2,...,or/and N is a strictly monotonically increasing function. ∎

Noting that, within equation (4-5), the parameters R_i and D_i can be replaced, WLOG, by the variable RD_i an alternate statement of Theorem 4-5 follows:

Corollary 4-5-1

The function defined implicitly between Φ and RD_i by $f_4(\Phi,RD_i)$, i= 1,2,...,or/and N, is a strictly monotonically increasing function.

Proof: Following from Theorem 4-3 and Theorem 4-4 the first derivative of $f_4(\Phi,RD_i)$, i= 1,2,...,or/and N can be stated as follows:

$$\sum_{i\in J} \frac{\delta\Phi}{\delta RD_i} = \frac{\sum_{i\in J} \frac{1}{2} RD_i^{-\frac{1}{2}} \left[2W_i/(C_i/W_i + 2\Phi)\right]^{1/2}}{\sum_{i=1}^{N} \left[2R_iD_iW_i/(C_i/W_i + 2\Phi)^3\right]^{1/2}} \tag{4-22}$$

Now since each element comprising equation (4-22) is positive, the resulting differential is always greater than zero and hence $f_4(\Phi,RD_i)$, i= 1,2,...,or/and N is a strictly monotonically increasing function. ∎

The results of the last four theorems establish those cases in which the monotonic nature of the function does not vary. The nature of these functions combined with their respective convexities, shown in the following section, provide the theoretical foundation for the algorithm presented in chapter 5. However, before this analysis is complete the function, $f_5(\Phi,C_i/W_i,R_i,D_i)$ i = 1,2,... or N, must be examined. Theorem A4-1, shown in Appendix 4, indicates that when Φ is established by the combined effect of each parameter from a single inventory item the function is monotonically increasing (decreasing) if $\Phi \geq (\leq) [(R_iD_i/2(R_i+D_i)] - (C_i/W_i)$.

4.4.2 Second Order Analysis

Due to the complexity of the implicitly generated second order derivatives it is not possible to determine the convexity of functions 4-10 through 4-14 by considering the Hessian. Theorem 3.3.3 [1, pp 91], however, can be used to circumvent this difficulty. This theorem simply states that, when $\mathbf{X}_1$ and $\mathbf{X}_2$ are vectors, f is convex iff

$$f(\mathbf{X}_2) \geq f(\mathbf{X}_1) + \nabla f(\mathbf{X}_1)^t (\mathbf{X}_2-\mathbf{X}_1)$$

and that f is concave iff

$$f(\mathbf{X}_2) \leq f(\mathbf{X}_1) + \nabla f(\mathbf{X}_1)^t (\mathbf{X}_2-\mathbf{X}_1)$$

This result leads to the following theorems.

Theorem 4-6

The function $f_1(\Phi,C_i/W_i)$, i = 1 to m, m $\leq$ N, where Φ is defined implicitly by C_i/W_i, is a convex function.

Proof: By Theorem 4-2 $f_1(\Phi,C_i/W_i)$ is a monotonically decreasing function. Then by Theorem 3.3.3 [1], $f_1(\Phi,C_i/W_i)$ i = 1 to m, m $\leq$ N is a convex function if:

$$f_1(\Phi_1,C_i/W_i+\Delta) \geq f_1(\Phi_2,C_i/W_i) + \nabla f_1(\Phi_2,C_i/W_i)^t\,(C_i/W_i+\Delta-C_i/W_i) \text{ or if}$$

$$f_1(\Phi_1,C_i/W_i) \geq f_1(\Phi_2,C_i/W_i-\Delta) + \nabla f_1(\Phi_2,C_i/W_i-\Delta)^t\,(C_i/W_i-C_i/W_i+\Delta) \tag{4-23}$$

where WLOG $\Delta \geq 0$ represents simultaneous and uniform shifts in selected C_i/W_i parameters. Focusing on the first of these expressions and recognizing that as long as $\exists$ $\Phi > 0$ such that the KKT condition of feasibility (2-8) is satisfied then

$$f_1(\Phi_2,C_i/W_i) = f_1(\Phi_1,C_i/W_i+\Delta).$$

It follows therefore that $f_1(\Phi,C_i/W_i)$ $i = 1$ to m, $m \leq N$ is a convex if:

$$\nabla f_1(\Phi_2,C_i/W_i)^t\,(\Delta) \leq 0. \tag{4-24}$$

Now since each of the m elements of vector Δ are positive and since each of the m elements of $\nabla f_1(\Phi_2,C_i/W_i)$, via equation 4-16, are negative then clearly equation 4-24 holds and $f_1(\Phi,C_i/W_i)$ is convex regardless of the value $m \leq N$ assumes. ∎

Theorem 4-7

The function $f_2(\Phi,R_i)$ $i = 1$ to m, $m \leq N$, where Φ is defined implicitly by R_i, is a concave function.

Proof: By Theorem 4-3 $f_2(\Phi,R_i)$ is a monotonically increasing function and by Theorem 3.3.3 [1], $f_2(\Phi,R_i)$ $i = 1$ to m, $m \leq N$ is a concave function if:

$$f_2(\Phi_1,R_i+\Delta) \leq f_2(\Phi_2,R_i) + \nabla f_2(\Phi_2,R_i)^t\,(R_i+\Delta-R_i) \text{ or if}$$

$$f_2(\boldsymbol{\Phi}_1,\mathbf{R}_i) \leq f_2(\boldsymbol{\Phi}_2,\mathbf{R}_i\text{-}\Delta) + \nabla f_2(\boldsymbol{\Phi}_2,\mathbf{R}_i\text{-}\Delta)^t\ (\mathbf{R}_i\text{-}\mathbf{R}_i\text{+}\Delta) \tag{4-25}$$

where WLOG $\Delta \geq 0$ represents simultaneous and uniform shifts in selected R_i parameters. Focusing on the first of these expressions and recognizing that as long as $\exists\ \Phi > 0$ such that KKT condition of feasibility (2-8) is satisfied then

$$f_2(\boldsymbol{\Phi}_2,\mathbf{R}_i) = f_2(\boldsymbol{\Phi}_1,\mathbf{R}_i\text{+}\Delta).$$

It follows therefore that $f_2(\Phi,R_i)$ $i = 1$ to m, $m \leq N$ is a concave if:

$$\nabla f_2(\boldsymbol{\Phi}_2,\mathbf{R}_i)^t\ (\Delta) \geq 0. \tag{4-26}$$

Now since each of the m elements of vector Δ are positive and since each of the m elements of $\nabla f_2(\boldsymbol{\Phi}_2,\mathbf{R}_i)$ via equation 4-18, are positive then clearly equation 4-26 holds and $f_2(\Phi,R_i)$ is concave regardless of the value $m \leq N$ assumes. ∎

Theorem 4-8

The function $f_3(\Phi,D_i)$ $i = 1$ to m, $m \leq N$, where Φ is defined implicitly by D_i, is a concave function.

Proof: Follows from proof of Theorem 4-7.

Theorem 4-9

The function $f_4(\Phi,R_i,D_i)$ $i = 1$ to m, $m \leq N$, where Φ is defined implicitly by both R_i and D_i, is a concave function.

Proof: By Theorem 4-5 $f_4(\Phi,R_i,D_i)$ is a monotonically increasing function and by Theorem 3.3.3 [1], $f_4(\Phi,R_i,D_i)$ $i = 1$ to m, $m \leq N$ is a concave function if:

$$f_4(\mathbf{\Phi_1,R_i}+\Delta,\mathbf{D_i}+\Delta) \leq f_4(\mathbf{\Phi_2,R_i,D_i}) + \nabla f_4(\mathbf{\Phi_2,R_i,D_i})^t\,[(R_i+\Delta-R_i),(D_i+\Delta-D_i)\ \text{or if}$$

$$f_4(\mathbf{\Phi_1,R_i,D_i}) \leq f_4(\mathbf{\Phi_2,R_i}-\Delta,\mathbf{D_i}-\Delta) + \nabla f_4(\mathbf{\Phi_2,R_i}-\Delta,\mathbf{D_i}-\Delta)^t\,[(R_i-R_i+\Delta),(D_i-D_i+\Delta)] \qquad (4\text{-}27)$$

where WLOG $\Delta \geq 0$ represents simultaneous and uniform shifts in selected R_i and D_i parameters. Focusing on the first of these expressions and recognizing that as long as $\exists$ $\Phi > 0$ such that KKT condition of feasibility (2-8) is satisfied then

$$f_4(\mathbf{\Phi_2,R_i,D_i}) = f_4(\mathbf{\Phi_1,R_i}+\Delta,\mathbf{D_i}+\Delta).$$

It follows therefore that $f_4(\Phi,R_i,D_i)$ $i = 1$ to m, $m \leq N$ is a concave if:

$$\nabla f_4(\mathbf{\Phi_2,R_i,D_i})^t\,(\Delta) \geq 0. \qquad (4\text{-}28)$$

Now since each of the m elements of vector Δ are positive and since each of the m elements of $\nabla f_4(\mathbf{\Phi_2,R_i,D_i})$, via equation 4-22, are positive then clearly equation (4-28) holds and $f_4(\Phi,R_i,D_i)$ is concave regardless of the value $m \leq N$ assumes. ∎

Corollary 4-9-1

The function $f_4(\Phi,RD_i)$ $i = 1$ to m, $m \leq N$, where Φ is defined implicitly by RD_i, is a concave function.

Proof: Follows from Corollary 4-5-1 and Theorem 4-7. See Theorem A4-3 in Appendix 4 for detailed proof.

As in section 4.4.1 the results of the last four theorems establish those cases in which the convexity of the selected function does not vary. Theorem A4-2, shown in

Appendix 4, indicates that when Φ is established by the combined effect of each parameter from a single inventory item, $f_5(\Phi,C_i/W_i,R_i,D_i)$ i = 1,2,..or N, that the function is concave (convex) if $\Phi \geq (\leq) [(R_iD_i/2(R_i+D_i)] - (C_i/2W_i)$.

4.5 Conclusions

The focus of this functional analysis has been on determining the effect on Φ generated by changes either in the C_i/W_i ratios depicted by equation (4-10), in the reorder rates (R_i) parameters characterize by equation (4-11) or in the demand rates (D_i) represented by equation (4-12). In both an analysis of the first order derivatives of these functions and the succeeding convexity study, the initial focus was on an establishment,

	FUNCTIONAL DESCRIPTION	
Function	Convexity	Monotonic Nature
$f1(\Phi,C_i/W_i)$ one parameter	convex	decreasing
$f_1(\Phi,C_1/W_1 .. C_m/W_m)$	convex	decreasing
$f_2(\Phi,R_i)$ one parameter	concave	increasing
$f_2(\Phi,R_1 .. R_m)$	concave	increasing
$f_3(\Phi,D_i)$ one parameter	concave	increasing
$f_3(\Phi,D_1 .. D_m)$	concave	increasing
$f_4(\Phi,R_1..R_m,D_1..D_m)$	concave	increasing
$f_5(\Phi,C_i/W_i,R_i,D_i)$ one item	mixed	mixed

Table 4-1 (Summary of Functional Analysis) using an implicit approach, of the functional descriptions of the single parameter functions studied by Maloney [26]. The intent of this foundation analysis was to validate the implicit approach to solving Problem (PΦ). With this validation completed, focus switched to authenticating the results of the empirical study where multi-parameters changed within a give parameter category.

A summary of the obtained results is contained in Table 4-1. It is the latter, multiple parameter relationships which ensure the effective extension, outlined in Chapter V, of the Ventura and Klein algorithm.

Chapter V

An Improved Bounding Algorithm

5.1 Introduction

To this point several factors have been established concerning the N-item, single constraint inventory system. The empirical study, considered in chapter III, identified numerous near-linear relationships between Φ^* and multiple parameter shifts. The detailed functional analysis, summarized in chapter IV, provided both slope and convexity information on those functions in which multiple parameters from the same system category undergo simultaneous perturbations.

This chapter will build on the monotonically decreasing, convex nature of $f_1(\Phi, C_i/W_i)$, defined in chapter IV for i = 1,2,...and/or N, to develop a process which provides, in closed form, an improved bounding algorithm for Φ^*. As was the case in Ventura and Klein [39], it will be shown that this improved algorithm rapidly converges from the lower of these bounds to an optimal Lagrangian value without the normal trial and error process described in Chapter II. Finally a comparative analysis will be presented. This analysis will indicate the relative efficiency of this algorithm to those presented by Ventura and Klein [39] and Ziegler [41].

5.2 Modified Initial Bounds

Maloney [26] identified an initial set of bounds around the optimal Lagrangian multiplier in Problem (PΦ) utilizing an equation which allowed direct calculation of these values. This calculation, formalized by Lewis [25], however, is only valid when the carrying costs, C_i, for each item of inventory are equal. Ventura and Klein [39] developed a similar expression for calculating initial bounds for the N-item inventory system in which W_i is free to assume any positive value. The following theorem establishes that expression and is included here for completeness.

Theorem 5-1

Let A be a N-item inventory for which a single constraint is active and whose C_i/W_i ratios are equal. Then Φ^* can be computed directly by:

$$\Phi^* = \tfrac{1}{2} \{ \sum_{i=1}^{N} (2R_iD_iW_i)^{1/2} / U\}^2 - (C/W)\} \qquad (5\text{-}1)$$

where each parameter is defined as in chapter II.

Proof: For the N-item inventory system in which W_i is free to assume any positive value it is known that $Q_i^* = [2R_iD_i/(C_i+2W_i\Phi^*)]^{1/2}$ from equation (2-11). Replacing Q_i^* in Problem (PΦ)'s condition of feasibility, equation (2-12), yields:

$$\sum_{i=1}^{N} W_i \, [2R_iD_i/(C_i+2W_i\Phi^*)]^{1/2} = U. \qquad (5\text{-}2)$$

Rearranging the terms of equation (5-2), replacing C_i/W_i with a constant, C/W, and solving for Φ^* results first in:

$$\sum_{i=1}^{N} \{2R_iD_iW_i/(C_i/W_i)+2\Phi^*)]^{1/2} = U \qquad (5\text{-}3)$$

then in

$$\sum_{i=1}^{N} \{2R_iD_iW_i/[(C/W)+2\Phi^*]\}^{1/2} = U, \tag{5-4}$$

and finally in equation (5-1). ∎

It should be noted, in examining equation (5-3), that W_i, i= 1,2,... N, no longer directly impacts Φ^* and that C^ can replace C/W which is constant. Considering these observations it must be concluded, as was reflected in Theorem 5-2, Corollary 5-2.1 and Corollary 5-2.2 [26] and Theorem 1 [39], that when δ is defined by equation (5-1) for C^ = Max (C_i/W_i) and when τ is defined similarly as C^ = Min (C_i/W_i) then $\delta < \Phi^* < \tau$. It follows, therefore, that δ and τ are initial bounds on Φ^*

5.3 Improved Bounds

With effective initial bounds identified Ventura and Klein [39] utilized equations (4-1) and (4-2) to develop improved bounds on Φ^*. Failing to consider the influence imposed by multiple items, however, the cord based upper bound generating portion of the Ventura/Klein algorithm can only be guaranteed to identify a valid upper bound in the two-item inventory case. This section provides the foundation needed to overcome this shortfall and suggest the improved bounding algorithm, presented in the following section. The following theorem establishes a methodology for this improvement which is based on the monotonic, convex nature of the multiple C_i/W_i function, $f_1(\Phi, C_i/W_i)$.

Let A be an N-item inventory system subject to a single constraint. Let $S_i = C_i/W_i$, for i = 1, ... , N and let A be arranged so that $S_1 \leq S_2 \leq S_3 \leq .... \leq S_N$. Notationally then A can be represented as $[\ S_1, S_2, S_3, ..., S_N\ ]$. Let $\Phi^* > 0$ be the optimal Lagrangian multiplier related to A. Let B_i and b_i represent a series of inventory systems, related to A, in which all except the S_i ratios remain unchanged and where each set of S_i ratios are defined as follows:

$$
\begin{array}{ll}
B_1 & \rightarrow [\ S_2, S_2, S_3, S_4, ..., S_N\ ] \\
B_2 & \rightarrow [\ S_3, S_3, S_3, S_4, ..., S_N\ ] \\
B_3 & \rightarrow [\ S_4, S_4, S_4, S_4, ..., S_N\ ] \\
\downarrow & \qquad\qquad\qquad\qquad \downarrow \\
B_{N-1} & \rightarrow [\ S_N, S_N, S_N, S_N, ..., S_N\ ]
\end{array}
$$

$$
\begin{array}{ll}
b_1 & \rightarrow [\ S_1, S_1, S_3, S_4, ..., S_N\ ] \\
b_2 & \rightarrow [\ S_1, S_1, S_1, S_4, ..., S_N\ ] \\
b_3 & \rightarrow [\ S_1, S_1, S_1, S_1, ..., S_N\ ] \\
\downarrow & \qquad\qquad\qquad\qquad \downarrow \\
b_{N-1} & \rightarrow [\ S_1, S_1, S_1, S_1, ..., S_1\ ].
\end{array}
$$

When optimum Lagrangian multipliers Φ_{Bi} and Φ_{bi} exist for each of these systems, the following relationships hold;

$$\Phi_{bN} > \Phi_{bN-1} > .... > \Phi_{b1} > \Phi^* > \Phi_{B1} > .... > \Phi_{BN-1} > \Phi_{BN}. \qquad (5\text{-}5)$$

Proof: Consider first the lower bounds expressed in equation (5-5). Since the discussion following Theorem 4-2 shows that $f(C_i/W_i, \Phi)$, i = 1,2, ...and/or N, is a monotonically decreasing function, then, by definition, as C_i/W_i increases Φ decreases. It follows, therefore, that since

$$S_1 < S_2 < S_3 < .... < S_N \quad \text{clearly}$$

$$\Sigma Q_A > \Sigma Q_{B1} > \Sigma Q_{B2} > .... > \Sigma Q_{BN} \quad \text{and}$$

$$\Phi^* > \Phi_{B1} > .... > \Phi_{BN-1} > \Phi_{BN}.$$

In considering the upper bounds found in equation (5-5), a similar argument holds. Again the monotonic nature of the function indicates that as C_i/W_i decreases Φ increases. It follows, therefore, that since

$$\Sigma Q_A < \Sigma Q_{b1} < \Sigma Q_{b2} < .... < \Sigma Q_{bN} \quad \text{then}$$

$$\Phi_{bN} > \Phi_{bN-1} > .... > \Phi_{b1} > \Phi^*$$

and equation (5-5) holds. ∎

The usefulness of the (N-1) sets of upper and lower bounds established by Theorem 5-2 can be illustrated by considering Figure 5-1. In this figure, Φ_{b3} and Φ_{B3} represent the initial set of upper and lower bounds defined, by equation 5-1, from inventory systems B_3 and b_3 in which all four C_i/W_i ratios first assume the value of C_N/W_N and then C_1/W_1 respectively. By moving along the monotonically decreasing, convex function defined by holding C_4/W_4 constant while the first three ratios decrease simultaneously, two new Lagrangian values can be identified. These values, Φ_{b2} and

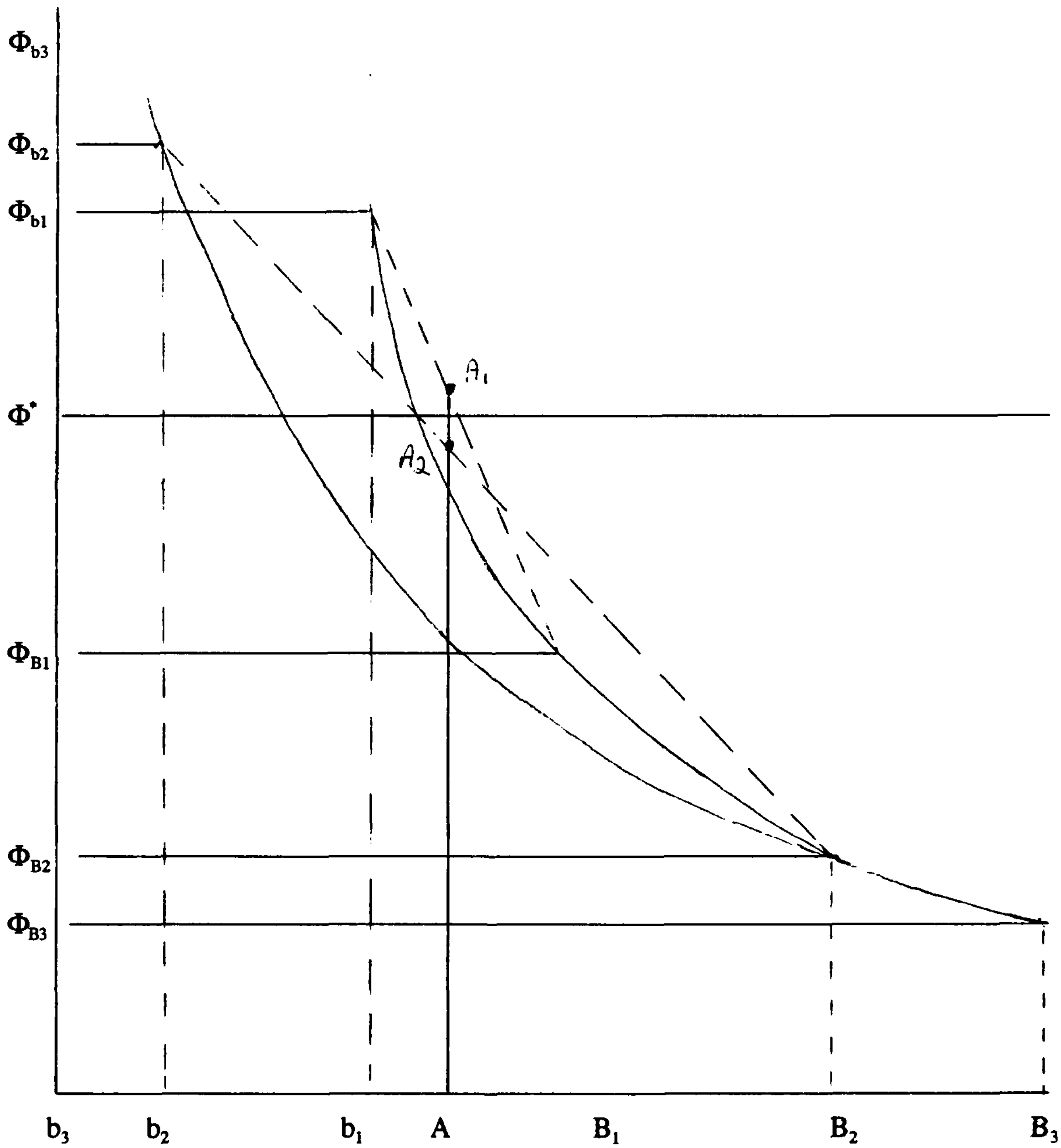

Figure 5-1 (Improved Bounds - A 4-Item Inventory Example)

Φ_{B2}, are linked to inventory systems b_2 and B_2 where the first three ratios each assume the value of C_1/W_1 and then C_3/W_3 respectively. These multipliers clearly represent improvements to both the upper and lower initial bounds. Repeating this process with inventory system B_2 and its newly identified multiplier, Φ_{B2}, two new multipliers are located which continue to tighten the existing bounds around Φ^*. These multipliers, Φ_{b1} and Φ_{B1}, are linked to inventory systems b_1 and B_1 where the first two ratios each assume the value of C_1/W_1 and then C_2/W_2, respectively, while the remaining ratios are held at their system A values. Repeated application of this methodology will identify ever tightening bounds on Φ^* until only a single parameter remains to be shifted back to its original value. In the above figure, this occurs when Φ_{b1} and Φ_{B1} have been located. An improved upper bound can, at this point, be found utilizing the cord defined between (b_1, Φ_{b1}) and (B_1, Φ_{B1}). In Figure 5-1, A_1, depicts this final upper bound. Ventura and Klein [39] utilized this method to locate their improved upper bound in the two-item inventory system. It must be noted, however, that this cord based method can only be applied during the final phase when multiple parameter shifts are considered. Application of this method prior to that point may or may not identify an upper bound on Φ^*. To illustrate this limitation consider the cord defined by connecting (b_2, Φ_{b2}) with (B_2, Φ_{B2}). The intersection of that cord with a vertical line through A, depicted by A_2, identifies a multiplier which is less than Φ^*.

5.4 An Improved Bounding Algorithm

The four-item inventory system illustration given in section 5.3 suggests the four step algorithm below. This algorithm identifies initial bounds and provides improvements to those bounds though a recursive process. A series of improved bounds are found which quickly converge to Φ^* for an N-item inventory system under a single resource constraint.

Step (0) Initial Conditions

(a) Sort inventory system so that the C_i/W_i ratios are in ascending order.

(b) Set t=0 and select $\varepsilon \geq 0$ as stopping criterion.

Step (1) Computation of Initial Bounds Φ_L^t and Φ_U^t

$$\Phi_L^t = ½ \{ \sum_{i=1}^{N} (2R_iD_iW_i)^{½} / U\}^2 - (C_N/W_N)\}$$

$$\Phi_U^t = ½ \{ \sum_{i=1}^{N} (2R_iD_iW_i)^{½} / U\}^2 - (C_1/W_1)\}$$

where U is the constraint level while $\Phi_L^\cdot$ and $\Phi_U^\cdot$ represents lower and upper bounds on Φ^* respectively.

Step (2) Improved Bounds

(a) Set $S_i = C_{N-t}/W_{N-t}$ for i = 1 to N-t,

$S = C_{N-t}/W_{N-t}$,

$\Phi = \Phi_L^t$ and

t = t + 1.

Then set $Z = C_{N-t}/W_{N-t}$ and flag = 0

(b) Computation of improved bound

$$\Phi = \Phi + \sum_{i \in 1}^{N-t} \frac{\partial \Phi}{\partial (C_i/W_i)} (Z - S)$$

(c) Computation of Recursive Update

$$S = \left[\frac{\sum_{i \in 1}^{N-t} [2R_iD_iW_i]^{1/2}}{U - \sum_{i=N-t}^{N} [2R_iD_iW_i/(S_i + 2\Phi)]^{1/2}} \right]^2 - 2\Phi$$

Set $S_i = S$ for i = 1 to N-t. If $(Z - S) \geq \varepsilon$ then go to 2(b). Otherwise continue.

(d) If $t < N-2$ and flag = 0 then set $\Phi_L^t = \Phi$, $S_i = C_{N-t}/W_{N-t}$ for i = 1 to N-t, $Z = C_1/W_1$, flag = 1 and go to 2(b).

If $t < N-2$ and flag = 1 then set $\Phi_U^t = \Phi$, $\Phi = \Phi_L^t$ and go to 2(a).

Otherwise continue.

Step (3) Optimizing Step

(a) If $(\Phi_U^t - \Phi_L^t) \leq \varepsilon$ then

$\Phi\wedge = (\Phi_U^t - \Phi_L^t)/2 \approx \Phi^*$ and stop, otherwise

(b) Evaluate estimate using

$$P = \sum_{i=1}^{N} \{2R_iD_iW_i/(C_i/W_i)+2\Phi\wedge)]^{1/2} - U$$

If $P > 0$ then set $\Phi_L^t = \Phi\wedge$ and go to 3(a),

If $P < 0$ then set $\Phi_U^t = \Phi\wedge$ and go to 3(a).

The apparent complexity of this algorithm, particularly in step (2), is misleading. Using the convention followed in Theorem 5-2 to define B_3, B_2, b_2 and b_3, Figure 5-2 portrays the sequence of events accomplished in this step of the algorithm. Following the identification of an initial lower bound, Φ_L^0, in step (1), step 2(b) generates the slope of a tangent line at (B_3,Φ_L^0) to the function defined by setting all C_i/W_i ratios equal to C_N/W_N and then simultaneously shifting all except the N^{th} ratio back towards the original value of the C_{N-1}/W_{N-1} ratio. An improved Lagrangian multiplier, Φ_{S1}, is then identified by the intersection of this tangent line with the vertical line passing through B_2. Step 2(c) utilizes Φ_{S1} to determine the location of S_1 between B_3 and B_2. When this sequence is repeated until S_1 coincides with B_2, [step 2(d)], an improved lower bound, $\Phi_L^1 = \Phi_{B2,}$ is identified in accordance with Theorem 5-2. With an improved lower bound established, the algorithm continues to move along the defined convex function locating first

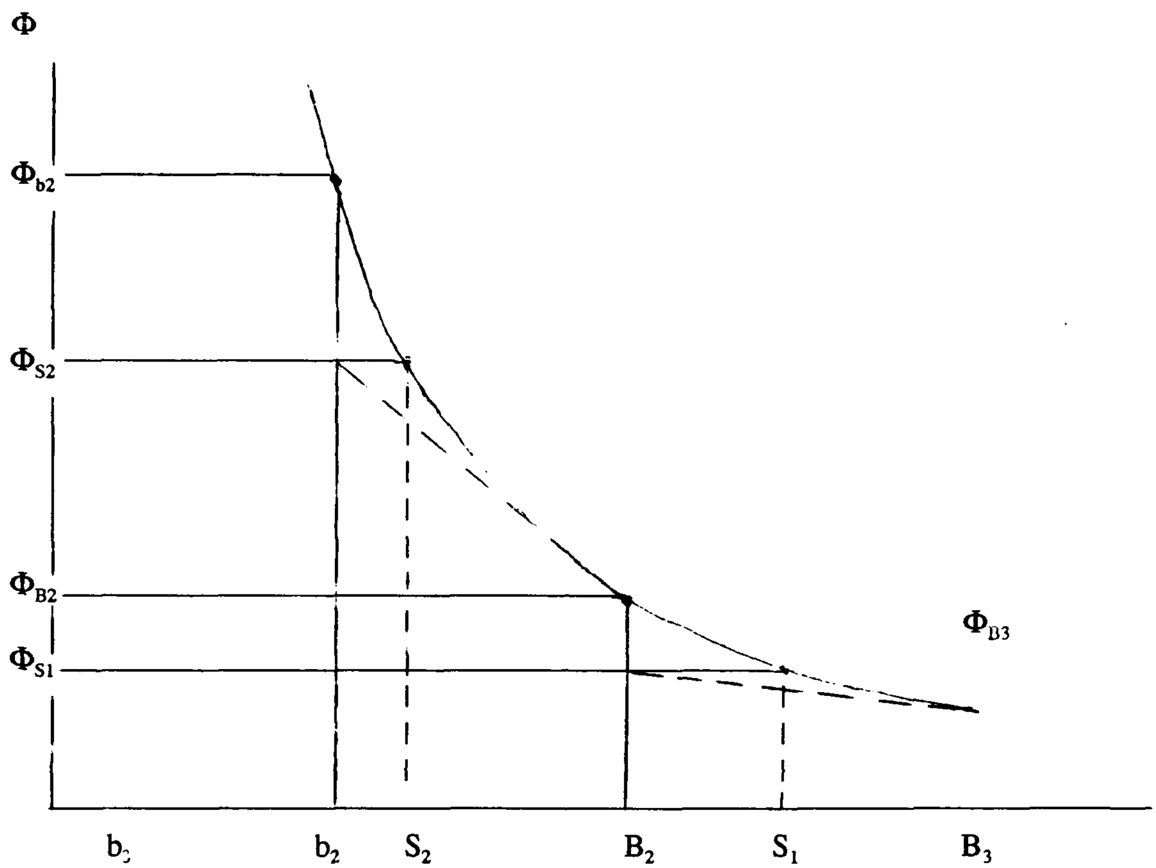

Figure 5-2 (Example of Bounding Algorithm)

S_2 and finally an improved upper bound, $\Phi_U{}^1 = \Phi_{b2}$. The algorithm identifies this new upper bound when S_2 coincides with b_2, again in accordance with Theorem 5-2.

Step (2) of this algorithm is repeated for each successive inventory system, B_3, B_2, etc., until N-2 sets of improved bounds have been identified. At this point only a single parameter remains to be considered. Step (3) utilizes this final set of bounding values and a simple bi-section technique to drive quickly to Φ^*. It must be pointed out that simply repeating steps 2(b) and 2(c) though one more recursion also drives to Φ^*. The suggested bi-section approach, however, computationally converges more rapidly to Φ^* over the small span which remains.

In concluding this section, note that while Theorem 5-2 proves the existence of a series of Lagrangian multipliers which sequentially close around Φ^*; it is the monotonically increasing, convex nature of the functions on which these values are found that allows each subsequent multiplier to be identified. Utilizing the recursive process described in Step 2, the identification of each pair of bounding values is guaranteed. Note also that the bisection search method, implemented as the Optimizing Step, ensures its convergence to Φ^* within any desired degree of accuracy. With $\varepsilon > 0$ defining that desired accuracy and the final set of bounding values, Φ_U^{N-1} and Φ_L^{N-1}, identified; $(½)^n \leq \varepsilon/(\Phi_U^t - \Phi_L^t)$ defines the number of bisection necessary for convergence to occur [1]. Combining this assurance of convergence with an expected tightness of Φ_U^{N-1} and Φ_L^{N-1} to Φ^*, the proposed algorithm should exhibit both speed and accuracy.

5.5 Numerical Example

To illustrate the computational efficiency of the above algorithm consider the following four-item inventory system in which the C_i/W_i ratios are already arranged in ascending order.

Carrying Costs	Re-order Costs	Demand Rates	Resource Required
2.5	64	94	96
15.75	71	95	100
5.0	73	46	25
23.5	87	35	51

Table 5-1 (Example: 4-Item Inventory System)

Utilizing the convention established by Theorem 5-2,

$$S_1 = C_1/W_1 = .2604167,$$
$$S_2 = C_2/W_2 = .1575,$$
$$S_3 = C_3/W_3 = .2,$$
$$S_4 = C_4/W_4 = .4607843,$$

and six related inventory systems are defined as follows:

$$B_3 = \{ S_4, S_4, S_4, S_4 \}$$
$$B_2 = \{ S_3, S_3, S_3, S_4 \}$$
$$B_1 = \{ S_2, S_2, S_3, S_4 \}$$

$$b_3 = \{ S_1, S_1, S_1, S_1 \}$$
$$b_2 = \{ S_1, S_1, S_1, S_4 \}$$
$$b_1 = \{ S_1, S_1, S_3, S_4 \}$$

Implementing the above algorithm on an IBM compatible personal computer using GW Basic and considering systems b3 and B3 in Step (1) produced the follow initial bounds for Φ^*:

Bound	C/W constant	Value
Lower	.4607843	3.7706
Upper	.25604167	3.9880

Table 5-2 (Initial Bounds)

Armed with these initial bounds, Tables 5-3 and 5-4 summarize the (N-2) iterations of step (2) needed to drive all but the final Ci/Wi ratios back to their original values. The opening conditions from which a new set of improved bounds was established [step 2(a)] are defined in the first line of Table 5-3. These conditions stem from system B_3 and its related initial lower bound. Similarly, the opening conditions necessary for determining a second set of improved bounds are defined in the first line of Table 5-4. These conditions come from system B_2 and the improved lower bound identified in Table 5-3. Simply put, Table 5-3 tracks the transition from system B_3 through system B_2 to

Trans	t	Target Ratio(Z)	S	Old Φ	New Φ	Updated S
0	0_1	-	.4607	-	3.7706	-
1	1	.2	.4607	3.7706	3.8783	.2010
2		.2	.2010	3.8783	$\underline{3.8788}^{L}$	.2000
3	1	.02604	.2	3.8788	$\underline{3.9512}^{U}$	.2649

Table 5-3 (First Improvements to Bounds)

system b_2 while Table 5-4 traces a similar transition from system B_2 through system B_1 and finally to system b_1.

Trans	t	Target Ratio(Z)	S	Old Φ	New Φ	Updated S
0	0_2		.2	-	3.8788	-
4	2	.1575	.2	3.8788	$\underline{3.8937}^{L}$	.15754
5	2	.02604	.1575	3.8937	3.9401	.02655
6	2	.02604	.0265	3.9401	$\underline{3.9403}^{U}$	.02604

Table 5-4 (Second Improvements to Bounds)

The "new Φ" values, marked by $\underline{( )}^L$, identified during transitions 2 and 4 represent two subsequent improvements to the initial lower bound on Φ^*. Similarly, those "New Φ" values marked by $\underline{( )}^U$, determined following transitions 3 and 6, constitute the desired upper bound improvements. After six transitions, the final bounds (italicized in Table 5-4), represent a 78.5% reduction in the bounds around Φ^* from the already tight initial bounds listed in Table 5-2.

In preparing this numerical example, Φ^* was determined to be 3.91625 using a classical line search technique. Employing this value as a gage, the bi-section algorithm (step (3)) converges rapidly. Starting with the final pair of improved bounds, 3.8937 and 3.9403, an approximate Φ^* value of 3.917 is obtained following a single bi-section. This approximation represents an absolute error of less .00075 when compared to the actual optimal value.

Several issues should be noted at this junction. First, no restriction need be placed on the Lagrangian multiplier value identified for each subproblem, B_3, B_2, B_1, b_3, b_2, b_1. Although the constraint is assumed to be binding for system A, that constraint does not have to be active for each subproblem. When such inactivity occurs, a negative lower bound will be identified. To establish valid bounds on Φ^* in such cases the lower bound simply is set to zero. Secondly, a comparison of the improved bounds contained in Tables 5-3 and 5-4 with the initial set of bounds shows that most of the indicated reduction occurred during the first recursion of the algorithm. The bounds listed in Table 5-3 represent a 66.7% improvement while those listed in Table 5-4 provide only an additional 11.8% improvement. This uneven reduction in the span between the bounding values, however, was expected since with each recursion the horizontal distance which the algorithm moves along the tangent line is diminished. Of particular note is the tightness of these bounding sets to Φ^*. This accuracy stems from the near linear nature of the functions from which the tangent lines are generated and guarantees that the bounds, thus computed, will fall consistently close to the Φ^*.

5.6 Comparative Analysis

This section summarizes a two phase analysis conducted to determine how the new algorithm compares to the algorithms presented by Ventura and Klein [39] and Ziegler [41], as well as with the classical solution approach. During phase one, the proposed multiple parameter algorithm, M1, was run setting each W_i equal to one so that an effective comparison could be made to the single parameter Ventura and Klein algorithm, M2. During the second phase a more detailed analysis was made in which the weights of algorithm M1 were allowed to assume any positive value. The focus of this second analysis was to compare algorithm M1 with both the Ziegler algorithm, M3, and the original Classical approach, M4. The Ventura and Klein algorithm was dropped from consideration during this phase since it is not effective for $W_i \neq 1$.

During phase one eight randomly generated inventory systems were examined. Table 5-5, below, clearly indicates the increased efficiency obtained when multiple parameter shifts are used to find Φ^*. As would be expected, the same multiplier was found in each case by both algorithms. Note, however, that as the size of the inventory increase, the CPU times differ at an increasing rate. In each case, the

Inventory Size	M2 Φ^*	M1 Φ^*	M2 CPU Time	M1 CPU Time
3	7.869	7.869	.828125	.71875
4	196.375	196.375	1.089844	.87890
4	127.629	127.629	2.261719	1.98828
5	248.247	248.247	1.210938	1.03906
10	241.004	241.004	4.054688	3.17578
15	277.434	277.434	6.039063	3.511719
20	292.309	292.308	9.5625	7.03125
30	299.687	99.687	19.82813	13.83984

Table 5-5 (Current Algorithm Compared to Ventura/Klein)

algorithm M1 required less CPU time to obtain Φ^*. For inventory systems which contain more that ten items, this time savings was considerable.

With the results of phase one in mind, phase two of this analysis used the classically determined Φ^* to compare algorithms M1 and M3 based on two criterion: Best Lower Bound and Best Upper Bound. This part of the comparison was made by determining the smallest absolute error between Φ^* and the upper and lower bounds generated by the respective algorithms. Specifically, the initial bounds produced by M1 were compared with the improved bounds provided by M3. Additional criterion used to round out the comparison of these algorithms included the tightness of the their bounding intervals, the average time each needed to determine those bounds and the average time each algorithm required to determine an approximate Φ^* value.

Utilizing the above criteria, 8400 randomly generated inventory systems were evaluated using a four factor, two level experimental design. Each of the four factors:

C_i, R_i, D_i and W_i were generated from uniform distributions characterized as either High

Configuration	C_i	R_i	D_i	W_i
1	Low	Low	Low	Low
2	High	High	High	High
3	High	Low	Low	High
4	Low	High	High	Low

Table 5-6 (4^2 Factor Experimental Design)

or Low based by the test configurations shown in Table 5-6. Each of these distribution are defined in Table 5-7 below.

Parameter	Low	High
C_i	(10,20)	(20,50)
R_i	(50,80)	(500,600)
D_i	(200,400)	(4000,6000)
W_i	(1,25)	(50,400)

Table 5-7 (Uniform Distributions of System Parameters)

For these test configurations, 100 inventory systems were evaluated for each possible combination of three general constraint levels and seven inventory sizes. The constraint levels utilized in this analysis were defined by $U = \alpha(\Sigma Q_i)$. A particular α value was drawn randomly from one of three uniform distributions defined on (.1,.3), (.3,.7) or (.7,.9). With constraint levels selected from these distinct ranges, the relative impact that constraint tightness has on the algorithms under study was measured. Similarly, the effects related to inventory size were gaged using systems of 2, 4, 8, 16, 21, 26 and 31 items. The result of the ensuing analysis is presented in the Tables 5-8, 5-9 and 5-10.

N	% Best Bound			% Shortest Time		Time Savings
	Low	Upper	Inter	Bounds	Φ^*	Class/Ziegler
2	97	100	100	95	99	.3862/ .1302
4	78	93	94	100	99.5	.6734/ .2365
8	55	82	74	100	100	1.3823/ .5835
16	22	70	38	100	100	3.0152/1.8510
21	14	68	31	100	100	3.2708/3.0634
26	14	54	22	100	100	4.5700/4.2024
31	17	51	24	100	100	5.9561/5.2651

Table 5-8 (Comparative Analysis for α Between .1 and .3)

N	% Best Bound			% Shortest Time		Time Savings
	Low	Upper	Inter	Bounds	Φ^*	Class/Ziegler
2	60	100	99	94	94	.3151/ .0704
4	31	66	54	100	99	.6342/ .1541
8	20	37	19	100	100	1.2714/ .4451
16	14	15	7	100	100	2.7122/1.6223
21	0	5	0	100	100	3.4746/2.7075
26	0	3	0	100	100	4.4139/4.1606
31	15	1	1	100	100	6.1314/5.4431

Table 5-9 (Comparative Analysis for α Between .3 and .7)

N	% Best Bound			% Shortest Time		Time Savings
	Low	Upper	Inter	Bounds	Φ^*	Class/Ziegler
2	7	55	28	98	87	.3250/ .0503
4	7	20	12	99	97	.6497/ .1112
8	7	1	0	100	100	1.3559/ .3567
16	9	0	0	100	100	3.1419/1.4474
21	10	0	0	100	100	3.8757/2.5746
26	10	0	0	100	100	4.8207/3.9654
31	0	0	0	100	100	5.9149/5.7060

Table 5-10 (Comparative Analysis for α Between .7 and .9)

Before drawing any conclusions concerning the results shown in these tables, note that the numbers listed under "% Best Bound" reflect the proportion of inventory systems for which the initial bounds obtained from algorithm M1 were tighter than the improved bound obtained by algorithm M3. Similarly, those figures specified under "% Shortest Time" indicate the percentage of systems in which algorithm M1 required less CPU time to obtain those bounds. The second column under this heading reflect the percentage of systems in which M1 required less CPU time to obtain Φ^* utilizing those bounds. Finally, values in the "Time Savings" column indicate the average CPU time saved by utilizing algorithm M1 to obtain Φ^*.

5.7 Comparative Analysis Conclusions

A number of conclusions can be drawn for the two part analysis described above. The comparison made during part one of the analysis, between algorithms M1 and M2, indicates that both algorithms converge effectively to Φ^*. However, from the time savings indicated in Table 5-6, M1 proves to be the more efficient algorithm. An initial source of this improved efficiency rest in the simplified slope generating equation utilized recursively in step (2b) of M1. This direct computation of slope replaced a multi-step procedure employed in M2. A second source of the indicated improvement is drawn from M1's ability to consider simultaneous multiple parameter movements. By moving multiple parameters along the convex function to their original values rather than handling each parameter in turn, the total number of transitions M1 needs to obtain Φ^* is greatly reduced.

A careful review of the results obtained during the second phase of this comparative analysis indicates three additional conclusions. The first of these conclusions arise from the results contained in Tables 5-9 and 5-10 for the two-item inventory system. Taken together these tables indicate that algorithm M1 obtained a tighter bounding interval 99.75 % of the time when $\alpha \in (.1,.7)$. These results mirror those obtained by Ventura and Klein [39] for $\alpha \in (.2,.7)$. This similarity was expected since when N = 2 both M1 and M2 are in effect single parameter algorithms.

The second and third conclusions to be drawn from this analysis involve the impact that increased inventory size and constraint tightness have on algorithms M1 and M3. Examining, in turn, Tables 5-9, 5-10 and 5-11 reveals clearly that as the inventory size increases the percentage of systems for which M1 provides the best bounding interval decreases. The magnitude of this decreased efficiency unquestionably increases as α approaches one. For example consider the bounding intervals for the 8-item inventory system. The 74 % efficiency obtained by M1 in Table 5-9, where $\alpha \in (.1,.3)$, was reduced to 19 % and then to 0 % in Tables 5-10 and 5-11 where $\alpha \in (.3,.7)$ and $\alpha \in$

(.7,.9) respectively. From this discussion it must be concluded that the proposed multiple parameter algorithm provides better bounds on Φ^* only when $\alpha \in (.1,.7)$ and the inventory size is less than 8.

		AMRE		ARE	
N	α Range	M1	M3	M1	M3
	1	.5368	11.9394	.0623	1.4074
2	2	1.6743	12.1314	.3974	1.5261
	3	6.0589	14.0067	1.4634	1.6707
	1	.7433	5.0469	.1302	.8263
4	2	1.9441	6.6377	.7241	.8311
	3	4.0715	6.7434	2.2882	.8991
	1	.8031	1.9262	.2104	.4205
8	2	1.7725	1.5626	.9578	.4690
	3	4.3187	1.2181	2.2758	.3850
	1	.8362	.8260	.3004	.2353
16	2	1.8210	.8417	1.0421	.2869
	3	4.8195	.7882	2.3438	.2443
	1	.8067	.4229	.2901	.1317
21	2	1.7598	.3866	1.1494	.1121
	3	4.2896	.6629	2.3634	.2194
	1	.7785	.3899	.3017	.1067
26	2	1.7842	.4031	1.1640	.1007
	3	4.4579	.6478	2.4010	.1894
	1	.8609	.5983	.3576	.1847
31	2	1.8135	.4863	1.2221	.2205
	3	4.6908	.3008	2.3954	.0892

Table 5-11 (Comparison of Relative Errors)

This apparent limitation, however, must be viewed in light of each algorithm's computational efficiency and the relative errors contained in Table 5-11. This table provides an indication of how the initial bounds obtained by M1 and the final bounds derived from M3 are effected as an inventory's size is increased and when $\alpha \rightarrow 1$. The Average Maximum Relative Errors (AMRE), displayed in this table, were obtained by summing the max $[|(\Phi^* - \Phi_L)|, |(\Phi_U - \Phi^*)|]$ values found for each test configuration (Table 5-7) and then dividing by four. The Average Relative Errors, ARE, were determined by summing the relative errors, $[(\Phi_U - \Phi_L)/ \Phi^*)]$, from a given test configuration and dividing by the number of systems considered.

It should be evident from the ARE values displayed in this table that the exceptionally narrow bounds achieved by algorithm M1 for inventories comprised of eight or less items widen as N increases. This growth, however, appears to slow as the size of the inventory expands so that both the ARE and AMRE for systems comprised of 16, 21, 26 and 31 items vary only slightly. Equally apparent, from Table 5-12, is a gradual narrowing of the final bounds achieved by algorithm M3 as N increases. From these observation it must be concluded that, although algorithm M3 provides the best bounding interval for systems comprised of more than eight items, the bounds provided by algorithm M1 continue to be fairly tight throughout. For this reason, the computational efficiency of these algorithms, measured by the CPU times required by each, must be considered.

In appraising these times, Tables 5-9 through 5-11 indicate that for over 99 % of the 8400 inventory systems considered, algorithm M1 found its bounds more rapidly than

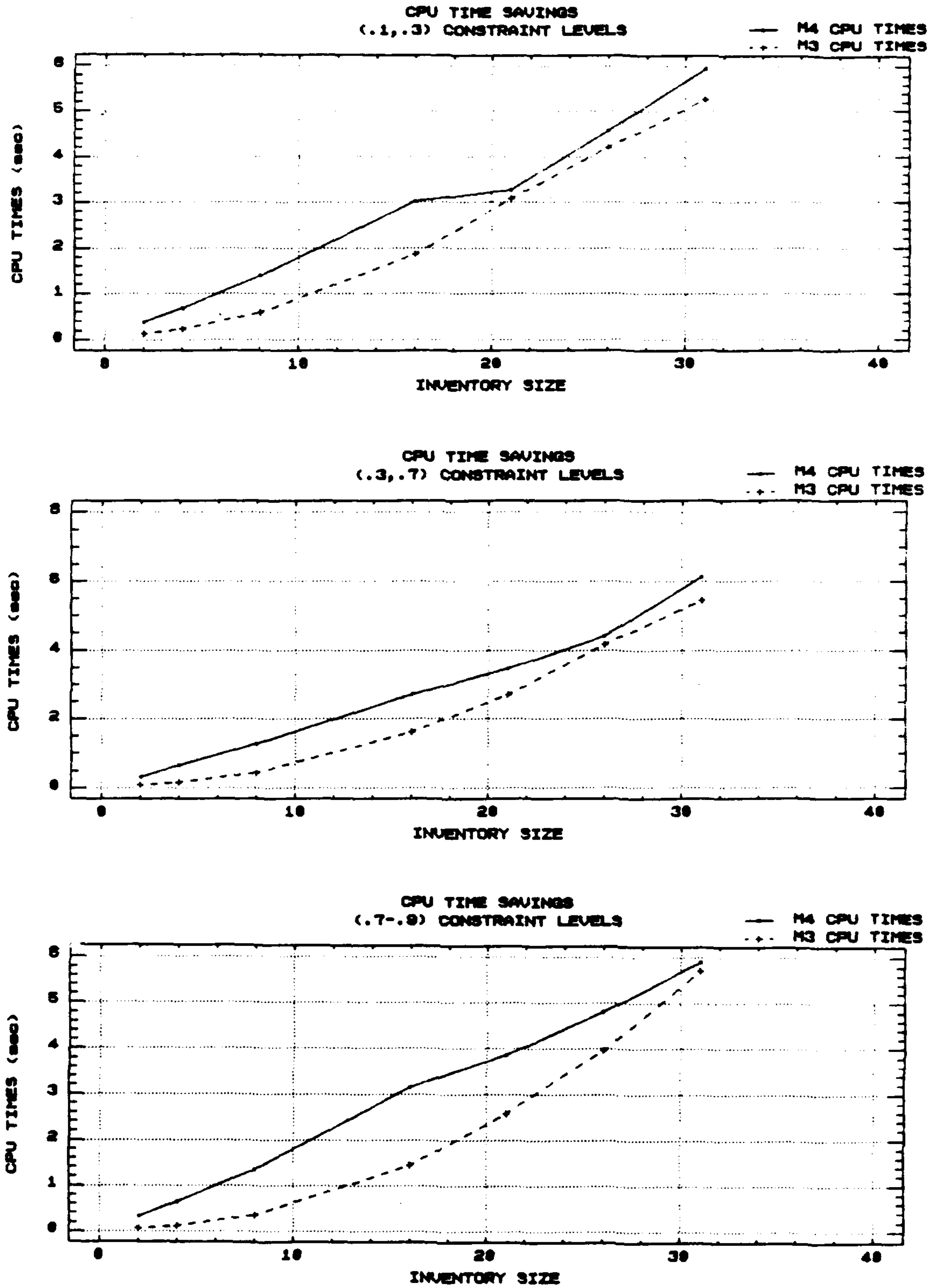

Figure 5-3 (CPU Time Savings)

did algorithm M3. In obtaining estimates for Φ^*, algorithm M1 again out performed M3 acquiring those estimates more swiftly for 98.83% of the systems considered. It should be noted that only for systems comprised of two and four items does algorithm M3 out perform M1. This, however, occurred in less than 5% of such systems. The magnitude of the time savings secured by utilizing algorithm M1 rather than either M3 or the classical approach, M4, to determine the optimal Lagrangian multiplier related to Problem (PΦ) is obvious from the plots shown in Figure 5-3.

To summarize, four conclusions must be drawn from the results of this comparative analysis:

1) The proposed multiple parameter algorithm effectively extends the Ventura and Klein single parameter algorithm to the N-item inventory system in which constraint weights, W_i, are free to assume any positive value. The efficiency of the former effort was also improved.

2) If the desired application of these algorithms is to obtain effective bounds on Φ^* without regard to the CPU time required, then the proposed algorithm should only be used when the imposed constraint is extremely tight [($\alpha \in (.1,.7)$)] and when the inventory's size is less than eight.

3) If, however, the desired application is to obtain those bounds when CPU time is critical, then the proposed algorithm can be used effectively.

4) If the desired application is to locate Φ^* rather than its bounds, the proposed algorithm should be favored over both the Ziegler algorithm and the Classical approach.

Chapter VI

A Multi-Period Application

6.1 Introduction

The success achieved using the Implicit Algorithm, described in chapter V, in obtaining an optimal solution to the single period constrained inventory problem (PΦ) suggests that similar results can be secured when a multiple period constrained inventory system is considered. Many authors have addressed the partially constrained multiple period inventory problem applying, in most cases, some form of dynamic programming to identify order quantities for each period in the horizon. The goal of such an approach is to eliminate periods of constraint within the horizon by shifting peek demands into less restricted adjacent periods. Of the algorithms recently presented, those most noted have been developed by H.M. Wagner and T.M.Whitin as well as E.A. Silver [13,18,21, 29,38, etc.]. These algorithms, however, are generally applied to production problems in which a restricted production rate rather than storage and/or budget limitations is of concern. Algorithms dealing with these latter constraints, represented by those presented by Paul and Page [30], Kaspi and Rosenblatt [22], Ziegler [41] and Ventura and Klein [39], focus solely on the single period problem. No consideration of those cases where constrained conditions extend over multiple periods of the horizon or where established dynamic programming algorithms fail to identify feasible solutions has been found.

Building on the theory established in chapter IV, this chapter presents an algorithm which utilizes existing system parameters along with their related Lagrangian multiplier, Φ^*, to forecast the multiplier, Φ^*_{new}, needed to optimize constrained inventory systems defined when various system parameters shift from period to period. Applying this algorithm recursively over the horizon of an inventory system can, therefore, identify a series optimal solutions when constrained conditions extend over several consecutive periods of the horizon. Additionally, the proposed algorithm can be employed to provide an initial feasible solution when dynamic programming solutions are infeasible due to prolonged system restrictions. A description of this algorithm and a simple numerical example will be given. This is then followed by the results of a comparative analysis conducted to determine the relative efficiency of the proposed Horizon algorithm to that achieved by applying the Implicit Algorithm, Ziegler algorithm and the Classical solution technique sequentially to each period of an inventory system's horizon.

6.2 A Theoretical Foundation

The results of the functional analysis, described in chapter IV, reveal that a function defining Φ implicitly by any combination of C_i/W_i ratios is monotonically decreasing and convex. Similarly, where Φ is defined implicitly by a collection of R_i's, an assortment of D_i's or a combination of both R_i's and D_i's the resulting function is monotonically increasing and concave. Considering these functional characteristics, upper and lower bounds on Φ^*_{new} can quickly be obtain when both Φ^* and the various parameter shifts occurring within a given inventory system are known. Figures 6-1 and 6-2, respectively depict, cases where parameter shifts occur only among a system's C_i/W_i ratios and where those shifts occur only among the reorder costs. This illustrates the theoretical foundation on which the Horizon algorithm will be based.

The solid curves in these figures represent functions in which Φ is defined implicitly by selected system parameters undergoing uniform and simultaneous shifts. For example point A, in figure 6-1, is defined by uniformly shifting each selected C_i/W_i ratio

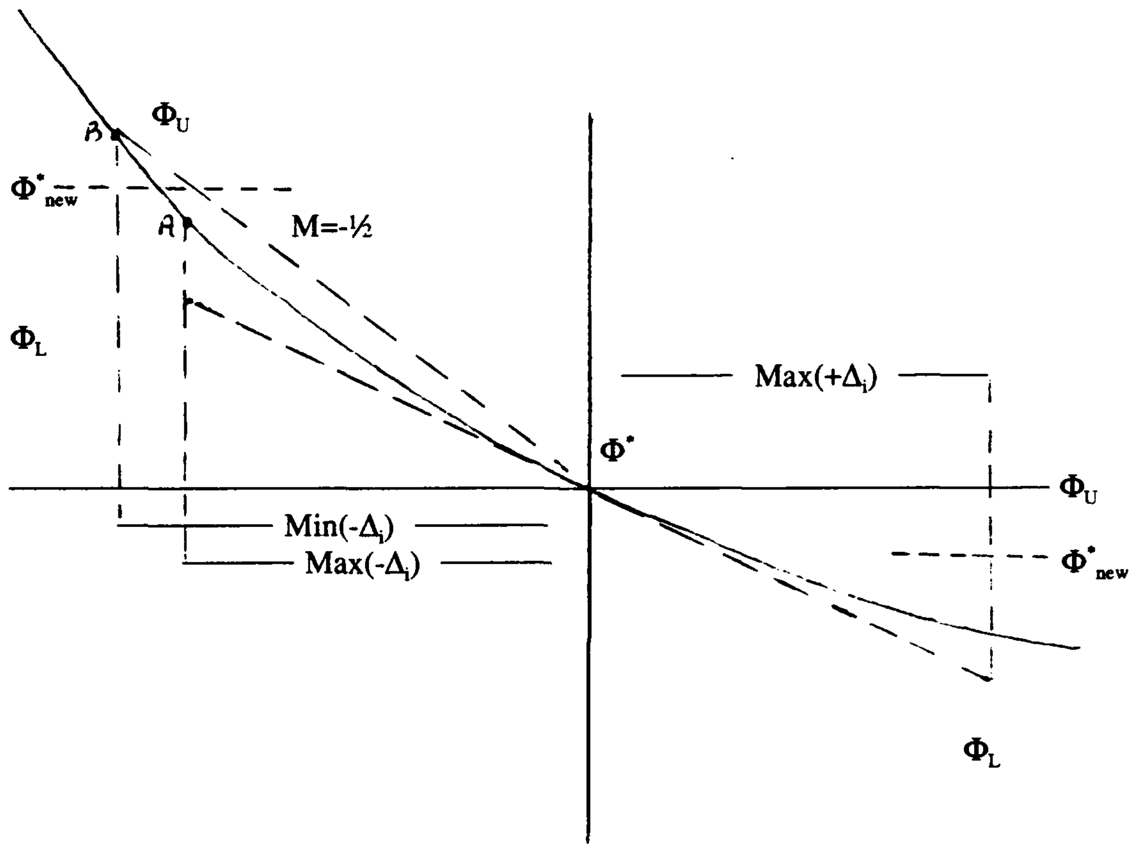

Figure 6-1 (Bounds on Ci/Wi Shifts)

by Δ = Max $[C_i/W_i - \Delta_i - C_i/W_i]$, the smallest shift occurring within the inventory system. Similarly point B is identified by applying Δ = Min $[C_i/W_i - \Delta_i - C_i/W_i]$, the largest ratio shift within the system, to the same C_i/W_i ratios. Thus defined, Theorem 5-2 establishes Φ_A and Φ_B, respectively, as a lower and an upper bound on Φ^*_{new}. The time consuming double application of the Implicit algorithm required to obtain these values can, however, be circumvented simply by utilizing the convex nature of the function and recalling from

Theorem 4-1 that, when the C_i/W_i ratio for each item carried in inventory are shifted equally and simultaneously, the resulting function is a straight line with a slope of -½. These characteristics suggest that an effective lower bound, Φ_L, on Φ_A and thus on Φ^*_{new} can be quickly identified by projecting the tangent line, generated at Φ^*, a distance of max[$-\Delta_i$] when system ratios decrease or a distance of max[$+\Delta_i$] when those ratios increase. Similarly, the function's convexity assures that Φ^* provides the needed upper bound on Φ^*_{new} when shifting ratios increase and that a linear projection from Φ^* exhibiting a slope of -½ for a distance Δ = Min[$-\Delta_i$] provides the desired upper bound when those parameters decrease.

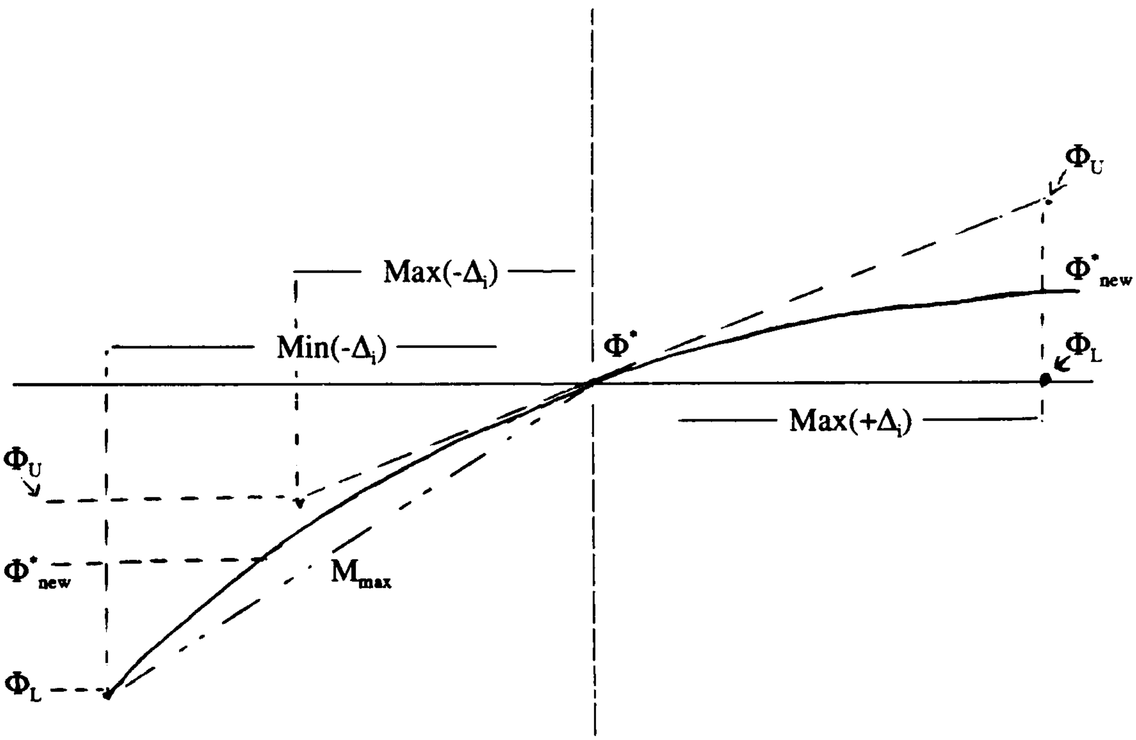

Figure 6-2 (Bounds On Reorder or Demand Shifts)

Following the logic used above to establish effective bounds on Φ^*_{new} when increasing or decreasing shifts occur in selected C_i/W_i ratios, the concave nature of those functions defining Φ implicitly by shifts in reorder costs and/or demand rates (Theorems 4-7, 4-8 and 4-9) assures the existence of similar bounding values when shifts occur in these parameters. For shifts in reorder cost parameters, as depicted in Figure 6-2, an effective upper bound, Φ_U, on Φ^*_{new} can be quickly identified by simply projecting the tangent line, generated at Φ^*, a distance of max[$-\Delta_i$] when system parameters decrease or a distance of max[$+\Delta_i$] when those parameters increase. The concavity of these functions ensures that Φ^* serves effectively as a lower bound on Φ^*_{new} when those parameters increase. Establishing an effective lower bound on Φ^*_{new} when decreasing parameters are encountered, however, requires some discussion since the function defined when all reorder and/or all demand parameters shift uniformly and concurrently can not be shown to be linear. On the surface, this non-linearity suggest that $\Phi_L=0$ is the only viable lower bound when parameters decrease. The near linear nature of such functions displayed during the empirical study documented in chapter III, however, suggest an alternate lower bound. Correlation coefficient values ranging between .999 and 1.000 imply that projecting a distance of Δ=Min ($-\Delta_i$) along a tangent line, generated at Φ^* when all reorder and/or demand parameters shift, will in most cases identify a lower bound on Φ^*_{new}. Only when shifts occur in all reorder and/or demand parameters will this approach fail to provide the desired lower bound. In such cases, however, the near linear nature of the function guarantees that this Φ_L will approximately equal Φ^*_{new} as long as parameter shifts are relatively small.

6.3 Horizon Algorithm

The discussion detailed in section 6.2 provides the insight needed to develop versions of the Horizon algorithm which can effectively solve multiple period inventory systems in which a variety of parameter shifts occur. Horizon algorithm - Version one identifies Φ^*_{new} when shifts occurring only in the C_i/W_i ratios on a N-item inventory. Horizon algorithm - Version two obtains the desired optimal multiplier when those shifts occur either in that system's reorder or demand parameters. Horizon algorithm - Version three handles those cases where shifts occur simultaneously in both reorder and demand parameters. Combining variation one recursively with variation three provides a means of handling those systems in which shifts occur simultaneously among all three parameter categories.

HORIZON ALGORITHM -- Variation 1 (C_i/W_i Ratio Perturbations)

Step (0) Set N=# of periods in horizon, set i=0 and compute Φ^*_i utilizing the Implicit algorithm (Chapter V).

Step (1) Set i=i+1, If C_i/W_i ratios decreasing go to step (3), otherwise continue.

Step (2) Establish initial bounds on Φ^*_{i+1} by setting:

$\Phi_U = \Phi^*_i$ and

$\Phi_L = \Phi^*_i$ - Max$(+\Delta_i)$*M

where $\Delta_i = (C_i/W_i + \Delta_i) - C_i/W_i$ and from equation (4-17)

$$M = \sum_{i \in J} \frac{\partial \Phi}{\partial (C_i/W_i)}.$$

The set J contains the indices of those parameters undergoing change.

Go to step (4).

Step (3) Establish initial bounds on Φ^*_{i+1} by setting:

$\Phi_L = \Phi^*_i + \text{Max}(-\Delta_i)*M$

$\Phi_U = \Phi^*_i + .5 * \text{Min}(-\Delta_i)$

where $\Delta_i = C_i/W_i - (C_i/W_i - \Delta_i)$ and M is defined as in step (2)

Step (4) Apply Bisection search to initial bounds established in steps (2) or (3) to obtain Φ^*_{i+1} and set i=i+1. If i < n go to step (1), otherwise stop.

HORIZON ALGORITHM -- Variation 2 (R_i or D_i Parameter Perturbations)

Step (0) Set N=# of periods in horizon, set i=1 and compute Φ^*_i utilizing the Implicit algorithm (Chapter V).

Step (1) Set i=i+1, If parameters decreasing go to step (3), otherwise continue.

Step (2) Establish initial bounds on Φ^*_{i+1} by setting:

$\Phi_L = \Phi^*_i$ and

$\Phi_U = \Phi^*_i + \text{Max}(+\Delta_i)*M$

where, when reorder parameters shift, $\Delta_i = (R_i + \Delta_i) - R_i$ and from equation (4-18)

$$M = \sum_{i \in J} \frac{\partial\Phi}{\partial(R_i)}$$

or when demand parameters shift, $\Delta_i = (D_i + \Delta_i) - D_i$ and from equation (4-19),

$$M = \sum_{i \in J} \frac{\partial\Phi}{\partial(D_i)}.$$

The set J contains the indices of those parameters undergoing change.

Go to step (4).

Step (3) Establish initial bounds on Φ^*_{i+1} by setting:

$$\Phi_U = \Phi^*_i - \mathrm{Max}(-\Delta_i)*M$$

$$\Phi_L = \Phi^*_i - \mathrm{Min}(-\Delta_i)*M_{max},$$

where $\Delta_i = R_i - (R_i - \Delta_i)$ and as defined as in step (2),

$$M_{max} = \sum_{i=1}^{N} \frac{\partial\Phi}{\partial(R_i)}$$ when reorder parameters shift or

where $\Delta_i = D_i - (D_i - \Delta_i)$ and as defined as in step (2),

$$M_{max} = \sum_{i=1}^{N} \frac{\partial\Phi}{\partial(D_i)}$$ when demand parameter shift.

Step (4) If $\sum_{i=1}^{N} \{2R_iD_iW_i/[(C/W)+2\Phi_L]\}^{1/2} < U$

then set $\Phi_L = 0$

Step (5) Apply Bisection search to initial bounds established in steps (2) or (3) to obtain Φ^*_{i+1} and set i=i+1. If i < n go to step (1), otherwise stop.

HORIZON ALGORITHM -- Variation 3 (R_i and D_i Parameter Perturbations)

Step (0) Set N=# of periods in horizon, set i=1 and compute Φ^*_i utilizing the Implicit algorithm (Chapter V).

Step (1) Set i=i+1, If parameters decreasing go to step (3), otherwise continue.

Step (2) Establish initial bounds on Φ^*_{i+1} by setting:

$\Phi_L = \Phi^*_i$ and

$\Phi_U = \Phi^*_i + \mathrm{Max}(+\Delta_i)*M$

where,

$\Delta_i = (\Delta_{Ri})(\Delta_{Di}) + Di(\Delta_{Ri}) + R_i(\Delta_{Di})$ and from equation (4-19A)

$$M = \sum_{i \in J} \frac{\partial \Phi}{\partial (R_i, D_i)}$$

Go to step (4).

Step (3) Establish initial bounds on Φ^*_{i+1} by setting:

$$\Phi_U = \Phi^*_i + \mathrm{Max}(-\Delta_i)*M$$

$$\Phi_L = \Phi^*_i + \mathrm{Min}(-\Delta_i)*M_{max}$$

where $\Delta_i = (\Delta_{Ri})(\Delta_{Di}) - Di(\Delta_{Ri}) - R_i(\Delta_{Di})$ and from equation (4-19A)

M is defined as in step (2),

$$M_{max} = \sum_{i=1}^{N} \frac{\partial\Phi}{\partial(R_i,D_i)}$$

Step (4) If $\sum_{i=1}^{N} \{2R_iD_iW_i/[(C/W)+2\Phi_L]\}^{1/2} < U$

then set $\Phi_L = 0$

Step (5) Apply Bisection search to initial bounds established in steps (2) or (3) to obtain Φ^*_{i+1} and set i=i+1. If i < n go to step (1), otherwise stop.

With the bounding values on Φ^*, identified by these algorithms at Steps 2 and 3, assured by the monotonic convexity of each implicitly defined function, the convergence of each algorithm is guaranteed by the bisection search method implemented at Step 5. As explained in section 5.4, with $\varepsilon > 0$ defining a desired degree of accuracy and Φ_U and Φ_L identified, $(1/2)^n \leq \varepsilon/(\Phi_U - \Phi_L)$ defines the number of bisection necessary for convergence to occur [1]. Combining this assurance of convergence with an expected tightness of Φ_U and Φ_L to Φ^*, the proposed algorithms should exhibit both speed and accuracy.

6.4 Numerical Example

To illustrate the computational efficiency of the proposed Horizon algorithm consider the four item inventory system, presented in Table 6-1, in which the carrying cost parameters for the first three items shift during each interval of a two period horizon.

Carrying Costs	Re-order Costs	Demand Rates	Resource Required
17.0	536	4515	18
17.75	512	4204	18
20.0	584	5775	4
18.5	546	5061	2

Table 6-1 (Example: 4-Item Inventory System)

If these system parameters decrease during the first period by 10, 13 and 10 respectively and then increase by 17, 10 and 16 respectively during the second period, then the Horizon algorithm yields the following results:

First Recursion

Step (0) N=2, Φ^*_0 = 6.130301 (computed use Implicit algorithm)

Step (1) i=1, decreasing ratios

Step (3) Since Max($-\Delta_i$) = -10/18 = -.5555, Min($-\Delta_i$) = -10/4 = -2.5 and

M = -½{(2)(536)(4515)(18)/[(17/18)+(2)(6.130301)]^3}^.5
- ½ {(2)(512)(4204)(18)/[(17.75/18)+(2)(6.130301)]^3}^.5
- ½ {(2)(584)(5775)(4)/[(20/4)+(2)(6.130301)]^3}^.5

{(2)(536)(4515)(18)/[(17/18)+(2)(6.130301)]^3}^.5
{(2)(512)(4204)(18)/[(17.75/18)+(2)(6.130301)]^3}^.5
{(2)(584)(5775)(4)/[(20/4)+(2)(6.130301)]^3}^.5
{(2)(546)(5061)(2)/[(18.5/2)+(2)(6.130301)]^3}^.5

M = -225.0528 / 483.4303 = -.4655

Then initial bounds on Φ^*_{new} are:

Φ_L = 6.130301 + (-.5555)(-.4655) = 6.3889

Φ_U = 6.130301 + (-.5)(-2.5) = 7.3803

Step (4) $\Phi^*_{new} = \Phi^*_1$ = 6.577886 (obtained via bisection search)

Second Recursion

Step (1) i=2, increasing ratios

Step (2) Since Max($-\Delta_i$) = -10/18 = -.5555, Min($-\Delta_i$) = -16/4 = -4.0 and

M = -½{(2)(536)(4515)(18)/[(7/18)+(2)(6.577886)]^3}^.5
- ½ {(2)(512)(4204)(18)/[(4.75/18)+(2)(6.577886)]^3}^.5
- ½ {(2)(584)(5775)(4)/[(10/4)+(2)(6.577886)]^3}^.5

{(2)(536)(4515)(18)/[(7/18)+(2)(6.577886)]^3}^.5
{(2)(512)(4204)(18)/[(4.75/18)+(2)(6.577886)]^3}^.5
{(2)(584)(5775)(4)/[(10/4)+(2)(6.577886)]^3}^.5
{(2)(546)(5061)(2)/[(18.5/2)+(2)(6.577668)]^3}^.5

M = -225.0801 / 481.5078 = -.46744

Then initial bounds on Φ^*_{new} are:

$$\Phi_L = 6.577886 + (-4.0)(-.46744) = 4.70809$$

$$\Phi_U = 6.577886$$

Step (4) $\Phi^*_{new} = \Phi^*_2 = 5.9780$ (obtained via bisection search)

Two things should be noted from this numerical example. First, recognize that the slope of the tangent line at Φ^* is calculated using system parameters prior to any shifts occurring. For example, during the first recursion the tangent line at Φ^*_0 is computed using $C_1/W_1 = 17/18$, $C_2/W_2 = 17.75/18$ and $C_3/W_3 = 20/4$ rather than the shifted values of 7/18, 4.75/18 and 10/4 respectively. Similarly, during the second recursion, that slope is figured using values of 7/18, 4.75/18 and 10/4 rather than the altered values of 24/18, 14.75/18 and 26/4. The magnitudes in which parameters shift only come into play only when determining the upper and lower bounds during steps 2 and 3 of the proposed algorithm. Second, it should be noted that even when parameter shifts are large, the relative error of the resulting bounds remains fairly tight. For example parameter reductions, ranging from 50 to over 70 %, produced a relative error just over 15 % following the initial recursion of the Horizon algorithm while augmentations ranging from 160 to over 240 % yielded an error of only 31 % during its second recursion. The tightness of these relative errors, even when large shifts in system parameters occur, suggest the efficiency which the proposed algorithm demonstrates in the following comparative analysis.

Four Inventory Sizes: 5, 15, 30 and 50

Three System Categories: C_i/W_i, R_i and D_i

Ten Inventory Horizons:
5 with less than ⅓ parameters shifting
5 with more than ⅔ parameters shifting

Ten Periods in each Horizon

Φ^*_Z, Φ^*_I and Φ^*_C determined from scratch for each period in an horizon

Φ^*_H computed for period i from Φ^*_{i-1} and known parameter shifts between period i-1 and i

Figure 6-3 (Analysis Design)

6.5 Comparative Analysis

From the results displayed in the above numerical example, the proposed Horizon algorithm seems to provide excellent bounds around each Φ^*_{new} needed to solve a multiple period inventory problem in which constrained conditions persist throughout the horizon. The purpose of the ensuing analysis is to determine the relative efficiency of this algorithm to the results obtained when the Implicit, Ziegler and Classical algorithms each are used to determine Φ^* separately for each constrained period of an extended horizon.

As shown in Figure 6-3, this analysis was conducted by examining ten multiple period inventory systems for each of the twelve inventory size and system category combinations indicated. The initial inventory system for each of these horizons was randomly generated using the LOW - HIGH - HIGH - LOW parameter configuration defined in Tables 5-7 and 5-8. The ten inventory systems comprising each horizon were then determined by applying, sequentially, 10 % shifts to each of the selected system parameters. The increasing or decreasing nature of these perturbations was also randomly determined. For example, if the number of carrying cost parameters undergoing change were selected randomly from the first ⅓ of a 10 item inventory system, an initial horizon might be determined from the following parameter values:

Parameter	Init	1	2	3	4	5	6	7	8	9	10
1	17	15.3	13.77	14.87	13.38	14.72	16.19	17.81	19.59	17.63	15.86
2	20	18	16.2	17.82	16.03	17.64	19.40	21.34	23.48	21.13	19.02
3	18.5	16.65	14.98	16.48	14.83	16.32	17.95	19.74	21.72	19.54	17.58

when all other system parameters remain unchanged from period to period.

For each of the resulting 1200 inventory systems, the optimal Lagrangian multiplier identified by the Ziegler algorithm, Φ^*_Z, the Implicit algorithm, Φ^*_I, and the Classical algorithm, Φ^*_C, were each determined independently. The optimal Lagrangian multiplier for the Horizon algorithm, Φ^*_H, however, was computed for each period using both the Φ^* optimizing the previous period's system and the known parameter shifts occurring during that period.

In an effort to determine if the number of parameters undergoing change impact the results obtained by the various algorithms under consideration while providing the means needed to compare the efficiency exhibited by each, the Average Time to Optimal, the Average Maximum Error (AME) and the Average Relative Error (ARE) were computed for each algorithm utilizing the following expressions:

$$\text{Average Time to Optimal} = [\sum_{i=1}^{50} \text{Time to } \Phi^*_i] / 5$$

$$\text{AME} = \{\sum_{i=1}^{50} \text{Max}[(\Phi^* - \Phi_L),(\Phi_U - \Phi^*)]\}/50$$

$$\text{ARE} = [\sum_{i=1}^{50} (\Phi_U - \Phi_L) / \Phi^*]/50$$

For each of the noted inventory size and system category combinations these statistics were established first for five Lower Range Horizons in which less than ⅓ of the system parameters underwent change and then for five Higher Range Horizons in which more than ⅔ of the parameters were modified. The Average Time to Optimal statistic, therefore, measures the average time required by these algorithms to establish the complete set of solutions needed to optimize each horizon within these Lower or Higher Range Horizon sets. The AME and ARE statistics, on the other hand, gauge the average error which was encountered as bounds were established around those Lagrangian multipliers needed to optimize each of the 50 systems constituting those same horizon sets. Clearly that algorithm achieving the smallest ARE while exhibiting the narrowest AME in the shortest CPU time must be considered the most efficient.

6.6 Analysis Results and Conclusions

LOWER RANGE					
N	P	CLASSICAL	ZIEGLER	IMPLICIT	HORIZON
5	1	13.22813	10.92813	8.431249	5.04375
	3	12.77188	10.23125	8.725001	7.223438
	4	14.86719	10.60313	8.014063	7.76875
15	1	41.77031	36.46719	23.95781	13.80625
	3	44.75157	38.63906	23.78594	20.49844
	4	47.79405	37.14936	23.37493	21.85731
30	1	83.80205	101.9279	47.18601	27.75811
	3	79.1921	104.0385	46.90156	39.99444
	4	79.57188	102.4519	46.09209	40.28213
50	1	118.7525	229.7293	75.00332	45.47129
	3	137.2934	233.5248	78.6295	66.23145
	4	155.0195	236.484	78.6957	68.99961
HIGHER RANGE					
N	P	CLASSICAL	ZIEGLER	IMPLICIT	HORIZON
5	1	12.89219	11.03281	8.082813	6.607813
	3	13.66406	10.94219	8.096875	7.845313
	4	15.7	11.39688	8.782812	8.748438
15	1	39.02031	38.02344	23.8375	18.90469
	3	42.82032	38.39844	24.28438	21.44531
	4	37.38398	36.97001	23.62403	21.71398
30	1	76.43828	103.3676	47.75415	37.30401
	3	96.16982	103.896	48.26612	43.66612
	4	74.67891	97.2418	44.23047	40.54922
50	1	126.5617	229.5254	77.74434	62.19707
	3	137.6395	237.9875	78.21328	70.6043
	4	153.8551	239.4485	79.40352	74.24805

Table 6-2 (CPU Time to Optimal Solution Comparison)

LOWER RANGE				
N	P	ZIEGLER	IMPLICIT	HORIZON
5	1	3.080632	1.58994	.3054634
	3	1.2047	2.613391	.1665774
	4	2.145741	1.299445	.4286291
15	1	.4993691	3.845297	.0159614
	3	1.100063	3.416093	.4894976
	4	2.165528	2.635204	.8400979
30	1	.4439501	4.110094	.0193058
	3	.5036907	3.573534	.3863216
	4	.3457469	3.341647	.322058
50	1	.1264408	2.38723	.0235899
	3	.4299531	5.75841	.383842
	4	.4790868	4.853921	.5428595
HIGHER RANGE				
N	P	ZIEGLER	IMPLICIT	HORIZON
5	1	2.572356	.9914414	.0665258
	3	2.268833	1.522478	.4069725
	4	4.6392	3.12515	1.162146
15	1	1.2827	3.733086	.1670194
	3	2.031989	4.217268	.6066523
	4	.3113536	2.499456	.3506118
30	1	.9994894	5.250293	.2556258
	3	1.441224	5.569216	.8428677
	4	.2022089	1.836559	.2474792
50	1	.4721026	3.9761	.1960983
	3	.617644	3.956522	.5482505
	4	.7908976	5.453927	.9146324

Table 6-3 (Bounding Interval - Average Maximum Error Comparison)

LOWER RANGE				
N	P	ZIEGLER	IMPLICIT	HORIZON
5	1	.6469662	.4501467	.0031258
	3	.4649393	1.162966	.0646857
	4	.4043682	.2264123	.0651284
15	1	.1166436	.9722205	.0053985
	3	.3343205	1.01634	.0672408
	4	.1743851	.5740883	.0698297
30	1	.1007918	.9145961	.0054767
	3	.1506514	.8631471	.0660557
	4	.1165651	.9041912	.0879659
50	1	.0711698	1.193059	.0098275
	3	.0567860	1.048549	.0681019
	4	.0711860	.8698786	.0841274

HIGHER RANGE				
N	P	ZIEGLER	IMPLICIT	HORIZON
5	1	.7333079	.8095734	.026282
	3	.4190447	1.528829	.059291
	4	.4923482	.8489205	.112727
15	1	.1966155	1.580501	.0271217
	3	.180413	1.732007	.0857475
	4	.1881088	1.623098	.137754
30	1	.0986909	1.896508	.0512733
	3	.0827869	1.664853	.0780258
	4	.0828171	2.040882	.1535839
50	1	.0391378	2.054192	.0776186
	3	.0806787	1.725076	.0778811
	4	.1060811	1.720387	.128317

Table 6-4 (Bounding Interval - Average Relative Error Comparison)

Tables 6-2, 6-3 and 6-4 exhibit the efficiency characteristic of the proposed Horizon algorithm when uniform shifts occur in a single system parameter category. The CPU times achieved by the Horizon algorithm are significantly faster, on average, than those exhibited by the Classical, Ziegler and Implicit algorithms.

In considering Tables 6-3 and 6-4 note that both the AME and ARE values obtained for the Horizon algorithm are consistently smaller than those errors exhibited by either the Ziegler or Implicit algorithms. AME values, ranging between [.0159, 1.162146], for the Horizon algorithm reflect maximum errors which are generally 2.57 and 4.158 times smaller than those errors generated, respectively, by the Ziegler and Implicit algorithms. Similarly, ARE values, fluctuating between [.003125, .137754], for the Horizon algorithm reflect relative errors which are 5.15 and 13.57 times tighter than the other two algorithms. The effectiveness of the proposed algorithm, suggested by the compactness of these errors, is strengthened by the realization that even in the four instances (once during the Lower Range horizons) that the Ziegler algorithm achieved better AME or ARE values than did the proposed algorithm, the differences between those results averaged only .068 and .036 respectively.

After further examination of Tables 6-2, 6-3 and 6-4, it must be concluded that, some degradation in the efficiency of the proposed algorithm occurs as both the number of shifting parameters and the inventory size increase. The impact of this degradation is clearly displayed in Table 6-5. Note that, when the inventory size was less than 30, the Horizon algorithm achieved the best bounding interval over 85 % of the time. If, however, the inventory size was greater than 15 that efficiency dropped to less than 60%.

LOWER RANGE N	ZIEGLER	IMPLICIT	HORIZON
5	9	2	139
15	21	6	123
30	33	0	117
50	55	0	95
HIGHER RANGE **N**	**ZIEGLER**	**IMPLICIT**	**HORIZON**
5	0	11	139
15	37	0	113
30	77	4	79
50	82	0	68

Table 6-5 (Best Bounding Interval)

Similarly, the 79 % efficiency obtained by the proposed algorithm when the number of shifting parameters was low decreased to 66.5 % when the number of shifting parameters was high. In those cases where both the inventory size and the number of shifting parameters was large, the Horizon algorithm achieved the best bounds only 49 % of the time. The apparent magnitude of this reduced efficiency can, however, be minimized considering the ARE and AME values obtained when the Horizon algorithm failed to identify the best bounds. In such cases, judging from these statistics, the Horizon algorithm identified only slightly expanded bounds while the CPU time it required to obtain those bounds was greatly reduced.

It must be concluded from the above discussion that, although some degradation occurred, the proposed Horizon algorithm provides an efficient means for solving a multiple period inventory system in which parameter shifts occur in only one parameter category. Before concluding this analysis, however, consideration must be given to those cases where random shifts occur concurrently in more than one parameter category. To gauge the algorithm's efficiency under such circumstances, an analysis paralleling that described in section 6-3 was run in which random shifts occurred simultaneously in both reorder and demand parameters utilizing Version 3 of the proposed algorithm. The results of this analysis, displayed in Appendix 5, suggest that the proposed algorithm continues to provide an effective means for handling the multiple period inventory system with CPU times 2.54 and 1.059 times better than the Ziegler and Implicit algorithms respectively. This apparent effectiveness is again strengthened by the realization that when the Ziegler algorithm achieved better AME or ARE values than did the proposed algorithm, the differences between those results continue to be minimal, averaging only .809 and .1103 respectively.

By comparing the results obtained when random shifts occur concurrently in both reorder and demand parameters with those results produced when shifts occurred only in a single parameter category, the following conclusions can be made:

1) When the desire is to identify effective bounds around each of the needed Φ^* values when shifts occur either in a single system category or in both the reorder and demand categories, the Horizon algorithm can be effectively used when $N < 30$ and when

less than ⅓ of the parameters shift. When N > 15 and more than ⅔ of the parameters shift then the Ziegler algorithm provides the tightest bound. However, when effective bounds are needed and CPU time is restricted, utilizing the Horizon algorithm should be sufficient.

2) When parameter shifts occur simultaneously in all three system categories (carrying cost, reorder cost and Demand) a sequential application of the Implicit algorithm provides the most efficient means for obtaining the desired Φ^* values. Although not analyzed here, it is projected, considering the CPU time presented in Tables 6-1 and A5-1, that the proposed application of both version 1 and 3 of the Horizon algorithm will consistently require more CPU than does the Implicit algorithm for this scenario.

3) When parameter shifts occur only in a single system category of a constrained multiple period inventory system then versions one and two of the Horizon algorithm effectively obtain that set of Φ^* values needed to optimize the system. Neither the inventory size nor number of shifting parameters degrade this efficiency to a significant degree.

4) When parameter shifts occur concurrently in both the reorder and demand parameter categories of a constrained multiple period inventory system then Version three of the Horizon algorithm effectively obtains that set of Φ^* values needed to optimize the system. In such cases, the noted decline in the bounding efficiency of the Horizon algorithm, although more pronounced, continues to be insignificant.

Chapter VII

A Multiple Constraint Inventory Problem

The Dual Constraint Case

7.1 Introduction

With Chapters IV, V and VI presenting first the theory and then the resulting algorithms needed to effectively handle the single constraint inventory problem ($P\Phi$), this chapter begins an examination of the multiple constraint inventory problem $P(\Phi,\Theta)$. Recognizing the potential complexity of such problems, the focus of this initial research will again be on the fundamental EOQ inventory model defined by equation (2-1). Here, however, both a carrying cost constraint and a storage space restriction are imposed.

7.1.1 Selection of Initial Constraint Set

In selecting these constraints as the basis for this initial study, two concerns were satisfied:

1) Problem Realism: The desire during this study was to examine as realistic a problem as possible while minimizing the complexity of the computational effort. The linear nature and the general applicability of these two constraints within most real-world inventory systems satisfy both these concerns.

2) Potential for Expansion: As a foundation study, the desire was to select an initial constraint set which exhibited a good potential for extension. The combination of both a budget and a space constraints satisfy this concern in two ways. First, their linear nature suggest that a direct extension from the two linear

constraint case to those cases involving three or more linear constraints might be possible. Second, since the Lagrangian multiplier associated with a non-linear constraint can often be stated in terms of one or more of a system's linear constraints [12],[20],[37], it may also be possible to deal initially with the linear portion of the constraint set and then calculate directly the Lagrangian multiplier associated with the non-linear constraint. In either case, the initial examination of a multiple constraint inventory system in which both a carrying cost limitation and a storage space restriction imposed should provide desired potential for expansion.

7.1.2 Re-statement of Dual Constraint Problem

Recall from Chapter II that such an inventory system can be stated as follows:

$$(P) \qquad \text{Min } TC = \sum_{i=1}^{N} (1/2\, b_iQ_i + R_iD_i/Q_i) \tag{7-1}$$

Subject To:

$$\sum_{i=1}^{N} W_iQ_i \leq W \tag{7-2}$$

$$\sum_{i=1}^{N} b_iQ_i \leq B \tag{7-3}$$

$$-Q_i \leq 0 \quad i=1,2,....N \tag{7-4}$$

where W_i defines the storage space required per unit of inventory and where b_i defines the cost associated with holding each item in inventory. As before, W and B set limits on the maximum inventory space available and the inventory budget allowed respectively. Letting Φ and Θ be nonnegative Lagrangian multipliers for equation (7-2) and (7-3), respectively, the Lagrangian function associated with Problem (P) can be restated as:

$$P(\Phi,\Theta) = \sum_{i=1}^{N}(1/2\ b_iQ_i + R_iD_i/Q_i) + \Phi(\sum_{i=1}^{N}W_iQ_i - W) + \Theta(\sum_{i=1}^{N}b_iQ_i - B).$$

In considering problem $P(\Phi,\Theta)$, it should be recalled, again from Chapter II, that the classical solution procedure presented by Buchan and Koeningsberg [4] as well as Tersine [38], entails a five step process which assumes, with equal probability, that either none, one or both of the constraints imposed on the inventory system will be binding at the optimal solution. With no means available to predict which of these situations will occur, the classical algorithm must consider, in turn, each possible case in order to ensure that the resulting solution is optimal. With this deficiency in mind, the goal for Chapter VII is to establish an algorithm which can effectively determine that portion of the constraint set which will be binding at the optimal solution of problem $P(\Phi,\Theta)$. In those cases where only one constraint remains active, this algorithm will identify that binding restriction and then use the single constraint Implicit algorithm to obtain the optimal solution. In those cases where both constraints are active, this algorithm will effectively estimate the Lagrangian multipliers needed to solve problem $P(\Phi,\Theta)$.

Following a description of the data collection procedure, a presentation of the results for an empirical study conducted in support of this research effort is given in

Section 7.2. Section 7.3 will establish the necessary theoretical foundation and Section 7.3 provides a detailed description of the proposed Dual Constraint algorithm. Since the efficiency exhibited by the Implicit algorithm has been well established in the preceding chapters, the effectiveness of the proposed algorithm will be measured, in section 7.6, by determining how well the resulting Φ^* and Θ^* estimates match the Lagrangian multipliers needed to optimize Problem P(Φ,Θ) when both constraints are binding.

7.2 Empirical Study - Dual Constraints

In order to provide a solid foundation for the Dual Constraint algorithm, an empirical study was used to examine the changes generated in the Lagrangian multipliers, Φ and Θ, when shifts occurred either in the budget constraint level, B, or the space constraint level, W. Since the results obtained when B was modified mirrored the results obtained when W was shifted, this section will present only that portion of the study in which the space constraint was held constant while the budget constraint level was systematically modified. As will be shown, these results provided the insight needed to establish a theoretical basis on which the proposed algorithm was developed. In order to minimize the magnitude of the computational effort required while still providing a clear picture of the desired functional relationships, the ensuing investigation was limited to 3-item inventory systems. The constraint levels, B and W, associated with each of these systems were established utilizing the following equations:

$$W = P_1 * \sum W_i Q_i \quad \text{for } i = 1 \text{ to } N \text{ and} \tag{7-5}$$

$$B = P_2 * \sum b_i Q_i \quad \text{for } i = 1 \text{ to } N \tag{7-6}$$

where $\sum b_iQ_i$ and $\sum W_iQ_i$ represented, respectively, the inventory budget and storage space required to hold equations (7-2) and (7-3) as equalities. The reduction factors, P_1 and P_2, were defined between [0,1]. Once P_1 and P_2 were selected, equations (7-5) and (7-6) clearly specified the pressure exerted on Problem $P(\Phi,\Theta)$ by each of its associated constraints. For example, when both P_1 and P_2 were each set equal to one, problem $P(\Phi,\Theta)$ became unconstrained with both Φ and Θ equal zero. If, on the other hand, P_1 and P_2 each assumed values close to zero, then problem $P(\Phi,\Theta)$ is severely constrained and either Φ and/or Θ take on values much greater than zero.

As a basis for this study, initial inventory systems were randomly generated for each of the four test configurations defined in Tables 7-1 and 7-2. The initial reduction

Configuration	C_i	R_i	D_i	W_i
1	Low	Low	Low	Low
2	High	High	High	High
3	High	Low	Low	High
4	Low	High	High	Low

Table 7-1 (Test Configurations)

Parameter	Low	High
C_i	(10,20)	(20,50)
R_i	(50,80)	(500,600)
D_i	(200,400)	(4000,6000)
W_i	(1,25)	(50,400)

Table 7-2 (Uniform Distributions of System Parameters)

factors, P_1 and P_2, needed to fully define these constrained inventory systems were identified by randomly selecting an initial reduction factor, $P_I \in [.1,.9]$ for each and then setting $P_1 = P_2 = P_I$. From each of these initial inventory systems, the empirical data needed to identify those relationships which exist between the associated Lagrangian multipliers and shifts which occur in the system's budget constraint level were collected utilizing a data generating process comprised of a System Generation Step, a System Solution Step and a Function Refinement Step.

The System Generation step of this process produced nine related inventory systems by systematically modifying the reduction factors, P_1 and P_2, assigned to an initial inventory system following the scheme displayed in Table 7-3. Note that, in defining each of these systems, the space constraint level W remained constant at its initial level while the pressure imposed on Problem $P(\Phi,\Theta)$ by the budget constraint gradually decreased.

SYSTEM	1	2	3	4	5	6	7	8	9
P_1	P_I	————————————————————						—>	P_I
P_2	.1	.2	.3	.4	.5	.6	.7	.8	.9

Table 7-3 (Phase I System Generation Scheme)

Once defined, each of these nine related systems were solved utilizing the General Interactive Optimizer, Super GENO, during the System Solution Step. By focusing not only on the desired optimal Lagrangian multipliers, Φ^* and Θ^*, but also on the slack generated within each system constraint, an initial picture of the resulting functional relationships was obtained. As will be seen, the data generated during this step tentatively identified a critical range of P_2 values in which both of the constraints associated with $P(\Phi,\Theta)$ remain binding at the optimal solution.

In order to refine this initial picture of the desired functional relationships, additional inventory systems are generated from within the critical range of P_2 values identified by the System Solution Step. Reducing the step size used to generate these systems from .1 to .02, the Function Refinement Step effectively pinpointed the P_2 values, P_{2L} and P_{2U} between which both of the constraints associated with $P(\Phi,\Theta)$ were always binding at the optimal solution. By employing this data generation scheme to specify $P_2 \in [P_{2L},P_{2U}]$, the three distinct regions which comprise the desired functional relationships were clearly defined.

Realizing that "a picture is worth a thousand words" and the benefit that an example can at times provide, consider the implementation of the above data generation process displayed in Tables 7-5, 7-6 and 7-7 for the randomly selected inventory system displayed in Table 7-4 for which P_I was set at .4.

Carry Cost	Reorder Cost	Demand Rate	Space Consumed
18.25	57	360	16
13.5	72	217	10
16	58	395	1

Table 7-4 (Sample Low-Low-Low-Low 3-Item Inventory System)

Following the shifting scheme shown in Table 7-3, Table 7-5 displays the space and budget constraint levels used to define the nine inventory systems formed during the System Generating Step. Note that in defining these nine systems, the space constraint

SYSTEM	1	2	3	4	5	6	7	8	9
W	555.22	———	———	———	———	———	——>	555.22	
B	237.11	474.23	711.34	948.46	1185.57	1422.69	1659.81	1896.92	2134.04

Table 7-5 (Example of the Shifts Generated In Constraint Levels During Phase I)

level, W, was held constant at 555.22. In accordance with equation (7-5), this constraint level was determined by applying the initial reduction factor, P_I =.4, to an unconstrained space requirement of 1388.05. Similarly the budget constraint level, identified for system 1 as 237.11, was computed by multiplying the unconstrained budget requirement of 2371.10 by the reduction factor P_2 =.1.

Reduction Factors					
Space	Budget	Φ^*	Θ^*	B Slack	W Slack
.4	.1	49.4998	0	0	425.882
|	.2	11.9999	0	0	296.545
|	.3	5.0555	0	0	167.209
|	**.4**	**2.6249**	**0**	**0**	**37.873**
|	**.5**	**.5896**	**2.1207**	**0**	**0**
|	**.6**	**0**	**2.9979**	**68.019**	**0**
|	.7	0	|	305.134	0
|	.8	0	|	542.250	0
.4	.9	0	2.9979	779.366	0

Table 7-6 (Step 2 Results For Low-Low-Low-Low Inventory System)

Table 7-6 displays both the Lagrangian multipliers and slack values obtained as the nine systems defined in Table 7-5 were evaluated during the System Solution Step. The Φ^* and Θ^* values, displayed in this table, clearly illustrate the three distinct functional relations which exist between these multipliers and the indicated shifts applied to both the budget and space constraint levels imposed on Problem $P(\Phi,\Theta)$. In this specific example, $P_2 \in (.4,.6)$ defines a critical region in which either one or both of the constraints imposed on Problem $P(\Phi,\Theta)$ remained binding. Clearly, where $P_2 \leq .4$, only the budget constraint exerts pressure on the optimal solution while for $P_2 \geq .6$, only the space constraint defines that solution. Clearly when the reduction factor, P_2, falls below the indicated critical region, the Φ^* value producing a budget slack equal to zero also satisfies the space constraint. Similarly, when P_2 lies beyond that critical region, the Θ^* value optimizing the space constraint also satisfies the budget constraint. The data generated by step two of this process can, therefore, be used to effectively describe the desired functional relationships as long as P_2 does not fall within an inventory system's critical region. Further investigation, however, was needed in order to describe the functional relationships which exist within that region.

Table 7-7 presents the Lagrangian multipliers and slack values obtained when the Function Refinement Step was applied to the critical region identified in Table 7-6. Here it should be noted that the tentative Critical Region has been reduced from $P_2 \in [.4,.6]$ to a range of P_2 values $\in (.4,.58)$ in which no slack was generated in either the budget constraint or the space constraint. Note also that the Φ^* needed to optimize $P(\Phi,\Theta)$ gradually decreased from its single budget constraint value of 2.6249 to zero while the

Reduction Factors					
Space	Budget	Φ^*	Θ^*	B Slack	W Slack
.4	.4	2.6249	0	0	37.873
|	.42	2.3344	.2742	0	0
|	.44	1.8216	.4966	0	0
|	.46	1.2709	1.2069	0	0
|	.48	.8787	1.7260	0	0
|	.5	.5896	2.1207	0	0
|	.52	.3702	2.4314	0	0
|	.54	.1998	2.6825	0	0
|	.56	.0645	2.8934	0	0
|	.58	0	2.9979	20.595	0
.4	.6	0	2.9979	68.019	0

Table 7-7 (Step 3 Results For Low-Low-Low-Low Inventory System)

Θ^* steadily increased from zero to its single space constraint value of 2.9979.

Before attempting to draw any general conclusions concerning the functional relationships in the empirical data, consideration must be given to the other system configurations examined. Utilizing the graphics package, STATGRAPHIC, the data generated for initial inventory systems from each configuration produced the plots given in the following pages. In each of these graphs, the solid line depicts the Lagrangian multiplier, Θ^*, associated with the budget constraint while the dashed line represents the Lagrangian multiplier, Φ^*, related to the space constraint of the selected inventory system. Plotting the Θ^* and Φ^* values shown in Tables 7-6 and 7-7 against their corresponding reduction factors, Figure 7-1, for example, clearly displays the three distinct functional regions noted earlier.

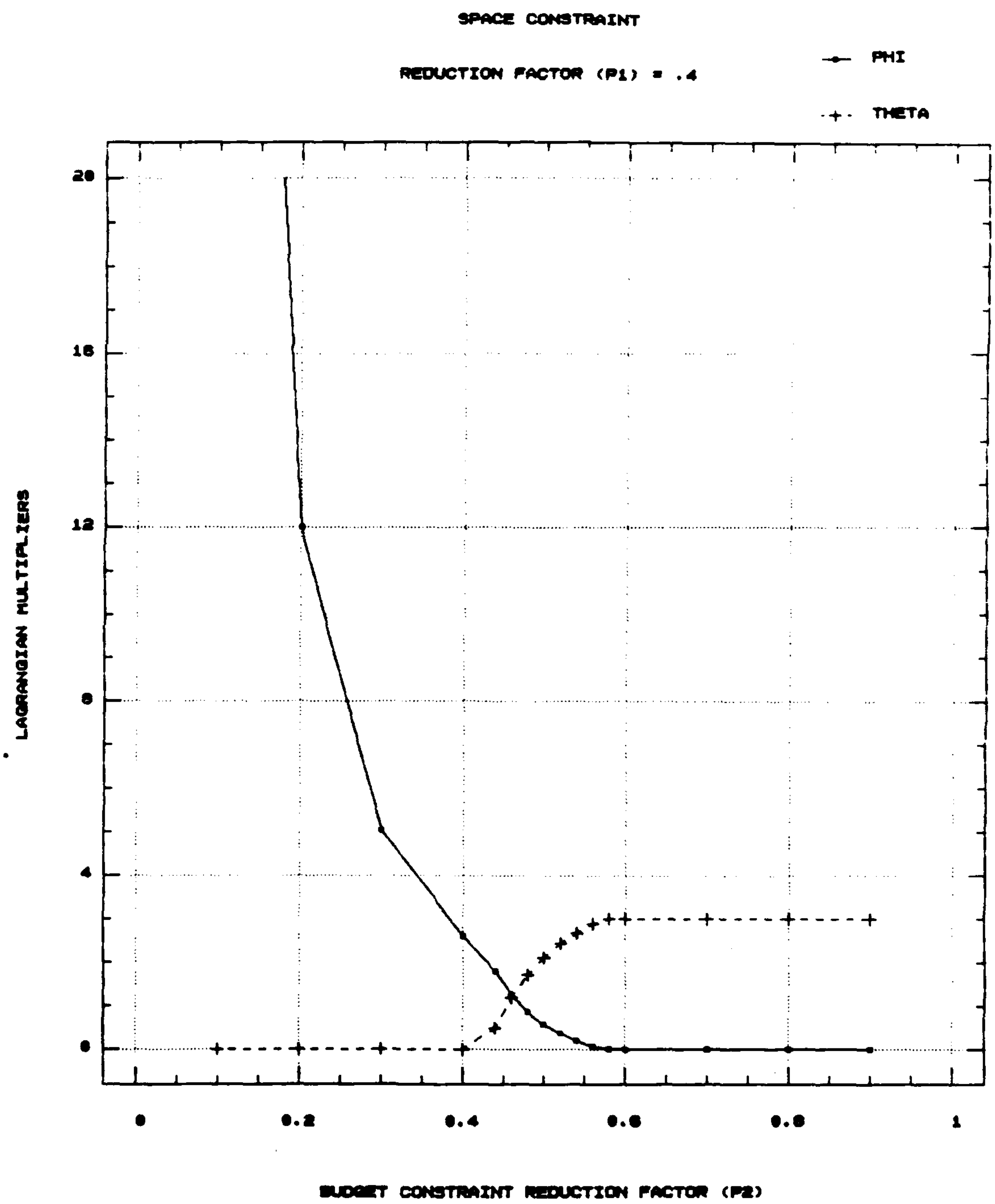

Figure 7-1 (Dual Constraint - Low-Low-Low-Low System Configuration)

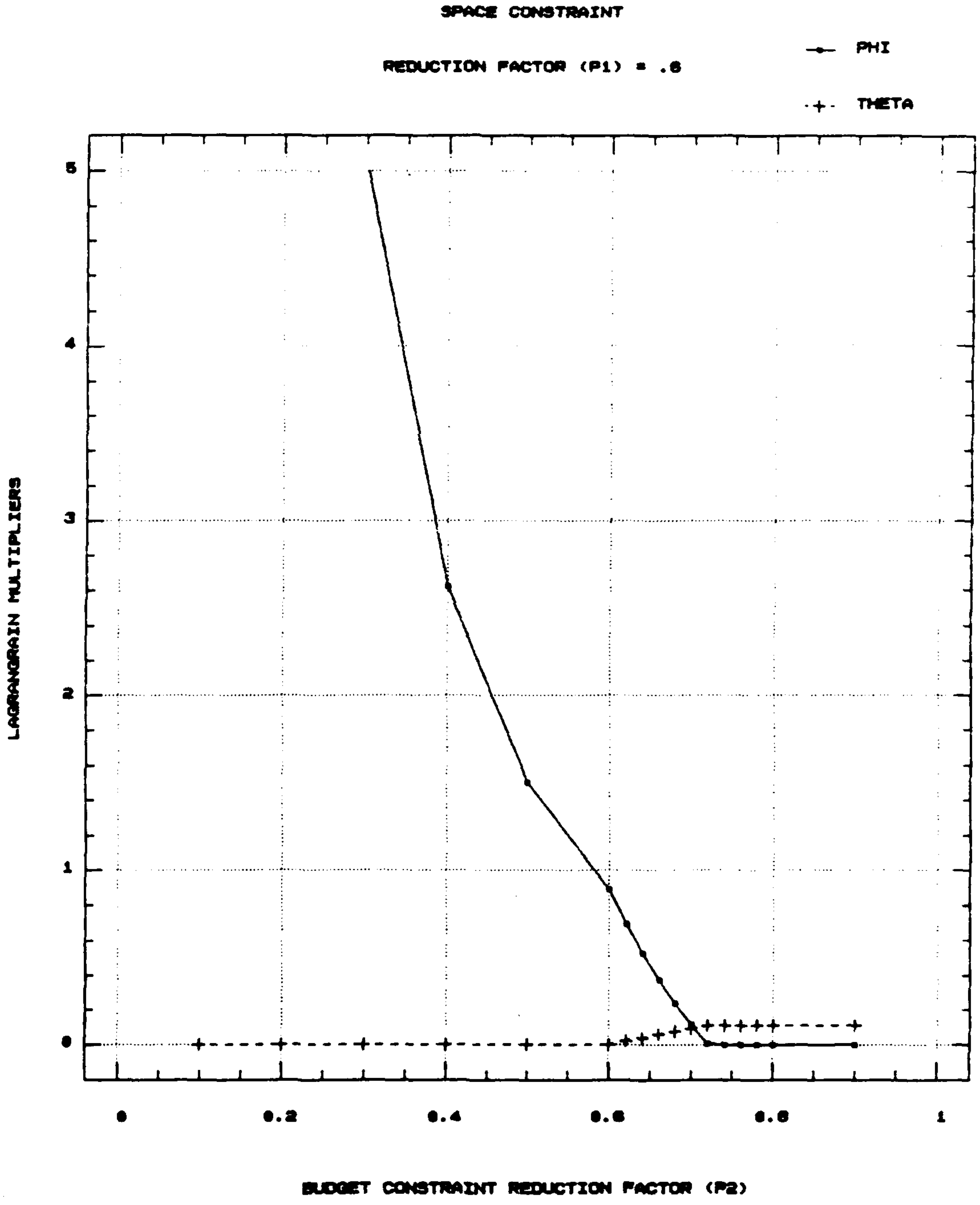

Figure 7-2 (Dual Constraint - High-High-High-High System Configuration)

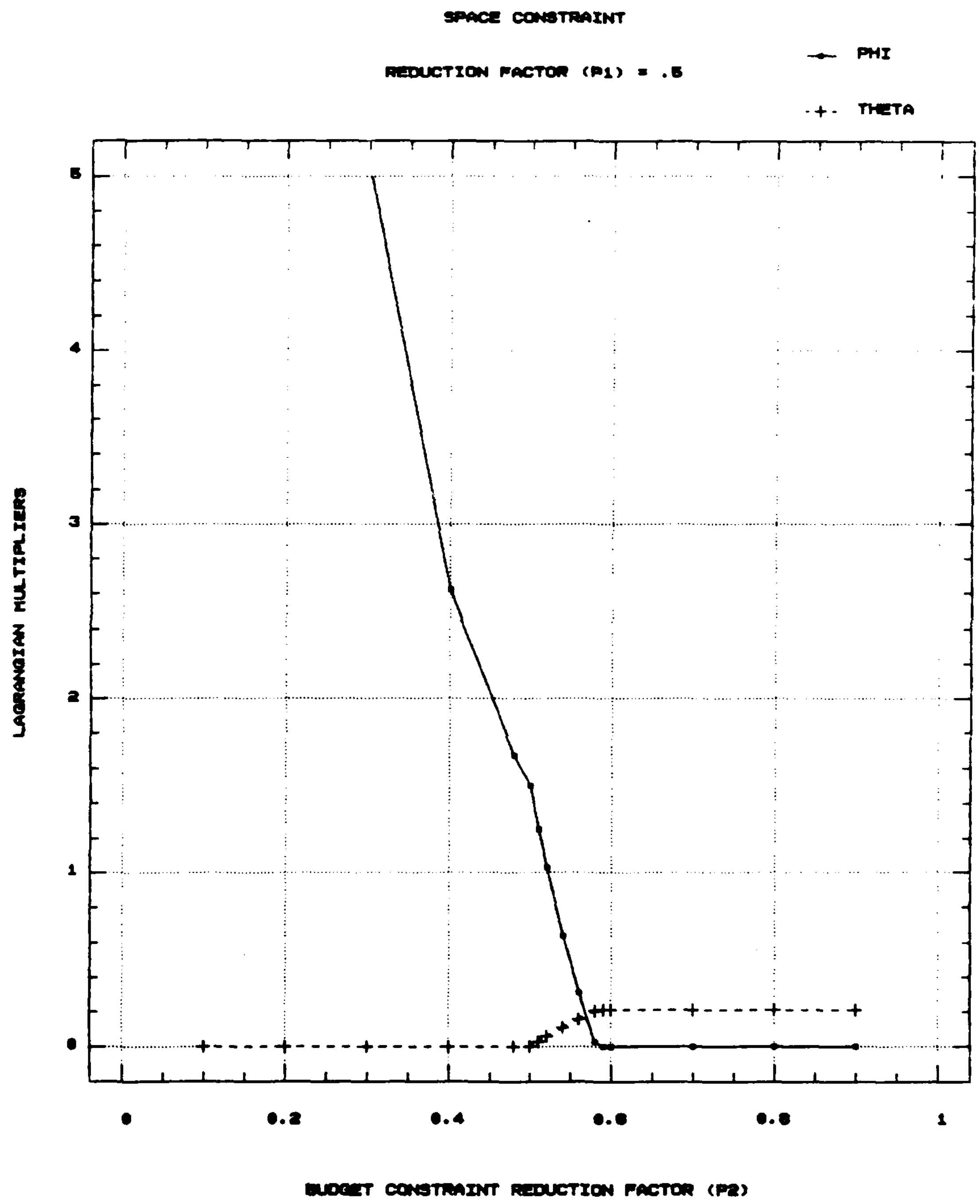

Figure 7-3 (Dual Constraint - High-Low-Low-High System Configuration)

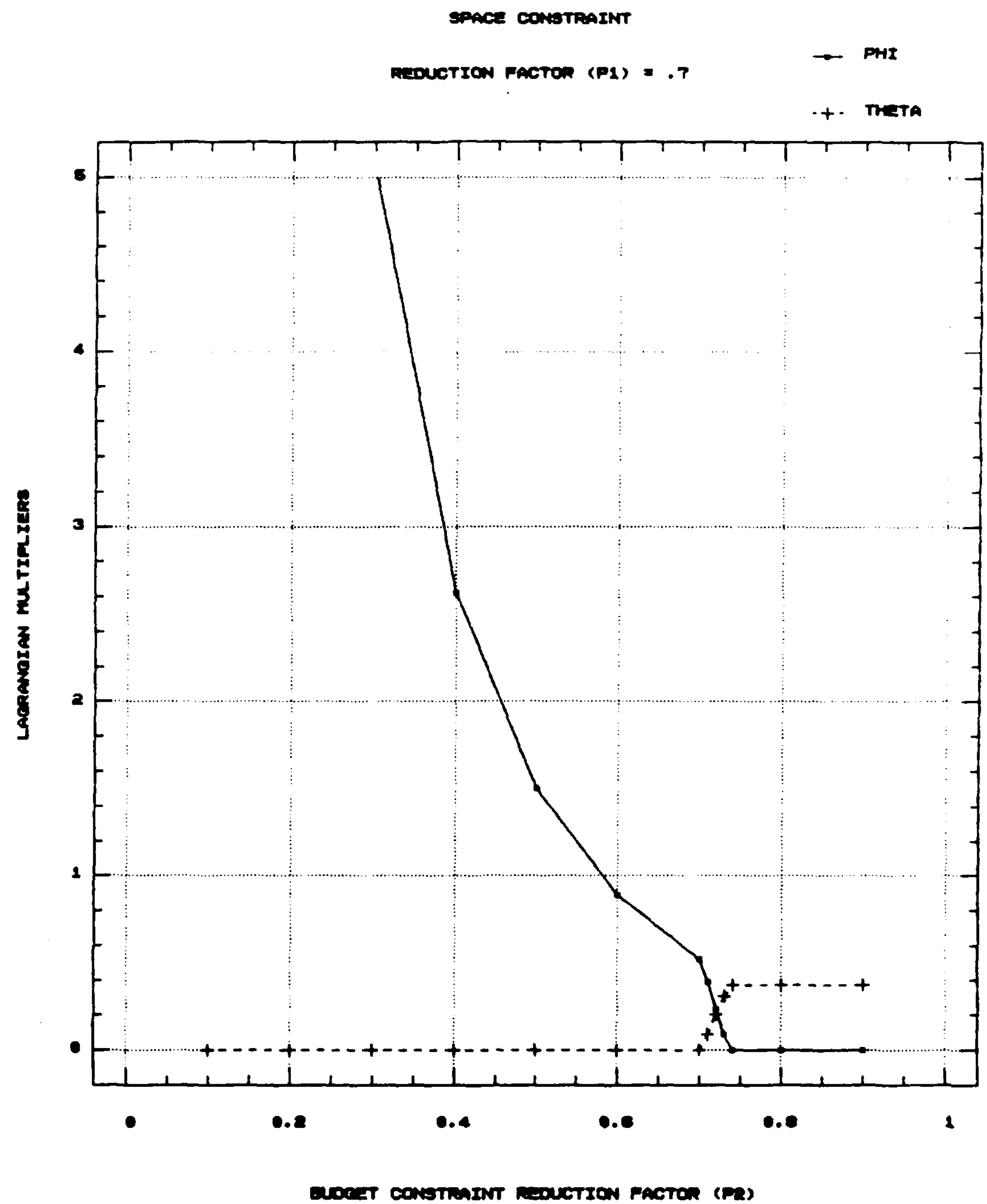

Figure 7-4 (Dual Constraint - Low-High-High-Low System Configuration)

In view of Figures 7-1 through 7-4 and the supporting empirical data displayed in Tables 7-4 and 7-5 as well as in Tables A6-1, A6-2 and A6-3 presented in Appendix 6, it can be concluded that the functional relationships which exist between a shifting budget constraint level and the resulting Φ^* and Θ^* values exhibit the same characteristics regardless of the range from which system parameters are selected. Several general conclusion can, therefore, be made concerning the constrained problem $P(\Phi,\Theta)$.

1) The size of the Critical Region is small. With the proportion of inventory systems falling into the Critical Regions exhibited in the above figures ranging from 4 % for the LHHL system configuration to 16 % for the LLLL system configuration, it is apparent that in most cases either the Φ^* for problem $P(\Phi)$ or the Θ^* problem $P(\Theta)$ will optimize problem $P(\Phi,\Theta)$.

2) As suggested by Tersine [38], when both Φ^* and Θ^* greater than zero comprise an optimal solution, these values fall consistently close to zero. For the 22 systems which fell within the indicated Critical Regions of this empirical study, the value of these multipliers ranged between zero and 2.8934 while the average Φ^* and Θ^* values were only .6106 and .6997, respectively.

3) Within the Critical Region of an inventory system both constraints are active and hence the inequalities defined by equations (7-2) and (7-3) become strict equalities.

4) Within the Critical Region of an inventory system, Φ^* decreases from its single budget constraint solution obtained at the region's lower bound to zero at its upper bound while Θ simultaneously increases from zero to the single space constraint solution. The apparent near linear nature of these partial functions also suggest that effective estimates

of both Φ^* and Θ^* may be obtained using a simple linear approximation.

5) When $P_2 \leq P_1$, the single budget constraint solution to problem ($P\Phi$) is optimal for problem $P(\Phi,\Theta)$. It is apparent, in this study, that the lower bound of the critical region occurs when $P_2 = P_1$.

6) When $P_2 > P_1$ the single space constraint solution to Problem ($P\Theta$) is usually optimal for Problem $P(\Phi,\Theta)$. The exact P_2 value which defines the upper bound of an inventory system's Critical Region, however, is dynamically determined by the parameters of that specific system.

7) Stemming from conclusion (5) and the slack values displayed in this study, the following conditions hold at the lower bound of an inventory system's Critical Region: $\Phi^* > 0$, $\Theta^* = 0$ and

$$\sum_{i=1}^{N} W_i Q_i = W$$

$$\sum_{i=1}^{N} b_i Q_i = B$$

8) Similarly, at the upper bound of an inventory system's Critical Region the following conditions hold : $\Phi^* = 0$, $\Theta^* > 0$ and as indicated above both budget and space constraints are equalities.

7.3 Theoretical Foundation

Combining the results of the above dual constraint empirical study with the KKT conditions identified in section 2.3, this section establishes the following assertions concerning the constrained inventory problem $P(\Phi,\Theta)$ when P1, P2 and $P \in [0,1]$:

1) When $P_2 \leq P_1$ the single budget constraint solution to Problem ($P\Phi$) is optimal for Problem $P(\Phi,\Theta)$,

2) When $P_2 \geq P$ the single space constraint solution to Problem ($P\Theta$) is optimal for Problem $P(\Phi,\Theta)$ and

3) When $P_1 < P_2 < P$, Problem $P(\Phi,\Theta)$ is minimized only if $\exists\ \Phi^* > 0$ and $\Theta^* > 0$.

Let: Q_i represent the unconstrained order quantity of the i^{th} inventory item defined by equation (1-2),

Q_i^C represent the constrained order quantity of the i^{th} inventory item defined by equation (2-24),

$Q_{i\Phi}$ represent the constrained order quantity for the i^{th} inventory item defined by equation (2-24) utilizing the Φ^* optimizing the single budget constraint problem $P(\Phi)$ and setting $\Theta^* = 0$,

and $Q_{i\Theta}$ represent the constrained order quantity for the i^{th} inventory item defined by equation (2-24) utilizing the Θ^* optimizing the single space constraint problem $P(\Theta)$ while setting $\Phi^* = 0$.

The following proofs are based on the recognition that, in accordance with the KKT conditions of feasibility:

$$\Phi \left\{ \sum_{i=1}^{N} b_i Q_i^C - C \right\} = 0 \tag{7-7}$$

and

$$\Theta \left\{ \sum_{i=1}^{N} W_i Q_i^C - W \right\} = 0, \tag{7-8}$$

only one of the following situations can exist when the constrained Problem $P(\Phi,\Theta)$ is minimized.

A) $\Phi = 0$ and $\Theta = 0$ while $\sum b_iQ_i^C - C \leq 0$ and $\sum W_iQ_i^C - W \leq 0$,

B) $\Phi \geq 0$ and $\Theta = 0$ while $\sum b_iQ_i^C - C = 0$ and $\sum W_iQ_i^C - W \leq 0$,

C) $\Phi = 0$ and $\Theta \geq 0$ while $\sum b_iQ_i^C - C \leq 0$ and $\sum W_iQ_i^C - W = 0$,

D) $\Phi \geq 0$ and $\Theta \geq 0$ while $\sum b_iQ_i^C - C = 0$ and $\sum W_iQ_i^C - W = 0$.

Theorem 7-1 (Assertion one)

Let P_1 and $P_2 \in [0,1]$ be reduction factors $\ni$ the constraint set of Problem $P(\Phi,\Theta)$ becomes:

$$\sum W_iQ_i^C \leq W = P_1 \{ \sum W_iQ_i \} \text{ and} \tag{7-9}$$

$$\sum b_iQ_i^C \leq C = P_2 \{ \sum b_iQ_i \} \tag{7-10}$$

for i= 1 to N. If $\exists\, \Phi^* > 0 \ni$ equation (7-10) holds as an equality then when $P_2 \leq P_1$, Θ^* must equal 0.

Proof: Using the direct calculation suggested by Rosenblatt [33], the $\Phi^* > 0$ holding equation (7-10) as an equality can be stated as follows:

$$\Phi^* = ½ \{ (1/C)[\sum b_iQ_i]\}^2 -½. \tag{7-11}$$

From equation (7-10), $C = P_2 \{ \sum b_iQ_i \}$, and equation (7-11) can be restated as:

$$\Phi^* = ½ [(1/P_2)^2 - 1]. \tag{7-12}$$

Substituting this latter expression for Φ^* into equation (7-9) yields:

$$\Sigma\ W_iQ_i^C \leq \Sigma\ W_iQ_{i\Phi} = \Sigma\ W_i\ [2R_iD_i/C_i(1/P_2)^2]^{1/2} = P_2\ \{\ \Sigma\ W_iQ_i\ \} \leq W = P_1\ \{\ \Sigma\ W_iQ_i\ \}. \tag{7-13}$$

If, therefore, $P_2 \leq P_1$ then $\Sigma\ W_iQ_i^C \leq W$ and following from situation (B), $\Theta^* \equiv 0$ provides the desired optimal solution. ■

Note that with $\Phi^* > 0$ fixed via equation (7-12), then equation (7-10) holds as an equality. The introduction of any $\Theta^* > 0$ decreases the reorder quantities associated with the constrained problem $P(\Phi,\Theta)$ inducing increased slack values in both the budget and space constraint. It follows, therefore, that when $P_2 \leq P_1$ such solutions, although feasible, are not optimal.

Theorem 7-2 (Assertion 2)

Let P1, P2 and $P \in [0,1]$ be reduction factors and let the constraint set of Problem $P(\Phi,\Theta)$ be defined by equations (7-9) and (7-10). If $\exists\ \Theta^* > 0 \ni$ equation (7-9) holds as an equality and P is defined such that

$$P\ \Sigma\ b_iQ_i = \Sigma\ b_iQ_{i\Theta},\ \text{for } i=1 \text{ to } N \tag{7-14}$$

then when $P_2 \geq P$, Φ^* must equal 0.

Proof: From equation (7-10) and KKT condition (7-7) it follows that when $P(\Phi,\Theta)$ is constrained:

$$\Sigma\ b_iQ_i^C \leq \Sigma\ b_iQ_{i\Theta} = P\ \{\ \Sigma\ b_iQ_i\ \} \leq C = P_2\ \{\ \Sigma\ b_iQ_i\ \} \tag{7-15}$$

in order to maintain feasibility. If, therefore $P_2 \geq P$ then $\sum C_iQ_i^C \leq C$ and following from situation (C), $\Phi^* \equiv 0$ provides an optimal solution.

■

In considering the proof of Theorem 7-2, recognize the validity of equation (7-14). Since at the upper bound of the Critical Region for $P(\Phi,\Theta)$ $\exists$ $\Theta > 0 \ni$ when $\Phi = 0$ both equation (7-9) and (7-10) hold as equalities then at this point $Q_i^C = Q_{i\Theta}$. It follows, therefore, from equation (7-9) that $\exists$ $P_2 = P \ni P \sum b_iQ_i = \sum b_iQ_{i\Theta}$. Note also that with Θ^* > 0 fixed so that equation (7-9) holds as an equality, the introduction of any $\Phi^* > 0$ decreases the reorder quantities associated with the constrained Problem $P(\Phi,\Theta)$ introducing increased slack values into both the budget and space constraints. It follows, therefore, that when $P_2 \geq P$ such solutions, although feasible, are again not optimal.

Theorem 7-3 (Assertion 3)

Let P_1, P_2 and $P \in [0,1]$ be reduction factors. Let P be defined by equation (7-14) and let the constraint set of Problem $P(\Phi,\Theta)$ be defined by equations (7-9) and (7-10). If $P_1 < P_2 < P$ then $\exists$ $\Theta^* > 0$ and $\Phi^* > 0 \ni$ both equation (7-9) and equation (7-10) hold as equalities.

Proof: From equation (7-13), defined in the proof of Theorem 7-1, when $P_2 > P_1$ and $\exists$ $\Phi^* > 0$ such that equation (7-10) holds as an equality:

$$\sum W_iQ_i^C \leq \sum W_iQ_{i\Phi} = P_2 \{ \sum W_iQ_i \} > P_1 \{ \sum W_iQ_i \} = W. \qquad (7\text{-}16)$$

In such cases, since in equation (7-16) $\sum W_iQ_i^C$ must be less than or equal to W in order to maintain feasibility, clearly $\Theta^* = 0$ defines an infeasible solution. This being the case, situation (B) can be eliminated from consideration. Similarly, from equation (7-15), defined in the proof of Theorem 7-2, when $P_2 < P$ and $\exists\ \Theta^* > 0$ such that equation (7-9) holds as an equality:

$$\sum b_i Q_i^C \leq \sum b_i Q_{i\Theta} = P\ \{\ \sum b_i Q_i\ \} > P_2\ \{\ \sum b_i Q_i\ \} = C \qquad (7\text{-}17)$$

Here, since $\sum b_i Q_i^C$ must be less than or equal to C in order to maintain feasibility, clearly $\Phi^* = 0$ defines an infeasible solution. This again being the case, situation (C) can also be eliminated from consideration. Therefore, by elimination situations (B) and (C) from consideration, when $P_1 < P_2 < P$ either problem $P(\Phi,\Theta)$ is unconstrained with both Φ^* and Θ^* equal to zero following from situation (A) or, following from situation (D), $\exists\ \Theta^* > 0$ and $\Phi^* > 0 \ni$ both equation (7-9) and equation (7-10) hold as equalities.

■

Collectively, these proofs not only establish:

1) That the single budget constraint solution, Φ^*, to Problem ($P\Phi$) is optimal for the constrained problem $P(\Phi,\Theta)$ when $P_2 \leq P_1$,

2) That the single space constraint solution, Θ^*, to Problem ($P\Theta$) is optimal for the constrained problem $P(\Phi,\Theta)$, when $P_2 \geq P$ and

3) That $\Phi^* > 0$ and $\Theta^* > 0$ define the optimal solution for the constrained problem $P(\Phi,\Theta)$ when $P_1 < P_2 < P$;

but also specify $P_2 = P_1$ and $P_2 = P$ as the lower and upper bounds of the Critical Region needed to predict when one or both of the constraints imposed on problem $P(\Phi,\Theta)$ will be binding at the optimal solution.

7.4 The Dual Constraint Algorithm

Based on the theoretical foundation developed in section 7.3, the following three step algorithm effectively determines that portion of the constraint set which is binding at the optimal solution of the constrained problem $P(\Phi,\Theta)$ and then either solves directly for or estimates the needed Lagrangian multipliers.

Step 1) Determine the reduction factors P_1 and P_2:

$$P_1 = W / \Sigma W_i Q_i \quad \text{for } i=1 \text{ to } N$$
$$P_2 = B / \Sigma b_i Q_i \quad \text{for } i=1 \text{ to } N$$

If both P_1 and $P_2 \geq 1$ Stop. Problem $P(\Phi,\Theta)$ is unconstrained.

If only $P_2 \leq 1$ Stop. Problem $P(\Phi,\Theta) \equiv P(\Phi)$ and $\Phi^* = ½ [(1/P_2)^2 - 1]$ is optimal,

If only $P_1 \leq 1$ Stop. Problem $P(\Phi,\Theta) \equiv P(\Theta)$ and Θ^* computed using the Implicit algorithm described in Chapter V is optimal.

If both P_1 and $P_2 \leq 1$ and if $P_2 \leq P_1$ stop,

$\Phi^* = ½ [(1/P_2)^2 - 1]$ and $\Theta^* = 0$ are optimal,

Otherwise go to Step (2).

Step 2a) Compute the single space constraint solution, Θ^*, to Problem $(P\Theta)$ using the Implicit algorithm described in Chapter V.

2b) Compute

$$P = \sum_{i=1}^{N} b_i Q_{i\Theta} \ / \ \sum_{i=1}^{N} b_i Q_i$$

where $Q_{i\Theta} = \{2R_iD_i \ / \ [b_i + 2W_i\Theta^*]\}^{1/2}$

2c) If $P_2 \geq P$ then stop, Θ^* alone is optimal,

Otherwise go to Step 3.

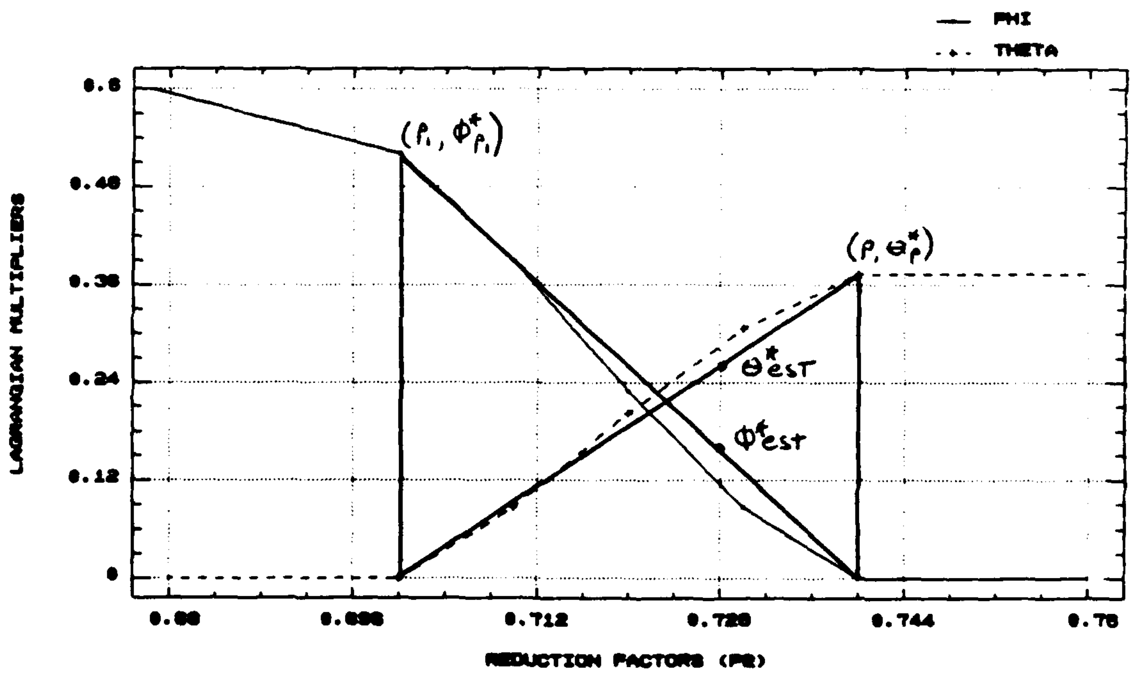

Figure 7-5 (Expanded Critical Region)

Step 3a) Compute $\Phi^*_{P1} = ½\ [\ (1/P_1)^2 - 1\ ]$ and set $\Theta^*_P = \Theta^*$.

3b) Compute Critical Region slopes:

$$M_1 = \Phi^*_{P1} / (P_1 - P)$$

$$M_2 = -\Theta^*_P / (P_1 - P)$$

3c) Compute Lagrangian multiplier estimates, Φ^*_{est} and Θ^*_{est}, using:

$$\Phi^*_{est} = M_1\ (P_2 - P_1) + \Phi^*_{P1} \quad \text{and}$$

$$\Theta^*_{est} = M_2\ (P_2 - P) + \Theta^*_P$$

Stop.

The potential effectiveness of the Lagrangian multiplier estimating portion of this algorithm, Step 3, can be seen by the near linear nature of the partial functions identified during the foregoing empirical study. Figure 7-5 depicts the generation of these estimated values.

7.5 Numerical Example

To illustrate the Dual Constraint algorithm consider, again, the inventory system displayed in Table 7-4 in which a space constraint level was set at 555.2183. By letting the budget constraint level be set first at 474.2317, then at 1659.8111 and finally at 1233.0025, the ability of the proposed algorithm to effectively determine that portion of the constraint set which will be binding at the optimal solution of the constrained problem $P(\Phi,\Theta)$ will be demonstrated. Note here that the unconstrained space resource, defined by $\Sigma\ W_iQ_i$, equal 1388.0459; while the unconstrained budget requirement, defined by $\Sigma\ b_iQ_i$, equals 2371.1588. With these totals in mind, implementing the Dual Constraint

Algorithm for each of the defined budget constraint levels produced the following results:

Example 1 ($W = 555.2183$ and $B = 474.2317$)

Step 1) $P_1 = W / \Sigma W_iQ_i = 555.2183 / 1388.0459 = .4$

$P_2 = B / \Sigma b_iQ_i = 474.2317 / 2371.1588 = .2$

Since here $P_2 \leq P_1$ stop, $\Phi^* = ½ [(1/.2)^2 - 1] = 12$ and $\Theta^* = 0$ are optimal.

Example 2 ($W = 555.2183$ and $B = 1659.8111$)

Step 1) $P_1 = W / \Sigma W_iQ_i = 555.2183 / 1388.0459 = .4$

$P_2 = B / \Sigma b_iQ_i = 1659.8111 / 2371.1588 = .7$

Here $P_2 > P_1$ so Step (2) is needed.

Step 2a) Utilizing the Implicit algorithm $\Theta^* = 2.99799$.

2b) Where $Q_{1\Theta} = [2(57)(360)/(18.25 + 2(16)(2.99799))]^{½} = 18.9584$,

$Q_{2\Theta} = [2(72)(217)/(13.5 + 2(10)(2.99799))]^{½} = 20.6248$ and

$Q_{2\Theta} = [2(58)(395)/(16 + 2(1)(2.99799))]^{½} = 45.6412$; then

$P = [(18.25)(18.9584) + (13.5)(20.6248) + (16)(45.6412)] / 2371.1588 =$

$1354.6883/2371.1588 = .5713$

2c) Since $P_2 = .7 \geq P$ then stop, $\Theta^* = 2.99799$ and $\Phi^* = 0$ are optimal, Example 3 ($W = 555.2183$ and $B = 1233.0025$)

Step 1) $P_1 = W / \sum W_iQ_i = 555.2183 / 1388.0459 = .4$

$P_2 = B / \sum b_iQ_i = 1233.0025 / 2371.1588 = .52$

Here $P_2 > P_1$ so Step (2) is needed.

Step 2a) Utilizing the Implicit algorithm $\Theta^* = 2.99799$.

2b) $P = 1354.6883/2371.1588 = .5713$

2c) Since $P_2 = .52 < P$ then Step (3) is needed.

Step 3a) $\Phi^*_{P1} = ½ [(1/.4)^2 - 1] = 2.625$ and $\Theta^*_P = 2.99799$.

3b) Compute Critical Region slopes:

$M_1 = 2.625 / (.4 - .5713) = -15.3239$

$M_2 = -2.99799 / (.4 - .5713) = 17.5014$

3c) Compute Lagrangian multiplier estimates, Φ^*_{est} and Θ^*_{est}:

$\Phi^*_{est} = (-15.3239)(.52 - .4) + 2.625 = .7861$

$\Theta^*_{est} = (17.5014)(.52 - .5713) + 2.99799 = 2.1001$

Stop.

7.6 Evaluation of the Dual Constraint Algorithm

Displaying the Lagrangian multipliers obtained in section 7.5 along side the optimal multipliers identified using Super GENO, Table 7.8 illustrates the potential effectiveness of the Dual Constraint algorithm. With the required Lagrangian multipliers either calculated in closed form or accuracy obtained utilizing the Implicit algorithm presented in Chapter V, the proposed algorithm precisely optimizes problem $P(\Phi,\Theta)$ when only a single constraint remained binding. The overall effectiveness of the Dual

Example	Proposed Algorithm	GENA Solution	BSlack/ WSlacks
1	$\Phi = 12$ $\Theta = 0$	$\Phi = 11.9999$ $\Theta = 0$	0 - 296.545
2	$\Phi = 0$ $\Theta = 2.9979$	$\Phi = 0$ $\Theta = 2.9979$	- 20.595 0
3	$\Phi_{est} = .7861$ $\Theta_{est} = 2.1001$	$\Phi = .3702$ $\Theta = 2.4314$	- 105.9647 - 18.2416

Table 7-8 (Example Dual Constraint Algorithm - GENA Solution Comparison)

Constraint algorithm, however, can be measured only by determining the amount of error Φ^*_{est} and Θ^*_{est} introduce. To gage the magnitude of such errors, the relative error statistics defined by equations (7-17) and (7-18) for the budget and space constants were employed.

$$RE_{budget} = \left(\sum_{i=1}^{N} b_i Q_i^C - B \right) / B \tag{7-17}$$

$$RE_{space} = \left(\sum_{i=1}^{N} W_i Q_i^C - W \right) / W \tag{7-18}$$

With relative errors of -.0859 and -.0328 thus calculated from the estimated Lagrangian multipliers shown in Table 7-8, the effectiveness of the proposed algorithm seems certain.

To fully evaluate the usefulness of these estimated Lagrangian multipliers, the relative errors introduced into 800 randomly generated inventory systems were examined. To strengthen the results of this analysis one hundred 5-item and one hundred 30-item inventory systems were randomly generated for each of the test configurations shown in Table 7-1. For each of these systems: first $P_1 \in (0,1)$ was selected randomly, then P was determined utilizing equation (7-14) and finally $P_2 \in (P_1,P)$ was selected randomly; so that in each case Φ^*_{est} and Θ^*_{est} values were required. A sample of the resulting relative errors as well as the extent of each system's Critical Region is exhibited in

System Type	Inv Size	Maximum Error	Average RE_{budget}	Average RE_{space}	Region Size Min	Max
LLLL	5	-.1154	-.02110	-.00893	.0019	.1498
	30	-.1114	-.02512	-.00944	.0192	.1161
HHHH	5	-.0806	-.01409	-.00873	.0046	.1152
	30	-.0555	-.01272	-.00767	.0143	.0943
HLLH	5	-.0571	-.00854	-.00572	.0048	.1376
	30	-.0664	-.01531	-.00948	.0135	.0704
LHHL	5	-.0963	-.01663	-.00783	.0018	.1500
	30	-.1246	-.02079	-.00817	.0206	.1159

Table 7-9 (Relative Error Analysis and Critical Region Size)

Appendix 7. The Maximum Relative Error generated as systems from each type and size were examined along with the Average Relative Errors introduced within both the budget and space constraints and the span over which the Critical Regions for these systems extended are summarized in Table 7-9. With every relative error less than zero and with more than 93 % of the 800 inventory systems displaying errors of less than .5%, it would appear that the estimated Lagrangian multipliers, Φ^*_{est} and Θ^*_{est}, are both near optimal and feasible. It is also apparent, from the errors displayed in Table 7-9, that the size of the inventory system has little impact. Observe further that the maximum span of P_2 values within which both Lagrangian multiplier estimates are needed is negligible. In fact 95 % of all examined inventory systems exhibited Critical Regions spanning less than .1. With these observations in mind, it can be concluded that the overall efficiency of the proposed algorithm is excellent.

7.7 The Dual Constraint Algorithm Extended

As stated the proposed algorithm effectively handles the problem $P(\Phi,\Theta)$ when its constraint set is comprised of a budget and a space constraint. Before concluding this chapter, the necessary modifications for this algorithm to effectively deal with $P(\Phi,\Theta)$ when its constraint set is comprised of either two budget or two space constraints will be addressed.

When two budget restrictions constitute the constraint set only a single modification is required. Since, in such cases, the second budget constraint's optimal Lagrangian multiplier can now be calculated directly simply by replacing the Implicit algorithm's generation of Θ^* with the direct calculation, $\Theta^* = ½ [(1/P_1)^2 - 1]$, in Step 2a. All other steps of the algorithm remain unchanged.

When, however, two storage space restrictions constitute the constraint set modifications are needed in Steps 1 and 3a. Since the direct calculation of Φ^* is no longer possible, when $P_2 \leq P_1$, the Implicit algorithm must be used to obtain the desired single space constraint solution in Step 1. Similarly, when $P_1 < P_2 < P$, the direct calculation of Φ^*_{P1}, in Step 3a, must also be obtained utilizing the Implicit algorithm. In implementing this single constraint algorithm, the constraint level associated with the first space constraint must be set equal to the constraint level associated with the second in order to identify the lower bound of the system's critical region.

7.8 Summary

With the stated goal of this chapter to provide a realistic foundation on which an expanded examination of the multiple constraint inventory problem could be made, the focus of this study was on an inventory within which both a carrying cost limitation and a storage space restriction were imposed. In establishing this foundation, five milestones were reached:

1) The three distinct regions characteristic of the functional relationships which exist between a shifting budget constraint level and the resulting Θ^* and Φ^* values associated with problem $P(\Phi,\Theta)$ have been identified and illustrated,

2) The theoretical foundation needed to effectively define these regions for any system in which two linear constraints are imposed was establish,

3) A methodology for effectively determining that portion of the constraint set which is binding at the optimal solution was developed,

4) A procedure was formulated to quickly obtain Φ^*_{est} and Θ^*_{est} when both constraints are found to be binding at the optimal solution of problem

(Φ,Θ), and

5) The proposed Dual Constraint algorithm was developed, demonstrated and evaluated.

The effectiveness displayed by this algorithm, in section 7.6, and its adaptability to a variety of linear constraint combinations, shown in section 7.7, clearly demonstrate the potential value of the proposed Dual Constraint algorithm.

Chapter VIII

A Real World Application

8.1 Introduction

The purpose of this chapter is to summarize the efforts made to implement the Dual Constraint Algorithm proposed in chapter VII within Tyree Parts & Hardware, a small hardware company located at 2126 East Business Loop, Columbia Missouri. In accomplishing this goal, section 8.2 will describe the two-bin inventory system currently used within this company to maintain an inventory system comprised of over 15000 line items. The overall ineffectiveness of this system will be illustrated by tracing its impact on 32 large volume items from 1988 through 1990. Section 8.3 will outline the steps necessary to identify both the carrying and re-order cost parameters needed to implement an unconstrained EOQ model within this system. By tracking its impact on the same 32 items examined in section 8.2, the savings potential of the EOQ model will be highlighted. Building on these results, section 8.4 will define both a budget and a space constraint which the management of Tyree Parts & Hardware wished to imposed on this segment of its inventory. With these constraint level specified, the potential benefit gained from the implementation of the Dual Constraint Algorithm will then be demonstrated. Finally, section 8.5 will summarize both the conclusions and recommendations which grew out of this effort.

8.2 A Two-Bin Inventory System

As has been mentioned, Tyree Parts & Hardware currently utilizes a two bin inventory system to determine both the size and frequency of replenishment orders for each of its 15000 line items. As a basis of this system, Tyree's management employs a six month review period and vendor recommended stockage levels. These stockage levels, established considering such factors as minimum order levels and price breaks, govern the resulting replenishment pattern for each item carried in inventory. During each semiannual review, an order for a given line item is placed when the number of those items remaining on-hand falls to less than half its recommended stockage level. These vendor recommended levels, therefore, not only define the size but also establish the frequency of those replenishments. If, for example, the vendor recommended stockage level for a type 1 water heater is 25, then upon review, an order for 25 of these heaters is placed only when the quantity on-hand falls below 13. If, on the other hand, the quantity on-hand at the time of review is greater than 13 then no order is placed.

As should be expected, the management of Tyree Parts & Hardware has found this two-bin inventory system particularly effective for low dollar items which individually consume little inventory space such as clamps, screws or bolts. This efficiency, however, does not hold when the aggregate inventory is considered. Historical records reveal that, from 1986 through 1990, this simple inventory policy produced a 59 % increase in the aggregate inventory investment. With more than $ 623,000 tied up in its inventory at the end of 1990, Tyree Parts & Hardware has increased its aggregate investment in inventory by over $ 254,000 in just four years. During this same period, a marked increase in the total space needed to house the resulting inventory quickly consumed all available inventory space. At present, approximately 70 % of those items which consume large

Item #	WATER HEATERS	Space	Suggested Re-Order	Desired Safety
1	ZER6STR	6,601.6	4	2
2	ZER10STR	6,733.6	4	2
3	ZER21STR	9,240.8	8	2
4	ZEFU 90-30 STR	16,832.8	10	2
5	ZHEFR 90-32 STR	16,599.9	10	2
6	ZHEFR 90-42 DTR	20,802.4	5	2
7	ZHEFR 90-52 DTR	25,059.4	3	2
8	ZMHG 90-333T NAT	21,617.3	15	4
9	ZMHG 90-333T LP	21,617.3	20	10
10	ZMHG 90-433T NAT	28,497.0	5	2
11	ZMHG 90-433T LP	27,213.8	10	2
12	EQF 250 QUICK FLO	3,811.5	5	2
13	EQF 400 QUICK FLO	3,811.5	4	2
14	SP 1-6C	5,132.9	4	2
15	SP 1-10C	7,596.5	7	2
16	SP 1-17C	10,086.0	7	2
17	SP 1-20C	12,054.0	13	2
	MISCELLANEOUS			
18	HYDRANT	1,675.0	5	2
19	LS2 48	3,189.4	3	2
20	LS2 410	3,937.5	2	2
21	Q1PC 100G	810.0	10	10
22	Q2PC 100G	1,587.0	10	10
23	Q3PC 100G	2,500.0	20	10
24	Q4PC 100G	3,920.0	20	10
	BATH TUBS			
25	T2470-00	23,256.0	1	1
26	T2473-00	22,134.0	2	1
27	T2472-06	23,256.0	1	1
28	T2473-06	22,848.0	2	1
29	T2300-00	27,608.0	10	1
30	T2301-00	28,985.0	16	4
31	T2300-06	28,985.0	5	1
32	T2301-06	29,837.5	10	4

Table 8-1 (Inventory Space Required In Cubic Inches)

volumes of inventory space must be held in temporary external storage areas. As a result, an average annual loss approaching $ 6,000 or 1 % of the aggregate inventory investment has been incurred from pilferage and weather damage. With these annual losses in mind, the needed to reduce both the aggregate inventory investment and size has been recognized.

In order to provide a comparative basis with which to evaluated an implementation of the EOQ based Dual Constraint Algorithm while highlighting the actual ineffectiveness of the current two-bin inventory system, the thirty two large volume inventory items listed in Table 8-1 were identified. Utilizing historical records of the product costs, Table A8-1, and periodic demands, Table A8-2, associated with these inventory items, Tables A8-3 through A8-8 (displayed in Appendix 8) depict the inventory activity which occurred during the six period horizon between the Spring of 1988 and the Fall of 1990. In reconstructing this inventory activity it was assumed that the "Suggested Re-order" quantities indicated in Table 8-1 were initially on-hand.

	Historical Demand		Current System	
Period	Min Space Requirement	Min Budget Requirement	Space Requirement	Budget Requirement
Spring 88	2811180.5	813.0	6647761.2	1017.9
Fall 88	1553270.0	521.9	5917018.4	886.7
Spring 89	3646354.5	1106.0	7764004.9	1303.4
Fall 89	1415732.1	742.4	5865409.5	1126.5
Spring 90	2396377.2	1020.2	7356772.6	1309.6
Fall 90	911420.4	661.5	5942650.6	1120.0

Table 8-2 (A Basic Resource Requirement Comparison)

Tracking the maximum inventory space required and computing the total cost of maintaining these 32 line items using equation (2-1), Table 8-2 exhibits these requirements along with the minimum budget and storage space resources necessary to satisfy exactly each line item's historical demand pattern. In comparing these budget and space requirements, it should be observed that a substantial excess inventory was maintained throughout the horizon. With these excesses consuming, on average, 67.75 % of the currently required inventory space and 28.07 % of the inventory investment, clearly there exist the potential for implementing an improved system for handling large volume line items.

8.3 Unconstrained EOQ Model Implemented

Before implementing the proposed Dual Constraint Algorithm within the indicated large volume portion of Tyree Parts and Hardware's inventory, a conversion had to be made from the current two-bin system to the unconstrained EOQ model defined in Chapter 1. By projecting an unconstrained EOQ based inventory policy over the same historical horizon examined in section 8.2, some reduction in both the inventory's size and budget investment should be realized. To accomplish this transformation, both the carrying cost and the re-order cost parameters associated with each inventory line item had to be specified. In designating a carrying cost fraction, to be applied uniformly to each of these inventory items, consideration was given: the cost of money tied up in inventory, the cost of inventory storage, the taxes paid on inventory, the losses due to damage or theft and the cost of inventory insurance as suggested by Naddor [28]. Distributing these costs, which in 1990 totaled approximately $ 31,474, over an inventory which exceeds 650,000 individual items suggested a carrying cost fraction of .05 per unit time. With this fraction identified, the carrying cost associated with each inventory item was computed by simply multiplying the vendor cost of each item by .05.

In defining the re-order cost parameters, consideration was given to both the clerical and administrative costs, the transportation costs and the general labor costs associated with the replenishment of those items needed to maintain an effective inventory. For the 32 line items displayed in Table 8-1, Table 8-3 shows the cost elements which jointly establish the "fixed" portion of these parameters.

Line Item	Average Unload Time	Labor Costs @ $ 4.5/hr	Average Admin Time	Admin Cost @ $ 12.5/hr	Total + Drop Charge (If Any)
Heaters	2 hrs	$ 9.00	3 hrs	$ 37.50	$ 96.50
Tubing	1 hr	$ 4.50	2 hrs	$ 25.00	$ 29.50
Hydrants	1 hr	$ 4.50	1 hr	$ 12.50	$ 17.00
LS Sheets	1.5 hrs	$ 6.75	2 hrs	$ 25.00	$ 31.75
Tubs	3 hrs	$ 13.50	3 hrs	$ 37.50	$ 101.00

Table 8-3 (Fixed Re-Order Cost Elements)

At each semiannual review, these "fixed" re-order costs were equally distributed to each related line item for which an order was to be placed. The actual re-order cost parameter was then determined by adding to these values a "variable" transportation cost defined as 15 % of each item's vendor cost.

To illustrate the calculation of both the carrying and re-order cost parameters as well as the desired unconstrained EOQ values, consider the inventory activity generated using the historical demands from the spring of 1988 for the 17 hot water heaters listed in Table 8-1. The carrying cost parameter of \$ 3.2425, displayed in Table 8-4 for inventory item 1, simply represents 5 % of the \$ 64.85 cost incurred by Tyree Parts and Hardware to obtain each ZER6STR hot water heater (See Table A8-1 for vendor costs).

Inventory Item	Carrying Cost	Re-Order Cost	Demand Rate	Space	Unconstrained EOQ
1	3.2425	16.6203	1.0000	6,601.6	3.0
2	3.5670	17.5938	0.0000	6,733.6	0.0
3	4.0870	19.1538	4.0000	9,240.8	6.0
4	4.5685	20.5983	5.0000	16,832.8	7.0
5	4.5685	20.5983	5.0000	16,599.9	7.0
6	4.8250	21.3678	1.0000	20,802.4	3.0
7	6.5135	26.4333	1.0000	25,059.4	3.0
8	5.4720	23.3088	8.0000	21,617.3	8.0
9	5.4720	23.3088	25.000	21617.3	15.0
10	6.1695	25.4013	8.0000	28497.0	8.0
11	6.2875	25.7553	4.0000	27213.8	6.0
12	4.3625	19.9803	0.0000	3811.5	0.0
13	4.6250	20.7678	0.0000	3811.5	0.0
14	2.9430	15.7218	1.0000	5132.9	3.0
15	3.0950	16.1778	4.0000	7596.5	6.0
16	3.2970	16.7838	3.0000	10086.0	6.0
17	3.5125	17.4303	5.0000	12054.0	7.0

Table 8-4 (Results of Unconstrained Application of EOQ Model to Spring 1988 Data)

The re-order cost parameter associated with this line item combined 15 % of the heater's \$ 64.85 vendor cost with a fix cost of \$ 96.50 distributed over the 14 heaters types which comprised that spring's replenishment order. It follows, therefore, that the re-order cost parameter for item 1 during the Spring of 1988 was ($ 96.5 ÷ 14) + [.15 (64.85)] or the indicated $ 16.6203. Substituting these parameters into equation (1-2) yielded a re-order quantity of 3.2018 which when rounded to the nearest whole number defined the indicated unconstrained EOQ value. The results obtained by repeating this process for each of the indicated line items are reflected in Table 8-4.

Incorporating these unconstrained EOQ values into an inventory policy which assumed no inventory shortages, an instantaneous replenishment rate and which maintained the management defined safety stock levels indicated in Table 8-1; projections of the inventory activity during the six period horizon between the Spring of 1988 and the Fall of 1990 were made. Displaying these projections, Tables A9-1 through A9-6 can be view in Appendix 9 along with a detailed explanation of this EOQ based inventory policy.

	Unconstrained EOQ		Current Two-Bin System	
Period	Space Requirement	Budget Requirement	Space Requirement	Budget Requirement
Spring 88	4480032.7	802.1	6647761.2	1017.9
Fall 88	3172713.8	606.6	5917018.4	886.7
Spring 89	4738885.4	1008.2	7764004.9	1303.4
Fall 89	2922055.7	769.6	5865409.5	1126.5
Spring 90	4213090.2	969.3	7356772.6	1309.6
Fall 90	2890925.1	720.6	5942650.6	1120.0

Table 8-5 (Unconstrained EOQ Verses Two-Bin Resource Requirement Comparison)

Tracking the maximum inventory space required and computing the total cost incurred by the resulting replenishment pattern, again using equation (2-1), Table 8-5 exhibits these requirements for both the two-bin inventory system and the suggested unconstrained EOQ based policy. In comparing these resource requirements, it should be observed that a substantial decrease in both inventory size and operating cost was achieved by implementing the suggested policy. Representing a 43.23 % reduction in the average size of inventory and a 27.91 % investment savings, implementation of this policy can be recommended as an initial step in effectively controlling this run away inventory.

8.4 Implementation of Dual Constraint Algorithm

The marked reductions achieved by implementing the suggested unconstrained EOQ based inventory policy, however, were not sufficient to eliminate completely the inventory excesses experienced over the last three years. In an effort to further minimize the annual loss incurred from the external storage of inventory items, a 3,333,618 cubic inch volume of internal space was reserved for the indicated 32 line item, large volume inventory. Simultaneous, in an effort to lessen its rapidly growing inventory investment, Tyree's management established a $ 500 limit on the carrying cost associated with the replenishments made during any single inventory period.

The degree to which these selected constraint levels impact the suggested unconstrained inventory policy is displayed in Table 8-6. In considering this table, note that the values listed as the "Total Projected Space Requirement" reflect the **maximum** storage space required by this large volume inventory; while those shown as the

"Projected EOQ Budget Requirement" track only the **additional** carrying cost incurred from the EOQ defined replenishments during each period. With space and budget

Period	Total Projected Space Requirement	Projected EOQ Budget Requirement	Space Reduction Factor	Budget Reduction Factor
Spring 88	**4480032.7**	**716.7**	**.744**	**.697**
Fall 88	3172713.8	420.2	1.051	1.189
Spring 89	**4738885.4**	**801.4**	**.703**	**.623**
Fall 89	**2922055.7**	**609.5**	**1.140**	**.820**
Spring 90	**4213090.2**	**763.3**	**.791**	**.655**
Fall 90	2890925.1	440.2	1.153	1.135

Table 8-6 (Unconstrained EOQ Based Policy - Constraint Pattern)

constraint reduction factors, computed by dividing the 3,333,618 cubic inch storage space constraint by each projected space requirement and the $ 500 investment limit by each projected budget requirement, the pressure imposed by these constraints becomes evident. Where reduction factors less than one identify those constraints which remain unsatisfied, clearly the unconstrained EOQ solutions are infeasible for four of the six periods examined.

In order to determine the constrained EOQ values needed to optimize this inventory horizon, the Dual Constraint algorithm was incorporated into step 4 of the EOQ based inventory policy described in appendix 9. Stepping sequentially through the six periods of this horizon, appendix 10 displays both a detail explanation of the necessary computations and the projected inventory activity which would be expected. Table 8-7 begins a summary of these computations by displayed the effective constraint levels, unconstrained resource requirements and the Θ^* defined budget requirements needed compute the reduction factors (P_1, P_2 and P) utilized within the Dual Constraint algorithm

Period	Effective Constraint Levels Budget	Effective Constraint Levels Space	Unconstrained Resource Requirement Budget	Unconstrained Resource Requirement Space	Total Budget Requirement (Single Space Constraint)
Spring 88	**500**	**2141679**	**716.7**	**3286917**	483.772
Fall 88	500	1953931	416.1	1444668	-------
Spring 89	**500**	**1769915**	**809.32**	**3238372**	488.829
Fall 89	**500**	**1945398**	**575.6**	**1509993**	-------
Spring 90	**500**	**1786877**	**719.95**	**2699965**	509.004
Fall 90	500	1854839	440.3	1007181	-------

Table 8-7 (Effective Constraint Levels Verses Resource requirements)

to determine whether none, one or both of the imposed constraints define the desired optimal solutions.

The "Effective Space Constraint Levels", shown in this table, represent that portion of the 3,333,618 cubic inch space restriction not consumed either by the inventory on-hand at the end of each period or those replenishments made to restore each safety stock to its desired level. The effective space constraint shown for the spring 1988 as 2,141,676 cubic inches, for example, was derived by subtraction the 1,191,939 cubic inches needed to handle the management defined safety stock from the total available inventory space.

The indicated "Unconstrained Resource Requirements" specify the resources needed satisfy each period's projected unconstrained EOQ. In determining the size of each unconstrained replenishment order, only that demand not satisfied by the non-safety stock, on-hand inventory from the previous period was considered. When, for example, the inventory on-hand at the end of a period exceeded the required safety stock level by 4 and the historical demand for the next period was 6; then an adjusted demand of 2 was utilized in equation (1-2) to compute the desired unconstrained EOQs. Thus computed these order quantities were then incorporated into equations (2-13) and (2-14) to obtain the values shown.

Finally, the "Total Budget Requirements" reflect the budget resources required to handle the resulting inventory when only the imposed space constraint was considered. These values were defined such that:

$$\text{Total Budget Requirement} = \sum_{i=1}^{N} b_i Q_{i\Theta}$$

$$\text{where} \quad Q_{i\Theta} = \{2R_iD_i \,/\, [b_i + 2W_i\Theta^*]\}^{1/2}.$$

As may be recalled from chapter VII, these Θ^* defined totals, in conjunction with the unconstrained budget requirements, determine whether the single space constraint or both the space and budget constraint impact a period's optimal solution.

Incorporating these resource requirements and constraint levels into the Dual Constraint algorithm, Table 8-8 first presents the resulting reduction factors: P_1, P_2 and P. In considering these values, note that the reduction factors P_1 and P_2 were computed by dividing the indicated "Effective Constraint Levels" by their corresponding "Unconstrained Resource Requirements" while, when needed, the reduction factor P was determined by dividing the period's "Total Budget Requirement" by its "Unconstrained Budget Resource Requirement". To illustrate the ease of these calculations, consider the reduction factors displayed for the Spring of 1988. The reduction factor P_1, shown as .651, was derived by simply dividing the 2,141,679 cubic inches of inventory space available at the start of that review period by the 3,286,917 cubic inches needed to store the unconstrained EOQ replenishments stemming from the adjusted demand pattern of that period. Similarly, the reduction factor P_2, shown as .698, was obtain by dividing the management imposed \$ 500 budget constraint by the \$ 716.7 investment needed to maintain those same unconstrained EOQ replenishments. Finally, the reduction factor P, shown as .675, was acquired by dividing the Θ^* defined total budget requirement of \$ 483.772 by the indicated \$ 716.79 unconstrained budget requirement.

Period	Reduction Factors P_1	P_2	P	Binding Constraints	Optimal Lagrangian Values Θ^*	Φ^*
Spring 88	**.651**	**.698**	**.675**	Space	.0001370858	0
Fall 88	1.352	1.20	----	None	0	0
Spring 89	**.546**	**.617**	**.604**	Space	.0002534664	0
Fall 89	**1.288**	**.868**	**----**	Budget	0	.16268351
Spring 90	**.661**	**.694**	**.707**	Both	.000104013	.18748288
Fall 90	1.849	1.13	----	None	0	0

Table 8-8 (The Dual Constraint Algorithm - Implemented)

Armed with these reduction factors, both the binding constraint(s) and the optimal Lagrangian multipliers, also shown in Table 8-8, were obtained. In considering the indicated binding constraints and the sequentially calculated Φ^* and Θ^* values, recall from the Dual Constraint algorithm that if:

a. only $P_2 \leq 1$ then Problem $P(\Phi,\Theta) \equiv P(\Phi)$ so that $\Phi^* = ½\, [\, (1/P_2)^2 - 1\,]$ is optimal,

b. only $P_1 \leq 1$ then Problem $P(\Phi,\Theta) \equiv P(\Theta)$ so that Θ^* computed using the Implicit algorithm described in Chapter V is optimal,

c. both P_1 and $P_2 \leq 1$ when $P_2 \leq P_1$ then $\Phi^* = ½\, [\, (1/P_2)^2 - 1\,]$ and $\Theta^* = 0$ are optimal,

d. both P_1 and $P_2 \leq 1$ when $P_2 > P$ then Θ^* alone is optimal while if

e. $P1 < P2 < P$ then both Φ^* and Θ^* greater than zero impact the optimal solution.

Collectively, these five rules eliminate the trial and error normally encountered with such dual constrained inventories by identifying the binding constraint(s) for each period of the horizon.

To illustrate the efficiency of these rules consider the reduction factors shown for the Spring of 1988. Since during this period P_2 was greater than P then, according to rule (d), the period's sole binding constraint was its space limitation. It follows therefore that $\Theta^* = .0001370858$, used to derive the period's reduction factor P, also defined the constrained EOQ values needed to optimize this initial period of the inventory horizon. Similarly, consider the reduction factors shown for the Fall of 1989. Since during this period P_2 was less than P_1 then, according to rule (a), the period's budget restriction was its sole binding constraint thereby allowing the direct computation of $\Phi^* = .16268351$.

For a detailed accounting of the computations underlying Table 8-8 see section A10.2 in appendix 10. A review of this computational effort underscores the potential efficiency of the Dual Constraint algorithm. For all but one of the horizon's four constrained periods, highlighted in Table 8-8, only a single multiplier had to be identified. Recognizing, that the identification of as many as eight multipliers may have been required to optimize this horizon utilizing the classical solution technique described in chapter II and that even then the optimal multipliers needed during period five would still not be known the potential efficiency of the Dual Constraint algorithm is apparent.

8.5 Conclusions and Recommendations

The effectiveness of the inventory pattern displayed in Tables A10-1 through A10-6, however, must be gauged first by how well the imposed constraint levels were met during the four constrained periods encountered within this horizon and then by the total cost incurred by its operation. In addressing the first of these gauges consider the maximum inventory space requirements and the additional budget investments displayed in Table 8-9. With relative errors ranging from .18% to only 1.38% it appears that by incorporating the Dual Constraint algorithm within the EOQ based inventory policy noted in section 8.3, both the budget and space constraints imposed in this inventory horizon have been effectively satisfied. Closer examination of the resulting constrained EOQ requirements, however, reveals that the constrained space requirement of 3,340,245.1 cubic inches shown for Spring of 1989 and the constrained budget requirement of \$ 501.1 shown for the Fall of 1989 actually exceed the stated constraint levels by 6,627.1 cubic inches and \$ 1.1 respectively. These excesses, representing only .19% and .22% relative errors, stem from the discrete nature of this inventory problem and were generated by the

Period	Constrained EOQ Dual Constraint Algorithm Space Requirement	Budget Requirement
Spring 88	**3324742.6**	484.9
Fall 88	2814331.0	415.1
Spring 89	**3340245.1**	485.6
Fall 89	2736061.8	**501.1**
Spring 90	**3327465.8**	**493.7**
Fall 90	2479635.8	437.1

Table 8-9 (Summary of Constrained EOQ Resource Requirements)

simple rounding technique used convert each computed EOQ values to its nearest whole number. With care these minor excesses can be eliminated.

In addressing the cost effectiveness of implementing the Constrained EOQ based policy within this large volume inventory, consider the average total costs displayed in Table 8-10 for each of the three inventory systems examined in this chapter. Reflecting not only the operating costs of each system, computed using equation (2-1), but also their estimated theft and shortage losses, these average total costs indicate a potential 75 % cost savings by implementing the Constrained EOQ based policy.

In projecting this potential cost savings note that the $ 3,000.00 semiannual theft loss shown for the Two-bin system reflects the estimated $6,000.00 annual loss noted earlier while the loss of $ 803.10 shown for the Unconstrained EOQ based policy reflects the 43.23% reduction in inventory size achieved by implementing that inventory system.

Note also that the "Ave Shortage Losses" shown in this table reflect the 30 % profit

Inventory Period	Two-Bin System	Unconstrained EOQ Based Sys	Constrained EOQ Based Sys
Operating Costs			
Spring 88	1017.90	802.10	782.20
Fall 88	886.70	606.60	575.90
Spring 89	1303.40	1008.20	979.20
Fall 89	1126.50	769.60	747.10
Spring 90	1309.60	969.30	922.10
Fall 90	1120.00	720.60	677.30
Ave Oper Costs	1127.36	812.73	780.62
Theft Losses	**3000.00**	803.10	0.00
Ave Shortage Loss	0.00	**46.91**	**264.58**
Average Total Cost	4127.36	1662.74	1045.19

Table 8-10 (Average Total Cost Comparisons)

margin maintained by Tyree Parts & Hardware. Inventory shortages which occurred while implementing the unconstrained EOQ based inventory policy amounted to a total loss of only $ 281.46 the over the entire horizon. However, as was expected, profit losses encountered as the constrained EOQ based policy was implemented were larger; amounting to $ 1587.48. In considering these potential losses, it should be noted that each is far overshadowed by the $ 3000 loss currently being experienced due to pilferage and weather damage each period.

In view of the effectiveness displayed by the Dual Constraint algorithm in identifying the constrained EOQs needed to optimize an inventory system and the potential cost savings indicated by the last two paragraphs, the following recommendations were made to the management of Tyree Parts & Hardware:

1) Utilize the EOQ based inventory policy described in appendix 9 to control the its unconstrained inventory. Judging from the average total costs, shown Table 8-10 for the 32 item large volume inventory, a cost savings of 40% can be expected. The current two-bin system should continued for all bulk items such as screw, bolts and brackets.

2) Incorporate the Dual Constraint algorithm as step 4 of the EOQ based inventory policy to control the constrained portion of its inventory. Tracking the various costs which comprise both the replenishment and carrying cost defined in section 8.3 should dictate periodic adjustments to the system parameter. As the EOQ defined inventory stabilizes the need for the currently defined safety stock level should diminish.

3) Use both the constrained and unconstrained EOQ based polices only as planning tools since the EOQ model underlying both these policy considers only a few of the pressures which impact a small business's inventory system. Noting the modification made in the recommended EOQs at the time an order is placed should allow ample planning time to adequately handle the resulting excess inventory.

Chapter IX

Summary and Future Research

9.1 Summary

The ground covered within this research effort has been considerable. As that ground was crossed a number of important discoveries were made. To begin, the linear relationships found during the multi-parameter empirical study, described in chapter III, established conditions under which an optimal Lagrangian multiplier, Φ^*, can be obtained directly from system parameters. When multiple parameters change within a known inventory system, in accordance with those conditions, the corresponding change in Φ^* can be obtained in closed form.

The functional analysis conducted in chapter IV provided both shape and character to the near-linear, multi-variable functions noted during this empirical study in which Φ^* is implicitly defined by the cost and demand parameters of an inventory system. From this information the Implicit algorithm was developed. This algorithm rapidly establishes effective bounds on the Φ^* needed to optimize the N-item, single constraint inventory problem (PΦ). A recursive process was also uncovered which converges to Φ^* without the usual guess work. A comparative analysis indicated that the proposed Implicit algorithm both obtains its bounds on Φ^* and converges more rapidly than the algorithms presented by Venture and Klein [39], Ziegler [41] and the classical solution technique. The initial goal of obtaining an improved process for locating Φ^* for the multi-item single constraint inventory problem has, therefore, been reached.

Building on the theoretical foundation underlying the Implicit algorithm, the

Horizon algorithm was also developed. Rather than assuming no knowledge of an inventory system, this algorithm utilizes both the system parameters and Φ^* of an existing inventory to obtain the multiplier, Φ^*_{new}, associated with an inventory defined when various parameters within the initial system shift. The recursive application of this algorithm proved effective in identifying the series of Lagrangian multipliers needed to optimize multi-period inventory horizons when constrained conditions extend over several consecutive periods. This algorithm proved particularly useful when the inventory was comprised of less than 30 line items or when less than one third of the system parameters shift.

With the second goal of this study completed, research focused on the dual constraint problem $P(\Phi,\Theta)$. The empirical study, presented in section 7.2, found that three distinct functional relationships emerge as the constraint levels of an inventory system shift. These relationships were used to establish the Dual Constraint algorithm which:

1) Determines directly that portion of a linear constraint set which defines the Lagrangian multipliers needed to optimize the dual constraint problem,
2) Obtains quickly the optimal multiplier when only a single constraint is found binding and
3) Effectively identifies Φ^* and Θ^* estimates when both constraints remain active.

By allowing the dual constraint problem $P(\Phi,\Theta)$ to be solved directly without using the trial and error method inherent within the classical approach, this algorithm accomplishes the third goal set for this research effort.

To conclude this study, a real world application of the algorithms developed during this study was attempted. Selecting a thirty two item, large volume portion of the inventory maintained by Tyree Parts & Hardware of Columbia Missouri and tracking its historical activity from 1988 through 1990, the potential benefit to be gained by implementing these algorithms was demonstrated. Displaying an approximate cost savings of 75 %, while effectively satisfying both an investment limitation and a storage space restriction imposed over a six period horizon, the advantages to be gained by implementing the results of this study were shown to be substantial.

9.2 Future Research

Two major areas of future research are suggested by the success achieved during the current research effort. First an extension of both the Implicit and Horizon algorithms should be made to more complex inventory models. Examination of those EOQ models which exhibit non-instantaneous replenishment and\or shortages costs, for example, would substantially expand the usefulness of these algorithms within real world inventory systems. Preliminary investigation of an EOQ model which includes inventory shortage costs indicates that near-linear relationships again emerge as various system parameters undergo change. The existence of these functional relationships strongly suggest the potential for success within this area of research.

A second area of future study is the expansion of the Dual Constraint algorithm. Considering the redundant nature which can be exhibited when multiple linear constraint are imposed on an inventory system and the dominance often exerted by one of several constraints, this extension should deal first with the dual constraint problem in which linear and non-linear constraints are paired. Since, when such constraints are active, the non-linear constraint's multiplier can often be express in term of the linear constraint's multiplier; this extension could follow directly from the work displayed in chapter VII. If successful, the foundation needed to extend the Dual Constraint algorithm to the N-constraint cases would be solidified.

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APPENDIX 1

The purpose of this appendix is to display plot showing the near-linear nature of the functional relationships found by Maloney [26] between Φ^* and selected single parameter shifts within a two item inventory system.

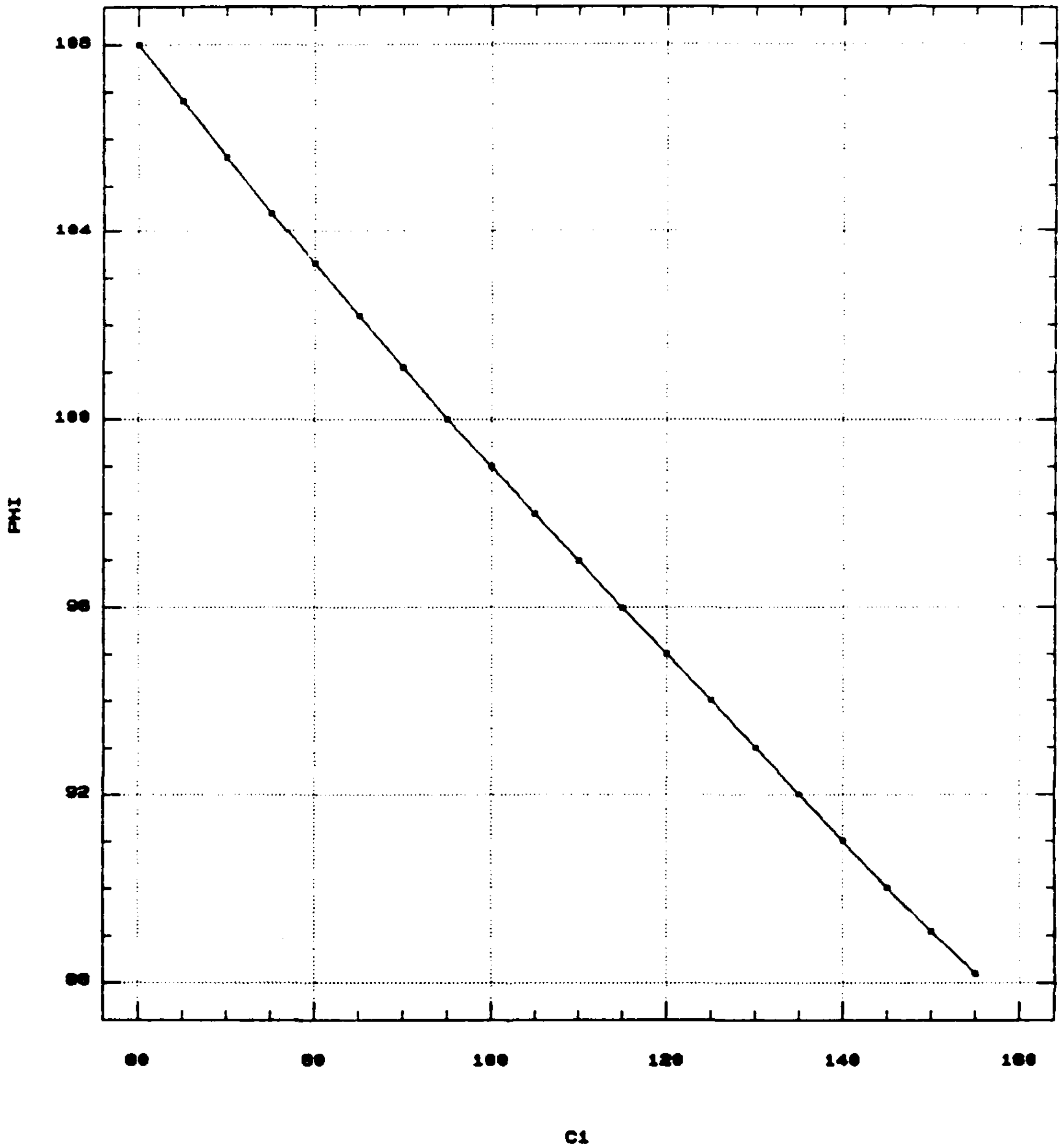

Figure A1-1 (Plot of PHI vs Single Carrying Cost)

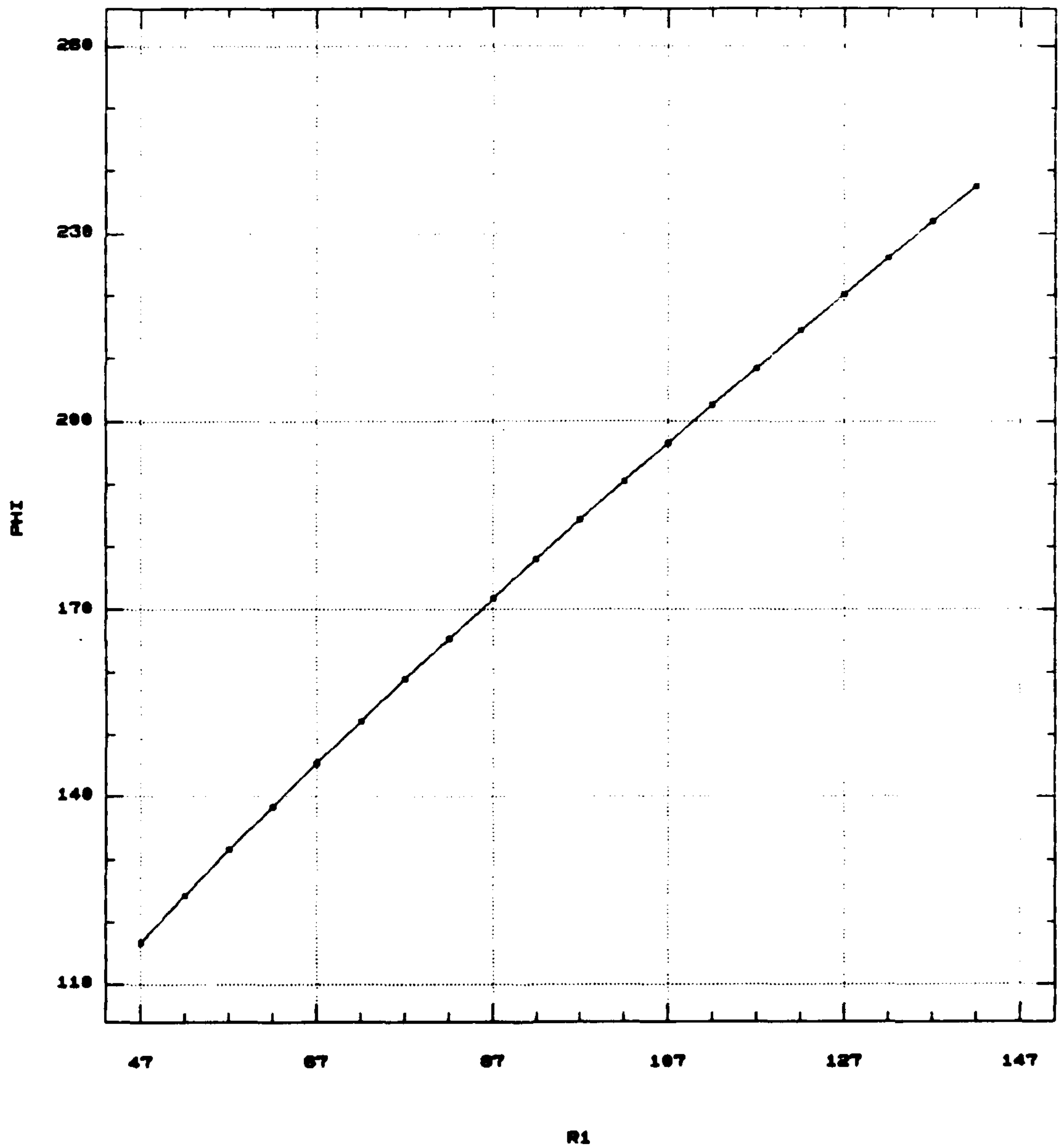

Figure A1-2 (Plot of PHI vs Single Re-order Cost)

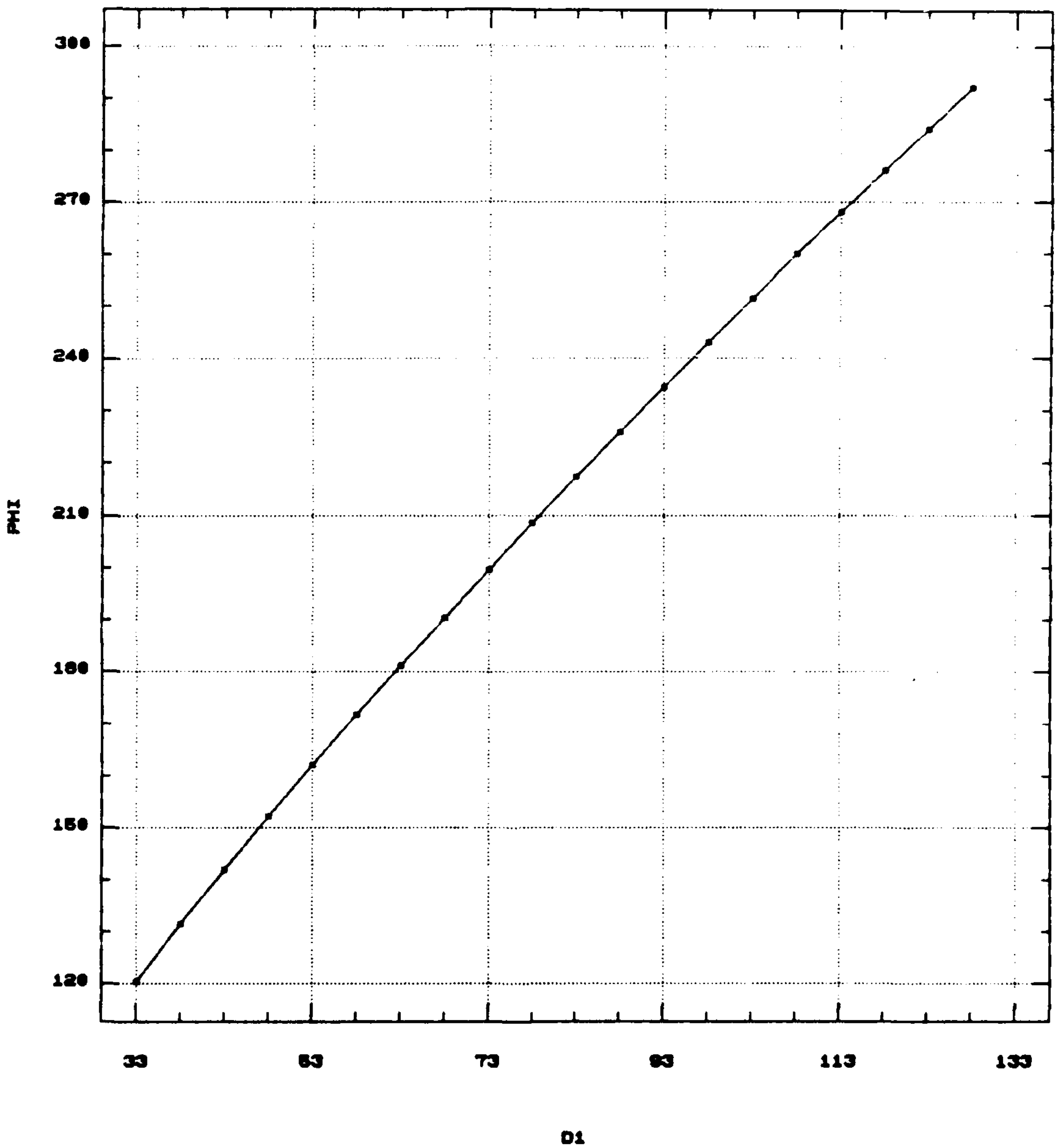

Figure A1-3 (Plot of PHI vs Single Demand Rate)

APPENDIX 2

The purpose of this appendix is to provide the data generated from a randomly selected four-item inventory system during the empirical study conducted for the single constraint Problem (PΦ). The data contained in the next two pages illustrates the system explosion described in chapter III. Each row represents a different data run in which twenty related systems were generated and analysis by adding one sequentially to each perturbed parameter. The column titled 'KEY' identifies those parameters shifted during each run. For example, in the run designated by (R 1 1 C 1 1) only item one's carrying cost was changed while all other system parameters were held constant. During the run designated by (R 1 4 C 1 2), however, all four carrying costs and all four re-order costs under went simultaneous changes while the remaining parameters were held constant.

	HOLDING	REORDER	DEMAND
Item #1	41	87	55
Item #2	5	92	95
Item #3	4	29	52
Item #4	30	7	79

CONSTRAINT SET AT 50.80544

KEY	SLOPE	CORR COEFF	INITIAL L	FINAL L
(R 1 1 C 1 1)	-0.0151	0.9939	9.6870	9.0070
(R 1 1 C 1 2)	0.0221	0.9954	10.0000	11.0000
(R 1 1 C 1 2)	0.0592	1.0000	10.1000	12.7700
(R 1 1 C 2 2)	0.0506	0.9996	10.0710	12.3530
(R 1 1 C 2 3)	0.1008	0.9981	10.2160	14.7600
(R 1 1 C 3 3)	0.0248	0.9948	10.0000	11.0530
(R 1 2 C 1 1)	-0.1435	0.9397	8.2040	1.5550
(R 1 2 C 1 2)	-0.0868	0.9596	8.8860	4.8820
(R 1 2 C 1 3)	0.0406	0.8046	9.4470	11.1700
(R 1 2 C 2 2)	0.1430	0.9996	10.5000	17.0000
(R 1 2 C 2 3)	0.3432	0.9957	11.0730	26.5230
(R 1 2 C 3 3)	0.1079	0.9999	10.3300	15.2000
(R 1 3 C 1 1)	-0.4574	0.9999	7.4400	0.5800
(R 1 3 C 1 2)	-0.1831	0.9975	8.7070	0.4710
(R 1 3 C 1 3)	0.1812	0.9294	9.6570	17.7700
(R 1 3 C 2 2)	0.2565	1.0000	11.0550	22.6000
(R 1 3 C 2 3)	0.6383	0.9940	12.0040	40.7300
(R 1 3 C 3 3)	0.1729	1.0000	10.6440	18.4200
(R 1 4 C 1 1)	-0.5009	1.0000	7.3000	2.2910
(R 1 4 C 1 2)	-0.0582	0.9971	9.4900	6.8800
(R 1 4 C 1 3)	0.5163	0.9715	10.6150	33.8100
(R 1 4 C 2 2)	0.4418	1.0000	12.0000	31.8800
(R 1 4 C 2 3)	1.0160	0.9925	13.1150	58.8100
(R 1 4 C 3 3)	0.1957	0.9999	10.7330	19.5340
(R 2 2 C 1 1)	-0.1341	0.9433	8.3120	2.0980
(R 2 2 C 1 2)	-0.1056	0.9593	8.7120	3.8500
(R 2 2 C 1 3)	-0.0466	0.8978	9.1170	6.9260
(R 2 2 C 2 2)	0.0788	0.9998	10.2060	13.7530
(R 2 2 C 2 3)	0.1893	0.9991	10.6250	19.1400
(R 2 3 C 1 1)	-0.4309	0.9999	7.5510	1.0900
(R 2 3 C 1 2)	-0.2284	0.9989	8.5240	0.5460
(R 2 3 C 2 2)	0.1784	0.9997	10.7600	18.7700
(R 2 3 C 2 3)	0.4320	0.9976	11.5310	31.0000

KEY	SLOPE	CORR COEFF	INITIAL L	FINAL L
(R 2 3 C 3 3)	0.1361	0.9997	10.5050	16.6500
(R 2 4 C 1 1)	-0.4735	1.0000	7.4020	0.3000
(R 2 4 C 1 2)	-0.1154	0.9999	9.3000	4.1300
(R 2 4 C 1 3)	0.2922	0.9710	10.2330	23.3400
(R 2 4 C 2 2)	0.3367	0.9997	11.6700	26.8640
(R 2 4 C 2 3)	0.7430	0.9961	12.6060	46.0300
(R 2 4 C 3 3)	0.1572	1.0000	10.6000	17.6800
(R 3 3 C 1 1)	-0.0518	0.9567	9.1700	6.7520
(R 3 3 C 1 2)	-0.0087	0.9487	9.6800	9.2740
(R 3 3 C 1 3)	0.0568	0.9975	10.0000	12.5100
(R 3 3 C 2 2)	0.0886	0.9982	10.3350	14.3100
(R 3 3 C 2 3)	0.1951	0.9994	10.6740	19.4600
(R 3 3 C 3 3)	0.0533	0.9986	10.1000	12.5100
(R 3 4 C 1 1)	-0.0665	0.9601	9.0400	5.9500
(R 3 4 C 1 2)	0.0601	0.9787	10.4050	13.1620
(R 3 4 C 1 3)	0.2174	0.9992	10.8560	20.7100
(R 3 4 C 2 2)	0.2258	0.9985	11.2300	21.4530
(R 3 4 C 2 3)	0.4451	0.9980	11.7060	31.7600
(R 3 4 C 3 3)	0.0709	0.9992	10.1850	13.4000
(R 4 4 C 1 1)	-0.0145	0.9822	9.6630	9.0000
(R 4 4 C 1 2)	0.0549	0.9649	10.4900	13.0500
(R 4 4 C 1 3)	0.1017	0.9949	10.6000	15.2500
(R 4 4 C 2 2)	0.1135	0.9956	10.6600	15.8320
(R 4 4 C 2 3)	0.1824	0.9998	10.7800	19.0340
(R 4 4 C 3 3)	0.0151	0.9973	9.8760	10.5700

APPENDIX 3

The purpose of this appendix is to provide extensions of the three proofs of linearity given by Maloney [26] to the N-item inventory case. Each of these proofs deal with restricted cases. The first considers the case where all C_i/W_i ratios within a system are equal while the latter two consider cases where, in addition $R_i = R$ and $D_i = D$.

Theorem A3-1

Let S_A and SB be two inventory systems such that

$$S_A = S(C/W, R_i, D_i) \text{ for } i = 1 \text{ to } N$$

$$S_B = S(C/W + \Delta, R_i, D_i) \text{ for } i = 1 \text{ to } N$$

Let S_A and S_B be subject to an active constraint U and let the Lagrangian multipliers Φ and θ exist for system S_A and S_B respectively. Then for all values a linear relationship between Φ and θ exist.

Proof: Let q_{Ai} and q_{Bi} be the EOQs for item i (i=1 to N) in systems SA and SB respectively. Then from equation (4-5) and KKT condition of feasibility equation (2-8) it follows that:

$$\Sigma\, q_{Ai} = \Sigma\, q_{Bi} \quad \text{so that,}$$

$$\sum_{i=1}^{N} [(2R_iD_iW_i)/(C/W + 2\Phi)]^{1/2} = \sum_{i=1}^{N} [(2R_iD_iW_i)/(C/W + \Delta + 2\theta)]^{1/2},$$

$$1/(C/W + 2\Phi)]^{1/2} \sum_{i=1}^{N} (2R_iD_iW_i)^{1/2} = 1/(C/W + \Delta + 2\Phi)^{1/2} \sum_{i=1}^{N} (2R_iD_iW_i)^{1/2},$$

$$[1/(C/W + 2\Phi)]^{1/2} = [1/(C/W + \Delta + 2\Phi)]^{1/2}.$$

Which yields the linear relationship

$$\theta = \Phi - ½\Delta. \qquad \blacksquare$$

Theorem A3-2

Let S_A and SB be two inventory systems such that

S_A = S(C/W,R,D) for i = 1 to N

S_B = S(C/W,R + Δ,D) for i = 1 to N

Let S_A and S_B be subject to an active constraint U and let the Lagrangian multipliers Φ and θ exist for system S_A and S_B respectively. Then for all values a linear relationship between Φ and θ exist.

Proof: The proof is similar to that of Theorem A3-1. This time, however,

$$\theta = \Phi\,(R + \Delta)/R + \tfrac{1}{2}\,(C/W)\Delta/R$$

■

Theorem A3-3

Let S_A and SB be two inventory systems such that

S_A = S(C/W,R,D) for i = 1 to N

S_B = S(C/W,R,D + Δ) for i = 1 to N

Let S_A and S_B be subject to an active constraint U and let the Lagrangian multipliers Φ and θ exist for system S_A and S_B respectively. Then for all values a linear relationship between Φ and θ exist.

Proof: The proof is similar to that of Theorem A3-1. This time, however,

$$\theta = \Phi\,(D + \Delta)/D + \tfrac{1}{2}\,(C/W)\Delta/D$$

■

APPENDIX 4

The purpose of this appendix is to complete the Implicit function analysis begun in section 4.4 by providing the following detailed proofs of:

the monotonic nature of $f_5(\Phi, C_i/W_i, R_i, D_i)$, i = 1, 2,.., or N, Theorem A4-1

the convexity displayed by $f_5(\Phi, C_i/W_i, R_i, D_i)$, i = 1, 2, ..., or N Theorem A4-2

and

the monotonic nature of $f_4(\Phi, RD_i)$, i = 1 to m, $m \leq N$. Theorem A4-3

Theorem A4-1

The function defined implicitly between Φ and each parameter of a single inventory item as $f_5(\Phi, C_i/W_i, R_i, D_i)$, i=1,2....,or N is monotonically increasing (decreasing) if $\Phi \geq (\leq) [R_iD_i/2(R_i+D_i)] - (C_i/2W_i)$.

Proof: Let $f_5(\Phi, C_i/W_i, R_i, D_i)$ =

$$\sum_{i=1}^{N} [(2R_iD_iW_i)/(C_i/W_i + 2\Phi)]^{1/2} - U. \tag{A4-1}$$

The summation of $\delta\Phi/\delta(C_i/W_i)$, $\delta\Phi/\delta R_i$ and $\delta\Phi/\delta D_i$ for the i^{th} item of inventory becomes:

$$\frac{-\frac{1}{2}[2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2} + \frac{1}{2} R_i^{-\frac{1}{2}} [2D_iW_i/(C_i/W_i + 2\Phi)]^{1/2} + \frac{1}{2} D_i^{-\frac{1}{2}} [2R_iW_i/(C_i/W_i + 2\Phi)]^{1/2}}{\sum_{i=1}^{N} [2R_iD_iW_i/(C_i/W_i + 2\Phi)^3]^{1/2}} \tag{A4-2}$$

From Theorem 1 [8, pp 165] function f_5 is monotonically increasing (decreasing) if equation (A4-2) is $\geq (\leq)$ zero. It follows, therefore, f_5 is monotonically increasing (decreasing) when:

$$-[2R_i^2D_i^2W_i]^{1/2} + [2D_i^2W_i(C_i/W_i + 2\Phi)^2]^{1/2} + [2R_i^2W_i(C_i/W_i + 2\Phi)^2]^{1/2} \geq (\leq) 0. \tag{A4-3}$$

Simplifying and rearranging equation (A4-3) reveals that function f_5 is monotonically increasing (decreasing) when:

$$\Phi \geq (\leq)\ [R_i D_i / 2(R_i + D_i)] - (C_i / 2W_i). \qquad \blacksquare$$

Theorem A4-2

The function defined implicitly between Φ and each parameter of a single inventory item as $f_5(\Phi, C_i/W_i, R_i, D_i)$, i=1,2....,or N is concave (convex) if:

$$\Phi \geq (\leq)\ [R_i D_i / 2(R_i + D_i)] - (C_i / 2W_i).$$

Proof: For $f_5(\Phi, C_i/W_i, R_i, D_i)$, i=1,2....,or N to be concave (convex), applying Theorem 3.3.3 [1],

$$f_5(\Phi_1, C_1/W_1+\Delta, R_1+\Delta, D_1+\Delta) \leq (\geq)\ f_5(\Phi_2, C_1/W_1, R_1, D_1) +$$
$$\nabla f_5(\Phi_2, C_1/W_1, R_1, D_1)^t (C_1/W_1+\Delta-C_1/W_1, R_1+\Delta-R_1, D_1+\Delta-D_1)$$

where WLOG i=1 and $\Delta \geq 0$. Recognizing that as along as $\exists\ \Phi > 0$ such that KKT condition (2-8) is satisfied then

$$f_5(\Phi_1, C_1/W_1+\Delta, R_1+\Delta, D_1+\Delta) = f_5(\Phi_2, C_1/W_1, R_1, D_1).$$

It follows, therefore that $f_5(\Phi, C_i/W_i, R_i, D_i)$, i=1,2....,or N is concave (convex) if:

$$\nabla f_5(\Phi_2, C_1/W_1, R_1, D_1)^t\ (\Delta, \Delta, \Delta) \geq (\leq)\ 0 \quad \text{so that}$$

$$\Delta(\partial\Phi/\partial(C_1/W_1) + \partial\Phi/\partial R_1 + \partial\Phi/\partial D_1) \geq (\leq)\ 0 \qquad \text{(A4-4)}$$

From the proof of Theorem A4-1 note that when $\Phi \geq [R_i D_i/2(R_i+D_i)] - (C_i/2W_i)$, equation (A4-2) is greater than zero. It follows, therefore, that the summation of partial derivatives which comprise each element of equation (A4-4) must also be greater than zero so that function f_5 is concave when $\Phi \geq [R_iD_i/2(R_i+D_i)] - (C_i/2W_i)$. From the proof of Theorem A4-1 note also that when $\Phi \leq [R_iD_i/2(R_i+D_i)] - (C_i/2W_i)$, equation (A4-2) is less than zero. It follows, therefore, that the summation of partial derivatives which constitute each element of equation (A4-4) must be less than zero so that function f_5 is convex when $\Phi \leq [R_iD_i/2(R_i+D_i)] - (C_i/2W_i)$. ■

Theorem A4-3

The function $f_4(\Phi,RD_i)$ $i = 1$ to m, $m \leq N$, where Φ is defined implicitly by RD_i, is a concave function.

Proof: Since from Corollary 4-5-1 $f_4(\Phi,RD_i)$ is a monotonically increasing function then, following Theorem 3.3.3 [1], $f_4(\Phi,RD_i)$ $i = 1$ to m, $m \leq N$ is a concave function if:

$$f_4(\Phi_1,\mathbf{RD}_i+\Delta) \leq f_4(\Phi_2,\mathbf{RD}_i) + \nabla f_4(\Phi_2,\mathbf{RD}_i)^t\,[(RD_i+\Delta-RD_i)] \text{ or if}$$

$$f_4(\Phi_1,\mathbf{RD}_i) \leq f_4(\Phi_2,\mathbf{RD}_i-\Delta) + \nabla f_4(\Phi_2,\mathbf{RD}_i-\Delta)^t\,[(RD_i-RD_i+\Delta)] \qquad \text{(A4-5)}$$

where WLOG $\Delta \geq 0$ represents simultaneous and uniform shifts in selected RD_i parameters. Focusing on the first of these expressions and recognizing that as long as $\exists$ $\Phi > 0$ such that KKT condition of feasibility (2-8) is satisfied then

$$f_4(\Phi_2,\mathbf{RD}_i) = f_4(\Phi_1,\mathbf{RD}_i+\Delta).$$

It follows therefore that $f_4(\Phi,RD_i)$ $i = 1$ to m, $m \leq N$ is a concave if:

$$\nabla f_4(\mathbf{\Phi_2},\mathbf{RD_i})^t\ (\Delta) \geq 0. \tag{A4-6}$$

Now since each of the m elements of vector Δ are positive and since each of the m elements of $\nabla f_4(\mathbf{\Phi_2},\mathbf{RD_i})$ via Corollary 4-5 are positive then clearly equation 4-25 holds and $f_4(\Phi,RD_i)$ is concave regardless of the value $m \leq N$ assumes.

Appendix 5

The purpose of this appendix is first to exhibit the validity of the Horizon algorithm by displaying the resulting optimal Lagrangian multipliers generated by each of the compared algorithms when inventory sizes were set at 15. The second part of this appendix presents the results obtained when Version 3 of the Horizon algorithm was used to solve inventory horizons in which shifts occurred simultaneously in both reorder and demand parameters.

Optimal Lagrangian Multipliers

INVENTORY SIZE AT 15 CATEGORY (Reorder)
SEED WAS 84573 ALPHA .2009115

H	CLASSIC	IMPLICIT	ZIEGLER	HORIZON	P	Max Slope
1	8.090002	8.099552	8.099546	8.09953	3	1.704985E-02
	8.420502	8.420479	8.420489	8.420478	4	1.705447E-02
	8.155405	8.155365	8.155371	8.155369	3	1.705312E-02
	7.8199	7.819886	7.819885	7.819896	4	1.708034E-02
	7.963331	7.963328	7.963319	7.963312	2	1.712086E-02
	7.881161	7.881137	7.881152	7.881168	1	1.709319E-02
	8.107661	8.10764	8.107649	8.107663	3	1.710963E-02
	7.775441	7.775433	7.775425	7.77545	4	1.707291E-02
	8.075005	8.074954	8.074942	8.074948	4	1.711625E-02
	7.744901	7.744946	7.744949	7.744962	4	1.707615E-02
2						
	27.64979	27.64977	27.64976	27.64977	1	5.153254E-02
	28.6426	28.64257	28.64259	28.64258	4	5.153917E-02
	27.54879	27.54876	27.54876	27.54876	4	5.156199E-02
	27.2816	27.28158	27.28155	27.28154	1	5.153992E-02
	27.02929	27.02923	27.02923	27.02921	1	5.153525E-02
	26.79001	26.79089	26.79088	26.7909	1	5.155383E-02
	25.8238	25.82375	25.82378	25.82378	4	5.159381E-02
	25.36389	25.36384	25.36381	25.36382	2	5.168929E-02
	25.7787	25.77868	25.77867	25.77866	2	5.180486E-02
	26.21729	26.21724	26.21722	26.21724	2	5.169843E-02
3						
	2.45046	2.450482	2.450473	2.450464	2	5.625628E-03
	2.487528	2.487531	2.487537	2.487507	1	5.629097E-03
	2.446669	2.446653	2.446674	2.446666	1	5.633128E-03
	2.483529	2.483519	2.483508	2.483512	1	5.628831E-03
	2.347	2.347008	2.347005	2.346985	4	5.632586E-03
	2.416628	2.416631	2.416621	2.416639	2	5.634216E-03
	2.309359	2.309338	2.309367	2.309373	3	5.640303E-03
	2.34518	2.345163	2.345174	2.345166	1	5.640355E-03
	2.382939	2.38294	2.382932	2.382953	1	5.64467E-03
	2.277868	2.277873	2.277864	2.277848	3	5.651552E-03
4						
	1.334484	1.33449	1.334472	1.334506	2	.0042274
	1.25359	1.253577	1.253596	1.253567	3	4.233917E-03
	1.227642	1.227648	1.227638	1.227632	1	4.242852E-03

H	CLASSIC	IMPLICIT	ZIEGLER	HORIZON	P	Max Slope
	1.27513	1.275112	1.275125	1.27515	2	4.250387E-03
	1.1979	1.197941	1.197941	1.197943	3	4.23816E-03
	1.22114	1.221131	1.221128	1.221132	1	4.251457E-03
	1.29211	1.292119	1.292104	1.292122	3	4.244626E-03
	1.38703	1.387033	1.387017	1.387023	4	4.234039E-03
	1.488162	1.488151	1.488161	1.488174	4	4.230773E-03
	1.427661	1.427669	1.427664	1.427676	2	4.232964E-03
5						
	2.602019	2.602039	2.602011	2.602015	4	5.772888E-03
	2.664928	2.664912	2.664921	2.66493	2	5.769477E-03
	2.700601	2.700577	2.700613	2.700591	1	5.772477E-03
	2.73819	2.738182	2.738195	2.738173	1	5.775409E-03
	2.878818	2.878825	2.878805	2.87881	4	5.780795E-03
	2.723899	2.723894	2.723876	2.723899	4	5.790914E-03
	2.763233	2.763214	2.76322	2.76322	1	5.780137E-03
	2.719861	2.719858	2.719875	2.719863	1	5.787778E-03
	2.789918	2.789896	2.789921	2.789927	2	5.779458E-03
	2.831319	2.831331	2.83131	2.831334	1	5.790524E-03
6						
	23.11679	23.11669	23.11669	23.11669	13	3.893094E-02
	25.1821	25.18203	25.18203	25.18204	12	3.894497E-02
	22.7787	22.77863	22.77863	22.77864	13	3.898167E-02
	24.9	24.94211	24.94212	24.94211	13	3.895525E-02
	27.17869	27.17864	27.17863	27.17864	12	3.891558E-02
	29.76569	29.76563	29.76564	29.76563	13	.039037
	27.06779	27.06769	27.0677	27.0677	12	3.908768E-02
	24.6238	24.62374	24.62374	24.62372	12	3.901038E-02
	22.5534	22.55339	22.55337	22.55336	11	3.895743E-02
	20.2763	20.27622	20.27623	20.27622	14	3.894312E-02
7						
	2.155913	2.155927	2.155907	20.27631	13	3.705458E-02
	1.9	1.9169	1.916896	1.916877	11	4.299146E-03
	1.677216	1.677226	1.67721	1.677208	14	4.265847E-03
	1.87086	1.870843	1.870861	1.870867	11	4.286413E-03
	1.638174	1.638172	1.638158	1.638178	13	4.289148E-03
	1.427051	1.427057	1.427054	1.427046	14	4.279929E-03
	1.615217	1.615202	1.615214	1.615238	13	4.268135E-03
	1.814411	1.814423	1.814418	1.814421	12	4.277573E-03
	1.587471	1.587475	1.587458	1.58748	13	4.284005E-03
	1.400012	1.400005	1.399995	1.400018	11	4.274182E-03
8						
	1.293139	1.293147	1.293147	1.400025	14	3.646981E-03
	1.13279	1.132811	1.132811	1.13279	12	3.139841E-03

H	CLASSIC	IMPLICIT	ZIEGLER	HORIZON	P	Max Slope
	.9000001	.9805597	.9805686	.9805691	13	3.140208E-03
	1.10271	1.102701	1.102705	1.028498	11	2.967404E-03
8	1.244524	1.244524	1.244519	1.244505	12	3.134354E-03
	1.081808	1.081815	1.0818	1.08179	13	3.138085E-03
	1.228289	1.228293	1.228292	1.228274	13	3.134999E-03
	1.0829	1.082919	1.082927	1.082927	11	3.137757E-03
	.9000001	.9505266	.9505393	.9505541	11	3.138307E-03
	.8156463	.8156376	.8156641	.779619	13	3.032382E-03
9						
	15.1349	15.13484	15.13485	15.13483	13	3.158994E-02
	13.59	13.59187	13.59191	13.59189	14	3.160791E-02
	14.79001	14.79887	14.79889	14.79886	11	.0316121
	13.28901	13.28961	13.28961	13.28963	14	.0315705
	14.47761	14.47759	14.47759	14.4776	11	.0315955
	13.1018	13.10175	13.10174	13.10177	12	3.158474E-02
	14.4417	14.44164	14.44164	14.44162	14	3.160241E-02
	13.14183	13.14183	13.14181	13.14184	11	3.159443E-02
	14.37901	14.37993	14.37991	14.37993	12	3.162122E-02
	15.66002	15.66003	15.66002	15.66001	11	3.159142E-02
10						
	7.91887	7.918875	7.918879	7.918866	11	1.461637E-02
	8.766445	8.766439	8.766445	8.766444	14	1.461773E-02
	9.590004	9.591376	9.591371	9.591368	11	1.462773E-02
	10.58102	10.58104	10.58103	10.58102	13	1.463621E-02
	9.584907	9.584879	9.584884	9.584873	11	1.464852E-02
	10.53641	10.5364	10.5364	10.5364	12	1.463453E-02
	9.451904	9.451885	9.451881	9.451905	13	1.464279E-02
	8.451642	8.451641	8.451619	8.451628	14	.0146318
	7.600701	7.600678	7.600678	7.600687	12	1.462018E-02
	6.83122	6.831219	6.831224	6.831232	12	.0146154

LOWER RANGE				
N	STATISTIC	ZIEGLER	IMPLICIT	HORIZON
5	TIME	11.48203	8.990626	8.353907
	AME	6.941858	3.4828	.9217744
	ARE	.876054	.9584746	.1101018
15	TIME	37.20937	23.28516	21.05
	AME	.60551	1.93297	.4455682
	ARE	.2079338	1.017191	.1614291
30	TIME	104.4781	48.96641	45.10704
	AME	.6636105	5.618967	1.121334
	ARE	.0955780	.8126926	.1460536
50	TIME	237.4266	81.05781	75.16719
	AME	.6614721	6.034627	1.377409
	ARE	.0463952	.8043341	.1379604

HIGHER RANGE				
N	STATISTIC	ZIEGLER	IMPLICIT	HORIZON
5	TIME	10.74141	8.582812	8.449219
	AME	3.517806	2.770939	.6564563
	ARE	.6304057	1.587542	.1700051
15	TIME	39.34375	24.51328	22.96406
	AME	1.606574	3.5807	.7560912
	ARE	.3233844	1.922109	.1862562
30	TIME	102.1328	48.1875	45.7
	AME	.2499031	4.043327	.7131383
	ARE	.0712604	1.740208	.2084213
50	TIME	239.8563	81.82188	80.39844
	AME	1.054366	6.604976	2.6561
	ARE	.048452	1.55747	.2108463

Table A5-1 (Statistics for Dual Parameter Category Shifts - Reorder and Demand)

Appendix 6

The purpose of this appendix is to display the results of the empirical study described in section 7-2. Mirroring the information presented in Tables 7-5, 7-6, and 7-7 for the LLLL system configuration, the following tables exhibit data concerning the HHHH, HLLH and LHHL system configurations.

Reduction Factors					
Space	Budget	Φ^*	Θ^*	B Slack	W Slack
.6	.1	49.4998	0	0	123798.76
|	.2	11.9999	0	0	99036.61
|	.3	5.0555	0	0	74277.46
|	.4	2.6249	0	0	49518.31
|	.5	1.4999	0	0	24759.16
|	**.6**	**.8888**	**0**	**0**	**0**
|	**.62**	**.6919**	**.0208**	**0**	**0**
|	**.64**	**.5210**	**.0401**	**0**	**0**
|	**.66**	**.3708**	**.0583**	**0**	**0**
|	**.68**	**.2372**	**.0758**	**0**	**0**
|	**.7**	**.1171**	**.0928**	**0**	**0**
|	**.72**	**.0076**	**.1096**	**0**	**0**
|	**.74**	**0**	**.1108**	**838.388**	**0**
|	.76	0	|	1742.795	0
|	.78	0	|	2647.205	0
|	.8	0	|	3551.624	0
.6	.9	0	.1108	8073.674	0

Table A6-1 (Empirical Data for High-High-High-High System Configuration)

Reduction Factors					
Space	Budget	Φ^*	Θ^*	B Slack	W Slack
.5	.1	49.4998	0	0	7759.66
|	.2	11.9999	0	0	5819.74
|	.3	5.0555	0	0	3879.83
|	.4	2.6249	0	0	1939.91
|	.48	1.6701	0	0	387.98
|	**.5**	**1.4999**	**0**	**0**	**0**
|	**.51**	**1.2520**	**.0304**	**0**	**0**
|	**.52**	**1.0274**	**.0591**	**0**	**0**
|	**.54**	**.6367**	**.1125**	**0**	**0**
|	**.56**	**.3091**	**.1615**	**0**	**0**
|	**.58**	**.0302**	**.2074**	**0**	**0**
|	**.59**	**0**	**.2127**	**25.696**	**0**
|	.6	0	|	59.342	0
|	.7	0	|	395.801	0
|	.8	0	|	735.261	0
.5	.9	0	.2127	1068.720	0

Table A6-2 (Empirical Data for High-Low-Low-High System Configuration)

Reduction Factors					
Space	Budget	Φ^*	Θ^*	B Slack	W Slack
.7	.1	49.4998	0	0	20009.34
|	.2	11.9999	0	0	16691.12
|	.3	5.0555	0	0	13372.89
|	.4	2.6249	0	0	10054.67
|	.5	1.4999	0	0	6736.44
|	.6	.8888	0	0	3418.22
|	**.7**	**.5204**	**0**	**0**	**99.99**
|	**.71**	**.3878**	**.0869**	**0**	**0**
|	**.72**	**.2285**	**.2016**	**0**	**0**
|	**.73**	**.0870**	**.3064**	**0**	**0**
|	**.74**	**0**	**.3726**	**88.614**	**0**
|	.8	0	|	1729.327	0
.7	.9	0	.3726	4463.848	0

Table A6-3 (Empirical Data for Low-High-High-Low System Configuration)

Appendix 7

Sample Data Generated for LLLL System Configuration

	SLACKB/% ERROR	SLACKS/% ERROR	P2 LOC/CRSZ
1	-18.24463000 -0.00710312	-3.23718300 -0.00174632	0.34704340 0.06134045
2	-7.50488300 -0.01362607	-3.56231700 -0.00771277	0.88591540 0.01572226
3	-34.13477000 -0.05273898	-12.30774000 -0.02130336	0.44831400 0.03186712
4	-51.43384000 -0.03493412	-20.07349000 -0.01511391	0.36802910 0.07763645
5	-60.77161000 -0.04050860	-14.46906000 -0.01679874	0.24409660 0.10706670
6	-13.11328000 -0.00471682	-3.75097700 -0.00138799	0.67292650 0.03497058
7	-1.41992200 -0.00128927	-1.58422900 -0.00148536	0.95359840 0.01349986
8	-86.74682000 -0.06672071	-11.13696000 -0.01354558	0.83754100 0.07826117
9	-16.49341000 -0.01433119	-4.31475900 -0.00560480	0.08642373 0.07160386
10	-8.45019600 -0.00351205	-10.81421000 -0.00321693	0.41223690 0.02443934
11	-0.92114260 -0.00087872	-0.65002450 -0.00071955	0.92951410 0.00575659
12	-108.82300000 -0.06183699	-17.73651000 -0.01844615	0.75356250 0.14303030

Appendix 8

Tyree Parts and Hardware - Historical Data

The initial purpose of this appendix is to display both the vendor costs incurred and the demand pattern experienced by Tyree Parts & Hardware during the six period horizon from the spring of 1988 through the fall of 1990 for the 32 large volume inventory item shown in Table 8-1. This historical data is presented in Tables A8-1 and A8-2 respectively. Table A8-3 through A8-8 exhibit the actual inventory activity experienced by this small hardware company during that same period utilizing it current two bin inventory system. These tables track the initial inventory on-hand, the size of each replenishment order and the final inventory on-hand for each of these six periods. The maximum space requirements encountered during each period is also

	S88	F88	S89	F89	S90	F90
Water Heaters						
ZER6STR	64.9	64.9	68.1	68.1	67.8	73.5
ZER10STR	71.3	71.3	71.3	71.3	71.3	71.3
ZER21STR	81.7	81.7	80.2	80.2	80.2	80.2
ZEFU 90-30 STR	91.4	85.4	89.7	89.7	89.7	89.7
ZHEFR 90-32 STR	91.4	85.4	89.7	89.7	89.7	98.6
ZHEFR 90-42 DTR	96.5	96.5	101.3	101.3	114.7	115.8
ZHEFR 90-52 DTR	130.3	130.3	130.3	127.8	138.6	138.6
ZMHG 90-333T NAT	109.4	102.3	117.3	157.3	159.3	159.3
ZMHG 90-333T LP	109.4	102.3	117.3	157.3	159.3	159.3
ZMHG 90-433T NAT	123.4	123.4	132.5	172.5	171.8	171.8
ZMHG 90-433T LP	125.8	117.5	132.5	172.5	171.8	171.8
EQF 250 QUICK FLO	87.3	87.3	87.3	87.3	87.3	92.0
EQF 400 QUICK FLO	92.5	92.5	92.5	92.5	97.0	97.0
SP 1-6C	58.9	58.9	58.9	58.9	62.4	62.4
SP 1-10C	61.9	64.9	61.9	61.9	65.6	65.6
SP 1-17C	65.9	65.9	65.9	65.9	69.9	69.9
SP 1-20C	70.3	70.3	70.3	70.3	74.5	74.5
MISCELLANEOUS						
HYDRANT	71.4	71.4	71.4	71.4	71.4	74.3
LS2 48	29.2	29.2	29.2	29.2	29.2	30.1
LS2 410	32.6	32.6	32.6	32.6	32.6	32.6
Q1PC 100G	24.4	24.4	28.5	28.5	28.5	28.5
Q2PC 100G	32.2	32.2	31.8	31.8	33.7	33.7
Q3PC 100G	30.4	35.4	35.4	35.4	37.6	37.6
Q4PC 100G	55.5	55.5	64.8	64.7	64.7	64.7
BATH TUBS						
T2470-00	84.6	84.6	85.6	85.6	89.1	89.1
T2473-00	84.6	84.6	85.6	85.6	89.1	89.1
T2472-06	94.2	94.2	94.2	94.2	98.0	98.0
T2473-06	90.0	90.0	98.0	98.0	98.0	98.0
T2300-00	80.6	80.6	81.6	81.6	85.0	85.0
T2301-00	80.6	80.6	80.6	80.6	85.0	85.0
T2300-06	85.7	85.7	89.8	89.8	89.8	89.8
T2301-06	85.7	85.7	89.8	89.8	89.8	89.8

Table A8-1 (Product Cost Data - 1988 through 1990)

WATER HEATERS	S88	F88	S89	F89	S90	F90
ZER6STR	1.0000	2.0000	4.0000	2.0000	1.0000	3.0000
ZER10STR	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
ZER21STR	4.0000	5.0000	4.0000	2.0000	3.0000	1.0000
ZEFU 90-30 STR	5.0000	3.0000	3.0000	5.0000	2.0000	1.0000
ZHEFR 90-32 STR	5.0000	4.0000	9.0000	7.0000	6.0000	0.0000
ZHEFR 90-42 DTR	1.0000	1.0000	3.0000	4.0000	4.0000	0.0000
ZHEFR 90-52 DTR	1.0000	0.0000	1.0000	1.0000	1.0000	0.0000
ZMHG 90-333T NAT	8.0000	16.000	13.000	11.000	5.0000	7.0000
ZMHG 90-333T LP	25.000	25.000	35.000	15.000	15.000	6.0000
ZMHG 90-433T NAT	8.0000	5.0000	5.0000	3.0000	4.0000	2.0000
ZMHG 90-433T LP	4.0000	4.0000	1.0000	2.0000	2.0000	5.0000
EQF 250 QUICK FLO	0.0000	0.0000	1.0000	2.0000	1.0000	2.0000
EQF 400 QUICK FLO	0.0000	0.0000	1.0000	2.0000	1.0000	2.0000
SP 1-6C	1.0000	1.0000	3.0000	2.0000	1.0000	4.0000
SP 1-10C	4.0000	5.0000	3.0000	4.0000	4.0000	3.0000
SP 1-17C	3.0000	6.0000	5.0000	7.0000	5.0000	6.0000
SP 1-20C	5.0000	3.0000	7.0000	8.0000	4.0000	6.0000
MISCELLANEOUS						
HYDRANT	0.0000	0.0000	1.0000	4.0000	3.0000	0.0000
LS2 48	0.0000	0.0000	2.0000	4.0000	7.0000	11.000
LS2 410	0.0000	0.0000	2.0000	4.0000	6.0000	4.0000
Q1PC 100G	0.0000	0.0000	5.0000	4.0000	5.0000	6.0000
Q2PC 100G	0.0000	6.0000	9.0000	7.0000	6.0000	14.000
Q3PC 100G	19.000	12.000	23.000	20.000	24.000	22.000
Q4PC 100G	0.0000	10.000	17.000	13.000	11.000	5.0000
BATH TUBS						
T2470-00	1.0000	0.0000	2.0000	0.0000	3.0000	0.0000
T2473-00	3.0000	0.0000	2.0000	0.0000	3.0000	0.0000
T2472-06	3.0000	0.0000	2.0000	0.0000	3.0000	0.0000
T2473-06	3.0000	0.0000	2.0000	0.0000	2.0000	0.0000
T2300-00	7.0000	0.0000	14.000	0.0000	4.0000	0.0000
T2301-00	11.000	0.0000	16.000	0.0000	17.000	0.0000
T2300-06	7.0000	0.0000	8.0000	0.0000	3.0000	0.0000
T2301-06	13.000	0.0000	16.000	0.0000	9.0000	0.0000

Table A8-2 (Historical Demand From 1988 through 1990)

Inventory Item	Initial On-Hand	Ordered	TOTAL	Final On-Hand
1	4.0	0.0	4.0	3.0
2	4.0	0.0	4.0	4.0
3	8.0	8.0	16.0	12.0
4	10.0	3.0	13.0	8.0
5	10.0	4.0	14.0	9.0
6	5.0	0.0	5.0	4.0
7	3.0	1.0	4.0	3.0
8	15.0	15.0	30.0	22.0
9	20.0	20.0	40.0	15.0
10	5.0	10.0	15.0	7.0
11	10.0	3.0	13.0	9.0
12	5.0	0.0	5.0	5.0
13	4.0	0.0	4.0	4.0
14	4.0	0.0	4.0	3.0
15	7.0	4.0	11.0	7.0
16	7.0	3.0	10.0	7.0
17	13.0	5.0	18.0	13.0
18	5.0	0.0	5.0	5.0
19	3.0	0.0	3.0	3.0
20	2.0	0.0	2.0	2.0
21	10.0	15.0	25.0	25.0
22	10.0	0.0	10.0	10.0
23	20.0	20.0	40.0	21.0
24	20.0	0.0	20.0	20.0
25	1.0	2.0	3.0	2.0
26	2.0	3.0	5.0	2.0
27	1.0	4.0	5.0	2.0
28	2.0	4.0	6.0	3.0
29	10.0	10.0	20.0	13.0
30	16.0	16.0	32.0	21.0
31	5.0	5.0	10.0	3.0
32	10.0	10.0	20.0	7.0
Total Inventory Space Required			6,647,761	

Table A8-3 (Results of Current Two-Bin Inventory System - Spring 1988)

Inventory Item	Initial On-Hand	Ordered	TOTAL	Final On-Hand
1	3.0	4.0	7.0	5.0
2	4.0	0.0	4.0	4.0
3	12.0	0.0	12.0	7.0
4	8.0	3.0	11.0	8.0
5	9.0	8.0	17.0	13.0
6	4.0	2.0	6.0	5.0
7	3.0	0.0	3.0	3.0
8	22.0	10.0	32.0	16.0
9	15.0	35.0	50.0	25.0
10	7.0	10.0	17.0	12.0
11	9.0	10.0	19.0	15.0
12	5.0	0.0	5.0	5.0
13	4.0	0.0	4.0	4.0
14	3.0	2.0	5.0	4.0
15	7.0	7.0	14.0	9.0
16	7.0	7.0	14.0	8.0
17	13.0	3.0	16.0	13.0
18	5.0	0.0	5.0	5.0
19	3.0	0.0	3.0	3.0
20	2.0	0.0	2.0	2.0
21	25.0	10.0	35.0	35.0
22	10.0	20.0	30.0	24.0
23	21.0	20.0	41.0	29.0
24	20.0	10.0	30.0	20.0
25	2.0	0.0	2.0	2.0
26	2.0	0.0	2.0	2.0
27	2.0	0.0	2.0	2.0
28	3.0	0.0	3.0	3.0
29	13.0	0.0	13.0	13.0
30	21.0	0.0	21.0	21.0
31	3.0	0.0	3.0	3.0
32	7.0	0.0	7.0	7.0
Total Inventory Space Required			5917018.4	

Table A8-4 (Results of Current Two-Bin Inventory System - Fall 1988)

Inventory Item	Initial On-Hand	Ordered	TOTAL	Final On-Hand
1	5.0	4.0000	9.0000	5.0
2	4.0	3.0000	7.0000	6.0
3	7.0	8.0000	15.0000	11.0
4	8.0	5.0000	13.0000	10.0
5	13.0	8.0000	21.0000	12.0
6	5.0	5.0000	10.0000	7.0
7	3.0	0.0000	3.0000	2.0
8	16.0	15.0000	31.0000	18.0
9	25.0	20.0000	45.0000	10.0
10	12.0	3.0000	15.0000	10.0
11	15.0	2.0000	17.0000	16.0
12	5.0	1.0000	6.0000	5.0
13	4.0	1.0000	5.0000	4.0
14	4.0	4.0000	8.0000	5.0
15	9.0	1.0000	10.0000	7.0
16	8.0	7.0000	15.0000	10.0
17	13.0	13.0000	26.0000	19.0
18	5.0	1.0000	6.0000	5.0
19	3.0	3.0000	6.0000	4.0
20	2.0	2.0000	4.0000	2.0
21	35.0	20.0000	55.0000	50.0
22	24.0	3.0000	27.0000	18.0
23	29.0	22.0000	51.0000	28.0
24	20.0	20.0000	40.0000	23.0
25	2.0	2.0000	4.0000	2.0
26	2.0	3.0000	5.0000	3.0
27	2.0	3.0000	5.0000	3.0
28	3.0	2.0000	5.0000	3.0
29	13.0	8.0000	21.0000	7.0
30	21.0	14.0000	35.0000	19.0
31	3.0	6.0000	9.0000	1.0
32	7.0	20.0000	27.0000	11.0
Total Inventory Space Required			7764004.9	

Table A8-5 (Results of Current Two-Bin Inventory System - Spring 1989)

Inventory Item	Initial On-Hand	Ordered	TOTAL	Final On-Hand
1	5.0	1.0000	6.0000	4.0
2	6.0	0.0000	6.0000	6.0
3	11.0	0.0000	11.0000	9.0
4	10.0	7.0000	17.0000	12.0
5	12.0	4.0000	16.0000	9.0
6	7.0	2.0000	9.0000	5.0
7	2.0	5.0000	7.0000	6.0
8	18.0	2.0000	20.0000	9.0
9	10.0	5.0000	15.0000	0.0
10	10.0	25.0000	35.0000	32.0
11	16.0	1.0000	17.0000	15.0
12	5.0	2.0000	7.0000	5.0
13	4.0	2.0000	6.0000	4.0
14	5.0	2.0000	7.0000	5.0
15	7.0	2.0000	9.0000	5.0
16	10.0	4.0000	14.0000	7.0
17	19.0	4.0000	23.0000	15.0
18	5.0	2.0000	7.0000	3.0
19	4.0	4.0000	8.0000	4.0
20	2.0	6.0000	8.0000	4.0
21	50.0	4.0000	54.0000	50.0
22	18.0	10.0000	28.0000	21.0
23	28.0	0.0000	28.0000	8.0
24	23.0	20.0000	43.0000	30.0
25	2.0	10.0000	12.0000	12.0
26	3.0	0.0000	3.0000	3.0
27	3.0	0.0000	3.0000	3.0
28	3.0	0.0000	3.0000	3.0
29	7.0	0.0000	7.0000	7.0
30	19.0	0.0000	19.0000	19.0
31	1.0	0.0000	1.0000	1.0
32	11.0	0.0000	11.0000	11.0
Total Inventory Space Required			5865409.7	

Table A8-6 (Results of Current Two-Bin Inventory System - Fall 1989)

Inventory Item	Initial On-Hand	Ordered	TOTAL	Final On-Hand
1	4.0	1.0000	5.0000	0.0
2	6.0	0.0000	6.0000	6.0
3	9.0	0.0000	9.0000	6.0
4	12.0	0.0000	12.0000	10.0
5	9.0	0.0000	9.0000	3.0
6	5.0	1.0000	6.0000	2.0
7	6.0	0.0000	6.0000	5.0
8	9.0	7.0000	16.0000	11.0
9	0.0	21.0000	21.0000	6.0
10	32.0	1.0000	33.0000	29.0
11	15.0	3.0000	18.0000	16.0
12	5.0	0.0000	5.0000	4.0
13	4.0	1.0000	5.0000	4.0
14	5.0	1.0000	6.0000	5.0
15	5.0	7.0000	12.0000	8.0
16	7.0	7.0000	14.0000	9.0
17	15.0	5.0000	20.0000	16.0
18	3.0	3.0000	6.0000	3.0
19	4.0	10.0000	14.0000	7.0
20	4.0	10.0000	14.0000	8.0
21	50.0	0.0000	50.0000	45.0
22	21.0	10.0000	31.0000	25.0
23	8.0	20.0000	28.0000	4.0
24	30.0	20.0000	50.0000	39.0
25	12.0	5.0000	17.0000	14.0
26	3.0	5.0000	8.0000	5.0
27	3.0	5.0000	8.0000	5.0
28	3.0	5.0000	8.0000	6.0
29	7.0	13.0000	20.0000	16.0
30	19.0	18.0000	37.0000	20.0
31	1.0	6.0000	7.0000	4.0
32	11.0	8.0000	19.0000	10.0
Total Inventory Space Required			7356772.6	

Table A8-7 (Results of Current Two-Bin Inventory System - Spring 1990)

Inventory Item	Initial On-Hand	Ordered	TOTAL	Final On-Hand
1	4.0	6.0000	10.0000	7.0
2	6.0	0.0000	6.0000	6.0
3	6.0	0.0000	6.0000	5.0
4	10.0	0.0000	10.0000	9.0
5	3.0	20.0000	23.0000	23.0
6	2.0	5.0000	7.0000	6.0
7	5.0	5.0000	10.0000	9.0
8	11.0	0.0000	11.0000	4.0
9	6.0	0.0000	6.0000	0.0
10	29.0	0.0000	29.0000	27.0
11	16.0	0.0000	16.0000	11.0
12	4.0	5.0000	9.0000	7.0
13	4.0	2.0000	6.0000	4.0
14	5.0	4.0000	9.0000	5.0
15	8.0	0.0000	8.0000	5.0
16	9.0	5.0000	14.0000	8.0
17	16.0	10.0000	26.0000	20.0
18	3.0	20.0000	23.0000	22.0
19	7.0	20.0000	27.0000	16.0
20	8.0	0.0000	8.0000	4.0
21	45.0	0.0000	45.0000	39.0
22	25.0	10.0000	35.0000	21.0
23	4.0	20.0000	24.0000	2.0
24	39.0	0.0000	39.0000	34.0
25	14.0	0.0000	14.0000	14.0
26	5.0	0.0000	5.0000	5.0
27	5.0	0.0000	5.0000	5.0
28	6.0	0.0000	6.0000	6.0
29	16.0	0.0000	16.0000	16.0
30	20.0	0.0000	20.0000	20.0
31	4.0	0.0000	4.0000	4.0
32	10.0	0.0000	10.0000	10.0
Total Inventory Space Required			5942650.6	5031230.3

Table A8-8 (Results of Current Two-Bin Inventory System - Fall 1990)

Appendix 9

The purpose of this appendix is two fold. First, the unconstrained EOQ based inventory policy implemented in section 8.3 will be explained. Second, the projected inventory activity generated based on the historical demand pattern from 1988 through 1990 will be presented.

Unconstrained EOQ Based Inventory Policy

This policy, which assumes no inventory shortages, an instantaneous replenishment rate and which maintains the management defined safety stock levels indicated in Table 8-1, utilizes the following steps in determining each items inventory activity during any given period of an horizon. In implementing this policy it is assumed that the desired safety stocks are in place at the beginning of the first period so that S^+ = Safety Stock and Final OH = 0 for each inventory item.

Step 1) Determine amount of replenishment, S^+, needed to return each item's safety stock to its desire level at the end of a period.

$$S^+ = \text{Safety Stock - Final OH inventory}$$

where $S^+ < 0$ set $S^+ = 0$, no safety stock replenishment needed.

Step 2) Determine effective on-hand inventory, I_{oh}, such that

$$I_{oh} = \text{Final OH} + S^+ - \text{Safety Stock}$$

where $I_{oh} < 0$ set $I_{oh} = 0$, no inventory on-hand.

Step 3) Compute the effective demand, D_{eff}.

$$D_{eff} = \text{Historical Demand} - I_{oh}$$

where $D_{eff} < 0$ set $D_{eff} = 0$, no replenishment is made.

Step 4) Calculate size of replenishment order.

$$\text{Order} = \text{unconstrained EOQ} + S^{+}$$

where the EOQ is computed using (1-2) and D_{eff}.

Step 5) Compute maximum inventory level, Total, for each item.

$$\text{Total} = \text{EOQ} + \text{Final OH} + S^{+}$$

Step 6) Determine new Final OH inventory.

$$\text{Final OH} = \text{Total} - \text{Historical Demand}$$

where Final OH < 0 set Final OH = 0.

To illustrate this inventory policy consider the values shown in Tables A9-1 and A9-2 for item #4. Where the desired safety stock level for this item is set at 2, the inventory activity during periods one and two of this horizon follows:

Period One
Historical Demand = 5

Step 1) $S^{+} = 2 - 0 = 2$

Step 2) $I_{oh} = 0 + 2 - 2 = 0$

Step 3) $D_{eff} = 5 - 0 = 5$

Step 4) Order = 7 + 0 = 7

Step 5) Total = 7 + 0 + 2 = 9

Step 6) Final OH = 9 - 5 = 4

Period Two
Historical Demand = 3

Step 1) $S^{+} = 2 - 4 = -2 \rightarrow 0$

Step 2) $I_{oh} = 4 + 0 - 2 = 2$

Step 3) $D_{eff} = 3 - 2 = 1$

Step 4) Order = 3 + 0 = 3

Step 5) Total= 3 + 4 + 0 = 7

Step 6) Final OH = 7 - 3 = 4

Note that an EOQ for each item will be computed at step 4 only if the inventory on=hand at the start of the period does not cover that period's demand. Note also that the actual replenishment order combines both the EOQ and S^+ so that at the start of each period the safety stock levels are fully restocked.

Projected Unconstrained EOQ Inventory Activity

Tracking the resulting inventory activity, Tables A9-1 through A9-6 display the Carrying cost (C), Re-order cost (R), Effective demand (D), the EOQ determined replenishment size (Order), maximum inventory level (Total) and the final on-hand inventory (Final OH) projected for each period of this horizon. The total inventory space needed to handle the maximum inventory projected for each period, along with the inventory investment demanded solely by each period's replenishment are also presented.

Inventory Item	C	R	D	Space	Order	Total	Final OH
1	3.2425	16.6203	1.0000	6,601.6	3.0	5.0	4.0
2	3.5670	17.5938	0.0000	6,733.6	0.0	2.0	2.0
3	4.0870	19.1538	4.0000	9,240.8	6.0	8.0	4.0
4	4.5685	20.5983	5.0000	16,832.8	7.0	9.0	4.0
5	4.5685	20.5983	5.0000	16,599.9	7.0	9.0	4.0
6	4.8250	21.3678	1.0000	20,802.4	3.0	5.0	4.0
7	6.5135	26.4333	1.0000	25,059.4	3.0	5.0	4.0
8	5.4720	23.3088	8.0000	21,617.3	8.0	12.0	4.0
9	5.4720	23.3088	25.000	21617.3	15.0	25.0	0.0
10	6.1695	25.4013	8.0000	28497.0	8.0	10.0	2.0
11	6.2875	25.7553	4.0000	27213.8	6.0	8.0	4.0
12	4.3625	19.9803	0.0000	3811.5	0.0	2.0	2.0
13	4.6250	20.7678	0.0000	3811.5	0.0	2.0	2.0
14	2.9430	15.7218	1.0000	5132.9	3.0	5.0	4.0
15	3.0950	16.1778	4.0000	7596.5	6.0	8.0	4.0
16	3.2970	16.7838	3.0000	10086.0	6.0	8.0	5.0
17	3.5125	17.4303	5.0000	12054.0	7.0	9.0	4.0
18	3.5700	27.7100	0.0000	1675.0	0.0	2.0	2.0
19	1.4600	20.2550	0.0000	3189.4	0.0	2.0	2.0
20	1.6305	20.7665	0.0000	3937.5	0.0	2.0	2.0
21	1.2210	16.4130	0.0000	810.0	0.0	10.0	10.0
22	1.6110	12.2080	0.0000	1587.0	0.0	10.0	10.0
23	1.5185	17.3855	19.000	2500.0	21.0	31.0	12.0
24	2.7750	15.7000	0.0000	3920.0	0.0	10.0	10.0
25	4.2300	25.3150	1.0000	23,256.0	3.0	4.0	3.0
26	4.2300	25.3150	3.0000	22,134.0	6.0	7.0	4.0
27	4.7075	26.7475	3.0000	23,256.0	6.0	7.0	4.0
28	4.5000	26.1250	3.0000	22,848.0	6.0	7.0	4.0
29	4.0300	24.7150	7.0000	27,608.0	9.0	10.0	3.0
30	4.0300	24.7150	11.000	28,985.0	12.0	16.0	5.0
31	4.2850	25.4800	7.0000	28,985.0	9.0	10.0	3.0
32	4.2850	25.4800	13.000	29,837.5	12.0	16.0	3.0
TOTAL SPACE REQUIREMENT (BUDGET)					**3288093.4 (716.7)**	**4480032.7**	

Table A9-1 (Results of Unconstrained Application of EOQ Model to Spring 1988 Data)

Inventory Item	C	R	D	Space	Order	Total	Final OH
1	3.2425	17.1505	0.0	6,601.6	0.0000	4.0	2.0
2	3.5670	17.5938	0.0	6,733.6	0.0000	2.0	2.0
3	4.0870	19.6840	3.0	9,240.8	5.0000	9.0	4.0
4	4.2700	20.2330	1.0	16,832.8	3.0000	7.0	4.0
5	4.2700	20.2330	2.0	16,599.9	4.0000	8.0	4.0
6	4.8250	21.8980	0.0	20,802.4	0.0000	4.0	3.0
7	6.5135	26.4333	0.0	25,059.4	0.0000	4.0	4.0
8	5.1140	22.7650	16.0	21,617.3	12.0000	16.0	0.0
9	5.1140	22.7650	25.0	21,617.3	15.0000	25.0	0.0
10	6.1695	25.9315	5.0	28,497.0	6.0000	8.0	3.0
11	5.8760	25.0510	2.0	27,213.8	4.0000	8.0	4.0
12	4.3625	19.9803	0.0	3,811.5	0.0000	2.0	2.0
13	4.6250	20.7678	0.0	3,811.5	0.0000	2.0	2.0
14	2.9430	16.2490	0.0	5,132.9	0.0000	4.0	3.0
15	3.2450	17.1580	3.0	7,596.5	6.0000	10.0	5.0
16	3.2970	17.3147	3.0	10,086.0	6.0000	11.0	5.0
17	3.5125	17.9605	1.0	12,054.0	3.0000	7.0	4.0
18	3.5700	27.7100	0.0	1,675.0	0.0000	2.0	2.0
19	1.4600	20.2550	0.0	3,189.4	0.0000	2.0	2.0
20	1.6305	20.6665	0.0	3,937.5	0.0000	2.0	2.0
21	1.2210	16.4130	0.0	810.0	0.0000	10.0	10.0
22	1.6110	12.2080	6.0	1,587.0	10.0000	20.0	14.0
23	1.7720	12.6910	10.0	2,500.0	12.0000	24.0	12.0
24	2.7750	15.7000	10.0	3,920.0	11.0000	21.0	11.0
25	4.2300	25.3150	0.0	23,256.0	0.0000	3.0	3.0
26	4.2300	25.3150	0.0	22,134.0	0.0000	4.0	4.0
27	4.7075	26.7475	0.0	23,256.0	0.0000	4.0	4.0
28	4.5000	26.1250	0.0	22,848.0	0.0000	4.0	4.0
29	4.0300	24.7150	0.0	27,608.0	0.0000	3.0	3.0
30	4.0300	24.7150	0.0	28,985.0	0.0000	5.0	5.0
31	4.2850	25.4800	0.0	28,985.0	0.0000	3.0	3.0
32	4.2850	25.4800	0.0	29,837.5	0.0000	4.0	4.0
TOTAL SPACE REQUIREMENT (Budget)					**1820880.2 (420.2)**	**3172713.8**	

Table A9-2 (Results of Unconstrained Application of EOQ Model to Fall 1988 Data)

Inventory Item	C	R	D	Space	Order	Total	Final OH
1	3.4045	15.8899	4.0	6,601.6	6.0	8.0	4.0
2	3.5670	16.4044	1.0	6,733.6	3.0	5.0	4.0
3	4.0110	17.7094	2.0	9,240.8	4.0	8.0	4.0
4	4.4835	19.1269	1.0	16,832.8	3.0	7.0	4.0
5	4.4825	19.1269	7.0	16,599.9	8.0	12.0	3.0
6	5.0660	20.8744	2.0	20,802.4	4.0	7.0	4.0
7	6.5135	25.2169	0.0	25,059.4	0.0	4.0	3.0
8	5.8640	23.2684	13.0	21,617.3	10.0	14.0	1.0
9	5.8640	23.2684	35.0	21,617.3	17.0	27.0	0.0
10	6.6260	25.5544	4.0	28,497.0	6.0	9.0	4.0
11	6.6260	25.5544	0.0	27,213.8	0.0	4.0	3.0
12	4.3625	18.7639	1.0	3,811.5	3.0	5.0	4.0
13	4.6250	19.5514	1.0	3,811.5	3.0	5.0	4.0
14	2.9430	14.5024	2.0	5,132.9	4.0	7.0	4.0
15	3.0950	14.9614	0.0	7,596.5	0.0	5.0	2.0
16	3.2970	15.5674	2.0	10,086.0	4.0	9.0	4.0
17	3.5125	16.2139	5.0	12,054.0	7.0	11.0	4.0
18	3.5700	27.7100	1.0	1,675.0	4.0	6.0	5.0
19	1.4600	20.2550	2.0	3,189.4	7.0	9.0	7.0
20	1.6305	20.7665	2.0	3,937.5	0.0	2.0	0.0
21	1.4250	11.6500	5.0	810.0	9.0	19.0	4.0
22	1.5905	12.1465	5.0	1,587.0	9.0	23.0	4.0
23	1.7720	12.6910	21.0	2,500.0	17.0	29.0	6.0
24	3.2380	17.0890	16.0	3,920.0	13.0	24.0	7.0
25	4.2800	25.4650	0.0	23,256.0	0.0	3.0	1.0
26	4.2800	25.4650	0.0	22,134.0	0.0	4.0	2.0
27	4.7075	26.7475	0.0	23,256.0	0.0	4.0	2.0
28	4.9000	27.3250	0.0	22,848.0	0.0	4.0	2.0
29	4.0800	24.8650	12.0	27,608.0	12.0	15.0	1.0
30	4.0300	24.7150	15.0	28,985.0	14.0	19.0	3.0
31	4.4875	26.0875	6.0	28,985.0	8.0	11.0	3.0
32	4.4875	26.0875	16.0	29,837.5	14.0	18.0	2.0
TOTAL SPACE REQUIREMENT (Budget)					**2816800.1 (801.4)**	**4738885.4**	

Table A9-3 (Results of Unconstrained Application of EOQ Model to Spring 1989 Data)

Inventory Item	C	R	D	Space	Order	Total	Final OH
1	3.4045	16.2447	0.0	6,601.6	0.0	4.0	2.0
2	3.5670	16.4044	2.0	6,733.6	4.0	8.0	8.0
3	4.0110	18.0642	0.0	9,240.8	0.0	4.0	2.0
4	4.4835	19.4817	3.0	16,832.8	5.0	9.0	4.0
5	4.4835	19.4817	6.0	16,599.9	7.0	10.0	3.0
6	5.0660	21.2292	2.0	20,802.4	4.0	8.0	4.0
7	6.3915	25.2057	0.0	25,059.4	0.0	3.0	2.0
8	7.8640	29.6232	11.0	21,617.3	9.0	13.0	2.0
9	7.8640	29.6232	15.0	21617.3	11.0	21.0	6.0
10	8.6260	21.9092	1.0	28497.0	3.0	7.0	4.0
11	8.6260	31.9092	1.0	27213.8	3.0	6.0	4.0
12	4.3625	19.1188	0.0	3811.5	0.0	4.0	2.0
13	4.6250	19.9062	0.0	3811.5	0.0	4.0	2.0
14	2.9430	14.8572	0.0	5132.9	0.0	4.0	2.0
15	3.0950	15.3162	4.0	7596.5	6.0	8.0	4.0
16	3.2950	15.9162	5.0	10086.0	7.0	11.0	4.0
17	3.5125	16.5687	6.0	12054.0	7.0	11.0	3.0
18	3.5700	27.7100	1.0	1675.0	4.0	9.0	5.0
19	1.4600	20.2550	0.0	3189.4	0.0	7.0	3.0
20	1.6305	20.7665	4.0	3937.5	10.0	12.0	8.0
21	1.4250	11.6500	0.0	810.0	0.0	14.0	10.0
22	1.5905	12.1465	3.0	1587.0	7.0	21.0	14.0
23	1.7720	12.6910	20.0	2500.0	17.0	27.0	7.0
24	3.2350	17.0890	13.0	3920.0	12.0	22.0	9.0
25	4.2800	25.4650	0.0	23,256.0	0.0	1.0	1.0
26	4.2800	25.4650	0.0	22,134.0	0.0	2.0	2.0
27	4.7075	26.7475	0.0	23,256.0	0.0	2.0	2.0
28	4.9000	27.3250	0.0	22,848.0	0.0	2.0	2.0
29	4.0800	24.8650	0.0	27,608.0	0.0	1.0	1.0
30	4.0300	24.7150	0.0	28,985.0	0.0	4.0	4.0
31	4.4875	26.0875	0.0	28,985.0	0.0	3.0	3.0
32	4.4875	26.0875	0.0	29,837.5	0.0	4.0	4.0
TOTAL SPACE REQUIREMENT					**1257267.5**	**2922055.7**	
(Budget)					**(609.5)**		

Table A9-4 (Results of Unconstrained Application of EOQ Model to Fall 1989 Data)

Inventory Item	C	R	D	Space	Order	Total	Final OH
1	3.3875	16.1937	1.0	6,601.6	3.0000	5.0	4.0
2	3.5670	16.4044	0.0	6,733.6	0.0000	8.0	8.0
3	4.0110	18.0642	3.0	9,240.8	5.0000	7.0	4.0
4	4.4835	19.4817	0.0	16,832.8	0.0000	4.0	2.0
5	4.4835	19.4817	5.0	16,599.9	7.0000	10.0	4.0
6	5.7365	23.2407	2.0	20,802.4	4.0000	8.0	4.0
7	6.9275	26.8137	1.0	25,059.4	3.0000	5.0	4.0
8	7.9625	29.9187	5.0	21,617.3	6.0000	10.0	5.0
9	7.9625	29.9187	15.0	21617.3	11.0000	21.0	6.0
10	8.5875	31.7937	2.0	28497.0	4.0000	8.0	4.0
11	8.5875	31.7937	0.0	27213.8	0.0000	4.0	2.0
12	4.3625	19.1188	1.0	3811.5	3.0000	5.0	4.0
13	4.8500	20.5812	1.0	3811.5	3.0000	5.0	4.0
14	3.1195	15.3897	1.0	5132.9	3.0000	5.0	4.0
15	3.2805	15.8727	2.0	7596.5	4.0000	8.0	4.0
16	3.4945	16.5147	3.0	10086.0	5.0000	9.0	4.0
17	3.7230	17.2002	3.0	12054.0	5.0000	8.0	4.0
18	3.5700	27.7100	0.0	1675[illegible]	[illegible].0000	5.0	2.0
19	1.4600	20.2550	6.0	31[illegible]9.4	13.0000	16.0	9.0
20	1.6305	20.7665	0.0	3937.5	0.0000	8.0	2.0
21	1.4250	11.6500	5.0	810.0	9.0000	19.0	4.0
22	1.6860	12.4330	2.0	1587.0	5.0000	19.0	3.0
23	1.8785	13.0105	24.0	2500.0	18.0000	28.0	4.0
24	3.2325	17.0725	11.0	3920.0	11.0000	21.0	10.0
25	4.4550	25.9900	3.0	23,256.0	6.0000	7.0	4.0
26	4.4550	25.9900	2.0	22,134.0	5.0000	7.0	4.0
27	4.9000	27.3250	2.0	23,256.0	5.0000	7.0	4.0
28	4.9000	27.3250	1.0	22,848.0	3.0000	5.0	3.0
29	4.2500	25.3750	4.0	27,608.0	7.0000	8.0	4.0
30	4.2500	25.3750	17.0	28,985.0	14.0000	18.0	1.0
31	4.4875	26.0875	1.0	28,985.0	3.0000	6.0	3.0
32	4.4875	26.0875	9.0	29,837.5	10.0000	14.0	5.0
TOTAL SPACE REQUIREMENT (Budget)					**2565643.2 (763.3)**	**4213090.2**	

Table A9-5 (Results of Unconstrained Application of EOQ Model to Spring 1990 Data)

Inventory Item	C	R	D	Space	Order	Total	Final OH
1	3.6770	17.4643	1.0	6,601.6	3.0000	7.0	4.0
2	3.5670	17.1343	0.0	6,733.6	0.0000	8.0	8.0
3	4.0110	18.4663	0.0	9,240.8	0.0000	4.0	3.0
4	4.4835	19.8838	1.0	16,832.8	3.0000	5.0	4.0
5	4.9320	21.2293	0.0	16,599.9	0.0000	4.0	4.0
6	5.7920	23.8093	0.0	20,802.4	0.0000	4.0	3.0
7	6.9275	27.2158	0.0	25,059.4	0.0000	4.0	3.0
8	7.9625	30.3208	6.0	21,617.3	7.0000	12.0	5.0
9	7.9625	30.3208	6.0	21617.3	7.0000	17.0	11.0
10	8.5875	32.1958	0.0	28497.0	0.0000	4.0	2.0
11	8.5875	32.1958	5.0	27213.8	6.0000	8.0	3.0
12	4.6000	20.2333	0.0	3811.5	0.0000	4.0	2.0
13	4.8500	20.9833	0.0	3811.5	0.0000	4.0	2.0
14	3.1195	15.7918	2.0	5132.9	4.0000	8.0	4.0
15	3.2805	16.2748	1.0	7596.5	3.0000	7.0	4.0
16	3.4945	16.9168	4.0	10086.0	6.0000	10.0	4.0
17	3.7230	17.6023	4.0	12054.0	6.0000	10.0	4.0
18	3.7130	28.1390	1.0	1675.0	4.0000	6.0	5.0
19	1.5035	20.3855	4.0	3189.4	10.0000	19.0	8.0
20	1.6305	20.7665	4.0	3937.5	10.0000	12.0	8.0
21	1.4250	11.6500	2.0	810.0	6.0000	20.0	4.0
22	1.6860	12.4330	11.0	1587.0	13.0000	26.0	2.0
23	1.8785	13.0105	22.0	2500.0	17.0000	27.0	2.0
24	3.2325	17.0725	5.0	3920.0	7.0000	17.0	2.0
25	4.4550	25.9900	0.0	23,256.0	0.0000	4.0	4.0
26	4.4550	25.9900	0.0	22,134.0	0.0000	4.0	4.0
27	4.9000	27.3250	0.0	23,256.0	0.0000	4.0	4.0
28	4.9000	27.3250	0.0	22,848.0	0.0000	3.0	3.0
29	4.2500	25.3750	0.0	27,608.0	0.0000	4.0	4.0
30	4.2500	25.3750	0.0	28,985.0	0.0000	4.0	4.0
31	4.4875	26.0875	0.0	28,985.0	0.0000	3.0	3.0
32	4.4875	26.0875	0.0	29,837.5	0.0000	5.0	5.0
TOTAL SPACE REQUIREMENT (Budget)					**813197.7 (440.2)**	**2890925.1**	

Table A9-6 (Results of Unconstrained Application of EOQ Model to Fall 1990 Data)

Appendix 10

A10.1 Purpose

The purpose of this appendix is first to provide, in detail, the computations necessary to implement the Dual Constraint algorithm within the large volume inventory carried by Tyree Parts & Hardware from 1988 through 1990 as identified in Chapter VIII. In presenting these calculations, each period's effective space constraint level, W, was determined by subtracting both the space required to handle the final inventory on-hand at the end of a period and any replenishment needed to return the management specified safety stocks to their desired levels from the 3,333,618 cubic inches of total inventory space. With these constraint levels identified, the second portion of this appendix will exhibit the projected inventory active.

A10.2 Detailed Calculations

With initial constraint levels set at 3,333,618 cubic inches and $500.00 for the ·pace and budget constraints, respectively, and the management designated stockages levels assumed to be initially on-hand, the calculations needed to implement the Dual Constraint algorithm defined in section 7.4 follows:

a) Spring 1988

Step 1) Determine the reduction factors P_1 and P_2:

$$P_1 = W / \sum W_iQ_i = 2{,}141{,}679 / 3{,}286{,}917 = .651$$
$$P_2 = B / \sum b_iQ_i = 500 / 716.7 = .698.$$

Since Both reduction factors are less than one and P2 > P_1 then must go to step 2.

Note that since the safety stock consumes 1,191,939 of the 3,333,618 cubic inches of storage space available, here W = 2,141,679. Note also that the 3,286,917 cubic inch space and the \$ 716.70 budget requirements, computed using equations (2-13) and (2-14) respectively, represent the projected unconstrained resource requirements for this period.

Step 2a) $\Theta^* = .0001370858$. Implicit algorithm used to identify this multiplier.

Step 2b)

$$P = \sum_{i=1}^{N} b_i Q_{i\Theta} \; / \; \sum_{i=1}^{N} b_i Q_i = 483.77 / 716.7 = .675$$

where $\sum b_i Q_{i\Theta} = \sum b_i \{2R_i D_i / [b_i + 2W_i \Theta^*]\}^{1/2}$

Since $P_2 > P$ then stop Θ^* alone is optimal.

Incorporating Θ^* into step 4 of the EOQ based inventory policy defined in appendix 9, Table A10-1 displays the resulting inventory activity. Note, from this table, that the 766,403.3 cubic inches required to house the remaining inventory at the end of period one combined with the space needed to handle the inventory needed to replenish the safety stocks prior to the start of period two specifies an effective space constraint level of 1,953,931 cubic inches for period two.

b) Fall 1988

Step 1) Determine the reduction factors P_1 and P_2:

$$P_1 = W / \sum W_i Q_i = 1,953,931 / 1,444,668 = 1.352$$

$P_2 = B / \sum b_iQ_i = 500 / 416.1 = 1.20.$

Since both reduction factors are greater than one stop. Inventory during this period not constrained. The resulting inventory activity, shown in Table A10-2, suggests a 1,769,915 cubic inch space constraint level for period three of this horizon.

c) Spring 1989

Step 1) Determine the reduction factors P_1 and P_2:

$P_1 = W / \sum W_iQ_i = 1{,}769{,}915 / 3{,}238{,}372 = .546$
$P_2 = B / \sum b_iQ_i = 500 / 809.32 = .617.$

Since both reduction factors are less than one and $P_2 > P_1$ then must go to step 2.

Step 2a) $\Theta^* = .0002534664$. Implicit algorithm used to identify this multiplier.

Step 2b)

$$P = \sum_{i=1}^{N} b_iQ_{i\Theta} \, / \, \sum_{i=1}^{N} b_iQ_i = 488.82 / 809.32 = .675$$

where $\sum b_iQ_{i\Theta} = \sum b_i\{2R_iD_i / [b_i + 2W_i\Theta^*]\}^{1/2}$

Since $P_2 > P$ then stop Θ^* alone is optimal.

The resulting inventory activity, shown in Table A10-3, suggests a 1,945,398 cubic inch space constraint level for period four of this horizon.

d) Fall 1989

Step 1) Determine the reduction factors P_1 and P_2:

$P_1 = W / \Sigma W_iQ_i = 1{,}945{,}398 / 1{,}509{,}993 = 1.288$
$P_2 = B / \Sigma b_iQ_i = 500 / 575.6 = .86865.$

Since only $P_2 \leq 1$ Stop. Problem $P(\Phi,\Theta) \equiv P(\Phi)$ and

$\Phi^* = ½ [(1/P_2)^2 - 1] = ½ [(1/.868)^2 - 1] = .1626$ alone is optimal.

The resulting inventory activity, shown in Table A10-4, suggests a 1,786,877 cubic inch space constraint level for period five of this horizon.

e) Spring 1990

Step 1) Determine the reduction factors P_1 and P_2:

$P_1 = W / \Sigma W_iQ_i = 1{,}786{,}877 / 2{,}699{,}965 = .661814875$
$P_2 = B / \Sigma b_iQ_i = 500 / 719.95 = .694483702.$

Since both reduction factors are less than one and $P_2 > P_1$ then must go to step 2.

Step 2a) $\Theta^* = .0001469606$. Implicit algorithm used to identify this multiplier.

Step 2b)

$$P = \sum_{i=1}^{N} b_iQ_{i\Theta} \Big/ \sum_{i=1}^{N} b_iQ_i = 508.79 / 719.95 = .7079724$$

where $\Sigma b_iQ_{i\Theta} = \Sigma b_i\{2R_iD_i / [b_i + 2W_i\Theta^*]\}^{½}$

Since $P_2 < P$ then must go to step 3 of algorithm.

Step 3a) Compute $\Phi^*_{P1} = ½ [(1/P_1)^2 - 1] = ½ [(1/.661)^2 - 1] = .641555303$

and set $\Theta^*_P = .0001469606$.

Step 3b) Compute Critical Region slopes:

$M_1 = \Phi^*_{P1} / (P_1 - P) = .641555303 / (.661814875 - .7079724)$

$= -13.89925702$.

$M_2 = -\Theta^*_P / (P_1 - P) = -.0001469606 / (.661814875 - .7079724)$

$= .003183879$.

3c) Compute Lagrangian multiplier estimates, Φ^*_{est} and Θ^*_{est}, using:

$\Phi^*_{est} = M_1 (P_2 - P_1) + \Phi^*_{P1} = -13.89925702 (.694483702 - .661814875)$

$+ .641555303 = .18748288$ and

$\Theta^*_{est} = M_2 (P_2 - P) + \Theta^*_P = .003183879 (.694483702 - .7079724)$

$+ .0001469606 = .000104013$.

The resulting inventory activity obtained by utilizing these Lagrangian multiplier estimates, shown in Table A10-5, suggests a 1,854,839 cubic inch space constraint level for period six of this horizon.

f) Fall 1990

Step 1) Determine the reduction factors P_1 and P_2:

$P_1 = W / \Sigma W_iQ_i = 1,854,839 / 1,007,181 = 1.849$
$P_2 = B / \Sigma b_iQ_i = 500 / 440.3 = 1.13$.

Since both reduction factors are greater than one stop. Inventory during this period not constrained. The resulting inventory activity is shown in Table A10-6.

A10.3 Projected Inventory Activity

Table A10-1 through A10-6 display the projected inventory activity resulting from the incorporation of the proposed Dual Constraint algorithm into step 4 of the EOQ based inventory policy described in appendix 9. In considering these tables note that the effective demand, Eff Demand, represents the period's demand not met by the non-safety stock on-hand at the end of previous period. The Lost Sales represent those historical demands not met during each period.

Item #	Effect Demand	Init Safe Stk	Size of Order	Total	Final On	Lost Sales
1	1.0	2.0	3.0	5.0	4.0	
2	0.0	2.0	0.0	2.0	2.0	
3	4.0	2.0	5.0	7.0	3.0	
4	5.0	2.0	5.0	7.0	2.0	
5	5.0	2.0	5.0	7.0	2.0	
6	1.0	2.0	2.0	4.0	3.0	
7	1.0	2.0	2.0	4.0	3.0	
8	8.0	4.0	6.0	10.0	2.0	
9	25.0	10.0	10.0	20.0	0.0	-5
10	8.0	2.0	5.0	7.0	0.0	-1
11	4.0	2.0	4.0	6.0	2.0	
12	0.0	2.0	0.0	2.0	2.0	
13	0.0	2.0	0.0	2.0	2.0	
14	1.0	2.0	3.0	5.0	4.0	
15	4.0	2.0	5.0	7.0	3.0	
16	3.0	2.0	4.0	6.0	3.0	
17	5.0	2.0	5.0	7.0	2.0	
18	0.0	2.0	0.0	2.0	2.0	
19	0.0	2.0	0.0	2.0	2.0	
20	0.0	2.0	0.0	2.0	2.0	
21	0.0	10.0	0.0	10.0	10.0	
22	0.0	10.0	0.0	10.0	10.0	
23	19.0	10.0	17.0	27.0	8.0	
24	0.0	10.0	0.0	10.0	10.0	
25	1.0	1.0	2.0	3.0	2.0	
26	3.0	1.0	4.0	5.0	2.0	
27	3.0	1.0	4.0	5.0	2.0	
28	3.0	1.0	4.0	5.0	2.0	
29	7.0	1.0	5.0	6.0	0.0	-1
30	11.0	4.0	7.0	11.0	0.0	
31	7.0	1.0	5.0	6.0	0.0	-1
32	13.0	4.0	7.0	11.0	0.0	-2
Total Space Requirement			2132793.3	3324732.6	766403.3	
Total Budget Requirement			484.9	836.7	281.0	

Table A10-1 (Results of Dual Constraint Algorithm - Spring 88)

Item #	Eff Demand	Init Safe Stk	Size of Order	Total	Final On	Lost Sales
1	0.0	4.0	0.0	4.0	2.0	
2	0.0	2.0	0.0	2.0	2.0	
3	4.0	3.0	6.0	9.0	4.0	
4	3.0	2.0	5.0	7.0	4.0	
5	4.0	2.0	6.0	8.0	4.0	
6	0.0	3.0	0.0	3.0	2.0	
7	0.0	3.0	0.0	3.0	3.0	
8	16.0	2.0	12.0	16.0	0.0	
9	25.0	0.0	15.0	25.0	0.0	
10	5.0	0.0	6.0	8.0	3.0	
11	4.0	2.0	6.0	8.0	4.0	
12	0.0	2.0	0.0	2.0	2.0	
13	0.0	2.0	0.0	2.0	2.0	
14	0.0	4.0	0.0	4.0	3.0	
15	4.0	3.0	7.0	10.0	5.0	
16	4.0	3.0	6.0	9.0	3.0	
17	3.0	2.0	6.0	8.0	5.0	
18	0.0	2.0	0.0	2.0	2.0	
19	0.0	2.0	0.0	2.0	2.0	
20	0.0	2.0	0.0	2.0	2.0	
21	0.0	10.0	0.0	10.0	10.0	
22	6.0	10.0	10.0	20.0	14.0	
23	12.0	8.0	13.0	23.0	11.0	
24	10.0	10.0	11.0	21.0	11.0	
25	0.0	2.0	0.0	2.0	2.0	
26	0.0	2.0	0.0	2.0	2.0	
27	0.0	2.0	0.0	2.0	2.0	
28	0.0	2.0	0.0	2.0	2.0	
29	0.0	0.0	0.0	1.0	1.0	
30	0.0	0.0	0.0	4.0	4.0	
31	0.0	0.0	0.0	1.0	1.0	
32	0.0	0.0	0.0	4.0	4.0	
Total Space Requirement			1434643.8	2814331.1	1261061.1	
Total Budget Requirement			415.1	814.7	380.8	

Table 10-2 (Results of Dual Constraint Algorithm - Fall 88 Quantity)

Item #	Eff Demand	Init Safe Stk	Size of Order	Total	Final On	Lost Sales
1	4	2.0	4.0	6.0	2.0	
2	1	2.0	2.0	4.0	3.0	
3	2	4.0	3.0	7.0	3.0	
4	1	4.0	2.0	6.0	3.0	
5	7	4.0	4.0	8.0	0.0	-1
6	3	2.0	3.0	5.0	2.0	
7	0	3.0	0.0	3.0	2.0	
8	13	0.0	6.0	10.0	0.0	-3
9	35	0.0	10.0	20.0	0.0	-10
10	4	3.0	3.0	6.0	1.0	-15
11	0	4.0	0.0	4.0	3.0	
12	1	2.0	2.0	4.0	3.0	
13	1	2.0	2.0	4.0	3.0	
14	2	3.0	3.0	6.0	3.0	
15	0	5.0	0.0	5.0	2.0	
16	[illegible]	3.0	4.0	7.0	2.0	
17	4	5.0	3.0	8.0	1.0	
18	1	2.0	4.0	6.0	5.0	
19	2	2.0	5.0	7.0	5.0	
20	2	2.0	5.0	7.0	5.0	
21	5	10.0	8.0	18.0	13.0	
22	5	14.0	7.0	21.0	12.0	
23	22	11.0	13.0	24.0	1.0	
24	16	11.0	10.0	21.0	4.0	
25	1	2.0	2.0	4.0	2.0	
26	1	2.0	2.0	4.0	2.0	
27	1	2.0	2.0	4.0	2.0	
28	1	2.0	2.0	4.0	2.0	
29	14	1.0	6.0	7.0	0.0	-7
30	16	4.0	6.0	10.0	0.0	-6
31	8	1.0	5.0	6.0	0.0	-2
32	16	4.0	7.0	11.0	0.0	-5
Total Space Requirements			1776542.5	3330245.1	673924.5	
Total Budget Requirements			485.6	963.3	266.7	

Table 10-3 (Results of Dual Constraint Algorithm - Spring 1989)

Item #	Eff Demand	Init Safe Stk	Size of Order	Total	Final On	Lost Sales
1	2.0	2.0	4.0	6.0	4.0	
2	0.0	3.0	0.0	3.0	3.0	
3	1.0	3.0	3.0	6.0	4.0	
4	4.0	3.0	5.0	8.0	3.0	
5	7.0	0.0	7.0	9.0	2.0	
6	4.0	2.0	5.0	8.0	4.0	
7	1.0	2.0	2.0	5.0	4.0	
8	11.0	0.0	8.0	12.0	1.0	
9	15.0	0.0	9.0	19.0	4.0	
10	3.0	1.0	4.0	6.0	3.0	
11	1.0	3.0	2.0	5.0	3.0	
12	1.0	3.0	3.0	6.0	4.0	
13	1.0	3.0	3.0	6.0	4.0	
14	1.0	3.0	3.0	6.0	4.0	
15	1.0	2.0	3.0	5.0	1.0	
16	7.0	2.0	7.0	9.0	2.0	
17	8.0	1.0	8.0	10.0	2.0	
18	1.0	5.0	3.0	8.0	4.0	
19	1.0	5.0	5.0	10.0	6.0	
20	1.0	5.0	4.0	9.0	5.0	
21	1.0	13.0	4.0	17.0	13.0	
22	5.0	12.0	8.0	20.0	13.0	
23	20.0	1.0	15.0	25.0	5.0	
24	13.0	4.0	10.0	20.0	7.0	
25	0.0	2.0	0.0	2.0	2.0	
26	0.0	2.0	0.0	2.0	2.0	
27	0.0	2.0	0.0	2.0	2.0	
28	0.0	2.0	0.0	2.0	2.0	
29	0.0	0.0	0.0	1.0	1.0	
30	0.0	0.0	0.0	4.0	4.0	
31	0.0	0.0	0.0	1.0	1.0	
32	0.0	0.0	0.0	4.0	4.0	
Total Budget Requirements			501.1	996.2	445.0	
Total Space Requirements			1301980.3	2736061.8	1320329.6	

Table 10-4 (Results of Dual Constraint Algorithm - Spring 1989)

Item #	Eff Demand	Init Safe Stk	Size of Order	Total	Final On	Lost Sales
1	0.0	2.0	0.0	4.0	3.0	
2	0.0	3.0	0.0	3.0	3.0	
3	1.0	4.0	2.0	6.0	3.0	
4	1.0	3.0	2.0	5.0	3.0	
5	6.0	2.0	5.0	7.0	1.0	
6	2.0	4.0	3.0	7.0	3.0	
7	0.0	4.0	0.0	4.0	3.0	
8	5.0	1.0	4.0	8.0	3.0	
9	15.0	4.0	8.0	18.0	3.0	
10	3.0	3.0	3.0	6.0	2.0	
11	1.0	3.0	2.0	5.0	3.0	
12	0.0	4.0	0.0	4.0	3.0	
13	0.0	4.0	0.0	4.0	3.0	
14	0.0	4.0	0.0	4.0	3.0	
15	4.0	1.0	5.0	7.0	3.0	
16	5.0	2.0	5.0	7.0	2.0	
17	4.0	2.0	4.0	6.0	2.0	
18	1.0	4.0	3.0	7.0	4.0	
19	3.0	6.0	7.0	13.0	6.0	
20	3.0	5.0	6.0	11.0	5.0	
21	2.0	13.0	5.0	18.0	13.0	
22	3.0	13.0	5.0	18.0	12.0	
23	24.0	5.0	14.0	24.0	0.0	
24	11.0	7.0	8.0	18.0	7.0	
25	2.0	2.0	3.0	5.0	2.0	
26	2.0	2.0	3.0	5.0	2.0	
27	2.0	2.0	3.0	5.0	2.0	
28	1.0	2.0	2.0	4.0	2.0	
29	4.0	1.0	4.0	5.0	1.0	
30	17.0	4.0	9.0	13.0	0.0	-4
31	3.0	1.0	4.0	5.0	2.0	
32	9.0	4.0	6.0	10.0	1.0	
Total Space Requirements			1780724.4	3327465.8	1047028.6	
Total Budget Requirements			493.7	1044.2	388.5	

Table 10-5 (Result of Dual Constraint Algorithm - Fall 1989)

Item #	Eff Demand	Init Safe Stk	Size of Order	Total	Final On	Lost Sales
1	2	3.0	4.0	7.0	4.0	
2	0	3.0	0.0	3.0	3.0	
3	0	3.0	0.0	3.0	2.0	
4	0	3.0	0.0	3.0	2.0	
5	0	1.0	0.0	2.0	2.0	
6	0	3.0	0.0	3.0	2.0	
7	0	2.0	0.0	3.0	2.0	
8	7	3.0	7.0	11.0	4.0	
9	6	3.0	7.0	17.0	11.0	
10	2	2.0	4.0	6.0	4.0	
11	4	3.0	5.0	8.0	3.0	
12	1	3.0	3.0	6.0	4.0	
13	1	3.0	3.0	6.0	4.0	
14	3	3.0	6.0	9.0	5.0	
15	2	3.0	4.0	7.0	4.0	
16	6	2.0	8.0	10.0	4.0	
17	6	2.0	8.0	10.0	4.0	
18	0	4.0	0.0	4.0	3.0	
19	7	6.0	14.0	20.0	9.0	
20	1	5.0	5.0	10.0	6.0	
21	3	13.0	7.0	20.0	14.0	
22	12	12.0	13.0	25.0	11.0	
23	22	0.0	17.0	27.0	5.0	
24	5	7.0	7.0	17.0	12.0	
25	0	2.0	0.0	2.0	2.0	
26	0	2.0	0.0	2.0	2.0	
27	0	2.0	0.0	2.0	2.0	
28	0	2.0	0.0	2.0	2.0	
29	0	1.0	0.0	1.0	1.0	
30	0	0.0	0.0	4.0	4.0	
31	0	1.0	0.0	2.0	2.0	
32	0	1.0	0.0	4.0	4.0	
Total Space Requirements			1000856.8	2479635.8	1568215.5	
Total Budget Requirements			437.1	956.2	559.4	

Table 10-6 (Results of Dual Constraint Algorithm - Fall 1989)

Vita

Billy Mike Maloney was born July 27, 1954, in Marshall, Texas. After attending public schools in Texas, he received the following degrees: B.S. in Engineering from The United States Military Academy, West Point, New York (1978); M.S. in Industrial Engineering from the University of Missouri at Columbia (1987); Ph.D. in Industrial Engineering from the University of Missouri at Columbia (1992). Married to the former Betty F. Hinkle, Texas, he presently hold the rank of Major in the United States Army.