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To: Dr. E. P. Rood

From: D. W. Hubbard and G. Trevino

Subject: Contract Title: Free Surface Turbulence
Contract Number: N0001485-K-0236
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Scientific Officers: C. Lee, J. Fein, and E. P. Rood

Final technical report for the project--1 March 1985 to 30 September 1988

The turbulence characteristics of a water jet flowing near a free surface were studied. The focus of the project was to develop two-point closure methods which may make possible a "universal" description of the turbulence in the region where the jet interacts with the free surface.

The interactions of a planar jet with an air-water interface were determined by measuring turbulent velocity fluctuations and pressure fluctuations using hot-film anemometry techniques. A two-dimensional jet was formed by pumping water through a rectangular slit into a channel filled with still water. The raw data are sampled time records of the anemometer output voltages obtained at different positions in the jet. These time records together with the sensor calibration information are analyzed using digital signal processing techniques. The double and triple velocity correlation functions and pressure-velocity correlation functions as well as the mean velocities, Reynolds stress, and turbulence intensities were calculated.

These calculated results were used to develop a two-point turbulence closure scheme for the jet flow. The closure model contains the effects of small non-homogeneities and anisotropy, since there is a preferred direction in the jet flow. Because of the proximity to the free surface, the flow field is not symmetric about the axis of the slit. By means of a projective transformation, the jet flow turbulence results can be used to predict the turbulence in the zone of interaction between a wake and a free surface.

SUMMARY OF THE EXPERIMENTAL WORK

The chronology of the project is as follows. During the first year, the flow system was designed and built, velocity measurement techniques were developed, and velocity sensor calibration equipment and methods were developed. The unique feature of this calibration method was devising a systematic method for dealing with sensor fouling during long periods of continuous operation of the hot-film sensors. This work was the subject of the M. S. thesis written by Eric J. Hine. During the second year, techniques for measuring turbulent pressure fluctuations were devised. A commercially available sensor was used, but extensive development work was required to adapt it to operation in water. This sensor system was used along with a standard hot-film anemometer system to measure two-point pressure-velocity correlations. These results were the subject of the M. S. thesis written by Lisa R. Harris. During the third year, the techniques developed were applied to an extensive study of the jet flow. The mean velocities and jet boundaries were measured along with single-point correlations--turbulent intensities and Reynolds stresses. Two-point correlations were also measured. The single-point and two-point measurements are the subject of the M. S. thesis written by Eric Menning.

SIGNIFICANT EXPERIMENTAL RESULTS

A. Development of Experimental Techniques

Sensor Calibration One important accomplishment has been to develop a method for allowing for fouling of hot-film sensors when making measurements in untreated water. Fouling cannot be avoided, but it has been shown that the decrease of anemometer output voltage with time while making measurements in fouling conditions is linear except for an initial "seasoning" period. By calibrating the sensor periodically during the course of a series of experiments, and by keeping careful account of the time at which individual measurements were made, a sensor calibration relationship for each individual measurement can be calculated. The sensor calibration was done using a laminar jet apparatus supplied with the same untreated water which was being used for the jet flow experiments.

Pressure Fluctuation Measurement Another important accomplishment has been the successful application of a bleed type pressure sensor to measuring turbulent pressure fluctuations in water. Until recently, pressure sensors sensitive enough to measure turbulent pressure fluctuations were not available, so the effect of the pressure terms could not be included in two-point turbulence closure models for nonisotropic turbulence. Jones (1981) and Spencer(1970) report the development of a bleed-type pressure sensor based on a hot-film anemometer system. A hot-film velocity probe connected to a constant temperature anemometer circuit is enclosed in a tube open to the pressure to be measured at the downstream end. The upstream end is connected to a constant pressure source of fluid, and there is a constant flow of fluid from the constant pressure supply, past the velocity sensor, and into the flow field where the pressure is being measured. The velocity of the fluid flowing through the tube is related to the pressure difference, so the downstream pressure can be related directly to the anemometer output. It is well-known that hot-film velocity probes are very sensitive to changes in the cooling velocity in the neighborhood of the sensor. This means that pressure measurements based on the velocity determined using a hot-film sensor are equally as sensitive.

Pressure sensors such as described above are available commercially from TSI, Inc. These pressure sensors were originally developed for pressure measurements in gas flows using helium or nitrogen as the bleed fluid. Such sensors had never been applied to pressure measurements in liquids, so some development work was necessary. The bleed fluid must be non-conducting to prevent corrosion of the sensor, and the flow through the sensor must be laminar. For measurements in water, the bleed fluid selected is ethyl alcohol which is miscible with the main water flow. The ethanol must be supplied at constant pressure to the pressure sensor. In the system devised for doing this, ethanol flows to the sensor from a reservoir which is pressurized with nitrogen. The ethanol is pumped by a gear pump into the reservoir from a small supply tank. There is a filter in the line to remove any particles which may be in the ethanol. The pressure is maintained constant by a system of bleeds and pressure regulating valves.

To calibrate the pressure sensor, the sensor is placed in still water, and the bridge output is recorded as the pressure upstream in the reservoir is changed. Since the flow rate through the sensor depends on the pressure <u>difference</u>, it does not matter whether the upstream or downstream pressure is changed during calibration.

The pressure fluctuation measurements required an exacting and delicate experimental technique, and the sensor was rather fragile. The sensor failed twice during the course of the work and had to be returned to the manufacturer for repair. In November 1987, it failed again and could not be repaired. This means that we never were able to obtain data suitable for some of the calculations we wanted to do.

B. Single-Point Results

The single-point results were obtained using a crossed-sensor probe so that both longitudinal and transverse fluctuations could be calculated. The results presented were obtained by calculating the mean of time series records which had been converted to instantaneous velocity values by applying the appropriate calibration equation. These single-point quantities are presented as functions of position in the jet. The results presented are not synoptic--measurements were made at different times in a stationary turbulent field. The complete results are included in the thesis written by Eric Menning which is attached as an appendix to this report. The highlights of the work are summarized below. The notation used to describe the system is shown in Figure 1.

<u>Mean Velocities</u> The mean velocity profiles at different positions downstream from the source for a submergence ratio of H/D = 6.9 are shown in Figure 2. The dimensional data show that the jet decays rapidly, and distortion due to the presence of the free surface is evident. The center line of the jet is displaced toward the surface as the distance downstream increases. At the point farthest downstream, the profile is not well-defined. The jet boundary seems to have intersected the surface at approximately

x/D = 25. The mean velocity data are summarized in dimensionless form in Figure 3 where the mean velocities have been made dimensionless by dividing by the maximum velocity, and the vertical position has been made dimensionless by dividing by the length scale defined in the next paragraph. This figure shows that the jet decays in a self-similar way, because the profiles nearly coincide for all axial positions shown. The distortion due to the presence of the free surface, and the displacement of the jet center line toward the free surface are evident in this figure. This displacement is plotted in Figure 4 which shows that in terms of dimensionless coordinates, the displacement of the jet center line approaches an asymptote as x/D becomes larger. The effect of the proximity to the surface on the mean velocity profile at a single axial position(x/D = 11.0) is shown in Figure 5. The jet centerline is displaced toward the free surface from the nozzle centerline, and the submergence of the jet seems to have a small influence on this displacement. Velocity measurements close to the surface are rather uncertain because of waves present as the jet intersects the free surface.

The decay of the centerline mean velocities in a turbulent jet is described by--

$$(U_m/U_0)(x/D)^{0.5} = (2/\pi)^{0.25}(1/c)^{0.5} \qquad (1)$$

c is Reichardt's constant which must be determined experimentally. Mih(1975) reports c = 0.140. Equation (1) is plotted in Figure 6 along with the measured centerline mean velocities obtained in several different jet flow experiments at different submergence ratios. The velocities plotted are the velocities at the nozzle centerline. Because of the distortion due to the presence of the free surface, the jet centerline may not coincide with the nozzle centerline. This effect could cause some of the data scatter observed. There is general agreement with the work of Mih as represented by Equation (1).

Jet Spreading The distance from the jet center line downward to a point, y^*, where the mean velocity is one-half the maximum mean velocity is a measure of the jet spreading. The lower half of the jet was used for this measurement, in order to reduce the influence of the free surface so that the results might be compared with the work of others. Figure 7 shows the results for a submergence ratio of H/D = 6.9. The jet expands linearly according to

$$y^*/D = -\,0.054\,(x/D) - 0.094 \qquad (2)$$

(Correlation Coefficient = 0.994)

This equation is obtained when the three points closest to the jet source are included, and it coincides with the results others have reported for free jets. The spreading is also approximately linear when all four points are included in the analysis. The equation for this is

$$y^*/D = -0.134\ (x/D) + 0.728 \qquad (3)$$
(Correlation Coefficient = 0.986)

There are two effects that make this situation studied different from a true free jet. The confinement by the air-water interface has an undetermined effect on the spreading of a jet. The jet was also confined from below by the solid bottom of the tank which was located at $H_o/D = 39.4$ in this case. The point farthest downstream may have been influenced by back flow along the tank bottom, though no backflow was observed directly. Gutmark and Wygnanski(1976) report results obtained for measurements with free air jets in air which indicate the following relationship.

$$y^* \sim 0.1\ (x/D - x_o/D) \qquad (4)$$

The effect of the free surface as was shown above is to distort the centerline of the jet so that the distance from the centerline to the surface becomes smaller as x increases. This could account for the lower coefficient of (x/D) in equation (2).

<u>Turbulence Quantities</u> Turbulent velocity fluctuations were calculated by subtracting the mean velocity from each instantaneous velocity value in the time series which were calculated from the anemometer output data as described above. The turbulent velocity fluctuation results were combined in the appropriate ways and time-averaged to give the Reynolds stresses and turbulent intensities.

1) Reynolds stress

The Reynolds stress depends on axial position as illustrated in Figure 8 where the results are shown for a submergence ratio of 6.9. Lines drawn in the figures show approximate trends and are not "best fit" lines. These dimensional data show the distortion of the mean velocity profile due to the presence of the free surface. As the jet spreads out downstream from the source, the upper and lower parts of the profile become less and less antisymmetrical. If no free surface were present, one expects the Reynolds stress profile to be antisymmetrical about the centerline. In Figure 9, the dimensionless Reynolds stress is plotted as a function of the dimensionless transverse position. This shows that as the jet moves downstream, similarity of the turbulence is approximately maintained in the part of the jet below the centerline away from the free surface but not maintained in the part of the jet above the centerline toward the free surface. As the jet moves downstream, the position of the minimum Reynolds stress moves toward the surface. Based on the Reynolds stress results, one cannot say anything definitive about the position of the centerline of the jet. As the jet moves downstream, the magnitude of the Reynolds stress changes first increasing and then decreasing.

2) Turbulence Intensity

The turbulence intensities are plotted in Figure 10(axial turbulence intensity) and Figure 11(transverse turbulence intensity) in terms of dimensionless position. The submergence ratio which applies to these data is again 6.9. As the jet moves downstream, the dimensionless intensity profiles are reduced in size, though the magnitude of the maxima and the general shape are approximately the same at all axial positions studied. This indicates that in terms of the length scale y^* similarity of the turbulence is not preserved in the planar jet flow. This is a finding different from that based on the Reynolds stress results for which similarity was observed in the lower half of the jet.

The uncertainty is rather large for the turbulence intensities and for the Reynolds stresses as is seen from the data scatter of the plotted points. The tests for similarity require plotting dimensionless values which are calculated by multiplying together two velocity fluctuation values which in turn are calculated by subtracting a mean velocity from a measured velocity--two numbers which differ by at most by 10 to 20 per cent. This product is then divided by another quantity calculated in the same way. The uncertainly in the measured velocities is approximately 3 to 15 percent. This means that the uncertainty in the calculated turbulence results is 30 to 60 per cent. This is much larger than the uncertainty in the mean velocities because it is likely that the uncertainties associated with each individual velocity measurement are normally distributed and that the time averaging process keeps the uncertainty of the final calculated mean velocity approximately the same as the uncertainty of an individual measurement. Since a 12-bit A/D converter was used with a 16-bit microcomputer for data logging, the relative uncertainty associated with the electronic data processing was 0.00024(1/4096). All the uncertainty in the velocity measurements comes from sensor calibration and from not being able to produce a true stationary jet flow.

3) Influence of the Free Surface

From the intensity results reported, it is not possible to say anything definite about the effect of the proximity of the free surface on the turbulence intensity. Others have postulated that eddies are damped by the free surface and that, compared to a jet which is well-submerged, the transverse intensity should be reduced and the axial intensity should be increased. Intensity results for three different submergence ratios are shown in Figure 12. The intensity profiles are distorted as the submergence decreases. The scatter in the results is large enough to mask the characteristic shape with two maxima, but the magnitude of the intensity values scaled by maximum velocity does not depend on submergence. Figure 13 shows the effect of submergence on the Reynolds stress. The profiles are distorted as the distance to the free surface decreases. The part of the Reynolds stress profile nearest the free surface is most strongly affected. As the jet is submerged farther from the free surface, the characteristic concave outward shape of the Reynolds stress profile is restored.

4) Pressure Fluctuations

The pressure probe proved to be sensitive enough to detect pressure fluctuations of approximately 2000 Pascals. This is illustrated in Figure 14 where the difference between the pressure at the sensor tip and the reference pressure is plotted as a function of time. Power spectra of the pressure signals are presented in Figure 15. They show a large contribution at low frequencies with a peak at 32 Hertz. When the probe is pointed directly upstream, this peak is eliminated. The peak is greatly reduced when the probe is pointed directly downstream. We could not identify any noise or vibration in the system of this frequency, but without more evidence, we should not infer that this peak indicates some phenomenon related to the turbulence. These spectra were originally calculated to give information about the best mounting angle for the pressure probe to avoid an effect of impact in the output of the instrument. We found that the best position for reducing the effect of impact and avoiding the effect of the wake of the probe support was 135^{o} where zero degrees indicates that the probe is pointed directly upstream.

5) Autocorrelation Functions

Taylor's hypothesis allows transforming spatial and temporal correlations for homogeneous flows--those which have constant mean velocity. The autocorrelations--velocity/velocity correlations with one of the velocities displaced in time--were calculated using the single-point data , so that we could determine the magnitude of the error which would be made in applying Taylor's hypothesis to our non-homogeneous jet flow. The autocorrelations show the expected shape and indicate something about the time scales associated with the turbulence. The details of the autocorrelation results are discussed in the attached thesis by E. Menning--pages 35 to 41.

C. Two-Point Results

The two-point measurements were to be one of the main features of the whole project. The information is essential for developing a turbulence closure model that is more universal than single-point closure models such as the k-epsilon model. Flow features in turbulence which persist over a time long compared to the time required for a turbulent fluctuation--coherent structures--have been identified in many different flow fields. Because of this, it makes sense to try to describe turbulence in terms of the range of interactions between two points in the flow. A governing equation for the transport of the two-point correlation function can be written, but it contains triple correlations and pressure-velocity correlations. The triple correlations must be modelled for all flows, and the pressure-velocity correlations must be modelled for non-isotropic flows. These model functions must be evaluated from experimental data. Once the model functions are available, they are used in solving the differential equation for the two-point velocity correlation which in turn can be used to calculate the velocity field.

One objective of the work was to obtain experimental turbulent velocity data which would be suitable for calculating the two-point pressure-velocity correlations, $\langle pu_i' \rangle$, the two-point double velocity correlation, $\langle u_i u_j' \rangle$, and the two-point triple velocity correlations, $\langle u_i u_j u_k' \rangle$. These measurements were not successful, because of the experimental uncertainties which affected the results. The same uncertainties as found in the single-point measurements were present with the two-point measurements. The uncertainty introduced by subtracting the mean value from all velocity values which appeared in calculating the Reynolds stress and turbulent intensity also appears in the two-point correlation calculations. Using two hot-film sensors introduced further uncertainty, because the interactions in the turbulence took place over a short range in the jet flow. The wake of the upstream sensor seemed to interfere with the measurements obtained from the down stream sensor when the two sensors were placed along the same axis. These difficulties together with the fact that the relative uncertainty is increased when the turbulent velocities are multiplied together in calculating the correlations combined to decrease the signal-to-noise ratio to the point where no definitive results were obtained from the two-point correlation measurements.

The two-point spatial correlation values calculated are not accurate enough to be used to verify the results of closure modelling. However, the results are consistent with measurements made by others. The complete discussion of the double and triple spatial velocity correlations appears on pages 35 to 50 in the attached thesis.

D. Closure Modelling Work

During the three years of the project, the theoretical and modelling work has been focussed on developing a model which describes jet turbulence in terms of parameters which depend only on Reynolds number and are therefore fundamental turbulence parameters. To develop a two-point closure for anisotropic turbulence, it is necessary to model two terms in the two-point velocity correlation equation.

- $\langle pu' \rangle$ related to anisotropy(this is zero for isotropic turbulence)
- $\langle uuu' \rangle$ related to the presence of scale independent energy transfer modes or "preserved" structure

These quantities may be modelled in terms of the invariants of the separation position vector and a unit vector in the direction of the flow. Using this latter vector is our method for introducing anisotropy into the model. For example, the pressure velocity correlation can be expressed as--

$$\langle pu_i' \rangle = B_1(\underline{r} \cdot \underline{r},\ \underline{r} \cdot \underline{b},\ t)\underline{r}_i + B_2(\underline{r} \cdot \underline{r},\ \underline{r} \cdot \underline{b},\ t)\underline{b}_i$$

where $\underline{r}$ is the separation vector and $\underline{b}$ is a unit vector in the direction of flow, and

$$p = p(\underline{x},\ t) \qquad \text{and} \qquad u_i' = u_i(\underline{x} + \underline{r},\ t)$$

As an approximation for the model being developed, let there be "small" anisotropy. This allows separating the model into two functions--an isotropic part which depends on the separation vector and an anisotropic part which depends on the preferred direction.

$$\langle pu_i' \rangle = B_1(\underline{r} \cdot \underline{r},\ t)\underline{r}_i + B_2(\underline{r} \cdot \underline{b},\ t)\underline{b}_i$$

This model has been chosen, because it provides a simple way to include an explicit dependence on anisotropy and is distinctly different from the corresponding expression for the isotropic case. It has been determined that B_1 is and even function of $\underline{r}$ and that B_2 is an odd function of $\underline{b}$. Because of mass conservation, B_1 and B_2 are not independent but are related by

$$B_1(\underline{r} \cdot \underline{r},\ t) = -r^{-3} \int r^2 \frac{\partial B_2}{\partial(\underline{r} \cdot \underline{b})}\, dr$$

For $\underline{b} = (1,0,0)$ and $\underline{r} = (r,0,0)$, the expression for the pressure-velocity correlation gives--

$$\langle p\ u' \rangle = B_1\ r + B_2$$

$$\langle p\ v' \rangle = \langle p\ w' \rangle = 0$$

The triple velocity correlation is modelled as a function of an energy transfer parameter, $k(\underline{r})$. It has been determined that a reasonable form for k is--

$$k(\underline{r}) = a_1\ (r/\eta)^3 \exp[\ -\underline{a_2}(r/\eta)^2]$$

where η is a "characteristic" length scale, a_2 defines the value of (r/η) for which energy transfer is a maximum, and a_1 defines the corresponding amount of energy transferred. All of the functions described are also functions of time, so they are related to the decay of the turbulence as the water moves along in the jet. This decay is modelled as the sum of the decay of an isotropic part and the decay of an anisotropic part with no interaction between the two parts.

Because of the experimental difficulties encountered during the course of the project, we do not yet have any suitable data for the pressure-velocity correlations or the triple velocity correlations to use for developing the closure model for a non-isotropic flow. The model for two-point correlations of turbulent velocity fluctuations has been developed using the two-point triple velocity correlations for isotropic turbulence reported by Stewart(1951). This theoretical model contains two parameters, a and b, which are pure numbers. Since the only number peculiar to the analysis of turbulence is the Reynolds number, a and b cannot be arbitrary parameters but must depend on Reynolds number in some well-prescribed way. A formal representation could be written

$$a \sim F(N_{Re}) \quad \text{and} \quad b \sim G(N_{Re})$$

F(...) and G(...) may be different for different turbulent flow situations. One preliminary result of fundamental importance is that the ratio, a/b, is insensitive to changes in Reynolds number. This suggests an energy cascade autonomy or "self-governance" which has not been reported in the literature. The properties of a and b are as follows.

1) $a \neq 0$ and $b \neq 0$
2) $a < b$

It also appears that (b - a) is a measure of how much energy is transferred in the fluid by eddy entrainment, and (b + a) is a measure of how much energy is transferred in the fluid by eddy deformation.

E. Publications and Conference Presentations

Besides the four theses listed below under the heading of participants, the following publications have appeared as a result of research supported under this project. Copies of these publications are included in the appendix.

Trevino, G. 1989 "Isotropic Analysis of Grid Turbulence", Int Journal Eng Sci, 27, pp. 1463-1471

Trevino, G. 1989 "On the Invariant Functions of the Turbulence Bispectrum", Int Journal Eng Sci, 27, pp. 529-537

Trevino, G. 1989 "Energy Transfer in Turbulence", Physics of Fluids A, 1, pp. 2061-2064

Trevino, G. In Press "On the Theory of Isotropic Decay", Appl Mech Rev,

The following papers have been presented at technical meetings and conferences.

Hubbard, D. W.(with E. J. Hine and G. Trevino), "Experimental Techniques for Measuring Turbulent Velocity Correlations in a Jet Flow Near an Air-Water Interface", Paper 6AM6-3, 22nd Annual Meeting, Society for Engineering Science, Penn State University(October 1985)

Hubbard, D. W.(with E. J. Hine, L. R. Heydenburg, and G. Trevino), "Turbulence in Water Jets: A Two-Point Closure", Paper 131e, 1986 Annual Meeting, American Institute of Chemical Engineers, Miami Beach(November 1986)

Trevino, G. "The Invariant Functions of the Turbulence Bispectrum", International Conference on Fluid Mechanics, Beijing, China(July 1987

Hubbard, D. W. (with L. R. Heydenburg), "Pressure-Velocity Correlation Measurements in Turbulent Liquid Flows", 14th Annual Mid-Western Universities Fluid Mechanics Retreat, Rochester, Indiana(April 1987)

Trevino, G. "On the Theory of Isotropic Decay", Pan American Conference of Applied Mechanics, Rio de Janeiro, Brazil(January 1989)

F. Participants

Eric J. Hine, Graduate Research Assistant, M.S. Thesis entitled "Two-Point Third-Order Velocity Correlations in a Planar Jet Bounded by a Free Surface" completed and M.S. Degree granted in June 1986.

Lisa R. Harris, Graduate Research Assistant, M.S. thesis entitled "Two-Point Pressure-Velocity Correlations in a Planar Jet Bounded by a Free Surface" completed and M.S. degree granted in August 1987.

Eric Menning, Graduate Research Assistant, M.S. Thesis entitled "Pressure and Velocity Correlations in a Plane turbulent Water jet" completed and M.S. Degree awarded in November 1988.

Jiajin Qu, Graduate Research Assistant, Ph.D. Thesis entitled "Decay Laws of Isotropic Turbulence" completed and Ph.D. Degree awarded in June 1990.

Christine M. Cross, Undergraduate Research Assistant

Michael Bednar, Undergraduate Research Assistant

Craig Harris, Undergraduate Research Assistant

G. Plans for Future Work

Our main plan for future work for this jet flow turbulence project is to improve the precision of the experimental data so that the closure model can be developed for non-isotropic flows. We need to control the flow better to provide truly stationary turbulence. The jet flow we used for our measurements was produced by pumping water into a reservoir of still water. Using a centrifugal pump caused small fluctuations in the flow because of speed variations due to minor frequency variations in the line power. This difficulty can be overcome by using a stand-pipe water supply system which is available in the Department of Civil and Environmental Engineering at Michigan Tech. This stand-pipe is approximately 50 feet high and has an over flow weir and a pumping system to maintain a constant hydrostatic head. If such a system were connected to the jet flow nozzle unit, the upstream pressure would be constant and there should be no fluctuations in the mean flow. This supply system would also eliminate any difficulty of temperature changes in the water due to viscous dissipation when the nozzle is supplied directly by a centrifugal pump.

The turbulent jet is supposed to be two-dimensional, but there can be entrainment at the sides leading to transverse velocity gradients. A refinement which would eliminate this difficulty would be to design shields for the sides of the nozzle so that water would be entrained only from above and from below.

The calibration system we developed during the course of this project is effective, but the procedure is cumbersome and time consuming. We have some ideas for devising an in-situ sensor calibration device which would be a moveable laminar jet. Using this kind of calibration device would allow leaving the sensor in position for the frequent periodic calibration which is necessary when making measurements in water. Calibration data would be obtained with the calibration jet in one position, and the proper relationship between velocity and sensor output would be determined by the well-established equations giving the response of yawed sensor elements.

The separation distance necessary to measure non-zero two-point velocity correlation functions turned out to be quite small. This made it difficult to mount the sensors directly in alignment because of the arrangement of the supporting structures. With the close spacing needed, the wake of the upstream sensor interferes with the signal of the downstream sensor. This makes the output of the downstream sensor noisy. Both the alignment and proximity problems could be eliminated by used a laser doppler anemometer(LDA) as the upstream sensor. This is a nonintrusive device which does not disturb the flow. The downstream sensor could be a crossed-sensor hot film probe just as was used in the present study. This idea was first suggested by Thomas Sweam of the Naval Research Laboratory. There is no LDA unit available at Michigan Tech.

The bleed type pressure sensor proved to be sensitive enough to measure the pressure fluctuations which occurred in the jet flow, but the device was not rugged enough for long term application in liquid flows. If the pressure-velocity correlation measurements are continued, a more rugged pressure sensor is needed. A piezoelectric pressure sensor might be appropriate.

Computer power increases all the time, and people at various laboratories are able to simulate instantaneous turbulent velocity fields by solving the Navier-Stokes equations. Results are available right now for low Reynolds number flows in simple systems. It would be interesting to try to extend this direct simulation method to the case of a jet flow interacting with a free surface. If that effort was successful, the results could be used to develop the two-point closure scheme. These results could in turn be compared with experimental data. The simulation might be more effective for model development, because the velocity values are synoptic and closely spaced if the numerical analysis grid is fine enough.

H. References

Gutmark, E. and I. Wygnanski 1976 "The Planar Turbulent Jet", J Fluid Mech, 73, pp. 470-471

Jones, B. G. 1981 "A Bleed-Type Transducer for In-Stream Fluctuating Static pressure Sensing", TSI Quarterly, 7, pp. 5-8

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Spencer, B. W. 1970 "Measurement of Fluctuating Pressure", Tech. Bull. 34, TSI, Inc., St. Paul, Minn.

Stewart, R. W. 1951 "Triple Velocity Correlations in Isotropic Turbulence", Proc Camb Phil Soc, 47, pp.146-157

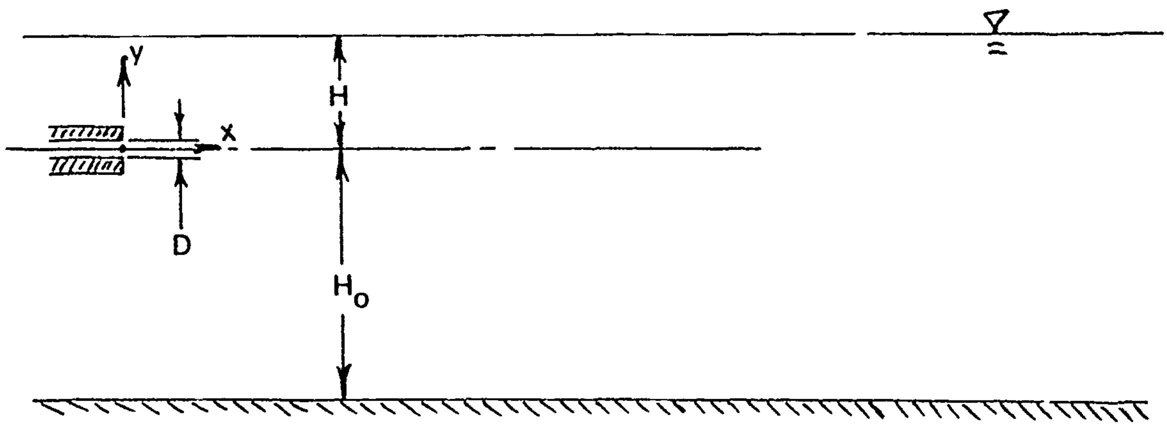


Figure 1. Jet Geometry and Notation

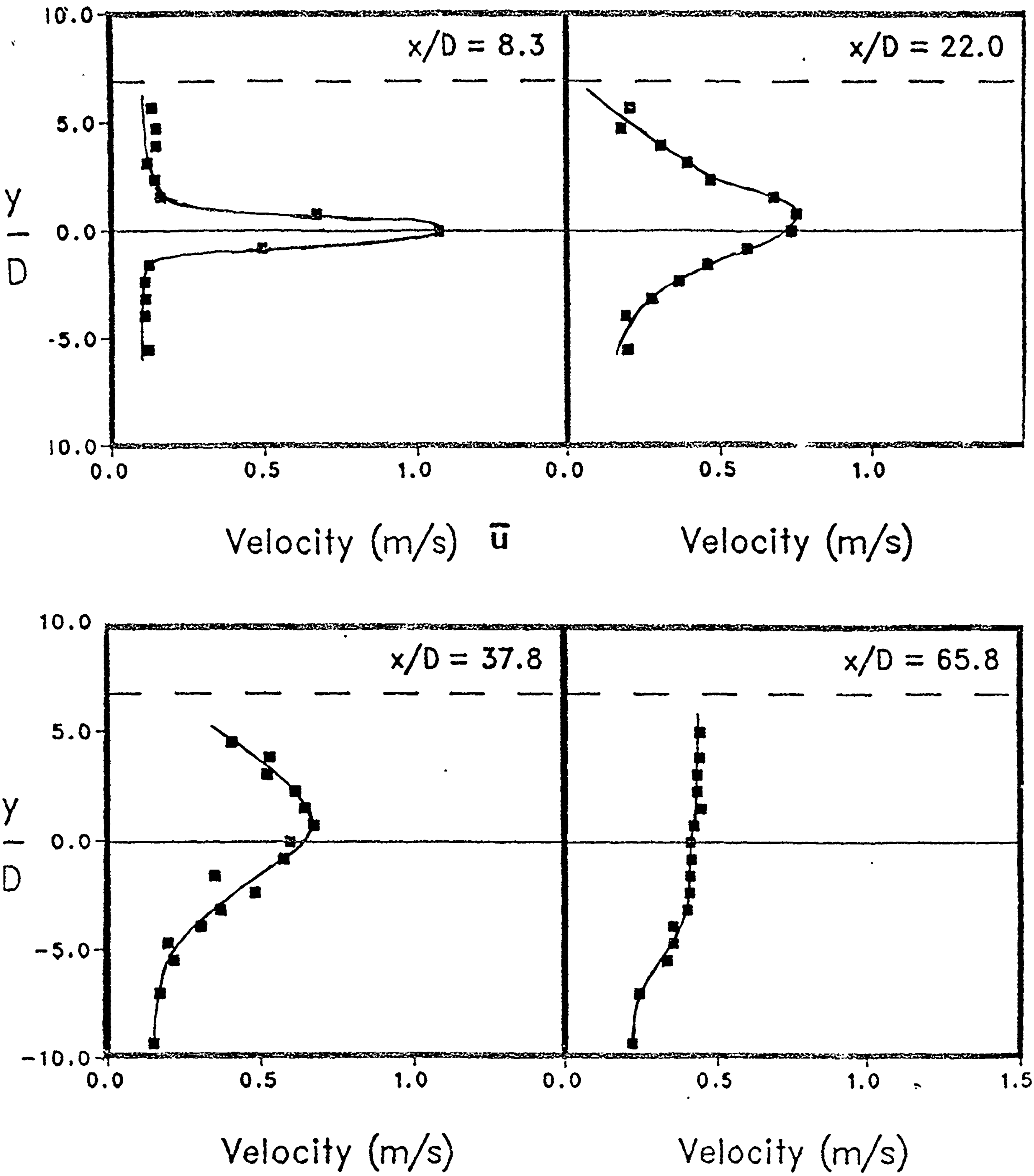


Figure 2. Effect of axial position on mean velocity profiles. (D = 6.35 mm, H/D = 6.9)

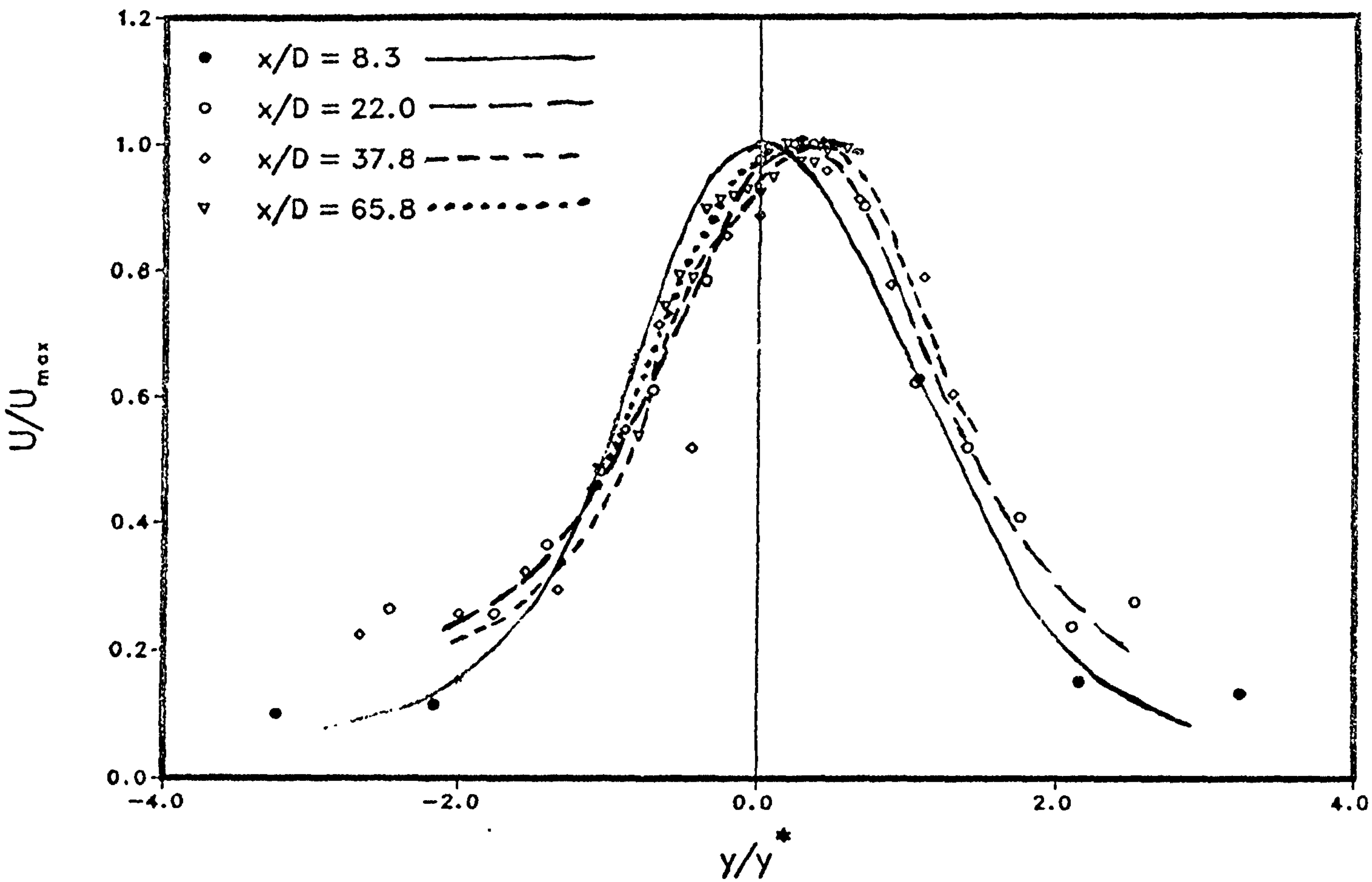


Figure 3. Dimensionless mean velocity profiles. (D = 6.35, H/D = 6.9)

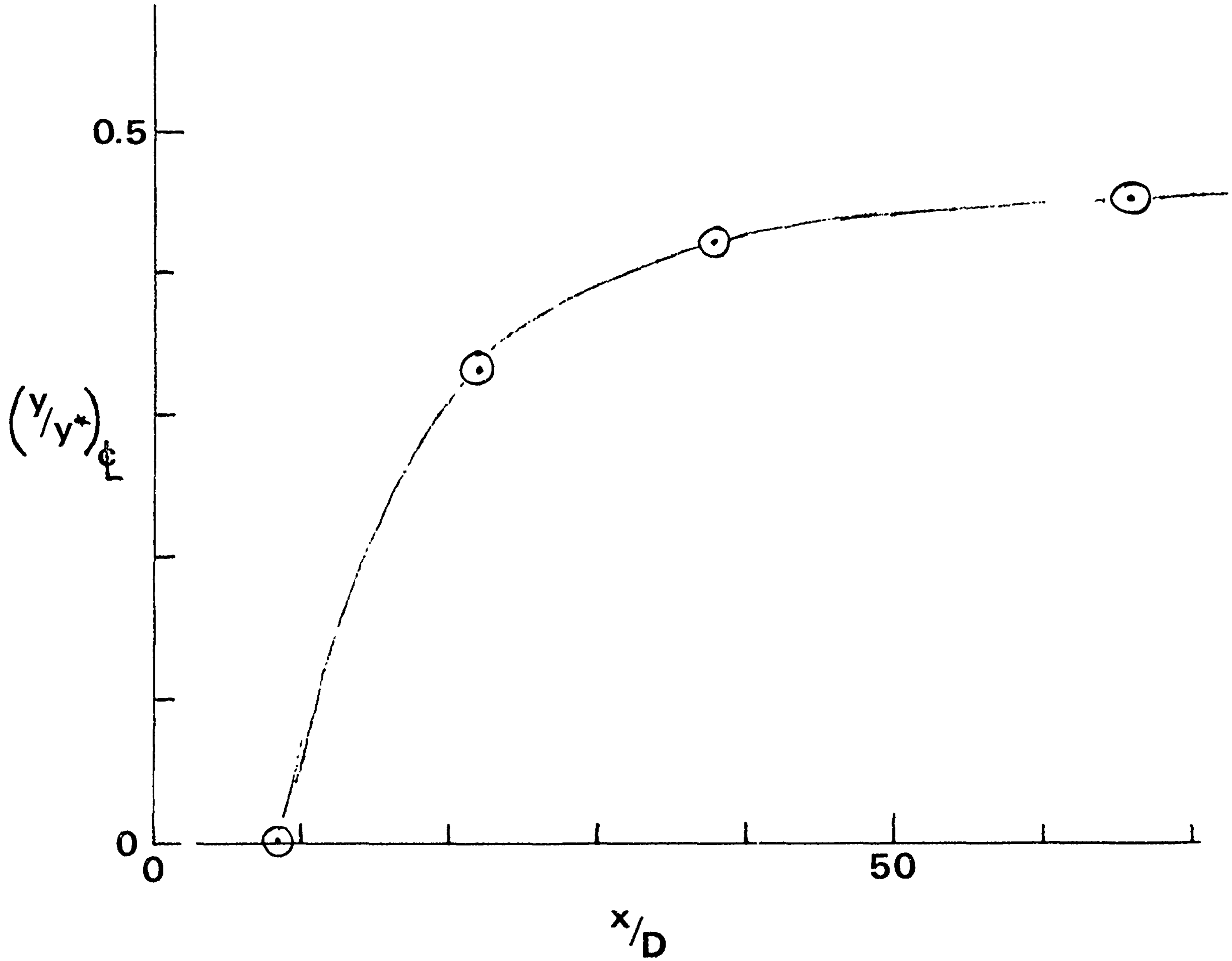


Figure 4. Effect of Axial Position on Jet Centerline Displacement

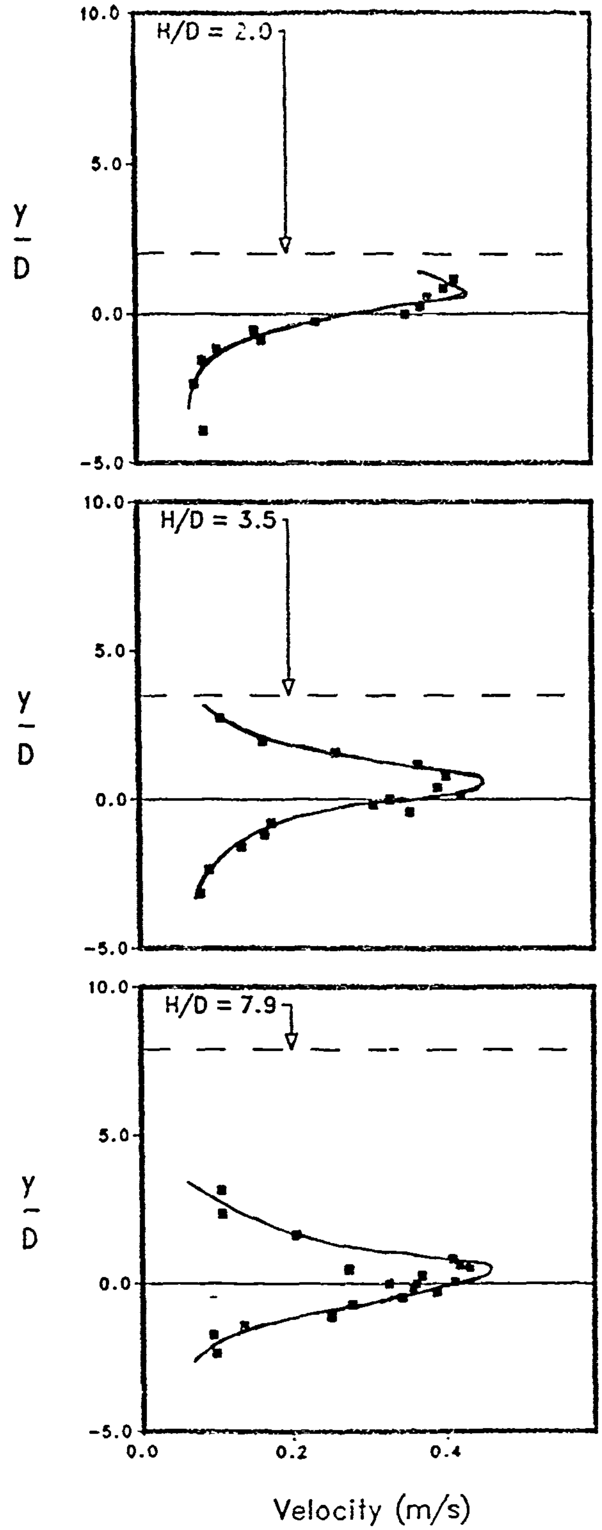


Figure 5. Effect of the Free Surface on the Mean Velocity (D = 12.7 mm, x/D = 11.0)

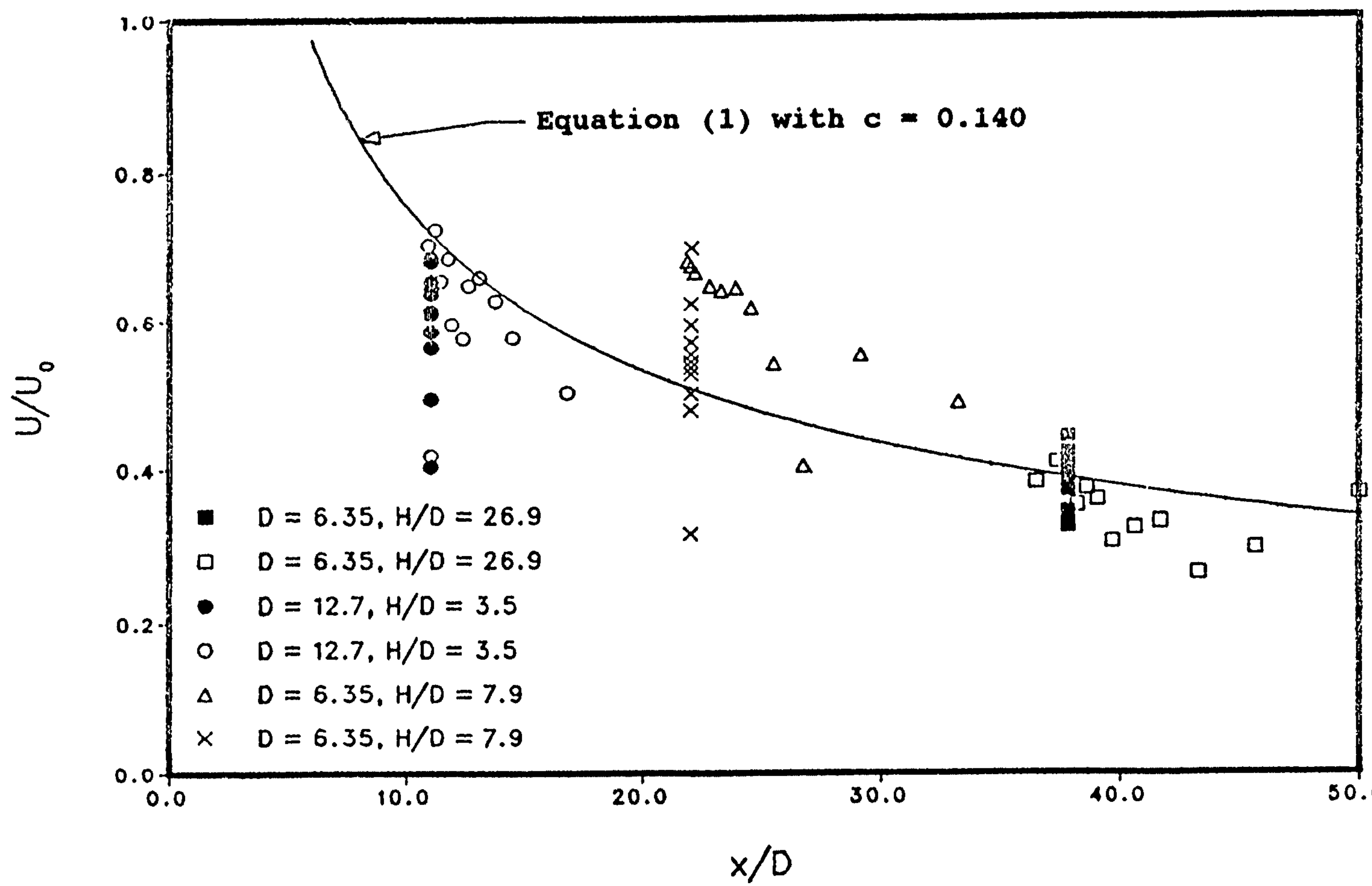


Figure **6**. Centerline mean velocities.

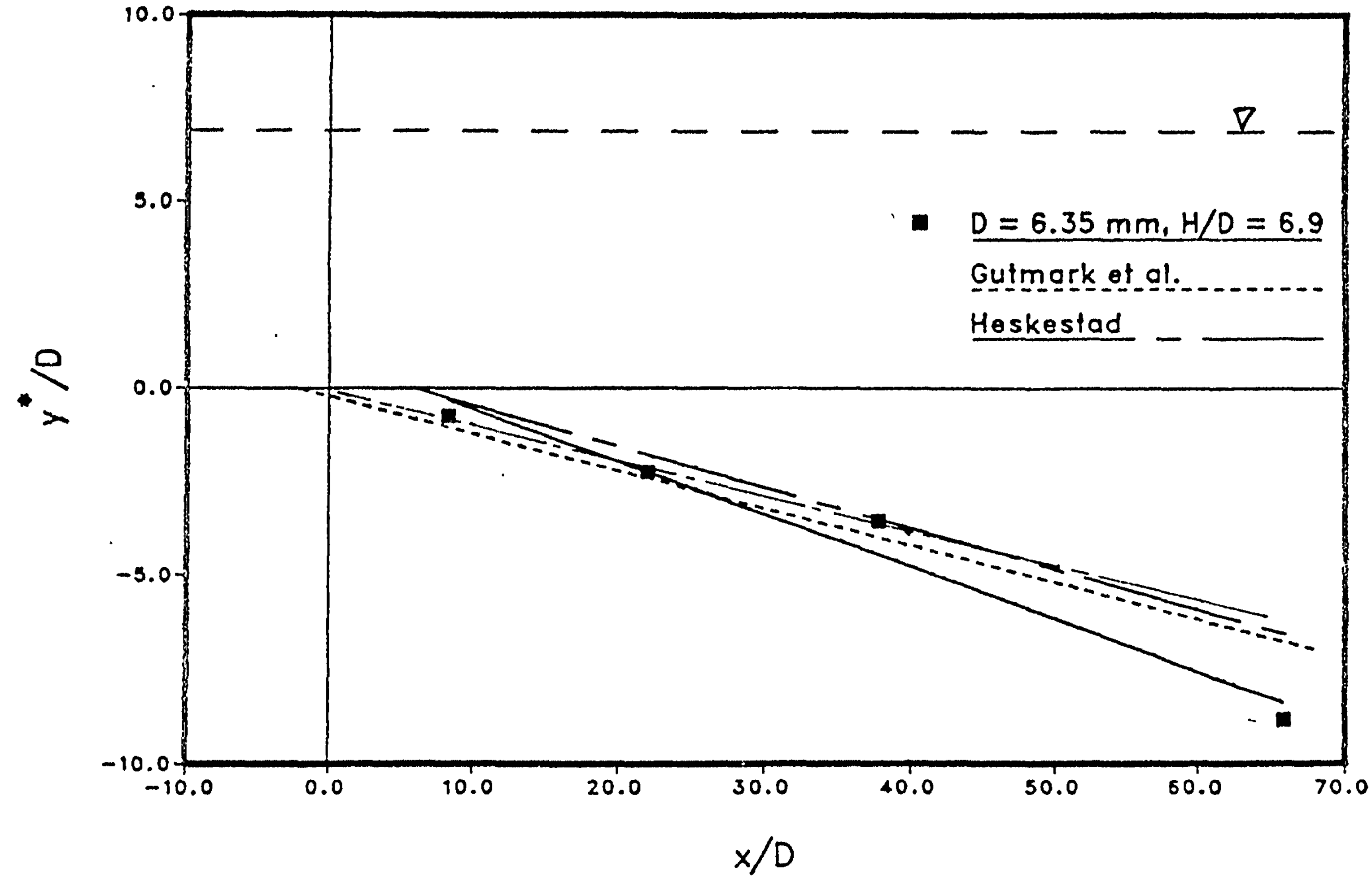


Figure 7. Spread of the jet. **$(H_o/D) = 39.4$**

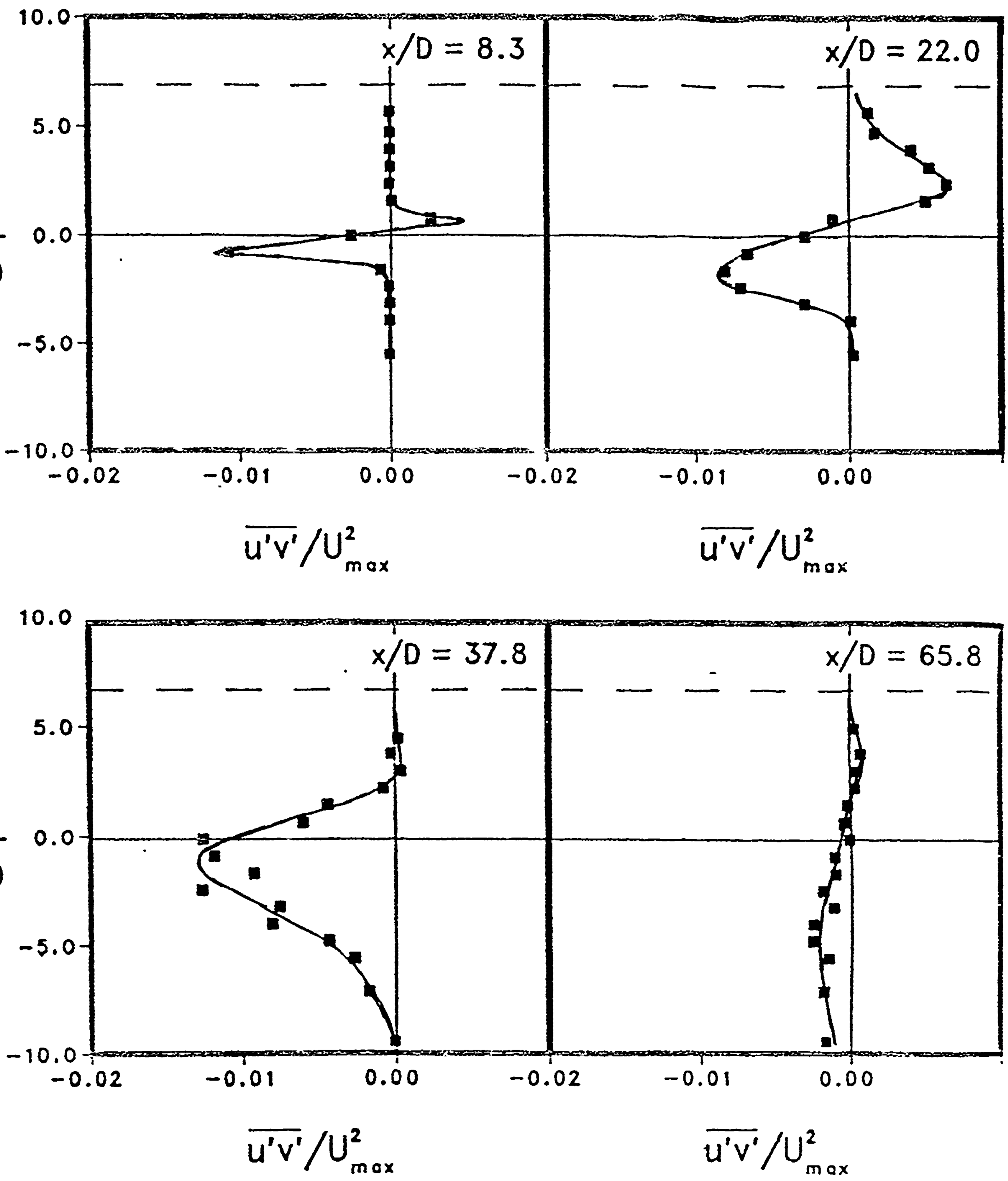


Figure 8. Effect of axial position on Reynolds stress profiles. (D = 6.35 mm, H/D = 6.9)

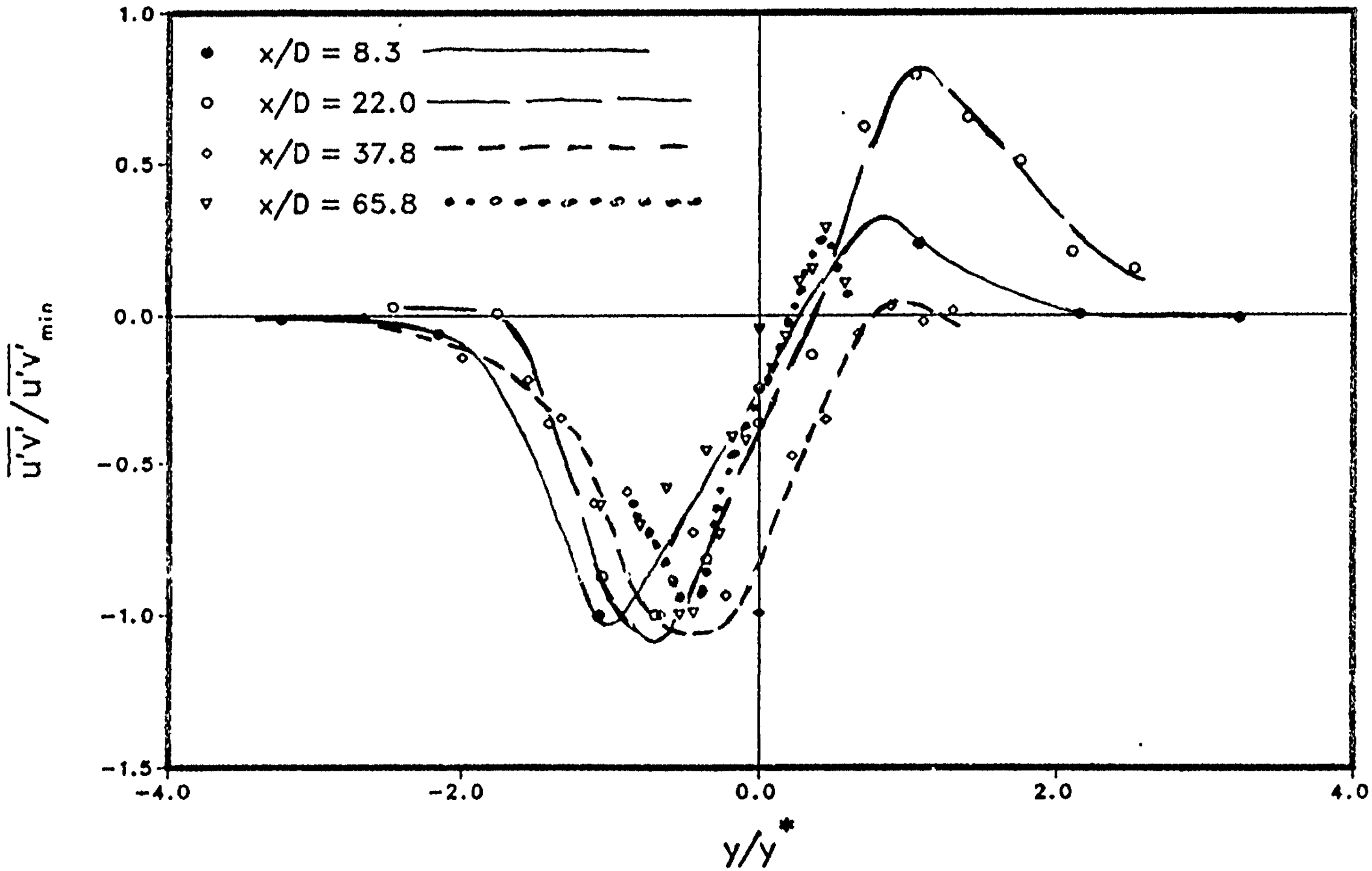


Figure 9. Dimensionless Reynolds stress profiles. (D = 6.35, H/D = 6.9)

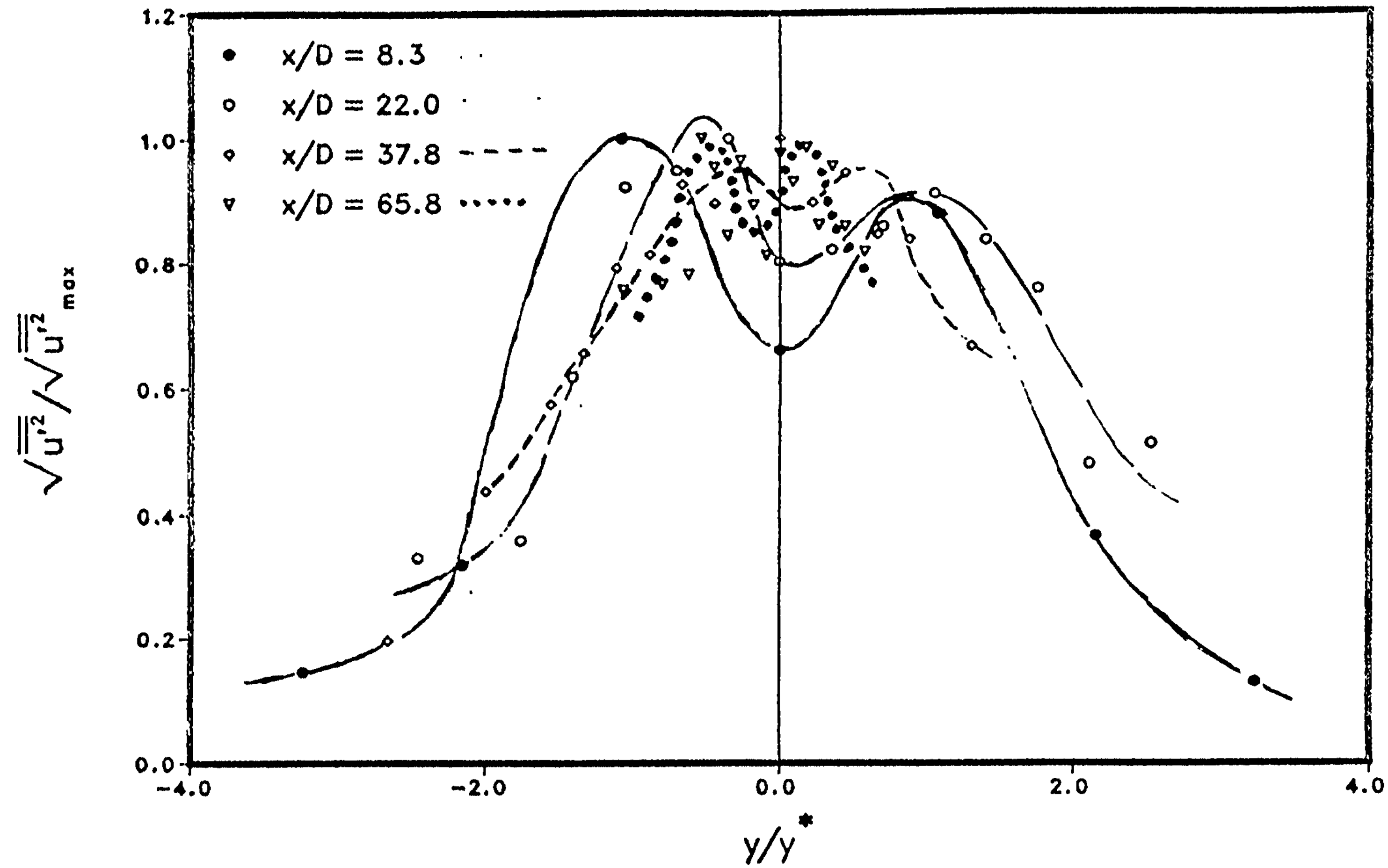


Figure **10**. Dimensionless axial turbulence intensity profiles. (D = 6.35, H/D = 6.9)

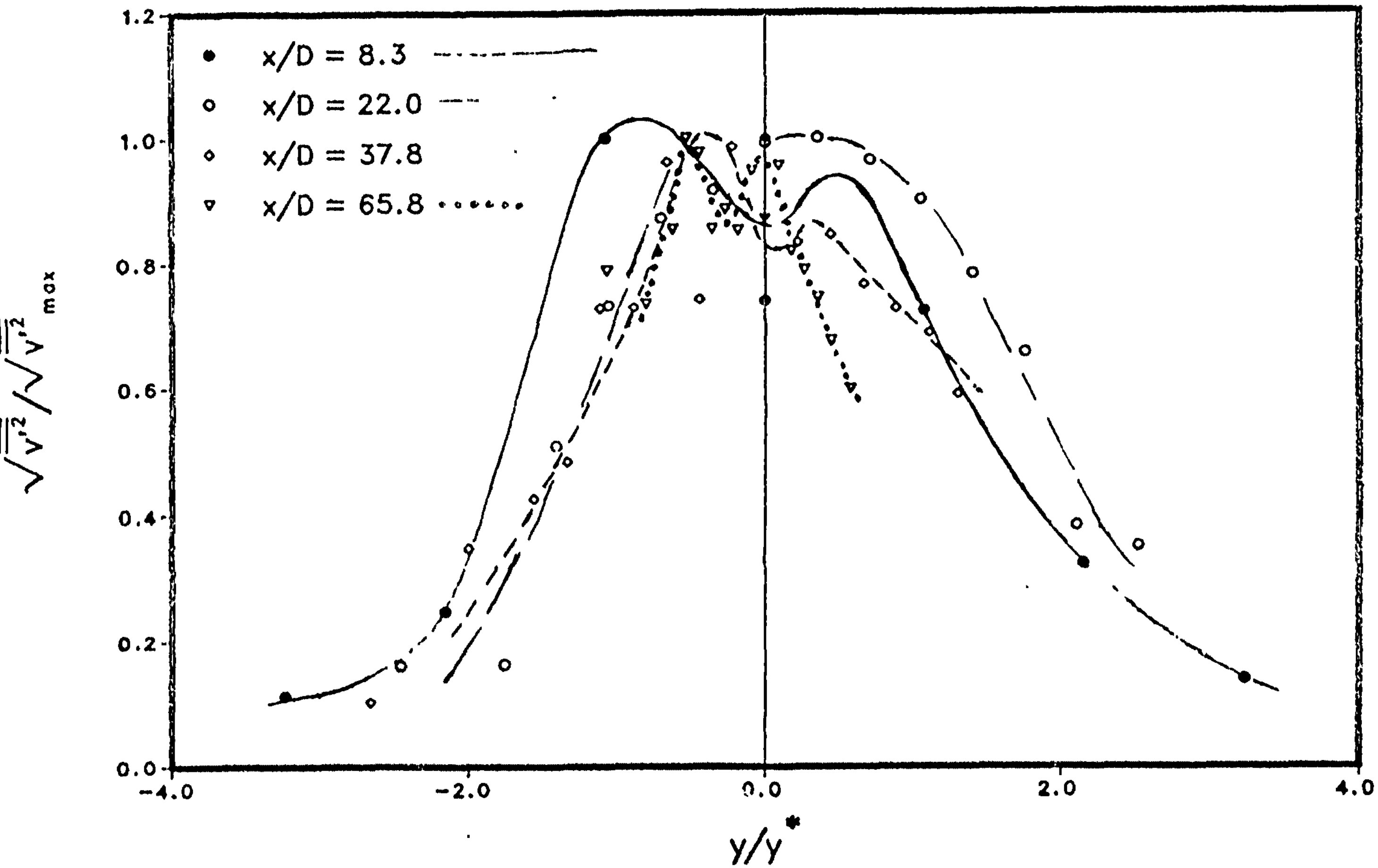


Figure **11.** Dimensionless transverse turbulence intensity profiles. (D = 6.35, H/D = 6.9)

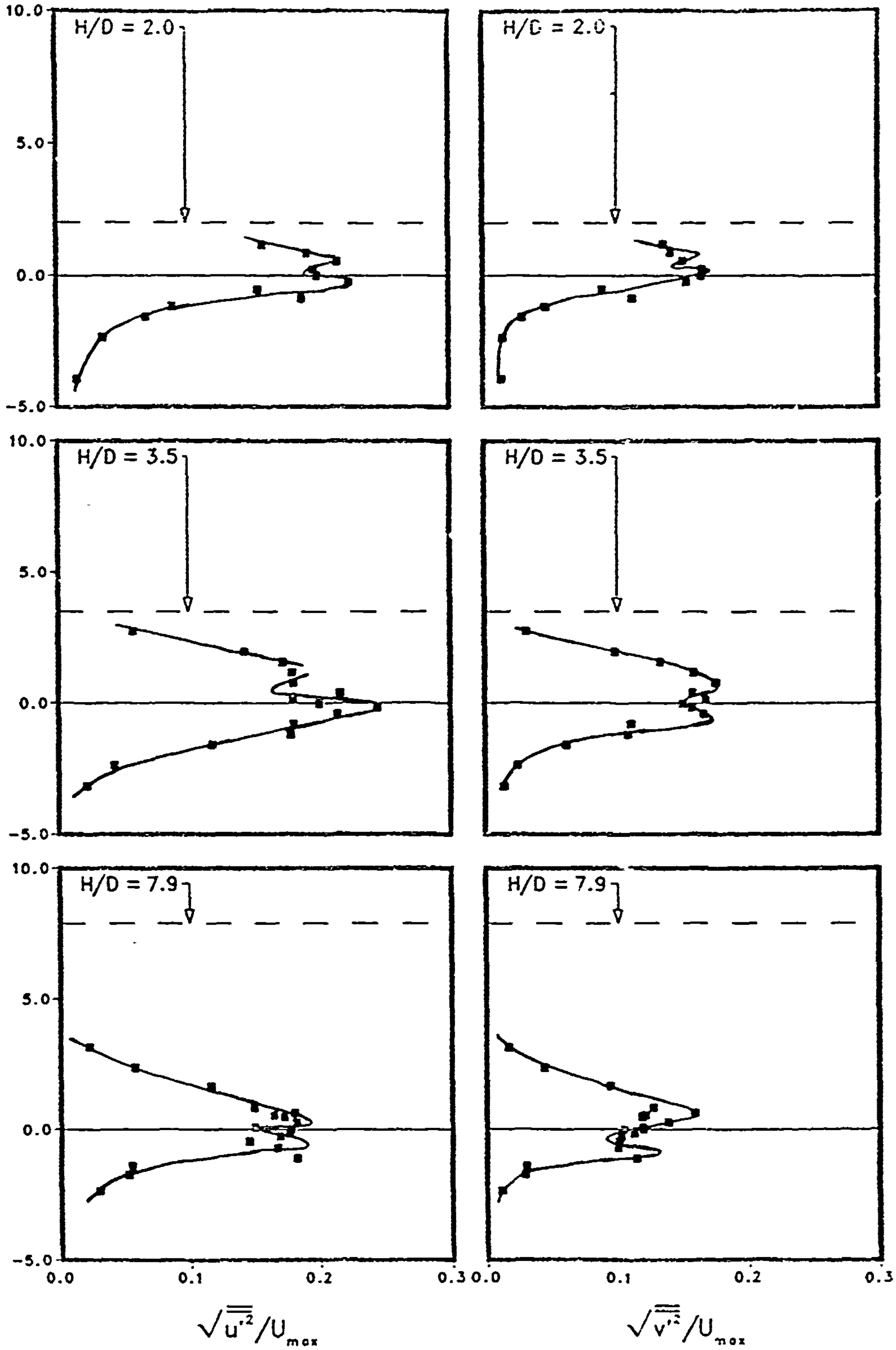


Figure 12. Effect of the Free Surface on Axial and Transverse Turbulent Intensity (D = 12,7 mm, x/D = 11.0)

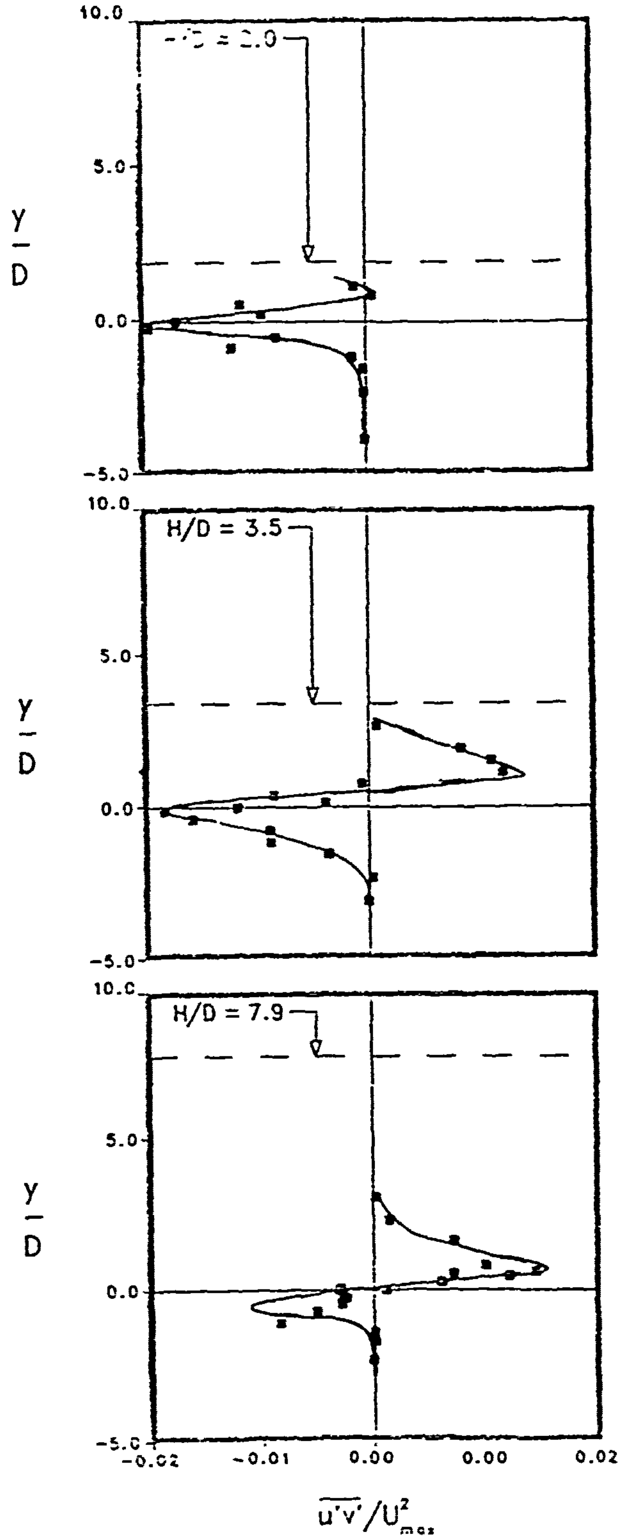


Figure 13. Effect of the Free Surface on the Reynolds Stress (D = 12.7 mm, x/D = 11.0)

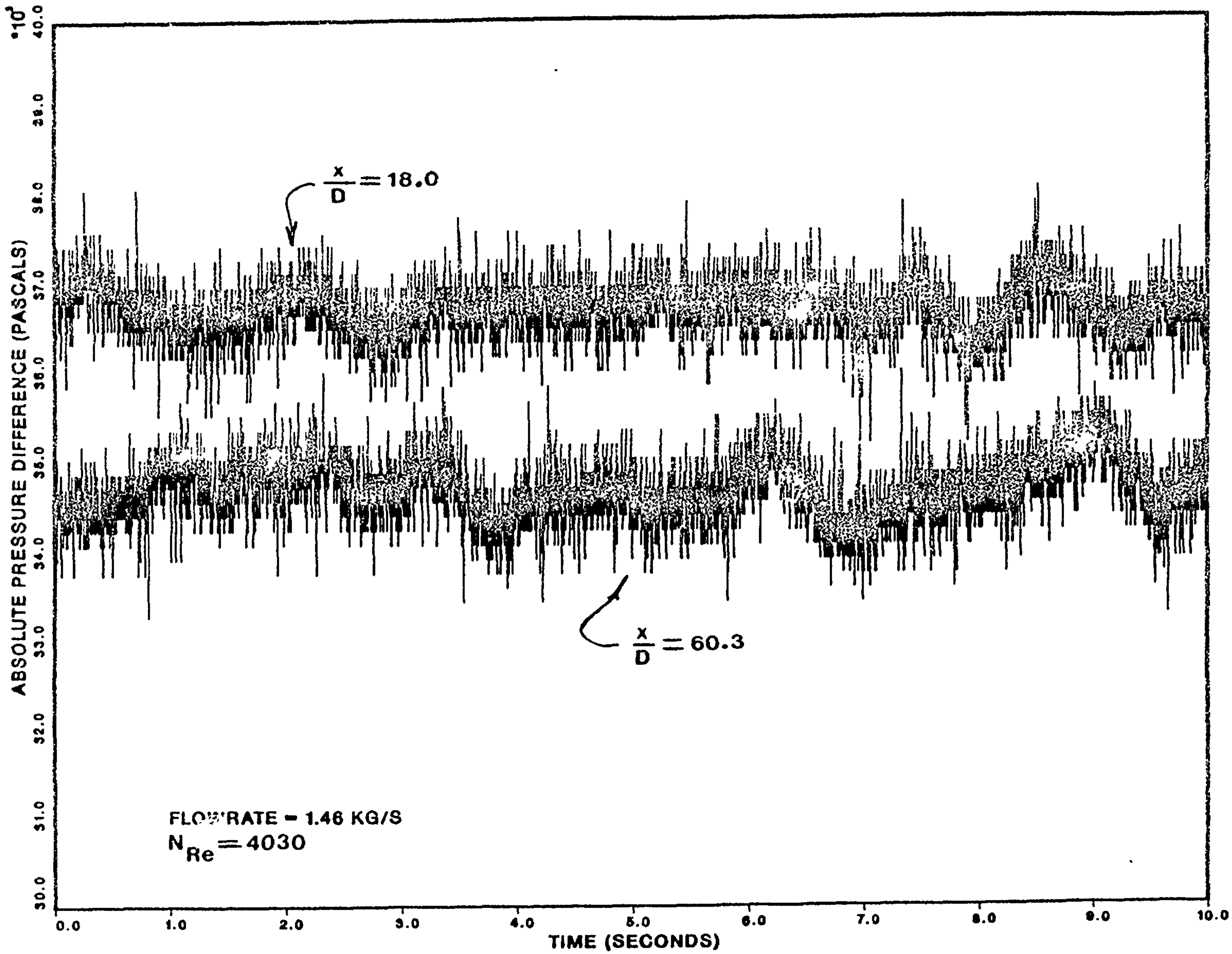
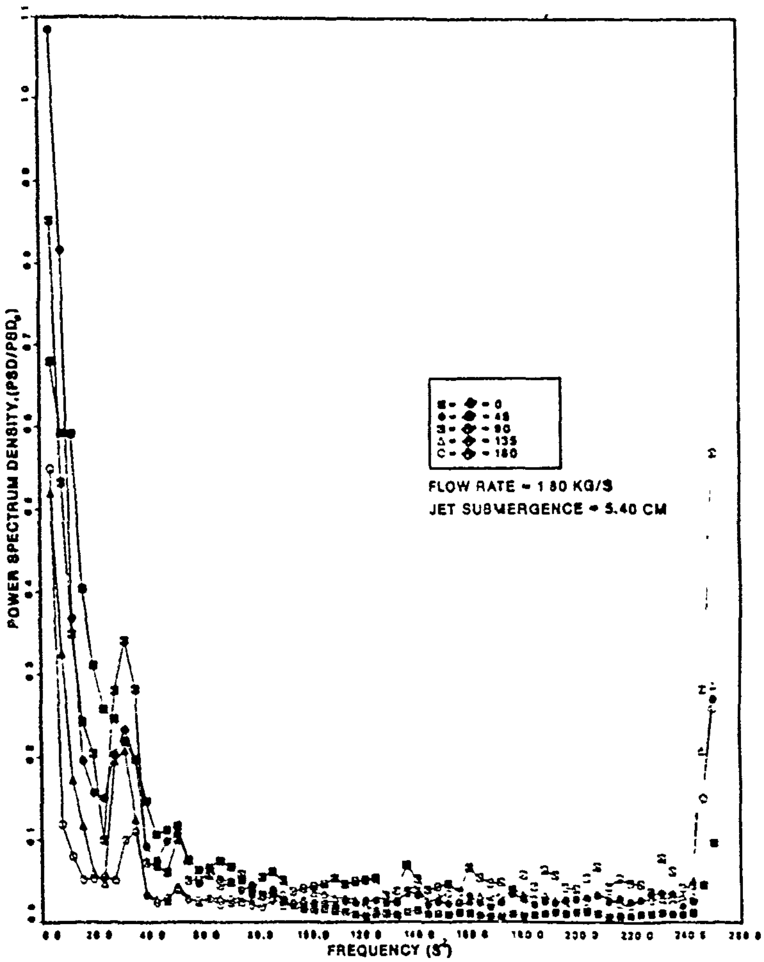


Figure 14. Instantaneous Pressure at the Jet Centerline

FIGURE 13. POWER SPECTRUM DENSITY

AS A FUNCTION OF FREQUENCY



APPENDICES

A. Copies of published papers

B. M.S. thesis written by Eric Menning

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ON THE INVARIANT FUNCTIONS OF THE TURBULENCE BISPECTRUM

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Abstract—It is established that the invariant functions of the turbulence bispectrum can themselves be represented in terms of basis functions which are independent of time. These basis functions are also scale-independent, and therefore identical for large eddies and small eddies alike. Dynamical implications of this result are discussed.

"*What we observe is not nature itself but nature exposed to our method of questioning*" [1].

1. INTRODUCTION

Several papers concerning the application of the bispectrum to the analysis of noise and turbulence have been presented to the community [2–13]. The accepted definition of the bispectrum is very simple; it is the double Fourier-transform of the triple-moment function, and for the homogeneous turbulence is typically written as

$$B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t) = \int \exp\{-\iota(\mathbf{k}_1 \cdot \mathbf{q} + \mathbf{k}_2 \cdot \mathbf{r})\} S_{ijl}(\mathbf{q}, \mathbf{r}, t)\, \mathrm{d}\mathbf{q}\, \mathrm{d}\mathbf{r}, \tag{1}$$

where

$$S_{ijl}(\mathbf{q}, \mathbf{r}, t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{q}, t) u_l(\mathbf{x} + \mathbf{r}, t)\rangle.$$

The bispectrum is the three-point generalization of the power-spectrum concept already fully developed for two-point moments, and is accordingly an energy-transfer spectrum. It provides information about the turbulent flow which is not obtainable through the customary energy-spectrum; in particular it supplies knowledge about the degree of nonlinearity. The bispectral tensor, $B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t)$, can be written in terms of the invariant functions formed from the wave-number vectors, $\mathbf{k}_1$ and $\mathbf{k}_2$, and for isotropic turbulence characteristically expresses as (cf. [14])

$$B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t) = \sum_{N=1}^{14} A_N F_{ijl}^{(N)}(\mathbf{k}_1, \mathbf{k}_2). \tag{2}$$

In equation (2) the $F_{ijl}^{(N)}(\mathbf{k}_1, \mathbf{k}_2)$ are necessarily real, but the $A_N(\cdots)$ may be complex depending on the nature of $B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t)$, cf. eqn. (1); also the functional form of each $F_{ijl}^{(N)}(\mathbf{k}_1, \mathbf{k}_2)$ is independent of the corresponding $A_N(\cdots)$, which are strictly scalar functions of the invariants, $\mathbf{k}_1 \cdot \mathbf{k}_1$, $\mathbf{k}_1 \cdot \mathbf{k}_2$, $\mathbf{k}_2 \cdot \mathbf{k}_2$, and the time t. In fact the functional form of each $F_{ijl}^{(N)}(\mathbf{k}_1, \mathbf{k}_2)$ is determined from purely invariant theoretic constraints while that of the $A_N(\cdots)$ is determined strictly from fluid dynamical deliberations. Proudman and Reid [15] have elegantly shown that indice symmetry conditions together with conservation of mass reduce the complete specification of $B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t)$ to two (2) independent functions, say $A_1 \sim A^{(1)}$ and $A_2 \sim A^{(2)}$.

The purpose of this note is to establish that for self-similar decay each of the $A^{(N)}$, $N = 1, 2$, can themselves be represented in terms of a finite number of basis functions which are not only scale-independent, but also independent of time; they are indeed independent of initial conditions, and the temporal decay does not at all affect them. This result suggests that during the attendant decay there are some aspects of energy-transfer which remain fully preserved (invariant), and are therefore in some sort of universal equilibrium. It is compatible with recent experimental findings that "turbulence" is indeed characterized by a remarkable degree of structure and order [16], although quantitative evidence of the herein intended structure remains yet to be reported in the literature. This "invariance within decay" is a paradox which rightfully belongs to the sub-structure of turbulence, but it can nonetheless be explained in traditional terms provided the proper formulism is posed. Invariance in any phenomenon is important because it fosters the search for those properties which are unique to the phenomenon itself, properties which are independent of the particular choice of reference system; it asserts that the substantive laws of a given field should be expressible in a manner which is completely independent of the explicit scales chosen to measure the field variables. For turbulence, which characteristically transfers and dissipates energy in a manner (seemingly) consistent with the scale-sizes present within the flow, this feature is fundamental, subtle, and new; it is strikingly "nonfluidic" in flavor. From a dimensional analysis viewpoint, the existence of such structure is not to be unexpected since energy-transfer is by nature inherent in turbulence decay, but length scales are for the most part subjective in the sense that the scientist is virtually free to choose for analysis any particular length scale of practical interest. Typical length scales whose growth during decay is commonly studied are the integral scale, Λ, defined as

$$\Lambda(t) = \int_0^\infty f(r, t)\,\mathrm{d}r,$$

$f(r, t) \sim$ longitudinal correlation function of turbulence, and the dissipation scale, λ, defined as $\lambda(t) = \{\{-\partial^2 f/\partial r^2\}_{r=0}\}^{-1/2}$, although other equally valid length scales could readily be conceived and likewise studied; examples of such are

$$L_1 = \Lambda^{-1}(t)\int_0^\infty r f(r, t)\,\mathrm{d}r$$

and

$$L_2 = \left\{\Lambda^{-1}(t)\int_0^\infty r^2 f(r, t)\,\mathrm{d}r\right\}^{1/2}.$$

Since a turbulence Reynolds number can be formed as

$$\mathrm{Re} \sim \begin{pmatrix}\text{Appropriate}\\ \text{velocity scale}\end{pmatrix}\begin{pmatrix}\text{Length}\\ \text{scale}\end{pmatrix}\Big/(\text{Viscosity}),$$

the proposed result infers that there is something about energy-transfer in a self-similar turbulent flow which is independent of Reynolds number, *regardless of how the flow is scaled.* The dimensional analysis approach to turbulence invariably assumes that a turbulent flow can be made independent of Reynolds number only if it is "properly" scaled. For the case of turbulence generated by, say a wire mesh, the present implies that there is then something about energy-transfer which is independent of grid size (gauge invariance), and therefore identical for all such generated flows. From an analytical dynamics viewpoint the presence of such structure is also not to be unexpected since the presence of sufficient symmetry in a given dynamical system almost invariably guarantees related constants of the system motion (conservation laws). The knowledge that certain functions of a particular system actually remain preserved during a considered motion can be of great help not only in simplifying the governing equations, but also in eventually leading to their solution. At the very least it can lead to new insights. It was indeed the attendant recognition that in isotropic decay there is inauspiciously no natural length scale imposed on turbulence which prompted Karman and Howarth [17] to initiate an investigation to determine if closure could be obtained solely from the assumption of self-similarity (self-preservance). The bispectrum, therefore, in addition to its use in studying such mechanisms as spectral-transfer and vorticity production, appears to also be a tool for analytical characterization (and study) of preserved structure during the decay of self-similar fluids. Because the ensuing has its genesis in topology-related research [18], it is accordingly intended only as a description, not as a physical theory, of turbulent energy-transfer.

2. THE CASE FOR SELF-SIMILARITY

Since the assumption of self-similarity is vital to the specific results of this communication, it is therefore necessary to first establish under what conditions such behavior would be expected to persist in, say, a grid-turbulence or even a jet-turbulence. Historically, the assumption of self-similarity during turbulence decay has been principally a mathematical device, employed solely to secure definite and explicit results. Nonetheless its existence and subsequent analysis can be equally justified from purely dynamical system grounds. From such a perspective, decaying turbulence is a time-varying (nonlinear) dynamical system. And even though time-varying systems can be regarded as "filters" which select eigenfunctions in the most pure sense rather than just simple sinusoids (i.e. they discriminate against the complex exponentials as the natural eigenfunctions for related spectral decompositions), the natural tendency in their analysis is to treat them as systems whose properties do not change "appreciably" with respect to time. The underlying motivation for this approach is that classical modal analysis is not explicitly restricted to systems which are time-invariant only; it is only necessary that the differential equation governing the system as well as the boundary conditions be separable into a function of time alone and in this case a function of non-dimensionalized length alone, i.e. that temporal changes in the system be in some sense "uniform".

For purposes of mean-value determination, the most fundamental mathematical tool available to the turbulence practitioner is the Reynolds time-average. In any real flow inhomogeneity is always present and therefore the best estimate to the mean quantities of a steady-state flow are provided by averages such as

$$Q_{uu}(x_1, x_2) = \langle u(x_1, t)u(x_2, t)\rangle \approx \frac{1}{T}\int_0^T u(x_1, t)u(x_2, t)\,\mathrm{d}t,$$

$$Q_{uuu}(x_1, x_2) = \langle u^2(x_1, t)u(x_2, t)\rangle \approx \frac{1}{T}\int_0^T u^2(x_1, t)u(x_2, t)\,\mathrm{d}t,$$

etc., where, in practice, measurements of flow quantities are most often taken along the centerline of the (spatial) decay. In studies of turbulence it is customary to introduce a velocity scale defined as $\sigma_u(x) = \{Q_{uu}(x, x)\}^{1/2}$. This scale permits $u(x, t)$ to be rewritten as $u(x, t) = \sigma(x)\xi(x, t)$, where $\xi(x, t)$ is a normalized turbulence velocity such that $R(x, x) = \langle \xi(x, t)\xi(x, t)\rangle \equiv 1$ for all x. Note that $\sigma(x)$, in contradistinction to a length scale, is natural to the turbulence itself in the sense that a particular turbulence will automatically define its own peculiar $\sigma(x)$. With this velocity scale in hand, $Q_{uu}(\cdots)$ and $Q_{uuu}(\cdots)$ immediately rewrite as $Q_{uu}(x_1, x_2) \rightarrow C(x, r) = \sigma(x - r/2)\sigma(x + r/2)R(x, r)$ and $Q_{uuu}(x_1, x_2) \rightarrow D(x, r) = \sigma^2(x - r/2)\sigma(x + r/2)S(x, r)$, where $x = (x_1 + x_2)/2$ and $r = x_2 - x_1$. The coordinates, x and r, can be considered as uncoupled *principal* or *natural* coordinates for the two physical coordinates, x_1 and x_2. Unlike modal analysis of dynamic systems, however, the principal coordinates here are not abstract but indeed always have well-defined physical meaning. Choosing a generic length scale, $\eta(x)$, it is further customary to write $R(\cdots) \sim R(r/\eta)$, meaning that the algebraic form of $R(\cdots)$ remains unchanged during the spatial decay, but the scaling along the abscissa (r-axis) has stretched according to the growth of $\eta(\cdots)$ with downstream distance from the grid (or jet). The physical condition necessary for this feature to manifest is that energy must cascade in such a manner that all length scales of the turbulence grow with x at approximately the same rate, and therefore knowledge of the growth of one length scale essentially provides knowledge of the growth of all others; accordingly, there is no accompanying loss of "information" in this process of energy transfer. By expanding $\sigma(x \pm r/2)$ in a Taylor's series about $r = 0$, the result is

$$\sigma(x \pm r/2) \approx \sigma(x) \pm \left.\frac{\partial \sigma}{\partial r}\right\}_{r=0} (r/2) + \cdots \approx \sigma(x)\{1 \pm r/2L + \cdots\}$$

where

$$L \sim L(x) = \left\{\sigma^{-1}(x)\left\{\frac{\partial \sigma}{\partial r}\right\}_{r=0}\right\}^{-1}$$

is a length scale over which $\sigma(\cdots)$ may be assumed to vary "appreciably". Since $R(\cdots)$ scales with $\eta(x)$, the net effect on $C(x, r)$ of the different length scales, $L(x)$ and $\eta(x)$, can be characterized by writing $C(x, r) \sim \sigma^2(x)\{1 - (r/2\eta)^2(\eta/L)^2\}R(x, r)$. If then $L \gg \eta$, $C(x, r) =$

$\sigma^2(x)R(r/\eta)$ for those values of r for which $r \leq \eta$. Since

$$\lim_{r\to\infty} \{R(x, r)\} = 0,$$

if L is prohibitively larger than η, $C(x, r) \simeq \sigma^2(x)R(r/\eta)$ is true for all values of r. Remote though these collective conditions are from anything which could readily be established in practice, the virtual impossibility of their manifestation constitutes no objection to their employment in a purely theoretical investigation. It is indeed one of the more interesting features of any physical theory that it invokes idealized behavior, yet the eventual results obtained from same reveal substantial knowledge about real phenomena. Note that self-similarity is always true for extremely small values of r. The formulation is ultimately aligned with isotropic theory by expressing downstream distance, x, as $x \simeq \bar{U}t$, where $\bar{U}$ is a (constant) characteristic mean-flow velocity (typically a centerline velocity).

3. FORMULATION

For simplicity consider the case where $\mathbf{k}_2 \simeq \alpha\mathbf{k}_1$; $\alpha \sim$ scalar. For this case the $A^{(N)}$ become $A^{(N)}(\mathbf{k}_1\cdot\mathbf{k}_1, \mathbf{k}_2\cdot\mathbf{k}_2, \mathbf{k}_1\cdot\mathbf{k}_2, t) \sim A^{(N)}(k_1, k_2, t)$, $N = 1, 2$, where $k_\mu = |\mathbf{k}_\mu|$, $\mu = 1, 2$. For self-similar decay the time-dependence of the $A^{(N)}$ is carried through appropriate scale-factors, viz. $\sigma \sim \sigma(t)$ and $\eta \sim \eta(t)$; $\sigma \sim$ velocity scale-factor defined through $\sigma^3(t) = \int B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t)\, d\mathbf{k}_1\, d\mathbf{k}_2$, while $\eta \sim$ subjectively defined length scale factor. Accordingly, $A^{(N)}(k_1, k_2, t) \simeq \sigma^3(t)\phi^{(N)}(\eta k_1, \eta k_2) \equiv \sigma^3(t)\phi^{(N)}(\kappa_1, \kappa_2)$, where $\kappa_1 = \eta k_1$ and $\kappa_2 = \eta k_2$ are time-dependent dimensionless coordinates. Proudman and Reid (loc. cit.) have established that $B_{ijl}(\cdots)$ must satisfy certain symmetries in the arguments, k_1 and k_2, viz. that $B_{ijl}(\mathbf{k}_1, \mathbf{k}_2, t) \equiv B_{ijl}(\mathbf{k}_2, \mathbf{k}_1, t) \equiv B_{jli}(\mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2, t)$. This symmetry is therefore also required of the scalar functions, $\phi^{(N)}(\kappa_1, \kappa_2)$; in particular, it must follow that $\phi^{(N)}(\kappa_1, \kappa_2) \equiv \phi^{(N)}(\kappa_2, \kappa_1) \equiv \phi^{(N)}(\kappa_2, -\kappa_1 - \kappa_2)$. If each $\phi^{(N)}(\cdots)$ is expanded in a Taylor's series about $\kappa_1 = \kappa_2 = 0$, the generic result is

$$\phi(\kappa_1, \kappa_2) \simeq \phi(0, 0) + \phi_{\kappa_1}(0, 0)\kappa_1 + \phi_{\kappa_2}(0, 0)\kappa_2 + \cdots,$$

where $\phi_{\kappa_1}(\cdots)$ and $\phi_{\kappa_2}(\cdots)$ denote respective appropriate partial derivatives. Suppose now that the Taylor's approximation is restricted to the zeroth-order case, viz. the case $\phi(\kappa_1, \kappa_2) \simeq \phi(0, 0) = \text{constant}$. It is obvious that $\phi(0, 0)$, because it is a constant, will automatically satisfy the required argument symmetry. Denote $\phi(0, 0)$ by $H_0(\kappa_1, \kappa_2)$. Suppose now that the Taylor's approximation is restricted to the first-order case, viz the case

$$\phi(\kappa_1, \kappa_2) \simeq H_0(\kappa_1, \kappa_2) + \phi_{\kappa_1}(0, 0)\kappa_1 + \phi_{\kappa_2}(0, 0)\kappa_2.$$

Since $H_0(\kappa_1, \kappa_2) \equiv \text{constant}$ already satisfies the required symmetry, imposition of the κ_1, κ_2-invertibility on $\phi(\kappa_1, \kappa_2)$ then requires that $\phi_{\kappa_1}(0, 0) \equiv \phi_{\kappa_2}(0, 0)$. However, imposition of the remaining symmetry, viz. $\phi(\kappa_1, \kappa_2) \equiv \phi(\kappa_2, -\kappa_1 - \kappa_2)$, demands that $\phi_{\kappa_1}(0, 0) \equiv 0$. Therefore, there is no linear homogeneous polynomial in κ_1, κ_2 that under any circumstances satisfies the entire required κ_1, κ_2-symmetry. Denote $\{\phi_{\kappa_1}(0, 0)\kappa_1 + \phi_{\kappa_2}(0, 0)\kappa_2\}$ by $H_1(\kappa_1, \kappa_2)$. Continuing to apply the same procedure to the second-order approximation, viz.

$$\phi(\kappa_1, \kappa_2) \simeq H_0(\kappa_1, \kappa_2) + H_1(\kappa_1, \kappa_2) + (1/2)\phi_{\kappa_1\kappa_1}(0, 0)\kappa_1^2 + \phi_{\kappa_1\kappa_2}(0, 0)\kappa_1\kappa_2 + (1/2)\phi_{\kappa_2\kappa_2}(0, 0)\kappa_2^2,$$

produces $H_2(\kappa_1, \kappa_2) = (\text{any constant})(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2)$. The procedure applied to the third-order approximation yields $H_3(\kappa_1, \kappa_2) = (\text{any constant})(\kappa_1^2\kappa_2 + \kappa_1\kappa_2^2)$. Strikingly, though, the procedure applied to the fourth-order approximation reveals that $H_4(\kappa_1, \kappa_2) = (\text{any constant})(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2)^2$, while the procedure applied to the fifth-order approximation supplies $H_5(\kappa_1, \kappa_2) = (\text{any constant})(\kappa_1^2 + \kappa_1\kappa_2 + \kappa_2^2)(\kappa_1^2\kappa_2 + \kappa_1\kappa_2^2)$, i.e. $H_4(\cdots)$ and $H_5(\cdots)$ can be constructed from $H_2(\cdots)$ and $H_3(\cdots)$. These collective results are not as surprising as they appear, since the existence of a finite set of polynomial basis functions for any sequence of systematically generated algebraic forms was established in an elegant theorem by David Hilbert in 1890 [19]. The essence of the theorem formulated by him is that if an infinite sequence of algebraic forms (in the case at hand the sequence is $H_0(\cdots)$, $H_1(\cdots)$, $H_2(\cdots), \ldots$) is constructed according to some general over-riding rule (the "rule" here is the dominating κ_1, κ_2-symmetry), then there comes a point in the construction where the set of available independent such forms is exhausted and each form constructed thereinafter is merely a repetition/modification of what has already preceded. What this means for the $\phi^{(N)}(\kappa_1, \kappa_2)$ is that $H_6(\kappa_1, \kappa_2)$ must be constructable from some combination of $H_2(\kappa_1, \kappa_2)$ and $H_3(\kappa_1, \kappa_2)$, e.g. $H_6(\cdots\cdot) \sim H_2^3(\cdots\cdot) + H_3^2(\cdots\cdot)$, while $H_7(\cdots\cdot)$ must be of the form, $H_7(\cdots\cdot) = (\text{some constant})\{H_1^2(\cdots\cdot)H_3(\cdots\cdot)\}$. No other fundamental form different from $H_2(\cdots\cdot)$ or $H_3(\cdots\cdot)$ may occur in the formulation.

4. DYNAMICAL IMPLICATIONS

Dynamical implications of the foregoing are best comprehended by examining the role that the bispectrum plays in the decay of total turbulent energy. Keeping in mind that inertia forces induce energy flow between *different* wavenumbers for the *same* velocity component, it is therefore accordingly necessary to analyze only the simple high Reynolds number case

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} \simeq 0, \qquad u \sim u(x, t).$$

The two-point correlation method duly yields

$$\frac{\partial \langle uu' \rangle}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r}\{\langle uu'^2 \rangle - \langle u^2 u' \rangle\} \simeq 0,$$

where $u' \sim u(x + r, t)$, and the bispectrum is thus introduced through

$$\langle uu'^2 \rangle = \int B(k_1, k_2, t) e^{i(k_1+k_2)r} \, dk_1 \, dk_2$$

and

$$\langle u^2 u' \rangle = \int B(k_1, k_2, t) e^{ik_2 r} \, dk_1 \, dk_2;$$

ultimately,

$$\frac{\partial \Phi(k, t)}{\partial t} \simeq W(k, t)$$

where $\Phi(k, t) = \int \langle uu' \rangle e^{-ikr} \, dr$ and $W(k, t) = (ik/2) \int \{B(k, k', t) - B(k - k', k', t)\} \, dk'$. A dynamical equation for the bispectrum (bispectral decay equation) evolves through

$$\frac{\partial u}{\partial t} \simeq \nu \frac{\partial^2 u}{\partial x^2};$$

in particular,

$$\frac{\partial B(k_1, k_2, t)}{\partial t} + 2\nu(k_1^2 + k_1 k_2 + k_2^2) B(k_1, k_2, t) = 0.$$

An immediate feature of this equation is the explicit presence in it of the algebraic form, $H_2(k_1, k_2) \sim (k_1^2 + k_1 k_2 + k_2^2)$; recall that the necessary presence of this form was earlier established on purely kinematical grounds yet the dynamics of the phenomenon establishes its presence independent of kinematics. The fact that $H_2(\cdots)$ emerges from two rather independent lines of thinking suggests that between $H_2(\cdots)$ and $H_3(\cdots)$, $H_2(\cdots)$ plays the stronger role in the transfer of energy from wave-number k_1 to wave-number k_2 (or vice versa). The solution to the bispectral decay equation is

$$B(k_1, k_2, t) = \beta(k_1, k_2) e^{-2\nu(k_1^2 + k_1 k_2 + k_2^2)t}$$

where $\beta(\cdots)$ is determined from initial conditions, and must be complex (i.e. have both real and imaginary parts). This solution illustrates an interesting but not unexpected result, viz. that (for "low" Reynolds number) the decrease of energy-transfer with time occurs at a higher rate for the eddies with the larger wave-numbers; i.e. energy-transfer for the smallest eddies decays due to viscous dissipation much faster than that for the largest ones. Note that for $k_1 = k_2 = 0$, $B(\cdots) \equiv$ constant. Since $\beta(\cdots)$ must also be expressible in terms of the formulated fundamental forms, the form suggested here is

$$\beta(k_1, k_2) = \frac{\beta_0 + \beta_1(k_1^2 k_2 + k_1 k_2^2)^2}{\beta_2 + \beta_3(k_1^2 + k_1 k_2 + k_2^2)}.$$

This form substituted into the bispectral decay equation provides the right k-dependence for $\Phi(k, t)$; keep in mind that β_N, $N = 0, 1, 2, 3$, may all be complex.

For purposes of the present analysis, however, it is best to consider $B(\cdots)$ as a function of the kinematic invariant, $\alpha = k_2/k_1 \equiv \kappa_2/\kappa_1$. Accordingly,

$$B(k_1, k_2, t) \rightarrow B(k, \alpha, t) = \frac{\beta_0 + \beta_1(\alpha + \alpha^2)^2 k^6}{\beta_2 + \beta_3(1 + \alpha + \alpha^2)k^2} e^{-2\nu(1+\alpha+\alpha^2)k^2 t}$$

The forms, $h_I(\alpha) = 1 + \alpha + \alpha^2$ and $h_{II}(\alpha) = \alpha + \alpha^2$, can be thought of as fully invariant (i.e. scale-independent, time-independent, and Reynolds number-independent) "modes" of energy transfer; note that there are only two such modes. In each mode the transfer resembles a simple one degree-of-freedom system (the modes being discrete and independent of one another), even though energy is transferred through a continuum of wave-number frequencies. Also note that for "small" α, $h_I(\alpha) \gg h_{II}(\alpha)$, while for "large" α, $h_I(\alpha) \equiv h_{II}(\alpha)$. This result is consistent with the widely held notion that the effect of a big eddy on a small eddy is composed of two parts, one a deformation and the other an entrainment, with the second part dominating when the eddies in question differ widely in size. The implications here are that $h_I(\alpha) \sim$ entrainment while

$$h_{II}(\alpha) \sim \begin{cases} \text{entrainment for "large" } \alpha, \\ \text{deformation for "small" } \alpha. \end{cases}$$

Furthermore, for "small enough" k, i.e. the larger scale-structure of the turbulence,

$$B(k, \alpha, t) \simeq \frac{\beta_0}{\beta_2 + \beta_3 h_I(\alpha) k^2} e^{-2\nu h_I(\alpha) k^2 t},$$

indicating that large-scale energy-transfer occurs in only one mode, viz. the entrainment mode. However, since entrainment between two eddies of different size can occur only when the larger eddy totally dominates the smaller, not only do the "big slow eddies" interact very weakly with the remainder of the turbulence (a phenomenon already well-known and understood), but the present establishes that they also interact very weakly (if at all) with each other, i.e. eddies of roughly the same size cannot entrain one another. This is perhaps what accounts for the fact that the turbulence spectrum, $\Phi(k, t)$, has a somewhat permanent form around $k \simeq 0$. Because the mode, $h_{II}(\alpha)$, does not participate in the large-scale transfer of energy, the dynamical system (i.e. the turbulent fluid), for this wavenumber domain behaves like a degenerate or semi-definite system; from this perspective $h_{II}(\alpha)$ is like an ignorable coordinate and the corresponding "constant of the motion" is the permanence of the large-scale turbulence structure. For the larger values of k, however,

$$B(k, \alpha, t) \simeq \frac{\beta_1 h_{II}^2(\alpha) k^4}{\beta_3 h_I(\alpha)} e^{-2\nu h_I(\alpha) k^2 t}$$

indicating that for the smaller scale-structure *both* modes, one mode (viz. $h_{II}(\alpha)$) determined from pure kinematics only, participate in the transfer. This condition persists even for the smallest scales. The different modal behavior for small and large scales reveals that even though kinematic similarity is assumed, formulated, and pursued, the system nonetheless refuses to behave dynamically similar. The presence of the mode, $h_I(\alpha)$, in the respective expressions for large- and small-scale energy-transfer alike supports the assertion made earlier that this mode plays the stronger role in the transfer of energy.

5. DEEPER IMPLICATIONS

Since self-similar turbulence must decay in a fashion which preserves the scale- and time-independent forms, $h_I(\alpha)$ and $h_{II}(\alpha) = h_I(\alpha) - 1$, and since the bispectrum for any such decay must be algebraically expressible in terms of $h_I(\alpha)$ and $h_{II}(\alpha)$, then all universal ramifications of same are embodied in the single absolutely invariant function, $h_I(\alpha)$, and eventually in the invariant, α. The physical implication of this result is best comprehended by recalling a simple experiment in physics which culminates in a precise operational definition of the *mass* of a body. The experiment involves placing a spring between two isolated masses, A and B, pushing them together so that the spring is compressed (the spring is not rigidly attached to either of the masses), and then releasing them. By repeating the experiment a number of times, each time with a spring of different constant, it will be found that although the recoil velocities of A and B will be different each time, the ratio, v_A/v_B, will be identical for all cases, i.e. $v_A/v_B \equiv$ invariant, regardless of the amount of initial energy in the system. The fact that this result provides a means for unambiguously defining the mass of a body, viz. $m_B/m_A \equiv v_A/v_B$, and that it is consistent with the principle of linear momentum conservation, is well established. Put simply, the invariance of v_A/v_B means that energy cannot be transferred from the potential energy of the spring to the kinetic energy of the masses in an arbitrary fashion but must indeed be transferred in a manner consistent with linear momentum conservation. In thermodynamics, a correspondingly simple experiment will establish that in a Carnot cycle the heat, Q_H, flowing in at a high temperature, T_H, is related to the heat, Q_L, flowing out at a lower temperature, T_L, by the relation

$$\frac{Q_H}{Q_L} = \frac{T_H}{T_L} \equiv \text{Invariant;}$$

the invariance of this ratio is a consequence of the principle of conservation of entropy, i.e., heat is not transferred arbitrarily through the engine. In geometrical optics the fact that the sines of the angles of incidence and refraction stand in a constant ratio to each other, regardless of the magnitude of each, leads to the formulation of *Snell's Law*. In the study of relativity, Mermin [20] has invoked the invariance of the ratio of collinear lengths for both the fixed and moving observers to cleverly derive the Lorentz transformation equations *without* making use of the principle of the constancy of the velocity of light. It appears, therefore, that the invariance of $\alpha = k_2/k_1$, as herein formulated, is a manifestation of some conservation law or principle which governs the transfer of energy in a decaying turbulence. Such a conservation law must describe, for self-similarity, the means of (scale-independent) "communication" between turbulence events of different scale-sizes such that in the limit, events distinguished by a great disparity in scale-size should exert minimal influence on one another. Moreover, such a principle is consistent with the traditional wisdom that turbulence does not transfer energy arbitrarily but indeed does so in some well-prescribed fashion. If we think of a "parcel" of turbulence of a particular scale-size as a dynamical system which receives energy from those parcels of turbulence of larger scale-sizes, dissipates some of this energy, and transfers the rest to those parcels of smaller scale-size, then a possible such law is

$$\frac{E_T(k_1)}{E_R(k_1, k_2)} \simeq \left(\frac{k_2}{k_1}\right)^{\gamma} = \left(\frac{\kappa_2}{\kappa_1}\right)^{\gamma};$$

in this expression $E_T(k_1)$ is the total energy transferred by a turbulence parcel of scale-size, k_1^{-1}, to all other turbulence parcels of smaller scale-size, $E_R(k_1, k_2)$ is that portion of $E_T(k_1)$ which is eventually *received* by a pracel of scale-size, k_2^{-1}, γ is a non-adjustable constant greater than zero, and $k_2 > k_1$ (see Fig. 1). The t-dependence of both $E_T(\cdots)$ and $E_R(\cdots)$ is explicitly deleted in the above expression in order to emphasize the time-independence of the ratio, $E_T(\cdots)/E_R(\cdots)$. The ultimate consequences of such a law (or others like it) need to be fully investigated.

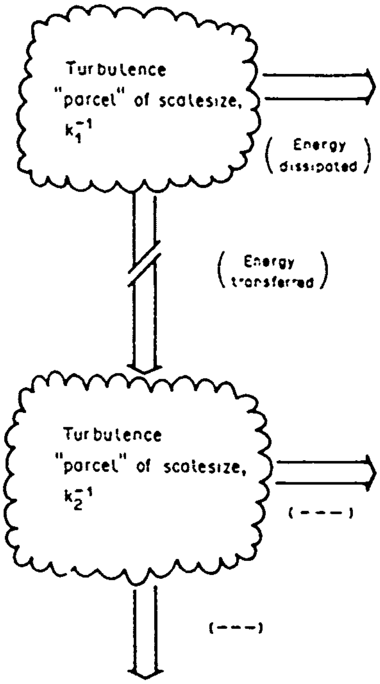

Fig. 1. Dynamical system schematic of a turbulence parcel of scale-size, k_1^{-1}, dissipating energy and transferring energy to turbulence parcels of smaller scale-sizes.

6. CONCLUDING REMARKS

In closing it remains to affirm that regardless of the nature of the decay—whether self-similar or non-self-similar, isotropic or non-isotropic—some aspect(s) of the turbulence will always be preserved. This follows from the nature of scaling effects (transformations) in general that regardless of how a geometric figure may be deformed, something about it nevertheless remains intact and unchanged during the deformation process [21]. The geometric figure in question here is the shape of the longitudinal correlation function $f(r, t)$, of turbulence, and how this function evolves in time determines where the first evidence of preserved structure can be found. For self-similar decay, viz. $f(r, t) \sim f(r/\eta)$, where $\eta \sim \eta(t)$ only, invariant structure, as herein formulated, is found in the energy-transfer mechanism. This invariance accordingly admits a representation of this mechanism in terms of a finite number of transfer modes which are independent of the magnitudes of the coordinates originally chosen to describe the motion. In each mode the transfer resembles a one-degree-of-freedom system (with no interplay between the modes), even though energy is transferred and dissipated in the entire system through a continuum of wavenumber frequencies. For a more general strain of decay, say where possibly $\eta \sim \eta(t, r)$, preserved structure will necessarily be found only in the higher-order moments [18]. This means that independent of adventitious disturbances, preserved structure of some sort will exist in any and all types of decay; furthermore, it can be definitely concluded that the degree of particularly endowed structure is not affected by the "age" of the flow.

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ISOTROPIC ANALYSIS OF GRID-TURBULENCE

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Abstract—The principles of self-similar isotropic turbulence are applied to the analysis of grid-decay In particular it is theoretically established that energy-transfer occurs in only two distinct invariant modes. The decay constant of third-order correlations is posed in terms of the "amounts" of each mode present in the decay. This constant is determined empirically, and is found to be relatively insensitive to changes in Re The related decay of turbulence intensity is computed.

1. INTRODUCTION

The theory of isotropic turbulence centers upon solving the dynamical equation,

$$\frac{\partial(\sigma^2 f)}{\partial t} = \sigma^3\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)k + 2\nu\sigma^2\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)f, \tag{1}$$

to explicitly obtain the time dependence of a peculiar length scale, $\eta(t)$, in terms of the time-decay of the velocity scale, $\sigma(t)$. The velocity scale is a parameter which is "natural" to the turbulence itself (typically a turbulence rms value) while $\eta(t)$, on the other hand, is a "subjective" parameter in the sense that the analyst is virtually free to choose for analysis any length scale of particular interest. Typical length scales whose growth during decay is studied in the literature are the *dissipation scale*, defined as

$$\lambda(t) = \left[-\frac{\partial^2 f}{\partial r^2}\right]^{-1/2}_{r=0},$$

or the *integral scale*, defined as

$$\Lambda(t) = \int_0^\infty f(r, t)\,\mathrm{d}r.$$

The only natural length scale of turbulence is characteristically that imposed by the mechanism which generates the turbulence, say for example, the mesh size in a grid-turbulence or the jet diameter in a jet-turbulence, but this scale is (almost) always constant and essentially inappropriate for decay analyses. In a practical scenario, $\sigma(\cdots)$ is invariably approximated from steady-state turbulence measurements as

$$\sigma^2(x) \approx \frac{1}{T}\int_0^T u^2(x, t)\,\mathrm{d}t,$$

where x is the downstream distance from the turbulence-generating mechanism; further, $x = \bar{U}t$, $\bar{U}$ = convective mean-flow velocity. Similarly, $f(\cdots)$ and $k(\cdots)$ are approximated as

$$\sigma^2(x)f(r, x) \approx \frac{1}{T}\int_0^T u(x, t)u(x + r, t)\,\mathrm{d}t$$

and

$$\sigma^3(x)k(r, x) \approx \frac{1}{T}\int_0^T u^2(x, t)u(x + r, t)\,\mathrm{d}t;$$

in all these approximations, u is the x-component of the three-dimensional turbulence velocity, $\mathbf{u} = (u, v, w)$, T is a long averaging time, and all measurements are typically taken along the flow centerline where because of the attendant zero mean-flow gradients the isotropic theory is most applicable.

In order to solve equation (1) from a purely theoretical perspective, some rather *ad hoc* but nonetheless plausible assumption needs to be made about the function, $k \equiv k(r, x) \to k(r, t)$. An especially useful one which has on occasion been invoked [1] is that of self-similarity in both $f(r, t)$ and $k(r, t)$, historically, this assumption requires that $f(\cdots)$ and $k(\cdots)$ both scale in the same way with a length scale which is a function of time alone (and completely independent of r), but it will be shown momentarily that such a requirement is strictly unnecessary and $f(\cdots)$ and $k(\cdots)$ may in fact each scale with totally different length scales without this condition affecting the fluid decay. Evidence that $f(\cdots)$ and $k(\cdots)$, in grid-turbulence, do indeed scale in different ways has already been provided by Stewart [2] when he plotted values of $h(\cdots)$ vs. r/M, M ~ mesh size; Stewart's graphic description of the large-scale behavior of $h(r/M)$ is that it "... falls remarkably slowly toward zero, and does not disappear until well beyond the position at which the double correlation becomes undetectable." Since $k(\cdots) \equiv -2h(\cdots)$ the same description is therefore applicable to $k(\cdots)$. The decay equation which emerges from the self-similarity assumption is

$$\frac{d(\sigma^2 C_\eta \eta)}{dt} = \sigma^3 C_k + 8C_f \nu \sigma^2 \eta^{-1}, \tag{2}$$

where C_η is a constant which is unity if $\eta \equiv \Lambda$ and greater than unity if $\eta \equiv \lambda$. C_f is a constant defined as

$$C_f = \int_0^\infty \theta^{-1} \frac{df}{d\theta} d\theta, \qquad \theta = r/\eta,$$

and

$$C_k = 4 \int_0^\infty r^{-1} k(r, t)\, dr.$$

Note that for self-similarity, $k(r, t) \sim k(r/L)$, L ~ any time-dependent only length scale, and accordingly

$$\int_0^\infty r^{-1} k(r/L)\, dr \equiv \int_0^\infty (r/L)^{-1} k(r/L)\, d(r/L). \tag{3}$$

Also note that

$$C_\eta \equiv \int_0^\infty f(\theta)\, d\theta$$

is not independent of C_f. The subsequent decay law is found by assuming $\eta(t) \sim t^c$ and $\sigma(t) \sim t^d$, from which it follows that for self-similarity the only possible values of c and d are $c = 0.5 = -d$. This same result was originally established by Dryden [3]. A peculiar consequence of this result is that the turbulence Re, defined as $Re = \sigma\eta/\nu$, is constant throughout the decay.

On occasion, equation (1) is addressed in its inviscid (infinite Re) form, viz.

$$\frac{\partial(\sigma^2 f)}{\partial t} = \sigma^3 \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) k, \tag{4}$$

yielding for the self-similar case the family of solutions, $c = 1 + d$, $Re \sim O(t^{1+2d})$. However, such a family of solutions should be viewed with caution, since it is inconsistent with the fundamental precept of turbulence that mass conservation together with momentum conservation duly provides four equations and four unknowns, thus rigorously requiring one solution only to the decay problem. Furthermore, since for "small" r, $k(\cdots) \sim O(r^3)$, any solution to the inviscid equation cannot be valid for the small-scale structure of the turbulence (and therefore not a viable solution at all); recall that for small r, $f(r, t) \simeq 1 - (r/\eta)^2$.

Because the theory of turbulence strictly prohibits the presence in the analysis of any adjustable or "tuneable" constants, a complete solution of the dynamical equation is therefore not achieved until a value for C_k is determined, or at least a viable method for so determining it explicitly formulated. In other words, since the only flow-related number unique to turbulence is Re, any other flow-related number(s) present in a given turbulence analysis necessarily has(have) to ultimately depend through some well-prescribed method on Re. The fomulation of such a method and its implications for grid-turbulence are presented in the following. The fact that C_k is formed from an integral of $k(\cdots)$ fortunately suggests that aside from some fundamental limiting constraints, viz.

$$\lim_{r\to 0}\{k(\cdots)\}\sim O(r^3)$$

and

$$\lim_{r\to\infty}\{k(\cdots)\}=0,$$

the exact functional form of $k(\cdots)$, although important is not devastating. Since,

$$C_k\sim\int_0^\infty r^{-1}k(\cdots)\,\mathrm{d}r,$$

$k(\cdots)$ also does not have to be integrable over all positive values of r; i.e. because $k(\cdots)$ is an *odd* function,

$$\int_{-\infty}^{\infty}k(\cdots)\,\mathrm{d}r\equiv 0$$

while

$$\int_0^\infty k(\cdots)\,\mathrm{d}r$$

itself may be undefined. Furthermore, because of the role it plays in decay,

$$\int_0^\infty r^{-1}k(\cdots)\,\mathrm{d}r$$

must always be less than zero.

2. FORMULATION

The transfer of energy in a system obeying nonlinear equations is a dynamical process explicitly dependent upon the equations. Bounds on the transfer, however, can be imposed from pure kinematics only, although not much more can be said about the functional form of the transfer without employing dynamical arguments. The first step in the formulation is therefore to invoke kinematics to establish said bounds and accordingly develop a representative algebraic form for $k(r,t)\sim k(r/\eta)$; recall that the exact algebraic form of $k(\cdots)$ is not central to the problem. To fundamentally accomplish this while simultaneously saying as little as possible about the transfer mechanism itself, it is here proposed to kinematically describe the three-point correlation,

$$Q(q,r,x)=\frac{1}{T}\int_0^T u(x,t)u(x+q,t)u(x+r,t)\,\mathrm{d}t,$$

where again $u(x,t)$ is only one component of the steady-state three-dimensional turbulence velocity field and x a one-dimensional position vector. In a strict nonhomogeneous turbulence, such as that generated by (say) a grid, the correlation, $Q(q,r,x)$, accordingly scales as

$$Q(q,r,x)=\sigma(x)\sigma(x+q)\sigma(x+r)s(q/\eta,r/\eta).$$

However, the self-similar theory assumes that the length scale, l, which scales $\sigma(\cdots)$ is "large" in comparison to η; large enough that

$$\sigma(x)\sigma(x+q)\sigma(x+r)s(q/\eta,r/\eta)=\sigma(x)\sigma\left\{x\left[1+\left(\frac{q/\eta}{x/l}\right)\left(\frac{\eta}{l}\right)\right]\right\}\sigma\left\{x\left[1+\left(\frac{r/\eta}{x/l}\right)\left(\frac{\eta}{l}\right)\right]\right\}$$
$$\times s(q/\eta,r/\eta)\simeq\sigma^3(x)s(q/\eta,r/\eta).$$

Through its dependence on q and r, $Q(\cdots)$ analytically described how a turbulence eddy of scale-size, q, "interacts with" (transfers energy to) a turbulence eddy of scale-size, r, or vice-versa; i.e. $Q(\cdots)$ quantities how turbulence eddies of differing size "communicate" with one another and thus measures the "strength" of the nonlinearity. For this reason, $Q(\cdots)$ must approach zero as the difference between q and r grows without bound.

A simple property of $Q(q, r, x)$ is (cf. Ref. [4]) $Q(q, r, x) \equiv Q(r, q, x) \equiv Q(-q, r-q, x)$; this q, r-argument symmetry must therefore also be satisfied by $s(\cdots)$. If $s(\cdots)$ is then expanded in · Taylor's series in q and r, the formal result is

$$s(q/\eta, r/\eta) \simeq s(0,0) + a_1 q + b_1 r + a_2 q^2 + c_2 qr + b_2 r^2 + \cdots,$$

where the a_n, b_n, c_n $n = 1, 2, 3, \ldots,$ represent the appropriately evaluated partial derivatives of $s(\cdots)$. Following the procedure outlined in Refs [5] and [6], it can be readily shown that due to the required argument symmetry in $s(\cdots)$, aside from $s(0,0)$ which automatically satisfies the required symmetry, the only algebraic forms which can appear in the expansion are the homogeneous polynomials,

$$F_1(q, r) = q^2 - qr + r^2 \quad \text{and} \quad F_2(q, r) = 2q^3 - 3q^2 r - 3qr^2 + 2r^3,$$

and those higher-order homogeneous polynomials which can be formed from $F_1(\cdots)$ and $F_2(\cdots)$, e.g. $F_1^2(\cdots)$, $F_1(\cdots)F_2(\cdots)$, $F_2^2(\cdots)$, etc. No other fundamental algebraic form different from $F_1(\cdots)$ and $F_2(\cdots)$ satisfies the required symmetry and can therefore not appear in the expansion of $s(\cdots)$; specifically, the (q, r)-dependence of $s(q/\eta, r/\eta)$ can be fully characterized by $F_1(\cdots)$ and $F_2(\cdots)$, i.e. as far as q and r only are concerned, $s(\cdots) \sim s\{F_1(q, r), F_2(q, r)\}$.

The physical significance of $F_1(\cdots)$ and $F_2(\cdots)$ is determined by writing $F_1(q, r) = (\alpha^2 - \alpha + 1)r^2 = h_1(\alpha)r^2$ and $F_2(q, r) = 2(\alpha^3 - 1.5\alpha^2 - 1.5\alpha + 1)r^3 = 2h_2(\alpha)r^3$, where $\alpha = q/r$ is a scale-independent parameter which is an invariant of the decay, i.e. during the decay $\alpha = q/r \to (q/\eta)/(r/\eta) \equiv \alpha$, or in words, as the big eddies are "scaled down" into little eddies in a unilaterally self-similar fashion, the ratio of two eddies remains preserved and unaffected. Consequently, the forms $h_1(\alpha)$ and $h_2(\alpha)$ are also invariants of the decay, and upon further reflection are seen to be identical for both homogeneous turbulence and nonhomogeneous turbulence alike. It is instructive to momentarily divert here to point out that in a 1981 communication, Deissler [7] concluded that even for a general nonhomogeneous turbulence, certain terms in the two-point stochastic-averaged Navier–Stokes equation can be interpreted as "transfer" terms. He accomplished this by introducing a *third point*, defined as $x_n = x + n(x' - x)$, $0 \leq n \leq 1$, into the two-point geometry of the flow and cleverly deducing that the appropriately formulated "turbulence self-interaction" terms produce zero total contribution to the decay of $\langle uu' \rangle$; $u \sim u(x, t)$, $u' \sim u(x', t)$, $x' = x + r$. The impact of his result (duly pointed out by him) is that turbulence self-interaction plays the same role in nonhomogeneous turbulence that it does in homogeneous turbulence, viz. to transfer energy from one part of wave-number space to another without changing the total amount. Furthermore he makes perfectly clear that the interpretation of self-interaction as energy-transfer is correspondingly conceptually independent of the condition of homogeneity; recall that the condition of homogeneity is typically invoked in making the "transfer" interpretation. More fundamentally, though, Deissler's result suggests that the two types of turbulence, through the energy-transfer mechanism, are somehow "unified". In the present, the forms $h_1(\alpha)$ and $h_2(\alpha)$, because of their insensitivity to the condition of homogeneity/nonhomogeneity, are collectively one manifestation of this unity. Because the entire transfer mechanism can be characterized in terms of these time-independent, scale-independent forms, these forms then can be thought of as fully invariant dynamical "modes" of energy-transfer. Note that there are only *two* such modes, and that the "two-mode" behavior is a direct consequence of argument symmetry only in $s(\cdots)$, while the formulated scale-independence is a consequence of the imposed self-similarity. Accordingly, it is not at all unreasonable to envision the herein considered transfer-modes analogous to the modes of classical heat-transfer, viz. *conduction*, *convection*, and *radiation*, save that here there are two distinct transfer modes only in contradistinction to the three of heat-transfer. The two-mode behavior is consistent with the already known [8] phenomenon that the effect of a large eddy on a smaller eddy is composed of two parts—one a *deformation*, the other an *entrainment*. As in classical modal analysis, $h_1(\alpha)$ and $h_2(\alpha)$ are

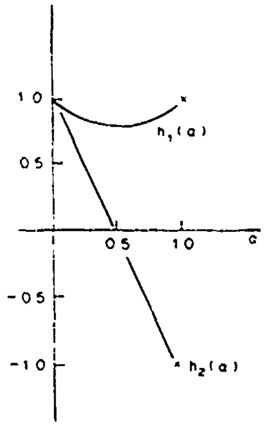

Fig. 1. Schematic of invariant energy-transfer modes

intended to remain intact and unaltered during the decay and an orthogonality condition between them is found by setting $\beta = \alpha - 0.5$, whereupon $h_1(\alpha) \to f_1(\beta) = \beta^2 + 0.75$, $h_2(\alpha) \to f_2(\beta) = \beta^3 - 2.25\beta$, and

$$\int_{-\theta}^{+\theta} f_1(\beta) f_2(\beta)\, d\beta \equiv 0 \quad \text{for any } \theta.$$

Plots of $h_1(\alpha)$ and $h_2(\alpha)$ for $0 \le \alpha (=q/r) \le 1$ are shown in Fig. 1. From purely dimensional arguments, then it follows that $s\{\cdots\} \equiv s\{B(t)h_1(\alpha)r^2, 2A(t)h_2(\alpha)r^3\}$ where $A(t) = a^3/\eta^3$, $B(t) = b^2/\eta^2$, $(a, b) \sim$ flow constants which depend only on Re, and thus on initial conditions. and the explicit algebraic expression for $s\{\cdots\}$ varies from flow to flow, i.e. $s\{\cdots\}$ will have a given form for grid-turbulence, another form for jet-turbulence, etc.

It is a rather simple analysis to deduce that the flow constants, a and b, define explicitly "how much" of each transfer-mode is present in the decay. Since for self-similarity Re = constant, it follows that a = constant = $F_a(\mathrm{Re})$, b = constant = $F_b(\mathrm{Re})$, where $F_a(\cdots)$ and $F_b(\cdots)$ have to be somehow postulated and their collective peculiar consequences compared with available data. The $F_a(\cdots)$ and $F_b(\cdots)$ proposed here are such that

$$b^2 - 2a^3 = \mathrm{Re}^1$$

and

$$b^2 + 2a^3 = \mathrm{Re}^2,$$

viz.

$$a \simeq 0.63(\mathrm{Re}^2 - \mathrm{Re}^1)^{0.33}$$

and

$$b \simeq 0.71(\mathrm{Re}^2 + \mathrm{Re}^1)^{0.5}.$$

3 APPLICATION TO GRID-TURBULENCE

The formulated theory was formally applied to the grid-turbulence results of Stewart (*loc. cit.*). First, a test for self-similarity in his data was performed by determining whether the two-point transfer function for all three of his cases (1. Re = 5300; 2. Re = 21,200; 3. Re = 42,400; $\mathrm{Re} = \bar{U}M/\nu$) could collectively be reasonably approximated by a single function of r^3 and r^2 (or combinations of r^3 and r^2) only, i.e. no terms in "r" or r to a fractional power; recall that for the two-point case, $q \equiv 0$. The result of that test is

$$k(\cdots) \simeq -\frac{2\beta(r/M)^3}{[1 + \gamma(r/M)^2]^2},$$

where the particular values of (β, γ) for each of his Re are specified in Figs 2–4. Thus, for theoretical analyses of grid-decay, a viable approximation of two-point third-order correlations

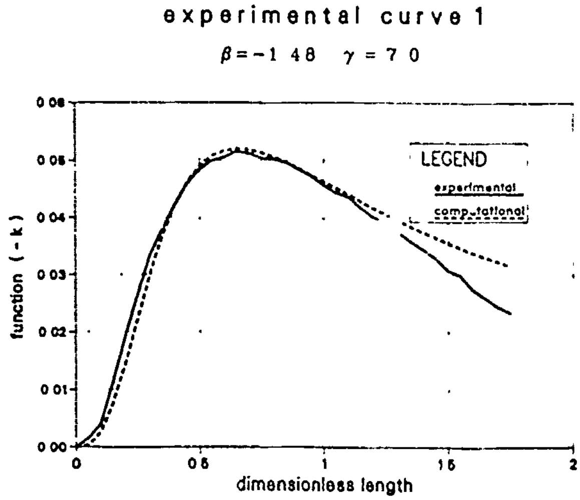

Fig 2. Triple-velocity curve, $k(\cdot\cdot\cdot)$, obtained by Stewart, broken line is theoretical curve (Re = 5300).

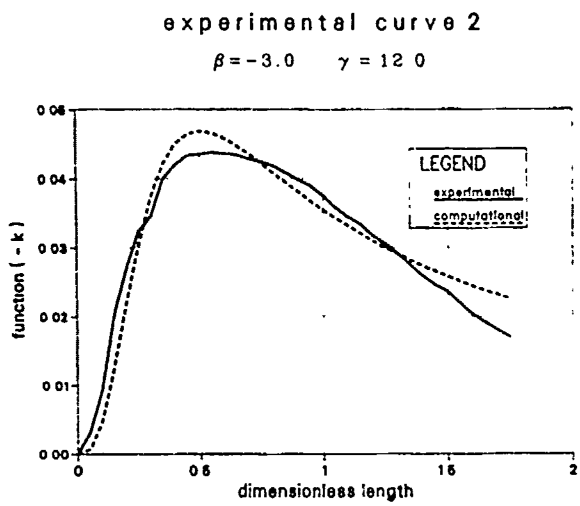

Fig. 3. Triple-velocity curve, $k(\cdot\cdot\cdot)$, obtained by Stewart; broken line is theoretical curve (Re = 21,200)

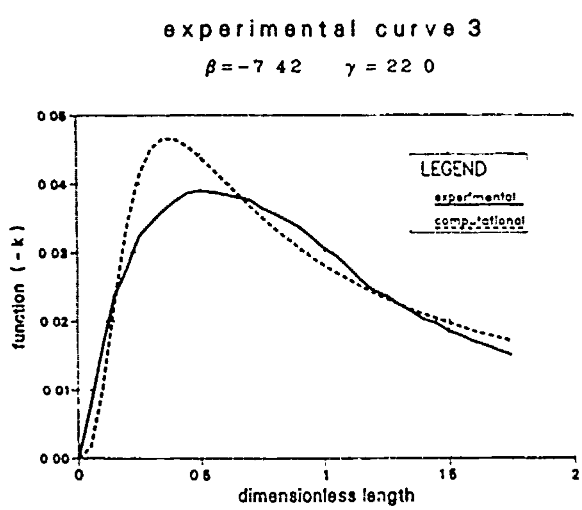

Fig. 4. Triple-velocity curve, $k(\cdot\cdot\cdot)$, obtained by Stewart, broken line is theoretical curve (Re = 42,400).

is

$$k(r,t) \equiv \frac{-2a^3(r/\eta)^3}{[1+b^2(r/\eta)^2]^2},$$

where (a, b) can be quickly determined from the Stewart data by visually approximating the value of dimensionless length, (r/η) in the theory and (r/M) in the data, at which $k(r/\eta)$ reaches its maximum value, viz. at $(r/\eta) = \sqrt{3}/b$, and similarly the corresponding maximum value of $k(r/\eta)$, viz. $k_{max} = k(\sqrt{3}/b) \approx 0.65(a/b)^3$. For the three available cases these values are accordingly

$$a \approx \begin{Bmatrix} 1.06 \\ 1.18 \\ 1.35 \end{Bmatrix}, \qquad b \approx \begin{Bmatrix} 2.47 \\ 2.88 \\ 3.46 \end{Bmatrix},$$

providing the values of

$$(a/b) \approx \begin{cases} 0.43, & \mathrm{Re} = 5300 \\ 0.41, & \mathrm{Re} = 21{,}200. \\ 0.39, & \mathrm{Re} = 42{,}400 \end{cases}$$

Note that although b, through the relation, $k(\sqrt{3}/b) \equiv k_{max}$, explicitly defines that point along the abscissa where $k(\cdots)$ peaks, through the "normalized" expression,

$$K(br/\eta) = \frac{1}{2}(b/a)^3 k(br/\eta) = -\frac{(r/\eta_1)^3}{[1+(r/\eta_1)^2]^2}, \qquad \eta_1 = \eta/b,$$

b also defines how the length scale, η, which is inherently peculiar to $f(\cdots)$ only, is itself "scaled" by the Re-dependent b to produce the length scale, η_1, peculiar to $k(\cdots)$. Also note that the series expansion,

$$K(br/\eta) \approx -(br/\eta)^3[1-(br/\eta)^2]^2 + \cdots$$

is not bilinear in $f(r/\eta)$ and is in fact quadratic in $f(r/\eta)$ only for $b \approx 0.7$ which as suggested by the above data is true only for extremely low Re. Bilinearity in $f(\cdots)$ of the leading term in an infinite series expansion of $k(\cdots)$ has been (rather authoritatively) claimed to the author by Kraichnan [9]. Note further the virtual Re-independence of (a/b), a feature already suggested by the scale-independence of C_k (cf. equation (3)); this feature and its implications will be addressed later. The appropriate values of y and z in the postulated $F_a(\cdots)$ and $F_b(\cdots)$ are then $y \approx 0.17$ and $z \approx 0.25$, so that for *any* Re (for grid-turbulence only) $a = F_a(\mathrm{Re}) \approx 0.63(\mathrm{Re}^{0.25} - \mathrm{Re}^{0.17})^{0.33}$ and $b = F_b(\mathrm{Re}) \approx 0.71(\mathrm{Re}^{0.25} + \mathrm{Re}^{0.17})^{0.5}$. Actually, the specific values of y and z for each case are:

$$\begin{array}{lll} \mathrm{Re} = 5300; & y \approx 0.17, & z \approx 0.25 \\ \mathrm{Re} = 21{,}200; & y \approx 0.17, & z \approx 0.25 \\ \mathrm{Re} = 42{,}000; & y \approx 0.18, & z \approx 0.27; \end{array}$$

these values suggest that $z \approx 1.5y$. Values of the flow constants as defined by $F_a(\mathrm{Re})$ and $F_b(\mathrm{Re})$, as well as the ratio of a over b, for representative values of Re are tabulated in Table 1. The therein specified (a/b) are "counter-intuitive" in the sense that knowing a priori there are two modes manifesting in the flow, the natural inclination is to suspect some sort of "equi-partition" among them to the extent that ideally $a \approx b$ or equivalently $(a/b) \approx 1$.

Table 1

Re	10^4	10^5	10^6	10^7	10^8
a	1.38	1.72	2.14	2.64	3.25
b	3.54	4.61	6.01	7.87	10.33
a/b	0.39	0.37	0.36	0.34	0.31

Assuming through that a characteristic value of (a/b) is roughly ≈ 0.4, the corresponding value of C_k is then

$$C_k = -8(a/b)^3 \int_0^\infty \frac{\omega^2}{(1+\omega^2)^2} \mathrm{d}\omega = -2\pi(a/b)^3 \simeq -0.13\pi.$$

4. CONCLUDING REMARKS

In concluding, it remains to examine some of the fundamental implications of the formulated theory. First, the complete solution of the r-integrated Karman–Howarth equation, equation (2), for grid-turbulence is now possible. With the empirically determined value of $C_k \approx -0.41$ in hand, the solution for $\sigma(t) = \gamma_1 t^{-0.5}$ and $\eta(t) \equiv \lambda(t) = \gamma_2 t^{0.5}$ is determined through $(\gamma_2/\gamma_1)C_\eta = -2C_k - 16C_f \,\mathrm{Re}^{-1}$, $\mathrm{Re} = \sigma\eta/\nu$. Acceptable values of C_η and C_f are found by assuming $f(r/\eta) \approx \exp\{-r^2/2\eta^2\}$, yielding $C_\eta = -C_f = \sqrt{\pi/2} \approx 5/4$; accordingly,

$$\gamma_1 \approx \left(\frac{5\,\mathrm{Re}}{\pi\,\mathrm{Re} + 80}\right)\gamma_2,$$

where $\gamma_2 = (10\nu)^{0.5}$. As to the constant, C_k, and its virtual insensitivity to changes in Re, the physical explanation of this behavior appears to be that even though the "scales" of the largest turbulence eddies initially created by a turbulence-generating source are roughly of the same order to magnitude of the physical dimensions† of said source (sort of like a "spatial-frequency" resonance), the particular mechanism through which these eddies are "scaled down" and directed to the sink provided by viscous dissipation is defined not by the source, sink, or fluid, but rather by the mechanisms of turbulence itself; i.e., the turbulent fluid, to a large extent, exhibits what could be described as "energy-cascade autonomy" or "self-governance". Keep in mind that the fluid has a strong inherent tendency to return to the state in which it existed before the turbulence was created, and that therefore the artificially produced eddies, now in a state of excitement, are eager to "unload" (on the smaller eddies) whatever portion of their energy they themselves cannot dissipate; this "unloading" process apparently depends only on turbulence mechanisms. More specifically, when an eddy of scale-size r transfers some of its total energy, either by entrainment or deformation, to a variety of eddies of smaller scale-size q, dissipates the remainder of its energy (itself eventually disappearing unless, of course, it is fed energy from those eddies of larger scale-size), the "transfer" part of such behavior, and its effectiveness, is governed by turbulence mechanisms only, and like a Carnot engine in thermodynamics, is independent of the "working fluid".

Second, the empirically derived expression for $k(r/\eta)$ suggests that

$$s(q/\eta, r/\eta) \approx \frac{-2a^3 h_2(\alpha)(r/\eta)^3}{[1 + b^2 h_1(\alpha)(r/\eta)^2]^2}.$$

Note that for small enough r, $s(\cdots) \approx -2a^3 h_2(\alpha)(r/\eta)^3$ while for larger r,

$$s(\cdots) \approx \frac{-2a^3 h_2(\alpha)(r/\eta)^3}{b^4 h_1^2(\alpha)(r/\eta)^4}.$$

This result establishes that only one mode participates in the transfer of energy for the small scale-structure of the turbulence while both modes participate in the large scale-structure. This means that even though kinematic self-similarity is assumed, formulated, and pursued, the system (unsurprisingly) refuses to nonetheless behave dynamically self-similar.

Third, a few remarks about self-similarity. This rather idealized condition is tantamount to saying that the two "scaling-effects" of turbulence, viz. $\sigma(\cdots)$ and $\eta(\cdots)$, are both simply functions of time alone, i.e. $\sigma \sim \sigma(t\text{-only})$ and $\eta \sim \eta(t\text{-only})$, and that as the turbulence decays all terms in the dynamical equation consequently maintain the same relative balance. While

† Actually, they are somewhat smaller owing to some dissipation of energy by the generating source.

within the scope of the isotropic theory the t-dependence only of turbulence intensity may be completely justifiable, the t-dependence only of the length scale is unfortunately not. In fact the question of different length scales for turbulence, viz. the dissipation scale, λ, for the small-scale structure and the integral scale, Λ, for the large-scale structure (*non self-similarity*), has received considerable attention in the literature, and the related issue of a single universal length scale has never been fully resolved. The approach to a multiple length scales formulation proposed here (for future investigation) is to instead regard $\eta(\cdots)$ as a function of both r and t, i.e. $\eta(\cdots) \sim \eta(r, t)$. Since the length scale is an even function of r, a possible expression for $\eta(r, t)$ is then

$$\eta(r, t) \approx \lambda + (\Lambda - \lambda)(r/\Lambda)^2,$$

where λ and Λ are those determined from the self-similar theory, such that for $r \ll \Lambda$, $\eta(r, t) \approx \lambda$ while for $r \approx \Lambda$, $\eta(r, t) \approx \Lambda$. Note that this formulation automatically provides for differing length scales for those values of $\lambda < r < \Lambda$. Correspondingly, $k(r, t) \rightarrow k\{r/\eta(r, t)\}$ and

$$\frac{-2a^3(r/\eta)^3}{[1 + b^2(r/\eta)^2]^2} \rightarrow \frac{-2a^3(r/\lambda)^3[1 + (r/2\Lambda)^2]}{[1 + 2(r/2\Lambda)^2 + b^2(r/\lambda)^2 + (r/2\Lambda)^4]^2},$$

where $\Lambda \approx 1.25\lambda$.

Finally, the formulated thoery is not intended as a "closure" since the value of C_k had to be determined empirically; a bonafide closure to the turbulence problem will for self-similar decay theoretically *predict* the value of C_k.

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Energy transfer in turbulence

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It is theoretically established that energy transfer in nonhomogeneous turbulence occurs in two distinct scale-independent "modes." The physical significance of these modes is examined. Further, evidence is presented that suggests the existence of a conservation principle that governs the transfer of energy between turbulence eddies of differing size. A possible algebraic form of such a principle is formulated.

Recent work[1–3] has produced evidence suggesting that there exists in self-similar isotropic turbulence a "law" that governs the transfer of energy from an eddy of one scale size to an eddy of smaller scale size. This law is independent of momentum conservation (Navier–Stokes equation) and fully consistent with the "traditional wisdom" that a turbulent fluid does not transfer energy in an arbitrary fashion but indeed does so in some well-prescribed manner. Recall that in turbulence, because it is a nonlinear phenomenon, eddies of differing (but nonetheless comparable) size do indeed "communicate" with one another, and their respective behaviors can accordingly not be treated independently. The importance of such a law to the field of turbulence, in general, is twofold. First, it is widely held[4] that the transfer of energy in a system obeying nonlinear equations is a dynamical process (completely) dependent on those equations; yet the results of Refs. 1–3 suggest that, at least in the self-similar isotropic case, there are some aspects of energy transfer that are strictly *independent* of said equations and in fact answer to some other (as yet unformulated) calling. Second, once formulated, such a law will provide a fundamental method for specifying how turbulence energy cascades, and thus supply a unified "closure" for the analysis of turbulence decay. Specifically, the available evidence suggests that when an eddy of scale size r transfers energy to an eddy of scale size q, there is a universal feature of said transfer, which is fully independent of both q and r, and dependent only on the ratio, $\alpha = q/r$, of scale sizes. This feature manifests for the ideal self-similar isotropic case in two distinct "modes" of energy transfer, denoted here as $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$, which are jointly analogous to the *entrainment/deformation* effects typically observed[5] between eddies in traditional experimental studies of turbulence, e.g., grid turbulence, jet turbulence, etc. Accordingly, it is not inappropriate to envision such modal behavior of turbulence energy transfer in some sense analogous to the different modes of classical heat transfer, save that in turbulence there are only two modes whereas in heat transfer there are three. This point is made here in order to emphasize the complexity of the overall turbulence energy-transfer mechanism, and perhaps suggest the need to initially investigate the said mechanism mode by mode only rather than in both modes simultaneously. Kraichnan and Spiegel,[6] similarly, have already made the analogy between the transfer of kinetic energy in turbulence and the radiative transfer of electromagnetic energy in an inhomogeneous medium that both *reflects* and *absorbs*. The existence of such a law is predicated on the fact that during self-similar decay the "scaling down" process of a turbulent eddy is invariably analytically modeled by introducing a length scale η into the decay formulation such that the dimensionless variable q/η in effect kinematically captures and/or reflects the evolving state of an eddy from its inception (say, at the grid or jet) to its demise at that point where the fluid, as a result of viscous dissipation, returns to its original unagitated condition. (The distance of this point from the grid or jet is usually referred to as the "decay length" L, and L is converted into a "decay time" T by scaling out a characteristic flow velocity, viz., $T \simeq L/\overline{U}$, where $\overline{U} \sim$ centerline mean-flow velocity.) Note that in any such dimensionless formulation

$$\alpha = q/r \equiv (q/\eta)/(r/\eta),$$

regardless of whether $\eta \sim$ *dissipation scale* or $\eta \sim$ *integral scale*, etc. The natural scale independence of α automatically translates for turbulence into Re independence, and to the turbulence practitioner any feature of fluid turbulence that is independent of Re immediately signals properties of the turbulence that are unique and peculiar to turbulence mechanisms themselves and independent of the physical character of the fluid. In a practical scenario where the turbulence under consideration is usually in the steady-state condition, these unique and peculiar properties are also independent of reference-frame origin (homogeneous), even though the turbulence under investigation may itself be reference-frame origin dependent (nonhomogeneous). Such "homogeneity within nonhomogeneity" merits a special place in turbulence phenomenology and deserves a careful investigation of its ultimate ramifications.

Although formulated here for turbulence in general for the first time, the initial motivation for such an endeavor was furnished by Deissler when he concluded[7] that even for a general nonhomogeneous turbulence, certain terms in the two-point stochastic-averaged Navier–Stokes equation can be clearly interpreted as "transfer" terms. He accomplished this by introducing a *third point*, defined as

$$x_n = x + n(x' - x), \quad 0 \leqslant n \leqslant 1,$$

into the two-point geometry of the flow, and cleverly deducing that the appropriately formulated "turbulence self-interaction" terms produce zero total contribution to the decay of $\langle u_i u_j' \rangle$; $u_i \sim u_i(x,t)$, $u_j' \sim u_j(x',t)$, $x' = x + r$. The impact of this result (duly pointed out by Deissler) is that turbulence self-interaction plays the same role in nonhomogeneous tur-

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bulence that it does in homogeneous turbulence, viz., to transfer energy from one eddy to another (smaller) eddy without changing the total amount of mechanical energy present in the flow. Furthermore, he makes clear that the interpretation of self-interaction as "energy transfer" is accordingly conceptually independent of the condition of homogeneity, recall[8] that the condition of homogeneity is typically invoked in making the energy-transfer interpretation. More fundamentally, though, Deissler's result suggests that the two types of turbulence, through the self-interaction mechanism, are somehow "indistinguishable." In this Brief Communication a correspondingly similar scheme is employed to show that for nonhomogeneous turbulence the algebraic form of the self-interaction tensor, $Q_{ijl}(\cdots) = \langle u_i u_j u_l' \rangle$, when properly formulated, can also be characterized in terms of two invariant "modes" of energy transfer, which are identical for both homogeneous and nonhomogeneous turbulence alike. The impact of this new result is that it strengthens the unity perceived in the Deissler scheme and suggests the existence of an energy-transfer law for all types of turbulence; the latter implies that the premier results of Refs. 1–3 are indeed a genuinely unique feature of turbulence, and not an artifact of the attendant simplified theory.

For simplicity consider only the u component of a three-dimensional turbulence velocity field, and further restrict the analysis to the case where $x = (x,0,0)$, where $x \sim$ downstream distance from the grid or jet. In its simplest form, then, the energy-transfer mechanism of turbulence is most effectively modeled through the three-point correlation

$$Q(x_1,x_2,x_3) \equiv \langle u(x_1,t)u(x_2,t)u(x_3,t)\rangle;$$

the t dependence of $Q(\cdots)$ is suppressed by restricting the analysis to *steady state* turbulence only. Note that by introducing the separation variables, $q = x_2 - x_1$, $r = x_3 - x_1$, and the spatial location variable, $\bar{x} = (x_1 + x_2 + x_3)/3$, $Q(\cdots)$ immediately converts into $S(\bar{x},q,r)$. Through its dependence on q and r, $S(\cdots)$ analytically describes how a turbulence eddy of size q "interacts with" (transfers energy to) an eddy of size r or vice versa; i.e., $S(\cdots)$ reveals the strength of the nonlinearity of turbulence by quantifying how turbulence eddies of differing size communicate with one another. In studies of turbulence, it is customary to introduce a velocity scale (typically a rms value) defined in practice as

$$\sigma(x) = \left(\frac{1}{T}\int_0^T u^2(x,t)\,dt\right)^{1/2}.$$

This scale permits $Q(\cdots)$ to be naturally written as

$$Q(x_1,x_2,x_3) = \sigma(x_1)\sigma(x_2)\sigma(x_3) \times \langle \xi(x_1,t)\xi(x_2,t)\xi(x_3,t)\rangle,$$

where $\sigma(x)\xi(x,t) = u(x,t)$, or in terms of $(\bar{x},q,r)$ as

$$S(\bar{x},q,r) = \sigma(\bar{x} - q/3 - r/3)\sigma(\bar{x} + 2q/3 - r/3) \times \sigma(\bar{x} - q/3 + 2r/3)s(\bar{x},q,r),$$

where $s(\cdots) = \langle \xi(x_1,t)\xi(x_2,t)\xi(x_3,t)\rangle$ is a normalized three-point correlation and $x_1 = \bar{x} - q/3 - r/3$, $x_2 = \bar{x} + 2q/3 - r/3$, and $x_3 = \bar{x} - q/3 + 2r/3$. Introducing the correlation length scale, $\eta \sim \eta(x)$, results in

$$S(x,q,r) \simeq \sigma(\bar{x} - q/3 - r/3)\sigma(\bar{x} + 2q/3 - r/3) \times \sigma(\bar{x} - q/3 + 2r/3)s[q/\eta(\bar{x}), r/\eta(\bar{x})],$$

and because of the explicit presence of q and r in $\sigma(\quad)$, the spatial evolution of $S(\cdot)$ is *not* of a self-similar nature, self-similarity follows only when the length scale that "scales" $\sigma(\cdots)$ is "large," in comparison to $\eta(\bar{x})$.[1] Note that the form of velocity scaling employed here is quite different from that reported in Refs. 9 and 10, in the sense that for $q \equiv 0$, viz., $Q(x_1,x_1,x_3) = \langle u^2(x_1)u(x_1 + r)\rangle$, $S(\bar{x},0,r) \equiv \sigma^2(\bar{x} - r/3)\sigma(\bar{x} + 2r/3)k\{r/\eta(\bar{x})\}$ and *not* $\sigma^2(x_1)\sigma(x_1 + r)k[r/\eta(x_1)]$, as implied therein. The form employed here duly reflects the maximum possible amount of symmetry (w.r.t. x_1,x_2,x_3) in the definition of $\bar{x}$ while simultaneously retaining the classical definitions of q and r. The rather "unsymmetric" velocity scaling of Refs. 9 and 10 is perhaps why their triple correlation curves do not pass through the origin of coordinates, and are in fact all shifted to the left to the origin, thus suggesting a unilaterally positive value for $k(0)$. A significant nonzero value of $k(0)$ means that the probability density function (pdf) of $u(x,t)$ is markedly skewed about $u \equiv 0$, and is accordingly inconsistent with the known[11] approximately normal distribution of turbulence velocity at any one point in the flow. The scaling of Refs. 9 and 10 follows from that employed herein by imposing the transformations $\bar{x} \to x_1 + r/3$ for the rms scale and $\bar{x} \to x_1$ for the length scale.

It is a simple property of $Q(\cdots)$ that $Q(x_1,x_2,x_3) \equiv Q(x_1,x_3,x_2) \equiv Q(x_2,x_1,x_3)$, or equivalently that $S(\bar{x},q,r) \equiv S(\bar{x},r,q) \equiv S(\bar{x}, -q,r, -q)$. Since the product $[\sigma(\bar{x} - q/3 - r/3)\sigma(\bar{x} + 2q/3 - r/3)\sigma(\bar{x} - q/3 + 2r/3)]$ automatically satisfies this symmetry, it follows that $s(\cdots)$ itself must also satisfy the same symmetry, i.e., $s(\bar{x},q,r) \equiv s(\bar{x},r,q) \equiv s(\bar{x}, -q,r - q)$. If $s(\cdots)$ is thus expanded in a Taylor's series in q and r, the result is

$$s(\bar{x},q,r) \simeq s(\bar{x},0,0) + a_1(\bar{x})q + b_1(\bar{x})r + a_2(\bar{x})q^2 + c^2(\bar{x})qr + b_2(\bar{x})r^2 + \cdots,$$

where the $a_n(\bar{x})$, $b_n(\bar{x})$, $c_n(\bar{x})$, $n = 1,2,3\ldots$, represent the appropriately evaluated partial derivatives of $s(\bar{x},q,r)$. It can be readily shown[12,13] that because of the required q,r argument symmetry in $s(\bar{x},q,r)$, aside from $s(\bar{x},0,0)$, which automatically satisfies the symmetry the only algebraic forms that can appear in the Taylor's expansion are the homogeneous polynomials, $F(\bar{x},q,r) = a_2(\bar{x})(q^2 - qr + r^2)$ and $G(\bar{x},q,r) = a_3(\bar{x})(2q^3 - 3q^2r - 3qr^2 + 2r^3)$, and those higher-order homogeneous polynomials, which can be formed from $F(\cdots)$ and $G(\cdots)$, viz., $F^2(\cdots)$, $F(\cdots)G(\cdots)$, $G^2(\cdots)$, etc. No other fundamental algebraic form different from $F(\cdots)$ and $G(\cdots)$ satisfies the required symmetry and can therefore not appear in the expansion. Specifically, the q,r dependence of $s(\bar{x},q,r)$ can thus be fully characterized by $F(\cdots)$ and $G(\cdots)$ only, i.e., $s(\bar{x},q,r) \sim s\{F(\bar{x},q,r), G(\bar{x},q,r)\}$. From purely dimensional grounds, then, it follows that $a_2(\bar{x}) = b^2/\eta^2(\bar{x})$ and $a_3(\bar{x}) = a^3/\eta^3(\bar{x})$, where a and b are dimensionless numbers, hence

$$s\{\cdots\} \sim s\{b^2(q^2 - qr + r^2)/\eta^2(\bar{x}),$$
$$2a^3(q^3 - 1.5q^2r - 1.5qr^2 + r^3)/\eta^3(\bar{x})\}.$$

The physical significance of $F(\cdots)$ and $G(\cdots)$ is established by writing them as

$$F(\bar{x},q,r) = b^2(\alpha^2 - \alpha + 1)r^2/\eta^2(\bar{x}),$$
$$G(\bar{x},q,r) = 2a^3(\alpha^3 - 1.5\alpha^2 - 1.5\alpha + 1)r^3/\eta^3(x),$$

and $\alpha = q/r$, such that $\alpha^2 - \alpha + 1 \equiv h_{\rm I}(\alpha)$ and $\alpha^3 - 1.5\alpha^2 - 1.5\alpha + 1 \equiv h_{\rm II}(\alpha)$ are scale-independent invariant forms. Since $s\{\cdots\}$ can be fully characterized in terms of the $\bar{x}$ independent $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$, these algebraic forms are then the homogeneous invariant "modes" of nonhomogeneous energy transfer referred to earlier. The *entire* nonhomogeneous transfer of energy, at *any* spatial location defined by $\bar{x} = (x_1 + x_2 + x_3)/3$, from an eddy of size $r = x_3 - x_1$, to an eddy of size $q = x_2 - x_1$, can be constructed from $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$, while the related "spatial decay" of energy transfer leaves $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$ preserved and unaffected. The physical significance of a and b is thus that they explicitly define "how much" of each mode is present in the flow; because they are dimensionless both a and b must depend on Re. The mode interpretation of $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$ is reinforced by introducing $\beta \equiv \alpha - 0.5$, whereupon $h_{\rm I}(\alpha) \to f_{\rm I}(\beta)$, $h_{\rm II}(\alpha) \to f_{\rm II}(\beta)$, and noting that

$$\int_{-\theta}^{+\theta} f_{\rm I}(\beta) f_{\rm II}(\beta)\, d\beta \equiv 0$$

for any θ (orthogonality). The interesting feature of this formulation is that depending on the (as yet unknown) exact algebraic form of $s(\cdots)$, for "small enough" r,

$$s(\bar{x},q,r) \simeq s(\bar{x},0,0) + b^2(q^2 - qr + r^2)/\eta^2(\bar{x})$$
$$\sim s(\bar{x},0,0) + b^2h_{\rm I}(\alpha)r^2/\eta^2(\bar{x})$$

or

$$s(\bar{x},q,r) \sim s(\bar{x},0,0) + 2a^3h_{\rm II}(\alpha)r^3/\eta^3(\bar{x}),$$

while for "larger" r

$$s(\bar{x},q,r) \sim s(\bar{x},0,0) + b^2h_{\rm I}(\alpha)r^2/\eta^2(\bar{x})$$
$$+ 2a^3h_{\rm II}(\alpha)r^3/\eta^3(\bar{x}) + \cdots;$$

this means that for the smaller eddy sizes only one mode participates in the transfer, while for the larger eddies both modes participate. More specifically, even though $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$ are themselves "immune" to the effects of scale size, there is nonetheless some mechanism manifesting in the flow which denies joint participating of both modes in the transfer of energy for all scale sizes in the flow This different modal communication between eddies for the large eddy structure versus the small eddy structure has long been suspected by turbulence practitioners, but insubstantially documented in the literature.

Since turbulence must transfer energy in a fashion that preserves the scale-independent forms, $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$, and since $Q(\cdots)$ for any turbulence must be algebraically expressible in terms of $h_{\rm I}(\alpha)$ and $h_{\rm II}(\alpha)$, then all universal ramifications of same are embodied in these absolutely invariant functions, and ultimately in the invariant α. The physical implication of this resut is best comprehended by recalling a simple experiment in physics which culminates in a precise operational definition of the *mass* of a body. The experiment involves placing a spring between two isolated masses, A and B, pushing them together so that the spring is compressed (the spring is not rigidly attached to either of the masses), and then releasing them. By repeating the experiment a number of times, each time with a spring of different constant, it will be found that although the recoil velocities of A and B will be different each time, the ratio v_A/v_B will be identical for all cases, i.e., $v_A/v_B \equiv$ invariant, regardless of the amount of initial energy in the system. The fact that this result provides a means for unambiguously defining the mass of a body, viz., $m_B/m_A \equiv v_A/v_B$, and that it is consistent with the principle of linear momentum conservation, is well established. Put simply, the invariance of v_A/v_B means that energy cannot be transferred from the potential energy of the spring to the kinetic energy of the masses in an arbitrary fashion, but must indeed be transferred in a manner consistent with linear momentum conservation. In thermodynamics, a correspondingly simple experiment will establish that in a Carnot engine the heat Q_H, flowing into the engine from a high temperature reservoir of fixed temperature T_H, is related to the heat Q_L, flowing out to a low temperature reservoir of fixed temperature T_L, by the relation

$$Q_H/Q_L = T_H/T_L \equiv \text{invariant};$$

the invariance of this ratio is a consequence of the principle of conversation of entropy, i.e., heat is not transferred arbitrarily through the engine. In geometrical optics the fact that the sines of the angles of incidence and refraction stand in a constant ratio to each other, regardless of the magnitude of each, leads to the formulation of *Snell's law*. It appears, therefore, that the invariance of $\alpha = q/r$ is a manifestation of some conservation law or principle which governs the transfer of energy in turbulence. Such a conservation law must describe the means of (scale-independent) communication between turbulence eddies of different size such that in the limit, eddies distinguished by a great disparity in size should exert minimal influence on one another. Moreover, such a principle is consistent with accepted wisdom that turbulence does not transfer energy arbitrarily but indeed does so in some well-prescribed fashion. If a turbulence eddy of a particular size is regarded as a dynamical system which receives energy from those eddies of larger size, dissipates some of this energy, and transfers the rest to those eddies of smaller size, then a possible such law is

$$E_T(r)/\Delta E_R(r,q) \simeq (r/q)^\gamma;$$

in this expression $E_T(r)$ is the total energy transferred by an eddy of size r to all other eddies of smaller size, $\Delta E_R(r,q)$ is that portion of $E_T(r)$ that is eventually *received* by an eddy of size q, γ is a nonadjustable constant greater than zero, and $r > q$. The unavoidable $\bar{x}$ dependence of both $E_T(\cdots)$ and $\Delta E_R(\cdots)$ is explicitly deleted in the above expression in order to emphasize the location independence of the ratio $E_T(\cdots)/\Delta E_R(\cdots)$. The total energy received by an eddy of size q is thus

$$E_R(q,\bar{x}) = \sum_{n=1}^{N} \Delta E_R(r_n,q,\bar{x}),$$

where r_n, $n = 1,2,3 \ldots N$ are all greater than q. The ultimate consequences of such a law (or others like it) need to be fully investigated.

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APPENDIX B

Pressure and Velocity Correlations in a Plane Turbulent Water Jet

by

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A THESIS

submitted in partial fulfillment of the requirements

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