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ORBIT PERTURBATIONS DUE TO AN AXI-SYMMETRIC GRAVITATIONAL FIELD, ANALYZED OVER EXTENDED PERIODS OF TIME

by

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SUMMARY

The author's untruncated orbital theory for perturbations due to the Earth's zonal harmonics, previously developed to cover all terms associated with J_2^2 and J_l, is being extended to cover the secular and long-periodic effects associated with J_2^3 and $J_2 J_l$. If J_2 and J_l $(l > 2)$ are regarded as first-order and second-order respectively, this means that formal third-order errors will no longer build up to second order over a timescale of order up to $1/J_2$ in angular measure. The extended theory has been partially checked out for an Earth model involving just J_2 and J_3.

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R. H. Gooding*

The author's untruncated orbital theory for perturbations due to the Earth's zonal harmonics, previously developed to cover all terms associated with J_2^2 and J_l, is being extended to cover the secular and long-periodic effects associated with J_2^3 and J_2J_l. If J_2 and J_l ($l > 2$) are regarded as first-order and second-order respectively, this means that formal third-order errors will no longer build up to second order over a timescale of order up to $1/J_2$ in angular measure. The extended theory has been partially checked out for an Earth model involving just J_2 and J_3.

INTRODUCTION

The author's earlier work[1] on eccentricity-untruncated perturbations due to the zonal harmonics J_2 and J_3 was subsequently[2] generalized to J_l and presented at the 1989 Astrodynamics Conference, more detailed versions[3,4] of Refs. 1 and 2 also being available. Refs. 3 and 4 were conceived as Parts 1 and 2 of a trilogy, Part 3 being envisaged as the full account of the work that is introduced in the present paper.

The main feature of the author's approach has been the particular way in which $\delta\zeta$, the short-period perturbation in the osculating element ζ (generic for the usual elements a, e, i, Ω, ω and M), is separated from the mean element $\bar{\zeta}$, where $\zeta = \bar{\zeta} + \delta\zeta$. The $\bar{\zeta}$, which have long-periodic as well as secular variation, are defined in such a way that the remnant $\delta\zeta$ can be combined into compact and non-singular perturbations $(\delta r, \delta b, \delta w)$ in a particular system of spherical polar coordinates (with $r = \bar{r} + \delta r$ etc), together with the corresponding perturbations in velocity $(\delta\dot{r}, \delta\dot{b}, \delta\dot{w})$; analysis for the general J_l leads[2] to unique definitions for the quasi-constants implicitly present in the $\bar{\zeta}$, the 'constant' for $\bar{a}$ being felicitously such as to make the mean semi-major axis exactly constant and identical with the quantity a' defined[1-6] such that $-\mu/2a'$ is the energy integral. Thus the theory of Refs. 1 and 2 may be regarded as having two components: one provides an algorithm for the variation of the $\bar{\zeta}$, thereby constituting an integration of formulae derived for the $\dot{\bar{\zeta}}$; the other component provides formulae for $\delta r, \delta b, \delta w$ (together with $\delta\dot{r}, \delta\dot{b}, \delta\dot{w}$). A recent paper[5] (presented at the 1990 Astrodynamics Conference) showed how the theory can be given logical completeness by three particular extensions: first (somewhat academically) the formulae can be generalized to cover the tesseral harmonic J_{lm}, so long as the Earth's rotation is ignored; second, the formulae for conceptual J_l

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(and by further extension, J_{lm}) with negative l can be obtained (with an application to lunisolar perturbations); third, the Fortran-implemented algorithm can be extended to cover hyperbolic orbits.

Unlike Ref. 2, the present paper is mainly concerned with the first component of the theory – the $\bar{\zeta}$ variation. In regard to the second component, it is noted that the $\delta\zeta$, and hence also δr, δb and δw, are essentially Poisson series in the generic angle $k\omega' + jv$, where k and j are integers, $\omega' = \omega - \frac{1}{2}\pi$ and v is true anomaly; they are obtained by integration of Lagrange's planetary equations with $\bar{v}$ as integration variable, terms with $j = 0$ being introduced (as required) as the integration 'constants'. Terms with $j = 0$ *before* the integration define the $\dot{\bar{\zeta}}$, though they actually arise as $d\bar{\zeta}/d\bar{v}$, there being no fundamental distinction between long-periodic variation ($k \neq 0$), occurring for all ζ other than a, and secular variation ($k = 0$), occurring for Ω, ω and M only (since the planetary equations involve $\sin(k\omega' + jv)$ for a, e and i, but $\cos(k\omega' + jv)$ for Ω, ω and M). To propagate the $\bar{\zeta}$, special action is required to cope with the obvious singularities associated with $e = 0$ and $\sin i = 0$, and the more esoteric ones associated with $\sin^2 i = 0.8$ (the critical inclinations); but[1] this action can effectively be taken at the subroutine-coding level and does not involve the abandonment of classical elements.

The existing theory has been regarded as 'formally complete to second order', relative to J_2, on the basis that $J_l \approx J_2^2$ for $l > 2$ and that all terms with coefficient J_2^2 are included[1,3]. The implication is that the errors are of third order, but this is only true, in practice, for orbital arcs limited to the order of one radian. Over a timescale with $nt = O(1/J_2)$, in fact, where n is the mean motion (and $n^2a^3 = \mu$), the modelled first- and second-order variations in $\bar{\zeta}$ build up to zero- and first-order effects respectively, and the unmodelled (formal) third-order variation leads to second-order error. For accurate modelling over quite modest periods of time, therefore, there is a need to incorporate the secular and long-periodic perturbations that are formally of third order, i.e. terms that have J_2^3 or J_2J_l as a coefficient. The third-order perturbations for $\bar{\Omega}$ and $\bar{\omega}$ would involve terms that are quadratic in t, induced by the second-order long-periodic variation in e and i, and effectively leading to first-order effects over a timescale with $nt = O(1/J_2)$, were it not for the inclusion[1] of the appropriate 'induced components' within the second-order theory. It is instructive to understand why induced terms were required for $\bar{\Omega}$ and $\bar{\omega}$ but not for $\bar{M}$: they originate from the first-order (secular) formulae for $\dot{\bar{\Omega}}$ and $\dot{\bar{\omega}}$ (see equation (7)), but the corresponding formula for the perturbation in $\bar{M}$ (for J_2 only[6]) happens to be null; to first order, in fact, $\dot{\bar{M}} = n'$ (where $n'^2a'^3 = \mu$).

The derivation of the secular and long-period components of the third-order variation, associated with coefficients J_2^3 and J_2J_l (for $l = 3$ in particular), is the subject of the present paper, the technique employed being essentially the same as was used[3] for all the perturbations associated with J_2^2. A complete set of formulae has not been obtained, however, and success has been limited for reasons that will emerge.

As in Ref. 2, it is convenient to conclude the introduction with some remarks on notation. The quasi-elements ψ, ρ and L are useful, as before, where $\dot{\psi} = \dot{\omega} + c\dot{\Omega}$ (for

$c = \cos i$, and we also write $s = \sin i$), $\dot{\rho} = \dot{\sigma} + q\dot{\psi}$ (where σ is such that $M = \sigma + \int$, $\int$ being shorthand for $\int n\,dt$, and $q^2 = 1 - e^2$) and $\dot{L} = \dot{M} + q\dot{\psi}$. For conciseness we also write f for s^2, g for $1 - \frac{5}{4}f$ and h for $1 - \frac{3}{2}f$. The use of p $(= aq^2)$ and u $(= v + \omega)$ is normal, but we also define $\gamma = pc^2$, $P = p/r$ $(= 1 + e\cos v)$ and $u' = u - \frac{1}{2}\pi$. As the paper is almost entirely concerned with *mean* elements, we again omit the bars on the right-hand sides of equations whenever this is possible without causing confusion. Finally, we follow Ref. 2 in the use of C_j^k and S_j^k for the cosine and sine of $jv + ku'$, sometimes suppressing the superfix, and we introduce γ_k and σ_k as concise versions of C_{-k}^k and S_{-k}^k, that is, as $\cos k\omega'$ and $\sin k\omega'$ respectively. (NB: this notation should not be confused with the other meanings of γ and σ.)

FURTHER BACKGROUND

Following Ref. 2 we express the potential due to J_l as $U_l = \sum U_l^k$ with

$$U_l^k = -K_l(\mu/p)\,A_{lk}\,P^{l+1}\cos ku', \tag{1}$$

where $K_l = J_l(R/p)^l$, R being the Earth's equatorial radius, and A_{lk} is an inclination function (which in Ref. 2 subsumed K_l, but separation is preferable now); here $l \geq 2$ but it will often be assumed that $l \neq 2$; further, $\frac{3}{2}K_2$ will (as in previous papers) be expressed as K. The index k has the same parity as l and $0 \leq k \leq l$, but two remarks apply here. First, though A_{lk} is only defined when l and k have the same parity, a parallel function, $A_{l\kappa}$, was found useful when the short-periodic quantity δb was analyzed, where κ has the opposite parity to l; Ref. 2 relates both the A_{lk} and the $A_{l\kappa}$ to a single family of inclination functions, the $A_l^k(i)$, where k now has *either* parity. Second, though the restriction to non-negative k may seem natural (and avoids the duplication of terms in various formulae), it leads to an unnatural factor of 2 in the definition of A_{lk} when $k > 0$, and in the extension to tesseral harmonics[5] a distinction between positive and negative values is essential; in the present analysis we effectively allow for negative k, with $A_{lk} = A_{l,-k}$, but it is only for $k = -1$ that such quantities figure in the final results. In addition to A_{lk}, we need A'_{lk} and A''_{lk}, the derivatives with respect to i, though it is more convenient to replace A'_{lk} by $A^{\cdot}_{lk} = c^{-1}s\,A'_{lk}$. We will often suppress the first suffix in all these functions, writing A_k etc.

To facilitate the substitution of the appropriate partial derivative of U_l^k in each planetary equation, we introduce the eccentricity functions B_{lj}, defined by

$$P^{l-1} \equiv (1 + e\cos v)^{l-1} = \sum B_{lj}\cos jv, \tag{2}$$

where overwhelming advantage derives from allowing negative j, with $B_{lj} = B_{l,-j}$. The series effectively runs from $-\infty$ to $+\infty$, with $B_{lj} = 0$ when $|j| \geq l$. We also require the derivatives, B'_{lj}, and (as with the A functions) we will often suppress the first suffix; a useful formula for B'_{lj}, not given in previous papers, is

$$B'_{lj} = \tfrac{1}{2}(l-1)(B_{l-1,j+1} + B_{l-1,j-1}). \tag{3}$$

In view of the frequent requirement for $B_{l-1,j}$ in the analysis for J_l we write this in abbreviated form as $B_{\cdot,j}$.

The B_{lj} are defined with P^{l-1} in equation (2), though it is P^{l+1} that occurs in (1), in anticipation of the change of variable in the planetary equations, the relation between the variables being

$$\frac{d\bar{v}}{dt} = \bar{n}\,\bar{q}^{-3}\,\bar{p}^2 + O(J_2^2, J_l)\,, \tag{4}$$

where $l > 2$ since the absence of a term in J_2 has already been remarked. After the change of variable, the integration for a formal second-order solution proceeds entirely straightforwardly, as far as the terms in J_l in the solution are concerned (these terms being first-order in the second-order quantity J_l), and fairly straightforwardly (though laboriously) for the terms in J_2^2. The short-periodic terms constitute the $\delta\zeta$, from which δr, δb and δw are derived, and are only of interest here in that they are needed to feed back for the third-order solution. The remaining terms constitute the secular and long-periodic perturbations, and in Ref. 2 their combination was expressed via formulae for the rates of change $\dot{\bar{\zeta}}$. However, it must not be overlooked that the 'averaged' parts of the planetary equations, after the change of variable, correspond to integrated quantities that are linear in $\bar{v}$ rather than $\bar{M}$, so that there are additional perturbations terms, short-periodic in nature, given by $(\dot{\bar{\zeta}}/\bar{n})(\bar{v} - \bar{M})$. The resulting difficulty was handled in Refs. 1 and 3 by the concept of 'semi-mean' elements, $\tilde{\zeta}$, that can be obtained from the $\bar{\zeta}$ by incorporation of these additional terms.

Three of the general $\dot{\bar{\zeta}}_{lk}$ formulae (associated with U_l^{+}) are repeated from Refs. 2 and 4 (with bars on the right-hand sides omitted for the first time). These formulae, in particular, are required in the sequel, and the first two formulae may be expressed via a single equation. Thus

$$-\bar{e}\,\bar{c}\,\dot{\bar{e}}_{lk} = \bar{q}^2\,\bar{s}\,\dot{\bar{i}}_{lk} = k\,K_l\,n\,q^2\,c\,A_k B_k\,\sigma_k \tag{5}$$

and

$$\dot{\bar{M}}_{lk} = K_l\,n\,e^{-1}\,q^3\,A_k\,B_k'\,\gamma_k\;; \tag{6}$$

for $\dot{\bar{L}}_{lk}$, as given in Ref. 2, we replace $e^{-1}\,q^3\,B_k'$, in (6), by $-(2l-1)\,q\,B_k$.

The complete secular and long-periodic solution associated with J_2^2 (from Refs. 1 and 3) is also repeated, as the formulae involved are special cases of the formulae to be associated with J_2J_l: they are 'special', rather than 'particular', as the general formulae give double the correct results on setting $l = 2$, since (as will become clearer) they cover J_lJ_2 as well as J_2J_l! As the general formulae will be expressed in terms of KK_l, rather than K_2K_l, we express the J_2^2 formulae, somewhat artificially, in terms of KK_2 (where $K = \frac{3}{2}K_2$); the well-known first-order formulae may be written

$$\dot{\bar{\Omega}}_{2,0} = -K\,n\,c \qquad \text{and} \qquad \dot{\bar{\omega}}_{2,0} = 2\,K\,n\,g\,. \tag{7}$$

As the notation $\dot{\bar{\zeta}}_{2lk}$ will be used to refer to the terms in γ_k or σ_k that emerge in the solution for J_2J_l, we express the secular part of the known solution for J_2^2 via the three formulae

$$\dot{\bar{\Omega}}_{2,2,0} = -\tfrac{1}{16} KK_2 \, n \, c \, [4(1-f) + e^2(4+5f)] \, , \tag{8}$$

$$\dot{\bar{\omega}}_{2,2,0} = \tfrac{1}{64} KK_2 \, n \, [2(64 - 180f + 95f^2) + e^2(56 - 36f - 45f^2)] \tag{9}$$

and

$$\dot{\bar{M}}_{2,2,0} = -\tfrac{1}{32} KK_2 \, n \, q^3(8 - 8f - 5f^2) \, . \tag{10}$$

It is possible to dispense with equation (10) by modifying Kepler's third law for mean elements[1,6], but the 'additional perturbation' (term in $\bar{\nu} - \bar{M}$) resulting from $\dot{\bar{M}}_{2,2,0}$ will still be required. The formulae from which the long-period perturbations may be derived are

$$\bar{e}\,\bar{c}\;\dot{\bar{e}}_{2,2,2} = -\bar{q}^2\,\bar{s}\;\dot{\bar{i}}_{2,2,2} = \tfrac{1}{16} KK_2 \, n \, e^2 q^2 \, c \, f\,(14-15f)\sigma_2 \, , \tag{11}$$

$$\dot{\bar{\Omega}}_{2,2,2} = \tfrac{1}{8} KK_2 \, n \, e^2 \, c(7 - 15f)\gamma_2 \, , \tag{12}$$

$$\dot{\bar{\Psi}}_{2,2,2} = \tfrac{1}{32} KK_2 \, n \, (2 + 5e^2) f\,(14 - 15f)\gamma_2 \tag{13}$$

and

$$\dot{\bar{L}}_{2,2,2} = \tfrac{7}{32} KK_2 \, n \, e^2 \, q\, f(14 - 15f)\gamma_2 \, . \tag{14}$$

The short-period perturbations will not be given again here, but two errors in Ref. 1 should be noted: the factors $\bar{c}$ and $\bar{s}$ were omitted from its equation (43), the formula for δb; and the terms with overall factor $4\bar{f}$ in its equation (44), the formula for δw, should all be negated (the wrong sign was attached).

To amplify the remarks in the Introduction concerning 'induced terms', it can now be stated that these terms arise from the variation of $\bar{e}$ and $\bar{i}$ (via $\bar{K}$, $\bar{c}$ and $\bar{g}$) in equations (7), the equations for this variation being (5) and (11).

$J_2 J_l$ PERTURBATIONS

The analysis for each element, ζ, is separated into two distinct parts or 'halves', which lead to the two components of the final formula for $\dot{\bar{\zeta}}_{2lk}$. In the first half we substitute the first-order solution associated with J_2, involving terms in $\bar{K}$, in the planetary equation associated with J_l, whilst in the second half we substitute the existing solution associated with J_l in the equation associated with J_2. (The analysis also covers $l = 2$, though there is then only one set of terms, which is why this case is special rather than particular.) The index k is associated with J_l, not J_2, so for J_2 it is best not to think of any separation between '$k = 0$' and '$k = 2$'; but it is also best not to think of k as irrevocably associated with U_{lk}, in the second half-analysis, as k has to be redefined during the development.

We start by considering general aspects of just the first half-analysis. In principle, every element has a first-order representation, such as $\Omega = \bar{\Omega} + \Delta\Omega$, where $\Delta\Omega$ consists of $\delta\Omega$ and a secular term, both having $\bar{K}$ as a factor, and these representations can be substituted in the planetary equation for ζ due to J_l. Because of the axial symmetry, however, Ω is actually the one element that does not occur in the formula for $\dot{\zeta}$ and for which substitution is therefore not required; thus only for ω is the secular component relevant, and this is taken account of separately, as we shall see, so we can revert to the notation typified by $\delta\Omega$. The contribution to $\dot{\bar{\zeta}}_{2lk}$ arising from $\delta\Omega$ we conceptually denote by D_Ω, but the form taken by the planetary equations makes it preferable to work with contributions D_p, D_P, D_u, D_i, D_e and D_v, derived from the perturbations δp etc that are combinations of the (first-order) perturbations in the elements (so that 'D_e' is not the same as the D_e that would arise if we stayed entirely with the elements themselves). When ζ is i or Ω, we do not need D_e and D_v, and the other four D contributions are based on the following formulae taken directly from Ref. 3:

$$\delta p = -\tfrac{1}{3}Kp\,[f(eC_1^2 + 3C_0^2 + 3eC_{-1}^2) - 2h]\,, \tag{15}$$

$$\delta P = -\tfrac{1}{12}KP\,[f(3eC_1^2 + 10C_0^2 + 11eC_{-1}^2) - 4h(e\cos v + 3)]\,, \tag{16}$$

$$\delta u = \tfrac{1}{12}K\,[2e(1-f)S_1^2 + (6-7f)S_0^2 + 2e(3-5f)S_{-1}^2 + 4e(5-6f)\sin v] \tag{17}$$

and

$$\delta i = -\tfrac{1}{6}Kcs(eC_1^2 + 3C_0^2 + 3eC_{-1}^2 - 3)\,. \tag{18}$$

The 'constant' in equation (15) is $2h$, rather than $3f$ (which might be expected in view of (18)), because the constants in δa, δe and δi are such that $\delta\gamma = \frac{2}{3}k\gamma(1-3f)$ as opposed to being identically zero (where $\gamma = pc^2$).

In regard to the secular component of $\Delta\omega$, this makes no *direct* contribution to $\dot{\bar{\zeta}}_{2lk}$, since its effect is taken into account at the first-order level by the evaluation of short-period perturbations *after* the mean elements have been propagated, so that $\bar{\omega}$ has its value at time t. There is an *indirect* contribution that must not be overlooked, however, arising from the expression for the 'constant' $\delta\zeta_{lk(c)}$, the complete set of these expressions being given in Ref. 2. Allowance has to be made for the fact that $\delta\zeta_{lk(c)}$ is not really constant, specifically (when $k \neq 0$) because of the variation of $k\,\bar{\omega}$. To counteract the effect of this variation (at third order) we incorporate a contribution to $\dot{\bar{\zeta}}_{2lk}$ in which $\bar{\gamma}_k$ or $\bar{\sigma}_k$ (in $\delta\zeta_{lk(c)}$) is replaced by $k\,\dot{\bar{\omega}}_{2,0}\,\bar{\sigma}_k$ or $-k\,\dot{\bar{\omega}}_{2,0}\,\bar{\gamma}_k$ respectively, where $\dot{\bar{\omega}}_{2,0}$ is given by equation (7). Essentially the same principle would apply to the short-period perturbations in a complete third-order analysis, with the terms in C_j^k or S_j^k (occurring in the $\delta\zeta$ due to J_l) leading to terms in $k\,\dot{\bar{\omega}}_{2,0}\,S_j^k$ or $-k\,\dot{\bar{\omega}}_{2,0}\,C_j^k$ respectively: this principle was used, with a detailed explanation, in the $J_2{}^2$ analysis of Ref. 3.

In the second half-analysis the rôles of J_2 and J_l are in principle reversed, but expressions for the general δp (more precisely δp_{lk}) etc are not so simple as equations (15)-(18), owing in part to the particular form taken by each constant; each expression may be derived from the equations of Ref. 2, however. The expression for δp is

$$\delta p = -K_l\, A_{lk}\, p \left\{ 2k \sum_{j \neq k} (j+k)^{-1} B_j C_j + [(l+1)B_k - (l-1)B_{-,k}]C_{-k} \right\}. \quad (19)$$

The expression for δP may then be derived from its identity with $(P/p)(\delta p - P\delta r)$, where $\delta r\ (= \delta r_{lk})$ is given by equation (61) of Ref. 2. Similarly, δu is given by $\delta w - c\, \delta\Omega$, where δw is given by equation (78) of Ref. 2, together with the special terms given by (85)-(88), and $\delta\Omega$ may be derived from equations (38) and (84).

Finally (because there has been no time to take the general analysis beyond $\zeta = \Omega$), δi may similarly be derived from equations (49) and (83) of Ref. 2 the result being

$$\delta i = -K_l\, c\, s^{-1} \left\{ k\, A_{lk} \sum_{j \neq k} (j+k)^{-1} B_j C_j - \tfrac{1}{2} A'_{lk} B_k\, C_{-k} \right\}. \quad (20)$$

As in the first half-analysis, we must also correct for the fact that the first-order 'constants' are not really constant, because of effects arising from the long-period variation in e, i and ω, as well as the secular variation in ω. The relevant 'constants' may be taken from equations (148), (149), (151), (153) and (164) of Ref. 3, being $-\frac{1}{24}Ke(9f\gamma_2 - 20h)$ for e, $\frac{1}{2}Kcs$ for i, zero for Ω, $\frac{3}{8}Kf\sigma_2$ for ω, and $-\frac{3}{8}Kqf\sigma_2$ for M. There is an important further source of terms in $\dot{\zeta}_{2lk}$, however, which only appears in the second half-analysis and which we must deal with before considering the analysis for the individual elements.

The terms referred to stem from the $O(J_l)$ component of equation (4), which complicates the change of integration variable from t to $\bar{v}$, a further complication being that it is really $\tilde{v}$, the semi-mean equivalent of $\bar{v}$, that is the new variable. Now $\tilde{v}$ is derived from $\tilde{e}$ and $\tilde{M}$ by the usual Keplerian procedure, so that

$$\frac{d\tilde{v}}{dt} = \frac{\partial \tilde{v}}{\partial \tilde{e}} \frac{d\tilde{e}}{dt} + \frac{\partial \tilde{v}}{\partial \tilde{M}} \frac{d\tilde{M}}{dt}, \quad (21)$$

where $\quad \partial v/\partial e = q^{-2} \sin v\, (2 + e\cos v), \quad \partial v/\partial M = q^{-3} (1 + e \cos v)^2. \quad (22)$

Also, $\dot{\zeta} = \bar{q}^{-3}\, \bar{P}^2\, \dot{\zeta}$, whilst $\dot{e}$ and $\dot{M}$ (residual to n') are given (for the assumed restriction to U_{lk}) by equations (5) and (6), so (21) leads to

$$\frac{d\tilde{v}}{dt} = n\, q^{-3} P^2 \left\{ 1 - K_l\, A_{lk}\, e^{-1} [k\, B_{lk} \sin v\, (1+P)\, \sigma_k - B'_{lk}\, P^2\, \gamma_k] \right\}. \quad (23)$$

It is the terms in K_l that specify the $O(J_l)$ component of equation (4). We invert (23) to obtain $dt/d\tilde{v}$ and thence the modified formula for changing the integration variable. On expanding the terms in K_l, we get the factor that must be applied to each second-half planetary equation as a source of additional terms in $\dot{\zeta}_{2lk}$. The factor may be expressed as

$$-\tfrac{1}{4}\left(q^{-3}P^{2}\right)^{-1} K_l\, e^{-1} A_k\,[(eB'_k + kB_k)(eC_{-k+2} + 4C_{-k+1}) + 2B'_k\,(2+e^2)C_{-k} + (eB'_k - kB_k)\,(4C_{-k-1} + eC_{-k-2})]\,. \quad (24)$$

When ζ is Ω or ω, there is a final source of terms, consideration of which will be postponed until we come to the second half of the analysis for Ω.

We now proceed to the analysis for individual ζ, none being required for a. We start with i, for which the analysis is the most straightforward and from which the result for e can also be derived. At the time of writing, the only other analysis completed is for Ω.

Analysis for Inclination

The planetary equation for the first half of the analysis is

$$di/dt \;=\; k\,K_l\;n\,q^{-3}\,c\,s^{-1}\,A_{lk}\,P^{l+1}\,S_0^k\,, \qquad (25)$$

where the right-hand side is exact as it stands, the elements being osculating not mean. Since $n\,q^{-3} = (\mu/p^3)^{1/2}$, the right-hand side is a function of p, P, u and i only, with p implicit in K_l as well as $n\,q^{-3}$. Hence we obtain the contributions to $\dot{\bar{i}}_{2lk}$ denoted by D_p, D_P, D_u and D_i and expressed in terms of δp etc from equations (15)-(18). In each contribution (after differentiating (25) with respect to p, P, u or i as appropriate) we effectively replace $q^{-3}P^{l+1}\,S_0^k$ by $\Sigma\,B_{lj}\;S_j^k$ on making use of equations (2) and (4). Thus D_p is given by the terms of $-k\,(l+\frac{3}{2})\,K_l\,n\,p^{-1}\,c\,s^{-1}\,A_k\,(\Sigma\,B_j\,S_j)\,\delta p$ that are free of short-periodic variation, that is, by

$$D_p \;=\; \tfrac{1}{12}k(2l+3)\,KK_l\,c\,s^{-1}\,A_k\,\{f(eB_{k+3} + 3B_{k+2} + 3eB_{k+1})\sigma_{k+2} - 4hB_k\sigma_k \;+ f(3eB_{k-1} + 3B_{k-2} + eB_{k-3})\sigma_{k-2}\}\,, \quad (26)$$

on completion of the algebra.

Formulae for D_P, D_u and D_i may be obtained in the same way from, respectively, the terms of $k(l+1)K_l\,n\,c\,s^{-1}\,P^{-1}\,A_k\,(\Sigma B_j\,S_j)\,\delta P$, $k^2K_l\,n\,c\,s^{-1}A_k\,(\Sigma B_j\,C_j)\,\delta u$ and $kK_l\,n\,(A'_k - f^{-1}A_k)(\Sigma\,B_j\,S_j)\delta i$, but they are not listed here. (The coefficient of δi takes this form because the derivative of $c\,s^{-1}A_k$ is $c\,s^{-1}A'_k - f^{-1}A_k$.) Finally, the effect of $\delta i_{lk(c)}$ is available at once from equation (20), on interpreting C_{-k} as γ_k and replacing it by $2k\,K\,n\,g\,\sigma_k$ in accord with the rationale that has been noted.

We now have to combine five results, of which one is D_p as given by equation (26). We seek a final formula with σ_k common to all terms, and there is no inherent difficulty in this since the four contributing formulae effectively apply for *all* k. Thus we can redefine k (as forecast) such that the quantity $kA_k\,B_{k+3}\sigma_{k+2}$ in (26), for example, is replaced by $(k-2)\,A_{k-2}B_{k+1}\sigma_k$. At this point we can conveniently introduce some notation that will simplify the presentation of results generally, not just for the element i.

Since terms (in each half-analysis, and for all elements) arise that involve $A_{k+2}B_{k+1}$, $A_{k+2}B_k$, $A_{k+2}B_{k-1}$, A_kB_{k+1}, A_kB_k, A_kB_{k-1}, $A_{k-2}B_{k+1}$, $A_{k-2}B_k$ and $A_{k-2}B_{k-1}$, we denote (arbitrary) coefficients of these nine quantities by λ_{++}, λ_{+o}, λ_{+-}, λ_{o+}, λ_{oo}, λ_{o-}, λ_{-+}, λ_{-o}, and λ_{--}, respectively. We also require, for some elements, similar coefficients for $\grave{A}_{k+2}B_{k+1}$ etc and for $A''_{k+2}B_{k+1}$ etc, and we denote these by μ_{++} etc and ν_{++} etc, respectively. Finally, we may need the coefficient of $A_kB_{-,k}$, and we denote this by $\lambda_{(-)}$. We now introduce the generic quantity Q, defined by

$$Q = \lambda_{++}A_{k+2}B_{k+1} + \ldots + \mu_{++}\grave{A}_{k+2}B_{k+1} + \ldots + \nu_{++}A''_{k+2}B_{k+1} + \ldots + \lambda_{(-)}A_kB_{-,k}, \quad (27)$$

to permit formulae to be expressed mainly in terms of the particular Q that is appropriate.

In terms of the foregoing notation we shall eventually be able to write (with both halves of the analysis covered) the formula we seek as

$$\dot{\bar{i}}_{2lk} = \tfrac{1}{24}k\,K K_l\,n\,c\,s^{-1}Q_i\,\sigma_k. \quad (28)$$

For the first half only, on combining equation (26) with four other results, we require

$$\begin{aligned}
(\lambda_{++},\lambda_{+o},\lambda_{+-}) &= -k^{-1}(k+2)\{e[6(k+1)-f(l+10k+27)],\\
&\qquad [6(k+1)-f(2l+7k+22)],\ e[2(k+1)-f(l+2k+7)]\},\\
(\lambda_{o+},\lambda_{oo},\lambda_{o-}) &= 2\{e[2(l+5k+1)-3f(l+4k+1)],\ 2[2l-3-3lf],\\
&\qquad e[2(l-5k+1)-3f(l-4k+1)]\},\\
(\lambda_{-+},\lambda_{-o},\lambda_{--}) &= k^{-1}(k-2)\{e[2(k-1)+f(l-2k+7)],\\
&\qquad [6(k-1)+f(2l-7k+22)],\ e[6(k-1)+f(l-10k+27)]\},\\
(\mu_{++},\mu_{+o},\mu_{+-}) &= -2k^{-1}(k+2)(1-f)(3e,3,e),\\
(\mu_{o+},\mu_{oo},\mu_{o-}) &= [0,6(6-7f),0],\ (\mu_{-+},\mu_{-o},\mu_{--}) = -2k^{-1}(k-2)(1-f)(e,3,3e)
\end{aligned}$$

and all $\nu = \lambda_{(-)} = 0$.

The factor k^{-1}, appearing in most of these expressions, is eliminated when we combine with the second-half results.

The planetary equation for the second half of the analysis is

$$di/dt = K\,n\,q^{-3}\,c\,s\,P^3\sin 2u', \quad (29)$$

which leads to the contributions D_p, D_P, D_u and D_i given by the short-period-free terms of $-\frac{7}{2}K\,n\,p^{-1}\,c\,s\,P\,S_0^2\,\delta p$, $3\,K\,n\,c\,s\,S_0^2\,\delta P$, $2\,K\,n\,c\,s\;P\,C_0^2\,\delta u$ and $K\,n\,(1-2f)\,P\,S_0^2\,\delta i$; δp and δi, here, are given by equations (19) and (20). The product $P\,S_0^2$ (and similarly $P\,C_0^2$) expands to $\frac{1}{2}(eS_1^2 + 2S_0^2 + eS_{-1}^2)$, the expansion being appropriate to the δP contribution, as well as the other three, since the factor P can be extracted from δP. But $\delta p, \delta P, \delta u$ and δi involve terms in C_j^k and S_j^k, the coefficients of which can be developed as combinations of $A_{lk}B_{lj}, A_{lk}B_{l-1,j}$ and $\grave{A}_{lk}B_{lj}$ (for particular values of j related to k), after which a term such as $A_{lk}B_{lj}S_1^2C_j^k$ can be re-expressed as $\frac{1}{2}A_kB_j\ (S_{j+1}^{k+2} - S_{j-1}^{k-2})$; extraction of the short-period-free portion and redefinition of k then lead to $\frac{1}{2}\,(A_{k\text{-}2}\,B_{k+1} - A_{k+2}\,B_{k\text{-}1})\;\sigma_k$ (since $B_{\cdot j} = B_{\text{-}j}$) for this term.

The full second-half analysis for D_p, D_P, D_u and D_i generates considerable algebra, which will be omitted, leading to a result that (as for the first half) can be expressed via the λ's and μ's. There would also be terms in $A_{k+2}\,B_{\text{-},k+1}$ and $A_{k\text{-}2}\,B_{\text{-},k\text{-}1}$, were it not that these are cancelled by terms we are about to derive. Following the general remarks on J_2J_l perturbations, we have two additional sources of terms to incorporate with the second-half contributions to $\bar{i}_{2lp}$. First, the first-order 'constant', viz $\frac{1}{2}K\,c\,s$, for i is responsible for contributions arising from the long-periodic variation of e and i, as given by equation (5) – the variation of ω is irrelevant as ω does not appear in this particular constant. These contributions combine to $-\frac{1}{2}k\,KK_l\,n\,c\,s^{-1}\,(1-6f)\,A_k\,B_k\,\sigma_k$.

The other additional source is the one associated with expression (24) that only arises in the second half-analysis. On multiplying this expression by the right-hand side of equation (29), and then picking out the short-period-free terms, we get

$$\frac{3}{8}KK_l\,n\,c\,s\,A_k\,[(kB_k + eB_k')\,\sigma_{k\text{-}2} + (kB_k - eB_k')\,\sigma_{k+2}]\,.$$

On redefinition of k and simplification via equation (3), this leads to $\frac{3}{8}(l-1)\,KK_l\,n\,e\,c\,s$ $(A_{k+2}\,B_{\text{-},k+1} - A_{k\text{-}2}\,B_{\text{-},k\text{-}1})\sigma_k$, which involves the cancelling terms that have been alluded to.

The second-half values of the λ's and μ's are now available. On combining them with the first-half values, we have the final solution for $\bar{i}_{2lk}$, expressed via equations (28) and (27). The final formulae for the λ's and μ's are as follows:

$$\begin{aligned}(\lambda_{++},\lambda_{+0},\lambda_{+-}) &= -\{e[6(k+2)-f(l+10k+21)],\ [6(k+2)-f(2l+7k+16)],\\ &\qquad e[2(k+2)-f(l+2k+5)]\}\,, \qquad (30a)\end{aligned}$$

$$\begin{aligned}(\lambda_{0+},\lambda_{00},\lambda_{0-}) &= 2\{e[2(l+5k+1)-3f(l+4k+1)],\ 2[2(l-3)-3f(l-6)],\\ &\qquad e[2(l-5k+1)-3f(l-4k+1)]\}\,, \qquad (30b)\end{aligned}$$

$$\begin{aligned}(\lambda_{-+},\lambda_{-0},\lambda_{--}) &= \{e[2(k-2)+f(l-2k+5)],\ [6(k-2)+f(2l-7k+16)],\\ &\qquad e[6(k-2)+f(l-10k+21)]\}\,, \qquad (30c)\end{aligned}$$

$$\begin{aligned}(\mu_{++},\mu_{+0},\mu_{+-}) &= -2(1-f)\,(3e,3,e)\,, \quad \mu_{00} = 6(6-7f)\,,\\ (\mu_{-+},\mu_{-0},\mu_{--}) &= -2(1-f)(e,3,3e)\,. \qquad (30d)\end{aligned}$$

Results for particular J_l may be derived by substituting in equations (30). For $l = 2$ we have a special situation, as already noted, such that the correct result, given by equation (11), is produced by either half-analysis on its own and equations (30) give double this result. But this remark posits an important point that arises for all even l: the value of $A_{l,0}$, to be used as $A_{k\text{-}2}$ when $k = 2$, must be doubled, to include the 'unnatural factor of 2' (see the Introduction) that normally applies only to A_{lk} with $k \neq 0$; this allows for the restriction of results to $k \geq 0$ and the consequent absence of $A_{l,0}$ in the rôle of A_{k+2} for $k = -2$. (This doubling does not apply to the use of $A_{l,0}$ as A_k for $k = 0$, but this is irrelevant for $\zeta = i$, since $\sigma_0 = 0$; the factor of 2 present in $A_{l,2}$ makes it unnecessary to include $A_{l,-2}$.)

Finally, we substitute for $l = 3$ in equations (30), with $k = 3$ and 1, so that (27) and (28) provide results for J_3, but first we note a point for odd l in general. It is that for $k = 1$ we require $A_{l,3}$, $A_{l,1}$ and $A_{l,-1}$; although $A_{l,-1} = A_{l,1}$, we cannot just combine the two, because the corresponding λ's will be quite different. As the pair of formulae for J_3 are new, we express them with σ_3 and σ_1 replaced by $\cos 3\omega$ and $-\cos\omega$, respectively. Then

$$\left.\begin{aligned} \dot{\bar{i}}_{2,3,3} &= -\tfrac{5}{64} KK_3\, n\, e^3 c f\, (20 - 21f) \cos 3\omega \\ \dot{\bar{i}}_{2,3,1} &= \tfrac{1}{32} KK_3\, n\, e\, c\, \left[2(8 - 15f + 5f^2) + 5e^2 f\, (6 - 7f)\right] \cos\omega \end{aligned}\right\} . \tag{31}$$

These formulae were originally derived by a specific analysis for J_2J_3, and have been checked out by inclusion in the author's Fortran test program.

Analysis for Eccentricity

The analysis for $\dot{\bar{e}}_{2lk}$ would be considerably more complicated than for $\dot{\bar{i}}_{2lk}$ if we proceeded *ab initio* in the same way, but there is a much shorter route that makes use of the exact constancy of the osculating quantity $\gamma (= pc^2)$, which is a measure of the angular momentum about the axis of symmetry. Now

$$d\gamma/dt = c\,(q^2 \dot{a} - 2a\,e\,c\;\dot{e} - 2a\,q^2 s\;\dot{i}\;) , \tag{32}$$

so we can get $\dot{\bar{e}}_{2lk}$ from

$$\dot{\bar{e}}_{2lk} = -e^{-1} q^2 c^{-1} s\; [\,\dot{\bar{i}}_{2lk} + (2p\,c\,s)^{-1}\,\dot{\bar{\gamma}}_{2lk}\,] . \tag{33}$$

It remains, therefore, to do the two half-analyses for the mean quantity $\bar{\gamma}$, the variation of which stems entirely from the general non-zero 'constant' $\delta\gamma_{lk(c)}$, to be derived for the first half, and the specific value $\frac{2}{3}K\gamma(1 - 3f)$, given after equation (18) and required for the second half.

From the expressions for $\delta a_{lk(c)}$, $\delta e_{lk(c)}$ and $\delta i_{lk(c)}$ in Ref. 2, we can derive

$$\delta\gamma_{lk(c)} = -K_l\gamma\;\{A_k\,[\,(l+1)B_k - (l-1)B_{-k}] + \dot{A_k}\,B_k\}\,\gamma_k , \tag{34}$$

from which the first-half contribution to γ_{2lk} is obtained on replacing γ_k by $2kKng\;\sigma_k$. From the value $\frac{2}{3}K\,\gamma(1 - 3f)$, similarly, using equation (5), we derive the second-half contribution such that in combination with the effect of (34) we get

$$\dot{\bar{\gamma}}_{2lk} = -\tfrac{1}{6}k\,KK_l\,n\,\gamma\{[4(3l-7)-3f\,(5l-19)]\,A_k B_k + 12g\,[A_k^{\cdot} B_k-(l-1)\,A_k B_{\cdot k}\}\,\sigma_k\,. \tag{35}$$

Now equation (35) can be interpreted via values of λ_{oo}, μ_{oo} and $\lambda_{(\cdot)}$. If it were not for these, the required values of the λ's and μ's could all be taken as identical with the values resulting from the i analysis since, in view of equations (28) and (33), we naturally write

$$\dot{\bar{e}}_{2lk} = -\tfrac{1}{24}k\,KK_l\,n\;e^{-1}\,q^2\,Q_e\,\sigma_k\,. \tag{36}$$

Thus the only values, to be substituted in (27), that are different from those given by equation (30) are

$$\left.\begin{aligned}\lambda_{oo} &= -2[8(l-2)-3f\,(3l-7)]\,,\\ \mu_{oo} &= 12(1-f) \quad\text{and}\quad \lambda_{(\cdot)} = 24(l-1)\,g\,.\end{aligned}\right\} \tag{37}$$

For $l=2$ we get the result (doubled) given by equation (11), whilst for $l=3$ we obtain the new formulae

$$\left.\begin{aligned}\dot{\bar{e}}_{2,3,3} &= \tfrac{5}{64}\,KK_3\,n\,e^2q^2\,s\,f\,(20-21f)\cos 3\omega\\ \dot{\bar{e}}_{2,3,1} &= \tfrac{1}{32}\,KK_3\,n\,q^2\,s\;[2(8-41f+40f^2)-5e^2f\;(6-7f)]\cos\omega\end{aligned}\right\}. \tag{38}$$

As with the results for i, given by (31), these formulae were originally derived by a specific analysis and the Fortran test program has shown them to be correct.

Analysis for Nodal Right Ascension

For Ω we proceed as for i, with a complication in the second half of the analysis. The planetary equation for the first half is

$$d\Omega/dt = -K_l\,n\,q^{-3}\,s^{-1}\,A_{lk}'\;P^{l+1}\,C_0^k\,. \tag{39}$$

Formulae for D_p, D_P, D_u and D_i can then be obtained from the short-period-free terms of $(l+\tfrac{3}{2})K_l\,n\,p^{-1}\,s^{-1}\,A_k'\;(\Sigma\,B_jC_j)\,\delta p$, $-(l+1)\,K_l\,n\,s^{-1}\,P^{-1}A_k'\;(\Sigma\,B_jC_j)\,\delta P$, $k\,K_l\,n\,s^{-1}\,A_k'\;(\Sigma\,B_jS_j)\,\delta u$ and $-K_l\,n\,s^{-1}\,(A_k''-c\,s^{-1}A_k')\,(\Sigma\,B_jC_j)\,\delta i$. Finally, the effect of $\delta\Omega_{lk(c)}$, given by equation (84) of Ref. 2, is $-k^2\,KK_l\,n\,c\,s^{-2}\,g\;A_kB_k\,\gamma_k$.

We express the final formula for $\dot{\bar{\Omega}}_{2lk}$ as

$$\dot{\bar{\Omega}}_{2lk} = \tfrac{1}{24}KK_l\,n\,c\,f^{-1}\,Q_\Omega\,\gamma_k\,, \tag{40}$$

with Q_Ω to be taken from the appropriate version of equation (27); the appearance of $c\,f^{-1}$, rather than s^{-1}, reflects the usage of $A_k^{\cdot}$ rather than A_k'. On combining all the first-half contributions, and after the algebra involved in the redefinition of k etc, we find we require

$(\mu_{++},\mu_{+0},\mu_{+-}) = \{e[6(k+1)-f(l+10k+21)], [6(k+1)-f(2l+7k+16)], e[2(k+1)-f(l+2k+5)]\}$, $(\mu_{0+},\mu_{00},\mu_{0-}) = -2\{e[2(l+5k+1)-3f(l+4k+1)], 2[(2l-3)-3f(l-1)], e[2(l-5k+1)-3f(l-4k+1)]\}$, $(\mu_{-+},\mu_{00},\mu_{--}) = -\{e[2(k-1)+f(l-2k+5)], [6(k-1)+f(2l-7k+16)], e[6(k-1)+f(l-10k+21)]\}$,

$(\nu_{++},\nu_{+0},\nu_{+-}) = 2f(3e,3,e)$, $(\nu_{0+},\nu_{00},\nu_{0-}) = (0,-12f,0)$, $(\nu_{-+},\nu_{-0},\nu_{--}) = 2f(e,3,3e)$, $\lambda_{00} = -24k^2 g$ and all other $\lambda = 0$.

The planetary equation for the second half of the analysis is

$$\delta\Omega/dt = -K\,n\,q^{-3}\,c\,P^3\,(\cos 2u' + 1)\,, \tag{41}$$

and this leads to D_p, D_P, D_u and D_i given by the short-period-free terms of $\frac{7}{2}K\,n\,p^{-1}\,c\,P\,(C_0^2+1)\delta p$, $-3K\,n\,c\,(C_0^2+1)\delta P$, $2Kn\,c\,P\,S_0^2\,\delta u$ and $Kn\,c\,P\,(C_0^2+1)\delta i$, to be combined via the redefinition of k as usual (laborious detail omitted). Since the first-order constant is zero for Ω, there is no additional contribution associated with equation (5). We do have the source associated with equation (24), however; on applying this to (41) and picking out the short-period-free terms, we get

$$\tfrac{1}{8}KK_l\,n\,e^{-1}c\,A_k\,[3e(eB_k' + kB_k)\gamma_{k-2} + 4(2+3e^2)B_k'\gamma_k + 3e(eB_k' - kB_k)\gamma_{k+2}]\,.$$

The preceding expression introduces an apparent singularity, since the coefficient of γ_k involves a non-cancelling factor e^{-1} outside the square brackets. There is a final contribution to the second-half $\dot{\Omega}_{2lk}$, however, which did not arise for i or e.

This final contribution derives from the first-order secular variation given by $\dot{\Omega}_{2,0}$ in equation (7), with a similar effect arising in the second half-analysis when $\zeta = \omega$ and deriving from $\dot{\omega}_{2,0}$. This quantity is a function of $\bar{e}$ (via $\bar{K}$), $\bar{i}$ (via $\bar{c}$) and $\bar{n}$. The long-periodic variations of $\bar{e}$ and $\bar{i}$ were already accounted for (as has been noted) in the second-order theory, via the so-called 'induced components', errors that are $O(KK_l\,n^2\,t^2)$ being thereby avoided. Now the false assumption that $\bar{n} = n'$ only involves $O(KK_l\,n\,t)$ effects; these were not previously relevant but are of precisely the order covered in the present paper.

The point is that $\dot{\Omega}_{2,0}$ originates in the term $-\bar{K}\bar{c}$ in $d\bar{\Omega}/d\bar{v}$, and this integrates to $-\bar{K}\bar{c}\,\bar{v}$, which is handled as the secular term $-\bar{K}\bar{c}\,n't$, with $-\bar{K}\bar{c}\,(\bar{v}-\bar{M})$ as a separate short-periodic effect. But $-\bar{K}\bar{c}\,\bar{v}$ has an $O(\bar{K}_l)$ component deriving from equations (5) and (6) via (21) and (22). Equations (5) and (21) lead to no long-term effect, but (6) and (22) lead to our required 'final contribution' of $-\frac{1}{2}KK_l\,n\,e^{-1}\,c\,(2+e^2)\,A_k\,B_k'\,\gamma_k$ on taking the mean value of $q^{-3}P^2$ to be $\frac{1}{2}q^{-3}(2+e^2)$. In combination with the preceding contribution, this effectively changes the coefficient of $B_k'\,\gamma_k$ (within the square brackets) from $4(2+3e^2)$ to $8e^2$, thus eliminating the apparent singularity.

It remains to combine all the contributions in the second half-analysis, to redefine k, and to combine with the results of the first half-analysis. In terms of equations (40) and (27) we finally derive:

$$(\lambda_{++},\lambda_{+o},\lambda_{+-}) = 2f\{e(l-8k-21),\ -(2l+3k+16),\ -e(3l+5)\}\,, \tag{42a}$$

$$(\lambda_{o+},\lambda_{oo},\lambda_{o-}) = -6\{ef(l-5k-2),\ [4k^2-f(5k^2+4l-8)],\ ef(l+5k-2)\}\,, \tag{42b}$$

$$(\lambda_{-+},\lambda_{-o},\lambda_{--}) = -2f\{e(3l+5),\ (2l-3k+16),\ -e(l+8k-21)\}\,, \tag{42c}$$

$$(\mu_{++},\mu_{+o},\mu_{+-}) = \{e[6(k+3)-f(l+10k+33)],\ [6(k+3)-f(2l+7k+28)],\ e[2(k+3)-f(l+2k+9)]\}\,, \tag{42d}$$

$$(\mu_{o+},\mu_{oo},\mu_{o-}) = -2\{e[2(l+5k+1)-3f(l+4k+1)],\ 2(2l-3-3fl),\ e[2(l-5k+1)-3f(l-4k+1)]\}\,, \tag{42e}$$

$$(\mu_{-+},\mu_{-o},\mu_{--}) = -\{e[2(k-3)+f(l-2k+9)],\ [6(k-3)+f(2l-7k+28)],\ e[6(k-3)+f(l-10k+33)]\}\,, \tag{42f}$$

$$(\nu_{++},\nu_{+o},\nu_{+-},\nu_{o+},\nu_{oo},\nu_{o-},\nu_{-+},\nu_{-o},\nu_{--}) = 2f(3e,3,e,0,-6,0,e,3,3e) \tag{42g}$$

and $\lambda_{(-)} = 12(l-1)f(4+3e^2)\,.$ (42h)

For $l=2$, equations (42), substituted in (27) and (40), give the results (doubled) known from equations (8) and (12), with $k=0$ and $k=2$ respectively; for $k=2$, as already noted, we require the values of $A_{2,0}, A'_{2,0}$ and $A''_{2,0}$ to be doubled, whilst for $k=0$ we use these values undoubled and do not require $A_{2,-2}$ etc. For $l=3$, with $k=3$ and $k=1$, we get the new results given by

$$\left.\begin{aligned} \dot{\Omega}_{2,3,3} &= -\tfrac{25}{64}\,KK_3\,n\,e^3\,c\,s\,(4+7f)\sin 3\omega \\ \dot{\Omega}_{2,3,1} &= \tfrac{1}{32}\,KK_3\,n\,e\,c\,s^{-1}\,[2(8+99f-185f^2)+5e^2f(18-35f)]\sin\omega \end{aligned}\right\}\,. \tag{43}$$

Equation (43) conforms with the results of the specific analysis originally carried out, but is unfortunately not validated by the test program. There are effectively seven numerical coefficients in the equations and computer runs for different values of e and i indicate that five of them are correct. The dubious integers are 99 and −185, with some evidence that the correct values might be 51 and −125, which would imply an error, in the square brackets of $\dot{\Omega}_{2,3,1}$, proportional to fg. At the time of writing, it is not known whether the error is a real one or an artefact of the algorithm used in the test program. (The two leading coefficients in equations (43), contributing $4\sin 3\omega$ and $16\sin\omega$ apart from the overall factors at the beginning of the expressions, could actually have been written down without any analysis, since they are mandated by the corresponding coefficients in equations (31), to avoid singularity.)

Analysis for Perigee Argument and Mean Anomaly

The general analysis for the last two elements has not been completed at the time of writing. Specific results for J_2J_3 have been derived, but (as for Ω) they are not validated by the test program and the formulae will not be quoted. (The formulae for $\dot{\Psi}_{2,3,3}$ and

$\dot{\bar{\psi}}_{2,3,1}$, from which $\dot{\bar{\omega}}_{2,3,3}$ and $\dot{\bar{\omega}}_{2,3,1}$ follow, involve four and nine numerical coefficients, respectively; two of the four, and three of the nine, are certainly correct from their mandatory association with the coefficients appearing in equation (38).) In the present paper we just remark on some aspects of the general analysis.

The analysis for the quasi-element ψ is distinctly simpler than for ω itself, since the planetary equation is simpler for $\dot{\psi}$ than for $\dot{\omega}(= \dot{\psi} - c\dot{\Omega})$ and the analysis for $\dot{\Omega}$ has already been done. We must allow for the variation of c ($= \cos i$), but this merely involves the incorporation of contributions given by $s\dot{\Omega}\,\delta i$, where $\dot{\Omega}$ as given by (39) and δi by (18) are appropriate for the first half-analysis, and similarly for the second. The analysis is necessarily more laborious than for i and Ω, however, as shown by the planetary equation for the first half-analysis, which (cf equations (25) and (39)) is

$$d\psi/dt = -\tfrac{1}{4}K_l\, n\, q^{-3} A_{lk} P^{l+1} \left\{ (l^+ + k)\, C_2^k + 2(l^+ + 2k)e^{-1} C_1^k + 2l^+ C_0^k + 2(l^+ - 2k)e^{-1} C_{-1}^k + (l^+ - k)\, C_{-2}^k \right\}, \quad (44)$$

where $l^+ = l + 1$ (cf equation (39) of Ref. 2).

Equation (44) leads to the usual quantities denoted by D_p, D_P, D_u and D_i, but we now also need D_e and D_v; the former is associated with the two appearances of e^{-1} in (44), and the latter with the implicit appearance of v in C_j^k, which becomes explicit (if $j \neq 0$) when we write C_j^k in full. The expressions for δe and δv (see equations (148) and (182) of Ref. 3) are more complicated than the four quantities given by equations (15)-(18), so we cannot expect that, after redefinition of k in the analysis, all the coefficients of γ_k in $\dot{\bar{\psi}}_{2lk}$ can be related to just B_{k+1}, B_k and B_{k-1}; the number of B's required in each half-analysis in fact rises from three to eleven, but on combination of the two halves it reduces to three again. This avoids the occurrence of non-zero $\dot{\bar{\psi}}_{2lk}$ with $k > l$.

Analysis for the sixth element in practice involves the quasi-element L rather than M, and this in turn involves the planetary equation for the quasi-element ρ rather than σ, where $\dot{\sigma} = \dot{\rho} - q\,\dot{\psi}$. The analysis for ρ proceeds in the same way as for ψ, allowance being made for the variation of q in $q\,\dot{\psi}$, and then a final analysis is required for the integral of n, the quantity defined such that $\dot{L} = \dot{\rho} + \dot{\int}$. The results for $\dot{\bar{\rho}}_{2lk}$ and $\dot{\bar{\int}}_{2lk}$, individually, do not reduce to zero for $k = l + 2$, even when the pairs of half-analysis are combined, but on combining these combinations we do get results for $\dot{\bar{L}}_{2lk}$ in our standard form.

Now $\int = n't + \int \delta n\, dt$ and we may conveniently write $\delta n = n'(K\hat{n} + K_l\hat{n}_l + KK_l\hat{n}_{2l})$ to the accuracy that concerns us (with additional terms in K^2 and K^3 when we come to J_2^3 perturbations). Expressing δa similarly, we require that $n^2a^3 = \mu = n'^2a'^3$, from which we obtain

$$\hat{n} = -\tfrac{3}{2}\hat{a}, \quad \hat{n}_l = -\tfrac{3}{2}\hat{a}_l, \quad \hat{n}_{2l} = -\tfrac{3}{2}\left(\hat{a}_{2l} - \tfrac{5}{2}\hat{a}\hat{a}_l\right). \quad (45)$$

The second term in the expression for $\hat{n}_{2l}$ may be regarded as an easily-evaluated correction term, so we consider only the main term. Thus we require to evaluate, as the principal part of $\dot{\bar{I}}_{2lk}$, the third-order component of $-(3n/2a)\,\delta a$. Changing the integration variable as usual, we have

$$dI/d\bar{v} = -(3/2a)\,q^3 P^{-2}\,\delta a\,, \tag{46}$$

so we need to express the required components of δa with P^2 as a factor.

We start from the exact equation

$$\delta a = -2a' K_l\, q^{-2} A_{lk}\, P^{l+1}\, C_0^k\,, \tag{47}$$

all quantities (other than a') on the right-hand side being osculating, not mean. When these quantities are expressed for the first-order solution, we can derive the usual first-half part of $\dot{\bar{I}}_{2lk}$, via the components D_p, D_P, D_u, D_i and D_e; the need for D_e stems from the factor q^{-2} in (47). There is no difficulty in either half of the analysis, but the details are omitted as the complementary analysis for $\dot{\bar{p}}_{2lk}$ has not been completed.

$J_2{}^3$ PERTURBATIONS

The analysis for $J_2{}^3$ perturbations follows the same principles as for J_2J_l perturbations, the second-order solution for ζ being fed back into the planetary equation. The analysis is more tedious, however, and there is no opportunity for cross-check between general and particular results. At the time of writing, formulae have been obtained for $\zeta = i, e$ and Ω, but none has been successfully validated by the test program.

Some revision of notation is called for, such that the suffix 3, in particular, implies a power of J_2 (or in practice $\bar{K}$) rather than a value of l. In general, we return to the notation of Ref. 3, writing the known short-periodic component of the second-order solution as ($\delta\zeta =$) $\bar{K}\zeta_1 + \bar{K}^2\zeta_2$, where ζ_1 and ζ_2 are Poisson series in $\cos jv$, C_j and Γ_j (or the corresponding sines), where (with bars suppressed) $C_j = \cos(jv + 2\omega)$ and $\Gamma_j = \cos(jv + 4\omega)$; in the earlier notation of this paper, C_j and Γ_j would be written $-C^2_{j-2}$ and C^4_{j-4}. (As in Ref. 3, the notations ζ_1 and ζ_2 are extended to any quantity that is a function of the orbital elements.) We seek formulae for the third-order mean rates of change, which will be expressed as $\dot{\bar{\zeta}}_{30}$ for the secular component and $\dot{\bar{\zeta}}_{32}$ for the long-periodic component; the latter notation is used because the long-period rate of change is proportional to $\cos 2\bar{\omega}$ or $\sin 2\bar{\omega}$, terms in $4\bar{\omega}$ being absent.

As with the J_2J_l analysis, it is helpful to express the planetary equations in terms of the quantities p, P, u, i, v and e, rather than the original elements; since the analysis has not been carried to $\zeta = \omega$ and $\zeta = M$, we only need p, P, u and i here. The notation D_p etc is no longer helpful, however, as we effectively require 14 basic contributions to $\dot{\bar{\zeta}}_{30}$ and $\dot{\bar{\zeta}}_{32}$: four associated with p_2, P_2, u_2 and i_2; four with $p_1{}^2, P_1{}^2, u_1{}^2$ and $i_1{}^2$;

and six with the cross-products $p_1 P_1$ etc. There are also contributions associated with the 'constants' in ζ_1 and ζ_2, and with the $O(J_2^2)$ part of equation (4).

Because p and P occur in the same way in four planetary equations ($\zeta = a$ and $\zeta = M$ are the exceptions), namely, via $K\,n\,q^{-3}\,P^2$, it is convenient to set up a grouping of the five basic contributions associated with p_2, P_2, p_1^2, P_1^2 and $p_1 P_1$. It is also convenient to introduce, for p and P only, a 'normalized notation' such that $\hat{p}_1 = p_1/p$ etc; this is non-trivial for P_1 and P_2, since P is itself a Poisson series, given by equation (2) with $l = 2$, by which P_1 and P_2 must be exactly divisible for $\hat{P}_1$ and $\hat{P}_2$ to have useful meanings. The formula for $\hat{P}_1$ is given at once by equation (187) of Ref. 3, which also includes all the other ζ_1 and ζ_2 that we need, an exception being P_2, which is given by evaluation of

$$P_2 = e_2 \cos v - e\, v_2 \sin v - \tfrac{1}{2} v_1 (2e_1 \sin v + e\, v_1 \cos v) . \qquad (48)$$

(It needs 38 separate cosine terms to express each term of (48), but most of these cancel out on combination; the remaining 18 then reduce to 13 when $P \equiv 1 + e \cos v$ is divided out – cf (51b) below – and this cancellation is responsible for the absence of $4\bar{\omega}$ terms in $\dot{\zeta}_{32}$.) We use the expansions (correct to second order)

$$(p/\bar{p})^{-7/2} = 1 - \tfrac{7}{2} \bar{K} \hat{p}_1 - \tfrac{7}{2} \bar{K}^2 (\hat{p}_2 - \tfrac{9}{4} \hat{p}_1^2) \qquad (49)$$

and

$$(P/\bar{P})^3 = 1 + 3\bar{K} \hat{P}_1 + 3\bar{K}^2 (\hat{P}_2 + \hat{P}_1^2) , \qquad (50)$$

from which it follows that the grouping of terms we require is given by

$$3(\hat{P}_2 + \hat{P}_1^2) - \tfrac{7}{2}(\hat{p}_2 - \tfrac{9}{4}\hat{p}_1^2 + 3\hat{p}_1 \hat{P}_1) ; \qquad (51a)$$

this leads to an expression of the form

$$\frac{1}{2304} \{ f^2(173e^2 \Gamma_6 + \ldots + 327e^2 \Gamma_2) - 4f \text{ (terms in } C_4, \ldots, C_0)$$
$$+ 16 \text{ (terms in } \cos 2v, \cos v \text{ and } 1)\} . \qquad (51b)$$

There are nine remaining 'basic contributions' to be covered, for each ζ separately: two lead to a particular combination of u_2 and u_1^2; two lead, similarly, to a combination of i_2 and i_1^2; one comes from $u_1 i_1$; and the last four lead to the product of an appropriate combination of u_1 and i_1 with the fixed combination of p_1 and P_1 given by

$$\tfrac{7}{2} \hat{p}_1 - 3\hat{P}_1 = \tfrac{1}{12} \{ f(5e\, C_3 + 12C_2 + 9e\, C_1) - 4h(3e \cos v + 2) \} . \qquad (52)$$

In regard to the contributions to $\dot{\zeta}_{30}$ and $\dot{\zeta}_{32}$ due to the non-constancy of the 'constants' in ζ_1 and ζ_2, we start with the ζ_1 constants, which were listed after equation (20). Each constant has a variation due to the variation in $\bar{e}$ (which affects the suppressed $\bar{K}$ in $\bar{K}\zeta$) and $\bar{I}$, as specified by equation (11), and cancellation of this variation is one source of the required contribution. Since the constants in ω_1 and M_1 (though not in i_1 or Ω_1) involve $\bar{\omega}$, we also have to cover the effects of $\bar{\omega}$ variation, with a complication associated with the first-order component of this variation, but these effects are not required in the incomplete analysis of the present paper. In dealing with the 'constant' part of ζ_2, on the other hand, it is only the variation of $\bar{\omega}$ that is relevant – it is dealt with in exactly the same way as in $\delta\zeta_{lk(c)}$ in the second half of the analysis of the J_2J_l perturbations.

To get the contribution to $\dot{\bar{\zeta}}_{30}$ and $\dot{\bar{\zeta}}_{32}$ due to the $O(J_2^2)$ part of equation (4), we apply equations (21) and (22) again. This time we use expressions for $\dot{\bar{e}}$ and $\dot{\bar{M}}$ given by equations (10), (11), (13) and (14). From $\dot{\bar{M}}_{2,2,0}$ we get the factor, analogous to (24), given by

$$\frac{1}{96} K^2 (8 - 8f - 5f^2) [e^2 \cos 2v + 4e \cos v + (2 + e^2)] , \qquad (53)$$

whilst from $\dot{\bar{e}}_{2,2,2}$ and $\dot{\bar{M}}_{2,2,2}$ we get

$$-\frac{1}{48} K^2 f (14 - 15f)[e^2 C_2 + 4eC_1 + (2 + e^2)C_0] . \qquad (54)$$

When ζ is Ω or ω, there is a final source of terms in $\dot{\bar{\zeta}}_{30}$ and $\dot{\bar{\zeta}}_{32}$, associated (just as in the second half of the J_2J_l analysis) with the interpretation of n in equations (7).

Analysis for Inclination

Lagrange's planetary equation has been given by (29), which can also be written as

$$di/dt = - \bar{K}(\bar{n}\,\bar{q}^{-3}\bar{P}^2)\,(p/\bar{p})^{-7/2}\,(P/\bar{P})^3\,\bar{P}\,c\,s\,S_2 . \qquad (55)$$

Then the change of variable, specified by equation (4), leads to

$$di/d\bar{v} = - \bar{K}(p/\bar{p})^{-7/2}\,(P/\bar{P})^3\,\bar{P}\,c\,s\,S_2 . \qquad (56)$$

We have developed $(p/\bar{p})^{-7/2}$ and $(P/\bar{P})^3$ via equations (49)-(51), so for the basic contributions to $\dot{\bar{i}}_{32}$ (there is no $\dot{\bar{i}}_{30}$) it remains to develop

$$c\,s = \bar{c}\,\bar{s} + \bar{K}(1 - 2\bar{f})i_1 + \bar{K}^2 [(1 - 2\bar{f})i_2 - 2\bar{c}\,\bar{s}\,i_1^2] \qquad (57)$$

and

$$S_2 = \bar{S}_2 + 2\bar{K}\,\bar{C}_2\,u_1 + 2\bar{K}^2 (\bar{C}_2\,u_2 - \bar{S}_2\,u_1^2) . \qquad (58)$$

We now work with equation (56) to derive all the contributions to $\dot{\bar{i}}_{32}$, expressing each contribution with the overall factor $(1/2304)\,K^3 n\,e^2\,c\,s\,\sin 2\omega$ suppressed.

From the grouped terms represented by equations (51), with the factor $-K^3 P\,c\,s\,S_2$ applied and the short-period-free component then picked out, we obtain $-3(448 - 1536f + 1279f^2)$. From the product of (52) with $K^3P\,[(1 - 2f)\,S_2\,i_1 + 2c\,s\,C_2\,u_1]$ we obtain $4(728 - 2169f + 1677f^2)$. From the product of $-K^3P$ with the combination of $2c\,s\,(C_2\,u_2 - S_2\,u_1^2)$, $S_2[(1 - 2f)i_2 - 2c\,s\,i_1^2]$ and $2(1 - 2f)\,C_2\,u_1\,i_1)$, we obtain $(5408 - 11724f + 5601f^2)$. And the combination of these three results, representing the sum of all the basic contributions, is $8(872 - 1974f + 1059f^2)$.

The 'constant' in i_1, viz $\frac{1}{2}c\,s$, leads (via $\dot{\bar{e}}_{2,2,2}$ and $\dot{\bar{i}}_{2,2,2}$) to $-48(14-99f+90f^2)$, and the C_0 component of the constant in i_2, viz[3] $-\frac{1}{72}e^2\,c\,s\,(9+f)C_0$, leads (via $\dot{\bar{\omega}}_{2,0}$) to $-32(36-41f-5f^2)$. And the combination of these two results is $-16(114-379f+260f^2)$.

Finally, the product of $-K\,P\,c\,s\,S_2$ with the factor given by (53) leads to $-36(8-8f-5f^2)$, there being no contribution from (54).

On combining all contributions we get

$$\dot{\bar{i}}_{32} = \frac{1}{576}K^3\,n\,e^2\,c\,s\,(1216-2360f+1123f^2)\sin 2\omega\,. \tag{59}$$

Analysis for Eccentricity

As in the analysis for J_2J_l perturbations, we derive $\dot{\bar{e}}_{32}$ from $\dot{\bar{i}}_{32}$ via the exact constancy of $\gamma(=pc^2)$. To apply the equation analogous to (33) we require the formula for $\dot{\bar{\gamma}}_{32}$; as noted in Ref. 3, $\bar{\gamma}$ has no first- or second-order variation.

The variation of $\bar{\gamma}$ stems entirely from the non-constancy of the 'constants' in γ_1 and γ_2. The constant in γ_1 is $\frac{2}{3}\gamma\,(1-3f)$, which leads (via $\dot{\bar{e}}_{2,2,2}$ and $\dot{\bar{i}}_{2,2,2}$) to the contribution

$$\frac{1}{18}K^3\,n\,e^2\gamma f\,(14-15f)(5-9f)\sin 2\omega$$

to $\dot{\bar{\gamma}}_{32}$. The C_0 component of the constant in γ_2 is $-\frac{1}{36}e^2\gamma f\,(63-82f)C_0$, which leads (via $\dot{\bar{\omega}}_{2,0}$) to the contribution

$$-\frac{1}{36}K^3 n\,e^2\gamma f\,(63-82f)(4-5f)\sin 2\omega\,.$$

So

$$\dot{\bar{\gamma}}_{32} = -\frac{1}{36}K^3\,n\,e^2\gamma f\,(112-241f+140f^2)\sin 2\omega\,. \tag{60}$$

From equations (33), (59) and (60) it now follows that

$$\dot{\bar{e}}_{32} = -\frac{1}{576}K^3\,n\,e\,q^2 f\,(320-432f+3f^2)\sin 2\omega\,. \tag{61}$$

Analysis for Nodal Right Ascension

We are now looking for secular variation, represented by $\dot{\bar{\Omega}}_{30}$, as well as the long-period variation represented by $\dot{\bar{\Omega}}_{32}$. The analysis is very similar to that for i, except for a final contribution that stems from $\dot{\bar{\Omega}}_{2,0}$ and the non-identity of n' and $\bar{n}$, just as with J_2J_l perturbations.

The planetary equation has been given by (41) and can also be written as

$$d\Omega/dt = \bar{K}(\bar{n}\,\bar{q}^{-3}\bar{P}^2)\,(p/\bar{p})^{-7/2}\,(P/\bar{P})^3\,\bar{P}\,c\,(C_2-1)\,, \tag{62}$$

whence

$$d\Omega/d\bar{v} \;=\; \bar{K}(p/\bar{p})^{-7/2}(P/\bar{P})^3\,\bar{P}\,c\,(C_2-1)\,. \tag{63}$$

For the factors involving $p/\bar{p}$ and $P/\bar{P}$ we can use the grouped terms represented by equations (51), whilst for $c\,(C_2-1)$ we require, in analogy with (57) and (58),

$$c \;=\; \bar{c}-\bar{K}\,\bar{s}\,i_1-\bar{K}^2\left(\bar{s}\,i_2+\tfrac{1}{2}\,\bar{c}\,u_1{}^2\right) \tag{64}$$

and

$$C_2-1 \;=\; \bar{C}_2-1-2\bar{K}\,\bar{S}_2\,u_1-2\bar{K}^2\left(\bar{S}_2\,u_2+\bar{C}_2\,u_1{}^2\right). \tag{65}$$

We now express the contributions to $\dot{\Omega}_{30}$ with the overall factor $(^1/_{576})\,K^3\,n\,c$ suppressed, and to $\dot{\Omega}_{32}$ with the factor $(^1/_{2304})\,K^3\,n\;e^2c\;\cos 2\omega$ suppressed.

From the grouped terms, with the factor $K^3\,P\,c(C_2-1)$ applied, we obtain

$$-[16(40-144f+113f^2)+e^2(576-2038f+1831f^2)] \text{ and } 3(448-2816f+2785f^2)\,.$$

From the product of (52) with $K^3\,P\,[s(C_2-1)i_1+2c\,S_2\,u_1]$ we obtain

$$8[4(6-13f+6f^2)+e^2(58-159f+86f^2)] \text{ and } -8(400-1107f+711f^2)\,.$$

From the product of $K^3\,P$ with the combinations of $-2c\,(S_2\,u_2+C_2\,u_1{}^2)$, $-(C_2-1)(si_2+\frac{1}{2}ci_1{}^2)$ and $2sS_2\,u_1\,i_1$, we obtain

$$-[16(24-41f+5f^2)+e^2(176-2942f+2955f^2)] \text{ and } -(6752-13416f+5907f^2)\,.$$

Hence the sum of all basic contribution yields (with the same overall factors assumed)

$$-2[8(52-159f+106f^2)+3e^2(48-618f+683f^2)] \text{ and } -8(1076-1728f+405f^2)\,.$$

The 'constant' in Ω_1 is zero, so there is no contribution to $\dot{\Omega}_{30}$ or $\dot{\Omega}_{32}$ via $\dot{e}_{2,2,2}$ or $\dot{i}_{2,2,2}$. But the constant in Ω_2 is $-\frac{1}{144}c\,e^2\,(18-19f)\,S_0$, and this leads (via $\dot{\bar{\omega}}_{2,0}$) to a contribution to $\dot{\Omega}_{32}$ of $16(72-166f+95f^2)$ (multiplied by the usual factor).

We treat, finally, the contributions to $\dot{\Omega}_{30}$ and $\dot{\Omega}_{32}$ arising from the product of $K\,P\,c(C_2-1)$ with the factors given by (53) and (54), together with the contributions associated with the fact that the factor n in equation (7) ought to be interpreted as $n'+\dot{M}_{2,2,0}+\dot{M}_{2,2,2}$, but is in practice set to just n'. The two pairs of contributions can conveniently be combined since (it can be shown that) the effect of the second pair is to delete the factor $2+e^2$ in (53) and (54). As a result we obtain

$$-18e^2f(14-15f) \text{ and } 72(8-8f+5f^2)\,.$$

On combining all contributions, we derive

$$\dot{\Omega}_{30} \;=\; -\tfrac{1}{144}\,K^3n\,c\,[4(52-159f+106f^2)+3e^2(24-288f+319f^2)] \tag{66}$$

and

$$\dot{\Omega}_{32} = -\frac{1}{144} K^3 n\, e^2 c\,(430 - 662f + 85f^2) \cos 2\omega . \tag{67}$$

There is a conflict, associated with singularity, between the leading coefficients in equations (59) and (67); this has not yet been resolved.

Analysis for Perigee Argument and Mean Anomaly

The analysis for the lat two elements has not been embarked upon at the time of writing.

DISCUSSION

An orbital theory is of no practical value until it has been implemented in an accurate and efficient computer program for ephemeris generation. The author's untruncated second-order theory[1-4] for an orbit in an axi-symmetric gravitational field, or any non-rotating field, had been fully validated by such a computer program, the main features of which are as follows. First, a mean-orbit-based coordinate system was utilized as being ideal for the efficient representation of the short-period perturbations free of singularity. Secondly, singularity problems in the propagation of the mean elements ($\bar{\zeta}$) were avoided by two expedients: two of the $\dot{\bar{\zeta}}$ formulae are for $\zeta = \psi$ and L, rather than ω and M, with $\bar{s}\dot{\bar{\Omega}}$ and $\bar{e}\dot{\bar{\psi}}$ stored (rather than $\dot{\bar{\Omega}}$ and $\dot{\bar{\psi}}$); and the non-singular mean elements $\bar{s} \sin \bar{\Omega}$, $-\bar{s} \cos \bar{\Omega}$ and $\bar{c}$ are introduced locally in the propagation of $\bar{i}$ and $\bar{\Omega}$, to complete the avoidance of any problem when $\bar{s} \approx 0$ (with a similar procedure that pays off when $\bar{e} \approx 0$). Thirdly, the concept of a semi-mean element was employed as a device associated with the transformation of the integration variable from t to $\bar{\nu}$ whilst the propagation of the $\bar{\zeta}$ is still required in terms of t. Fourthly, the terms in $\bar{\Omega}$ and $\bar{\omega}$ that are induced by the second-order terms in $\bar{e}$ and $\bar{i}$, and are formally only of third order but responsible for quadratic variation with t, were included in the program. Fifthly, and to supplement the fourth feature, an option was introduced to 'rectify' the propagation of the $\bar{\zeta}$ from t_0 (epoch) to t by the use of intermediate epochs as way-stations; use of the option degrades the status of the theory from fully analytic to semi-analytical, but the efficacy of the 'induced' terms is such that the interval $t - t_0$ has to be equivalent to several hundred orbital revolutions before there is any gain from switching to the semi-analytical mode. Finally, the inverse of the algorithm that converts the $\bar{\zeta}$ into position and velocity was made highly efficient, and of essentially unlimited accuracy, by the use of a general iterative inversion procedure[7] (it typically gives 8-decimal accuracy after a single iteration!); this procedure is at the heart of the program's operation, since it permits the conversion of successive state vectors (components of position and velocity), as generated by an independent pure-numerical integration, into successive sets of mean elements which can be compared with sets propagated by the theory-based algorithm.

The last feature above may be amplified by two remarks. First, the inversion procedure is independent of the algorithm it inverts (apart from the fact that it calls the algorithm directly during each iteration), so that modifications to the algorithm (such as the adding of higher-order terms) do not affect the inversion procedure itself. Secondly, the existence of the procedure provides an intrinsic and definitive answer to the question: "Yes, but what actually *are* your mean elements?".

In view of the successful validation of the second-order theory, originally for just J_2^2 and J_3 but later[2] with the perturbations due to J_4 incorporated, no fundamental difficulty was anticipated in extending the theory to cover the secular and long-periodic effects of third order. (That the algebra would be long and tedious *was* anticipated, and the use of a computer-algebra package was considered but appeared to be impracticable.) After validated formulae had been found for $\bar{i}$ and $\bar{e}$ associated with the product J_2J_3, therefore, it was assumed that completion of the results for J_2J_3, the analysis for J_2^3, and the generalization from J_2J_3 to J_2J_l would all be straightforward. The intention was that the updated computer program would then be used for a full examination of the long-term accuracy of the extended theory. Since discrepancies, as yet unexplained, have been met in the analysis for both J_2J_3 and J_2^3, however, the work is incomplete and the present paper must be regarded as no more than an interim report.

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