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The main goal of this research is to develop a unified geometric-asymptotic-adaptive methodology for feedback design of nonlinear control systems. Such a methodology is needed because the existing differential geometric results are restrictive and often violated by small modeling errors. Effects of these errors can be analyzed asymptotically by singular perturbation methods, which however, are still lacking a clear geometric interpretation. Neither geometric, nor perturbational problem formulations can cope with large parametric uncertainty, for which an adaptive approach seems suitable. Conversely, both geometric and asymptotic techniques are to be merged into a methodology which eliminates their individual shortcomings.

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NONLINEAR SYSTEM DESIGN: ADAPTIVE FEEDBACK LINEARIZATION WITH UNMODELED DYNAMICS

Petar V. Kokotovic
Principal Investigator

TECHNICAL REPORT
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Abstract

The main goal of this research is to develop a unified geometric–asymptotic–adaptive methodology for feedback design of nonlinear control systems. Such a methodology is needed because the existing differential geometric results are restrictive and often violated by small modeling errors. Effects of these errors can be analyzed asymptotically by singular perturbation methods, which, however, are still lacking a clear geometric interpretation. Neither geometric, nor perturbational problem formulations can cope with large parametric uncertainty, for which an adaptive approach seems suitable. Conversely, both geometric and asymptotic techniques can become constructive steps in the design of an adaptive scheme and in the analysis of its robustness. In our research these three heretofore separate techniques are to be merged into a methodology which eliminates their individual shortcomings.

During the first two years of the proposed research, major advances have been made in our study of geometric–asymptotic properties of nonlinear dynamic systems with parametric uncertainties and in the development of a systematic design methodology for adaptive nonlinear control.

First, we have demonstrated that the phenomenon of controller and/or observer peaking is of fundamental importance for nonlinear feedback design, and that interference of peaking with uncertain nonlinearities can result in a drastic decrease of the stability region. Geometric–asymptotic conditions under which this type of interference can be avoided are being developed.

Second, we have shown that adaptive control methods can reduce the effects of parametric uncertainties without introducing high–gain loops, thus avoiding the danger of peaking. Our adaptive results are for "pure–feedback" systems with known nonlinearities. We have solved the adaptive tracking problem with full–state feedback. Our solution is in the form of a systematic recursive procedure called "backstepping"

Third, we have formulated, and partially solved a class of nonlinear output feedback problems by developing a *Design Toolkit* applicable to a wide range of systems. Among the tools developed so far is our *nonlinear damping* lemma, which allows us to compensate for the effects of the estimation error in observer and/or parameter estimators.

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Contents

1. Introduction

The main goal of this research is to develop a unified geometric–asymptotic–adaptive design methodology for nonlinear control systems.

The rigorous differential geometric results suffer from restrictive necessary and sufficient conditions which are often violated by small modeling errors. Effects of these errors can be analyzed asymptotically by singular perturbation methods, which, however, are still lacking a clear geometric interpretation. Neither geometric, nor perturbational problem formulations can cope with large parametric uncertainty, for which an adaptive approach seems suitable. Conversely, both geometric and asymptotic techniques are constructive steps in the development of an adaptive scheme and in the analysis of its robustness. These three parallel, but heretofore separate, research directions are now to be merged in order to compensate for their individual shortcomings.

During the two years of the proposed research, five major advances have been made in our study of geometric–asymptotic properties of nonlinear dynamic systems with parametric uncertainties, in the state-feedback adaptive nonlinear control and in the development of a design toolkit for output feedback control.

We report on these three sets of research accomplishments in the following three sections of the report. Section 2 briefly describes our results on perturbed zero dynamics and peaking. In Sections 3 and 4 we outline our new procedure for adaptive nonlinear control design. Section 5 reports on other Principal Investigator's activities. Full details of the reported results can be found in our publications listed at the end of the report.

2. Perturbed Zero Dynamics and Peaking [J1, J3, J6, J7]

Here we summarize our results on approximate feedback linearization and peaking. Feedback linearization designs were expected to preserve stability under small regular perturbations. However, our results on perturbed zero dynamics show that in many situations this is not the case. A robustness requirement for every design is to guarantee a desired region of stability. This requirement often contradicts other design specifications, such as disturbance attenuation or speed of response. As the feedback gain is increased to meet these specifications, the stability region may become intolerably small. We have shown that this is due to an *interference of controller peaking and unmodeled nonlinearities.*

To develop more robust approximate linearization techniques, we have investigated systems in which exact techniques fail. As a benchmark physical example we have analyzed a system in which a small centrifugal term renders the exact linearization impossible. Modifying or outright neglecting this term would result in a simplified system to which the exact technique is applicable.

A deeper insight into the effects of such regular perturbations has been gained by our analysis which shows that *the zero dynamics of regularly perturbed systems may be, and often are, singularly–perturbed.* A rather far reaching conclusion is that the exact zero dynamics are insufficient for a robust nonlinear design and that, in fact, a perturbed approximate design which avoids high-gain, may be more robust.

Another frequent appearance of high gain is by design: to make the linear part of the system faster with the expectation that a sufficiently fast exponentially decaying disturbance will be of negligible effect on the zero–dynamics, that is on the remaining nonlinear part of the system. However, this expectation is, in general, false, because of the *peaking phenomenon*. To analyze the destabilizing effects of peaking, we have addressed the problem of global stabilization for a class of cascade systems. In this problem, the first part of the cascade is a linear controllable system and the second part is a nonlinear system receiving the inputs from the states of the first part.

In linear systems, a peaking phenomenon occurs when high–gain feedback is used to produce eigenvalues with very negative real parts. Then some states peak to very large values, before they rapidly decay to zero. Such peaking states act as destabilizing inputs to the nonlinear part and may even cause some of its states to escape to infinity in finite time, as illustrated by simple examples.

We have given precise structural conditions for peaking and proceeded to show that the destabilizing effects of peaking can be reduced if the nonlinearities have sufficiently slow growth. Based on our detailed analysis of the peaking phenomenon we have examined the tradeoffs between linear peaking and nonlinear growth conditions. To provide for realistic trade–offs between performance and stability, we have introduced several new concepts (nonlinear overshoot function, semiglobal stability) and given a method for computing robustness bounds.

3. Adaptive State-Feedback Control of Nonlinear Systems
[J2, J4, C3, C5, C6, C8]

Until a few years ago, adaptive linear and geometric nonlinear methods belonged to two separate areas of control theory. They were helpful in the design of controllers for plants containing either unknown parameters or known nonlinearities, but not both. In the last few years the problem of *adaptive nonlinear control* was formulated to deal with the control of plants containing both unknown parameters and known nonlinearities. A realistic plan of attack for this challenging new problem has led us through a series of simpler problems, each formulated under certain structural assumptions, much as the assumption the unknown parameters either appear, or can be made to appear, linearly. For example, if the plant model contains not only θ_1 and θ_2, but also $e^{\theta_1\theta_2}$, it is to be "overparametrized" by introducing $\theta_3 = e^{\theta_1\theta_2}$ as an additional parameter.

While we have kept the linear parametrization assumption, the applicability of our adaptive designs to larger classes of nonlinear systems is achieved by removing additional restrictive assumptions imposed on system structure, allowed types of nonlinearities and signals available for measurement.

According to these restrictions, we have classified the existing adaptive schemes into *uncertainty–constrained schemes* and *nonlinearity–constrained schemes*.

Uncertainty–constrained schemes impose restrictions (*matching conditions*) on the location of unknown parameters, but can handle all types of nonlinearities.

Nonlinearity–constrained schemes do not restrict the location of unknown parameters. Instead, they impose restrictions on the nonlinearities of the original system, as well as on those appearing in the transformed error system.

Our major result, most favorably received by the research community, is that the limitations on nonlinearities can be removed for the so–called *pure–feedback systems,* they are the broadest class of nonlinear systems for which adaptive controllers can be systematically designed without imposing any growth constraints on system nonlinearities.

The geometric characterization of pure feedback systems identifies the *level of uncertainty and nonlinear complexity* as structural obstacles to adaptive feedback linearization. For an unknown parameter, the level of uncertainty is its "distance," in terms of the number of integrators, from the control input. The larger this distance is, the smaller is the number of state variables on which the nonlinearity multiplying this parameter is allowed to depend (nonlinear complexity).

Our new adaptive scheme for pure–feedback systems is designed by a systematic recursive procedure called *backstepping*. This procedure interlaces, at each step, the change of coordinates required for feedback linearization, and the construction of parameter update laws required for adaptation.

One of the most important stability and robustness properties of every adaptive system is the size of its region of attraction, relative to the size of the region that would have been achieved if all the parameters were known. When with the known parameters the stability and tracking properties are global, but the same properties of the adaptive scheme are only local, then the loss of globality is due to adaptation. To avoid this loss, some adaptive schemes require that the nonlinearities and some of their derivatives satisfy a linear growth condition which severely limits the applicability of these schemes. The class of systems for which our new adaptive scheme guarantees global regulation and tracking is much wider.

The region of attraction for the new adaptive scheme is *global* if the feedback linearization is global. A subclass of pure–feedback systems for which this global property is easy to establish are *strict-feedback systems*. For these systems the new adaptive scheme achieves both *global regulation and global tracking* of smooth bounded reference inputs. In contrast to the earlier schemes, these global results are obtained *without any growth constraints on system nonlinearities*.

4. Output Feedback Nonlinear Control [J5, J8, C1, C4, C7, C9, C10]

By far the most difficult problems in adaptive nonlinear control are those with incomplete state measurement.

In the linear case, the adaptive output–feedback designs follow either a direct model–reference path or an indirect path via adaptive observers. Current research on adaptive observers for nonlinear systems indicates that the indirect path may become promising for adaptive nonlinear control. However, the major stumbling block along this path continues to be its linear–like proof of stability which imposes restrictive conic conditions on the nonlinearities. Under such linear growth constraints the actual nonlinear problem is, in fact, not addressed. Our first two results on output feedback adaptive nonlinear control, have made similar linear growth constraints and can be subjected to the same type of criticism.

In our current research, we aim to formulate and solve truly nonlinear output–feedback adaptive problems. To this end, we have first addressed a class of problems for which an exponentially convergent observer is known to exist. We have shown that, in general, a "certainty equivalence" control, which employs the state estimates as if they were exact states, is not stabilizing and may even lead to explosive instabilities. The reason for this is an *observer-induced peaking phenomenon*. We have developed a "Design Toolkit" to counteract the effects of peaking and achieve stabilization. Thanks to this result, a systematic output feedback design is now possible for a much larger class of nonlinear systems than before. In our current research we are extending these breakthroughs and preparing for their applications.

5. Principal Investigator's Activities

Petar V. Kokotovic was the co–organizer (with Alan J. Laub) of an NSF-NASA workshop on Nonlinear Control, April 5-7, 1990, at Cliff House, UCSB. Many of the topics covered by this research grant were discussed at the workshop.

Another major event organized by P. Kokotovic, in his capacity as Grainger Professor at the University of Illinois, was the series of fifteen Grainger Lectures on "Foundations of Adaptive Control," September 28-October 1, 1990. A volume (more than 500 pages) of extended texts of these lectures has been published by Springer in 1991.

At the World Congress of IFAC in August 1990, P. Kokotovic received the IFAC's highest award—Quazza Medal—that has been given triennially since 1981. He was chosen to deliver the Bode Prize lecture at the 1991 IEEE Conference on Decision and Control.

RESEARCH PUBLICATIONS UNDER GRANT NO. AFOSR-91-0011 and GRANT NO. AFOSR-91-147

"Nonlinear Systems Design: Adaptive Feedback Linearization with Unmodeled Dynamics,"

P. Kokotovic, Principal Investigator

Journal Papers

J1 H.K. Khalil and P.V. Kokotovic, "On Stability Properties of Nonlinear Systems with Slowly-Varying Inputs," *IEEE Transactions on Automatic Control*, AC-36, pp. 229, February 1991.

J2 I. Kanellakopoulos, P. Kokotovic, and R. Marino, "An Extended Direct Scheme for Robust Adaptive Nonlinear Control," *Automatica*, 27, pp. 247-255, March 1991.

J3 H.J. Sussmann and P.V. Kokotovic, "The Peaking Phenomenon and the Global Stabilization of Nonlinear Systems," *IEEE Transactions on Automatic Control*, AC-36, pp. 424-439, April 1991.

J4 I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, "Systematic Design of Adaptive Controllers for Feedback Linearizable Systems," *IEEE Transactions on Automatic Control*, vol. AC-36, pp. 1241-1253, Nov. 1991.

J5 I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, "A Toolkit for Nonlinear Feedback Design," *Systems & Control Letters*, vol. 18, Feb. 1992.

J6 J. Hauser, S. Sastry, and P.V. Kokotovic, "Nonlinear Control via Approximate Input-Output Linearization: the Ball and Beam Example," to appear in *IEEE Transactions on Automatic Control*, vol. AC-37, 1992.

J7 A. Isidori, S. Sastry, P.V. Kokotovic, and C. Byrnes, "Singularly Perturbed Zero Dynamics of Nonlinear Systems," to appear in *IEEE Transactions on Automatic Control*, vol. AC-37, 1992.

J8 I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, "Adaptive Output-Feedback Control of Systems with Output Nonlinearities," to appear in *IEEE Transactions on Automatic Control*, vol. AC-37, 1992.

Conference Papers

C1 I. Kanellakopoulos, P.V. Kokotovic, and R.H. Middleton, "Observer-Based Adaptive Control of Nonlinear Systems under Matching Conditions," *Proceedings of the 1990 American Control Conference*, pp. 549-555, San Diego, CA, May 1990.

C2 I. Kanellakopoulos, P.V. Kokotovic, and R. Marino, "Adaptive Control Design for a Class of Nonlinear Systems," *Proceedings of the 1990 American Control Conference*, pp. 1713-1717, San Diego, CA, May 1990.

C3 P.V. Kokotovic and I. Kanellakopoulos, "Adaptive Nonlinear Control: A Critical Appraisal," *Proceedings of the Sixth Yale Workshop on Adaptive and Learning Systems*, pp. 1-6, New Haven, CT, August 1990.

C4 I. Kanellakopoulos, P.V. Kokotovic, and R.H. Middleton, "Indirect Adaptive Output-Feedback Control of a Class of Nonlinear Systems," *Proceedings of the 29th IEEE Conference on Decision and Control*, pp. 2714-2719, Honolulu, HI, December 1990.

C5 I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, "Systematic Design of Adaptive Controllers for Feedback Linearizable Systems," *Proceedings of the 1991 American Control Conference*, pp. 649-654, Boston, MA.

C6 P.V. Kokotovic, I. Kanellakopoulos, and A.S. Morse, "Adaptive Feedback Linearization of Nonlinear Systems," in *Foundations of Adaptive Control*, P.V. Kokotovic, Ed., Springer-Verlag, Berlin, 1991, pp. 311-346.

C7 I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, "Adaptive Output-Feedback Control of Systems with Output Nonlinearities," in *Foundations of Adaptive Control*, P.V. Kokotovic, Ed., Springer-Verlag, Berlin, 1991, pp. 495-525.

C8 D.A. Recker and P.V. Kokotovic, "Approximate Decoupling of Regularly Perturbed Nonlinear Systems," *Proceedings of the 1991 American Control Conference*, pp. 522-527, Boston, MA.

C9 I. Kanellakopoulos, P.V. Kokotovic, R. Marino, and P. Tomei, "Adaptive Control of Nonlinear Systems with Partial State Feedback," *Proceedings of the 1991 European Control Conference*, Grenoble, France, July 1991, pp. 1322-1327.

C10 I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, "Adaptive Output-Feedback Control of a Class of Nonlinear Systems," *Proceedings of the 30th IEEE Conference on Decision and Control*, pp. 1082-1087, Brighton, UK, 1991.

APPENDIX

Selected Publications Under Grant #AFOSR-91-0011 and GRANT #AFOSR-91-147

Systematic Design of Adaptive Controllers for Feedback Linearizable Systems

I. Kanellakopoulos, *Student Member, IEEE*, P. V. Kokotovic, *Fellow, IEEE*, and A. S. Morse, *Fellow, IEEE*

***Abstract*—A systematic procedure is developed for the design of new adaptive regulation and tracking schemes for a class of feedback linearizable nonlinear systems. The coordinate-free geometric conditions, which characterize this class of systems, do not constrain the growth of the nonlinearities. Instead, they require that the nonlinear system be transformable into the so-called "parametric-pure-feedback form." When this form is "strict," the proposed scheme guarantees global regulation and tracking properties, and substantially enlarges the class of nonlinear systems with unknown parameters for which global stabilization can be achieved. The main results of this paper use simple analytical tools, familiar to most control engineers.**

I. Introduction

Most of the research activity on adaptive control of nonlinear systems [1]–[15] is still focused on the full-state feedback case [1]–[13], although output-feedback results are beginning to appear [14], [15]. The full-state feedback case continues to be a challenge because of the severe restrictions of the two currently available types of schemes: the *uncertainty-constrained schemes* [1]–[4], [10], [11] assume restrictive *matching conditions*, and the *nonlinearity-constrained schemes* [5]–[9], [12] impose restrictions on the *type of nonlinearities*.

The systems to which *uncertainty-constrained schemes* can be applied may contain all types of smooth nonlinearities and are fully characterized by coordinate-free geometric conditions [2], [3], [11], which, unfortunately, are quite restrictive. On the other hand, the applicability of *nonlinearity-constrained schemes* is restricted by coordinate-dependent growth conditions on the nonlinearities, which may exclude even certain *linear* systems [13]. Less restrictive coordinate-free growth conditions, written in terms of a "*control Lyapunov function*," are used in the schemes of [6]–[8]. Unfortunately, the existence of such a Lyapunov function cannot be ascertained *a priori*.

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The new adaptive control scheme developed in this paper can be classified as the least restrictive uncertainty-constrained scheme available for feedback linearizable systems. It significantly extends the class of nonlinear systems for which adaptive controllers can be systematically designed.

Among the advantages of the new scheme are its conceptual clarity and wide applicability. Its stability proof, based on a straightforward Lyapunov argument, is particularly simple. The coordinate-free geometric conditions, characterizing the class of systems to which the new scheme is applicable, do not constrain the growth of the nonlinearities. Instead, they require that the nonlinear system be transformable into the so-called *parametric-pure-feedback form*. Furthermore, in the case of systems transformable into the more restrictive *parametric-strict-feedback form*, the new adaptive scheme guarantees *global* regulation and tracking properties.

The presentation is organized as follows: First, we address the regulation problem. In Section II we characterize the class of single-input nonlinear systems to which the new scheme is applicable. The design procedure is presented in Section III, and the simple proof of stability is given in Section IV. In Section V we give the conditions under which the stability of the closed-loop system is global. Then, in Section VI, we use the design procedure to solve the tracking problem for a class of input-output linearizable systems with exponentially stable zero dynamics. In Section VII we illustrate this procedure on some "benchmark" examples, and discuss its properties in comparison to previous results. Finally, some concluding remarks are given in Section VIII. The reader, unfamiliar with differential geometric results for nonlinear systems can follow the presentation starting with Section III and then omitting Propositions 5.3 and 6.3.

II. The Class of Nonlinear Systems

The adaptive regulation problem will first be solved for single-input feedback linearizable systems that are linear in the unknown parameters:

$$\dot{\zeta} = f_0(\zeta) + \sum_{i=1}^{p} \theta_i f_i(\zeta) + \left[g_0(\zeta) + \sum_{i=1}^{p} \theta_i g_i(\zeta) \right] u \tag{2.1}$$

where $\zeta \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $\theta = [\theta_1, \cdots, \theta_p]^T$ is the vector of unknown constant parameters, and f_i, g_i, $0 \le i \le p$, are smooth vector fields in a neighborhood of the origin $\zeta = 0$ with $f_i(0) = 0,\ 0 \le i \le p$, $g_0(0) \ne 0$.

The design of the adaptive scheme assumes that, using a parameter-independent diffeomorphism $x = \phi(\zeta)$, the system (2.1) can be transformed into the *parametric-pure-feedback form*:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta^{\mathrm{T}}\gamma_1(x_1, x_2) \\
\dot{x}_2 &= x_3 + \theta^{\mathrm{T}}\gamma_2(x_1, x_2, x_3) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \theta^{\mathrm{T}}\gamma_{n-1}(x_1, \cdots, x_n) \\
\dot{x}_n &= \gamma_0(x) + \theta^{\mathrm{T}}\gamma_n(x) + \left[\beta_0(x) + \theta^{\mathrm{T}}\beta(x)\right]u
\end{aligned} \tag{2.2}$$

with

$$\gamma_0(0) = 0, \gamma_1(0) = \cdots = \gamma_n(0) = 0, \qquad \beta_0(0) \ne 0. \tag{2.3}$$

Nc ›ary and sufficient conditions for the existence of such a d..ieomorphism are given in the following proposition.

Proposition 2.1: A diffeomorphism $x = \phi(\zeta)$, with $\phi(0) = 0$, transforming (2.1) into (2.2), exists in a neighborhood $B_x \subset U$ of the origin if and only if the following conditions are satisfied in U.

i) Feedback Linearization Condition: The distributions

$$\mathcal{G}^i = \operatorname{span}\left\{g_0, \operatorname{ad}_{f0} g_0, \cdots, \operatorname{ad}^i_{f0} g_0\right\}, \qquad 0 \le i \le n-1 \tag{2.4}$$

are involutive and of constant rank $i + 1$.

ii) Parametric-Pure-Feedback Condition:

$$g_i \in \mathcal{G}^0,$$

$$[X, f_i] \in \mathcal{G}^{j+1}, \qquad \forall X \in \mathcal{G}^j, \quad 0 \le j \le n-3, \qquad 1 \le i \le p. \tag{2.5}$$

Proof:

Sufficiency: As proved in [16], condition i) is sufficient for the existence of a diffeomorphism $x = \phi(\zeta)$ with $\phi(0) = 0$ which transforms the system

$$\dot{\zeta} = f_0(\zeta) + g_0(\zeta)u, \qquad f_0(0) = 0, g_0(0) \ne 0 \tag{2.6}$$

into the system

$$\begin{aligned}
\dot{x}_i &= x_{i+1}, \qquad 1 \le i \le n-1 \\
\dot{x}_n &= \gamma_0(x) + \beta_0(x)u
\end{aligned} \tag{2.7}$$

with

$$\gamma_0(0) = 0, \quad \beta_0(0) \ne 0. \tag{2.8}$$

Hence, in the coordinates of (2.7) we have

$$f_0 = x_2\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_{n-1}} + \gamma_0(x)\frac{\partial}{\partial x_n} \tag{2.9}$$

$$g_0 = \beta_0(x)\frac{\partial}{\partial x_n} \tag{2.10}$$

$$\mathcal{G}^i = \operatorname{span}\left\{\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_{n-i}}\right\}, \qquad 0 \le i \le n-1 \tag{2.11}$$

where $\partial/\partial x_1, \cdots, \partial/\partial x_n$ are the coordinate vector fields associated with the x-coordinates. Because of (2.11), the parametric-pure-feedback condition (2.5), expressed in the x-coordinates, states that

$$g_i \in \operatorname{span}\left\{\frac{\partial}{\partial x_n}\right\},$$

$$\left[\frac{\partial}{\partial x_j}, f_i\right] \in \operatorname{span}\left\{\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_{j-1}}\right\}, \qquad 3 \le j \le n, \qquad 1 \le i \le p. \tag{2.12}$$

But (2.12) can be equivalently rewritten as

$$g_i = \beta_i(x)\frac{\partial}{\partial x_n},$$

$$\begin{aligned}
f_i &= \gamma_{1,i}(x_1, x_2)\frac{\partial}{\partial x_1} + \gamma_{2,1}(x_1, x_2, x_3)\frac{\partial}{\partial x_2} + \cdots \\
&\quad + \gamma_{n-1,i}(x_1, \cdots, x_n)\frac{\partial}{\partial x_{n-1}} + \gamma_{n,i}(x_1, \cdots, x_n)\frac{\partial}{\partial x_n}, \qquad 1 \le i \le p.
\end{aligned} \tag{2.13}$$

Furthermore, since $\phi(0) = 0$ and $f_i(0) = 0,\ 1 \le i \le p$, we conclude from (2.13) that

$$\gamma_1(0) = \cdots = \gamma_n(0) = 0. \tag{2.14}$$

Combining (2.9), (2.10), (2.13), and (2.14), we see that in the x-coordinates the system (2.1) becomes (2.2).

Necessity: If there exists a diffeomorphism $x = \phi(\zeta)$ that transforms (2.1) into (2.2), one can directly verify that the coordinate-free conditions i) and ii) are satisfied for the system (2.2), and hence for the system (2.1). □

Remark 2.2: A special case of the parametric-pure-feedback condition (2.5) is the "extended-matching" condition

$$g_i \in \mathcal{G}^0, \quad f_i \in \mathcal{G}^1, \qquad 1 \le i \le p \tag{2.15}$$

introduced in [2], [3] and formulated in [1] as a "strong linearizability" condition. This is clear from the proof of Proposition 2.1: if (2.5) is replaced by (2.15), then (2.13) still holds, but with $\gamma_1 \equiv 0, \cdots, \gamma_{n-2} \equiv 0$. Then, the system (2.1) is expressed in the x-coordinates as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} \\
\dot{x}_{n-1} &= x_n + \theta^{\mathrm{T}}\gamma_{n-1}(x_1, \cdots, x_n) \\
\dot{x}_n &= \gamma_0(x) + \theta^{\mathrm{T}}\gamma_n(x) + \left[\beta_0(x) + \theta^{\mathrm{T}}\beta(x)\right]u.
\end{aligned} \tag{2.16}$$

□

Remark 2.3: The expressions given in (2.4) and (2.5) for the feedback linearization and parametric-pure-feedback conditions are convenient for the proof of Proposition 2.1, but they are *not minimal.* As shown in [17], [18], the equivalent minimal form of (2.4) is

$$\mathcal{G}^{n-2} \text{ is involutive and } \mathcal{G}^{n-1} \text{ has constant rank } n. \tag{2.17}$$

The minimal form of (2.5) is

$$\left[\mathrm{ad}_{f_0}^{j} g_0, f_i\right] \in \mathcal{G}^{j+1}, \qquad 0 \le j \le n-3, \qquad 1 \le i \le p. \tag{2.18}$$

The equivalence of (2.5) and (2.18) follows from the involutivity of $\mathcal{G}^{n-2}$. □

Remark 2.4: The term "parametric-pure-feedback" indicates that the nonlinearities multiplying unknown parameters are allowed to depend only on state variables that are "fed back" when the system is written in the x-coordinates. This term should not be confused with the term "pure-feedback systems" used in [19] to denote the class of feedback linearizable systems, nor with the "pure-feedback systems" for which the nonlinearity-constrained scheme of [5] was developed. □

III. Adaptive Scheme Design

Since the diffeomorphism $x = \phi(\zeta)$ does not depend on the unknown parameter vector θ, Proposition 2.1 gives an *a priori verifiable* characterization of the class of nonlinear systems to which the new adaptive scheme is applicable. Assuming that the transformation of (2.1) into (2.2) has been performed, the new adaptive scheme is designed for the parametric-pure-feedback system

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^{\mathrm{T}}\gamma_i(x_1, \cdots, x_{i+1}), \qquad 1 \le i \le n-1 \\ \dot{x}_n &= \gamma_0(x) + \theta^{\mathrm{T}}\gamma_n(x) + \left[\beta_0(x) + \theta^{\mathrm{T}}\beta(x)\right]u \end{aligned} \tag{3.1}$$

with

$$\gamma_0(0) = 0, \gamma_1(0) = \cdots = \gamma_n(0) = 0, \qquad \beta_0(0) \neq 0. \tag{3.2}$$

Recall that γ_0, β_0, and the components of β and γ_i, $1 \le i \le n$, are smooth nonlinear functions in B_x, a neighborhood of the origin $x = 0$.

The following step-by-step procedure was inspired by an idea contained in an early linear result of [20]. However, the intuition behind our nonlinear development becomes much clearer if the procedure is interpreted as the interlacing of the steps of the "chain of integrators" method [21]–[23] with the design of a new parameter estimator at each step.

Step 0: Define $z_1 = x_1$, and denote by $c_1, c_2, \cdots, c_n$ constant coefficients to be chosen later.

Step 1: Starting with

$$\dot{z}_1 = x_2 + \theta^{\mathrm{T}}\gamma_1(x_1, x_2) \tag{3.3}$$

and following the "chain of integrators" method, we see that, if x_2 were the control input and θ were known, the "control law"

$$x_2 = -c_1 z_1 - \theta^{\mathrm{T}}\gamma_1(x_1, x_2) \tag{3.4}$$

would stabilize and regulate the system (3.3). As θ is unknown, this "control law" is modified to its "certainty-equivalence" form

$$x_2 = -c_1 z_1 - \vartheta_1^{\mathrm{T}}\gamma_1(x_1, x_2) \tag{3.5}$$

where ϑ_1 is an estimate of θ. Then, (3.5) together with the update law

$$\dot{\vartheta}_1 = z_1\gamma_1(x_1, x_2) \tag{3.6}$$

would render the closed-loop system stable and would achieve the regulation of z_1. However, x_2 is not the control. Therefore, define the new state z_2 as the difference between the actual x_2 and its desired expression (3.5):

$$z_2 \triangleq c_1 z_1 + x_2 + \vartheta_1^{\mathrm{T}}\gamma_1(x_1, x_2). \tag{3.7}$$

To complete Step 1, substitute (3.7) into (3.3)

$$\begin{aligned} \dot{z}_1 &= -c_1 z_1 + z_2 + (\theta - \vartheta_1)^{\mathrm{T}}\gamma_1(x_1, x_2) \\ &\triangleq -c_1 z_1 + z_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(z_1, z_2, \vartheta_1) \end{aligned} \tag{3.8}$$

and rewrite the update law (3.6) for the parameter estimate of ϑ_1 in the form

$$\dot{\vartheta}_1 = z_1 w_1(z_1, z_2, \vartheta_1). \tag{3.9}$$

In addition to the "error system" (3.8) and the update law (3.9), Step 1 has introduced the new state variable z_2, which is to be regulated in Step 2.

Step 2: Using the definitions for z_1, z_2, and $\dot{\vartheta}_1$, write $\dot{z}_2$ as

$$\begin{aligned} \dot{z}_2 &= c_1\left[-c_1 z_1 + z_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(z_1, z_2, \vartheta_1)\right] \\ &\quad + x_3 + \theta^{\mathrm{T}}\gamma_2(x_1, x_2, x_3) \\ &\quad + z_1 w_1(z_1, z_2, \vartheta_1)^{\mathrm{T}}\gamma_1(x_1, x_2) \\ &\quad + \vartheta_1^{\mathrm{T}}\left[\frac{\partial\gamma_1}{\partial x_1}(x_2 + \theta^{\mathrm{T}}\gamma_1) + \frac{\partial\gamma_1}{\partial x_2}(x_3 + \theta^{\mathrm{T}}\gamma_2)\right] \\ &\triangleq \left(1 + \vartheta_1^{\mathrm{T}}\frac{\partial\gamma_1}{\partial x_2}\right)\left[x_3 + \theta^{\mathrm{T}}\gamma_2(x_1, x_2, x_3)\right] \\ &\quad + \varphi_2(z_1, z_2, \vartheta_1) + \theta^{\mathrm{T}}\psi_2(z_1, z_2, \vartheta_1). \end{aligned} \tag{3.10}$$

In this system, we will think of x_3 as our control input. As in Step 1, we need an estimate for θ. Since the update law for ϑ_1 has already been defined in (3.9), ϑ_1 cannot be used again. Therefore, let ϑ_2 be a *new* estimate of θ and define the new state z_3 as

$$\begin{aligned} z_3 &\triangleq c_2 z_2 + \left(1 + \vartheta_1^{\mathrm{T}}\frac{\partial\gamma_1}{\partial x_2}\right)\left[x_3 + \vartheta_2^{\mathrm{T}}\gamma_2(x_1, x_2, x_3)\right] \\ &\quad + \varphi_2(z_1, z_2, \vartheta_1) + \vartheta_2^{\mathrm{T}}\psi_2(z_1, z_2, \vartheta_1). \end{aligned} \tag{3.11}$$

Substitute (3.11) into (3.10) to obtain

$$\begin{aligned}\dot z_2 &= -c_2 z_2 + z_3 \\ &\quad + (\theta - \vartheta_2)^{\mathrm T}\Bigg[\psi_2(z_1, z_2, \vartheta_1) \\ &\quad + \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1(x_1, x_2)}{\partial x_2}\right)\gamma_2(x_1, x_2, x_3)\Bigg] \\ &\triangleq -c_2 z_2 + z_3 + (\theta - \vartheta_2)^{\mathrm T} w_2(z_1, z_2, z_3, \vartheta_1, \vartheta_2). \end{aligned} \tag{3.12}$$

Then, let the update law for the new estimate ϑ_2 be

$$\dot\vartheta_2 = z_2 w_2(z_1, z_2, z_3, \vartheta_1, \vartheta_2). \tag{3.13}$$

Step i $(2 \le i \le n-1)$: Using the definitions for $z_1, \cdots, z_i$ and $\dot\vartheta_1, \cdots, \dot\vartheta_{i-1}$, express the derivative of z_i as

$$\begin{aligned}\dot z_i &= \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1}{\partial x_2}\right)\cdots\left(1 + \vartheta_{i-1}^{\mathrm T}\frac{\partial\gamma_{i-1}}{\partial x_i}\right) \\ &\quad \cdot\left[x_{i+1} + \theta^{\mathrm T}\gamma_i(x_1, \cdots, x_{i+1})\right] \\ &\quad + \varphi_i(z_1, \cdots, z_i, \vartheta_1, \cdots, \vartheta_{i-1}) \\ &\quad + \theta^{\mathrm T}\psi_i(z_1, \cdots, z_i, \vartheta_1, \cdots, \vartheta_{i-1}) \end{aligned} \tag{3.14}$$

with φ_i, ψ_i appropriately defined smooth functions. Let ϑ_i be a *new* estimate of θ and define the new state z_{i+1} as

$$\begin{aligned} z_{i+1} &\triangleq c_i z_i + \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1}{\partial x_2}\right)\cdots\left(1 + \vartheta_{i-1}^{\mathrm T}\frac{\partial\gamma_{i-1}}{\partial x_i}\right) \\ &\quad \cdot\left[x_{i+1} + \vartheta_i^{\mathrm T}\gamma_i(x_1, \cdots, x_{i+1})\right] \\ &\quad + \varphi_i(z_1, \cdots, z_i, \vartheta_1, \cdots, \vartheta_{i-1}) \\ &\quad + \vartheta_i^{\mathrm T}\psi_i(z_1, \cdots, z_i, \vartheta_1, \cdots, \vartheta_{i-1}). \end{aligned} \tag{3.15}$$

Substitute (3.15) into (3.14) to obtain

$$\begin{aligned}\dot z_i &= -c_i z_i + z_{i+1} + (\theta - \vartheta_i)^{\mathrm T} \\ &\quad \cdot\left[\psi_i + \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1}{\partial x_2}\right)\cdots\left(1 + \vartheta_{i-1}^{\mathrm T}\frac{\partial\gamma_{i-1}}{\partial x_i}\right)\gamma_i\right] \\ &\triangleq -c_i z_i + z_{i+1} + (\theta - \vartheta_i)^{\mathrm T} \\ &\quad \cdot w_i(z_1, \cdots, z_{i+1}, \vartheta_1, \cdots, \vartheta_i). \end{aligned} \tag{3.16}$$

Then, let the update law for ϑ_i be

$$\dot\vartheta_i = z_i w_i(z_1, \cdots, z_{i+1}, \vartheta_1, \cdots, \vartheta_i). \tag{3.17}$$

Step n: Using the definitions for $z_1, \cdots, z_n$ and $\dot\vartheta_1, \cdots, \dot\vartheta_{n-1}$, express the derivative of z_n as

$$\begin{aligned}\dot z_n &= \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1}{\partial x_2}\right)\cdots\left(1 + \vartheta_{n-1}^{\mathrm T}\frac{\partial\gamma_{n-1}}{\partial x_n}\right) \\ &\quad \cdot\left[\beta_0(x) + \theta^{\mathrm T}\beta(x)\right]u \\ &\quad + \varphi_n(z, \vartheta_1, \cdots, \vartheta_{n-1}) + \theta^{\mathrm T}\psi_n(z, \vartheta_1, \cdots, \vartheta_{n-1}) \end{aligned} \tag{3.18}$$

with φ_n, ψ_n appropriately defined smooth functions. Let ϑ_n be a *new* estimate of θ and choose the control u as

$$u = \frac{1}{\bar\beta(x, \vartheta_1, \cdots, \vartheta_n)}\left[-c_n z_n - \varphi_n - \vartheta_n^{\mathrm T}\psi_n\right] \tag{3.19}$$

where

$$\begin{aligned}\bar\beta(x, \vartheta_1, \cdots, \vartheta_n) &\triangleq \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1}{\partial x_2}\right)\cdots\left(1 + \vartheta_{n-1}^{\mathrm T}\frac{\partial\gamma_{n-1}}{\partial x_n}\right) \\ &\quad \cdot\left[\beta_0(x) + \vartheta_n^{\mathrm T}\beta(x)\right]. \end{aligned} \tag{3.20}$$

Substitute (3.19) into (3.18) to obtain

$$\begin{aligned}\dot z_n &= -c_n z_n + (\theta - \vartheta_n)^{\mathrm T}\Bigg[\psi_n + \left(1 + \vartheta_1^{\mathrm T}\frac{\partial\gamma_1}{\partial x_2}\right)\cdots \\ &\quad \left(1 + \vartheta_{n-1}^{\mathrm T}\frac{\partial\gamma_{n-1}}{\partial x_n}\right)\beta(x)u\Bigg] \\ &\triangleq -c_n z_n + (\theta - \vartheta_n)^{\mathrm T} w_n(z, \vartheta_1, \cdots, \vartheta_n) \end{aligned} \tag{3.21}$$

where (3.19) is used in the definition of w_n. Finally, let the update law for the estimate ϑ_n be

$$\dot\vartheta_n = z_n w_n(z, \vartheta_1, \cdots, \vartheta_n). \tag{3.22}$$

Feasibility of this design procedure and the stability of the resulting closed-loop adaptive system are analyzed in the next section.

IV. Feasibility and Stability

The above design procedure has introduced a control law defined by (3.19)–(3.20) and a set of new coordinates $z_1, \cdots, z_n$ defined by (3.15). In order to ensure that the procedure is *feasible*, we construct in Proposition 4.1 an estimate of $\mathcal{F} \subset \mathbb{R}^{n(1+p)}$ of the *feasibility region* such that for all $(x, \vartheta_1, \cdots, \vartheta_n) \in \mathcal{F}$ the denominator in (3.19) is nonzero and the coordinate change (3.15) is one-to-one, onto, continuous, and has a continuous inverse.

Proposition 4.1: Suppose the parametric-pure-feedback form (3.1) of the system (2.1) exists in B_x, and let $B_\vartheta \subset \mathbb{R}^p$ be an open set such that $\theta \in B_\vartheta$ and

$$\left|1 + \vartheta_i^{\mathrm T}\frac{\partial\gamma_i(x)}{\partial x_{i+1}}\right| > 0, \qquad \forall x \in B_x,\ \forall\vartheta_i \in B_\vartheta,\ 1 \le i \le n-1 \tag{4.1}$$

$$\left|\beta_0(x) + \vartheta_n^{\mathrm T}\beta(x)\right| > 0, \qquad \forall x \in B_x,\ \forall\vartheta_n \in B_\vartheta. \tag{4.2}$$

Then, the set $\mathcal{F} = B_x \times B_\vartheta^n$ is a subset of the region in which the design procedure of Section III is feasible.

Proof: Obvious, since (4.1) and (4.2) guarantee that in $B_x \times B_\vartheta^n$ the denominator in (3.19) is nonzero and (3.15) is *uniquely* solvable for x_i. □

Remark 4.2: In general, the feasibility region is not global. However, this is not due to the adaptive scheme because even when the parameters θ are known, the feedback linearization of the system (3.1) can only be guaranteed for $\theta \in B_\theta \subset \mathbb{R}^p$, an open set such that

$$\left|1+\theta^{\mathrm{T}}\frac{\partial\gamma_i(x)}{\partial x_{i+1}}\right|>0,$$

$$\forall x\in B_x,\ \forall\theta\in B_\theta,\ 1\le i\le n-1 \quad (4.3)$$

$$|\beta_0(x)+\theta^{\mathrm{T}}\beta(x)|>0,\qquad \forall x\in B_x,\ \forall\theta\in B_\theta. \quad (4.4)$$

□

In the feasibility region, the adaptive system resulting from the design procedure can be expressed in the z-coordinates as

$$\begin{aligned}
\dot z_1 &= -c_1z_1+z_2+(\theta-\vartheta_1)^{\mathrm{T}}w_1(z_1,z_2,\vartheta_1)\\
&\ \ \vdots\\
\dot z_{n-1} &= -c_{n-1}z_{n-1}+z_n+(\theta-\vartheta_{n-1})^{\mathrm{T}}\\
&\qquad\cdot w_{n-1}(z_1,\cdots,z_n,\vartheta_1,\cdots,\vartheta_{n-1})\\
\dot z_n &= -c_nz_n+(\theta-\vartheta_n)^{\mathrm{T}}w_n(z,\vartheta_1,\cdots,\vartheta_n)\\
\dot\vartheta_i &= z_iw_i,\qquad 1\le i\le n.
\end{aligned} \quad (4.5)$$

A nice property of this system is that its stability can be established using the quadratic Lyapunov function

$$V(z,\vartheta_1,\cdots,\vartheta_n)=\frac{1}{2}z^{\mathrm{T}}z+\frac{1}{2}\sum_{i=1}^{n}(\theta-\vartheta_i)^{\mathrm{T}}(\theta-\vartheta_i). \quad (4.6)$$

The derivative of $V(z,\vartheta_1,\cdots,\vartheta_n)$ along the solutions of (4.5) is

$$\begin{aligned}
\dot V &= -\sum_{i=1}^{n}\left[c_iz_i^2+(\theta-\vartheta_i)^{\mathrm{T}}(z_iw_i+\dot\vartheta_i)\right]+\sum_{i=1}^{n-1}z_iz_{i+1}\\
&= -\sum_{i=1}^{n}c_iz_i^2+\sum_{i=1}^{n-1}z_iz_{i+1}.
\end{aligned} \quad (4.7)$$

At this point we can choose the coefficients $c_1,\cdots,c_n$ to guarantee that $\dot V$ is negative semidefinite. The choice $c_i\ge 2$, for all $i=1,\cdots,n$, yields

$$\dot V\le -\|z\|^2. \quad (4.8)$$

This proves the uniform stability of the equilibrium

$$z=0,\quad \vartheta_1=\theta,\cdots,\vartheta_n=\theta \quad (4.9)$$

of the adaptive system (4.5). To give an estimate Ω of the region of attraction of this equilibrium, we note that Ω must be a subset of our estimate $\mathscr{F}$ of the feasibility region. We also note that, by definition, the point $x=0$ is contained in B_x, $\theta\in B_\theta$, and $\gamma_1(0)=\cdots=\gamma_n(0)=0$. Combining these facts with the definitions of $z_1,\cdots,z_n$, $\dot\vartheta_1,\cdots,\dot\vartheta_n$, it is straightforward to show that the equilibrium (4.9) coincides with the point $x=0$, $\vartheta_1=\theta,\cdots,\vartheta_n=\theta$, and is therefore contained in $\mathscr{F}$. Let $\Omega(c)$ be the invariant set of (4.5) defined by $V<c$, and let c^* be the largest constant c such that $\Omega(c)\subset\mathscr{F}$. Then, an estimate Ω of the region of attraction is

$$\Omega=\Omega(c^*)=\{(z,\vartheta_1,\cdots,\vartheta_n): V(z,\vartheta_1,\cdots,\vartheta_n)<c^*\},$$

$$c^*=\arg\sup_{\Omega(c)\subset\mathscr{F}}\{c\}. \quad (4.10)$$

Remark 4.3: It can be expected that the above estimate is not tight because the choice of the unity gains in the update laws was made for simplicity. With some *a priori* knowledge about the shape of $\mathscr{F}$, different adaptation gains can be found so that Ω is maximized by a better fit of $\mathscr{F}$. □

Next, from the invariance theorem of LaSalle, we conclude that for all initial conditions $(z,\vartheta_1,\cdots,\vartheta_n)_{t=0}\in\Omega$, the adaptive system (4.5) has the following regulation properties:

$$\lim_{t\to\infty}z(t)=0,\quad \lim_{t\to\infty}\dot z(t)=0,\quad \lim_{t\to\infty}\dot\vartheta_i(t)=0,$$

$$1\le i\le n. \quad (4.11)$$

Finally, to establish that the original coordinates ζ are regulated to zero, we note that (4.1) guarantees, first, that the solution $x_2=\cdots=x_n=0$ of the system of equations

$$x_{i+1}+\theta^{\mathrm{T}}\gamma_i(0,x_2,\cdots,x_{i+1})=0,\qquad 1\le i\le n-1 \quad (4.12)$$

is *unique* in $B_x\times B_\theta$, and, second, that $x_1,\cdots,x_n$ can be expressed as smooth functions of $z,\vartheta_1,\cdots,\vartheta_n$ using (3.15). Combining these two facts with (4.11), we obtain

$$\lim_{t\to\infty}x_1(t)=0,\quad \lim_{t\to\infty}\dot x_i(t)=0,\qquad 1\le i\le n. \quad (4.13)$$

Using an induction argument, it is now shown that $x_i(t)\to 0$ as $t\to\infty$, $1\le i\le n$.

- For $i=1$, we have $x_1(t)\to 0$ as $t\to\infty$.
- For $i=k$, $2\le k\le n$, we assume that $x_j(t)\to 0$ as $t\to\infty$, $1\le j\le k-1$. Then, from (4.13) we have

$$\begin{aligned}
\lim_{t\to\infty}\dot x_{k-1}(t) &= \lim_{t\to\infty}\{x_k+\theta^{\mathrm{T}}\gamma_{k-1}(x_1,\cdots,x_{k-1},x_k)\}\\
&=0
\end{aligned} \quad (4.14)$$

and from the uniqueness of solutions of (4.12) we conclude that $x_k(t)\to 0$ as $t\to\infty$.

Hence, $x(t)\to 0$ as $t\to\infty$. Finally, since $x=\phi(\zeta)$ is a diffeomorphism with $\phi(0)=0$, regulation is achieved in the original coordinates ζ, namely

$$\lim_{t\to\infty}\zeta(t)=0. \quad (4.15)$$

The above facts prove the following result.

Theorem 4.4: Suppose that the system (2.1) satisfies Proposition 2.1 and that the design procedure of Section III is applied to its parametric-pure-feedback form (3.1). Then, the equilibrium (4.9) of the resulting adaptive system (4.5) is uniformly stable and its region of attraction includes the set Ω defined in (4.10). Furthermore, regulation of the state $\zeta(t)$ is achieved for all initial conditions in Ω. □

V. Global Regulation

There are strong theoretical and practical reasons for investigating whether the stability properties of an adaptive system can be made global in the space of the states and parameter estimates. Systems with a finite region of attraction may not possess a wide enough robustness margin for disturbances and unmodeled dynamics. Furthermore, it is usually hard to find nonconservative estimates of finite regions of attraction.

Another aspect of the global stability issue is the comparison of the proposed adaptive controller to its deterministic counterpart, that is, the controller that would be used if the parameter vector θ were known. Suppose that for all values of θ there exists a deterministic controller which achieves global stabilization and regulation of the system. If, with θ unknown, the proposed adaptive controller does not achieve the same global stability, this loss is clearly due to adaptation.

The stability result of Theorem 4.4 is not global. However, as pointed out in Remark 4.2, this is not due to adaptation, because for parametric-pure-feedback systems global stability may not be achievable even with θ known. In Proposition 5.3, we define the class of "parametric-strict-feedback" systems, for which a globally stabilizing controller exists when θ is known. We then prove that for this class of systems our adaptive scheme guarantees global stability when θ is unknown.

In order to characterize the class of parametric-strict-feedback systems, we use the following assumption about the part of the system (2.1) that does not contain unknown parameters.

Assumption 5.1: There exists a *global* diffeomorphism $x = \phi(\zeta)$, with $\phi(0) = 0$, transforming the system.

$$\dot{\zeta} = f_0(\zeta) + g_0(\zeta)u \tag{5.1}$$

into the system

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad 1 \le i \le n-1 \\ \dot{x}_n &= \gamma_0(x) + \beta_0(x)u \end{aligned} \tag{5.2}$$

with

$$\gamma_0(0) = 0, \quad \beta_0(x) \neq 0 \quad \forall x \in \mathbb{R}^n. \tag{5.3}$$

Remark 5.2: The *local* existence of such a diffeomorphism is equivalent to the feedback linearization condition (2.4). At present there are no necessary and sufficient conditions verifying the *global* validity of this assumption. Sufficient conditions for Assumption 5.1 are given in [24], while necessary and sufficient conditions for the case where $\beta_0(x) \equiv$ const. can be found in [25], [26]. □

Proposition 5.3: Under assumption 5.1, the system (2.1) is globally diffeomorphically equivalent through $x = \phi(\zeta)$ to the *parametric-strict-feedback system*

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^{\mathrm{T}}\gamma_i(x_1, \cdots, x_i), \quad 1 \le i \le n-1 \\ \dot{x}_n &= \gamma_0(x) + \theta^{\mathrm{T}}\gamma_n(x) + \beta_0(x)u \end{aligned} \tag{5.4}$$

if and only if the following condition holds globally.

Parametric-Strict-Feedback Condition:

$$\begin{gathered} g_i \equiv 0, \\ [X, f_i] \in \mathscr{G}^j, \quad \forall X \in \mathscr{G}^j, \quad 0 \le j \le n-2, \\ 1 \le i \le p \end{gathered} \tag{5.5}$$

with $\mathscr{G}^j$, $0 \le j \le n-1$, as defined in (2.4).

Proof: The proof is very similar to that of Proposition 2.1. First, because of the assumptions that the diffeomorphism $x = \phi(\zeta)$ is global and $\beta_0(x) \neq 0$ $\forall x \in \mathbb{R}^n$, the distributions $\mathscr{G}^j$, $0 \le j \le n-1$ are globally defined and can be expressed in the x-coordinates as

$$\mathscr{G}^j = \operatorname{span}\left\{\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_{n-j}}\right\}, \quad 0 \le j \le n-1. \tag{5.6}$$

To prove sufficiency, note that if the parametric-pure-feedback condition (2.5) of Proposition 2.1 is replaced by the parametric-strict-feedback condition (5.5), then (2.12) is replaced by

$$\begin{gathered} g_i \equiv 0 \\ \left[\frac{\partial}{\partial x_j}, f_k\right] \in \operatorname{span}\left\{\frac{\partial}{\partial x_n}, \cdots, \frac{\partial}{\partial x_j}\right\}, \quad 2 \le j \le n, \\ 1 \le i \le p. \end{gathered} \tag{5.7}$$

Thus, the expression for f_i in (2.13) becomes

$$\begin{aligned} f_i &= \gamma_{1,i}(x_1)\frac{\partial}{\partial x_1} + \gamma_{2,i}(x_1, x_2)\frac{\partial}{\partial x_2} + \cdots \\ &\quad + \gamma_{n-1,i}(x_1, \cdots, x_{n-1})\frac{\partial}{\partial x_{n-1}} + \gamma_{n,i}(x_1, \cdots, x_n)\frac{\partial}{\partial x_n}, \\ &\quad 1 \le i \le p. \end{aligned} \tag{5.8}$$

The necessity is again straightforward. □

The above proposition gives a geometric characterization of the class of systems for which the following global properties can be achieved.

Theorem 5.4: Suppose that the system (2.1) satisfies Proposition 5.3 and that the design procedure of Section III is applied to its parametric-strict-feedback form (5.4). Then, the equilibrium

$$z = 0, \quad \vartheta_1 = \theta, \cdots, \vartheta_n = \theta$$

of the resulting adaptive system is globally uniformly stable. Furthermore, regulation of the state $\zeta(t)$ is achieved:

$$\lim_{t\to\infty} \zeta(t) = 0 \tag{5.9}$$

for all initial conditions in $\Omega = \mathbb{R}^{n(1+p)}$.

Proof: When the adaptive design procedure (3.3)–(3.22) is applied to the system (5.4), then for all $\vartheta_i \in \mathbb{R}^p$, $1 \le i \le n$, the change of coordinates (3.15) is one-to-one, onto, continuous, and has a continuous inverse, and the control (3.19) is well-defined since

$$\frac{\partial \gamma_i}{\partial x_{i+1}}(x) = 0, \quad \beta(x) = 0, \quad \beta_0(x) \neq 0, \quad \forall x \in \mathbb{R}^n. \tag{5.10}$$

Hence, (4.1)–(4.2) are trivially satisfied on $\mathscr{F} = B_x \times B_\vartheta^n = \mathbb{R}^{n(1+p)}$, and from (4.10) we conclude that $\Omega = \mathbb{R}^{n(1+p)}$. □

Remark 5.5: The results of Sections II–V can be extended to multiinput systems. We do not present this extension here, but refer the reader to [27]. □

VI. Global Tracking

Every regulation result in Sections II–V has its tracking counterpart. For brevity, we restrict our presentation to the tracking version of the global regulation result in Section V. The counterparts of nonglobal regulation results can be obtained using the same Lyapunov function argument as in this section to determine an invariant set in which asymptotic tracking is guaranteed.

Consider the nonlinear system

$$\dot{\zeta} = f_0(\zeta) + \sum_{i=1}^{p} \theta_i f_i(\zeta) + g_0(\zeta)u$$
$$y = h(\zeta) \qquad (6.1)$$

where $\zeta \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $\theta = [\theta_1, \cdots, \theta_p]^T$ is the vector of unknown constant parameters, h is a smooth function on $\mathbb{R}^n$ with $h(0) = 0$, and the vector fields g_0, f_i, $0 \le i \le p$, are smooth on $\mathbb{R}^n$ with $g(\zeta) \neq 0\ \forall \zeta \in \mathbb{R}^n$, $f_i(0) = 0$, $0 \le i \le p$. We first formulate the input-output counterpart of assumption 5.1.

Assumption 6.1: There exist $n - \rho$ smooth functions $\phi_i(\zeta)$, $\rho + 1 \le i \le n$, such that the change of coordinates

$$\begin{aligned} x_1 &= h(\zeta) \\ x_2 &= L_{f_0}h(\zeta) \\ x_3 &= L_{f_0}^2 h(\zeta) \\ &\vdots \\ x_\rho &= L_{f_0}^{\rho-1}h(\zeta) \\ x_i &= \phi_i(\zeta), \qquad \rho + 1 \le i \le n \end{aligned} \qquad (6.2)$$

is a *global* diffeomorphism $x = \phi(\zeta)$ transforming the system

$$\dot{\zeta} = f_0(\zeta) + g_0(\zeta)u$$
$$y = h(\zeta) \qquad (6.3)$$

into the special normal form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{\rho-1} &= x_\rho \\ \dot{x}_\rho &= \gamma_0(x) + \beta_0(x)u \\ \dot{x}^r &= \Phi_0(y, x^r) \\ y &= x_1, \end{aligned} \qquad (6.4)$$

with

$$\gamma_0(0) = L_{f_0}^\rho h(0) = 0, \quad \Phi_0(0,0) = 0 \qquad (6.5)$$

$$\beta_0(x) = L_{g_0}L_{f_0}^{\rho-1}h(\zeta) \neq 0, \qquad \forall x \in \mathbb{R}^n. \qquad (6.6)$$

Remark 6.2: In order for (6.3) to be *locally* equivalent to (6.4), it is necessary and sufficient that the following conditions hold in a neighborhood of the origin $\zeta = 0$:

$$L_{g_0}L_{f_0}^i h \equiv 0, \qquad 0 \le i \le \rho - 2 \qquad (6.7)$$

$$L_{g_0}L_{f_0}^{\rho-1}h(0) \neq 0 \qquad (6.8)$$

$$\mathcal{G}^{\rho-1} \text{ is involutive and of constant rank } \rho. \qquad (6.9)$$

The sufficiency of these conditions is a consequence of Proposition 10 in [28]. The necessity can be easily established by verifying that (6.7)–(6.9) hold in the coordinates of (6.4).

Furthermore, as shown in [29, Corollary 5.7], (6.3) is *globally* equivalent to (6.4) if and only if the following conditions are satisfied for all $\zeta \in \mathbb{R}^n$:

$$L_{g_0}L_{f_0}^i h \equiv 0, \qquad 0 \le i \le \rho - 2 \qquad (6.10)$$

$$L_{g_0}L_{f_0}^{\rho-1}h \neq 0 \qquad (6.11)$$

$$\mathcal{G}^{\rho-1} \text{ is involutive and of constant rank } \rho. \qquad (6.12)$$

the manifold

$$M = \{\zeta \in \mathbb{R}^n : h(\zeta) = L_{f_0}h(\zeta) = \cdots = L_{f_0}^{\rho-1}h(\zeta)\} \qquad (6.13)$$

is connected, and the vector fields $\bar{g}_0, \mathrm{ad}_{\bar{f}_0}\bar{g}_0, \cdots, \mathrm{ad}_{\bar{f}_0}^{\rho-1}\bar{g}_0$ are complete where

$$\bar{g}_0 = \frac{1}{L_{g_0}L_{f_0}^{\rho-1}h} g_0, \quad \bar{f}_0 = f_0 - \frac{L_{f_0}^\rho h}{L_{g_0}L_{f_0}^{\rho-1}h} g_0. \qquad (6.14)$$

□

We are now ready to formulate the input-output counterpart of Proposition 5.3.

Proposition 6.3: Under Assumption 6.1, the system (6.1) is globally diffeomorphically equivalent to the *parametric-strict-feedback normal form*

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T\gamma_i(x_1, \cdots, x_i, x^r), \qquad 1 \le i \le \rho - 1 \\ \dot{x}_\rho &= \gamma_0(x) + \theta^T\gamma_\rho(x) + \beta_0(x)u \\ \dot{x}^r &= \Phi_0(y, x^r) + \sum_{i=1}^{p} \theta_i \Phi_i(y, x^r) \\ y &= x_1 \end{aligned} \qquad (6.15)$$

if and only if the following condition holds globally.

Parametric-Strict-Feedback Condition:

$$[X, f_i] \in \mathcal{G}^j, \qquad \forall X \in \mathcal{G}^j, \quad 0 \le j \le \rho - 2, 1 \le i \le p \qquad (6.16)$$

with $\mathcal{G}^j$, $0 \le j \le \rho - 1$, as defined in (2.4).

Proof: The proof follows closely that of Proposition 5.3. First, because of the assumptions that the diffeomorphism $x = \phi(\zeta)$ defined in (6.2) is global and that $\beta_0(x) \neq 0$ $\forall x \in \mathbb{R}^n$, the distributions $\mathcal{G}^j$, $0 \le j \le \rho - 1$, are globally defined and can be expressed in the x-coordinates as

$$\mathcal{G}^j = \mathrm{span}\left\{\frac{\partial}{\partial x_\rho}, \cdots, \frac{\partial}{\partial x_{\rho-j}}\right\}, \qquad 0 \le j \le \rho - 1. \qquad (6.17)$$

The sufficiency follows from the fact that, by (6.16) and

(6.17),

$$\left[\frac{\partial}{\partial x_j}, f_i\right] \in \text{span}\left\{\frac{\partial}{\partial x_\rho}, \cdots, \frac{\partial}{\partial x_j}\right\}, \quad 2 \le j \le \rho,\ 1 \le i \le p. \tag{6.18}$$

Thus, the expression for f_i in the x-coordinates is

$$\begin{aligned} f_i = {} & \gamma_{1,i}(x_1, x^r)\frac{\partial}{\partial x_1} + \gamma_{2,i}(x_1, x_2, x^r)\frac{\partial}{\partial x_2} + \cdots \\ & + \gamma_{\rho-1,i}(x_1, \cdots, x_{\rho-1}, x^r)\frac{\partial}{\partial x_{\rho-1}} \\ & + \gamma_{\rho,i}(x_1, \cdots, x_\rho, x^r)\frac{\partial}{\partial x_\rho} \\ & + \sum_{j=\rho+1}^{n} \Phi_{i,j}(x_1, x^r)\frac{\partial}{\partial x_j}, \qquad 1 \le i \le p. \end{aligned} \tag{6.19}$$

The necessity is again straightforward. □

Remark 6.4: To obtain the input-output counterpart of Proposition 2.1, one just needs to replace the feedback linearization condition (2.4) with conditions (6.7)–(6.9), and the parametric-pure-feedback condition (2.5) with

$$\begin{aligned} & g_i \in \mathcal{G}^0, \\ & [X, f_i] \in \mathcal{G}^{j+1}, \quad \forall X \in \mathcal{G}^j,\ 0 \le j \le \rho - 2, \\ & \qquad 1 \le i \le p. \end{aligned} \tag{6.20}$$

□

As in most tracking problems, we need an assumption about the stability of the x^r-subsystem of (6.15).

Assumption 6.5: The x^r-subsystem of (6.15) has the bounded-input bounded-state (BIBS) property with respect to y as its input.

It was shown in [9, Proposition 2.1] that the following conditions are sufficient for Assumption 6.5 to be satisfied:

i) the zero dynamics of (6.1) are globally exponentially stable, and

ii) the vector field $\Phi = \Phi_0 + \sum_{i=1}^{p}\theta_i\Phi_i$ in (6.15) is globally Lipschitz in x.

These conditions are more convenient for nonglobal results, where i) can be used to estimate the region of attraction via a converse Lyapunov theorem. However, they are too restrictive for global results. For example, the system $\dot{x}^r = -(x^r)^3 + y^2$ violates both i) and ii), but is easily seen to satisfy Assumption 6.5.

The control objective is to force the output y of the system (6.1) to *asymptotically track* a known reference signal $y_r(t)$, while keeping all the closed-loop signals bounded.

Assumption 6.6: The reference signal $y_r(t)$ and its first ρ derivatives are known and bounded.

To achieve the asymptotic tracking objective, the design procedure presented in Section III is modified as follows:

Step 0: Define

$$z_1 = x_1 - y_r. \tag{6.21}$$

Step 1: Starting with

$$\dot{z}_1 = x_2 + \theta^{\mathrm{T}}\gamma_1(x_1, x^r) - \dot{y}_r \tag{6.22}$$

let ϑ_1 be an estimate of θ and define the new state z_2 as

$$\begin{aligned} z_2 & \triangleq c_1 z_1 + x_2 + \vartheta_1^{\mathrm{T}}\gamma_1(x_1, x^r) - \dot{y}_r \\ & \triangleq c_1 z_1 + x_2 + \vartheta_1^{\mathrm{T}} w_1(z_1, x^r, y_r) - \dot{y}_r, \qquad c_1 \ge 2. \end{aligned} \tag{6.23}$$

Substitute (6.23) into (6.22) to obtain

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(z_1, x^r, y_r). \tag{6.24}$$

Then, let the update law for the parameter estimate ϑ_1 be

$$\dot{\vartheta}_1 = z_1 w_1(z_1, x^r, y_r). \tag{6.25}$$

Step 2: Using the definitions for z_1, z_2, and $\dot{\vartheta}_1$, write $\dot{z}_2$ as

$$\begin{aligned} \dot{z}_2 = {} & c_1\left[-c_1 z_1 + z_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(z_1, x^r, y_r)\right] \\ & + x_3 + \theta^{\mathrm{T}}\gamma_2(x_1, x_2, x^r) \\ & + z_1 w_1(z_1, x^r, y_r)^{\mathrm{T}}\gamma_1(x_1, x^r) \\ & + \vartheta_1^{\mathrm{T}}\left[\frac{\partial\gamma_1(x_1, x^r)}{\partial x_1}(x_2 + \theta^{\mathrm{T}}\gamma_1(x_1, x^r)) \right. \\ & + \frac{\partial\gamma_1(x_1, x^r)}{\partial x^r} \\ & \left. \cdot\left(\Phi_0(x_1, x^r) + \sum_{i=1}^{p}\theta_i\Phi_i(x_1, x^r)\right)\right] - \ddot{y}_r \\ \triangleq {} & x_3 + \varphi_2(z_1, z_2, x^r, \vartheta_1, y_r, \dot{y}_r, \ddot{y}_r) \\ & + \theta^{\mathrm{T}} w_2(z_1, z_2, x^r, \vartheta_1, y_r, \dot{y}_r). \end{aligned} \tag{6.26}$$

Let ϑ_2 be a *new* estimate of θ and define the new state z_3 as

$$\begin{aligned} z_3 \triangleq {} & c_2 z_2 + x_3 + \varphi_2(z_1, z_2, x^r, \vartheta_1, y_r, \dot{y}_r, \ddot{y}_r) \\ & + \vartheta_2^{\mathrm{T}} w_2(z_1, z_2, x^r, \vartheta_1, y_r, \dot{y}_r), \qquad c_2 \ge 2. \end{aligned} \tag{6.27}$$

Substitute (6.27) into (6.26) to obtain

$$\dot{z}_2 = -c_2 z_2 + z_3 + (\theta - \vartheta_2)^{\mathrm{T}} w_2(z_1, z_2, x^r, \vartheta_1, y_r, \dot{y}_r). \tag{6.28}$$

Then, let the update law for the new estimate ϑ_2 be

$$\dot{\vartheta}_2 = z_2 w_2(z_1, z_2, x^r, \vartheta_1, y_r, \dot{y}_r). \tag{6.29}$$

Step i ($2 \le i \le \rho - 1$): Using the definitions for $z_1, \cdots, z_i$ and $\dot{\vartheta}_1, \cdots, \dot{\vartheta}_{i-1}$, express the derivative of z_i as

$$\begin{aligned} \dot{z}_i = {} & x_{i+1} + \varphi_i(z_1, \cdots, z_i, x^r, \vartheta_1, \cdots, \vartheta_{i-1}, \\ & y_r, \cdots, y_r^{(i)}) \\ & + \theta^{\mathrm{T}} w_i(z_1, \cdots, z_i, x^r, \vartheta_1, \cdots, \vartheta_{i-1}, \\ & y_r, \cdots, y_r^{(i-1)}). \end{aligned} \tag{6.30}$$

Let ϑ_i be a *new* estimate of θ and define the new state z_{i+1}

as

$$\begin{aligned} z_{i+1} \triangleq c_i z_i + x_{i+1} &+ \varphi_i(z_1,\cdots,z_i,x^r,\vartheta_1,\cdots,\vartheta_{i-1},y_r,\cdots,y_r^{(i)}) \\ &+ \vartheta_i^{\mathrm{T}} w_i(z_1,\cdots,z_i,x^r,\vartheta_1,\cdots,\vartheta_{i-1}, \\ &y_r,\cdots,y_r^{(i-1)}), \qquad c_i \ge 2. \end{aligned} \tag{6.31}$$

Substitute (6.31) into (6.30) to obtain

$$\begin{aligned} \dot z_i = -c_i z_i + z_{i+1} + (\theta - \vartheta_i)^{\mathrm{T}} \\ \cdot w_i(z_1,\cdots,z_i,x^r,\vartheta_1,\cdots,\vartheta_{i-1},y_r,\cdots,y_r^{(i-1)}). \end{aligned} \tag{6.32}$$

Then, let the update law for ϑ_i be

$$\dot\vartheta_i = z_i w_i(z_1,\cdots,z_i,x^r,\vartheta_1,\cdots,\vartheta_{i-1},y_r,\cdots,y_r^{(i-1)}). \tag{6.33}$$

Step ρ*:* Using the definitions for $z_1,\cdots,z_\rho$ and $\dot\vartheta_1,\cdots,\dot\vartheta_{\rho-1}$, express the derivative of z_ρ as

$$\begin{aligned} \dot z_\rho = \beta_0(x)u + \varphi_\rho(z_1,\cdots,z_\rho,x^r,\vartheta_1,\cdots,\vartheta_{\rho-1}, \\ y_r,\cdots,y_r^{(\rho)}) \\ + \theta^{\mathrm{T}} w_\rho(z_1,\cdots,z_\rho,x^r,\vartheta_1,\cdots,\vartheta_{\rho-1}, \\ y_r,\cdots,y_r^{(\rho-1)}). \end{aligned} \tag{6.34}$$

Let ϑ_ρ be a *new* estimate of θ and choose the control u as

$$u = \frac{1}{\beta_0(x)}\left[-c_\rho z_\rho - \varphi_\rho - \vartheta_\rho^{\mathrm{T}} w_\rho\right], \qquad c_\rho \ge 2. \tag{6.35}$$

Substitute (6.35) into (6.34) to obtain

$$\begin{aligned} \dot z_\rho = -c_\rho z_\rho + (\theta - \vartheta_\rho)^{\mathrm{T}} w_\rho(z_1,\cdots,z_\rho,x^r, \\ \vartheta_1,\cdots,\vartheta_{\rho-1},y_r,\cdots,y_r^{(\rho-1)}). \end{aligned} \tag{6.36}$$

Finally, let the update law for the estimate ϑ_ρ be

$$\dot\vartheta_\rho = z_\rho w_\rho(z_1,\cdots,z_\rho,x^r,\vartheta_1,\cdots,\vartheta_{\rho-1},y_r,\cdots,y_r^{(\rho-1)}). \tag{6.37}$$

As was the case in the regulation result of Section V, the assumptions of Proposition 6.3 guarantee that the design procedure (6.21)–(6.37) is globally feasible. The resulting closed-loop adaptive system is given by

$$\begin{aligned} \dot z_1 &= -c_1 z_1 + z_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(z_1,x^r,y_r) \\ &\vdots \\ \dot z_{\rho-1} &= -c_{\rho-1} z_{\rho-1} + z_\rho + (\theta - \vartheta_{\rho-1})^{\mathrm{T}} w_{\rho-1} \\ &\quad \cdot(z_1,\cdots,z_{\rho-1},x^r,\vartheta_1,\cdots,\vartheta_{\rho-2},y_r,\cdots,y_r^{(\rho-2)}) \\ \dot z_\rho &= -c_\rho z_\rho + (\theta - \vartheta_\rho)^{\mathrm{T}} w_\rho \\ &\quad \cdot(z_1,\cdots,z_\rho,x^r,\vartheta_1,\cdots,\vartheta_{\rho-1},y_r,\cdots,y_r^{(\rho-1)}) \\ \dot x^r &= \Phi_0(y,x^r) + \sum_{i=1}^{p}\theta_i\Phi_i(y,x^r) \\ \dot\vartheta_i &= z_i w_i(z_1,\cdots,z_i,x^r,\vartheta_1,\cdots,\vartheta_{i-1},y^r,\cdots,y_r^{(i-1)}), \\ &\qquad 1 \le i \le \rho \\ y &= z_1 + y_r. \end{aligned} \tag{6.38}$$

The stability and tracking properties of (6.38) will be established using the quadratic function

$$V_\rho(z_1,\cdots,z_\rho,\vartheta_1,\cdots,\vartheta_\rho) = \frac{1}{2}\sum_{i=1}^{\rho}\left[z_i^2 + (\theta - \vartheta_i)^{\mathrm{T}}(\theta - \vartheta_i)\right]. \tag{6.39}$$

The derivative of V_ρ along the solutions of (6.38) with $c_i \ge 2$, $1 \le i \le \rho$, is

$$\begin{aligned} \dot V_\rho &= -\sum_{i=1}^{\rho}\left[c_i z_i^2 + (\theta - \vartheta_i)^{\mathrm{T}}(z_i w_i - \dot\vartheta_i)\right] + \sum_{i=1}^{\rho-1} z_i z_{i+1} \\ &= -\sum_{i=1}^{\rho} c_i z_i^2 + \sum_{i=1}^{\rho-1} z_i z_{i+1} \\ &\le -\sum_{i=1}^{\rho} z_i^2 \le 0. \end{aligned} \tag{6.40}$$

This proves that V_ρ is bounded. Hence $z_1,\cdots,z_\rho$ and $\vartheta_1,\cdots,\vartheta_\rho$ are bounded, and $\dot V_\rho$ is bounded and integrable. The boundedness of z_1 and y_r implies that y is bounded. Combining this with Assumption 6.5 proves that x^r is bounded. Therefore, the state vector of (6.38) is bounded. This fact, combined with (6.31) and Assumption 6.6, implies the boundedness of x, ζ, and u. Thus, the derivatives $\dot z_1,\cdots,\dot z_\rho$ are bounded, which implies that $\ddot V_\rho$ is bounded. Hence $\dot V_\rho \to 0$ as $t \to \infty$, which, combined with (6.40), proves that

$$\lim_{t\to\infty} z_i(t) = 0, \qquad 1 \le i \le \rho. \tag{6.41}$$

In particular, this means that asymptotic tracking is achieved:

$$\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty}\left[y(t) - y_r(t)\right] = 0. \tag{6.42}$$

These results are summarized as follows.

Theorem 6.7: Suppose that the system (6.1) satisfies Proposition 6.3 and Assumption 6.5, and that the design procedure (6.21)–(6.37) is applied to its parametric-strict-feedback normal form (6.15). Furthermore, suppose that the reference signal y_r satisfies Assumption 6.6. Then, all the signals in the resulting closed-loop adaptive system (6.38) are bounded and asymptotic tracking (6.42) is achieved for all initial conditions in $\mathbb{R}^{n+\rho p}$. □

Remark 6.8: Since in (6.15) we allowed γ_i, $i = 1,\cdots,\rho - 1$ to depend on x^r, we had to restrict Φ_i, $i = 0,\cdots,p$ not to depend on $x_2,\cdots,x_\rho$. If Φ_i also depended on $x_2,\cdots,x_\rho$ and Assumption 6.5 were modified to read "with respect to $x_1,\cdots,x_\rho$ as its inputs," the boundedness of y would not guarantee the boundedness of x^r, and the arguments after (6.40) would be invalid. However, if γ_i, $i = 1,\cdots,q-1$ $(1 \le q \le \rho)$, were restricted not to depend on x^r, then Φ_i could be allowed to depend on $x_2,\cdots,x_q$ also. In that case, the boundedness of $z_1,\cdots,z_\rho$ and Assumption 6.6 would guarantee the boundedness of $x_1,\cdots,x_q$, and hence the boundedness of x^r. The boundedness of $x_{q+1},\cdots,x_\rho$ would then follow from arguments similar to those after (6.40). This means that our design procedure can be easily modified to be applicable to systems of the form (6.1) which are globally diffeomorphically equivalent to the following *parametric-strict-feedback normal form*:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta^T\gamma_1(x_1)\\
&\vdots\\
\dot{x}_{q-1} &= x_q + \theta^T\gamma_{q-1}(x_1,\cdots,x_{q-1})\\
\dot{x}_q &= x_{q+1} + \theta^T\gamma_q(x_1,\cdots,x_q,x^r)\\
&\vdots\\
\dot{x}_{\rho-1} &= x_\rho + \theta^T\gamma_q(x_1,\cdots,x_{\rho-1},x^r)\\
\dot{x}_\rho &= \gamma_0(x) + \theta^T\gamma_\rho(x) + \beta_0(x)u\\
\dot{x}^r &= \Phi_0(x_1,\cdots,x_q,x^r) + \sum_{i=1}^{p}\theta_i\Phi_i(x_1,\cdots,x_q,x^r)\\
y &= x_1.
\end{aligned} \tag{6.43}$$

Using [29, Corollary 5.7], it is straightforward to show that there exists a parameter-independent global diffeomorphism $x = \phi(\zeta)$ transforming (6.1) into (6.43) if and only if in addition to (6.10), (6.11), (6.13), and (6.14), the following conditions are satisfied for all $\zeta \in \mathbb{R}^n$:

$$\mathcal{G}^{\rho-q} \text{ is involutive and of constant rank } \rho - q + 1 \tag{6.44}$$

$$[X, f_i] \in \mathcal{G}^j, \quad \forall X \in \mathcal{G}^j, \quad 0 \le j \le q-2, 1 \le i \le p \tag{6.45}$$

$$d(L_{f_i}L_{f_0}^j h) \in \operatorname{span}\{dh,\cdots,d(L_{f_0}^j h)\}, \quad 0 \le j \le \rho - q - 1, 1 \le i \le p. \tag{6.46}$$

□

Discussion and Examples

With the help of two examples, we now discuss some of the main features of the new adaptive scheme. The first example illustrates the systematic nature of the design procedure, while the second one compares the stability properties of the new scheme to those of the nonlinearity-constrained scheme of [9].

Example 7.1 (Regulation): We first consider a "benchmark" example of adaptive nonlinear regulation:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta x_1^2\\
\dot{x}_2 &= x_3\\
\dot{x}_3 &= u
\end{aligned} \tag{7.1}$$

where θ is an unknown constant parameter. This system violates both the geometric conditions of [1]–[3] and the growth assumptions of [5], [6], [9], [12]. In fact, the only available global result for this example was obtained in [7].

The system (7.1) is already in the form of (5.4) with $\beta_0 = 1$. Hence, this system satisfies the conditions of Theorem 5.4, which guarantees that the point $x = 0$, $\vartheta_1 = \vartheta_2 = \vartheta_3 = \theta$ is a globally stable equilibrium of the adaptive system. Moreover, for any initial conditions $x(0) \in \mathbb{R}^3$, $(\vartheta_1(0), \vartheta_2(0), \vartheta_3(0)) \in \mathbb{R}^3$, the regulation of the state $x(t)$ is achieved

$$\lim_{t\to\infty} x(t) = 0. \tag{7.2}$$

The design procedure of Section III, applied to (7.1), is as follows.

Step 0: Define $z_1 = x_1$.

Step 1: Let ϑ_1 be an estimate of θ and define the new state z_2 as

$$z_2 = 2z_1 + x_2 + \vartheta_1 z_1^2. \tag{7.3}$$

Substitute (7.3) into (7.0) to obtain

$$\dot{z}_1 = -2z_1 + z_2 + z_1^2(\theta - \vartheta_1). \tag{7.4}$$

Then, let the update law for ϑ_1 be

$$\dot{\vartheta}_1 = z_1^3. \tag{7.5}$$

Step 2: Using (7.3) and (7.5) write $\dot{z}_2$ as

$$\dot{z}_2 = 2(x_2 + \theta x_1^2) + x_3 + \vartheta_1 2z_1(x_2 + \theta x_1^2) + z_1^5. \tag{7.6}$$

Let ϑ_2 be a new estimate of θ, and define the new state

$$z_3 = 2z_2 + 2(x_2 + \vartheta_2 x_1^2)(1 + \vartheta_1 z_1) + z_1^5 + x_3. \tag{7.7}$$

Substitute (7.7) into (7.6) to obtain

$$\dot{z}_2 = -2z_2 + z_3 + 2z_1^2(1 + \vartheta_1 z_1)(\theta - \vartheta_2). \tag{7.8}$$

Then, let the update law for ϑ_2 be

$$\dot{\vartheta}_2 = 2z_2 z_1^2(1 + \vartheta_1 z_1). \tag{7.9}$$

Step 3: Using (7.3), (7.5), (7.7), and (7.8), write $\dot{z}_3$ as

$$\begin{aligned}
\dot{z}_3 = {}& 2[-2z_2 + z_3 + 2z_1^2(1 + \vartheta_1 z_1)(\theta - \vartheta_2)]\\
&+ 2[x_3 + 2x_1\vartheta_2(x_2 + \theta x_1^2)\\
&+ 2x_1^2 z_2 z_1^2(1 + \vartheta_1 z_1)](1 + \vartheta_1 z_1)\\
&+ 2(x_2 + \vartheta_2 x_1^2)[z_1^4 + \vartheta_1(x_2 + \theta x_1^2)]\\
&+ 5z_1^4(x_2 + \theta x_1^2) + u.
\end{aligned} \tag{7.10}$$

Let ϑ_3 be a new estimate of θ, and define the control u as

$$\begin{aligned}
u = {}& -2z_3 - 2[-2z_2 + z_3 + 2z_1^2(1 + \vartheta_1 z_1)(\vartheta_3 - \vartheta_2)]\\
&- 2[x_3 + 2x_1\vartheta_2(x_2 + \vartheta_3 x_1^2)\\
&+ 2x_1^2 z_2 z_1^2(1 + \vartheta_1 z_1)]\\
&\cdot(1 + \vartheta_1 z_1) - 2(x_2 + \vartheta_2 x_1^2)[z_1^4 + \vartheta_1(x_2 + \vartheta_3 x_1^2)]\\
&- 5z_1^4(x_2 + \vartheta_3 x_1^2).
\end{aligned} \tag{7.11}$$

Substitute (7.11) into (7.10) to obtain

$$\begin{aligned}
\dot{z}_3 = {}& -2z_3 + [4z_1^2(1 + \vartheta_1 z_1) + 4x_1^3\vartheta_2\\
&+ 2\vartheta_1(x_2 + \vartheta_2 x_1^2)x_1^2 + 5z_1^6](\theta - \vartheta_3).
\end{aligned} \tag{7.12}$$

Finally, let the parameter update law for ϑ_3 be

$$\dot{\vartheta}_3 = z_3\big[4z_1^2(1+\vartheta_1 z_1) + 4x_1^3\vartheta_2 + 2\vartheta_1(x_2+\vartheta_2 x_1^2)x_1^2 + 5z_1^6\big]. \quad (7.13)$$

The resulting adaptive system is

$$\begin{aligned}
\dot{z}_1 &= -2z_1 + z_2 + z_1^2(\theta - \vartheta_1)\\
\dot{z}_2 &= -2z_2 + z_3 + 2z_1^2(1+\vartheta_1 z_1)(\theta - \vartheta_2)\\
\dot{z}_3 &= -2z_3 + \big[4z_1^2(1+\vartheta_1 z_1) + 4x_1^3\vartheta_2 + 2\vartheta_1(x_2+\vartheta_2 x_1^2)x_1^2 + 5z_1^6\big](\theta - \vartheta_3)\\
\dot{\vartheta}_1 &= z_1^3\\
\dot{\vartheta}_2 &= 2z_2 z_1^2(1+\vartheta_1 z_1)\\
\dot{\vartheta}_3 &= z_3\big[4z_1^2(1+\vartheta_1 z_1) + 4x_1^3\vartheta_2 + 2\vartheta_1(x_2+\vartheta_2 x_1^2)x_1^2 + 5z_1^6\big]. \quad (7.14)
\end{aligned}$$

Using the Lyapunov function

$$V = \frac{1}{2}\big[z_1^2 + z_2^2 + z_3^2 + (\theta-\vartheta_1)^2 + (\theta-\vartheta_2)^2 + (\theta-\vartheta_3)^2\big] \quad (7.15)$$

it is straightforward to establish the global stability and regulation properties of (7.14). □

Example 7.2 (Tracking): Consider now the problem in which the output y of the nonlinear system

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta x_1^2\\
\dot{x}_2 &= u + x_3\\
\dot{x}_3 &= -x_3 + y\\
y &= x_1 \quad (7.16)
\end{aligned}$$

is required to asymptotically track the reference signal $y_r = 0.1\sin t$.

For the sake of comparison, let us first solve this problem using the scheme of [9]. This scheme employs the control

$$u = -x_3 + k_1(x_1 - y_r) + k_2(x_2 + \hat{\theta}_1 x_1^2 - \dot{y}_r) + \ddot{y}_r - 2\hat{\theta}_1 x_1 x_2 - 2\hat{\theta}_2 x_1^3 \quad (7.17)$$

where $\hat{\theta}_1, \hat{\theta}_2$, the estimates of θ, θ^2, respectively, are obtained from the update laws

$$\dot{\hat{\theta}}_1 = \frac{e_1\xi_1}{1+\xi_1^2+\xi_2^2}, \quad \dot{\hat{\theta}}_2 = \frac{e_1\xi_2}{1+\xi_1^2+\xi_2^2}. \quad (7.18)$$

Using a relative-degree-two stable filter $M(s)$, the variables e_1, ξ_1, ξ_2 in (7.18) are defined as

$$e_1 = y - y_r + \omega - \hat{\theta}_1\xi_1 - \hat{\theta}_2\xi_2 \quad (7.19)$$

$$\xi_1 = M(s)\big[2x_1x_2 + k_2x_1^2\big] \quad (7.20)$$

$$\xi_2 = M(s)\big[2x_1^3\big] \quad (7.21)$$

$$\omega = M(s)\big[\hat{\theta}_1(2x_1x_2 + k_2x_1^2) + \hat{\theta}_2(2x_1^3)\big]. \quad (7.22)$$

Simulations of this system were performed with

$$M(s) = \frac{1}{s^2+5s+6}, \qquad \theta = 1,\ k_1 = -6,\ k_2 = -5 \quad (7.23)$$

and all the initial conditions zero, except for $x_1(0)$, which was varied between 0 and 0.45. The results of these simulations, shown in Fig. 1, indicate that the response of the closed-loop system is bounded for $x_1(0)$ sufficiently small, that is, for $x_1(0) < 0.45$. However, for $x_1(0) \geq 0.45$, the response is unbounded. This behavior is consistent with the proof of Theorem 3.3 in [9], which guarantees boundedness for all initial conditions only under a global Lipschitz assumption. In the above system, the presence of the term x_1^2 leads to the violation of this assumption, and, as the simulations show, to unbounded response. Simulations with other schemes based on linear growth conditions [5], [12] show that the behavior illustrated by Fig. 1 is typical.

The unbounded behavior in Fig. 1 is avoided by the new scheme, which results in globally stable tracking. The design procedure in Section VII, applied to the system (7.16), results in the control

$$u = -x_3 - 3z_2 - 2(x_2 + \vartheta_2 x_1^2)(1+\vartheta_1 x_1) - z_1 x_1^4 + 2\dot{y}_r + \ddot{y}_r \quad (7.24)$$

and the update laws

$$\dot{\vartheta}_1 = z_1 x_1^2, \quad \dot{\vartheta}_2 = 2z_2 x_1^2(1+\vartheta_1 z_1) \quad (7.25)$$

where

$$\begin{aligned}
z_1 &= x_1 - y_r\\
z_2 &= 2(x_1 - y_r) + x_2 + \vartheta_1 x_1^2 - \dot{y}_r. \quad (7.26)
\end{aligned}$$

Theorem 6.7 establishes that uniform stability and asymptotic tracking are achieved for all $x_1(0), x_2(0), x_3(0), \vartheta_1(0), \vartheta_2(0)$. This is illustrated by simulations in Fig. 2. □

The above example illustrates an obvious advantage of the new scheme when applied to parametric-strict-feedback systems: it guarantees global stability for all types of smooth nonlinearities. For parametric-pure-feedback systems, when the feedback linearization is not global, the new scheme provides an estimate of the region of attraction. An advantage of the schemes in [1], [5]–[9], [12] is that they provide local results without assuming the parametric-pure-feedback form. However, estimates of the region of attraction are given only in [1], [6]–[8]. A quantitative comparison of the regions of attraction and robustness properties guaranteed by different schemes is a topic of future research.

VIII. Conclusions

The results of this paper have advanced in several directions our ability to control nonlinear systems with unknown constant parameters. The most significant progress has been made in solving the *global* adaptive regulation and tracking problems. The class of nonlinear systems for which these problems can be solved systematically has been substantially enlarged. The parametric-strict-feedback condition precisely

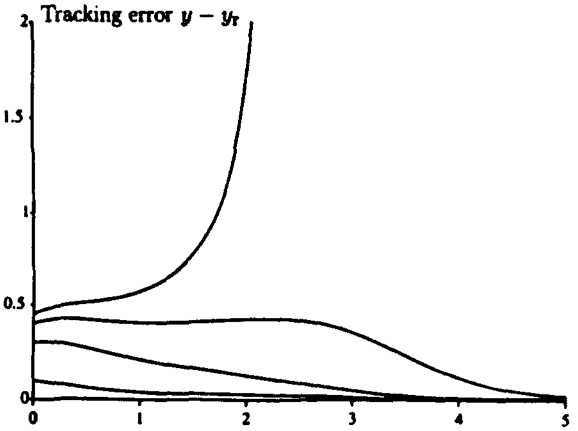

Fig. 1.

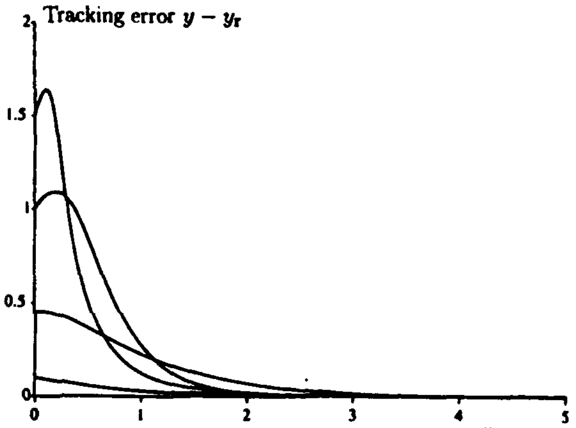

Fig. 2.

characterizes the class of systems for which the global results hold with any type of smooth nonlinearities. For the broader class of systems satisfying the parametric-pure-feedback condition, the regulation and tracking results may not be global, but are guaranteed in regions for which *a priori* estimates are given. It is crucial that the loss of globality, when it occurs, is not due to adaptation, but is inherited from the deterministic part of the problem. All these results are obtained using a step-by-step procedure which, at each step, interlaces a change of coordinates with the construction of an update law. Apart from the geometric conditions, this paper uses simple analytical tools, familiar to most control engineers.

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A Toolkit for Nonlinear Feedback Design

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Abstract

Motivated by several recent results, we assemble a set of basic tools which can be used to construct systematic procedures for nonlinear feedback design. As an illustration, we construct a backstepping procedure for observer-based global stabilization and tracking of a class of nonlinear systems.

Keywords. Design tools, nonlinear damping, integrator backstepping, observer-based design, global tracking.

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1 Introduction

The complexity of the nonlinear output-feedback problem challenges not only the researcher's knowledge of nonlinear geometric techniques [4, 16], but also his/her ability to employ, and often invent, a wide variety of other tools. This is particularly apparent in several recent results which make use of intricate combinations of diverse concepts: special classes of systems characterized by geometric conditions [8, 9, 15] or "Control Lyapunov Functions" [17, 5], strict positive real properties of some part of the system [10, 12] and filtered transformations to guarantee these properties [13, 15], means to deal with swapping terms [9] or to avoid them altogether [15], etc. Proofs combining these tools may appear too technical and discourage potential users.

A more systematic treatment, which supplements rigor with intuitive appeal, seems to be needed, and we make a step in this direction. In Section 2 we assemble a set of four simple tools for nonlinear feedback design, either with or without full-state measurement. The first two of these tools, "nonlinear damping" and "integrator backstepping," were used previously in adaptive and nonlinear control [3, 1, 22, 2, 10, 8, 14, 15, 5]. In Section 3 the tools of Section 2 are employed to give an alternative solution to an output-feedback problem recently solved by Marino and Tomei [15]. The dynamic part of the controller designed in Section 3 consists of only a nonlinear observer, while in [15] it also contains the filters required for the filtered transformations.

2 The Design Toolkit

Throughout this section it is assumed that a feedback control $u = \alpha(x)$ is known, which, when applied to the system

$$\dot{x} = f(x) + g(x)u\,, \quad x \in \mathbb{R}^n\,,\ u \in \mathbb{R}\,, \tag{2.1}$$

guarantees **global** boundedness of $x(t)$ and regulation of $\eta(x) \in \mathbb{R}^m$, that is, $\eta(x(t)) \to 0$ as $t \to \infty$. These properties are ascertained by the inequality

$$\frac{\partial V}{\partial x}(x)\left[f(x) + g(x)\alpha(x)\right] \leq -W(\eta(x)) \leq 0\,, \quad \forall x \in \mathbb{R}^n\,, \tag{2.2}$$

where $V(x)$ is positive definite and radially unbounded, and $W(\eta)$ is positive definite. It is further assumed that f, g, η, α, V and W are C^2 on $\mathbb{R}^n$.

Each of the four lemmas in this section employs $V(x)$ and $\alpha(x)$ to design a feedback control for a particular perturbed or augmented version of (2.1).

In the first lemma, the system (2.1) is perturbed by an unknown disturbance. As the following example demonstrates, even an exponentially decaying disturbance may cause explosive forms of unbounded behavior if it multiplies a term with significant nonlinear growth rate.

Example 1. Suppose that in the system

$$\dot{x} = x^3 + u + x^2 k e^{-t}, \tag{2.3}$$

the term $x^2 k e^{-t}$, where k is an unknown constant, is considered as a perturbation. If, because of the factor e^{-t}, this perturbation is neglected and the control for the unperturbed system $u = -x - x^3$ is applied to the perturbed system (2.3), the resulting feedback system is

$$\dot{x} = -x + x^2 k e^{-t}. \tag{2.4}$$

Since (2.4) is linear in $\frac{1}{x}$, its explicit solution is known:

$$x(t) = \frac{2x(0)}{(2 - kx(0))e^t + kx(0)e^{-t}}. \tag{2.5}$$

The denominator of (2.5) is zero at $t = \frac{1}{2} \ln \frac{kx(0)}{kx(0) - 2}$. It follows that, whenever $kx(0) > 2$, $x(t)$ escapes to infinity in finite time. □

It is clear from this example that additional control action is needed to attenuate the effects of the unknown disturbance. In the following lemma, a "nonlinear damping" term is designed to fulfill this task for a class of unknown disturbances which are in the span of the control.

Lemma NDM (Nonlinear Damping–Matched). *Consider the perturbed system*

$$\dot{x} = f(x) + g(x)\left[u + p(x)^{\mathrm{T}} d(x, \varepsilon)\right], \tag{2.6}$$

where $p(x), d(x,\varepsilon)$ are continuous and $d(x,0) \equiv 0$. Let the "disturbance generator"

$$\dot{\varepsilon} = q(x,\varepsilon), \quad q(x,0) \equiv 0, \quad \varepsilon \in \mathbb{R}^{\ell}, \tag{2.7}$$

satisfy the inequality

$$\frac{\partial \Omega}{\partial \varepsilon}(\varepsilon) q(x,\varepsilon) \leq -\|d(x,\varepsilon)\|^2 \tag{2.8}$$

for some positive definite radially unbounded function $\Omega(\varepsilon)$ *and for all* $\varepsilon \in \mathbb{R}^{\ell}$, $x \in \mathbb{R}^n$. *Then, the feedback control*

$$u = \alpha(x) - \frac{\partial V}{\partial x}(x)g(x)\|p(x)\|^2 \triangleq \alpha_{\text{NDM}}(x), \tag{2.9}$$

when applied to (2.6), guarantees global boundedness of $x(t)$ *and regulation of* $\eta(x)$.

Proof. Because of (2.2) and (2.8), the time derivative of $V_{\text{NDM}}(x,\varepsilon) = V(x) + \Omega(\varepsilon)$ for (2.7) and the perturbed system (2.6) with the feedback (2.9) is

$$\begin{aligned} \dot{V}_{\text{NDM}}(x,\varepsilon) &= \frac{\partial V}{\partial x}[f + g\alpha] - \left[\frac{\partial V}{\partial x}g\right]^2 \|p\|^2 + \frac{\partial V}{\partial x}gp^{\mathrm{T}}d + \frac{\partial \Omega}{\partial \varepsilon}q \\ &\leq -W(\eta) - \left[\frac{\partial V}{\partial x}g\right]^2 \|p\|^2 + \frac{\partial V}{\partial x}gp^{\mathrm{T}}d - \|d\|^2 \\ &\leq -W(\eta) - \frac{3}{4}\|d\|^2 - \left\|\frac{1}{2}d - \frac{\partial V}{\partial x}gp\right\|^2 \\ &\leq -W(\eta) - \frac{3}{4}\|d\|^2, \end{aligned} \tag{2.10}$$

which proves global boundedness of $x(t), \varepsilon(t)$. Furthermore, LaSalle's invariance theorem guarantees that $x(t), \varepsilon(t)$ converge to the largest invariant set of (2.6)–(2.7) on which $\dot{V}_{\text{NDM}}(x,\varepsilon) = 0$. This proves that the regulation of $\eta(x)$ is achieved and that the disturbance vanishes: $d(x(t),\varepsilon(t)) \to 0$ as $t \to \infty$. □

The control $\alpha_{\text{NDM}}(x)$ in (2.9) is designed by adding a term to the control $\alpha(x)$ for the unperturbed system. The implementation of this nonlinear damping term does not require that $d(x,\varepsilon), q(x,\varepsilon)$ or $\Omega(\varepsilon)$ be known: it is sufficient that they satisfy (2.8).

The nonlinear damping (2.9) is a variant of a design by Barmish, Corless and Leitmann [1]. Its effectiveness as a tool for output-feedback design was suggested by Sontag [20, 21] and demonstrated by Marino and Tomei [15]. A form of nonlinear damping is implicit in an early adaptive control result by Feuer and Morse [3].

Example 1 (cont'd). Using $p(x) = x^2$, $d(x,\varepsilon) = \varepsilon$, and the disturbance generator $\dot{\varepsilon} = -\varepsilon$, $\varepsilon(0) = k$, the perturbed system (2.3) is rewritten in the form (2.6) as

$$\dot{x} = x^3 + u + x^2\varepsilon. \tag{2.11}$$

With $\alpha(x) = -x - x^3$, $V(x) = \frac{1}{2}x^2$ and $\Omega(\varepsilon) = \frac{1}{2}\varepsilon^2$, Lemma NDM applies and the nonlinear damping feedback is

$$u = \alpha_{\text{NDM}}(x) = -x - x^3 - x^5. \tag{2.12}$$

With $V_{NDM} = \frac{1}{2}(x^2 + \varepsilon^2)$ it is easy to show that the resulting closed-loop system

$$\dot{x} = -x + x^2 k e^{-t} - x^5 \tag{2.13}$$

is globally exponentially stable. Clearly, the nonlinear damping term $-x^5$ prevented the unbounded behavior that would have been caused by $x^2 k e^{-t}$. □

In the second lemma, "integrator backstepping" is used to design a feedback control for the system obtained when (2.1) is augmented by an integrator.

Lemma IB (Integrator Backstepping). *Consider the augmented system*

$$\dot{x} = f(x) + g(x)\xi \tag{2.14a}$$

$$\dot{\xi} = u, \tag{2.14b}$$

where $\xi \in \mathbb{R}$ is available for measurement. Then, the feedback control

$$u = -(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x)\left[f(x) + g(x)\xi\right] - \frac{\partial V}{\partial x}(x)g(x) \triangleq \alpha_{IB}(x, \xi), \tag{2.15}$$

when applied to the system (2.14), guarantees global boundedness of $x(t), \xi(t)$ and regulation of $\eta(x), \xi - \alpha(x)$.

Proof. The backstepping idea is to first view ξ as the control and stabilize (2.14a) with $\alpha(x)$ from (2.2). Then, to account for the fact that ξ is not the control, the change of variables $z = \xi - \alpha(x)$ is introduced to transform (2.14) into

$$\begin{aligned} \dot{x} &= f(x) + g(x)(\alpha(x) + z) \\ \dot{z} &= u - \frac{\partial \alpha}{\partial x}(x)\left[f(x) + g(x)(\alpha(x) + z)\right]. \end{aligned} \tag{2.16}$$

Finally, (2.15) is designed to make the time derivative of $V_{IB}(x, z) = V(x) + \frac{1}{2}z^2$ nonpositive:

$$\dot{V}_{IB}(x, z) = \frac{\partial V}{\partial x}(f + g\alpha) + \frac{\partial V}{\partial x}gz - z^2 - \frac{\partial V}{\partial x}gz \leq -W(\eta) - z^2. \tag{2.17}$$

This proves global boundedness of $x(t)$, $z(t)$ and, by LaSalle's invariance theorem, regulation of $\eta(x)$ and z. □

Integrator backstepping has recently been used by several authors [22, 2, 10], and was one of the tools for the systematic design of adaptive nonlinear controllers by Kanellakopoulos, Kokotovic and Morse [8] and Jiang and Praly [5]. As a design tool, it often leads to global results not achievable by feedback linearization, as illustrated by the following example.

Example 2. Consider a system which is not globally feedback linearizable:

$$\dot{x} = x^2 + (1+x)\xi \tag{2.18a}$$

$$\dot{\xi} = u\,. \tag{2.18b}$$

Thinking of (2.18a) as a system controlled by ξ, we use $\alpha(x) = -x$, $V(x) = \frac{1}{2}x^2$ to satisfy (2.2). Recognizing that ξ is not the control, we let $z = \xi + x$ and transform (2.18) into

$$\begin{aligned} \dot{x} &= -x + (1+x)z \\ \dot{z} &= u - x + (1+x)z\,. \end{aligned} \tag{2.19}$$

For this system, $\alpha_{IB}(x,\xi)$ given by (2.15) is

$$u = -\xi - (\xi + 2x)(1+x) = -z + x - z(1+x) - x(1+x)\,. \tag{2.20}$$

With $V_{IB}(x,z) = \frac{1}{2}(x^2+z^2)$ it is easy to show that the system (2.19) controlled by (2.20) is globally exponentially stable. □

Lemmas NDM and IB can be combined into more sophisticated tools. One such combination, incorporating filtered transformations, was used by Marino and Tomei in [15]. Two additional combinations are given in Lemmas NDE and OIB below.

While in Lemma NDM the perturbation $p(x)^{\mathrm{T}}d(x,\varepsilon)$ was in the span of the control u, in Lemma NDE it precedes the control by one integrator.

Lemma NDE (Nonlinear Damping–Extended). *For the augmented perturbed system*

$$\dot{x} = f(x) + g(x)\left[\xi + p(x)^{\mathrm{T}}d(x,\varepsilon)\right] \tag{2.21a}$$

$$\dot{\xi} = u\,, \tag{2.21b}$$

under the assumptions of Lemma NDM, the feedback control

$$\begin{aligned} u \;=\; & -[\xi - \alpha_{NDM}(x)] + \frac{\partial \alpha_{NDM}}{\partial x}(x)\,[f(x) + g(x)\xi\,] - \frac{\partial V}{\partial x}(x)g(x) \\ & -[\xi - \alpha_{NDM}(x)]\left\| \frac{\partial \alpha_{NDM}}{\partial x}(x)g(x)p(x)\right\|^2 \triangleq \alpha_{NDE}(x,\xi)\,, \end{aligned} \tag{2.22}$$

guarantees global boundedness of $x(t), \xi(t)$ and regulation of $\eta(x), \xi - \alpha_{\text{NDM}}(x)$.

Proof. Viewing ξ as the control, the nonlinear damping feedback of Lemma NDM for (2.21a) is $\xi = \alpha_{\text{NDM}}(x)$, with $V_{\text{NDM}}(x,\varepsilon) = V(x) + \Omega(\epsilon)$. As in Lemma IB, $z = \xi - \alpha_{\text{NDM}}(x)$ is used to transform (2.21) into

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha_{\text{NDM}}(x) + g(x)p(x)^{\mathrm{T}}d(x,\varepsilon) + g(x)z \\ \dot{z} &= u - \frac{\partial \alpha_{\text{NDM}}}{\partial x}(x)\left[f(x) + g(x)\alpha_{\text{NDM}}(x) + g(x)z\right] - \bar{p}(x)^{\mathrm{T}}d(x,\varepsilon)\,, \end{aligned} \tag{2.23}$$

where $\bar{p}(x) \triangleq \frac{\partial \alpha_{\text{NDM}}}{\partial x}(x)g(x)p(x)$. In the absence of the term $\bar{p}(x)^{\mathrm{T}}d(x,\varepsilon)$, Lemma IB would result in the feedback control

$$u = -z + \frac{\partial \alpha_{\text{NDM}}}{\partial x}\left[f + g\alpha_{\text{NDM}} + gz\right] - \frac{\partial V}{\partial x}g \triangleq \bar{\alpha}_{\text{IB}}(x,z) \tag{2.24}$$

with $\bar{V}_{\text{IB}}(x,z,\varepsilon) = V_{\text{NDM}}(x,\varepsilon) + \frac{1}{2}z^2$. To account for the presence of $\bar{p}(x)^{\mathrm{T}}d(x,\varepsilon)$, we apply Lemma NDM once again and add a nonlinear damping term to (2.24):

$$u = \bar{\alpha}_{\text{IB}}(x,z) - \frac{\partial \bar{V}_{\text{IB}}}{\partial z}\left\|\bar{p}(x)\right\|^2 = \bar{\alpha}_{\text{IB}}(x,z) - z\left\|\frac{\partial \alpha_{\text{NDM}}}{\partial x}(x)g(x)p(x)\right\|^2. \tag{2.25}$$

When applied to (2.23), this control guarantees global boundedness of $x(t), z(t)$ and regulation of $\eta(x)$ and z. □

The tools we presented so far assumed full-state measurement. Suppose now that the system (2.1) is augmented by an integrator whose state is not measured, but is instead estimated by an observer. We consider (2.14) for the case $g(x) \equiv g \neq 0$:

$$\dot{x} = f(x) + g\xi \tag{2.26a}$$

$$\dot{\xi} = u\,. \tag{2.26b}$$

Following [11], an observer for this system is

$$\dot{\hat{y}} = -k_1(\hat{y} - y) + g_i\hat{\xi} + f_i(x) \tag{2.27a}$$

$$\dot{\hat{\xi}} = -k_2(\hat{y} - y) + u\,, \tag{2.27b}$$

where $y = x_i$ is a component of x such that $g_i \neq 0$ and k_1, k_2 are chosen to guarantee the exponential stability of the error system

$$\begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{\xi}} \end{bmatrix} = \begin{bmatrix} -k_1 & g_i \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{\xi} \end{bmatrix} \triangleq A_0 \begin{bmatrix} \tilde{y} \\ \tilde{\xi} \end{bmatrix}, \tag{2.28}$$

where $\tilde{y} = y - \hat{y}, \tilde{\xi} = \xi - \hat{\xi}$. Then, an observer-based feedback control for (2.26) is designed by backstepping the integrator (2.27b) in the observer.

Lemma OIB (Observed-Integrator Backstepping). *Consider the augmented system (2.26), where the unmeasured state ξ is estimated by the observer (2.27). Then, with $\alpha_1(x) \triangleq \alpha(x) - \frac{\partial V}{\partial x}(x)g$, the feedback control*

$$u = k_2(y-\hat{y}) - \left[\hat{\xi} - \alpha_1(x)\right] + \frac{\partial \alpha_1}{\partial x}(x)\left[f(x) + g\hat{\xi}\right] - \frac{\partial V}{\partial x}(x)g - \left[\hat{\xi} - \alpha_1(x)\right]\left[\frac{\partial \alpha_1}{\partial x}(x)g\right]^2, \quad (2.29)$$

when applied to the system (2.26), guarantees global boundedness of $x(t), \xi(t)$ and regulation of $\eta(x), \xi - \alpha_1(x)$.

Proof. Let us combine (2.26a) with the observer equation (2.27b) into a system:

$$\begin{aligned} \dot{x} &= f(x) + g\left[\hat{\xi} + \tilde{\xi}\right] \\ \dot{\hat{\xi}} &= -k_2(y - \hat{y}) + u\,, \end{aligned} \quad (2.30)$$

and treat $\tilde{\xi}$ as a disturbance generated by(2.28). The system (2.30) is in the form (2.21) with $\varepsilon = \left[\tilde{y}\ \tilde{\xi}\right]^{\mathrm{T}}$, $d(x,\varepsilon) = \tilde{\xi}$, $p(x) = 1$ and $\Omega(\varepsilon) = \varepsilon^{\mathrm{T}} P_0 \varepsilon$, $P_0 A_0 + A_0^{\mathrm{T}} P_0 = -I$. Hence, Lemma NDE applies and the feedback control (2.29) guarantees global boundedness of $x(t), \hat{\xi}(t), \varepsilon(t)$ and regulation of $\eta(x), \hat{\xi} - \alpha_1(x)$. Then, global boundedness of $\xi(t)$ and regulation of $\xi - \alpha_1(x)$ follow from $\xi = \hat{\xi} + \tilde{\xi} = z + \alpha_1(x) + \tilde{\xi}$ and $\tilde{\xi} \to 0$. □

3 A Backstepping Design Procedure

Employing the tools of Section 2 in a step-by-step fashion, we are able to construct backstepping procedures for nonlinear feedback design problems. With full-state feedback, such procedures have been constructed for partially linear composite systems in [18] and for a class of nonlinear systems containing unknown constant parameters in [8]. Here we design an observer-based controller for the class of nonlinear systems that can be transformed via a global diffeomorphism into the output-feedback form

$$\begin{aligned} \dot{\zeta} &= A\zeta + \varphi(y) + b\sigma(y)u \\ y &= c^{\mathrm{T}}\zeta = \zeta_1 \end{aligned} \quad (3.1)$$

$$A = \begin{bmatrix} 0 & & I \\ \vdots & & \\ 0 & \dots & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{n-\rho} \\ \vdots \\ b_0 \end{bmatrix},\ c^{\mathrm{T}} = [1\ \ 0 \dots 0],\ \varphi(y) = \begin{bmatrix} \varphi_1(y) \\ \vdots \\ \varphi_n(y) \end{bmatrix}, \quad (3.2)$$

where only y is available for measurement, $b_{n-\rho}s^{n-\rho}+\cdots+b_1 s+b_0$ is a Hurwitz polynomial, and $\varphi_1,\ldots,\varphi_n$, σ are smooth functions with $\sigma(y)\neq 0$ $\forall y\in\mathbb{R}$. This class of nonlinear systems, characterized via geometric conditions in [15, Theorem 5.1], is of interest because its state can be estimated using the observer

$$\dot{\hat{\zeta}} = A\hat{\zeta} + K_0(\hat{\zeta}_1 - y) + \varphi(y) + b\,\sigma(y)u\,, \tag{3.3}$$

where K_0 is chosen so that $A_0 = A - K_0c^{\mathrm{T}}$ in the error system $\dot{\tilde{\zeta}} = A_0\tilde{\zeta}$, $\tilde{\zeta} = \zeta - \hat{\zeta}$, is Hurwitz. Using this observer, we now design a feedback controller to force the output y of (3.1) to track a reference signal $y_r(t)$, which is given along with its first ρ derivatives $\dot{y}_r, \ddot{y}_r, \ldots, y_r^{(\rho)}$.

In Step 1, our step-by-step design applies Lemma OIB to the first two equations of (3.1). Each consecutive step adds one more equation from (3.3) and applies Lemmas IB and NDM. The procedure terminates at Step $\rho-1$, when the true control appears for the first time.

Step 1: We define the output error $z_1 = y - y_r \triangleq y - \alpha_0(y_r)$ and consider the second-order system

$$\begin{aligned} \dot{z}_1 &= \zeta_2 + \varphi_1(y) - \dot{y}_r \triangleq \zeta_2 + \beta_1(y, y_r, \dot{y}_r) && (3.4a)\\ \dot{\zeta}_2 &= \zeta_3 + \varphi_2(y)\,. && (3.4b)\end{aligned}$$

As in Lemma OIB, we replace (3.4b) with the second equation of the observer (3.3) to compose the system

$$\begin{aligned} \dot{z}_1 &= \hat{\zeta}_2 + \beta_1(y, y_r, \dot{y}_r) + \tilde{\zeta}_2 && (3.5a)\\ \dot{\hat{\zeta}}_2 &= \hat{\zeta}_3 + K_{02}(\hat{\zeta}_1 - y) + \varphi_2(y)\,. && (3.5b)\end{aligned}$$

If $\hat{\zeta}_2$ were the control, then Lemma NDM would result in

$$\hat{\zeta}_2 = -\beta_1(y, y_r, \dot{y}_r) - 2z_1 \triangleq \alpha_1(y, y_r, \dot{y}_r) \tag{3.6}$$

and $V_1(z_1, \tilde{\zeta}) = \frac{1}{2}z_1^2 + \tilde{\zeta}^{\mathrm{T}}P_0\tilde{\zeta}$, where $P_0A_0 + A_0^{\mathrm{T}}P_0 = -I$. Since $\hat{\zeta}_2$ is not the control, the new state $z_2 = \hat{\zeta}_2 - \alpha_1(y, y_r, \dot{y}_r)$ is introduced and (3.5) is rewritten as

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_2 - z_1 + \tilde{\zeta}_2 \\ \dot{z}_2 &= \hat{\zeta}_3 + K_{02}(\hat{\zeta}_1 - y) + \varphi_2(y) - \frac{\partial\alpha_1}{\partial y}(\hat{\zeta}_2 + \varphi_1(y) + \tilde{\zeta}_2) - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial \dot{y}_r}\ddot{y}_r \\ &\triangleq \hat{\zeta}_3 + \beta_2(y, \hat{\zeta}_1, \hat{\zeta}_2, y_r, \dot{y}_r, \ddot{y}_r) - \frac{\partial\alpha_1}{\partial y}\tilde{\zeta}_2\,, \end{aligned} \tag{3.7}$$

where in the definition of β_2 we have used the fact that $\frac{\partial \alpha_1}{\partial y}, \frac{\partial \alpha_1}{\partial y_r}, \frac{\partial \alpha_1}{\partial \dot{y}_r}$ are known functions of y, y_r and $\dot{y}_r$. Again, if $\hat{\zeta}_3$ were the control, then Lemmas IB and NDM would result in

$$\begin{aligned} \hat{\zeta}_3 &= -\beta_2(y, \hat{\zeta}_1, \hat{\zeta}_2, y_r, \dot{y}_r, \ddot{y}_r) - \left[1 + \left(\frac{\partial \alpha_1}{\partial y}\right)^2\right] z_2 - z_1 \\ &\triangleq \alpha_2(y, \hat{\zeta}_1, \hat{\zeta}_2, y_r, \dot{y}_r, \ddot{y}_r), \end{aligned} \tag{3.8}$$

and $V_2(z_1, z_2, \tilde{\zeta}) = V_1(z_1, \tilde{\zeta}) + \frac{1}{2}z_2^2 + \tilde{\zeta}^{\mathrm{T}} P_0 \tilde{\zeta}$.

<u>Step i ($2 \le i < \rho - 1$):</u> In Steps 2 through $i-1$ we designed $\alpha_1, \ldots, \alpha_i$. Now, in Step i, we apply Lemma IB to backstep another integrator. We introduce the new state $z_{i+1} = \hat{\zeta}_{i+1} - \alpha_i(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_i, y_r, \dot{y}_r, \ldots, y_r^{(i)})$ to obtain

$$\begin{aligned} \dot{z}_j &= -z_j + z_{j+1} - \frac{\partial \alpha_{j-1}}{\partial y}\tilde{\zeta}_2 - \left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j - z_{j-1}, \quad 1 \le j \le i \\ \dot{z}_{i+1} &= \hat{\zeta}_{i+2} + \beta_{i+1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{i+1}, y_r, \dot{y}_r, \ldots, y_r^{(i+1)}) - \frac{\partial \alpha_i}{\partial y}\tilde{\zeta}_2. \end{aligned} \tag{3.9}$$

Again, if $\hat{\zeta}_{i+2}$ were the control, Lemmas IB and NDM would result in

$$\begin{aligned} \hat{\zeta}_{i+2} &= -\beta_{i+1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{i+1}, y_r, \dot{y}_r, \ldots, y_r^{(i+1)}) - \left[1 + \left(\frac{\partial \alpha_i}{\partial y}\right)^2\right] z_{i+1} - z_i \\ &\triangleq \alpha_{i+1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{i+1}, y_r, \dot{y}_r, \ldots, y_r^{(i+1)}) \end{aligned} \tag{3.10}$$

and $V_{i+1} = V_i + \frac{1}{2}z_{i+1}^2 + \tilde{\zeta}^{\mathrm{T}} P_0 \tilde{\zeta}$.

<u>Step $\rho - 1$:</u> Finally, we backstep the last integrator before the actual control u appears. Following Lemma IB, we substitute $z_\rho = \hat{\zeta}_\rho - \alpha_{\rho-1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{\rho-1}, y_r, \dot{y}_r, \ldots, \dot{y}_r^{(\rho-1)})$ into (3.9) (with $i = \rho - 2$) and augment the resulting system with the $\dot{z}_\rho$-equation:

$$\begin{aligned} \dot{z}_j &= -z_j + z_{j+1} - \frac{\partial \alpha_{j-1}}{\partial y}\tilde{\zeta}_2 - \left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j - z_{j-1}, \quad 1 \le j \le \rho - 1 \\ \dot{z}_\rho &= b_{n-\rho}\sigma(y)u + \hat{\zeta}_{\rho+1} + \beta_\rho(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_\rho, y_r, \dot{y}_r, \ldots, y_r^{(\rho)}) - \frac{\partial \alpha_{\rho-1}}{\partial y}\tilde{\zeta}_2. \end{aligned} \tag{3.11}$$

Now the actual control u has appeared, and Lemmas IB and NDM result in the control law

$$u = -\frac{1}{b_{n-\rho}\sigma(y)}\left\{\hat{\zeta}_{\rho+1} + \beta_\rho(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_\rho, y_r, y_r, \ldots, y_r^{(\rho)}) + \left[1 + \left(\frac{\partial \alpha_{\rho-1}}{\partial y}\right)^2\right] z_\rho + z_{\rho-1}\right\}. \tag{3.12}$$

The derivative of

$$V_\rho = V_{\rho-1} + \frac{1}{2}z_\rho^2 + \tilde{\zeta}^{\mathrm{T}} P_0 \tilde{\zeta} = \frac{1}{2}\sum_{j=1}^{\rho} z_j^2 + \rho\,\tilde{\zeta}^{\mathrm{T}} P_0 \tilde{\zeta} \tag{3.13}$$

along the solutions of (3.11)-(3.12) is nonpositive:

$$\dot{V}_\rho \le -\sum_{j=1}^{\rho}\left[z_j^2 + \frac{3}{4}\left\|\tilde{\zeta}\right\|^2 + \left(\frac{\partial \alpha_{j-1}}{\partial y} z_j + \frac{1}{2}\tilde{\zeta}_2\right)^2\right] \le 0\,. \tag{3.14}$$

With this systematic procedure we have not only designed the control law (3.12), but have also set the stage for the following result:

Theorem 3.1 (Stability and Tracking). *For the nonlinear system (3.1), assume that $b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0$ is a Hurwitz polynomial, and that $y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho)}$ are bounded on $[0,\infty)$ and $y_{\mathrm{r}}^{(\rho)}(t)$ is piecewise continuous. Then, all the signals in the closed-loop system consisting of the system (3.1), the observer (3.3) and the control (3.12) are globally bounded, and, in addition,*

$$\lim_{t\to\infty}\left[y(t) - y_{\mathrm{r}}(t)\right] = 0\,. \tag{3.15}$$

Proof. Due to the piecewise continuity of $y_{\mathrm{r}}^{(\rho)}(t)$ and the smoothness of the nonlinearities, the solution of the closed-loop system exists. Let its maximum interval of existence be $[0, t_{\mathrm{f}})$. On this interval, the nonnegative function V_ρ is nonincreasing because of (3.14). Thus, $z_1, \ldots, z_\rho$ are bounded on $[0, t_{\mathrm{f}})$ by some constants depending only on the initial conditions of (3.1) and (3.3). The boundedness of all other signals on $[0, t_{\mathrm{f}})$ is established as follows. Since z_1 and y_{r} are bounded, y is bounded. The boundedness of $\tilde{\zeta}$ and $\hat{\zeta}_1 = y - \tilde{\zeta}_1$ imply that $\hat{\zeta}_1$ is bounded. Since z_2 is bounded, $\hat{\zeta}_2$ is bounded. In the same manner, it can be shown that $\hat{\zeta}_1, \ldots, \hat{\zeta}_\rho$ are bounded. Hence, $\zeta_1, \ldots, \zeta_\rho$ are bounded.

To prove the boundedness of $\zeta_{\rho+1}, \ldots, \zeta_n$, we use the fact (see, for example, [19, Theorem 2.1]) that there exists a similarity transformation $\bar{\zeta} = T\zeta$, with $\bar{\zeta}_1 = \zeta_1, \ldots, \bar{\zeta}_\rho = \zeta_\rho$, which results in

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + \varphi_1(y) \\
&\vdots \\
\dot{\zeta}_{\rho_1} &= \zeta_\rho + \varphi_{\rho-1}(y) \\
\dot{\zeta}_\rho &= \varphi_\rho(y) + b_{n-\rho}\sigma(y)u + a_1^{\mathrm{T}}\bar{\zeta} \\
\dot{\bar{\zeta}}^{\mathrm{r}} &= A_{\mathrm{r}}\bar{\zeta}^{\mathrm{r}} + \bar{\varphi}(y) \\
y &= \zeta_1\,,
\end{aligned} \tag{3.16}$$

where the eigenvalues of the $(n-\rho) \times (n-\rho)$ matrix A_r are the roots of the Hurwitz polynomial $b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0$. Now the boundedness of $\bar{\zeta}^r$, which follows from the boundedness of $\bar{\varphi}(y)$, together with the boundedness of $\zeta_1, \ldots, \zeta_\rho$, imply that $\bar{\zeta}$ is bounded. We conclude that $\zeta = T^{-1}\bar{\zeta}$ and $\hat{\zeta} = \zeta - \tilde{\zeta}$ are bounded. Since $b_{n-\rho}\sigma(y)$ is bounded away from zero, the feedback control u (3.12) is bounded.

We have thus shown that the state of the closed-loop system is bounded on its maximal interval of existence $[0, t_f)$. Hence, $t_f = \infty$.

To prove the convergence of the tracking error to zero, note that the boundedness of ζ, $\hat{\zeta}$, $\tilde{\zeta}$ and u, together with (3.13) and (3.14) imply that both $\dot{V}_\rho$ and $\ddot{V}_\rho$ are bounded, and, moreover, that $\dot{V}_\rho$ is integrable on $[0, \infty)$. Hence, $\dot{V}_\rho \to 0$ as $t \to \infty$, which proves that $z_1, \ldots, z_\rho \to 0$ as $t \to \infty$. Since $z_1 = y - y_r$, this proves (3.15). □

4 Concluding Remarks

Among the major contributions of geometric methods to the systematic design of nonlinear feedback systems over the last decade are conditions characterizing classes of nonlinear systems which are feedback linearizable or transformable into so-called "normal forms." Under such conditions, the feedback design problem either becomes linear or is greatly simplified due to the special properties of the normal forms.

Geometric results have their own limitations: they are often only locally valid, as in Example 2, or valid only in a disturbance-free setting, which excludes the system of Example 1. Tools like those assembled in this paper alleviate some of these limitations and appear as a valuable supplement to geometric methods. The backstepping procedure of Section 3 demonstrates that such tools are not just "tricks of the trade," but can also be used in a systematic fashion. In a more complicated adaptive setting, where the backstepping has to be performed not only under state observation, but also under parameter estimation, such tools are currently used to design fundamentally new adaptive schemes.

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Adaptive Output-Feedback Control of a Class of Nonlinear Systems*

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Abstract. For a class of single-input single-output nonlinear systems with unknown constant parameters, we construct a new systematic procedure for adaptive nonlinear control design, which requires only output, rather than full-state, measurement, and which yields global boundedness and tracking properties without imposing any type of growth constraints on the nonlinearities. The new procedure is applicable to nonlinear systems which can be expressed in the output-feedback canonical form. We give a coordinate-free characterization of this class of systems, and show that a single-link robotic manipulator with an elastically coupled dc-motor actuator belongs to this class and can thus be adaptively controlled via our new design procedure using only position measurement.

1 Introduction

In the last few years, the problem of adaptive nonlinear control was formulated to deal with the control of plants containing both unknown parameters and known nonlinearities. The motivation for this problem lies in the fact that many practically important systems (robotic manipulators with rigid or flexible joints, electric motors, automotive suspensions, chemical processes) are inherently nonlinear (due to the presence of gravitational, Coriolis or aerodynamic forces, flux-speed, flux-current or ion concentration products, and even hydraulic valve actuators) and typically contain unknown parameters which vary with the operating conditions (task-dependent load masses and torques, temperature-sensitive resistances and heat-transfer coefficients).

Many of the early results (see [7] for a unifying treatment and detailed references) yielded global properties only when growth conditions were imposed on the nonlinearities. Such growth conditions restrict the applicability of the corresponding schemes, and, in some cases, actually bypass the true nonlinear problem. Moreover, all of the early results employed the assumption of full-state measurement, which further restricted their applicability to practical situations.

In this paper, we construct a new systematic procedure for adaptive nonlinear control design, which requires only output measurement, and which yields global boundedness and tracking properties without imposing any type of growth constraints on the nonlinearities. This procedure is applicable to single-input single-output nonlinear systems which can be transformed into the *output-feedback canonical form*. This is the same class of systems to which the filtered-transformation-based design procedure of [6] is applicable.

The design procedure is constructed by interlacing in an intricate fashion several tools from our nonlinear toolkit [2, 4], which are presented in Section 2. Some of these tools (nonlinear damping, integrator backstepping, parameter-dependent estimation) were used previously in the adaptive and nonlinear literature. The tools of adaptive integrator backstepping and observed-integrator backstepping were the crucial ingredients of the design procedures of [3] and [4], respectively. Here we combine them to form the new tool of "adaptive observed-integrator backstepping" (Lemma 6), which is the crucial ingredient of our new design procedure, presented in Section 3. The boundedness and tracking properties of the resulting closed-loop system are established in Section 4.

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Since models of nonlinear systems are often derived from physical principles and given in specific coordinates, it may not always be obvious whether or not the nonlinear system at hand can be transformed into the output-feedback form to which our design procedure is applicable. Therefore, in Section 5 we use differential geometric conditions to derive a coordinate-free characterization of the corresponding class of nonlinear systems. Then, in Section 6, we show that the nonlinear system consisting of a single-link robotic manipulator and an elastically coupled dc-motor actuator (including the actuator dynamics) belongs to this class. Thus, our new design procedure is applicable to this system and, for all positive values of the elasticity constant and of the electrical and mechanical time constants of the motor, yields global boundedness and asymptotic position tracking using only position measurement. Finally, in Section 7, we give some concluding remarks about improvements and extensions of the new design procedure.

2 The Design Toolkit

In this section we present the tools that will be used as building blocks in the construction of the design procedure of Section 3. The proofs of the following lemmas were presented in [2, 4].

Definition 1 (η-g.r.f.) *A feedback control $u = \alpha_0(x)$ which, when applied to the system*

$$\dot{x} = f_0(x) + g(x)u\,, \quad x \in \mathbb{R}^n\,, \quad u \in \mathbb{R}\,. \tag{2.1}$$

guarantees global boundedness of $x(t)$ and regulation of $\eta(x) \in \mathbb{R}^m$, is called an η-globally regulating feedback (η-g.r.f.) *for (2.1). Furthermore, if these properties are ascertained by the inequality*

$$\frac{\partial V}{\partial x}(x)\left[f_0(x) + g(x)\alpha_0(x)\right] \le -W(\eta(x)) \le 0\,, \quad \forall x \in \mathbb{R}^n\,, \tag{2.2}$$

where $V(x)$ is positive definite and radially unbounded, and $W(\eta)$ is positive definite, we say that $\alpha_0(x)$ is an η-g.r.f. for (2.1) with respect to (w.r.t.) $V(x)$.

Lemma 1 (Nonlinear Damping) *Consider the perturbed system*

$$\dot{x} = f_0(x) + g(x)\left[u + p(x)^{\mathrm{T}}\epsilon(x,\varepsilon)\right]\,. \tag{2.3}$$

where $p(x), \epsilon(x,\varepsilon)$ are continuous and $\epsilon(x,0) \equiv 0$. Let the "disturbance generator"

$$\dot{\varepsilon} = q(x,\varepsilon)\,, \quad q(x,0) \equiv 0\,, \quad \varepsilon \in \mathbb{R}^\ell\,, \tag{2.4}$$

satisfy the inequality

$$\frac{\partial \Omega}{\partial \varepsilon}(\varepsilon)q(x,\varepsilon) \le -\|\epsilon(x,\varepsilon)\|^2 \tag{2.5}$$

for some positive definite radially unbounded function $\Omega(\varepsilon)$ and for all $\varepsilon \in \mathbb{R}^\ell$, $x \in \mathbb{R}^n$. If $\alpha_0(x)$ is an η-g.r.f. for (2.1) w.r.t. $V(x)$, then an η-g.r.f. for (2.3) is

$$\alpha_1(x) = \alpha_0(x) - d\,\frac{\partial V}{\partial x}(x)g(x)\|p(x)\|^2\,, \tag{2.6}$$

where $d > 0$ is a design constant.

Lemma 2 (Integrator Backstepping) *Consider the augmented system*

$$\begin{aligned} \dot{x} &= f_0(x) + g(x)\chi & (2.7a)\\ \dot{\chi} &= u\,, & (2.7b) \end{aligned}$$

where $\chi \in \mathbb{R}$ is available for measurement. If $\alpha_0(x)$ is an η-g.r.f. for (2.1) w.r.t. $V(x)$, then the feedback control $u = \alpha_2(x, z)$,

$$\alpha_2(x,z) = -z + \frac{\partial \alpha_0}{\partial x}(x)\left[f_0(x) + g(x)(\alpha_0(x) + z)\right] - \frac{\partial V}{\partial x}(x) g(x), \tag{2.8}$$

is an (η, z)-g.r.f. for (2.7), where

$$z = \chi - \alpha_0(x). \tag{2.9}$$

The tools we presented thus far assumed full-state measurement. Suppose now that the system (2.1) is augmented by an integrator whose state is not measured, but is instead estimated by an observer. We consider (2.7) for the case $g(x) \equiv g \neq 0$:

$$\begin{aligned} \dot{x} &= f_0(x) + g\chi & (2.10a)\\ \dot{\chi} &= u. & (2.10b) \end{aligned}$$

Following [5], an observer for this system is

$$\begin{aligned} \dot{\hat{y}} &= -k_1(\hat{y} - y) + g_i\hat{\chi} + f_{0,i}(x) & (2.11a)\\ \dot{\hat{\chi}} &= -k_2(\hat{y} - y) + u, & (2.11b) \end{aligned}$$

where $y = x_i$ is a component of x such that $g_i \neq 0$ and k_1, k_2 are chosen to guarantee the exponential stability of the error system

$$\begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{\chi}} \end{bmatrix} = \begin{bmatrix} -k_1 & g_i \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{\chi} \end{bmatrix} \triangleq A_0 \begin{bmatrix} \tilde{y} \\ \tilde{\chi} \end{bmatrix}, \tag{2.12}$$

where $\tilde{y} = y - \hat{y}$, $\tilde{\chi} = \chi - \hat{\chi}$. Then, an observer-based feedback control for (2.10) is designed by backstepping the integrator (2.11b) in the observer.

Lemma 3 (Observed-Integrator Backstepping) *Consider the augmented system (2.10), in which the unmeasured state χ is estimated by the observer (2.11). If $\alpha_0(x)$ is an η-g.r.f. for (2.1) w.r.t. $V(x)$, then the feedback control $u = \alpha_2(x, z)$,*

$$\begin{aligned} \alpha_2(x,z) &= -z + k_2(\hat{y} - y) + \frac{\partial \alpha_1}{\partial x}(x)\left[f_0(x) + g\left(\alpha_1(x) + z\right)\right] \\ &\quad - \frac{\partial V}{\partial x}(x) g - d_2 z\left[\frac{\partial \alpha_1}{\partial x}(x) g\right]^2, \end{aligned} \tag{2.13}$$

is an $(\eta, \chi - \alpha_1(x))$-g.r.f. for (2.10), where

$$z = \hat{\chi} - \alpha_1(x) \triangleq \hat{\chi} - \alpha_0(x) + d_1\frac{\partial V}{\partial x}(x) g, \tag{2.14}$$

and $d_1 > 0, d_2 > 0$ are design constants.

Let us now consider the nonlinear system

$$\dot{x} = f_0(x) + \theta f(x) + g(x)u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}, \tag{2.15}$$

where $\theta \in \mathbb{R}$ is an unknown constant parameter. Let us assume that there exists a feedback control $u = \alpha_0(x) + \theta\alpha(x)$, with $\alpha_0(x)$ and $\alpha(x)$ known, which, for every $\theta \in \mathbb{R}$, is an η-g.r.f. for (2.15) w.r.t. $V(x)$, with $V(x)$ known. That is, $\forall x \in \mathbb{R}^n$, $\forall \theta \in \mathbb{R}$, we have

$$\frac{\partial V}{\partial x}(x)\left[f_0(x) + \theta f(x) + g(x)(\alpha_0(x) + \theta\alpha(x))\right] \leq -W(\eta(x)) \leq 0, \tag{2.16}$$

where $V(x)$ is positive definite and radially unbounded, and $W(\eta)$ is positive definite. It is further assumed that f_0, f, g, η, α_0, α, V and W are C^2 on $\mathbb{R}^n$.

Lemma 4 (Adaptive Integrator Backstepping) *Consider the augmented system*

$$\begin{aligned} \dot{x} &= f_0(x) + \theta f(x) + g(x)\chi & (2.17a)\\ \dot{\chi} &= u, & (2.17b) \end{aligned}$$

where $\chi \in \mathbb{R}$ is available for measurement, and assume that $\alpha_0(x) + \theta\alpha(x)$ is an η-g.r.f. for (2.15) w.r.t. $V(x)$. Let ϑ_1 and ϑ_2 be two estimates of θ. Then, the feedback control $u = \alpha_2(x, z, \vartheta_1, \vartheta_2)$,

$$\begin{aligned} \alpha_2(x,z,\vartheta_1,\vartheta_2) &= -z + \left[\frac{\partial \alpha_0}{\partial x}(x) + \vartheta_1\frac{\partial \alpha}{\partial x}(x)\right] \\ &\quad \left[f_0(x) + \vartheta_2 f(x) + g(x)(\alpha_1(x,\vartheta_1) + z)\right] \\ &\quad - \gamma_1\frac{\partial V}{\partial x}(x) g(x)\alpha^2(x) - \frac{\partial V}{\partial x}(x) g(x), \end{aligned} \tag{2.18}$$

$$z = \chi - \alpha_1(x, \vartheta_1) = \chi - [\alpha_0(x) + \vartheta_1\alpha(x)], \tag{2.19}$$

is an (η, z)-g.r.f. for the system consisting of (2.17) and the update laws

$$\begin{aligned} \dot{\vartheta}_1 &= -\gamma_1\frac{\partial V}{\partial x}(x) g(x)\alpha(x) & (2.20)\\ \dot{\vartheta}_2 &= -\gamma_2\left[\frac{\partial \alpha_0}{\partial x}(x) + \vartheta_1\frac{\partial \alpha}{\partial x}(x)\right] f(x) z, & (2.21) \end{aligned}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$ are design constants.

Let us now consider the case in which (2.15) is augmented by an integrator whose state is not measured. As in Lemma 3, we consider (2.15) for the case $g(x) \equiv g \neq 0$:

$$\begin{aligned} \dot{x} &= f_0(x) + \theta f(x) + g\chi & (2.22a)\\ \dot{\chi} &= u. & (2.22b) \end{aligned}$$

The main difference between (2.22) and (2.10) is the presence of the unknown parameter θ in (2.22a). If this parameter were known, an observer of the form (2.10) would provide an exponentially convergent estimate of χ, which would then be used in the design of a g.r.f. for (2.22). Keeping in mind that our ultimate goal is not the estimation of χ itself, but rather the design of a g.r.f., we construct a "parameter-dependent" estimate of χ using the following lemma:

Lemma 5 (Parameter-Dependent Estimation) *Choose a component of x, $y = x_i$ such that $g_i \neq 0$, choose k_1, k_2 to guarantee the exponential stability of the matrix*

$$A_0 = \begin{bmatrix} -k_1 & g_i \\ -k_2 & 0 \end{bmatrix}. \tag{2.23}$$

and define the filters

$$\begin{aligned} &\dot{\xi}_{01} = f_{0,i}(x) + g_i\xi_{02} + k_1(y - \xi_{01}) \qquad &\dot{\xi}_{11} = f_i(x) + g_i\xi_{12} - k_1\xi_{11} \\ &\dot{\xi}_{02} = u + k_2(y - \xi_{01}) &\dot{\xi}_{12} = -k_2\xi_{11}. \end{aligned} \tag{2.24}$$

Then, the signal $\epsilon = \chi - (\xi_{02} + \theta\xi_{12})$ converges to zero.

We can now combine Lemmas 1, 4 and 5 into a new tool, which shows us how to design a g.r.f. for (2.22).

Lemma 6 (Adaptive Observed-Integrator Backstepping) *Consider the augmented system (2.22) and the filters (2.24), and assume that $\alpha_0(x) + \theta\alpha(x)$ is an η-g.r.f. for (2.15) w.r.t. $V(x)$. Let ϑ_1 and ϑ_2 be two estimates of θ. Then, the feedback control $u = \alpha_2(x, z, \xi_{01}, \xi_1, \vartheta_1, \vartheta_2)$*

$$\begin{aligned} \alpha_2 &= -z + k_2(\xi_{01} + \theta\xi_{11} - y) + \left[\frac{\partial \bar{\alpha}_1}{\partial x}(x) + \vartheta_1\frac{\partial \alpha}{\partial x}(x)\right]\left[f_0(x)\right. \\ &\quad \left. + \vartheta_2 f(x) + g(\alpha_1(x,\xi_1,\vartheta_1) + z)\right] - \gamma_1\frac{\partial V}{\partial x}(x) g(\alpha(x) - \xi_{12}) \\ &\quad - d_2 z\left(\left[\frac{\partial \bar{\alpha}_1}{\partial x}(x) + \vartheta_1\frac{\partial \alpha}{\partial x}(x)\right] g\right)^2, \end{aligned} \tag{2.25}$$

where $\bar{\alpha}_1(x) = \alpha_0(x) - d_1\frac{\partial V}{\partial x}(x) g$ and

$$z = \xi_{02} - \alpha_1(x, \xi_1, \vartheta_1) \triangleq \xi_{02} - \left[\alpha_0(x) + \vartheta_1(\alpha(x) - \xi_{12}) - d_1\frac{\partial V}{\partial x}(x) g\right] \tag{2.26}$$

is an (η, z)-g.r.f. for the system consisting of (2.22) and the update laws

$$\begin{aligned} \dot{\vartheta}_1 &= -\gamma_1\frac{\partial V}{\partial x}(x) g(\alpha(x) - \xi_{12}) & (2.27)\\ \dot{\vartheta}_2 &= -\gamma_2\left[\frac{\partial \bar{\alpha}_1}{\partial x}(x) + \vartheta_1\frac{\partial \alpha}{\partial x}(x)\right] f(x) z, & (2.28) \end{aligned}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, $d_1 > 0$, $d_2 > 0$ are design constants.

3 The Systematic Procedure

In this section we present the main result of the paper. We use the tools of Section 2, especially nonlinear damping (Lemma 1), adaptive integrator backstepping (Lemma 4) and adaptive observed-integrator backstepping (Lemma 6), to construct a fundamentally new systematic design procedure for nonlinear systems which can be transformed into the *output-feedback canonical form*:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_{0,1}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,1}(y) \\
\dot{x}_2 &= x_3 + \varphi_{0,2}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,2}(y) \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho + \varphi_{0,\rho-1}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,\rho-1}(y) \\
\dot{x}_\rho &= x_{\rho+1} + \varphi_{0,\rho}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,\rho}(y) + b_{n-\rho}\sigma(y)u \\
&\vdots \\
\dot{x}_n &= \varphi_{0,n}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,n}(y) + b_0\sigma(y)u \\
y &= x_1,
\end{aligned}
\tag{3.1}
$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $\varphi_{j,i}$, $0 \le j \le p$, $1 \le i \le n$, and σ are smooth nonlinear functions, and $\theta = [\theta_1, \ldots, \theta_p]^{\mathrm{T}} \in \mathbb{R}^p$, $b = [b_{n-\rho}, \ldots, b_0]^{\mathrm{T}} \in \mathbb{R}^{n-\rho+1}$ are vectors of unknown constant parameters.

We now make the following assumptions about the system (3.1):

Assumption 1 *The sign of $b_{n-\rho}$ is known.*

Assumption 2 *The polynomial $B(s) = b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0$ is known to be Hurwitz.*

Assumption 3 $\sigma(y) \neq 0 \;\forall y \in \mathbb{R}$.

These assumptions imply, in particular, that the system (3.1) has a globally well-defined and known strong relative degree ρ [1, Chapter 4], and that its zero dynamics [1, Chapter 4] are linear and exponentially stable.

Assuming that *only the output y is measured*, the control objective is to track a given reference signal $y_r(t)$ with the output y of the system (3.1), while keeping all of the signals in the closed-loop system globally bounded. For the adaptive controller which results from our design procedure to be implementable, we assume that

Assumption 4 *The reference signal $y_r(t)$ and its first ρ derivatives are known and bounded, and, in addition, $y_r^{(\rho)}(t)$ is piecewise continuous.*

This assumption is satisfied if $y_r(t)$ is the output of a linear stable reference model of relative degree $\rho_r \ge \rho$.

The first step in our systematic design procedure is the choice of filters which will provide "parameter-dependent estimates" of the unmeasured state variables $x_2, \ldots, x_n$. Following the development of Lemma 5, we rewrite the system (3.1) in the form

$$
\begin{aligned}
\dot{x} &= Ax + \varphi_0(y) + \sum_{j=1}^{p} \theta_j \varphi_j(y) + b\sigma(y)u \\
y &= c^{\mathrm{T}}x,
\end{aligned}
\tag{3.2}
$$

$$
A = \begin{bmatrix} 0 & & \\ \vdots & I_{(n-1)\times(n-1)} & \\ 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0_{(\rho-1)\times 1} \\ b_{n-\rho} \\ \vdots \\ b_0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0_{(n-1)\times 1} \end{bmatrix}
\tag{3.3}
$$

$$
\varphi_j(y) = (\varphi_{j,1}(y), \ldots, \varphi_{j,n}(y))^{\mathrm{T}}, \quad 0 \le j \le p. \tag{3.4}
$$

We then choose a gain vector K such that $A_0 = A - Kc^{\mathrm{T}}$ is Hurwitz, and define the filters

$$
\begin{aligned}
\dot{\xi}_0 &= A_0\xi_0 + Ky + \varphi_0(y) \\
\dot{\xi}_j &= A_0\xi_0 + \varphi_j(y), \quad 1 \le j \le p \\
\dot{v}_j &= A_0 v_j + e_{n-j}\,\sigma(x^{\mathrm{m}})u, \qquad 0 \le j \le n-\rho,
\end{aligned}
\tag{3.5}
$$

where e_i is the ith coordinate vector in $\mathbb{R}^n$. From (3.2) and (3.5) it follows that

$$
\dot{\varepsilon} = A_0\varepsilon, \quad \varepsilon \triangleq x - \left(\xi_0 + \sum_{j=1}^{p}\theta_j\xi_j + \sum_{j=0}^{n-\rho} b_j v_j\right), \tag{3.6}
$$

which implies that ε satisfies the disturbance conditions of Lemma 1 with $\Omega(\varepsilon) = \varepsilon^{\mathrm{T}}P_0\varepsilon$, $P_0A_0 + A_0^{\mathrm{T}}P_0 = -I$. In particular, ε converges exponentially to zero. We also note that the derivative of y can be expressed as

$$
\dot{y} = \xi_{0,2} + \varphi_{0,1}(y) + \sum_{j=1}^{p}\theta_j\left[\varphi_{j,1}(y) + \xi_{j,2}\right] + \sum_{j=0}^{n-\rho} b_j v_{j,2} + \epsilon. \tag{3.7}
$$

We are now ready to construct our systematic design procedure. The main idea of this procedure is to apply Lemma 6 to the first two equations of (3.1), and then, at each next step, to add one more equation and use Lemmas 4 and 1 to stabilize the resulting system. At each step, new design constants $c_i, d_i > 0$ and a new symmetric design matrix $\Gamma_i > 0$ are introduced.

<u>Step 1:</u> We define the output error $z_1 = x_1 - y_r \triangleq y - \alpha_0(y_r)$ and write $\dot{z}_1$ as

$$
\dot{z}_1 = x_2 + \varphi_{0,1}(x_1) + \sum_{j=1}^{p}\theta_j\varphi_{j,1}(x_1) - \dot{y}_r. \tag{3.8}
$$

Since x_2 is not measured, we use (3.7) to rewrite (3.8) as

$$
\dot{z}_1 = \xi_{0,2} + \varphi_{0,1}(y) + \sum_{j=1}^{p}\theta_j\left[\varphi_{j,1}(y) + \xi_{j,2}\right] + \sum_{j=0}^{n-\rho} b_j v_{j,2} - \dot{y}_r + \epsilon. \tag{3.9}
$$

A closer examination of the filters (3.5) reveals that the control u will first appear in the ρth derivative of $v_{n-\rho,2}$. Hence, $v_{n-\rho,2}$ is the variable that we should view as the control in (3.9). If $v_{n-\rho,2}$ were the control and the parameters $\theta_1, \ldots, \theta_p$, $b_{n-\rho}, \ldots, b_0$ were known, then, by Lemma 1, a z_1-g.r.f. for the system (3.9) would be

$$
\begin{aligned}
v_{n-\rho,2} &= -\frac{1}{b_{n-\rho}}\left[c_1 z_1 + d_1 z_1 + \xi_{0,2} + \varphi_{0,1}\right] \\
&\quad - \sum_{j=1}^{p}\frac{\theta_j}{b_{n-\rho}}\left[\varphi_{j,1} + \xi_{j,2}\right] - \sum_{j=0}^{n-\rho-1}\frac{b_j}{b_{n-\rho}}v_{j,2}.
\end{aligned}
\tag{3.10}
$$

To deal with the fact that $v_{n-\rho,2}$ is multiplied with the unknown coefficient $b_{n-\rho}$, we use (3.10) to rewrite (3.9) in the form

$$
\begin{aligned}
\dot{z}_1 &= -c_1 z_1 - d_1 z_1 + \epsilon \\
&\quad + b_{n-\rho}\left\{\frac{1}{b_{n-\rho}}\left[c_1 z_1 + d_1 z_1 + \xi_{0,2} + \varphi_{0,1}\right]\right. \\
&\quad \left. + \sum_{j=1}^{p}\frac{\theta_j}{b_{n-\rho}}\left[\varphi_{j,1} + \xi_{j,2}\right] + \sum_{j=0}^{n-\rho-1}\frac{b_j}{b_{n-\rho}}v_{j,2} + v_{n-\rho,2}\right\} \\
&\triangleq -c_1 z_1 - d_1 z_1 + \epsilon + b_{n-\rho}\left\{v_{n-\rho,2} + \bar{\theta}_0^{\mathrm{T}}\omega_1(z_1, C_1\xi, C_1 v, y_r, \dot{y}_r)\right\},
\end{aligned}
\tag{3.11}
$$

where, for $i = 1, \ldots, n-1$:

$$
C_i\xi \triangleq \left[\xi_{0,1}, \ldots, \xi_{0,i+1}, \ldots, \xi_{p,1}, \ldots, \xi_{p,i+1}\right] \tag{3.12}
$$

$$
C_i v \triangleq \left[v_{0,1}, \ldots, v_{0,i+1}, \ldots, v_{n-\rho-1,1}, \ldots, v_{n-\rho-1,i+1}, v_{n-\rho,1}, \ldots, v_{n-\rho,i}\right] \tag{3.13}
$$

$$
\bar{\theta}_0^{\mathrm{T}} \triangleq \left[\frac{1}{b_{n-\rho}}, \frac{\theta_1}{b_{n-\rho}}, \ldots, \frac{\theta_p}{b_{n-\rho}}, \frac{b_0}{b_{n-\rho}}, \ldots, \frac{b_{n-\rho-1}}{b_{n-\rho}}\right] \tag{3.14}
$$

$$
\begin{aligned}
\omega_1^{\mathrm{T}} &\triangleq \left[c_1 z_1 + d_1 z_1 + \xi_{0,2} + \varphi_{0,1},\right. \\
&\quad \left.\varphi_{1,1} + \xi_{1,2}, \ldots, \varphi_{p,1} + \xi_{p,2}, v_{0,2}, \ldots, v_{n-\rho-1,2}\right].
\end{aligned}
\tag{3.15}
$$

Let ϑ_1 be an estimate of $\bar\theta_0$ and denote $\tilde\vartheta_1 = \bar\theta_0 - \vartheta_1$. Then, from Lemmas 4 and 1, and (3.11), we know that if the update law for ϑ_1 is chosen as

$$\dot\vartheta_1 = \operatorname{sgn}(b_{n-\rho})\,\Gamma_1\omega_1(z_1, C_1\xi, C_1v, y_r, \dot y_r) \tag{3.16}$$

and if $v_{n-\rho,2}$ were the control, then

$$\begin{aligned} v_{n-\rho,2} &= -\vartheta_1^{\mathrm T}\omega_1(z_1, C_1\xi, C_1v, y_r, \dot y_r) \\ &\triangleq \alpha_1(z_1, C_1\xi, C_1v, \vartheta_1, y_r, \dot y_r) \end{aligned} \tag{3.17}$$

would be a z_1-g.r.f. for the system consisting of (3.11) and (3.16), w.r.t.

$$V_1 = \frac{1}{2}\left(z_1^2 + |b_{n-\rho}|\tilde\vartheta_1^{\mathrm T}\Gamma_1^{-1}\tilde\vartheta_1\right) + \frac{1}{d_1}\Omega(\varepsilon)\,. \tag{3.18}$$

Step 2: Since $v_{n-\rho,2}$ is not our control, we augment the system consisting of (3.11), and (3.16) by the $\dot v_{n-\rho,2}$-equation from (3.5) and use Lemmas 4 and 1 to backstep it. We define the new state z_2 as the difference between $v_{n-\rho,2}$ and its desired expression (3.17):

$$z_2 = v_{n-\rho,2} - \alpha_1(z_1, C_1\xi, C_1v, \vartheta_1, y_r, \dot y_r)\,, \tag{3.19}$$

and, using (3.5), (3.7), (3.11), (3.16) and (3.17), we write $\dot z_2$ as

$$\begin{aligned} \dot z_2 &= v_{n-\rho,3} + \psi_2(z_1, z_2, C_2\xi, C_2v, \vartheta_1, y_r, \dot y_r, \ddot y_r) \\ &\quad + \bar\theta^{\mathrm T}\omega_2(z_1, z_2, C_1\xi, C_1v, \vartheta_1, y_r, \dot y_r) \\ &\quad + \frac{\partial z_2}{\partial y}(z_1, C_1\xi, C_1v, \vartheta_1, y_r, \dot y_r)\epsilon\,, \end{aligned} \tag{3.20}$$

where $\dfrac{\partial z_2}{\partial y}$ denotes the partial derivative of the right-hand side of (3.19) with respect to y, and

$$\bar\theta = [\theta_1, \ldots, \theta_\rho, b_0, \ldots, b_{n-\rho}]^{\mathrm T}\,. \tag{3.21}$$

Let ϑ_2 be an estimate of $\bar\theta$ and denote $\tilde\vartheta_2 = \bar\theta - \vartheta_2$. From Lemmas 4 and 1, we know that if the update law for ϑ_2 is chosen as

$$\dot\vartheta_2 = \Gamma_2\omega_2(z_1, z_2, C_1\xi, C_1v, \vartheta_1, y_r, \dot y_r)z_2 \tag{3.22}$$

and if $v_{n-\rho,3}$ were the control, then

$$\begin{aligned} v_{n-\rho,3} &= -c_2z_2 - z_1 - \psi_2 - \vartheta_2^{\mathrm T}\omega_2 - d_2\left(\frac{\partial z_2}{\partial y}\right)^2 z_2 \\ &\triangleq \alpha_2(z_1, z_2, C_2\xi, C_2v, \vartheta_1, y_r, \dot y_r, \ddot y_r) \end{aligned} \tag{3.23}$$

would be a (z_1, z_2)-g.r.f. for the system consisting of (3.11), (3.16), (3.20), and (3.22) w.r.t.

$$V_2 = V_1 + \frac{1}{2}\left(z_2^2 + \tilde\vartheta_2^{\mathrm T}\Gamma_2^{-1}\tilde\vartheta_2\right) + \frac{1}{d_2}\Omega(\epsilon)\,. \tag{3.24}$$

Step i ($3 \le i \le \rho - 1$): In Steps 1 through $i-1$, we designed $\alpha_1, \ldots, \alpha_{i-1}$, $\dot\vartheta_1, \ldots, \dot\vartheta_{i-1}$ and we know that if $v_{n-\rho,i}$ were the control, then

$$\begin{aligned} v_{n-\rho,i} &= -c_{i-1}z_{i-1} - z_{i-2} - \psi_{i-1} - \vartheta_{i-1}^{\mathrm T}\omega_{i-1} + d_{i-1}\left(\frac{\partial z_{i-1}}{\partial y}\right)^2 z_{i-1} \\ &\triangleq \alpha_{i-1}(z_1, \ldots, z_{i-1}, C_{i-1}\xi, C_{i-1}v, \vartheta_1, \ldots, \vartheta_{i-1}, y_r, \ldots, y_r^{(i-1)}) \end{aligned} \tag{3.25}$$

would be a $(z_1, \ldots z_{i-1})$-g.r.f. for the system

$$\begin{aligned} \dot z_1 &= -(c_1 + d_1)z_1 + z_2 + \tilde\vartheta_1^{\mathrm T}\omega_1 + \epsilon \\ \dot z_j &= -\left[c_j + d_j\left(\frac{\partial z_j}{\partial y}\right)^2\right] z_j + z_{j+1} - z_{j-1} + \tilde\vartheta_j^{\mathrm T}\omega_j + \frac{\partial z_j}{\partial y}\epsilon\,, \\ &\qquad 2 \le j \le i-2\,, \\ \dot z_{i-1} &= v_{n-\rho,i} + \psi_{i-1} + \bar\theta^{\mathrm T}\omega_{i-1} + \frac{\partial z_{i-1}}{\partial y}\epsilon \\ \dot{\tilde\vartheta}_1 &= -\operatorname{sgn}(b_{n-\rho})\Gamma_1\omega_1 z_1 \\ \dot{\tilde\vartheta}_j &= -\Gamma_j\omega_j z_j\,, \quad 2 \le j \le i-1\,, \end{aligned} \tag{3.26}$$

w.r.t.

$$V_{i-1} = V_{i-2} + \frac{1}{2}\left(z_{i-1}^2 + \tilde\vartheta_{i-1}^{\mathrm T}\Gamma_{i-1}^{-1}\tilde\vartheta_{i-1}\right) + \frac{1}{d_{i-1}}\Omega(\epsilon)\,. \tag{3.27}$$

Now, in Step i, we augment (3.26) by the $\dot v_{n-\rho,i}$-equation from (3.5) and use Lemmas 4 and 1 to backstep it. We define the new state z_i as the difference between $v_{n-\rho,i}$ and its desired expression (3.25):

$$\begin{aligned} z_i &= v_{n-\rho,i} - \alpha_{i-1}(z_1, \ldots, z_{i-1}, C_{i-1}\xi, C_{i-1}v, \\ &\qquad \vartheta_1, \ldots, \vartheta_{i-1}, y_r, \ldots, y_r^{(i-1)})\,, \end{aligned} \tag{3.28}$$

and, using (3.5), (3.7), and (3.26), we write $\dot z_i$ as

$$\begin{aligned} \dot z_i &= v_{n-\rho,i+1} + \psi_i(z_1, \ldots, z_i, C_i\xi, C_iv, \vartheta_1, \ldots, \vartheta_{i-1}, y_r, \ldots, y_r^{(i)}) \\ &\quad + \bar\theta^{\mathrm T}\omega_i(z_1, \ldots, z_i, C_{i-1}\xi, C_{i-1}v, \vartheta_1, \ldots, \vartheta_{i-1}, y_r, \ldots, y_r^{(i-1)}) \\ &\quad + \frac{\partial z_i}{\partial y}(z_1, \ldots, z_{i-1}, C_{i-1}\xi, C_{i-1}v, \vartheta_1, \ldots, \vartheta_{i-1}, y_r, \ldots, y_r^{(i-1)})\epsilon\,. \end{aligned} \tag{3.29}$$

Let ϑ_i be a new estimate of $\bar\theta$ and denote $\tilde\vartheta_i = \bar\theta - \vartheta_i$. Then, from Lemmas 4 and 1, we know that if the update law for ϑ_i is chosen as

$$\dot\vartheta_i = \Gamma_i\omega_i(z_1, \ldots, z_i, C_{i-1}\xi, C_{i-1}v, \vartheta_1, \ldots, \vartheta_{i-1}, y_r, \ldots, y_r^{(i-1)})z_i\,, \tag{3.30}$$

and if $v_{n-\rho,i+1}$ were the control, then

$$\begin{aligned} v_{n-\rho,i+1} &= -c_iz_i - z_{i-1} - \psi_i - \vartheta_i^{\mathrm T}\omega_i - d_i\left(\frac{\partial z_i}{\partial y}\right)^2 z_i \\ &\triangleq \alpha_i(z_1, \ldots, z_{i-1}, C_i\xi, C_iv, \vartheta_1, \ldots, \vartheta_i, y_r, \ldots, y_r^{(i)}) \end{aligned} \tag{3.31}$$

would be a $(z_1, \ldots, z_i)$-g.r.f. for the system

$$\begin{aligned} \dot z_1 &= -(c_1 + d_1)z_1 + z_2 + \tilde\vartheta_1^{\mathrm T}\omega_1 + \epsilon \\ \dot z_j &= -\left[c_j + d_j\left(\frac{\partial z_j}{\partial y}\right)^2\right] z_j + z_{j+1} - z_{j-1} + \tilde\vartheta_j^{\mathrm T}\omega_j + \frac{\partial z_j}{\partial y}\epsilon\,, \\ &\qquad 2 \le j \le i-1\,, \\ \dot z_i &= v_{n-\rho,i+1} + \psi_i + \bar\theta^{\mathrm T}\omega_i + \frac{\partial z_i}{\partial y}\epsilon \\ \dot{\tilde\vartheta}_1 &= -\operatorname{sgn}(b_{n-\rho})\Gamma_1\omega_1 z_1 \\ \dot{\tilde\vartheta}_j &= -\Gamma_j\omega_j z_j\,, \quad 2 \le j \le i\,, \end{aligned} \tag{3.32}$$

w.r.t.

$$V_i = V_{i-1} + \frac{1}{2}\left(z_i^2 + \tilde\vartheta_i^{\mathrm T}\Gamma_i^{-1}\tilde\vartheta_i\right) + \frac{1}{d_i}\Omega(\varepsilon)\,. \tag{3.33}$$

Step ρ: This is the final step of our design procedure, in which we backstep the last equation separating us from the actual control u. Since $v_{n-\rho,\rho}$ is not the control, we augment (3.32) (with $i = \rho - 1$) by the $\dot v_{n-\rho,\rho}$-equation from (3.5) and use Lemmas 4 and 1 to backstep it. We define the new state z_ρ as the difference between $v_{n-\rho,\rho}$ and its desired expression (3.31) (with $i = \rho - 1$):

$$\begin{aligned} z_\rho &= v_{n-\rho,\rho} - \alpha_{\rho-1}(z_1, \ldots, z_{\rho-1}, C_{\rho-1}\xi, C_{\rho-1}v, \\ &\qquad \vartheta_1, \ldots, \vartheta_{\rho-1}, y_r, \ldots, y_r^{(\rho-1)})\,, \end{aligned} \tag{3.34}$$

and, using (3.5), (3.7), and (3.32), we write $\dot z_\rho$ as

$$\begin{aligned} \dot z_\rho &= \sigma(y)u + v_{n-\rho,\rho+1} + \psi_\rho(z_1, \ldots, z_\rho, C_\rho\xi, C_\rho v, \vartheta_1, \ldots, \vartheta_{\rho-1}, \\ &\quad y_r, \ldots, y_r^{(\rho)}) + \bar\theta^{\mathrm T}\omega_\rho(z_1, \ldots, z_\rho, C_{\rho-1}\xi, C_{\rho-1}v, \vartheta_1, \ldots, \vartheta_{\rho-1}, \\ &\quad y_r, \ldots, y_r^{(\rho-1)}) + \frac{\partial z_\rho}{\partial y}(z_1, \ldots, z_{\rho-1}, C_{\rho-1}\xi, C_{\rho-1}v, \vartheta_1, \ldots, \vartheta_{\rho-1}, \\ &\quad y_r, \ldots, y_r^{(\rho-1)})\epsilon\,. \end{aligned} \tag{3.35}$$

Let ϑ_ρ be a new estimate of $\bar\theta$ and denote $\tilde\vartheta_\rho = \bar\theta - \vartheta_\rho$. Then, from Lemmas 4 and 1, we know that if the update law for ϑ_ρ is chosen as

$$\dot\vartheta_\rho = \Gamma_\rho\omega_\rho(z_1, \ldots, z_\rho, C_{\rho-1}\xi, C_{\rho-1}v, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_r, \ldots, y_r^{(\rho-1)})z_\rho \tag{3.36}$$

then the control

$$u = \frac{1}{\sigma(y)}\left\{-c_\rho z_\rho - z_{\rho-1} - \psi_\rho - \vartheta_\rho^{\mathrm T}\omega_\rho - d_\rho\left(\frac{\partial z_\rho}{\partial y}\right)^2 z_\rho\right\} \tag{3.37}$$

is a $(z_1, \ldots, z_\rho)$-g.r.f. for the $(z_1, \ldots, z_\rho, \tilde\vartheta_1, \ldots, \tilde\vartheta_\rho)$-system w.r.t.

$$\begin{aligned} V_\rho &= V_{\rho-1} + \frac{1}{2}\left(z_\rho^2 + \tilde\vartheta_\rho^{\mathrm T}\Gamma_\rho^{-1}\tilde\vartheta_\rho\right) + \frac{1}{d_\rho}\Omega(\epsilon) \\ &= \frac{1}{2}\left(\sum_{j=1}^{\rho} z_j^2 + |b_{n-\rho}|\tilde\vartheta_1^{\mathrm T}\Gamma_1\tilde\vartheta_1 + \sum_{j=2}^{\rho}\tilde\vartheta_j^{\mathrm T}\Gamma_j^{-1}\tilde\vartheta_j\right) + \sum_{j=1}^{\rho}\frac{1}{d_j}\Omega(\varepsilon) \end{aligned} \tag{3.38}$$

Indeed, the $(z_1, \ldots, z_\rho, \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_\rho)$-system becomes with the choice of control (3.37)

$$
\begin{aligned}
\dot{z}_1 &= -(c_1 + d_1)z_1 + z_2 + \tilde{\vartheta}_1^{\mathrm{T}}\omega_1 + \epsilon \\
\dot{z}_j &= -\left[c_j + d_j\left(\frac{\partial z_j}{\partial y}\right)^2\right] z_j + z_{j+1} - z_{j-1} + \tilde{\vartheta}_j^{\mathrm{T}}\omega_j + \frac{\partial z_j}{\partial y}\epsilon, \\
&\qquad 2 \le j \le \rho - 1, \\
\dot{z}_\rho &= -\left[c_\rho + d_\rho\left(\frac{\partial z_\rho}{\partial y}\right)^2\right] z_\rho - z_{\rho-1} + \tilde{\vartheta}_\rho^{\mathrm{T}}\omega_\rho + v_1 + \frac{\partial z_\rho}{\partial y}\epsilon \\
\dot{\tilde{\vartheta}}_1 &= -\mathrm{sgn}(b_{n-\rho})\Gamma_1\omega_1 z_1 \\
\dot{\tilde{\vartheta}}_j &= -\Gamma_j\omega_j z_j, \quad 2 \le j \le \rho,
\end{aligned}
\tag{3.39}
$$

and the derivative of the partial Lyapunov function V_ρ defined in (3.38) along the solutions of (3.39) is nonnegative (since $c_j, d_j > 0$):

$$
\dot{V}_\rho = -\sum_{j=1}^{\rho}\left[c_j z_j^2 + d_j\left(\frac{\partial z_j}{\partial y}z_j - \frac{1}{2d_j}\epsilon\right)^2 + \frac{3}{4d_j}\|\varepsilon\|^2\right] \le 0. \tag{3.40}
$$

4 Boundedness and Tracking

With the above systematic design procedure we have not only designed an adaptive controller, but we have also set the stage for the following result:

Theorem 1 *Under Assumptions 1-4, all of the signals in the closed-loop adaptive system, which consists of the system (3.1), the filters (3.5), the parameter update laws for $\vartheta_1, \ldots, \vartheta_\rho$, and the control (3.37), are globally bounded, and, in addition,*

$$
\lim_{t\to\infty}[y(t) - y_r(t)] = 0. \tag{4.1}
$$

Proof. Due to the piecewise continuity of $y_r^{(\rho)}(t)$ and the smoothness of the nonlinearities in (3.1), the solution of the closed-loop adaptive system exists. Let its maximum interval of existence be $[0, t_f)$. On this interval, the nonnegative function V_ρ is nonincreasing because of (3.40). Thus, $z_1, \ldots, z_\rho$, $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_\rho$, and hence $\vartheta_1, \ldots \vartheta_\rho$, are bounded on $[0, t_f)$ by constants depending only on the initial conditions of the adaptive system. Furthermore, from (3.6), we know that the same is true of ε.

The boundedness of all other signals on $[0, t_f)$ is established as follows. Since z_1 and y_r are bounded, it follows that y is bounded. This implies that $\sigma(y)$ is bounded away from zero and, from (3.5), that $\xi_0, \ldots, \xi_\rho$ are bounded. From (3.5) we also see that

$$
v_{n-\rho-j,i} = \left[e_i^{\mathrm{T}}(sI - A_0)^{-1}e_{\rho+j}\right]\sigma(y)u, \quad 0 \le j \le n - \rho, \tag{4.2}
$$

where e_i is the ith coordinate vector in $\mathbb{R}^n$. Then, we express (3.1) in the differential equation form ($D = d/dt$)

$$
D^n y = \sum_{i=1}^{n} D^{n-i}\left[\varphi_{0,i}(y) + \sum_{j=1}^{p}\theta_j\varphi_{j,i}(y)\right] + \sum_{i=\rho}^{n} b_{n-i}D^{n-i}[\sigma(y)v]. \tag{4.3}
$$

Since y is bounded and, by Assumption 2, the polynomial $B(s) = \sum_{i=\rho}^{n} b_{n-i}s^{n-i}$ is Hurwitz, we conclude from (4.3) that $H_\rho(s)[\sigma(y)u]$ is bounded, where $H_i(s)$ denotes any exponentially stable transfer function of relative degree greater than or equal to i. By (4.2), this in turn implies that $F_j v_{n-\rho-j}$, $0 \le j \le n - \rho$, are bounded, where

$$
F_i v_j = [v_{j,1}, \ldots, v_{j,i+1}]. \tag{4.4}
$$

In particular, by (3.13), this implies that $C_1 v$ is bounded. From (3.19) we conclude that $v_{n-\rho,2}$ is bounded. Hence, by (4.2), $H_{\rho-1}(s)[\sigma(y)u]$ and thus $F_{j+1}v_{n-\rho-j}$, $0 \le j \le n - \rho$, are bounded. This again implies that $v_{n-\rho,3}$ is bounded. Continuing in the same fashion, we use (3.28), (3.34) and (4.2) to show that $H_i(s)[\sigma(y)u]$, $\rho - 2 \ge i \ge 1$, are bounded, which implies that v is bounded. Since $\sigma(y)$ is bounded away from zero, we conclude from (3.37) that u is bounded. Furthermore, from (3.6) we see that x is bounded.

We have thus shown that all of the signals in the closed-loop adaptive system are bounded on $[0, t_f)$ by constants depending only on initial conditions. Hence, $t_f = \infty$.

To prove the convergence of the tracking error to zero, we note that the boundedness of x, $\xi_1, \ldots, \xi_\rho$, v and u, together with (3.38), (3.40) and (4.2), implies that both V_ρ and $\dot{V}_\rho$ are bounded, and moreover, that $\dot{V}_\rho$ is integrable on $[0, \infty)$. Hence, $\dot{V}_\rho \to 0$ as $t \to \infty$, which proves that $z_1, \ldots, z_\rho \to 0$ as $t \to \infty$. Since $z_1 = y - y_r$, this proves (4.1). $\square$

5 The Class of Nonlinear Systems

Most models of nonlinear systems are derived from physical principles and given in specific coordinates. As we shall see in the robotic-arm example of Section 6, it may not always be obvious whether or not the nonlinear system at hand can be transformed into the output-feedback canonical form to which our design procedure is applicable. Therefore, in this section we derive a coordinate-free characterization of this form using differential geometric conditions which are necessary and sufficient for the existence (and also provide the guidelines for the construction) of a diffeomorphism which transforms the nonlinear system at hand into the output-feedback canonical form (3.1).

In the full-state feedback case, it is natural to look for parameter-independent diffeomorphisms, since one wants to be able to calculate the new state variables from the measurements of the original ones. On the other hand, when only the output is measured, the dependence of the diffeomorphism on the unknown parameters is not important. Therefore, we now give necessary and sufficient conditions for the system

$$
\begin{aligned}
\dot{\zeta} &= f(\zeta;\kappa) + g(\zeta;\kappa)u \\
y &= h(\zeta;\kappa),
\end{aligned}
\tag{5.1}
$$

where κ is a vector of unknown parameters, to be globally transformable into (3.1) via a possibly parameter-dependent diffeomorphism. The proof of the following theorem was given in [2].

Theorem 2 *The system (5.1) can be transformed via a global diffeomorphism $x = \phi(\zeta;\kappa)$ into the output-feedback canonical form (3.1) if and only if the following conditions are satisfied for all $\zeta \in \mathbb{R}^n$ and for the true value of the parameter vector κ:*

(i) $\mathrm{rank}\left\{dh, d(L_f h), \ldots, d\left(L_f^{n-1}h\right)\right\} = n$.

(ii) $\left[\mathrm{ad}_f^i r, \mathrm{ad}_f^{i+1} r\right] = 0, \quad 0 \le i \le n - 2$.

(iii) $\mathrm{ad}_f^n r = \sum_{i=0}^{n-1}\left[\varphi'_{0,n-i}(y) + \sum_{j=1}^{p}\theta_j\varphi'_{j,n-i}(y)\right](-1)^{n-i}\mathrm{ad}_f^i r,$

where $\varphi_{j,n-i}(y) = \int_0^y \varphi'_{j,n-i}(s)ds$, $0 \le i \le n - 1$, $0 \le j \le p$.

(iv) $\left[g, \mathrm{ad}_f^i r\right] = 0, \quad 0 \le i \le n - 2,$

(v) $g = \sigma(\cdot)\sum_{i=0}^{n-\rho} b_i(-1)^i\mathrm{ad}_f^i r$, *and*

(vi) the vector fields $r, \mathrm{ad}_f r, \ldots, \mathrm{ad}_f^{n-1} r$ are complete,

where σ is a smooth nonlinear function and r is the vector field defined by

$$
L_r L_f^i h = \begin{cases} 0, & i = 0, \ldots, n-2 \\ 1, & i = n-1. \end{cases} \tag{5.2}
$$

6 Application: Single-Link Flexible Robot

As an example of a mechanical system to which our new design procedure is applicable, let us now consider a single-link robotic manipulator whose rotary motion is controlled through an elastically coupled dc motor. If the effect of elastic coupling is modeled as a linear torsional spring, then the dynamic equations of the system are (cf. [1, p. 231])

$$
\begin{aligned}
J_1\ddot{q}_1 + F_1\dot{q}_1 + K\left(q_1 - \frac{q_2}{N}\right) + mgd\cos q_1 &= 0 \\
J_2\ddot{q}_2 + F_2\dot{q}_2 - \frac{K}{N}\left(q_1 - \frac{q_2}{N}\right) &= K_t i \\
LDi + Ri + K_b\dot{q}_2 &= u.
\end{aligned}
\tag{6.1}
$$

where q_1 and q_2 are the angular positions of the link and the motor shaft, $\imath$ is the armature current and u is the armature voltage. The inertias J_1, J_2, the viscous friction constants F_1, F_2, the elasticity constant K, the torque constant K_t, the back-EMF constant K_b, the armature resistance R and inductance L, the link mass M, the position of the link's center of gravity d, the transmission gear ratio N and the acceleration of gravity g can all be unknown.

We now assume that only the link position q_1 is measured. Then, we would like to see if the design procedure of Section 3 with $n = \rho = 5$, is applicable to this system. To this end, we first try the natural choice of state variables $\zeta_1 = q_1$, $\zeta_2 = \dot{q}_1$, $\zeta_3 = q_2$, $\zeta_4 = \dot{q}_2$, $\zeta_5 = \imath$. The dynamic equations (6.1) become

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= -\frac{mgd}{J_1}\cos\zeta_1 - \frac{F_1}{J_1}\zeta_2 - \frac{K}{J_1}\left(\zeta_1 - \frac{\zeta_3}{N}\right) \\
\dot{\zeta}_3 &= \zeta_4 \\
\dot{\zeta}_4 &= \frac{K}{J_2 N}\left(\zeta_1 - \frac{\zeta_3}{N}\right) - \frac{F_2}{J_2}\zeta_4 + \frac{K_t}{J_2}\zeta_5 \\
\dot{\zeta}_5 &= -\frac{R}{L}\zeta_5 - \frac{K_b}{L}\zeta_4 + \frac{1}{L}u \\
y &= \zeta_1 .
\end{aligned} \tag{6.2}$$

Clearly, (6.2) is not in the output-feedback form (3.1). However, there exists a different choice of coordinates which brings (6.1) into that form. To show this, we derive the input-output description of (6.2).

Differentiating y twice, we obtain $\zeta_2 = Dy$ and

$$D^2 y = -\frac{mgd}{J_1}\cos y - \frac{F_1}{J_1}Dy - \frac{K}{J_1}\left(y - \frac{\zeta_3}{N}\right), \tag{6.3}$$

which implies that

$$\zeta_3 = \frac{J_1 N}{K}\left(D^2 y + \frac{mgd}{J_1}\cos y + \frac{F_1}{J_1}Dy + \frac{K}{J_1}y\right) \tag{6.4}$$

$$\zeta_4 = D\zeta_3 = \frac{J_1 N}{K}\left(D^3 y + \frac{mgd}{J_1}D\cos y + \frac{F_1}{J_1}D^2 y + \frac{K}{J_1}Dy\right). \tag{6.5}$$

Differentiating (6.5) and substituting ζ_3 and ζ_4 from (6.4) and (6.5), we obtain

$$\begin{aligned}
\zeta_5 = \frac{J_1 J_2 N}{K_t K}\Bigg[&D^4 y + \left(\frac{F_1}{J_1} + \frac{F_2}{J_2}\right)D^3 y + \left(\frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2}{J_1 J_2}\right)D^2 y \\
&+ \frac{mgd}{J_1}D^2\cos y + \left(\frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 K}{J_1 J_2}\right)Dy \\
&+ \frac{mgdF_2}{J_1 J_2}D\cos y + \frac{mgdK}{J_1 J_2 N^2}\cos y\Bigg].
\end{aligned} \tag{6.6}$$

Finally, differentiating (6.6) and substituting ζ_4 and ζ_5 from (6.4) and (6.5), we arrive at the input-output description of (6.1):

$$\begin{aligned}
D^5 y = {} & \frac{K_t K}{J_1 J_2 N L}u - \left(\frac{R}{L} + \frac{F_1}{J_1} + \frac{F_2}{J_2}\right)D^4 y - \Bigg[\frac{R}{L}\left(\frac{F_1}{J_1} + \frac{F_2}{J_2}\right) \\
& + \frac{K_b}{L}\frac{K_t}{J_2} + \left(\frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2}{J_1 J_2}\right)\Bigg]D^3 y \\
& - \frac{mgd}{J_1}D^3\cos y - \Bigg[\frac{R}{L}\left(\frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2}{J_1 J_2}\right) \\
& + \frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 K}{J_1 J_2} + \frac{K_b}{L}\frac{F_1 K_t}{J_1 J_2}\Bigg]D^2 y \\
& - \left(\frac{R}{L} + \frac{F_2}{J_2}\right)\frac{mgd}{J_1}D^2\cos y - \Bigg[\frac{R}{L}\left(\frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 K}{J_1 J_2}\right) \\
& + \frac{K_b}{L}\frac{K_t K}{J_1 J_2}\Bigg]Dy - \left(\frac{K}{N^2} + \frac{RF_2}{L} + \frac{K_b K_t}{L}\right)\frac{mgd}{J_1 J_2}D\cos y \\
& - \frac{R}{L}\frac{mgdK}{J_1 J_2 N^2}\cos y .
\end{aligned} \tag{6.7}$$

From (6.7), it is clear that there exists a choice of variables which brings (6.1) into the form (3.1):

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta_1 x_1 \\
\dot{x}_2 &= x_3 + \theta_2 x_1 + \theta_3\cos x_1 \\
\dot{x}_3 &= x_4 + \theta_4 x_1 + \theta_5\cos x_1 \\
\dot{x}_4 &= x_5 + \theta_6 x_1 + \theta_7\cos x_1 \\
\dot{x}_5 &= \theta_8\cos x_1 + b_0 u ,
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
\theta_1 &= -\left(\frac{R}{L} + \frac{F_1}{J_1} + \frac{F_2}{J_2}\right) \\
\theta_2 &= -\left[\frac{R}{L}\left(\frac{F_1}{J_1} + \frac{F_2}{J_2}\right) + \frac{K_b}{L}\frac{K_t}{J_2} + \left(\frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2}{J_1 J_2}\right)\right] \\
\theta_3 &= -\frac{mgd}{J_1} \\
\theta_4 &= -\left[\frac{R}{L}\left(\frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2}{J_1 J_2}\right) + \frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 K}{J_1 J_2} + \frac{K_b}{L}\frac{F_1 K_t}{J_1 J_2}\right] \\
\theta_5 &= -\left(\frac{R}{L} + \frac{F_2}{J_2}\right)\frac{mgd}{J_1} \\
\theta_6 &= -\left[\frac{R}{L}\left(\frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 K}{J_1 J_2}\right) + \frac{K_b}{L}\frac{K_t K}{J_1 J_2}\right] \\
\theta_7 &= -\left(\frac{K}{N^2} + \frac{RF_2}{L} + \frac{K_b K_t}{L}\right)\frac{mgd}{J_1 J_2} \\
\theta_8 &= -\frac{R}{L}\frac{mgdK}{J_1 J_2 N^2} \\
b_0 &= \frac{K_t K}{J_1 J_2 N L} > 0 .
\end{aligned} \tag{6.9}$$

Hence, the design procedure of Section 3 is applicable to (6.8) and yields an adaptive controller that achieves bounded asymptotic position tracking from all initial conditions and for all positive values of the constants $K, K_t, K_b, J_1, J_2, R, L$. It is a tedious but straightforward task to verify that (6.2) satisfies the conditions of Theorem 2 and that the map from the physical coordinates of (6.2) to those of (6.8) is parameter-dependent.

7 Concluding Remarks

The results of this paper have advanced our ability to control nonlinear systems with unknown parameters using only output, rather than full-state, measurement. They have also demonstrated that the design tools of Section 2 are not just "tricks of the trade," but can be used in a systematic fashion. In fact, these tools were recently used in [2] to produce new design procedures for the more general case of partial-state measurement, thus generalizing the results of this paper. Finally, it is worth noting that the new design procedure can be modified in several ways to decrease the dynamic order of the resulting adaptive controller [2, Chapter 5].

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