DEVELOPMENT OF A STRATIFIED PLATE MODEL FOR COMPOSITE PANELS

THESIS

Alan L. Lesmerises, Capt, USAF

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Presented to the Faculty of the School of Engineering

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Master of Science in Astronautical Engineering

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Captain, USAF

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Preface

The purpose of this thesis was to develop a mechanical analysis tool for composite materials that models the matrix and fibers separately. We intended this tool to be suitable for viscoelastic analysis, since the aerodynamic heating anticipated for future high-speed aerospace applications is expected to cause either the matrix, the fibers, or both, of the most promising composites to behave viscoelastically. However, we saw other possible applications of this model, including being able to account for delaminations, incorporation of piezoelectric materials into a composite panel, and forming the basis of a new type of finite element.

Once we got started, we concluded that we needed a rigorous demonstration of the validity of this model, which required a slight change in direction during my research. Also, I encountered a few road blocks and blind alleys along the way. Both of these kept me from accomplishing as much as I had originally thought I would, but I feel that I have established a solid foundation upon which other research can build.

This research was sponsored by Mr. Jerome Pearson, WL/FIBGD, and Dr. T. Nicholas, WL/MLLN.

I want to thank my thesis advisor, LtCol. R.L. Bagley, for helping me find the way around those roadblocks and blind alleys, and for putting-up with some of my dumb mistakes. I also want to thank Drs. P.J. Torvik and A.N. Palazotto, as members of my thesis committee, for their contributions.

Alan L. Lesmerises

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Notation

A,B,C	Coefficients of Strata Theory Displacement Field Solution
a,b	x and y Dimensions of Panel
E	Elastic Modulus
e	Euler's Number (2.718281828459...)
F	Total Number of Fiber Strata
h	Total Stack Thickness
h_{fl}	Thickness of the l^{th} Fiber Stratum
h_{mk}	Thickness of the k^{th} Matrix Stratum
i,j,k,l	Summation Indices
K	Matrix Stiffness Coefficients
M	Total Number of Matrix Strata
N	Order of Z Polynomial Assumed
p,q	Mode Shape Parameters
P	Applied Load
$\mathcal{P}$	Load Coefficient
Q	Characteristic matrix of the Exact Stratified Plate Solution
S	Fiber Stiffness Coefficients
T	Kinetic Energy Term in Energy Analysis
t_p	Thickness of a Fiber Ply
u,v,w	Displacement in the x,y,z Directions
U,V,W	z-dependent functions in the assumed displacement field expressions for the Exact Stratified Plate Solution for u(x,y,z), v(x,y,z), and w(x,y,z)

W	Work Term in Energy Analysis
x,y,z	Coordinate Directions
α,β	Mapping Coefficients for Strata Theory z-Dependent portion of the Assumed Displacement Field
θ	Fiber Orientation Angle
ψ,ξ	The x- and y-dependent functions in the assumed displacement functions for u(x,y,z) and v(x,y,z) in Strata Theory
ν	Poisson's Ratio
σ,τ	Normal and Shear Stresses
ε,γ	Normal and Engineering Shear Strains
Π	Strain (Internal) Energy
π	Pi (3.14159265...)
[illegible]	Eigenvectors and Eigenvalues
CPT	Classic Laminated Plate Theory
ESP	Exact Stratified Plate
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
ST	Strata Theory

Subscripts

f,m	Fiber and Matrix-related material properties
i,j,k,l	Summation Indices
",x"	Derivative with respect to the x variable (similar for y and z)

See pg X for abstract

AD-A248206 Words/Phrases(4 words max) that match Thesaurus Entries

TEXT	THESAURUS
APPROACH	Approach
BENDING	Bending
BOUNDARY	Boundaries
COMPARISON	Comparison
DIFFERENTIAL EQUATIONS	Differential Equations
DISPLACEMENT	Displacement
ENERGY	Energy
EQUATIONS	Equations
FIBER	Fibers
FIBERS	Fibers
GRADIENTS	Gradients
LAYER	Layers
LAYERS	Layers
MATERIAL	Materials
MECHANICAL PROPERTIES	Mechanical Properties
MODEL	Models
PANEL	Panels
PANELS	Panels
PARTIAL DIFFERENTIAL EQUATIONS	Partial Differential Equations
PLATE	Plates
THEORY	Theory
THICKNESS	Thickness

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@23@ Approach, Bending, Boundaries, Comparison, Differential Equations, Displacement, Energy, Equations, Fibers, Gradients, Layers, Materials, Mechanical Properties, Models, *Panels, Partial Differential Equations, *Plates, Theory, Thickness.
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@27@ @1@ADA248206 @6@DEVELOPMENT OF A STRATIFIED PLATE MODEL FOR COMPOSITE PANELS. @9@Master's thesis. @10@Alan L. /Lesmerises @11@Mar 92 @12@111 @14@AFIT/GA/ENY/92M-02 @15@ @16@ @17@ @18@XF @19@WL* @20@u @21@ @22@ @26@Stratified Plate, Composite, Composite Model, Cross-Plies @27@A new model for composites represents a composite panel as an alternating stack of isotropic matrix and orthotropic fiber-dominated layers. An energy approach results in a set of partial differential equations and boundary conditions where the mechanical properties of the matrix and fibers appear separately. Two solution methods are developed. The first, called the "Strata Theory" is developed for general applications, while the second, called the "Exact Stratified Plate", is used as a benchmark for comparison to Strata Theory. Both solutions assume zero gradients of displacement through the thickness of a fiber layer. They are compared for an isotropic material and a two-ply composite with cross-plies, using simply-supported cases of a square plate and cylindrical bending. @33@ @35@012250 @END@
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* Matrix materials
* Fiber reinforced composites
Stiffness
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theses

Modulus of elasticity
Computer programs

Fld 25

Strata theory
Exact Stratified Plate method

Abstract

A new model for composites represents a composite panel as an alternating stack of isotropic matrix and orthotropic fiber-dominated layers. An energy approach results in a set of partial differential equations and boundary conditions where the mechanical properties of the matrix and fibers appear separately. Two solution methods are developed. The first, called "Strata Theory" is developed for general applications, while a second, called the "Exact Stratified Plate", is used as a benchmark for comparison to Strata Theory. Both solutions assume zero gradients of displacement through the thickness of a fiber layer. They are compared for an isotropic material and a two-ply composite with cross-plies, using simply-supported cases of a square plate and cylindrical bending.

DEVELOPMENT OF A STRATIFIED PLATE MODEL FOR COMPOSITE PANELS

I. Introduction

Background

A composite material typically consists of a relatively soft matrix material that contains stiffer fibers. The length of the fibers used may vary, depending on the application, from very short fibers (i.e. particles) to long continuous strands. These fibers endow the composite material with increased mechanical stiffness. The increase in stiffness depends primarily on the fiber density, properties, and orientation.

When the composite is formed into a panel, it is usually built-up as a series of layers of parallel fibers known as plies. If each fiber ply is modeled as an orthotropic plate, the panel as a whole could be modeled as a set of these orthotropic plate stacked upon one another. In fact, a commonly used method for the analysis of composite panels, known as Classic Laminated Plate Theory (CPT), models laminated composites this way. In CPT, the elastic properties of the matrix are "averaged" with those of the fibers, and the ability to separately model the matrix and the fiber materials is lost.

At high temperatures, many of the materials used for the matrix and fibers in today's composites can become viscoelastic. In such a case, there is no reason to expect that the fibers and matrix will become viscoelastic at the same time, or that the fibers and matrix would maintain the same elastic relationships once they do become viscoelastic. Therefore, it is essential that any type of vibration analysis where viscoelasticity will be considered is able to deal with the matrix and fiber elasticity terms separately.

Scope

There is a need for an alternate method of modeling composites, such that the matrix and fiber elasticity terms appear separately in the governing equations. Furthermore, a formulation suitable for viscoelastic stress and vibration analyses is needed.

This thesis details the development of a new model for the analysis of laminated composites called the Stratified Plate Model. A composite is modeled as an alternating stack of isotropic matrix strata and orthotropic fiber strata. As a result, the matrix and fiber properties have an identifiable and distinct contribution to the overall mechanical characteristics of the composite.

A new analysis technique, called Strata Theory, arises from the Stratified Plate Model and is intended for general applications. Strata Theory will be validated by comparing it with a more accurate formulation called the "Exact Stratified Plate".

Idealization of Fiber Layers

The Stratified Plate Model treats each fiber ply somewhat differently than the CPT model. First, each fiber is idealized [Figure 1.] with a square cross-section of equal cross-sectional area. The thickness of a fiber stratum h_f is then given as

$$h_f = \sqrt{\frac{\pi D_f^2}{4}} \qquad (1)$$

where D_f is the average diameter of the actual fibers. The shape of the fibers now define three distinct regions within the ply: a fiber-dominated stratum sandwiched by two thin matrix strata above and below [Figure 2.]. The matrix strata consist of purely homogeneous matrix material, while the fiber strata contains the idealized fibers and the matrix material between the fibers within the ply.

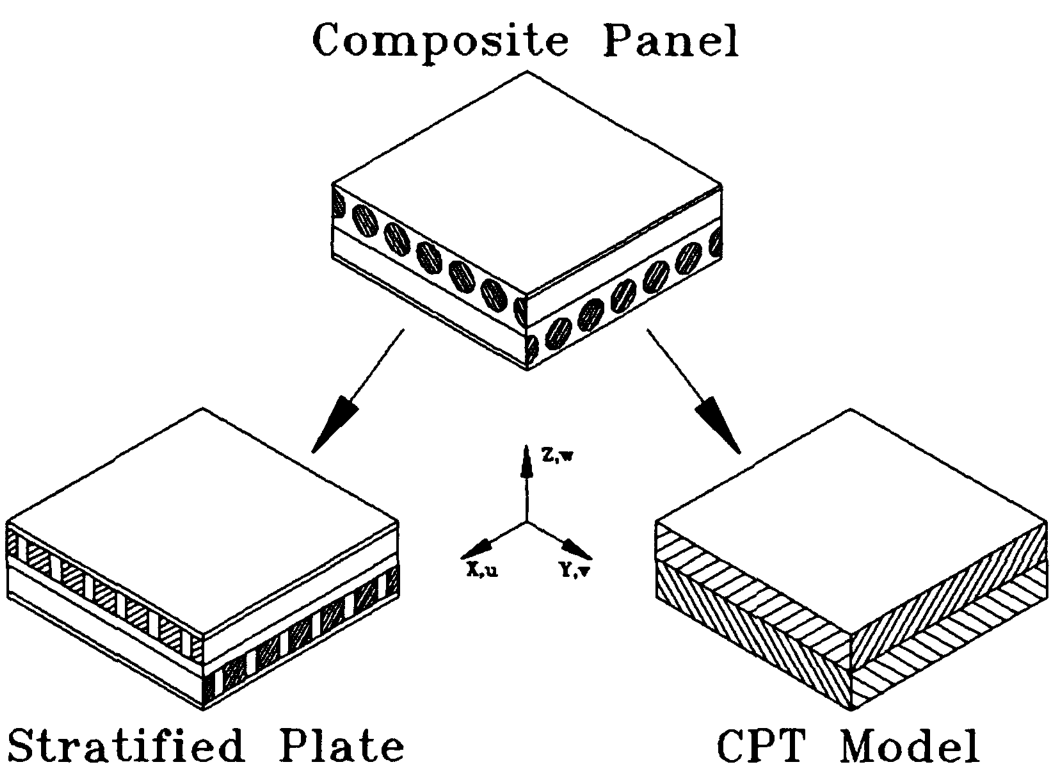

Figure 1. Comparison between Stratified Plate and CPT models of a composite panel

Notice that when the fibers are modeled this way, the idealized fibers span the entire thickness of the fiber stratum. Since the fiber's elastic modulus is typically at least

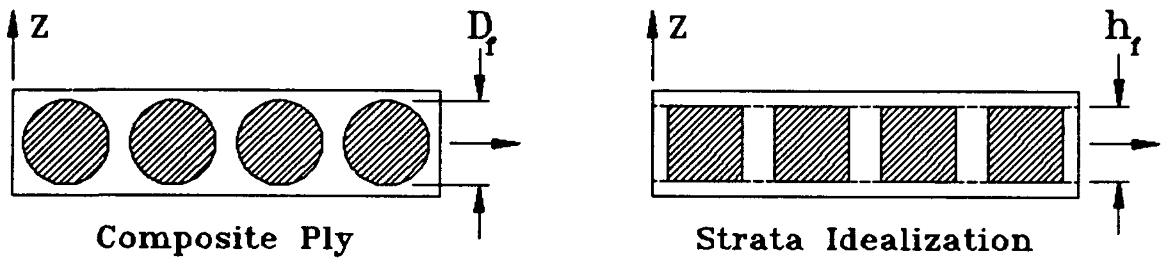

Figure 2. Strata Theory idealization of fibers

an order of magnitude larger than that of the matrix, it seems reasonable to assume that the z-direction displacement gradients through the fiber strata will be much smaller than those through the matrix. Indeed, this assumption is a key feature of the Stratified Plate Model. This will become important in the development of the governing partial differential equations (PDEs), as it will allow all the z-direction displacement derivatives through the fiber strata to be set to zero.

Development and Validation

The governing differential equations and associated boundary conditions are derived using the principle of minimum potential energy. The Strata Theory model is developed using an assumed displacement field. The Exact Stratified Plate model, on the other hand, finds the z-dependent portion of the displacement field through an eigenvalue formulation once a set of x- and y-dependent functions are found that satisfy the boundary conditions.

Exact orthotropic plate solutions have been found for composites in cylindrical bending and for rectangular bidirectional sandwich composite plates [8:398-411; 9:20-34]. These solutions were used for comparison with CPT. Both CPT and these exact orthotropic plate solutions model a composite as a set of stacked orthotropic plates. This is considerably different from the way the Stratified Plate Model represents a composite. These two different representations of a composite may predict radically different behaviors for the composites they model. As a result, it was felt that using results obtained from either CPT or the exact orthotropic plate methods as a basis of comparison for Strata Theory would not be appropriate. Strata Theory needed a different standard against which it could be compared, hence the development of the Exact Stratified Plate solution.

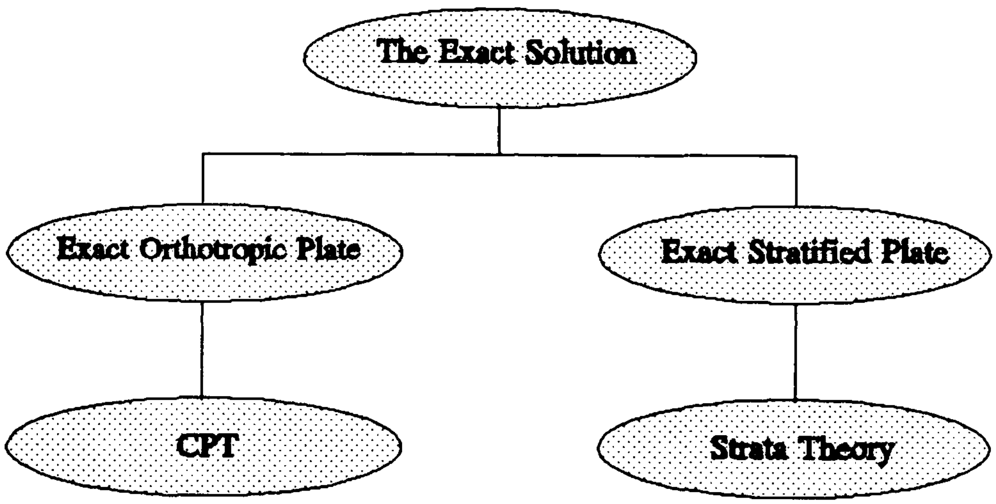

Figure 3. Relationships between the various models for composites

II. The "Strata Theory"

Basic Concepts and Assumptions

The Strata Theory, hereafter referred to as ST, is based on the Stratified Plate Model described above. The Stratified Plate Model assumes that the z-direction displacement gradients through the fibers are zero. This means that ϵ_z will be zero through the fibers. The ST model makes the additional assumption that ϵ_z is zero through the matrix strata as well. Since ST was intended to form the basis of a new vibration analysis tool and not for detailed stress and strain estimation, it seems to be a reasonable approximation that the contributions of ϵ_z from the matrix strata will be negligible.

Another approximation is that ST assumes that the displacement field can be represented as a polynomial in z, leaving the x and y dependencies as unknowns that need to be found.

The presence of fibers in a composite endow the composite with complex mechanical properties such that coupling may occur between the fiber plies. When there is coupling, it means that plane sections neither remain plane nor perpendicular to the midplane. In other words, what would start as a vertical line through the thickness of an unloaded, undeflected composite panel will become a complex 3-dimensional curve when the panel is deformed. To represent this complex 3-D curve, the following expressions for the displacements u, v, and w will be used for displacements through the fibers

$$u_f(x,y,z) = \sum_{i=0}^{N} z_f^{\,i}\psi_i(x,y)$$

$$v_f(x,y,z) = \sum_{i=0}^{N} z_f^{\,i}\zeta_i(x,y) \qquad (2)$$

$$w_f(x,y,z) = w(x,y)$$

while the displacement field through the matrix will use the following expressions for u, v, and w.

$$u_m(x,y,z) = \sum_{i=0}^{N} (\alpha_m+\beta_m z)^i \ \psi_i(x,y)$$

$$v_m(x,y,z) = \sum_{i=0}^{N} (\alpha_m+\beta_m z)^i \ \zeta_i(x,y) \qquad (3)$$

$$w_m(x,y,z) = w(x,y)$$

The term z_f that appears in the fiber displacement terms above is the z coordinate of the midplane of a fiber stratum referenced to the midplane of the entire stack [Figure 4.]. The values of z that are used in the matrix displacement field expressions must be a z coordinate that lies within a matrix stratum. Also, the w, ψ_i, and ζ_i terms that appear in the assumed displacement field expressions are the same for both the matrix and fibers.

The complex 3-D curve described above can be decomposed by taking projections onto the x-z and y-z planes. The in-plane displacements u and v are polynomials in z that approximate these projections, and the ψ_i and ζ_i terms are the x and y dependent coefficients of these polynomials in z. It should be noted that the number "N", the order of the assumed polynomial, is not directly related to the number of fiber plies. However, the more twisting and warping anticipated, the larger N should be to better approximate this deformation. Ultimately, N is dictated by the accuracy required, and the complexity of the problem.

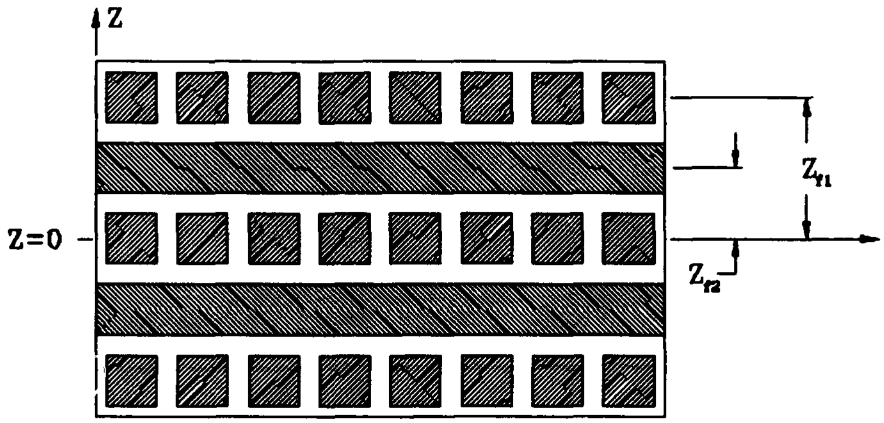

Figure 4. Definition of the fiber strata parameter z_f with respect to the z axis origin

The α's and β's that appear in the matrix displacement functions are mapping terms. Since the z gradients through the fibers are assumed to be zero, the x and y displacements are constant through a given fiber stratum. This implies that there must be continuity of in-plane displacements (u and v) from the bottom of one matrix stratum to the top of the next lower matrix stratum. This continuity of displacement is established by a linear mapping of the z coordinate of the lower surface of the ith matrix stratum and the z coordinate of the upper surface of the (i+1)th matrix stratum to the z coordinate of the midplane of the intervening fiber stratum, denoted by z_f.

$$\begin{aligned} \alpha_{m_i} + \beta_{m_i} z^{-}_{m_i} &= z_f \\ \alpha_{m_{i+1}} + \beta_{m_{i+1}} z^{+}_{m_{i+1}} &= z_f \end{aligned} \tag{4}$$

Each matrix stratum has its own α and β. Finding α and β for a given matrix stratum is a simple linear problem of 2 equations and 2 unknowns. The first equation maps the z coordinate of the upper surface of the matrix stratum to the midplane of the fiber stratum immediately above; the second equation maps the z coordinate of the lower surface of the matrix stratum to the midplane of the fiber stratum immediately below. For a matrix stratum at the top or bottom of the composite, the free surface coordinate (h/2 or -h/2) is used in place of z_f.

$$\begin{aligned} \alpha_{m_i} + \beta_{m_i} z_{m_i}^{+} &= z_{fu} \quad \textit{(upper fiber layer)} \\ \alpha_{m_i} + \beta_{m_i} z_{m_i}^{-} &= z_{fl} \quad \textit{(lower fiber layer)} \end{aligned} \tag{5}$$

When all the fiber plies of the original composite being modeled are of equal thickness, β is the same for all the matrix strata

$$\beta_m = \left(\frac{t_p}{t_p - h_f} \right) \tag{6}$$

where t_p is the thickness of the original fiber ply. Note that this is not a requirement for the ST approach. However, for the sake of simplicity, this assumption will be used for the remainder of the ST development.

These assumed displacement field expressions can now be substituted into the expressions for strain, which will later be used to determine the strain energy

$$\{\epsilon\}_{fk} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}_{fk} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ w_{,x} + u_{,z} \\ w_{,y} + v_{,z} \end{Bmatrix}_{fk} = \begin{Bmatrix} \sum_{i=0}^{N} z_{fk}^{i} \psi_{i,x} \\ \sum_{i=0}^{N} z_{fk}^{i} \zeta_{i,y} \\ 0 \\ \sum_{i=0}^{N} z_{fk}^{i} [\psi_{i,y} + \zeta_{i,x}] \\ w_{,x} + \sum_{i=0}^{N} i z_{fk}^{i-1} \psi_i \\ w_{,y} + \sum_{i=0}^{N} i z_{fk}^{i-1} \zeta_i \end{Bmatrix} \tag{7}$$

$$\{\epsilon\}_{ml} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}_{ml} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ w_{,x} + u_{,z} \\ w_{,y} + v_{,z} \end{Bmatrix}_{ml} = \begin{Bmatrix} \sum_{i=0}^{N}(\alpha_{ml}+\beta_{ml}z)^i \psi_{i,x} \\ \sum_{i=0}^{N}(\alpha_{ml}+\beta_{ml}z)^i \zeta_{i,y} \\ 0 \\ \sum_{i=0}^{N}(\alpha_{ml}+\beta_{ml}z)^i [\psi_{i,y}+\zeta_{i,x}] \\ w_{,x}+\sum_{i=0}^{N} i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}\psi_i \\ w_{,y}+\sum_{i=0}^{N} i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}\zeta_i \end{Bmatrix} \tag{8}$$

Stress and strain for the matrix strata are related through the expression $\{\sigma\}=[K]\{\epsilon\}$, where the elasticity matrix [K] for the matrix material is given by

$$[K]_{ml} = \frac{E_{ml}}{(1-2\nu_{ml})(1+\nu_{ml})} \begin{bmatrix} 1-\nu_{ml} & \nu_{ml} & \nu_{ml} & 0 & 0 & 0 \\ \nu_{ml} & 1-\nu_{ml} & \nu_{ml} & 0 & 0 & 0 \\ \nu_{ml} & \nu_{ml} & 1-\nu_{ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu_{ml}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu_{ml}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu_{ml}}{2} \end{bmatrix} \tag{9}$$

or in simplified notation

$$[K]_{ml} = \begin{bmatrix} K_{1_{ml}} & K_{2_{ml}} & K_{2_{ml}} & 0 & 0 & 0 \\ K_{2_{ml}} & K_{1_{ml}} & K_{2_{ml}} & 0 & 0 & 0 \\ K_{2_{ml}} & K_{2_{ml}} & K_{1_{ml}} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{3_{ml}} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{3_{ml}} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{3_{ml}} \end{bmatrix} \tag{10}$$

where

$$K_{1_{ml}} = \frac{E_{ml}(1-\nu_{ml})}{(1-2\nu_{ml})(1+\nu_{ml})} \qquad K_{2_{ml}} = \frac{E_{ml}\nu_{ml}}{(1-2\nu_{ml})(1+\nu_{ml})} \qquad K_{3_{ml}} = \frac{E_{ml}}{2(1+\nu_{ml})} \tag{11}$$

As discussed previously, the fiber strata are modeled as orthotropic layers. The elastic constants for such a layer are derived using a "rule of mixtures". For this derivation, the following fiber elasticity terms will be defined

$$\begin{aligned}
\tilde{S}_1 &= vol_f \frac{E_f(1-\nu_f)}{(1-2\nu_f)(1+\nu_f)} + vol_m \frac{E_m(1-\nu_m)}{(1-2\nu_m)(1+\nu_m)} \\
\tilde{S}_2 &= \frac{1}{vol_m} \times \frac{E_m(1-\nu_m)}{(1-2\nu_m)(1+\nu_m)} \\
\tilde{S}_3 &= vol_f \frac{E_f\nu_f}{(1-2\nu_f)(1+\nu_f)} + vol_m \frac{E_m\nu_m}{(1-2\nu_m)(1+\nu_m)} \\
\tilde{S}_4 &= \frac{E_m\nu_m}{(1-2\nu_m)(1+\nu_m)} \\
\tilde{S}_5 &= G_m \,/\, vol_m \\
\tilde{S}_6 &= vol_f\, G_f + vol_m\, G_m \\
\tilde{S}_7 &= G_m \,/\, vol_m
\end{aligned} \tag{12}$$

The terms vol_f and vol_m are the volume fractions of fibers and matrix in a given fiber stratum. This must not be confused with the global volume fractions of fibers and matrix, but there is a direct relationship between the two.

The elastic stiffness for each fiber stratum in its principal axis directions will have the form [2:35]

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_z \\ \tau_{12} \\ \tau_{1z} \\ \tau_{2z} \end{Bmatrix} = \begin{bmatrix} \tilde{S}_{1_f} & \tilde{S}_{3_f} & \tilde{S}_{3_f} & 0 & 0 & 0 \\ \tilde{S}_{3_f} & \tilde{S}_{2_f} & \tilde{S}_{4_f} & 0 & 0 & 0 \\ \tilde{S}_{3_f} & \tilde{S}_{4_f} & \tilde{S}_{1_f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{S}_{5_f} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_{6_f} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{S}_{7_f} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_z \\ \gamma_{12} \\ \gamma_{1z} \\ \gamma_{2z} \end{Bmatrix} \tag{13}$$

where the "1" direction is parallel to the fibers, and the "2" direction is perpendicular to the fibers in the plane of the fiber stratum. The coordinate transformation matrix $[\theta]$ for an orthotropic material is given by [2:48-49]

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_f & \sin^2\theta_f & 0 & -\sin2\theta_f & 0 & 0 \\ \sin^2\theta_f & \cos^2\theta_f & 0 & \sin2\theta_f & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}\sin2\theta_f & -\frac{1}{2}\sin2\theta_f & 0 & \cos2\theta_f & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_f & -\sin\theta_f \\ 0 & 0 & 0 & 0 & \sin\theta_f & \cos\theta_f \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_z \\ \gamma_{12} \\ \gamma_{1z} \\ \gamma_{2z} \end{Bmatrix} \tag{14}$$

This transformation can be used for stresses as well. This is now applied to the principal axis elasticity matrix by means of the similarity transformation

$$\{\sigma\}_f = [\theta][\tilde{S}][\theta]^{-1}\{\epsilon\}_f \tag{15}$$

where $[\theta]$ is the rotation matrix above. This produces the general constitutive relation [2:34]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} S_{1_f} & S_{3_f} & S_{10_f} & S_{5_f} & 0 & 0 \\ S_{3_f} & S_{2_f} & S_{11_f} & S_{6_f} & 0 & 0 \\ S_{10_f} & S_{11_f} & S_{12_f} & S_{13_f} & 0 & 0 \\ S_{5_f} & S_{6_f} & S_{13_f} & S_{4_f} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{7_f} & S_{9_f} \\ 0 & 0 & 0 & 0 & S_{9_f} & S_{8_f} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{16}$$

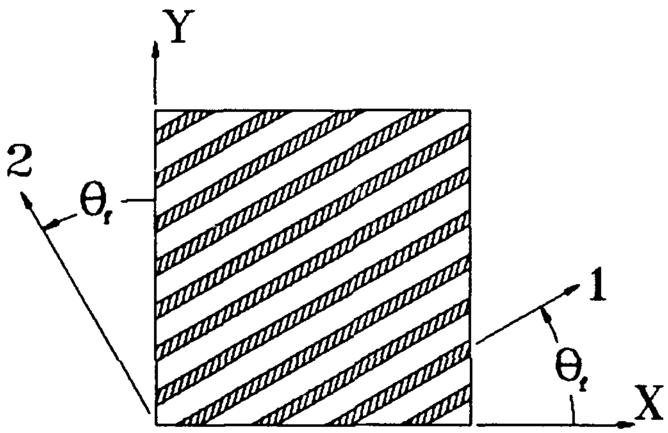

Figure 5. Definition of fiber orientation angle θ_f

Development of the PDEs

The development is based on the principle of minimum potential energy, given by

$$\int_0^b \int_0^a \left[-\sum_{l=1}^{M} \int_{z_{ml}^-}^{z_{ml}^+} \delta\Pi_{ml}\, dz - \sum_{k=1}^{F} \int_{z_{fk}^-}^{z_{fk}^+} \delta\Pi_{fk}\, dz + p(x,y)\, \delta w \right] dx\, dy = 0 \tag{17}$$

The expressions for the strain energy for the matrix and fiber strata are given by

$$\Pi_{ml} = \frac{1}{2}\{\epsilon\}_{ml}^{T}[K]_{ml}^{T}\{\epsilon\}_{ml}$$
$$\Pi_{fk} = \frac{1}{2}\{\epsilon\}_{fk}^{T}[S]_{fk}^{T}\{\epsilon\}_{fk} \tag{18}$$

where F = the number of fiber strata, and M = the number of matrix strata. The strain energy is derived using the assumed displacement field expressions, along with the [K] and [S] matrices given above. The work and strain energy expressions are combined, and the principle of minimum potential energy is applied.

Since the z dependence has already been assumed, the z derivatives that arise from the expressions for strain can be evaluated directly, and integration with respect to z can be performed. However, integration by parts is necessary to produce the boundary conditions, and to eliminate the x and y derivatives of the variations on w, ψ_j, and ζ_j. The governing PDEs look like higher order plate theory with coefficients that depend on material (matrix or fiber) properties. The PDEs that result are

$$j\beta_m K_3^{j-1} w_{,x} = \sum_{i=0}^{N} \left\{ \begin{array}{c} -ij\beta_m^2 K_3^{j+i-2}\psi_i + (S_1^{i+j}+K_1^{i+j})\psi_{i,xx} + 2S_5^{i+j}\psi_{i,xy} \\ +(S_4^{i+j}+K_3^{i+j})\psi_{i,yy} + S_5^{i+j}\zeta_{i,xx} + S_6^{i+j}\zeta_{i,yy} \\ +(S_3^{i+j}+S_4^{i+j}+K_2^{i+j}+K_3^{i+j})\zeta_{i,xy} \end{array} \right\} \qquad j=0,1,2,\dots,N \tag{19}$$

$$j\beta_m K_3^{j-1} w_{,y} = \sum_{i=0}^{N} \left\{ \begin{array}{c} -ij\beta_m^2 K_3^{j+i-2}\zeta_i + (S_2^{i+j}+K_1^{i+j})\zeta_{i,yy} + 2S_6^{i+j}\zeta_{i,yx} \\ +(S_4^{i+j}+K_3^{i+j})\zeta_{i,xx} + S_6^{i+j}\psi_{i,yy} + S_5^{i+j}\psi_{i,xx} \\ +(S_3^{i+j}+S_4^{i+j}+K_2^{i+j}+K_3^{i+j})\psi_{i,yx} \end{array} \right\} \qquad j=0,1,2,\dots,N \tag{20}$$

and

$$(K_3^0+S_7^0)w_{,xx}+2S_9^0w_{,xy}+(K_3^0+S_8^0)w_{,yy}+\sum_{i=0}^{N} i\beta_m K_3^{i-1}(\psi_{i,x}+\zeta_{i,y})+p(x,y) = 0 \qquad (21)$$

where

$$\begin{aligned} K_r^n &= \sum_{l=1}^{M} \int_{z_{ml}^-}^{z_{ml}^+} (\alpha_{ml}+\beta_{ml}z)^n K_{r_{ml}} dz \\ S_r^n &= \sum_{k=1}^{F} S_{r_{fk}} z_{fk}^n h_{fk} \end{aligned} \qquad (22)$$

There are 2N+2 sets of boundary conditions that result from this energy derivation, and they are

$$\sum_{i=0}^{N} \left[\begin{array}{l} \sum_{l=0}^{M} \int_{z_{ml}^-}^{z_{ml}^+} (\alpha_{ml}+\beta_{ml}z)^{i+j}(K_{1_{ml}}\psi_{i,x}+K_{2_{ml}}\zeta_{i,y})\ dz \\ +\sum_{k=1}^{F} z_{fk}^{i+j}[S_{1_{fk}}\psi_{i,x}+S_{3_{fk}}\zeta_{i,y}+S_{5_{fk}}(\psi_{i,y}+\zeta_{i,x})]\,h_{fk} \end{array} \right] \delta\psi_j \Bigg|_0^a = 0 \qquad (23)$$

$$\sum_{i=0}^{N} \left[\begin{array}{l} \sum_{l=0}^{M} \int_{z_{ml}^-}^{z_{ml}^+} (\alpha_{ml}+\beta_{ml}z)^{i+j}K_{3_{ml}}(\psi_{i,y}+\zeta_{i,-})\ dz \\ +\sum_{k=1}^{F} z_{fk}^{i+j}[S_{5_{fk}}\psi_{i,x}+S_{6_{fk}}\zeta_{i,y}+S_{4_{fk}}(\psi_{i,y}+\zeta_{i,x})]\,h_{fk} \end{array} \right] \delta\zeta_j \Bigg|_0^a = 0 \qquad (24)$$

$$\sum_{i=0}^{N} \left[\begin{array}{l} \sum_{l=0}^{M} \int_{z_{ml}^-}^{z_{ml}^+} (\alpha_{ml}+\beta_{ml}z)^{i+j}(K_{2_{ml}}\psi_{i,x}+K_{1_{ml}}\zeta_{i,y})\ dz \\ +\sum_{k=1}^{F} z_{fk}^{i+j}[S_{3_{fk}}\psi_{i,x}+S_{2_{fk}}\zeta_{i,y}+S_{6_{fk}}(\psi_{i,y}+\zeta_{i,x})]\,h_{fk} \end{array} \right] \delta\zeta_j \Bigg|_0^b = 0 \qquad (25)$$

$$\sum_{i=0}^{N}\left[\begin{array}{c}\displaystyle\sum_{l=0}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}(\alpha_{ml}+\beta_{ml}z)^{i+j}K_{3_{ml}}(\psi_{i,y}+\zeta_{i,x})\ dz\\ \displaystyle+\sum_{k=1}^{F}z_{fk}^{i+j}[S_{5_{fk}}\psi_{i,x}+S_{6_{fk}}\zeta_{i,y}+S_{4_{fk}}(\psi_{i,y}+\zeta_{i,x})]\,h_{fk}\end{array}\right]\delta\,\psi_j\Bigg|_0^b = 0 \tag{26}$$

$$\left[\begin{array}{c}\displaystyle\sum_{l=1}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}\sum_{i=1}^{N}i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}K_{3_{ml}}\psi_i+K_{3_{ml}}w_{,x}dz\\ \displaystyle+\sum_{k=1}^{F}[S_{7_{fk}}w_{,x}+S_{9_{fk}}w_{,y}]\,h_{fk}\end{array}\right]\delta\,w\Bigg|_0^a = 0 \tag{27}$$

$$\left[\begin{array}{c}\displaystyle\sum_{l=1}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}\sum_{i=1}^{N}i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}K_{3_{ml}}\zeta_i+K_{3_{ml}}w_{,y}dz\\ \displaystyle+\sum_{k=1}^{F}[S_{9_{fk}}w_{,x}+S_{8_{fk}}w_{,y}]\,h_{fk}\end{array}\right]\delta\,w\Bigg|_0^b = 0 \tag{28}$$

III. The "Exact Stratified Plate"

Basic Concepts and Assumptions

The Exact Stratified Plate model, hereafter referred to as ESP, is intended to be a standard against which the results of ST can be compared. Both ST and ESP are based on the same Stratified Plate Model, where it is assumed that z-direction displacement gradients are zero through the fiber strata. Further, ESP is somewhat similar to ST in that the differential equations are developed using the principle of minimum potential energy. However, that is where the similarity ends.

There are two primary differences between ESP and ST methods. The first is that ESP allows ϵ_z to be non-zero through the matrix. Recall that one of the first assumptions of ST was that ϵ_z could be neglected in both the fiber and matrix strata. This approximation seems reasonable, but lacks analytical justification. By allowing ϵ_z to be nonzero, ESP will be useful for estimating the effects of assuming $\epsilon_z=0$.

The second major difference between ESP and ST is that ESP is restricted to modeling only simply-supported composites with cross-plies. This limitation arises because of the choice of the assumed displacement field. For this analysis, ESP assumes that the form of the x and y dependence of the displacement field is the same as the ST displacement field expressions used for the simply-supported example problems. This is not really a drawback, it simply means that ESP and ST can only be directly compared for a simply-supported plate.

Another consequence of using this form of the assumed x and y dependence is that it cannot accommodate angle plies. In developing the governing differential equations, all coupling terms that arise from having angle plies would not have the same x and y dependence as the rest of the terms. If these coupling terms were nonzero, the x and y dependent functions would not result in the set of ODEs that result from this analysis. Therefore, only a composite with cross-plies can be modeled using the ESP approach.

For ST, it was assumed that the z dependence of u and v could be described by a polynomial in z. However, the ESP solution leaves the form of the z dependence through the matrix strata as unknown functions that are determined by the system of ODE's that result. Since this approach lets the problem dictate what the z dependence should be, ESP can be used to gauge how well the polynomial used in ST represents the more complex stress, strain, and displacement behavior that a composite would experience.

Development of the ODEs

All strain terms are treated as being nonzero for the matrix, but strains through the fibers are assumed to have no z dependence. Therefore, the expressions for the strains become

$$\{\epsilon\}_f = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}_f = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ 0 \\ u_{,y} + v_{,x} \\ w_{,x} \\ w_{,y} \end{Bmatrix}_f \qquad \{\epsilon\}_m = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}_m = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ u_{,z} + w_{,x} \\ v_{,z} + w_{,y} \end{Bmatrix}_m \tag{29}$$

The ESP solution is developed using the principle of minimum potential energy, just as with ST. The strain energy is initially developed using the expressions for the strains shown above. Integration-by-parts is performed to eliminate the derivatives of the variational terms. This results in a set of PDEs and boundary conditions. The integration-by-parts produces a set of stress/displacement boundary condition expressions for the 4 "edges" of the composite plate, as well as for the horizontal (x-y plane) surfaces of the composite. The resulting expression for the energy is

$$-\sum_{l=1}^{M}\delta U_{ml} - \sum_{k=1}^{F}\delta U_{fk} + \int_0^b\int_0^a p(x,y)\,\delta w\,dx\,dy = 0 \tag{30}$$

where p(x,y) is applied at the top and/or bottom surface ($\pm h/2$),

$$\begin{aligned}
\delta U_{ml} = & -\int_0^b\int_0^a\int_{z_{ml}^-}^{z_{ml}^+}\left\{\begin{array}{l}\left[K_{1_{ml}}u_{,xx}+K_{3_{ml}}u_{,yy}+K_{3_{ml}}u_{,zz}+(K_{2_{ml}}+K_{3_{ml}})(v_{,xy}+w_{,xz})\right]\delta u\\ +\left[K_{3_{ml}}v_{,xx}+K_{1_{ml}}v_{,yy}+K_{3_{ml}}v_{,zz}+(K_{2_{ml}}+K_{3_{ml}})(u_{,xy}+w_{,yz})\right]\delta v\\ +\left[K_{3_{ml}}w_{,xx}+K_{3_{ml}}w_{,yy}+K_{1_{ml}}w_{,zz}+(K_{2_{ml}}+K_{3_{ml}})(u_{,xz}+v_{,yz})\right]\delta w\end{array}\right\}dz\,dy\,dx\\
& + \int_0^b\int_{z_{ml}^-}^{z_{ml}^+}\left[K_{1_{ml}}u_{,x}+K_{2_{ml}}v_{,y}+K_{2_{ml}}w_{,z}\right]\delta u\Big|_0^a dzdy + \int_0^a\int_{z_{ml}^-}^{z_{ml}^+}K_{3_{ml}}(u_{,y}+v_{,x})\delta u\Big|_0^b dzdx\\
& + \int_0^a\int_{z_{ml}^-}^{z_{ml}^+}\left[K_{2_{ml}}u_{,x}+K_{1_{ml}}v_{,y}+K_{2_{ml}}w_{,z}\right]\delta v\Big|_0^b dzdx + \int_0^b\int_{z_{ml}^-}^{z_{ml}^+}K_{3_{ml}}(u_{,y}+v_{,x})\delta v\Big|_0^a dzdy\\
& + \int_0^a\int_0^b\left[K_{2_{ml}}u_{,x}+K_{2_{ml}}v_{,y}+K_{1_{ml}}w_{,z}\right]\delta w\Big|_{z_{ml}^-}^{z_{ml}^+} dydx + \int_0^b\int_{z_{ml}^-}^{z_{ml}^+}K_{3_{ml}}(u_{,z}+w_{,x})\delta w\Big|_0^a dzdy\\
& + \int_0^a\int_0^b K_{3_{ml}}(u_{,z}+w_{,x})\delta u\Big|_{z_{ml}^-}^{z_{ml}^+} dydx + \int_0^a\int_0^b K_{3_{ml}}(v_{,z}+w_{,z})\delta v\Big|_{z_{ml}^-}^{z_{ml}^+} dydx\\
& + \int_0^a\int_{z_{ml}^-}^{z_{ml}^+}K_{3_{ml}}(v_{,z}+w_{,y})\delta w\Big|_0^b dzdx
\end{aligned} \tag{31}$$

and

$$
\begin{aligned}
\delta U_{fk} = & -\int_0^b\int_0^a h_{fk}\left\{\begin{array}{l}\left[S_{1_{fk}}u_{,xx}+S_{4_{fk}}u_{,yy}+(S_{3_{fk}}+S_{4_{fk}})v_{,xy}\right]\delta u\\ +\left[S_{4_{fk}}v_{,xx}+S_{2_{fk}}v_{,yy}+(S_{3_{fk}}+S_{4_{fk}})u_{,xy}\right]\delta v\\ +\left[S_{7_{fk}}w_{,xx}+S_{8_{fk}}w_{,yy}\right]\delta w\end{array}\right\} dy\,dx \\
& + \int_0^b h_{fk}\left[S_{1_{fk}}u_{,x}+S_{3_{fk}}v_{,y}\right]\delta u\Big|_0^a dy + \int_0^a h_{fk}S_{4_{fk}}(u_{,y}+v_{,x})\delta u\Big|_0^b dx \\
& + \int_0^a h_{fk}\left[S_{3_{fk}}u_{,x}+S_{2_{fk}}v_{,y}\right]\delta v\Big|_0^b dx + \int_0^b h_{fk}S_{4_{fk}}(u_{,y}+v_{,x})\delta v\Big|_0^a dy \\
& + \int_0^b h_{fk}S_{7_{fk}}w_{,x}\delta w\Big|_0^a dy + \int_0^a h_{fk}S_{8_{fk}}w_{,y}\delta w\Big|_0^b dx
\end{aligned}
\tag{32}
$$

The term h_{fk} represents the thickness of the k^{th} fiber stratum, which arises from the integration with respect to z through a fiber stratum.

This energy expression results in three "field equations", a set of coupled PDEs, for each matrix stratum. The remaining terms form the boundary conditions for this set of PDEs (see discussion below).

As described above, the x and y portions of the displacements u, v, and w through the matrix are assumed to have the same form as the ST displacement field expressions for a simply-supported composite with cross-plies. The assumed displacement field and loading expressions are given by

$$
\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \\ P(x,y) \end{Bmatrix} = \begin{Bmatrix} U(z)\cos(px)\sin(qy) \\ V(z)\sin(px)\cos(qy) \\ W(z)\sin(px)\sin(qy) \\ \rho\sin(px)\sin(qy) \end{Bmatrix}
\tag{33}
$$

where $p=\pi/a$ and $q=\pi/b$. Since the displacements u, v, and w for the fibers are assumed to have no z dependence, U(z), V(z), and W(z) are constant through the fiber strata.

The form of the surface load "P" dictates this choice of the assumed displacement field. These displacement field expressions and the subsequent differential equations can be generalized for more complex surface loads using Fourier analysis.

When these displacement field expressions are substituted into the PDEs, the x and y derivatives can be evaluated directly, and a set of ODEs for U(z), V(z), and W(z) result. The functions that are used for the variational analysis are the U(z), V(z), and W(z) of the assumed displacement field, and the x- and y-dependent functions can be factored out to produce the following 3 second-order coupled ODE's for each matrix stratum "l".

$$\begin{aligned}
&-\left[p^2K_{1_{ml}}+q^2K_{3_{ml}}\right]U-\left[K_{2_{ml}}+K_{3_{ml}}\right]pqV+\left[K_{2_{ml}}+K_{3_{ml}}\right]pW'+K_{3_{ml}}U'' = 0\\
&-\left[K_{2_{ml}}+K_{3_{ml}}\right]pqU-\left[q^2K_{1_{ml}}+p^2K_{3_{ml}}\right]V+\left[K_{2_{ml}}+K_{3_{ml}}\right]qW'+K_{3_{ml}}V'' = 0\\
&-\left[p^2K_{3_{ml}}+q^2K_{3_{ml}}\right]W-p\left[K_{2_{ml}}+K_{3_{ml}}\right]U'-q\left[K_{2_{ml}}+K_{3_{ml}}\right]V'+K_{1_{ml}}W'' = 0
\end{aligned} \tag{34}$$

The solution of this set of ODEs is obtained by an associated Eigenvalue formulation.

The field equations for each matrix stratum "l" can be manipulated into matrix form to become

$$\begin{aligned}
&\begin{bmatrix} K_{3_{ml}} & 0 & 0 \\ 0 & K_{3_{ml}} & 0 \\ 0 & 0 & K_{1_{ml}} \end{bmatrix}\begin{Bmatrix} U'' \\ V'' \\ W'' \end{Bmatrix} + \begin{bmatrix} 0 & 0 & pK_{4_{ml}} \\ 0 & 0 & qK_{4_{ml}} \\ -pK_{4_{ml}} & -qK_{4_{ml}} & 0 \end{bmatrix}\begin{Bmatrix} U' \\ V' \\ W' \end{Bmatrix}\\
&+\begin{bmatrix} -p^2K_{1_{ml}}-q^2K_{3_{ml}} & -pqK_{4_{ml}} & 0 \\ -pqK_{4_{ml}} & -q^2K_{1_{ml}}-p^2K_{3_{ml}} & 0 \\ 0 & 0 & -K_{3_{ml}}(p^2+q^2) \end{bmatrix}\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}
\end{aligned} \tag{35}$$

where $K_4 = K_2 + K_3$. These can be recast as a set of six first-order coupled ODEs to get

$$\begin{bmatrix} 0 & 0 & 0 & K_{3_{ml}} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{3_{ml}} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{1_{ml}} \\ K_{3_{ml}} & 0 & 0 & 0 & 0 & pK_{4_{ml}} \\ 0 & K_{3_{ml}} & 0 & 0 & 0 & qK_{4_{ml}} \\ 0 & 0 & K_{1_{ml}} & -pK_{4_{ml}} & -qK_{4_{ml}} & 0 \end{bmatrix} \frac{d}{dz} \begin{Bmatrix} U' \\ V' \\ W' \\ U \\ V \\ W \end{Bmatrix}$$

$$- \begin{bmatrix} K_{3_{ml}} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{3_{ml}} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{1_{ml}} & 0 & 0 & 0 \\ 0 & 0 & 0 & p^2K_{1_{ml}}+q^2K_{3_{ml}} & pqK_{4_{ml}} & 0 \\ 0 & 0 & 0 & pqK_{4_{ml}} & q^2K_{1_{ml}}+p^2K_{3_{ml}} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{3_{ml}}(p^2+q^2) \end{bmatrix} \begin{Bmatrix} U' \\ V' \\ W' \\ U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{36}$$

If the solutions for U, V, and W are assumed to have the form of $U_0e^{\lambda z}$, the governing equations can be expressed as a system of 6 coupled algebraic equations

$$\begin{bmatrix} -K_{3_{ml}} & 0 & 0 & \lambda K_{3_{ml}} & 0 & 0 \\ 0 & -K_{3_{ml}} & 0 & 0 & \lambda K_{3_{ml}} & 0 \\ 0 & 0 & -K_{1_{ml}} & 0 & 0 & \lambda K_{1_{ml}} \\ \lambda K_{3_{ml}} & 0 & 0 & -p^2K_{1_{ml}}-q^2K_{3_{ml}} & -pqK_{4_{ml}} & \lambda pK_{4_{ml}} \\ 0 & \lambda K_{3_{ml}} & 0 & -pqK_{4_{ml}} & -q^2K_{1_{ml}}-p^2K_{3_{ml}} & \lambda qK_{4_{ml}} \\ 0 & 0 & \lambda K_{1_{ml}} & -\lambda pK_{4_{ml}} & -\lambda qK_{4_{ml}} & -K_{3_{ml}}(p^2+q^2) \end{bmatrix} \begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \\ U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{37}$$

Rewriting this in state-space form, this becomes

$$[[Q] + \lambda[I]]\{\vec{U}_0\} = \{\vec{0}\} \tag{38}$$

where

$$\{\vec{U}_0\} = \begin{Bmatrix} \lambda U_0 \\ \lambda V_0 \\ \lambda W_0 \\ U_0 \\ V_0 \\ W_0 \end{Bmatrix} \tag{39}$$

and

$$[Q] = \begin{bmatrix} 0 & 0 & p\dfrac{K_{4_{ml}}}{K_{3_{ml}}} & -\left(p^2\dfrac{K_{1_{ml}}}{K_{3_{ml}}}+q^2\right) & -pq\dfrac{K_{4_{ml}}}{K_{3_{ml}}} & 0 \\ 0 & 0 & q\dfrac{K_{4_{ml}}}{K_{3_{ml}}} & -pq\dfrac{K_{4_{ml}}}{K_{3_{ml}}} & -\left(p^2+q^2\dfrac{K_{1_{ml}}}{K_{3_{ml}}}\right) & 0 \\ -p\dfrac{K_{4_{ml}}}{K_{1_{ml}}} & -q\dfrac{K_{4_{ml}}}{K_{1_{ml}}} & 0 & 0 & 0 & -(p^2+q^2)\dfrac{K_{3_{ml}}}{K_{1_{ml}}} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

The eigenvalues of this system are two sets of three repeated roots. The two values of the roots are

$$\lambda = \pm\sqrt{p^2 + q^2} \tag{41}$$

Note that these roots are independent of the material stiffness coefficients.

The repeated roots means that an ordinary eigenvector solution technique cannot be employed here. However, Chen [1:53-57] outlines a more general method for generating an eigenvector matrix.

The basis of this method is as follows: let [A] be an nxn matrix with n eigenvalues (they do not have to be distinct), and f(λ) be a function defined on the spectrum of [A]. If a polynomial g(λ) is constructed such that g(λ) and f(λ) have the same values on the spectrum of [A], then f([A]) can be calculated using the polynomial defined by g([A]). When the n eigenvalues are distinct, f(λ) and g(λ) must be equated to solve for the coefficients in the polynomial g(λ). When there are repeated roots, derivatives of f(λ) and g(λ) must be equated as well.

For the eigenvalue problem given by ESP, this approach starts by defining the following terms

$$\begin{aligned} g(z,\lambda) &= \alpha_0(z) + \alpha_1(z)\lambda + \alpha_2(z)\lambda^2 + \alpha_3(z)\lambda^3 + \alpha_4(z)\lambda^4 + \alpha_5(z)\lambda^5 \\ f(z,\lambda) &= e^{\lambda z} \end{aligned} \tag{42}$$

where the α_i's are unknown functions of z, and λ is an eigenvalue as defined above.

Since there are repeated roots, the derivatives of f(z,λ) and g(z,λ) are equated such that

$$f(z,\lambda) = g(z,\lambda)\ ,\ \ \frac{d}{d\lambda}f(z,\lambda) = \frac{d}{d\lambda}g(z,\lambda)\ ,\ \textit{and}\ \ \frac{d^2}{d\lambda^2}f(z,\lambda) = \frac{d^2}{d\lambda^2}g(z,\lambda) \tag{43}$$

Next, these expressions are manipulated to solve for the α_i's. For this problem, when there are two sets of triple roots, the α_i's are found by the relation

$$
\begin{Bmatrix} f(z,\lambda) \\ f'(z,\lambda) \\ f''(z,\lambda) \\ f(z,-\lambda) \\ f'(z,-\lambda) \\ f''(z,-\lambda) \end{Bmatrix} = \begin{Bmatrix} e^{\lambda z} \\ z e^{\lambda z} \\ z^2 e^{\lambda z} \\ e^{-\lambda z} \\ z e^{-\lambda z} \\ z^2 e^{-\lambda z} \end{Bmatrix} = \begin{bmatrix} 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \lambda^5 \\ 0 & 1 & 2\lambda & 3\lambda^2 & 4\lambda^3 & 5\lambda^4 \\ 0 & 0 & 2 & 6\lambda & 12\lambda^2 & 20\lambda^3 \\ 1 & -\lambda & \lambda^2 & -\lambda^3 & \lambda^4 & -\lambda^5 \\ 0 & 1 & -2\lambda & 3\lambda^2 & -4\lambda^3 & 5\lambda^4 \\ 0 & 0 & 2 & -6\lambda & 12\lambda^2 & -20\lambda^3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \tag{44}
$$

Finally, exponentiation of a matrix [Q] can be obtained by the relation

$$
e^{[Q]z} = \alpha_0[I] + \alpha_1[Q] + \alpha_2[Q]^2 + \alpha_3[Q]^3 + \alpha_4[Q]^4 + \alpha_5[Q]^5 \tag{45}
$$

Assembling the Boundary Conditions

The PDEs are solved subject to boundary conditions along the four edges (at x=0, x=a, y=0, and y=b), at the top and bottom horizontal faces, as well as at each fiber stratum. At the fiber strata, these internal boundary conditions represent stresses τ_{xz}, τ_{yz}, and σ_z at the top and bottom surfaces of each matrix stratum. External loads (P) applied at the upper and lower surfaces of the composite are also incorporated into these boundary conditions.

When the x and y portions of the displacements u, v, and w through the matrix are assumed to have the same form as the ST displacement field expressions for a simply-supported composite with cross-plies, all the edge boundary conditions are satisfied exactly.

The assumption of no z-gradients of displacement through the fiber strata also implies a set of continuity boundary conditions. The values of u, v, or w are constant through a fiber stratum, therefore values of U, V, and W at the bottom of the matrix stratum just above this fiber stratum must be equal to the corresponding values at the top of the matrix stratum just below the fiber stratum.

Since U, V, and W are constant through each fiber stratum, integration through the fibers can be performed directly. The results of the integration through a fiber stratum are combined with the boundary condition terms of the matrix strata immediately above and below this fiber stratum. In other words, the sum of all terms that are evaluated at the i^{th} fiber stratum (the terms at the bottom of the upper matrix stratum, the fiber stratum itself, and the top of the matrix stratum below) must sum to zero. This forms a set of internal stress boundary conditions between the matrix strata.

For M matrix strata, there will be 3(M-1) internal stress boundary conditions, and the presence (or lack of) external loads at the upper and lower surfaces of the composite, i.e. at $z=h/2$ or $z=-h/2$, specify a total of 6 more stress boundary conditions. Additionally, it is necessary to specify 3(M-1) displacement (continuity) boundary conditions to tie the bottom surface of one matrix stratum to the top of the next matrix stratum below it. In all, there will be a total of 6M boundary conditions (both stress and continuity) specified for M matrix strata that will be used to solve for the functions U, V, and W.

The exponential expressions for U, V, and W describes how U, V, and W vary with z, subject to the boundary conditions specified in Equations (31) and (32). The boundary conditions are assembled into a series of stress balance equations for τ_{xz}, τ_{yz}, and σ_z. Equilibrium is established at the upper surface, at each fiber stratum, and finally at the lower surface. For a composite with two fiber plies, this would be given as

$$\left\{ \begin{matrix} 0 \\ 0 \\ \begin{pmatrix} Normal\ Stress \\ @\ h/2 \end{pmatrix} \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \begin{pmatrix} Normal\ Stress \\ @\ -h/2 \end{pmatrix} \end{matrix} \right\} = [\ BC\ ] \left\{ \begin{matrix} \vec{U}\,\big|_{z=h/2} \\ \vec{U}\,\big|_{z=z_{m1}^-} \\ \vec{U}\,\big|_{z=z_{m2}^+} \\ \vec{U}\,\big|_{z=z_{m2}^-} \\ \vec{U}\,\big|_{z=z_{m3}^+} \\ \vec{U}\,\big|_{z=-h/2} \end{matrix} \right\} \tag{46}$$

where

$$[BC] = \begin{bmatrix} M1_1 & M2_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -M1_1 & -M2_1 & M1_2 & M2_{f1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -M1_2 & -M2_2 & M1_3 & M2_{f2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -M1_3 & -M2_3 \end{bmatrix} \tag{47}$$

and

$$M1_l = \begin{bmatrix} K_{3_{ml}} & 0 & 0 \\ 0 & K_{3_{ml}} & 0 \\ 0 & 0 & K_{1_{ml}} \end{bmatrix} \qquad M2_l = \begin{bmatrix} 0 & 0 & pK_{3_{ml}} \\ 0 & 0 & qK_{3_{ml}} \\ -pK_{2_{ml}} & -qK_{2_{ml}} & 0 \end{bmatrix} \tag{48}$$

$$M2_{fk} = \begin{bmatrix} p^2S_{1_{fk}}+q^2S_{4_{fk}} & pq(S_{3_{fk}}+S_{4_{fk}}) & 0 \\ pq(S_{3_{fk}}+S_{4_{fk}}) & p^2S_{4_{fk}}+q^2S_{2_{fk}} & 0 \\ 0 & 0 & p^2S_{7_{fk}}+q^2S_{8_{fk}} \end{bmatrix} h_{fk} + M2_{k+1}$$

The submatrices $M1_l$ and $M2_l$ are the boundary conditions given in Equation (31) expressed in matrix form. Submatrix $M2_{fk}$ incorporates the interstrata boundary conditions imposed by the fiber strata given in Equation (32). Continuity of displacement across the fiber strata is established by the 3x3 identity submatrices, denoted by "I" above.

Determining the solution for U(z), V(z), and W(z) depends on finding a U_0, V_0, and W_0 for each matrix stratum. Another matrix must be constructed to specify the relationship between the values of U, V, and W at the upper and lower surfaces of a matrix stratum. The matrix shown below expresses U, V, and W at the top surface of each matrix stratum in terms of U, V, and W at the bottom of the matrix stratum. Expressed this way, the values of U_0, V_0, and W_0 that are found are equal to U, V, and W at the lower surface of the matrix stratum.

$$\begin{Bmatrix} \vec{U}\big|_{z=h/2} \\ \vec{U}\big|_{z=z_{m1}^-} \\ \vec{U}\big|_{z=z_{m2}^+} \\ \vec{U}\big|_{z=z_{m2}^-} \\ \vec{U}\big|_{z=z_{m3}^+} \\ \vec{U}\big|_{z=-h/2} \end{Bmatrix} = \begin{bmatrix} e^{[Q]\left(\frac{h}{2}-z_{m1}^-\right)} & [0] & [0] \\ [I]^{6\times 6} & [0] & [0] \\ [0] & e^{[Q]\left(z_{m2}^+-z_{m2}^-\right)} & [0] \\ [0] & [I]^{6\times 6} & [0] \\ [0] & [0] & e^{[Q]\left(z_{m3}^+ + \frac{h}{2}\right)} \\ [0] & [0] & [I]^{6\times 6} \end{bmatrix} \begin{Bmatrix} \vec{U}_0\big|_{m1} \\ \vec{U}_0\big|_{m2} \\ \vec{U}_0\big|_{m3} \end{Bmatrix} \tag{49}$$

where

$$\vec{U}_0\Big|_{ml} = \begin{Bmatrix} \lambda U_0 \\ \lambda V_0 \\ \lambda W_0 \\ U_0 \\ V_0 \\ W_0 \end{Bmatrix} \tag{50}$$

evaluated for matrix stratum "1". The displacements for any value of z are found using the expression

$$\vec{U}(z) = e^{[Q](z-z_m^-)}\,\vec{U}_0 \qquad (51)$$

where z_m^- is the z-coordinate of the lower surface of the matrix stratum in which "z" lies.

Equation (49) is now substituted into Equation (46), so that the external loads are expressed in terms of the unknowns U_0, V_0, and W_0. The coefficients U_0, V_0, and W_0 are found by inverting the matrix and then multiplying the inverse by the force vector.

IV. A Simply-Supported Square Plate with Cross-Plies

Strata Theory

The strata theory's assumed displacement field does not, at the outset, explicitly assume the x and y dependence of u, v, and w. The restriction of the problem to a simply-supported plate with cross-plies permits the use of sinusoidal functions to describe the x and y dependencies in the general PDEs. The x and y dependence can be assumed to have the form

$$\begin{Bmatrix} w(x,y) \\ \psi_i(x,y) \\ \zeta_i(x,y) \\ P(x,y) \end{Bmatrix} = \begin{Bmatrix} A\sin(px)\sin(qy) \\ B_i\cos(px)\sin(qy) \\ C_i\sin(px)\cos(qy) \\ \wp\sin(px)\sin(qy) \end{Bmatrix} \qquad i=1,2,\ldots N \tag{52}$$

where $p=\pi/a$ and $q=\pi/b$. The form of the surface load "P" dictates this choice of the assumed displacement field. These displacement field expressions and the subsequent differential equations can be generalized for more complex surface loads using Fourier analysis.

Considering a composite with cross-plies further simplifies the equations previously developed, since it says that $S_5=S_6=S_9=0$. When the assumed displacement field expressions are substituted into Equations (19), (20), and (21), the x and y dependence drops out entirely. This results in a system of algebraic equations in the unknown coefficients A, B_i, and C_i

$$0 = A[\, j\beta_m K_3^{j-1}]p + \sum_{i=0}^{N} \left\{ \begin{array}{l} B_i \left[\begin{array}{c} ij\beta_m^2 K_3^{j+i-2} + p^2 S_1^{i+j} + \\ p^2 K_1^{i+j} + q^2 S_4^{i+j} + q^2 K_3^{i+j} \end{array} \right] + \\ C_i \left[\, S_3^{i+j} + S_4^{i+j} + K_2^{i+j} + K_3^{i+j} \,\right] pq \end{array} \right\} \qquad j=0,1,2,\ldots,N \tag{53}$$

$$0 = A[\, j\beta_m K_3^{j-1}]q + \sum_{i=0}^{N} \left\{ \begin{array}{l} C_i \left[\begin{array}{c} ij\beta_m^2 K_3^{j+i-2} + q^2 S_2^{i+j} + \\ q^2 K_1^{i+j} + p^2 S_4^{i+j} + p^2 K_3^{i+j} \end{array} \right] + \\ B_i \left[\, S_3^{i+j} + S_4^{i+j} + K_2^{i+j} + K_3^{i+j} \,\right] pq \end{array} \right\} \qquad j=0,1,2,\ldots,N \tag{54}$$

and

$$\rho = A \left[(K_3^0 + S_7^0)p^2 + (K_3^0 + S_8^0)q^2 \right] + \sum_{i=0}^{N} (i\beta_m K_3^{i-1}) \left[B_i p + C_i q \right] \tag{55}$$

When this is expressed in matrix form, this system of equations can be decomposed into a matrix of constants multiplied by the vector of unknown coefficients A, B_i, and C_i.

Using the same displacement functions for u, v, and w in Equations (23) through (28), the associated boundary conditions are now given by

$$\sum_{i=0}^{N} \left[\begin{array}{c} \displaystyle\sum_{l=0}^{M} \int_{z_{ml}^-}^{z_{ml}^+} (\alpha_{ml} + \beta_{ml} z)^{i+j} (pK_{1_{ml}} B_i + qK_{2_{ml}} C_i)\, dz \\ \displaystyle + \sum_{k=1}^{F} z_{fk}^{i+j} (pS_{1_{fk}} B_i + qS_{3_{fk}} C_i) h_{fk} \end{array} \right] \sin(px)\sin(qy)\, \delta\psi_j \Bigg|_0^a = 0 \tag{56}$$

$$\sum_{i=0}^{N}\left[\begin{array}{c}\displaystyle\sum_{l=0}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}(\alpha_{ml}+\beta_{ml}z)^{i+j}K_{3_{ml}}(qB_i+pC_i)\ dz\\ \displaystyle+\sum_{k=1}^{F}z_{fk}^{i+j}S_{4_{fk}}(qB_i+pC_i)\,h_{fk}\end{array}\right]\cos(px)\cos(qy)\,\delta\zeta_j\Bigg|_0^a = 0 \tag{57}$$

$$\sum_{i=0}^{N}\left[\begin{array}{c}\displaystyle\sum_{l=0}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}(\alpha_{ml}+\beta_{ml}z)^{i+j}(pK_{2_{ml}}B_i+qK_{1_{ml}}C_i)\ dz\\ \displaystyle+\sum_{k=1}^{F}z_{fk}^{i+j}(pS_{3_{fk}}B_i+qS_{2_{fk}}C_i)\,h_{fk}\end{array}\right]\sin(px)\sin(qy)\,\delta\zeta_j\Bigg|_0^b = 0 \tag{58}$$

$$\sum_{i=0}^{N}\left[\begin{array}{c}\displaystyle\sum_{l=0}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}(\alpha_{ml}+\beta_{ml}z)^{i+j}K_{3_{ml}}(qB_i+pC_i)\ dz\\ \displaystyle+\sum_{k=1}^{F}z_{fk}^{i+j}S_{4_{fk}}(qB_i+pC_i)\,h_{fk}\end{array}\right]\cos(px)\cos(qy)\,\delta\psi_j\Bigg|_0^b = 0 \tag{59}$$

$$\left[\begin{array}{c}\displaystyle\sum_{l=1}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}\sum_{i=1}^{N}i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}K_{3_{ml}}B_i+pK_{3_{ml}}A\,dz\\ \displaystyle+\sum_{k=1}^{F}pS_{7_{fk}}A\,h_{fk}\end{array}\right]\cos(px)\sin(qy)\,\delta w\Bigg|_0^a = 0 \tag{60}$$

$$\left[\begin{array}{c}\displaystyle\sum_{l=1}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}\sum_{i=1}^{N}i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}K_{3_{ml}}C_i+qK_{3_{ml}}A\,dz\\ \displaystyle+\sum_{k=1}^{F}qS_{8_{fk}}A\,h_{fk}\end{array}\right]\sin(px)\cos(qy)\,\delta w\Bigg|_0^b = 0 \tag{61}$$

By the definition of a simply supported plate, $\delta u=0$ at $y=0$ and $y=b$, $\delta v=0$ at $x=0$ and $x=a$, and $\delta w=0$ at all four edges. This accounts for the boundary equations given in Equations (57), (59), (60), and (61). Equations (56) and (58) are satisfied because σ_x and σ_y are defined to be zero at the edges of a simply supported plate.

Specific Results

Comparison calculations were made for both isotropic plates and 2-ply composite lay-ups. The following values were assumed for these test cases

Table 1. Assumed values for comparison calculations (square plate)

<table>
<tr><td colspan="2">$E_f = 2.0\times10^7$</td><td colspan="2">$\nu_f = 0.35$</td><td colspan="2">$vol_f = \sqrt{2}/2$</td></tr>
<tr><td colspan="2">$E_m = 3.0\times10^5$</td><td colspan="2">$\nu_m = 0.40$</td><td colspan="2">$vol_m = 1-\sqrt{2}/2$</td></tr>
<tr><td colspan="2">$p = \pi/a$</td><td colspan="2">$a = 10$</td><td colspan="2">$\wp = 1$</td></tr>
<tr><td colspan="2">$q = \pi/b$</td><td colspan="2">$b = 10$</td><td colspan="2">$h = 1$</td></tr>
<tr><td colspan="3">$\theta_{f1} = 0°$</td><td colspan="3">$h_{f1} = \sqrt{2}/4$</td></tr>
<tr><td colspan="3">$\theta_{f2} = 90°$</td><td colspan="3">$h_{f2} = \sqrt{2}/4$</td></tr>
<tr><td colspan="3">$\alpha_{m1} = -(1+\sqrt{2})/2$</td><td colspan="3">$\beta_{m1} = 2+\sqrt{2}$</td></tr>
<tr><td colspan="3">$\alpha_{m2} = 0$</td><td colspan="3">$\beta_{m2} = 2+\sqrt{2}$</td></tr>
<tr><td colspan="3">$\alpha_{m3} = (1+\sqrt{2})/2$</td><td colspan="3">$\beta_{m3} = 2+\sqrt{2}$</td></tr>
</table>

Volume fraction terms vol_m and vol_f were derived by assuming global volume fractions of 50% fibers and 50% matrix. Recall that vol_m and vol_f refer to the fractions of matrix and fiber in an idealized fiber ply, whereas the term "global volume fraction" refers to the fractions of matrix and fibers that exist before the fiber plies are idealized and dictates the thicknesses of the matrix and fiber plies.

A fifth-order polynomial in z for the ST displacement field is assumed, i.e. N=5 in Equations (2) and (3), unless otherwise indicated. Also, the applied load in ESP is split half and half between the upper and lower surfaces of the plate. Loading the plate this way introduces symmetry into the ESP solution that approximates the way a load is applied in ST.

The values for h and $\mathcal{P}$ were chosen to normalize the governing equations. The x, y, and z coordinates, plate dimensions and displacements are normalized to the plate thickness (x/h, y/h, z/h, a/h, b/h, u/h, v/h, and w/h). Also, the elastic moduli are normalized to the applied pressure ($E_m/\mathcal{P}$ and $E_f/\mathcal{P}$).

The Isotropic Square Plate

Since the plate consists of matrix only and fibers are not present, E_f and ν_f are set to zero in the governing equations. When these terms are set to zero and N=1, the ST equations simplify to the Mindlin plate equations [7:33]. Application of the ESP approach to an isotropic plate is the same as for a composite, except that no inter-strata boundary conditions exists.

The complete set of displacements, stresses and strains are shown in Figures 6. through 21. The displacements are obtained directly for both ST and ESP. The strains are derived from the displacement information, and stresses are calculated using the constitutive stress-strain relations, given in Equations (9) and (16).

The following plots for the stresses, strains, and displacements include only the z-dependent portion of these parameters. In other words, the x- and y-dependent sine and cosine functions are divided out for the presentation of this data.

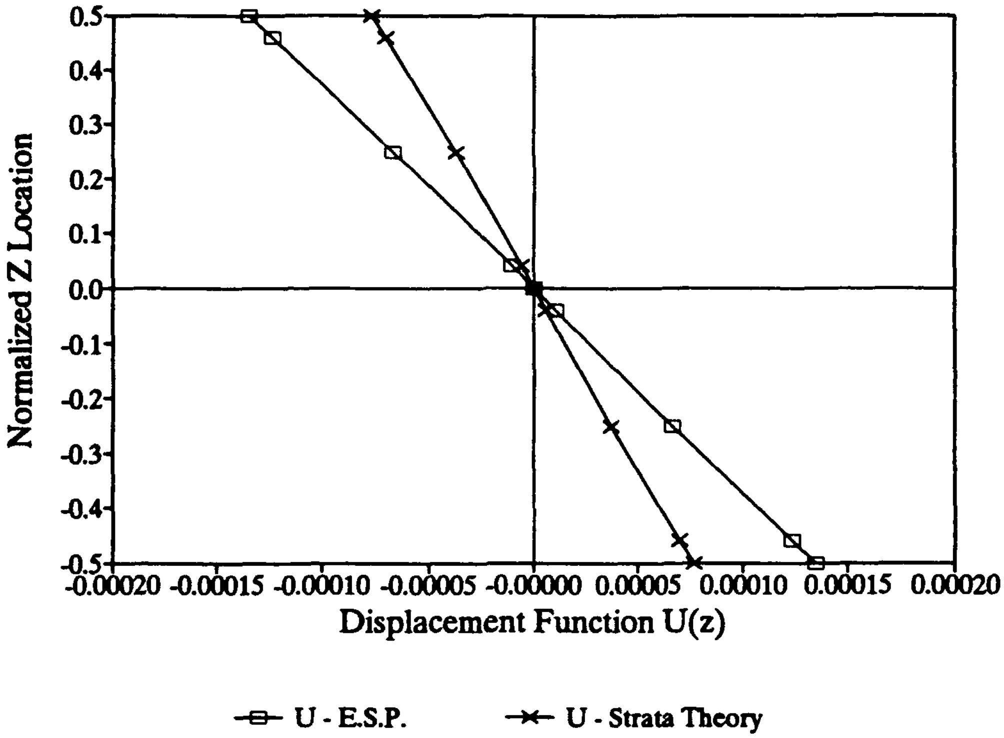

Figure 6. In-plane displacement u for square isotropic plate

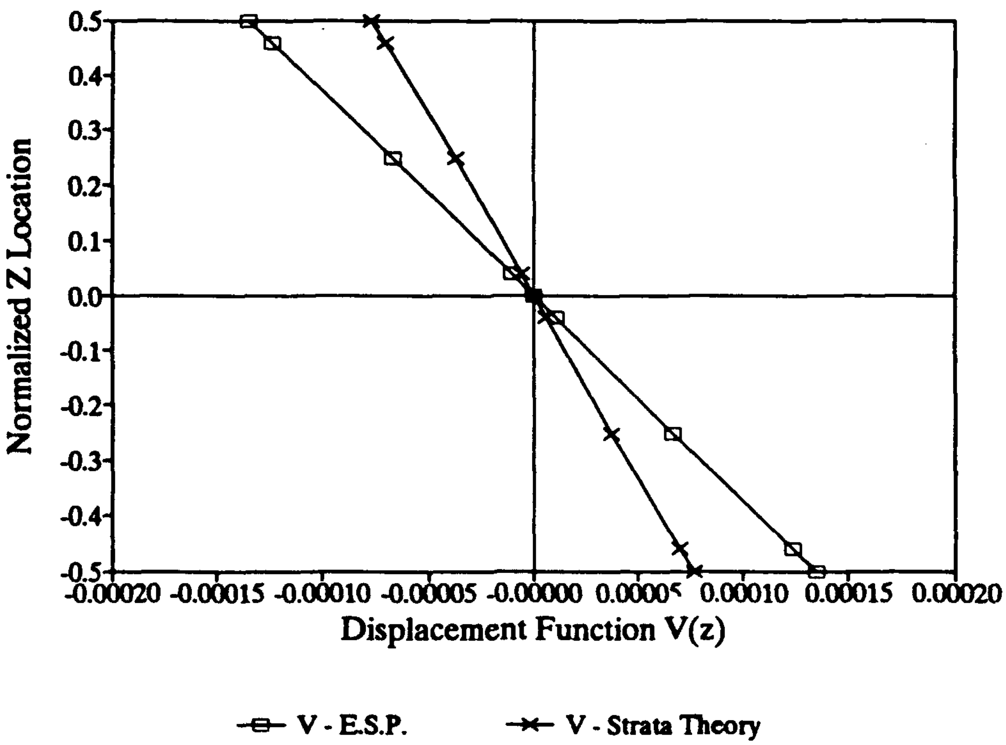

Figure 7. In-plane displacement v for square isotropic plate

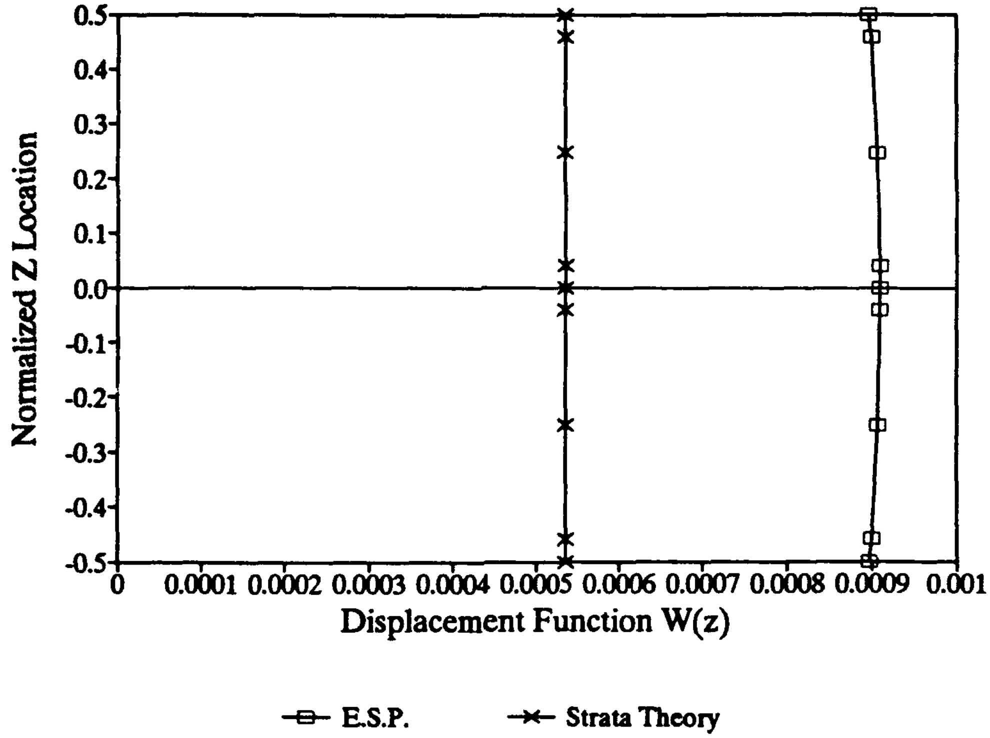

Figure 8. Out-of-plane displacement w for square isotropic plate

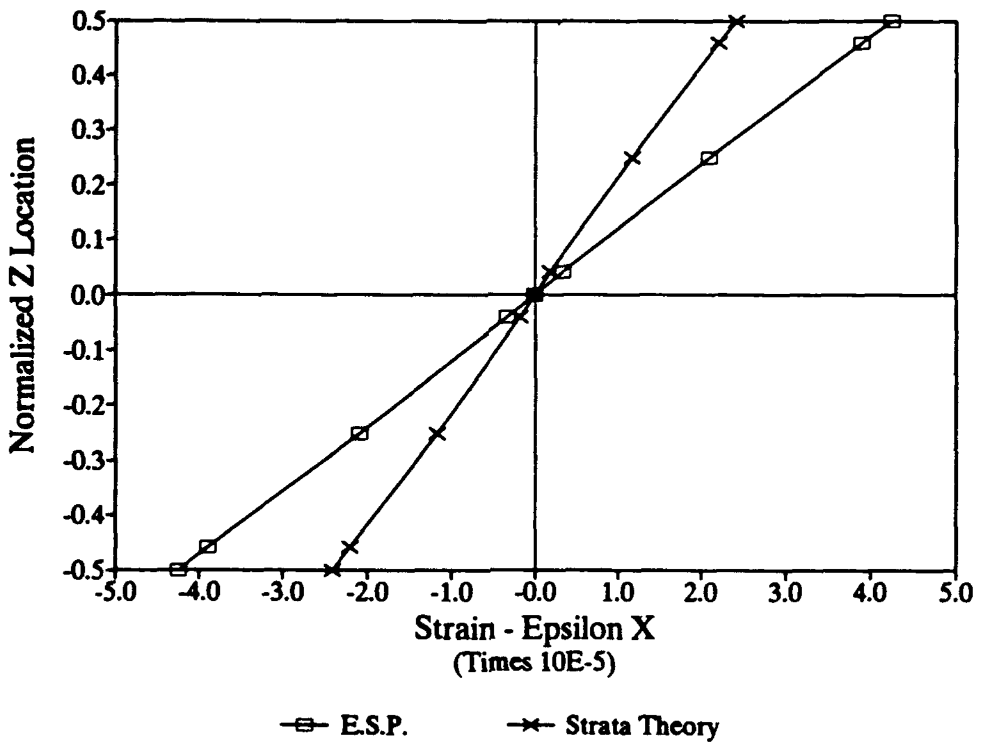

Figure 9. ϵ_x for square isotropic plate

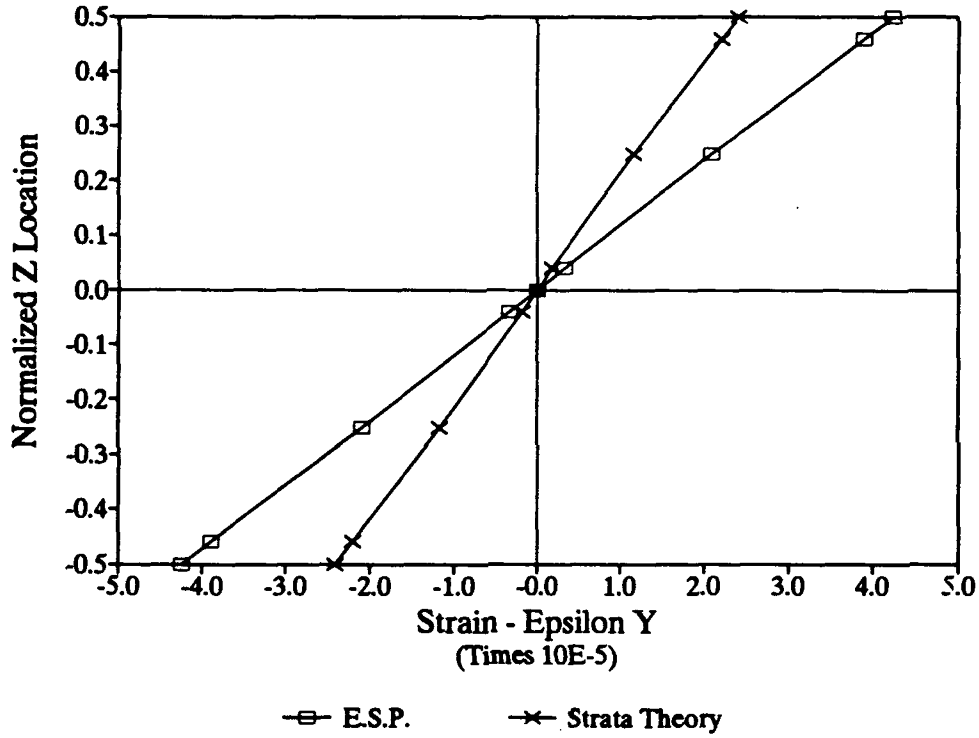

Figure 10. ϵ_y for square isotropic plate

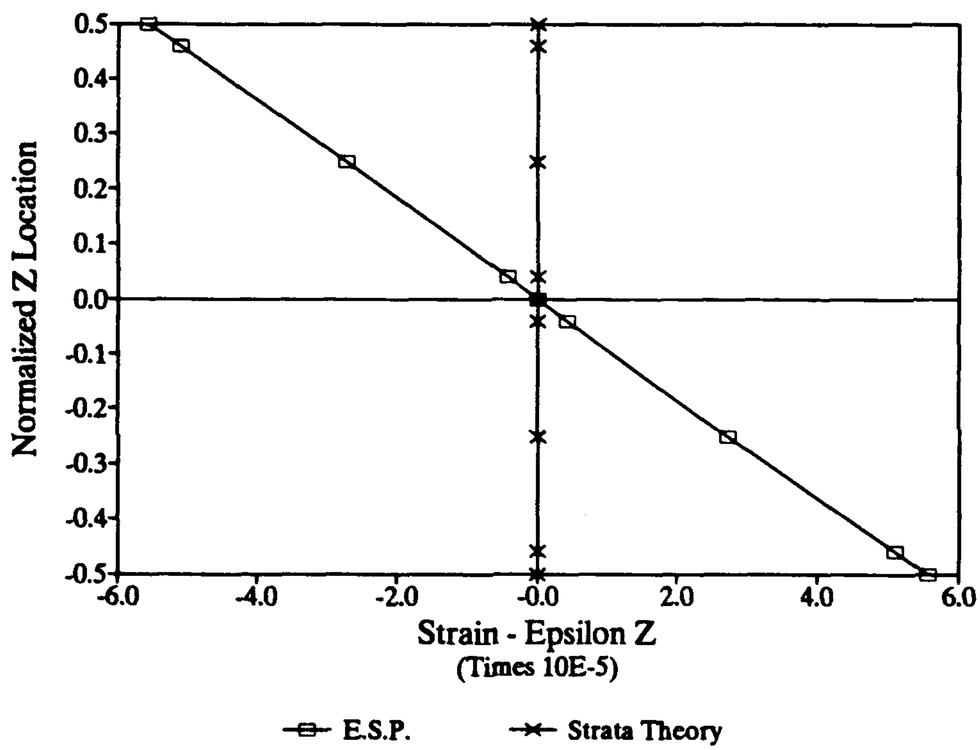

Figure 11. ϵ_z for square isotropic plate

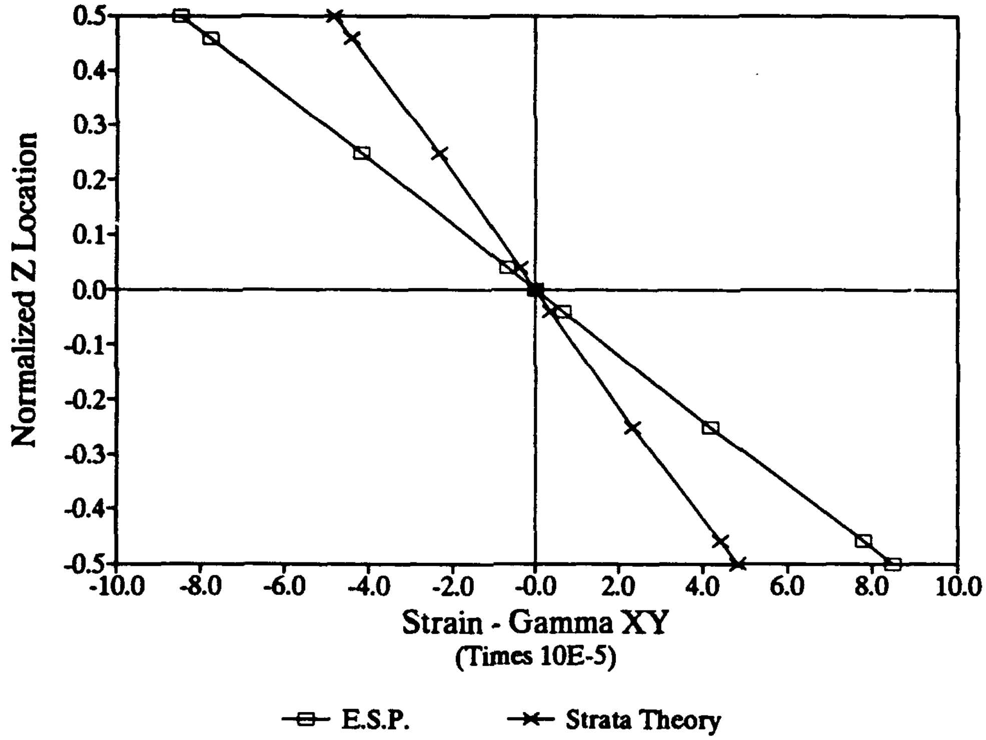

Figure 12. γ_{xy} for square isotropic plate

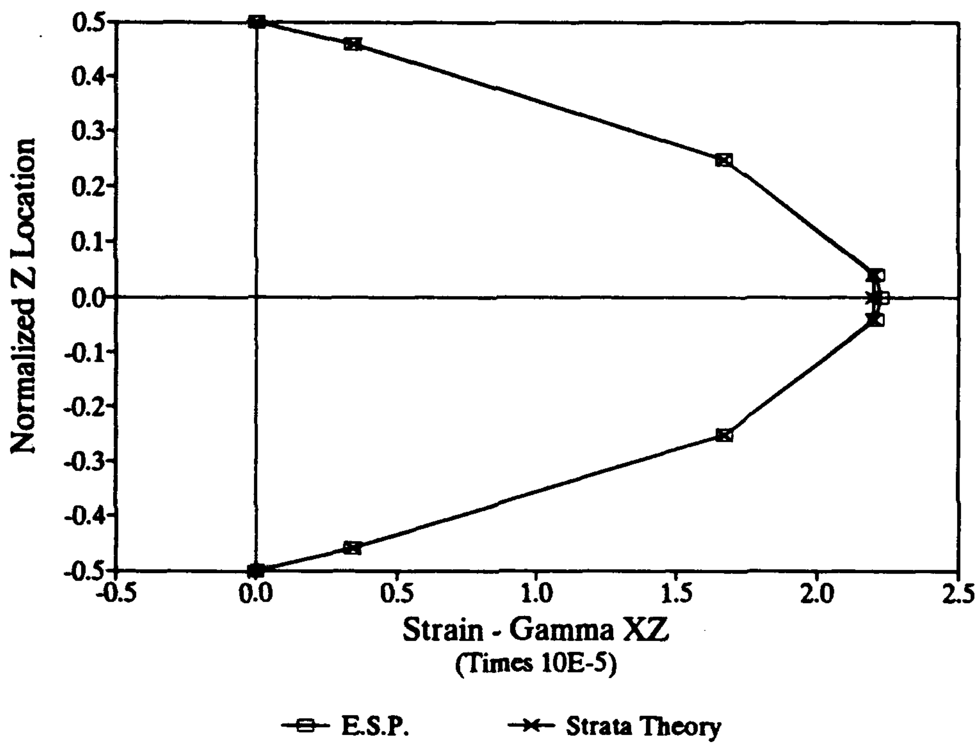

Figure 13. γ_{xz} for square isotropic plate

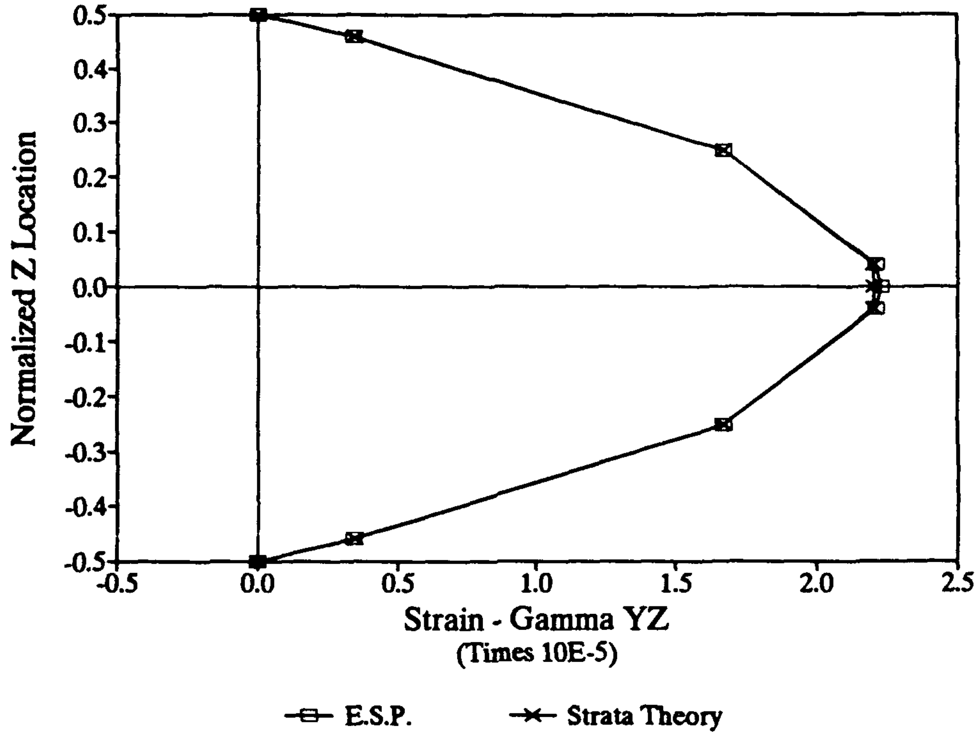

Figure 14. γ_{yz} for square isotropic plate

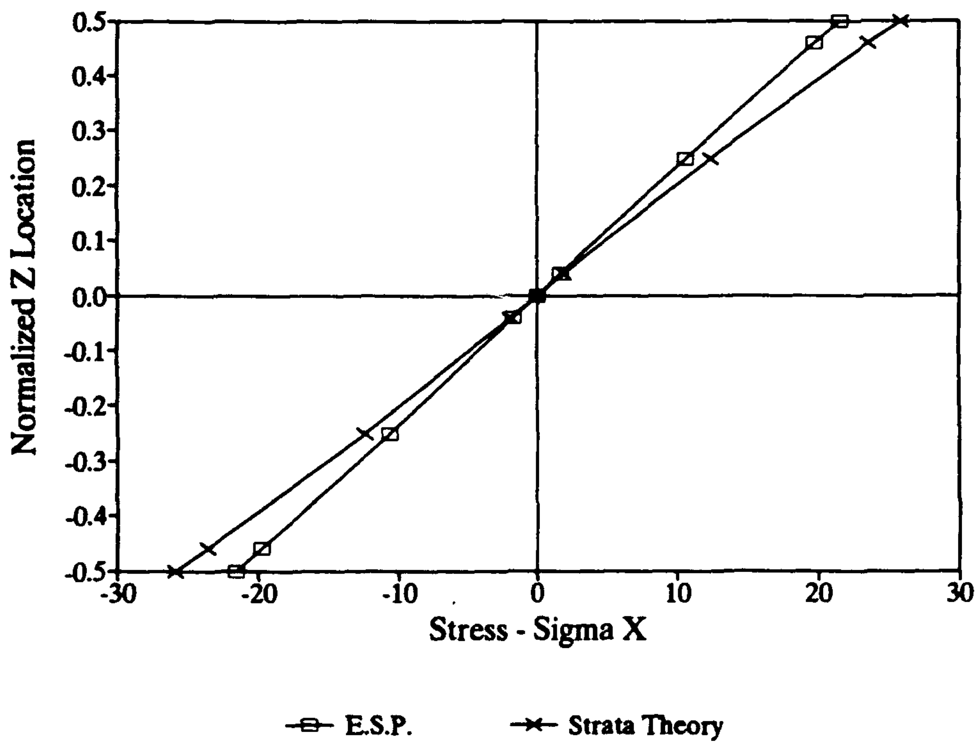

Figure 15. σ_x for square isotropic plate

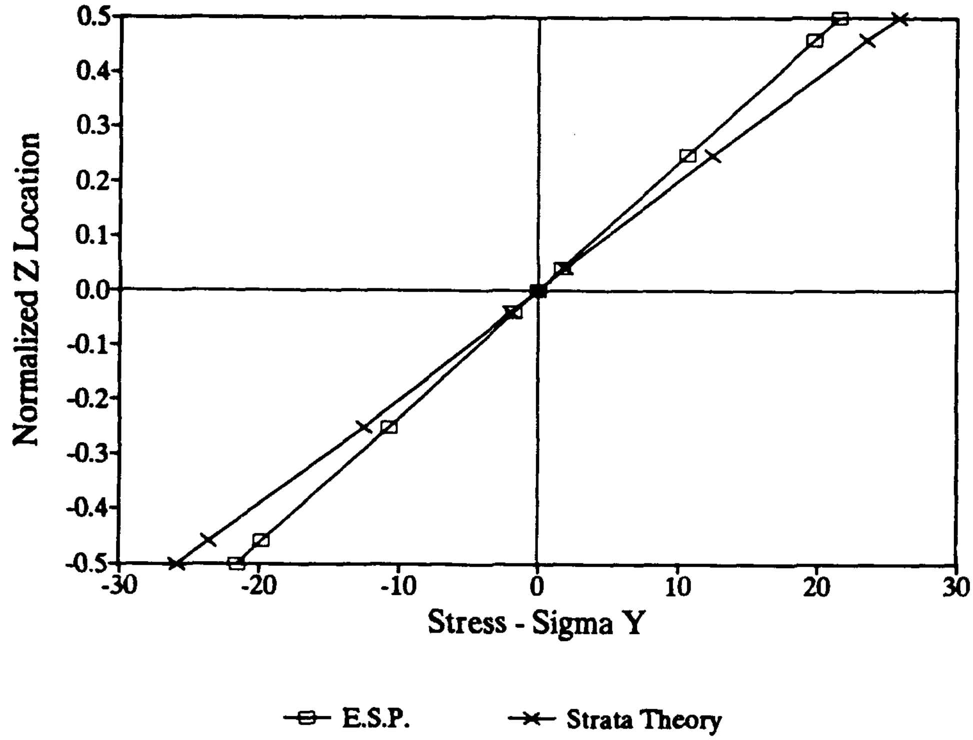

Figure 16. σ_y for square isotropic plate

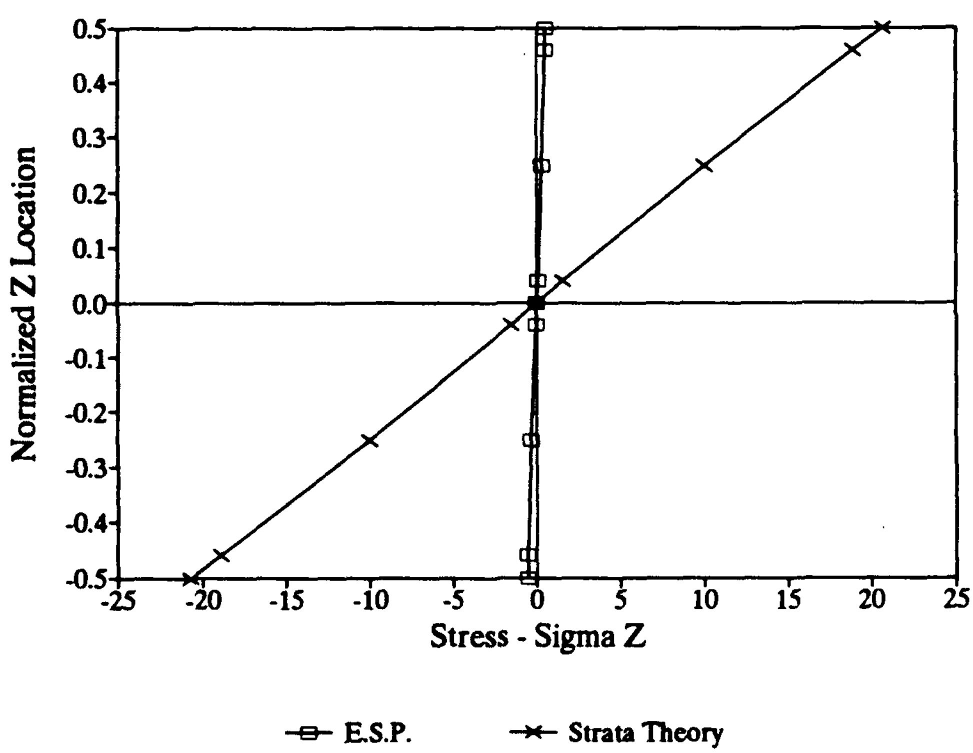

Figure 17. σ_z for square isotropic plate

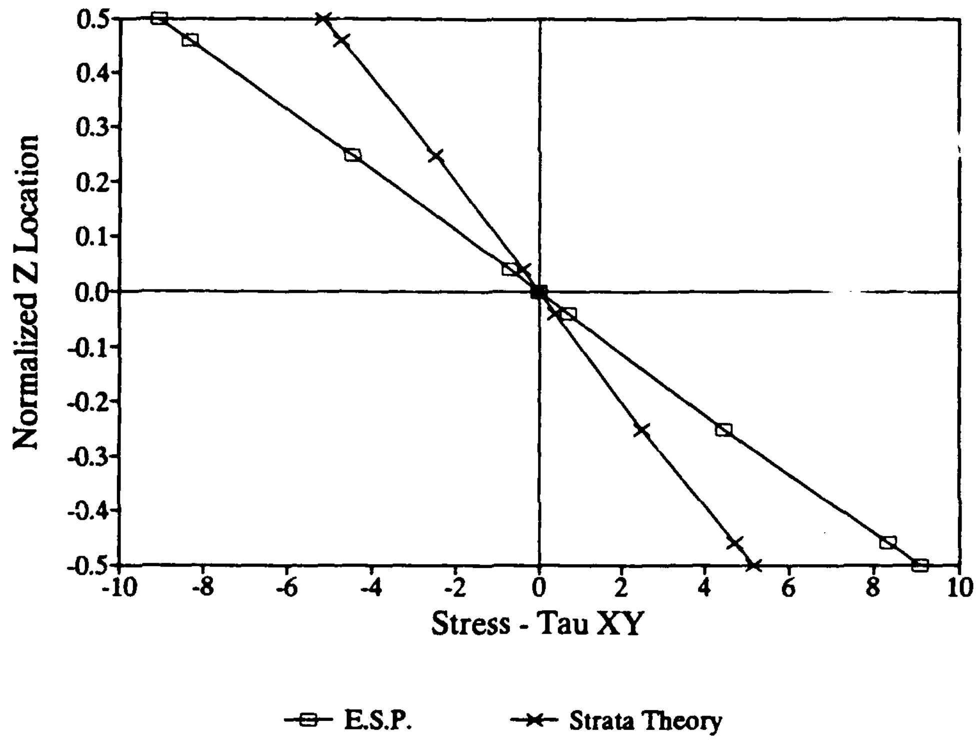

Figure 18. τ_{xy} for square isotropic plate

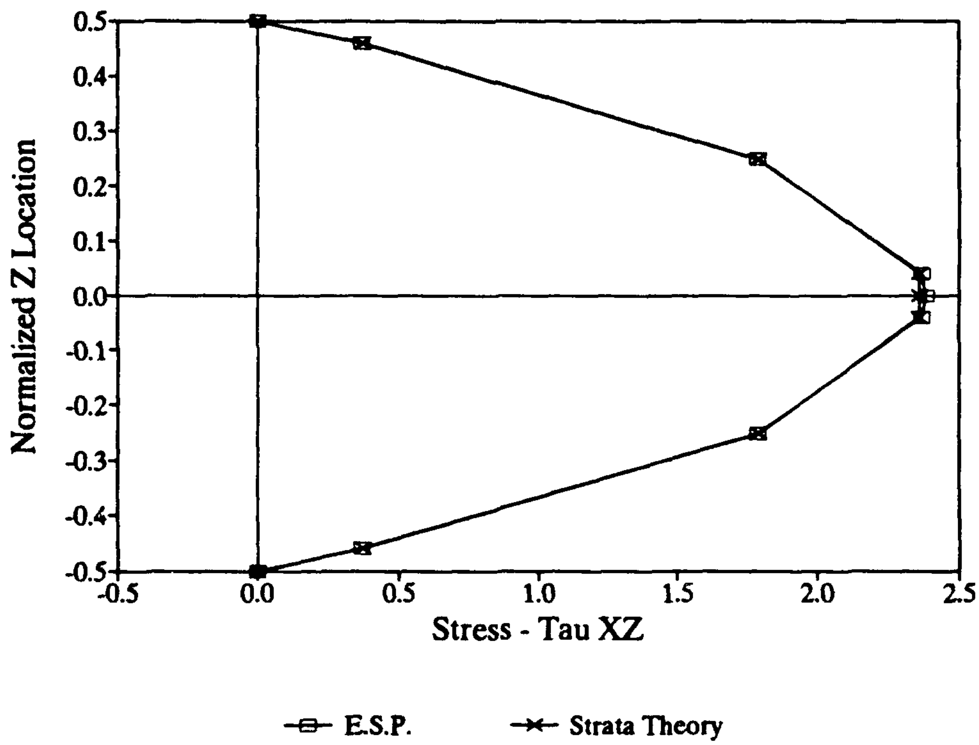

Figure 19. τ_{xz} for square isotropic plate

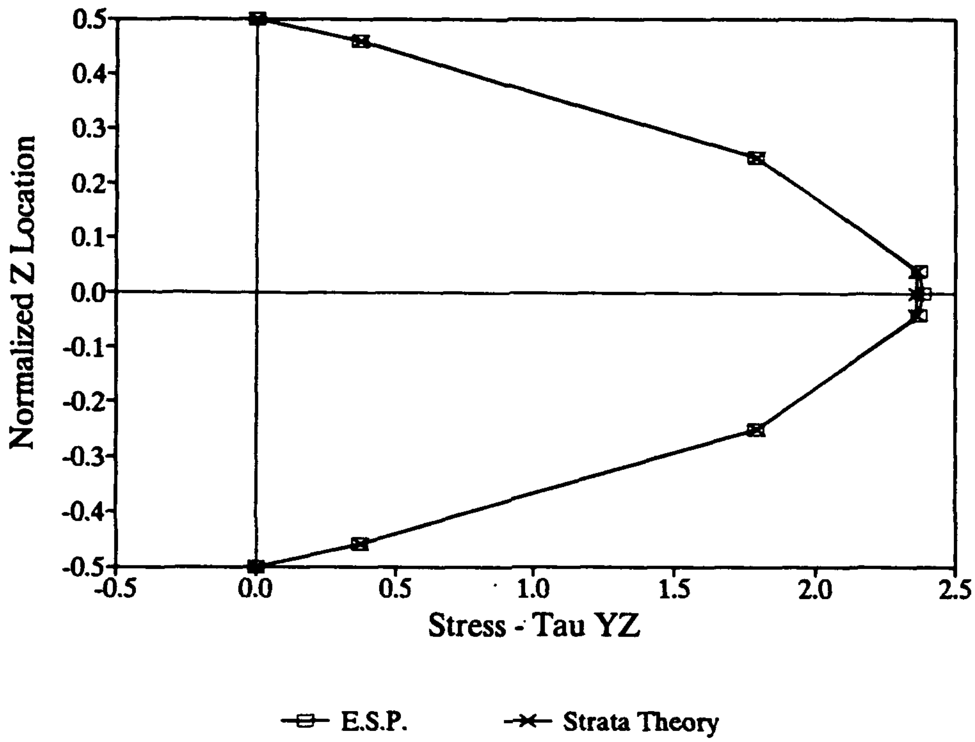

Figure 20. τ_{yz} for square isotropic plate

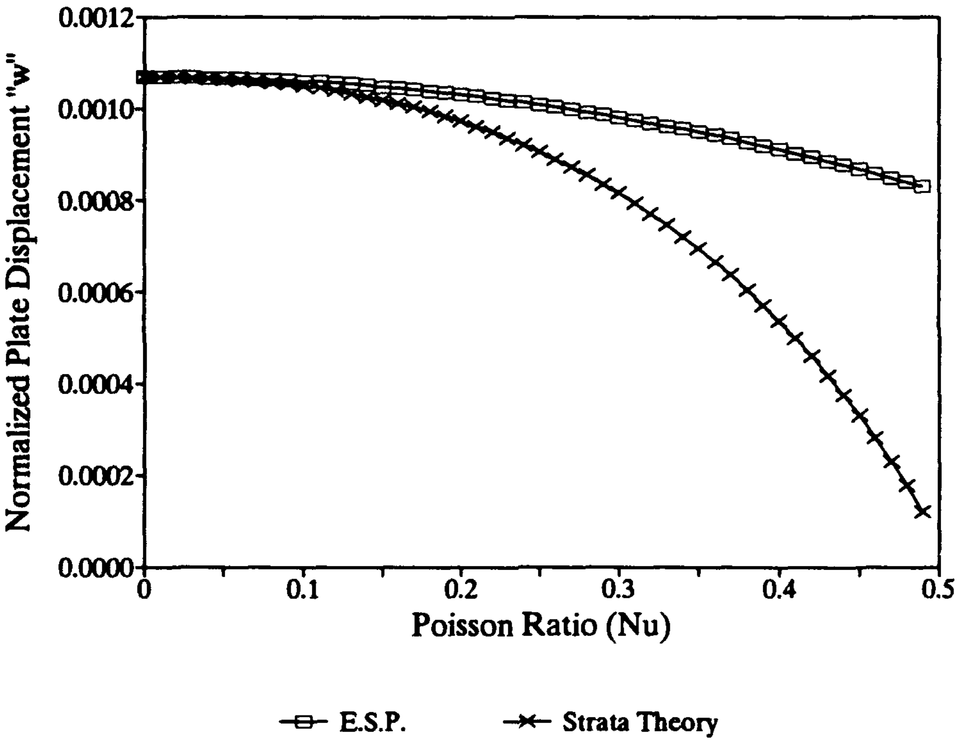

Figure 21. Displacement w of a square isotropic plate vs. ν

The following tables list the values of the parameters that were calculated for the preceding plots:

Table 2. Matrix strata elastic coefficients (square isotropic plate)

K_{1m}	6.4286E+005
K_{2m}	4.2857E+005
K_{3m}	1.0714E+005

Table 3. Strata Theory coefficients (square isotropic plate)

A	= 5.3574E-004		
B_0	= 0.0	C_0	= 0.0
B_1	= -1.4613E-004	C_1	= -1.4613E-004
B_2	= 0.0	C_2	= 0.0
B_3	= -2.8843E-005	C_3	= -2.8843E-005
B_4	= 0.0	C_4	= 0.0
B_5	= -1.7313E-006	C_5	= -1.7313E-006

Table 4. ESP displacement function coefficients (square isotropic plate)

λU_0	-2.6824E-004
λV_0	-2.6824E-004
λW_0	-2.7244E-005
U_0	-6.6316E-005
V_0	-6.6316E-005
W_0	9.0704E-004

Evaluated at z=-h/2

For an isotropic plate, there was a reasonable degree of agreement between ESP and ST. There was excellent agreement for τ_{xz}, τ_{yz}, γ_{xz}, and γ_{yz} when a cubic (or higher order) polynomial was used for ST. The assumed order of the polynomial N for ST determines the variation of γ_{xz} and γ_{yz} through the thickness. When N=1, ST predicts that γ_{xz}, γ_{yz}, τ_{xz} and τ_{yz} are constants. But as N is increased, i.e. N=3 or larger, ST and ESP agree to within 1% for these shear terms. The remaining stresses and strains as well as the displacements displayed the same general behavior for both ESP and ST, although the actual values calculated disagreed by nearly a factor of 2 in some cases.

The amount by which ESP and ST disagree appears to be primarily dependent upon the value of the Poisson's ratio (ν). When the Poisson's ratio ν is allowed to vary, it is immediately apparent from Figure 21. that ESP and ST agree exactly when $\nu=0$, and the two solutions diverge as ν increases. At $\nu=0.5$, which represents an incompressible material, ST predicts that w=0, whereas ESP predicts a non-zero z displacement. Intuitively, the ESP results seem more reasonable when one realizes that shear becomes the primary form of deformation for an incompressible material, and some displacement is still possible.

Clearly, the effect of shear is diminished in ST. ST assumes that $\epsilon_z=0$, while ESP expects a nonzero ϵ_z. Since the normal stresses are coupled through ν, elimination of one of the normal stresses, in this case ϵ_z, introduces an error that is not present in ESP.

The 2-Ply Square Composite Plate

The complete set of displacements, stresses and strains are shown in Figures 22. through 36. The displacements are obtained directly from both ST and ESP. The strains are derived from the displacement information, and stresses are calculated using the constitutive stress-strain relations, given in Equations (9) and (16).

The following plots for the stresses, strains, and displacements include only the z-dependent portion of these parameters. In other words, the x- and y-dependent sine and cosine functions are divided out for the presentation of this data.

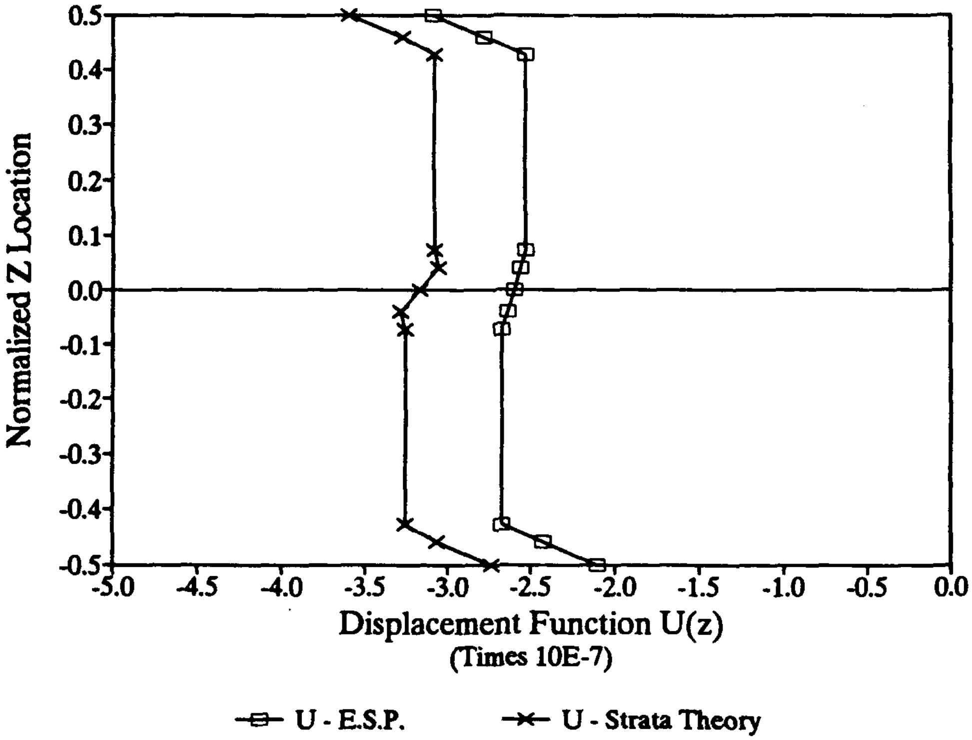

Figure 22. In-plane displacement u for square 2-ply composite plate

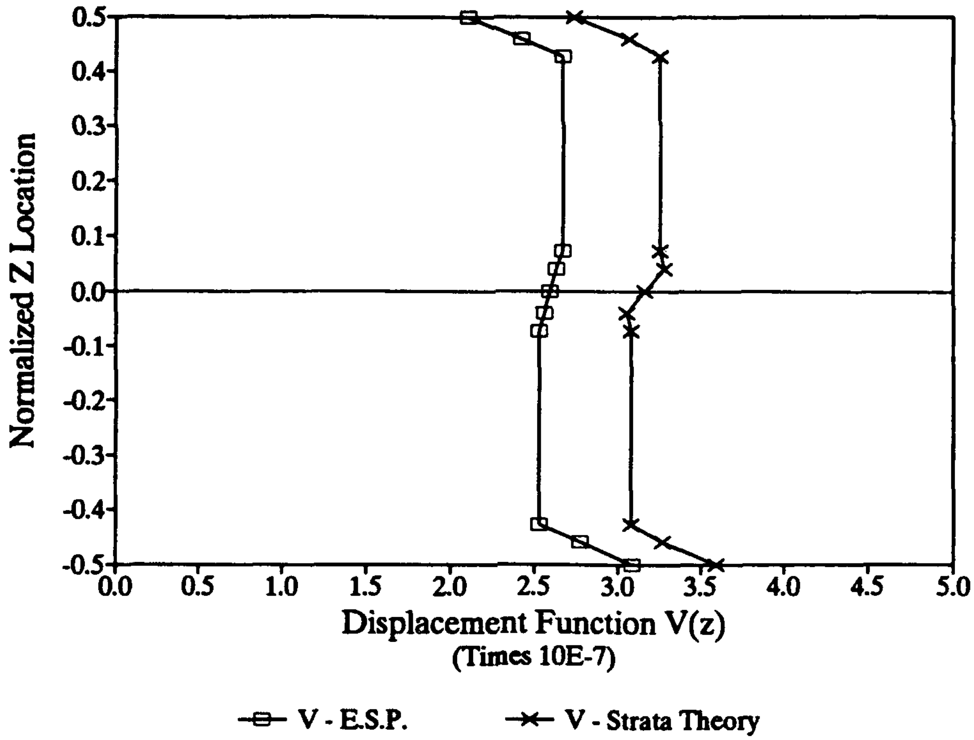

Figure 23. In-plane displacement v for square 2-ply composite plate

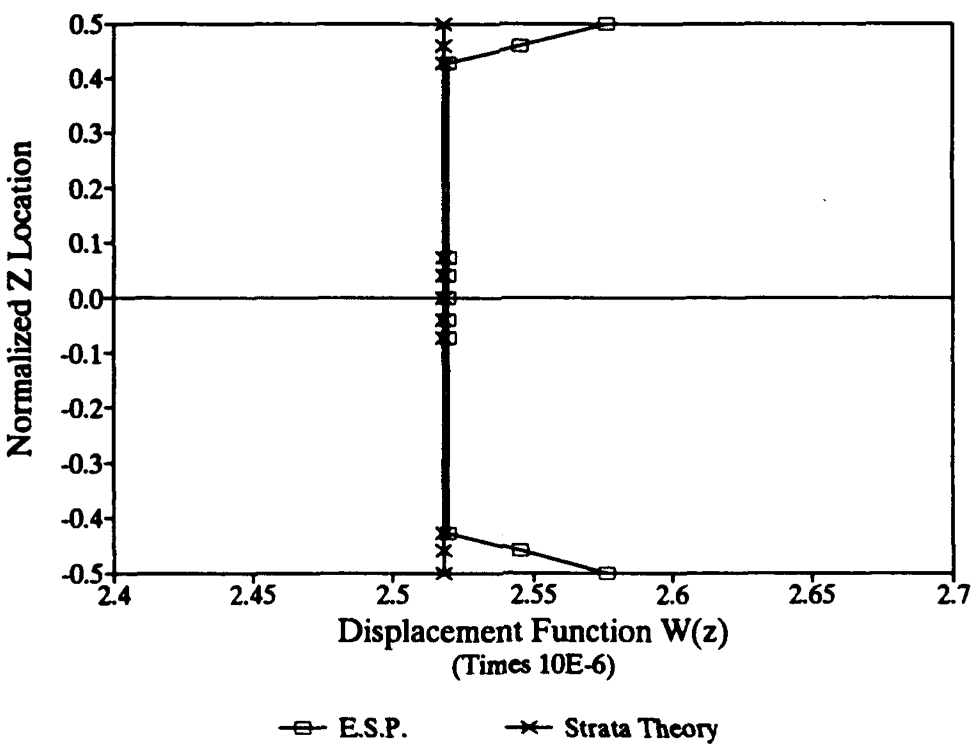

Figure 24. Out-of-plane displacement w for square 2-ply composite plate

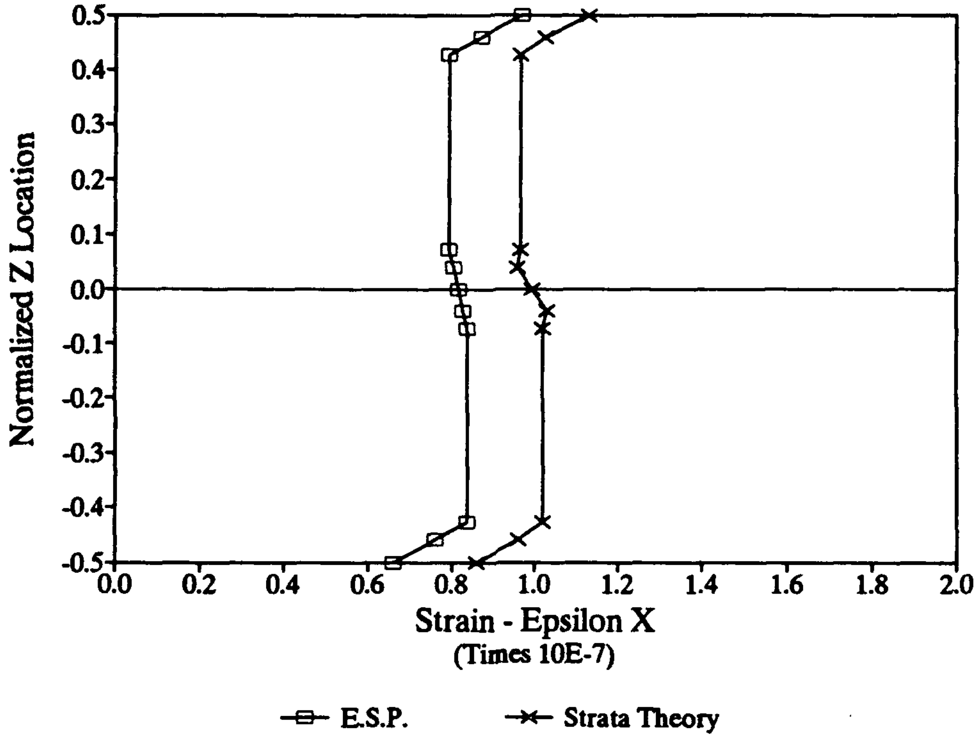

Figure 25. ϵ_x for square 2-ply composite plate

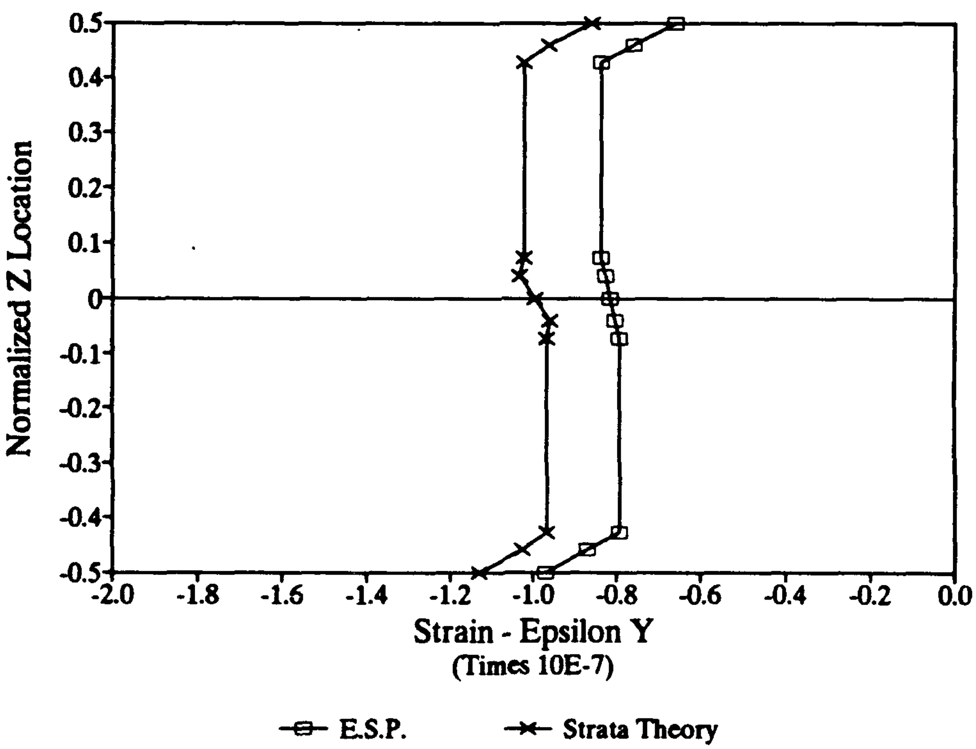

Figure 26. ϵ_y for square 2-ply composite plate

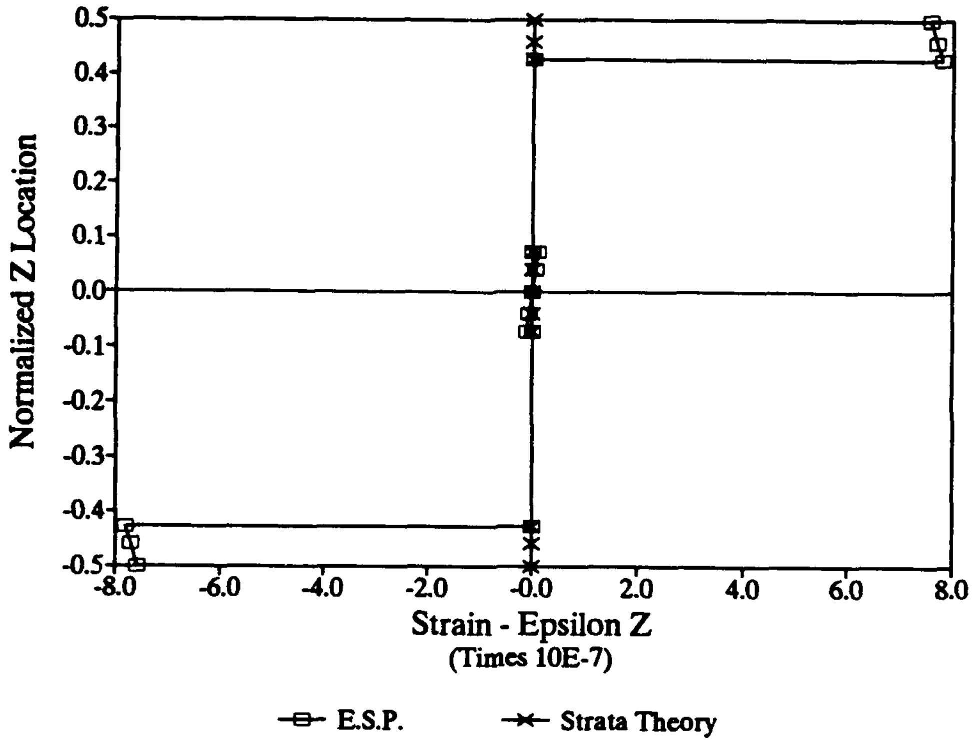

Figure 27. ϵ_z for square 2-ply composite plate

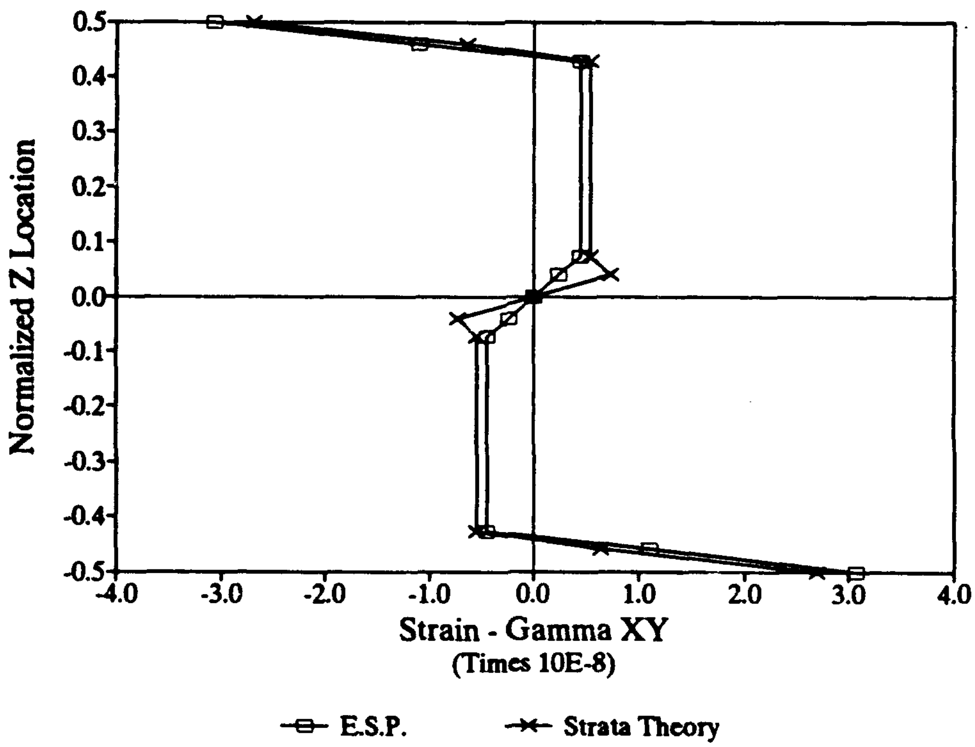

Figure 28. γ_{xy} for square 2-ply composite plate

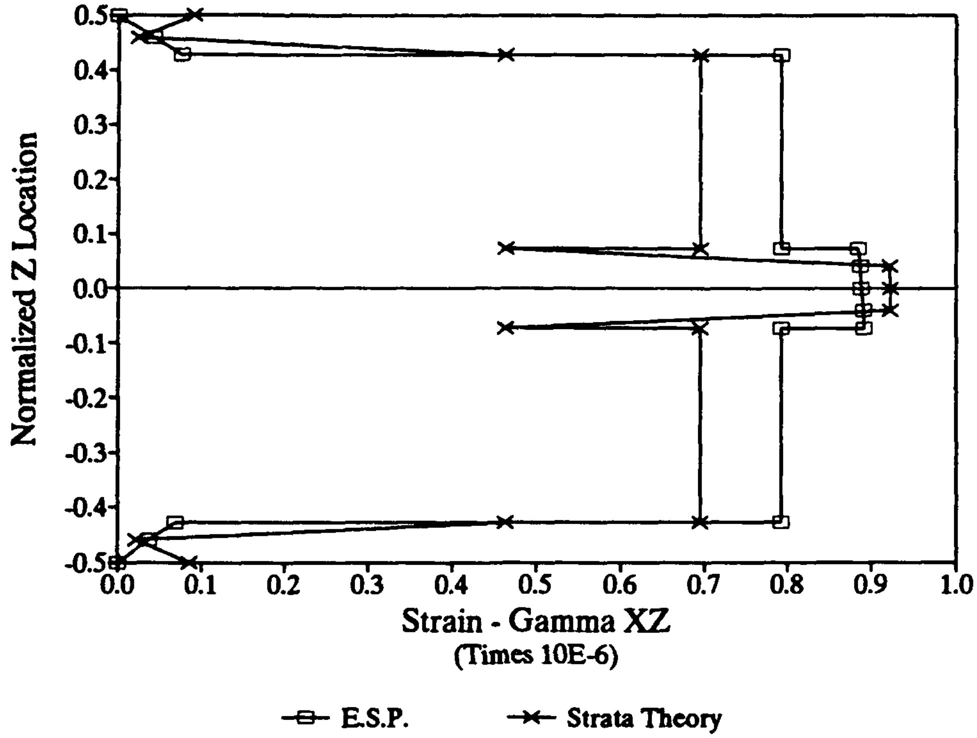

Figure 29. γ_{xz} for square 2-ply composite plate

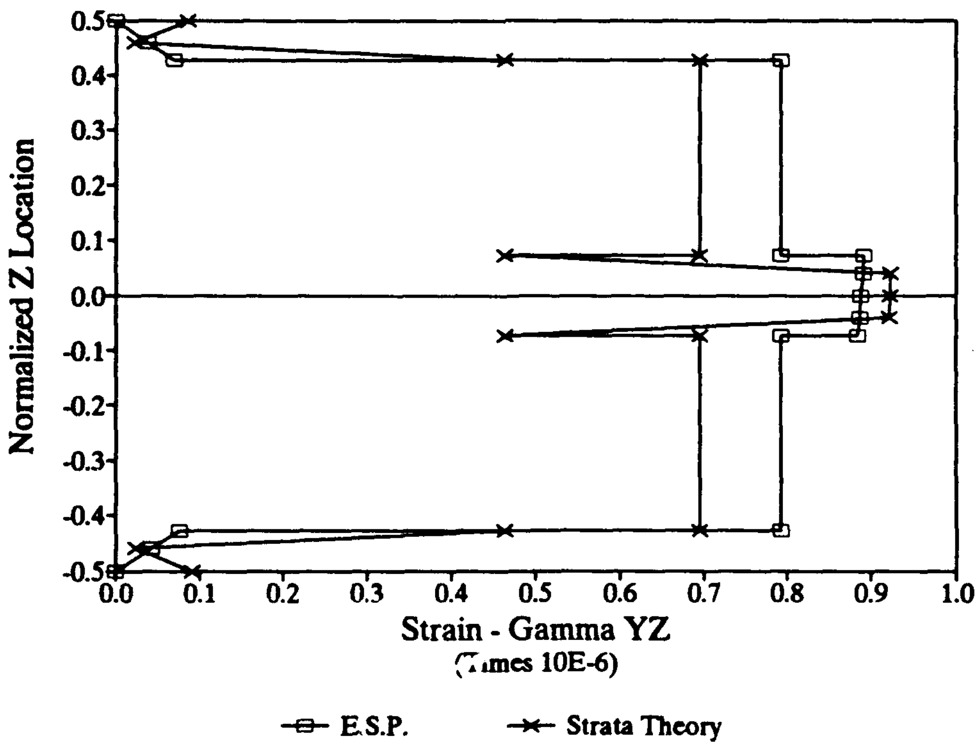

Figure 30. γ_{yz} for square 2-ply composite plate

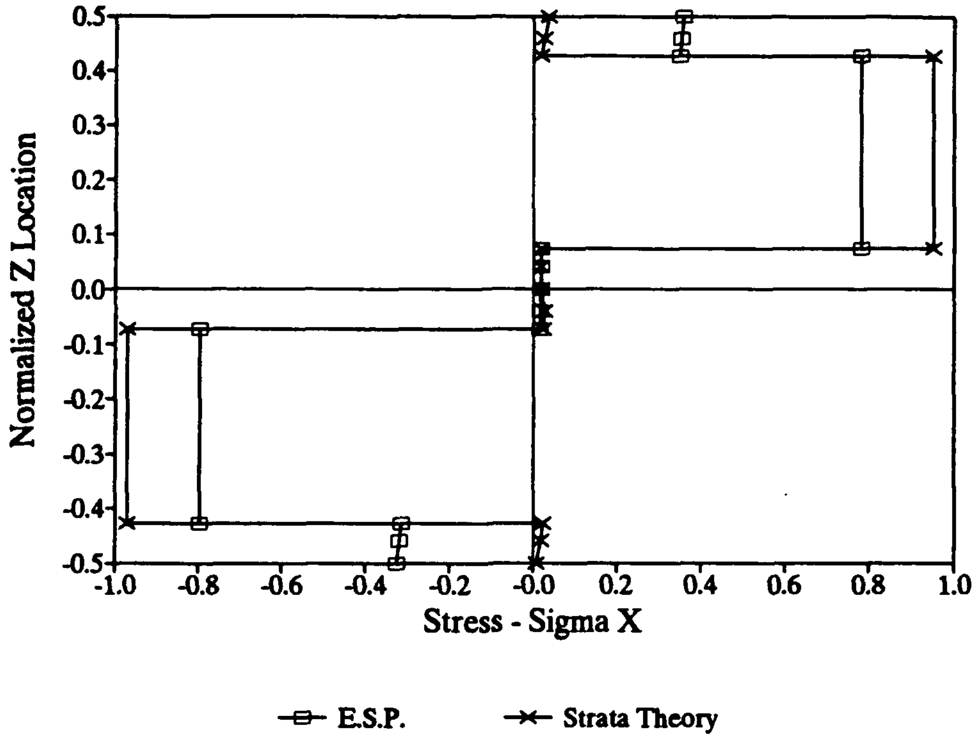

Figure 31. σ_x for square 2-ply composite plate

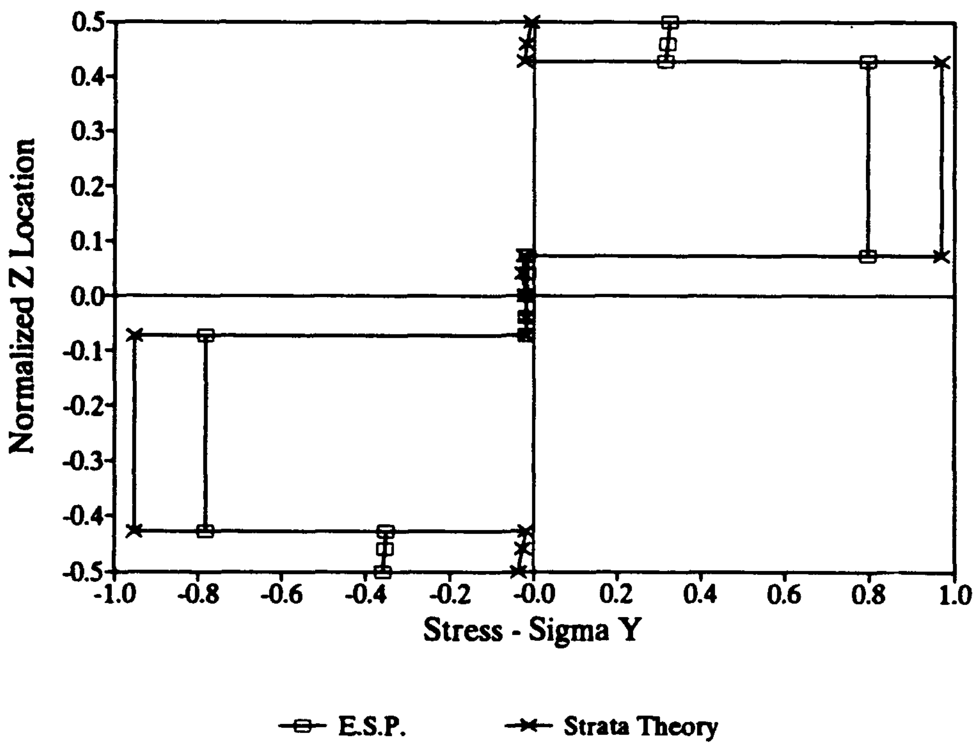

Figure 32. σ_y for square 2-ply composite plate

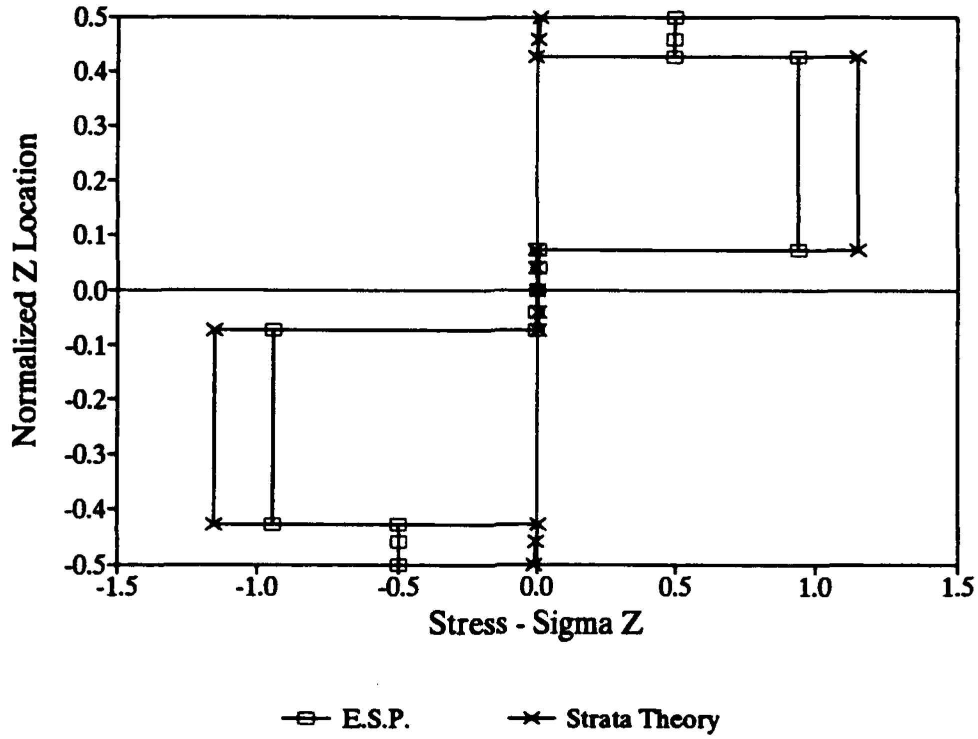

Figure 33. σ_z for square 2-ply composite plate

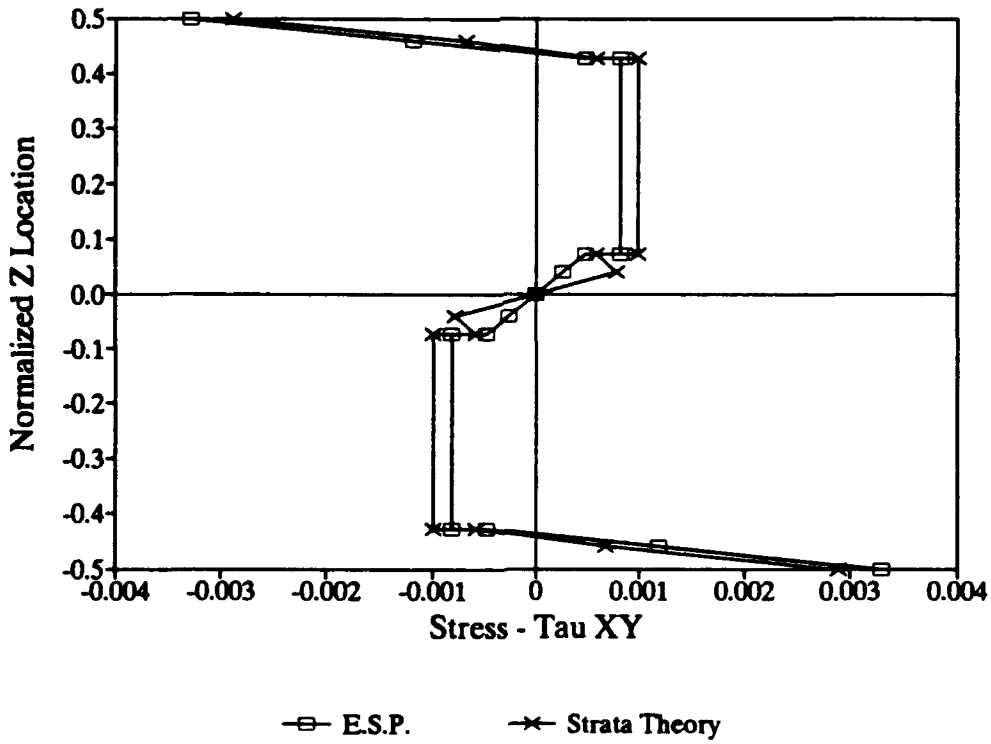

Figure 34. τ_{xy} for square 2-ply composite plate

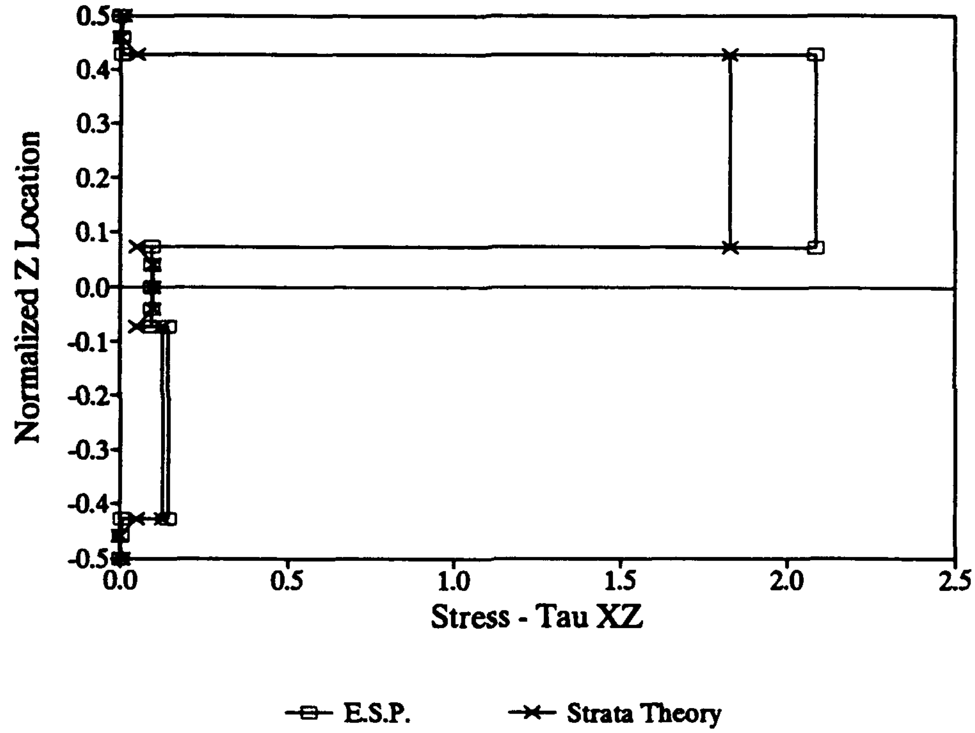

Figure 35. τ_{xz} for square 2-ply composite plate

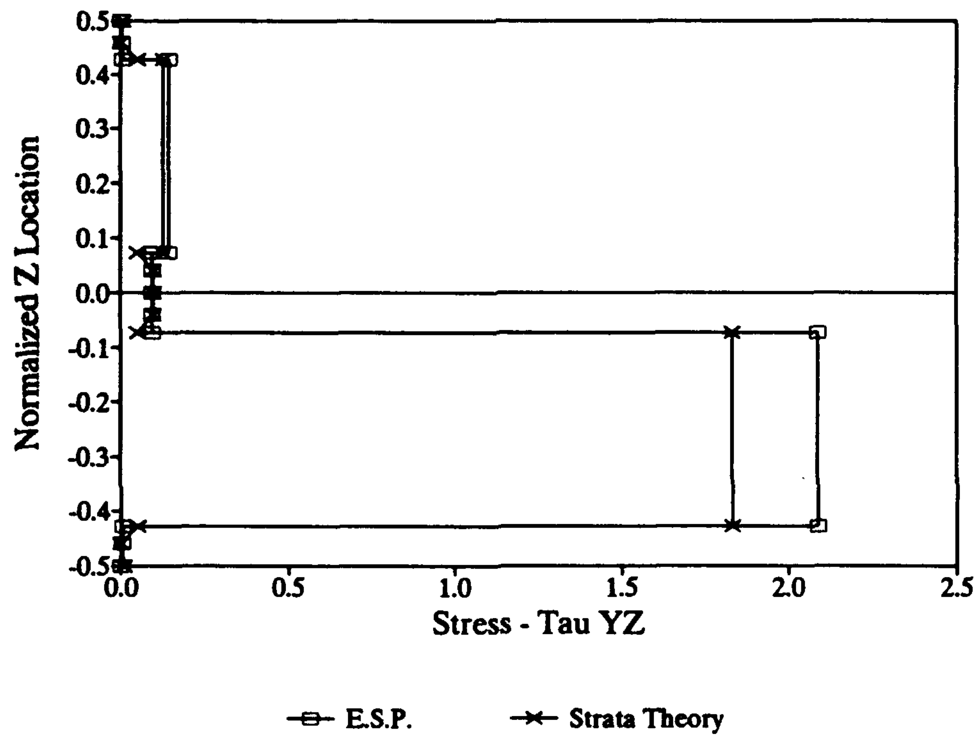

Figure 36. τ_{yz} for square 2-ply composite plate

The following tables list the values of the parameters that were calculated for the preceding plots:

Table 5. Matrix strata elastic coefficients (2-ply square plate)

K_{1m}	6.4286E+005
K_{2m}	4.2857E+005
K_{3m}	1.0714E+005

Table 6. Fiber strata elastic coefficients (2-ply square plate)

	Fiber Stratum 1	Fiber Stratum 1
S_{1f}	2.2886E+007	2.1949E+006
S_{2f}	2.1949E+006	2.2886E+007
S_{3f}	1.2347E+007	1.2347E+007
S_{4f}	3.6581E+005	3.6581E+005
S_{5f}	0.0000E+000	0.0000E+000
S_{6f}	0.0000E+000	0.0000E+000
S_{7f}	5.2692E+006	3.6581E+005
S_{8f}	3.6581E+005	5.2692E+006
S_{9f}	0.0000E+000	0.0000E+000
S_{10f}	1.2347E+007	4.2857E+005
S_{11f}	4.2857E+005	1.2347E+007
S_{12f}	2.2886E+007	2.2886E+007
S_{13f}	0.0000E+000	0.0000E+000

When fibers are present, the agreement between ST and ESP for the displacements u, v, and w are very good. Also, there is very good agreement between

Table 7. Strata Theory coefficients (2-ply square plate)

A	= 2.5179E-006		
B_0	= -3.1637E-007	C_0	= 3.1637E-007
B_1	= 1.0810E-007	C_1	= 1.0810E-007
B_2	= -3.2442E-010	C_2	= 3.2442E-010
B_3	= -1.3121E-006	C_3	= -1.3121E-006
B_4	= 2.1974E-009	C_4	= -2.1974E-009
B_5	= 2.1453E-006	C_5	= 2.1453E-006

Table 8. ESP displacement function coefficients (2-ply square plate)

	Matrix Stratum 1	Matrix Stratum 2	Matrix Stratum 3
λU_0	-7.1529E-007	1.0012E-007	-8.0927E-007
λV_0	-7.2279E-007	9.2623E-008	-8.0927E-007
λW_0	7.8046E-007	-9.7706E-009	-7.5731E-007
U_0	-2.5245E-007	-2.6659E-007	-2.1051E-007
V_0	2.6659E-007	2.5245E-007	3.0825E-007
W_0	2.5197E-006	2.5197E-006	2.5760E-006

Evaluated at the z coordinate at the bottom of each matrix stratum

ESP and ST in the general behavior for all the stresses and strains, with the obvious exception of ϵ_z. The polynomial form of the z dependence of the assumed displacement field is clearly evident in the plots of the shear stresses and shear strains.

In comparing the results of the 2-ply composite with the isotropic plate, it is obvious that the fibers dominate the behavior of the composite. While the general behavior of the ST results echo those of ESP, there is still some disagreement in the actual values calculated for displacements, stresses, and strains. Other test cases for a square plate were examined, and they revealed that the level of agreement between ESP and ST seems to be most highly dependent upon the Poisson's ratio for the fibers (ν_f), and the order of the polynomial (N) assumed for ST.

Just as with the isotropic plate, there is considerable disagreement in ϵ_z between ST and ESP for a composite panel, but this should be expected. In the matrix strata that comprise the top and bottom surfaces of the composite, note that ϵ_z increases slightly as z gets farther away from the free surfaces at $z=\pm h/2$. This demonstrates a coupling of the normal stresses. At the free surfaces of the composite, the matrix is allowed to deform freely through Poisson effects. However, at the matrix-fiber strata interface, the matrix is forced to match the displacements of the fibers, and so it cannot deform freely.

Because ϵ_z is allowed to be nonzero for ESP, ESP predicts that the presence of fibers results in significant jumps in ϵ_z through the thickness. Through-the-thickness shear through the fiber strata affects how ϵ_z varies for the ESP solution. Since a square plate is being considered here, and since the plies are rotated 90° with respect to each other, dimensional symmetry causes the fiber strata to absorb equal amounts of strain energy. If the aspect ratio of the plate (a/b) is allowed to increase, more strain energy is absorbed by the fiber stratum whose fibers are aligned with the shorter dimension of the plate than by a stratum whose fibers are parallel with the long dimension of the plate. This will be more clearly evident in the cylindrical bending case that follows.

Assuming that the aspect ratio b/a is constant, both ST and ESP predict that the z displacement (w) of a composite varies as a function of the plate width squared (a^2). In other words, the limit of w/a^2 as "a" goes to infinity is a constant. When the given load distribution is used, and a linear polynomial is assumed for ST (i.e. $N=1$), the solution for the coefficient "A" of the ST assumed displacement field predicts this limit to be

$$\lim_{a \to \infty} \frac{A}{a^2} = \frac{\rho}{\pi^2 \left[S^0_{7_f} + S^0_{8_f} \right]} \tag{62}$$

This is a fundamentally different behavior than that of an isotropic plate or for a set of stacked orthotropic plates. CPT, the exact orthotropic plate solutions developed by Pagano [8:398-411; 9:20-34], ESP, or ST (when applied to an isotropic plate) predicts that the limit of w/a^4 as "a" goes to infinity is a constant. Note the dependence on the characteristic plate width "a". Using the Kirchhoff plate assumptions for a thin isotropic plate (Navier's Solution), this limit is given as [11:548-549]

$$\lim_{a \to \infty} \frac{\bar{w}}{a^4} = \frac{3\rho(1-\nu^2)}{h^3 \pi^4 E}$$

$$\bar{w} = w\left(x = \frac{a}{2}, y = \frac{b}{2}, z = 0\right) \tag{63}$$

When "a" is allowed to get large, the ESP results corresponds to this value. Notice that the material property terms correspond to what would be a K_1 term for a plane stress elasticity problem.

The expression for an isotropic plate using ST becomes

$$\lim_{a \to \infty} \frac{A}{a^4} = \frac{3\rho(1-2\nu_m)(1+\nu_m)}{h^3 \pi^4 E_m (1-\nu_m)} \tag{64}$$

Similar expressions arise for CPT or the exact orthotropic plate solutions as well. Notice that the material property terms corresponds to an expression for a K_1 term that would be used for a plane strain elasticity problem.

V. *The Cylindrical Bending Case*

Cylindrical bending is a special case of the rectangular plate formulation developed above. The dimension in the y direction (given as "b") is assumed to be infinite, therefore v and derivatives with respect to y are zero. As a result, some strain terms are defined as being zero. Because "b" is infinite, ϵ_y, τ_{xy}, and τ_{yz} are zero throughout the composite.

Strata Theory

Using the fact that $b=\infty$, and that v and derivatives with respect to y equal 0, the PDEs in Equations (19) and (21) reduce to

$$j\beta_m K_3^{j-1} w_{,x} = \sum_{i=0}^{N} \left\{ -ij\beta_m^2 K_3^{j+i-2} \psi_i + (S_1^{i+j} + K_1^{i+j}) \psi_{i,xx} \right\} \qquad j=0,1,2,\ldots,N \tag{65}$$

and

$$(K_3^0 + S_7^0) w_{,xx} + \sum_{i=0}^{N} i\beta_m K_3^{i-1} \psi_{i,x} + p(x,y) = 0 \tag{66}$$

where

$$K_r^n = \sum_{l=1}^{M} \int_{z_{ml}^-}^{z_{ml}^+} (\alpha_{ml} + \beta_{ml} z)^n K_{r_{ml}} dz$$

$$S_r^n = \sum_{k=1}^{F} S_{r_{fk}} z_{fk}^n h_{fk} \tag{67}$$

These are the same definitions as used for the rectangular plate. Note that Equation (20) vanishes for the cylindrical bending case.

The boundary conditions that remain are

$$\sum_{i=0}^{N}\left[\sum_{l=0}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}(\alpha_{ml}+\beta_{ml}z)^{i+j}K_{1_{ml}}\psi_{i,x}\,dz+\sum_{k=1}^{F}z_{fk}^{i+j}S_{1_{fk}}h_{fk}\psi_{i,x}\right]\delta\psi_j\Bigg|_0^a=0 \tag{68}$$

$$\left[\sum_{l=1}^{M}\int_{z_{ml}^{-}}^{z_{ml}^{+}}\sum_{i=1}^{N}i\beta_{ml}(\alpha_{ml}+\beta_{ml}z)^{i-1}K_{3_{ml}}\psi_i+K_{3_{ml}}w_{,x}dz+\sum_{k=1}^{F}S_{7_{fk}}h_{fk}w_{,x}\right]\delta w\Bigg|_0^a=0 \tag{69}$$

Simply-Supported Cylindrical Bending with Cross-Plies

For simply-supported cylindrical bending, the x dependence of w, Ψ_i, and P can be assumed to have the following form

$$\begin{Bmatrix} w(x) \\ \psi_i(x) \\ P(x) \end{Bmatrix} = \begin{Bmatrix} A\sin(px) \\ B_i\cos(px) \\ \rho\sin(px) \end{Bmatrix} \tag{70}$$

where $p=\pi/a$.

These expressions are now used in the above PDEs, which results in a system of equations in the unknown coefficients A and B_i

$$0=A[\,j\beta_m K_3^{j-1}p]+\sum_{i=0}^{N}B_i\left[\,ij\beta_m^2K_3^{j+i-2}+p^2\left(S_1^{i+j}+K_1^{i+j}\right)\right] \qquad j=0,1,2,\ldots,N \tag{71}$$

and

$$\rho = A(K_3^0+S_7^0)p^2 + \sum_{i=0}^{N} B_i\left[i\beta_m K_3^{i-1} p\right] \tag{72}$$

When this is expressed in matrix form, this system of equations can be decomposed into a matrix of constants multiplied by the vector of unknown coefficients A and B_i.

Exact Stratified Plate

As with the ST approach, cylindrical bending is a special case of the rectangular plate formulation developed above. Again, the dimension in the y direction (given as "b") is assumed to be infinite, therefore v and derivatives with respect to y are zero. The assumed displacement field expressions are

$$\begin{Bmatrix} u(x,z) \\ w(x,z) \end{Bmatrix} = \begin{Bmatrix} U(z)\cos(px) \\ W(z)\sin(px) \end{Bmatrix} \tag{73}$$

where $p=\pi/a$.

When v and the y derivatives are set to zero in the rectangular plate formulation, and b is defined to be infinite, the ODEs that result are

$$\begin{bmatrix} K_{3_m} & 0 \\ 0 & K_{1_m} \end{bmatrix}\begin{Bmatrix} U'' \\ W'' \end{Bmatrix} + \begin{bmatrix} 0 & pK_{4_m} \\ -pK_{4_m} & 0 \end{bmatrix}\begin{Bmatrix} U' \\ W' \end{Bmatrix} + \begin{bmatrix} -p^2K_{1_m} & 0 \\ 0 & -p^2K_{3_m} \end{bmatrix}\begin{Bmatrix} U \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{74}$$

where $K_4=K_2+K_3$. When the 2 second-order ODEs are recast as a set of four first-order coupled ODEs, U and W are assumed to have the form of $U_0e^{\lambda z}$, and the governing equations are expressed as a system of 4 coupled algebraic equations. Rewritten in state-space form, they become

$$[[Q] + \lambda[I]]\{\vec{U}_0\} = \{0\} \tag{75}$$

where

$$\{\vec{U}_0\} = \begin{Bmatrix} \lambda U_0 \\ \lambda W_0 \\ U_0 \\ W_0 \end{Bmatrix} \tag{76}$$

and

$$[Q] = \begin{bmatrix} 0 & -\dfrac{pK_{4_m}}{K_{3_m}} & -\dfrac{p^2K_{1_m}}{K_{3_m}} & 0 \\ -\dfrac{pK_{4_m}}{K_{1_m}} & 0 & 0 & -\dfrac{p^2K_{3_m}}{K_{1_m}} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{77}$$

The eigenvalues of this system are 2 sets of repeated roots

$$\lambda = \pm p \tag{78}$$

The same method for generating an eigenvector matrix as described above for the rectangular plate is used [1:53-57], but it is tailored for a system with four eigenvalues. First, the functions $f(z,\lambda)$ and $g(z,\lambda)$ are defined as

$$\begin{aligned} g(z,\lambda) &= \alpha_0(z) + \alpha_1(z)\,\lambda + \alpha_2(z)\,\lambda^2 + \alpha_3(z)\,\lambda^3 \\ f(z,\lambda) &= e^{\lambda}z \end{aligned} \tag{79}$$

and these expressions are manipulated to solve for the α_i's as before.

$$\begin{Bmatrix} f(z,\lambda) \\ f'(z,\lambda) \\ f(z,-\lambda) \\ f'(z,-\lambda) \end{Bmatrix} = \begin{Bmatrix} e^{\lambda z} \\ ze^{\lambda z} \\ e^{-\lambda z} \\ ze^{-\lambda z} \end{Bmatrix} = \begin{bmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \\ 0 & 1 & 2\lambda & 3\lambda^2 \\ 1 & -\lambda & \lambda^2 & -\lambda^3 \\ 0 & 1 & -2\lambda & 3\lambda^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{80}$$

Finally, exponentiation of the matrix [Q] can be obtained by the relation

$$e^{[Q]z} = \alpha_0[I] + \alpha_1[Q] + \alpha_2[Q]^2 + \alpha_3[Q]^3 \tag{81}$$

The boundary conditions are constructed in the same fashion as for the rectangular plate, using a series of force balance equations for the x-z shear and normal stress. Equilibrium is established at the upper surface, at each fiber stratum, and finally at the lower surface.

The boundary conditions are setup the same as for the rectangular plate, given by

$$[BC] = \begin{bmatrix} M1_1 & M2_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -M1_1 & -M2_1 & M1_2 & M2_{f1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -M1_2 & -M2_2 & M1_3 & M2_{f2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -M1_3 & -M2_3 \end{bmatrix} \tag{82}$$

but the submatrices are defined as

$$M1_l = \begin{bmatrix} K_{3_{ml}} & 0 \\ 0 & K_{1_{ml}} \end{bmatrix}$$

$$M2_l = \begin{bmatrix} 0 & pK_{3_{ml}} \\ -pK_{2_{ml}} & 0 \end{bmatrix} \tag{83}$$

$$M2_{fk} = \begin{bmatrix} p^2 S_{1_{fk}} & 0 \\ 0 & p^2 S_{7_{fk}} \end{bmatrix} h_{fk} + M2_{k+1}$$

instead. These represent stresses τ_{xz} and σ_z at the top and bottom surfaces of the composite stack, and at the fiber strata.

Specific Results

Comparison calculations were made for both isotropic plates and 2-ply composite lay-ups. The following values were assumed for these test cases

Table 9. Assumed values for comparison calculations (cylindrical bending)

$E_f = 10^8$	$\nu_f = 0.25$	$vol_f = \sqrt{2}/2$
$E_m = 10^7$	$\nu_m = 0.25$	$vol_m = 1-\sqrt{2}/2$
$p = \pi/a$	$a = 10$	$\mathcal{P} = 1$
$q = 0$	$b = \infty$	$h = 1$
$\theta_{f1} = 0°$	$h_{f1} = \sqrt{2}/4$	
$\theta_{f2} = 90°$	$h_{f2} = \sqrt{2}/4$	
$\alpha_{m1} = -(1+\sqrt{2})/2$	$\beta_{m1} = 2+\sqrt{2}$	
$\alpha_{m2} = 0$	$\beta_{m2} = 2+\sqrt{2}$	
$\alpha_{m3} = (1+\sqrt{2})/2$	$\beta_{m3} = 2+\sqrt{2}$	

Volume fraction terms vol_m and vol_f were derived by assuming global volume fractions of 50% fibers and 50% matrix. Recall that vol_m and vol_f refer to the fractions of matrix and fiber in an idealized fiber ply, whereas the term "global volume fraction" refers to the fractions of matrix and fibers that exist before the fiber plies are idealized and dictates the thicknesses of the matrix and fiber plies.

A fifth-order polynomial in z for the ST displacement field is assumed, i.e. $N=5$ in Equations (2), unless otherwise indicated. Also, the applied load is split half and half between the upper and lower surfaces of the plate.

The values for h and $\mathcal{P}$ were chosen to normalize the governing equations. The x and z coordinates and the plate width are normalized to the plate thickness (x/h, z/h, and a/h), and the elastic moduli are normalized to the applied pressure ($E_m/\mathcal{P}$ and $E_f/\mathcal{P}$). Displacements are also normalized to the plate thickness (u/h and w/h).

The Isotropic Cylindrical Bending Case

Since the plate consists of matrix only and fibers are not present, E_f and ν_f are set to zero in the governing equations. The complete set of displacements, stresses and strains are shown in Figures 37. through 45. The displacements are obtained directly from both ST and ESP. The strains are derived from the displacement information, and stresses are calculated using the constitutive stress-strain relations, given in Equations (9) and (16).

The following plots for the stresses, strains, and displacements include only the z-dependent portion of these parameters. In other words, the x- and y-dependent sine and cosine functions are divided out for the presentation of this data.

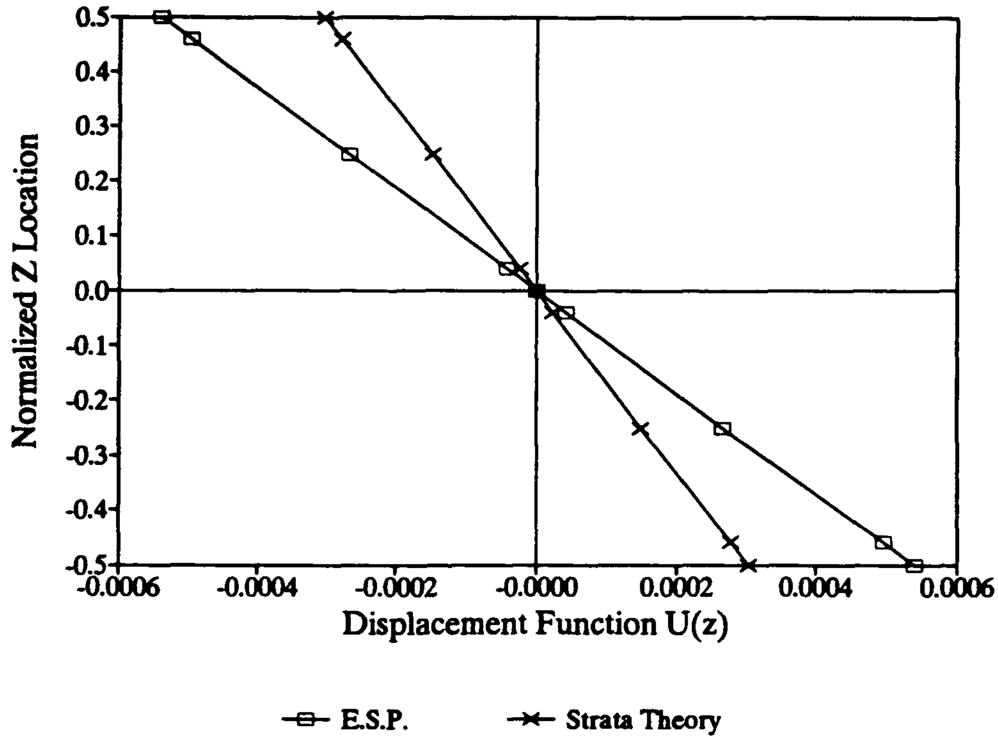

Figure 37. In-plane displacement u for cylindrical bending isotropic plate

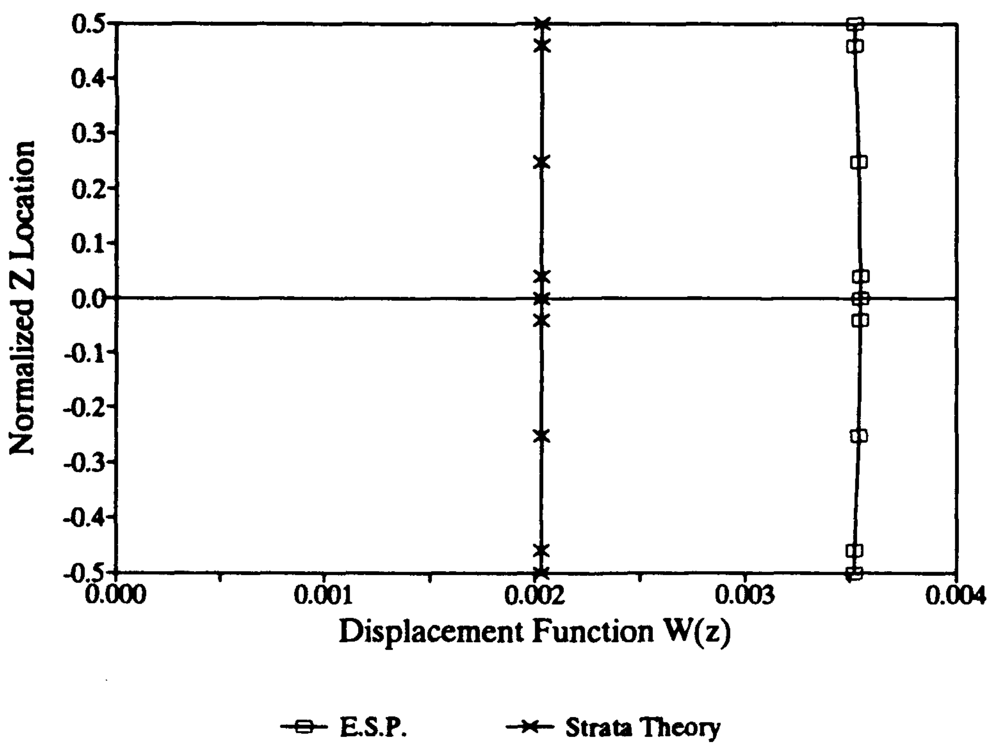

Figure 38. Out-of-plane displacement w for cylindrical bending isotropic plate

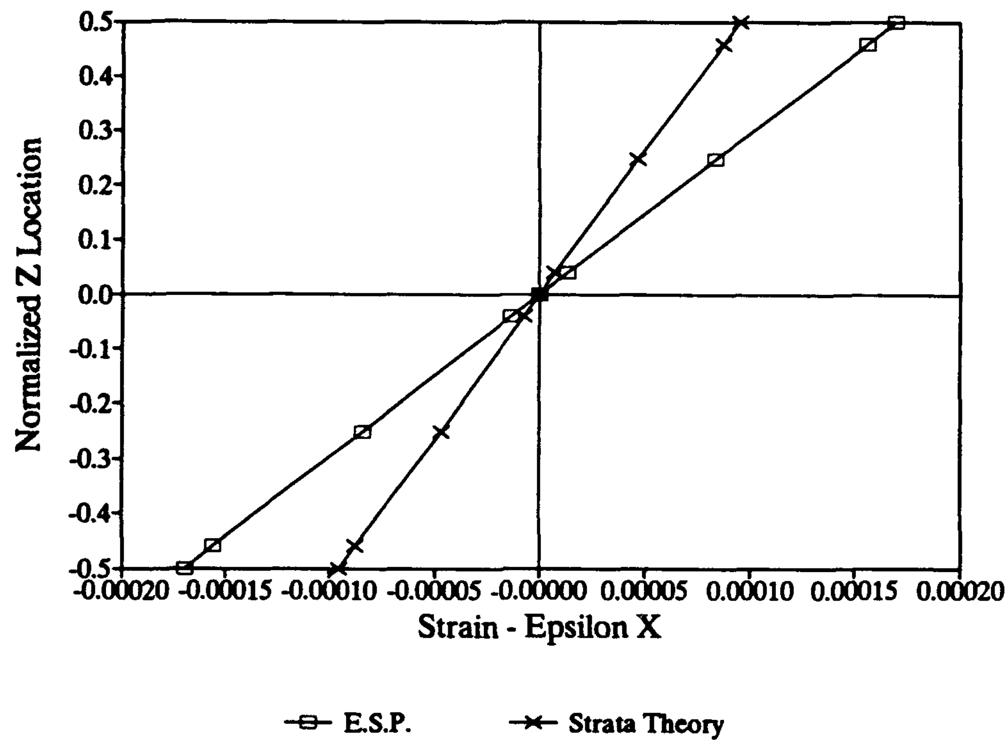

Figure 39. ϵ_x for cylindrical bending isotropic plate

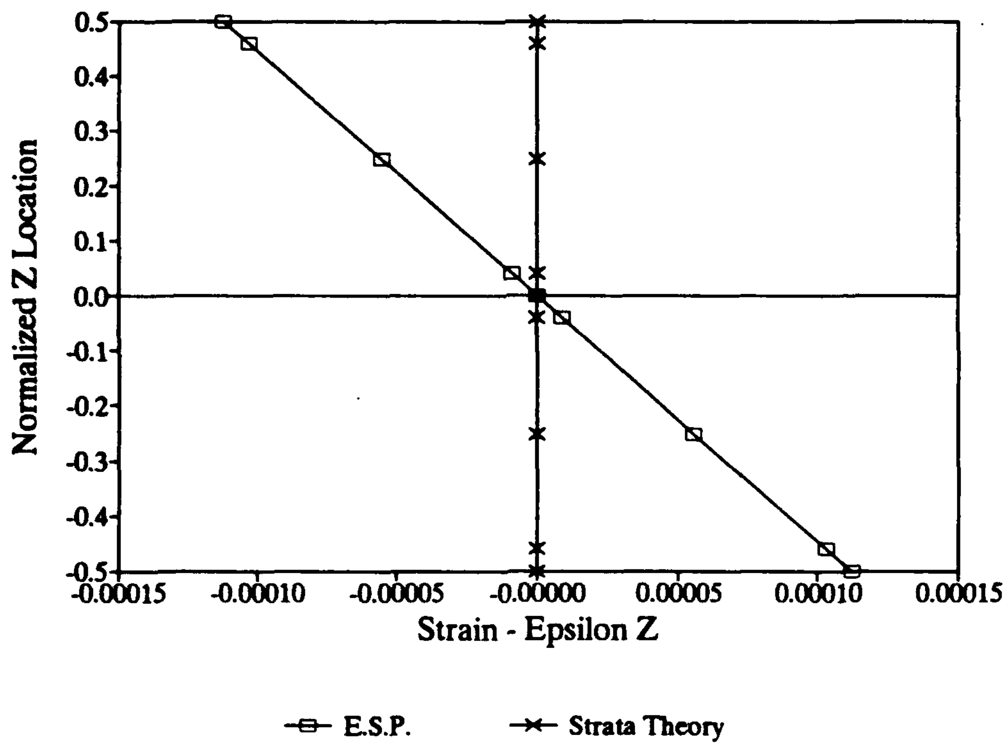

Figure 40. ϵ_z for cylindrical bending isotropic plate

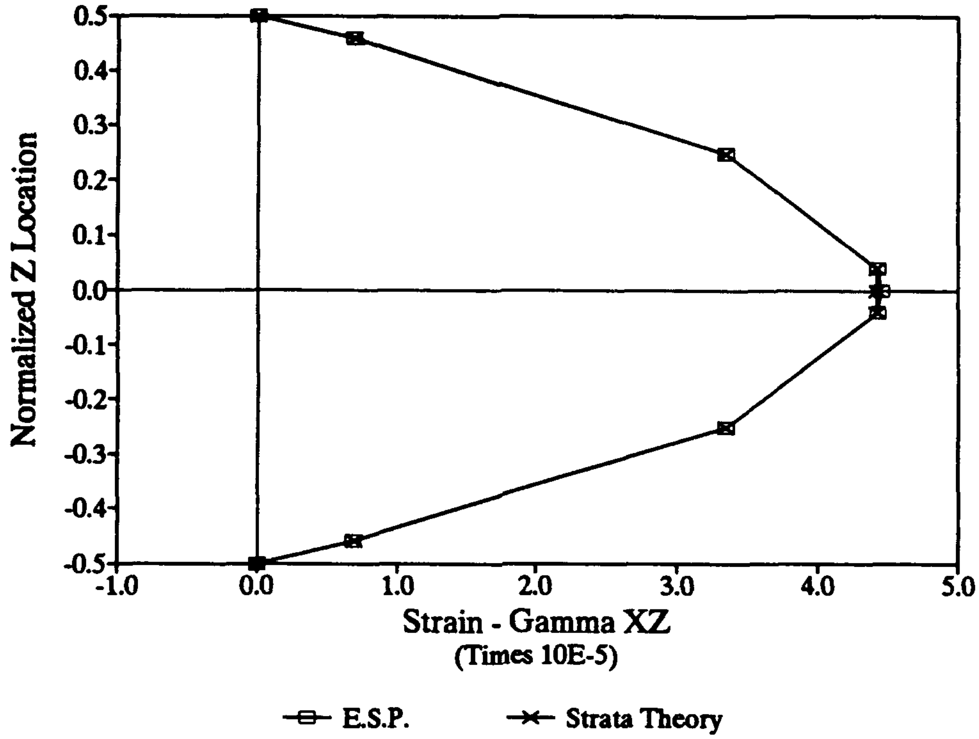

Figure 41. γ_{xz} for cylindrical bending isotropic plate

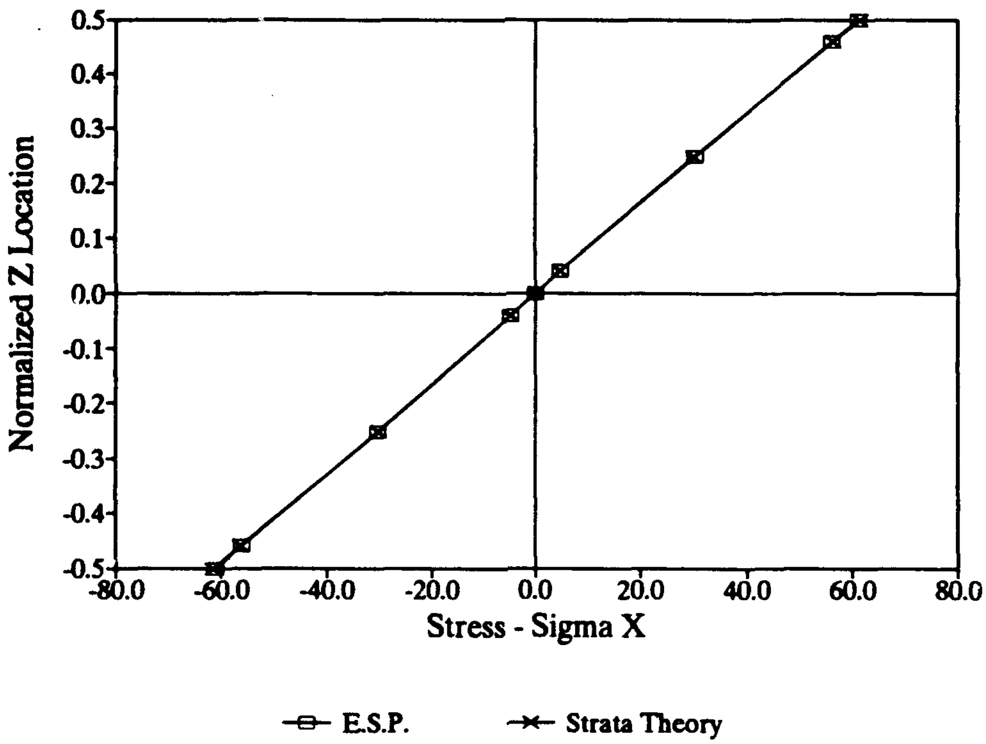

Figure 42. σ_x for cylindrical bending isotropic plate

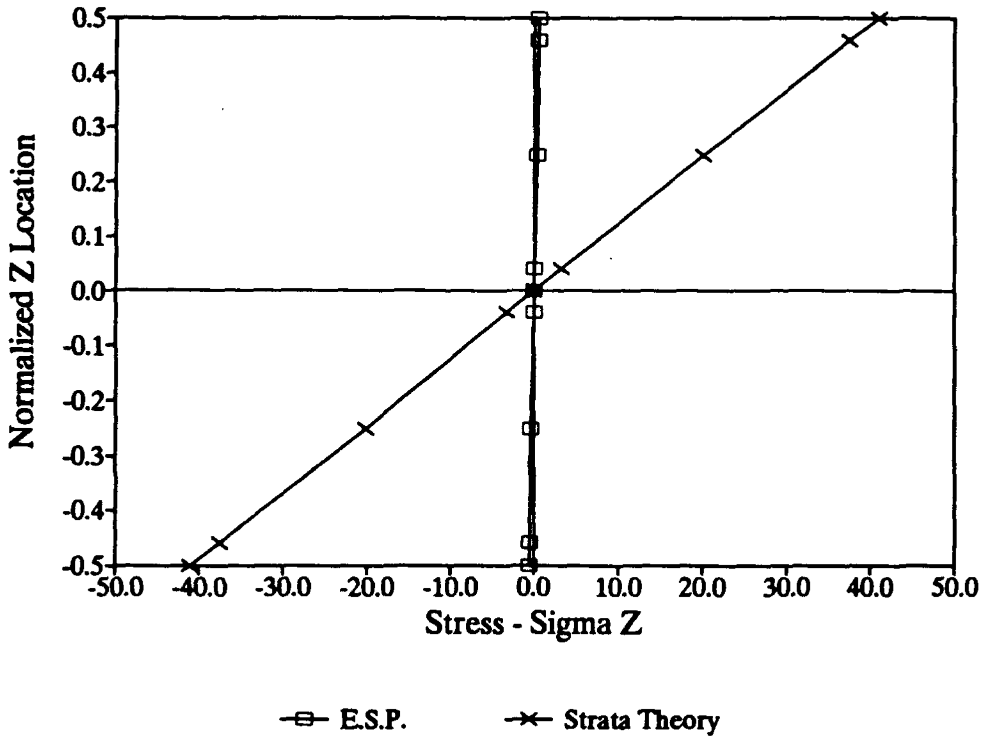

Figure 43. σ_z for cylindrical bending isotropic plate

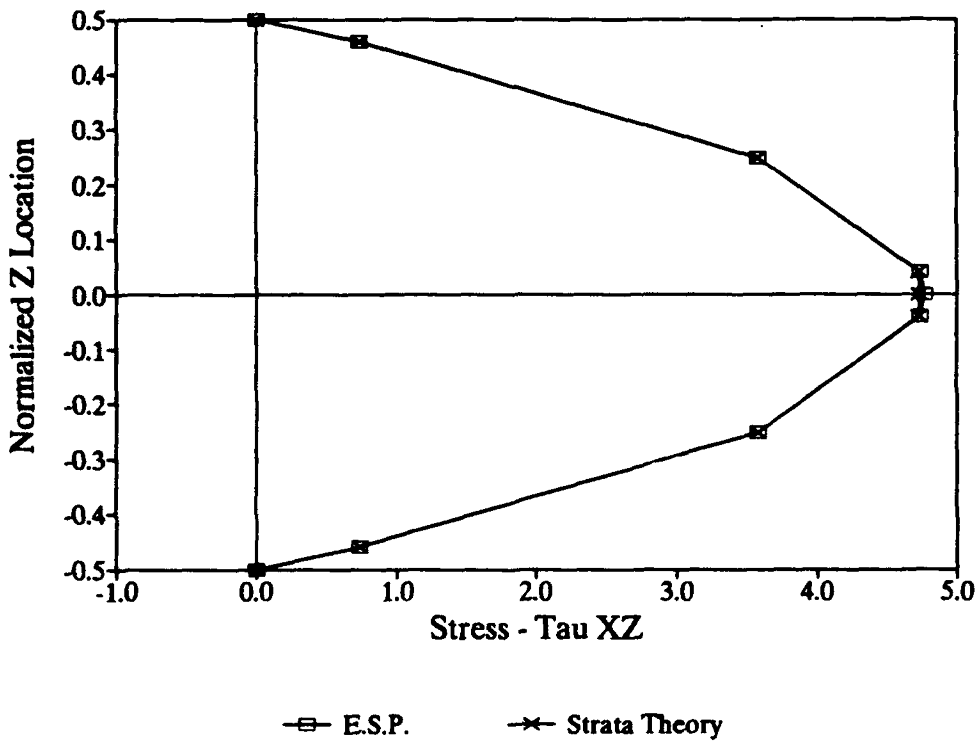

Figure 44. τ_{xz} for cylindrical bending isotropic plate

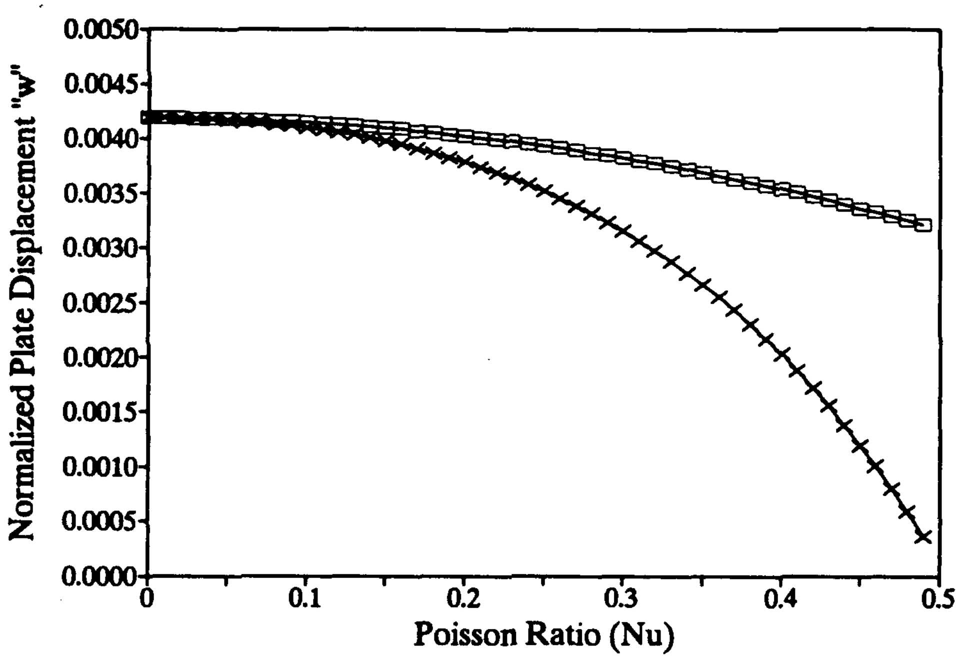

Figure 45. Displacement w for cylindrical bending isotropic plate vs. ν

The following tables list the values of the parameters that were calculated for the preceding plots:

Table 10. Matrix strata elastic coefficients (cylindrical bending isotropic plate)

K_{1m}	6.4286E+005
K_{2m}	4.2857E+005
K_{3m}	1.0714E+005

Just as with the rectangular plate, there was a reasonable degree of agreement between ESP and ST. There was excellent agreement for τ_{xz}, σ_x, and γ_{xz} when a cubic (or higher order) polynomial was used for ST. The remaining stresses and strains as well

Table 11. Strata Theory coefficients (cylindrical bending isotropic plate)

$$
\begin{aligned}
A &= 2.0297E\text{-}003 \\
B_0 &= 0.0 \\
B_1 &= -5.9320E\text{-}004 \\
B_2 &= 0.0 \\
B_3 &= -5.8545E\text{-}005 \\
B_4 &= 0.0 \\
B_5 &= -1.7452E\text{-}006
\end{aligned}
$$

Table 12. ESP displacement function coefficients (cylindrical bending isotropic plate)

λU_0	-1.1051E-003
λW_0	1.1245E-004
U_0	5.4064E-004
W_0	3.5178E-003

Evaluated at z=-h/2

as the displacements displayed the same general behavior for both ESP and ST, although the actual values calculated disagreed by nearly a factor of 2 in some cases.

The amount by which ESP and ST disagree appears to be primarily dependent upon the value of the Poisson's ratio (ν). When the Poisson's ratio ν is allowed to vary, it is immediately apparent from Figure 45. that ESP and ST agree exactly when $\nu=0$, and the two solutions diverge as ν increases.

Just as with the rectangular plate, ST assumes that $\epsilon_z=0$, while ESP expects a nonzero ϵ_z. Since the normal stresses are coupled through ν, elimination of ϵ_z introduces an error that is not present in ESP. Again, and there is poor agreement for ϵ_z and σ_z.

The 2-Ply Cylindrical Bending Case

The complete set of displacements, stresses and strains are shown in Figures 46. through 53. The displacements are obtained directly from both ST and ESP. The strains are derived from the displacement information, and stresses are calculated using the constitutive stress-strain relations, given in Equations (9) and (16).

The following plots for the stresses, strains, and displacements include only the z-dependent portion of these parameters. In other words, the x- and y-dependent sine and cosine functions are divided out for the presentation of this data.

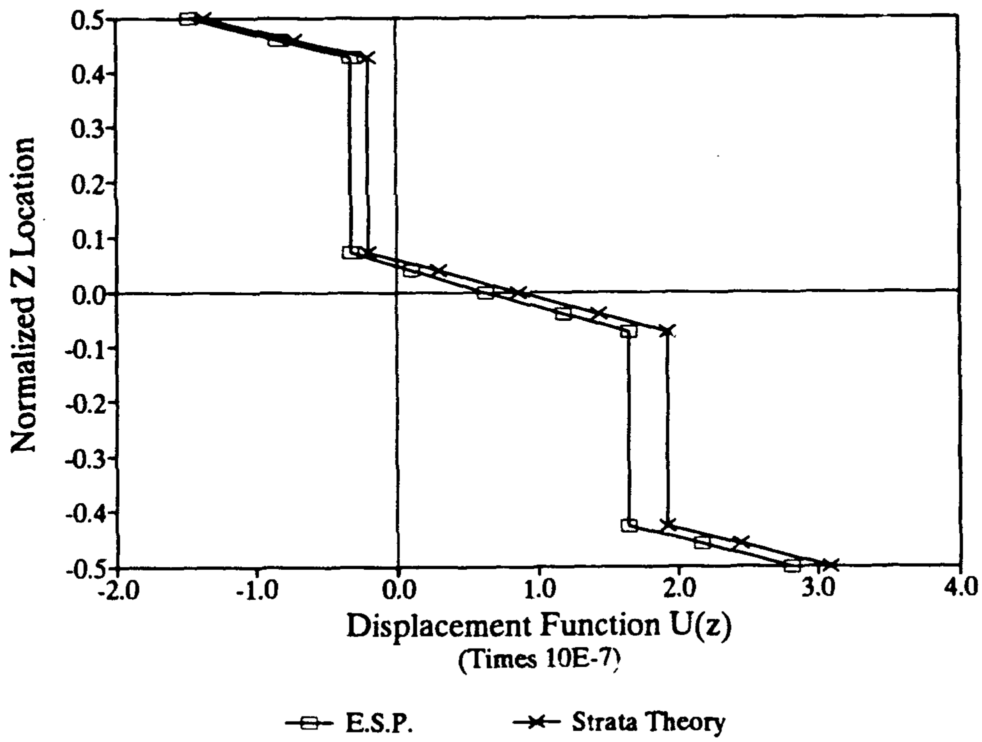

Figure 46. In-plane displacement u for cylindrical bending 2-ply composite plate

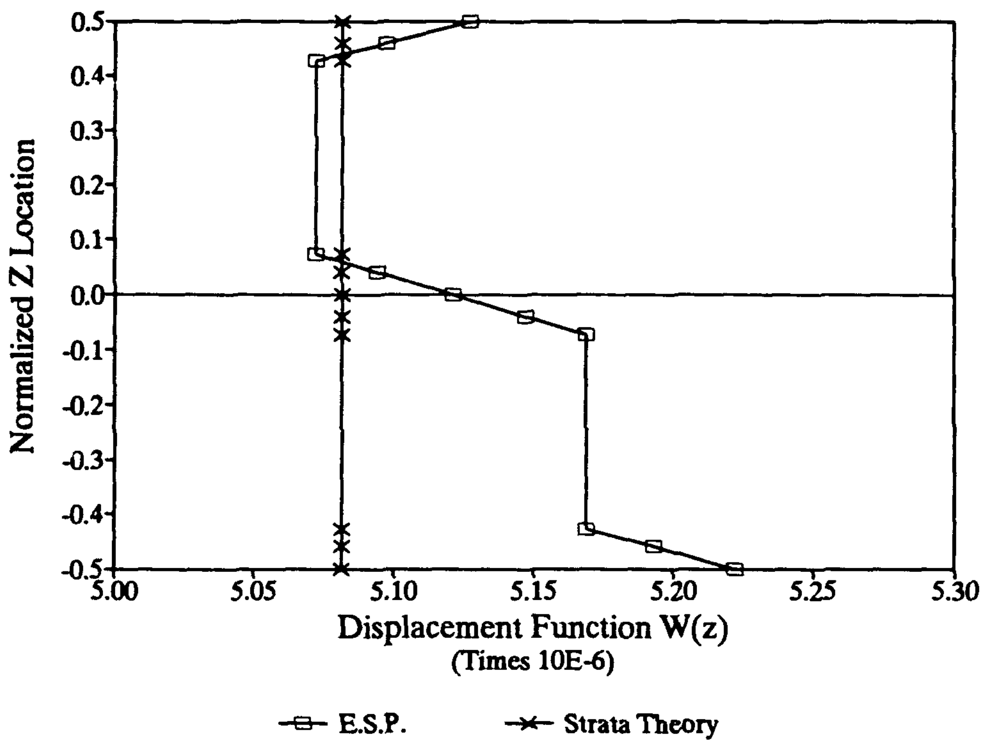

Figure 47. Out-of-plane displacement w for cylindrical bending 2-ply composite plate

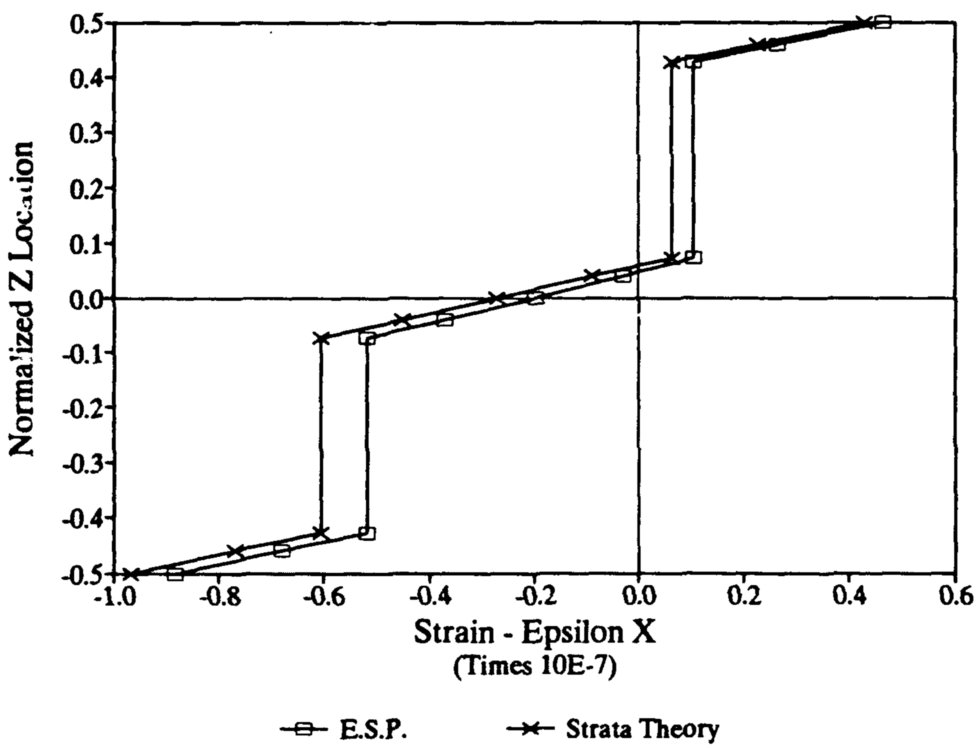

Figure 48. ϵ_x for cylindrical bending 2-Ply composite plate

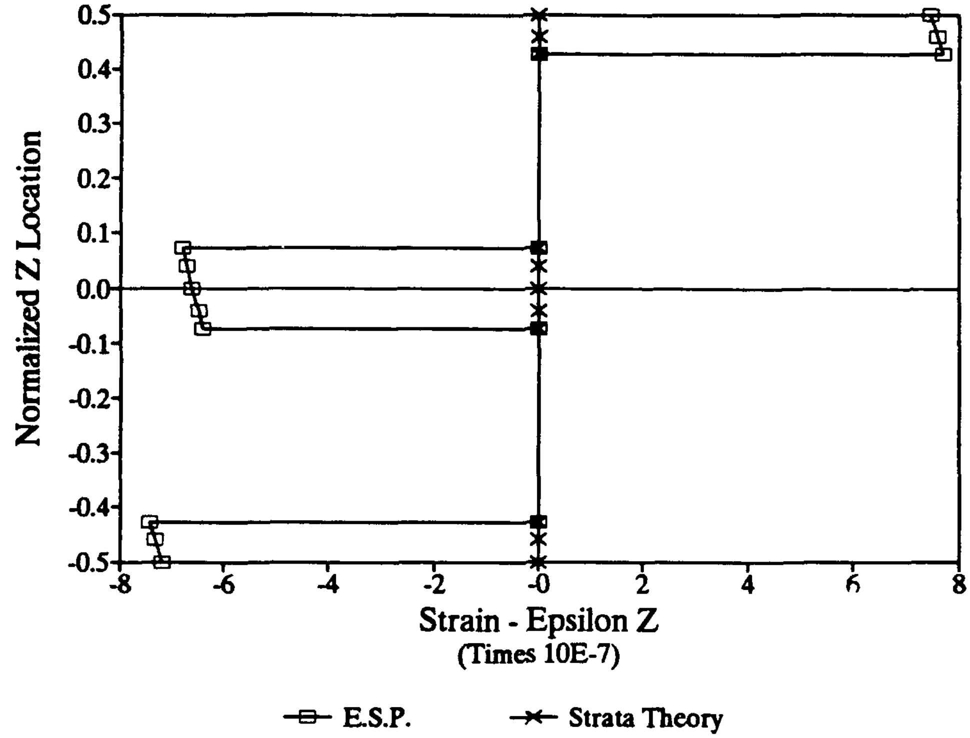

Figure 49. ϵ_z for cylindrical bending 2-Ply composite plate

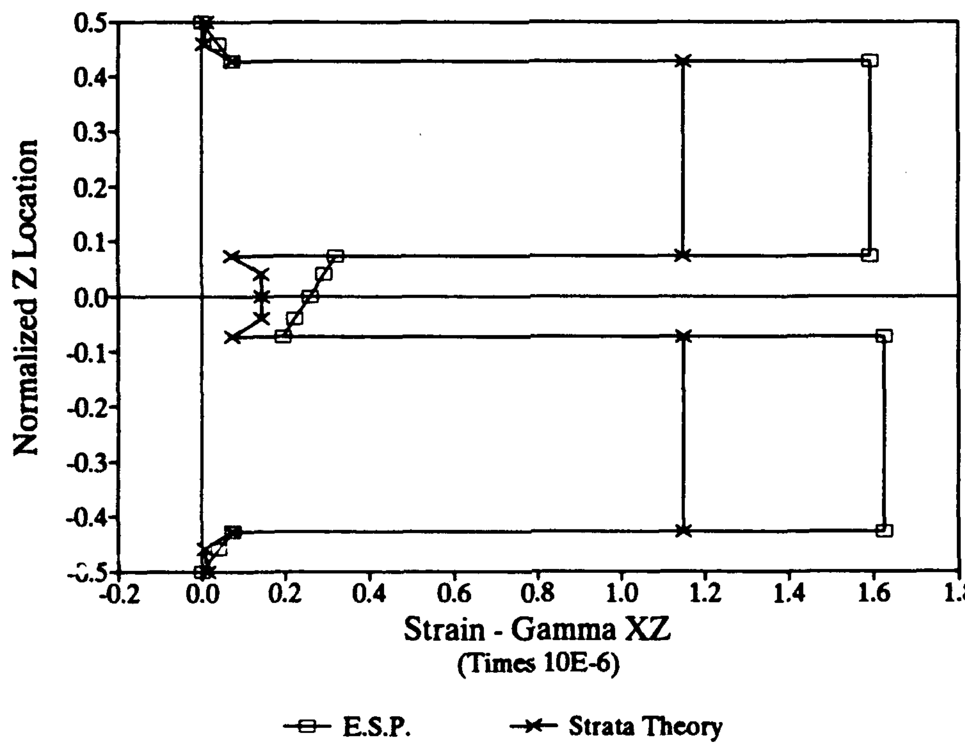

Figure 50. γ_{xz} for cylindrical bending 2-Ply composite plate

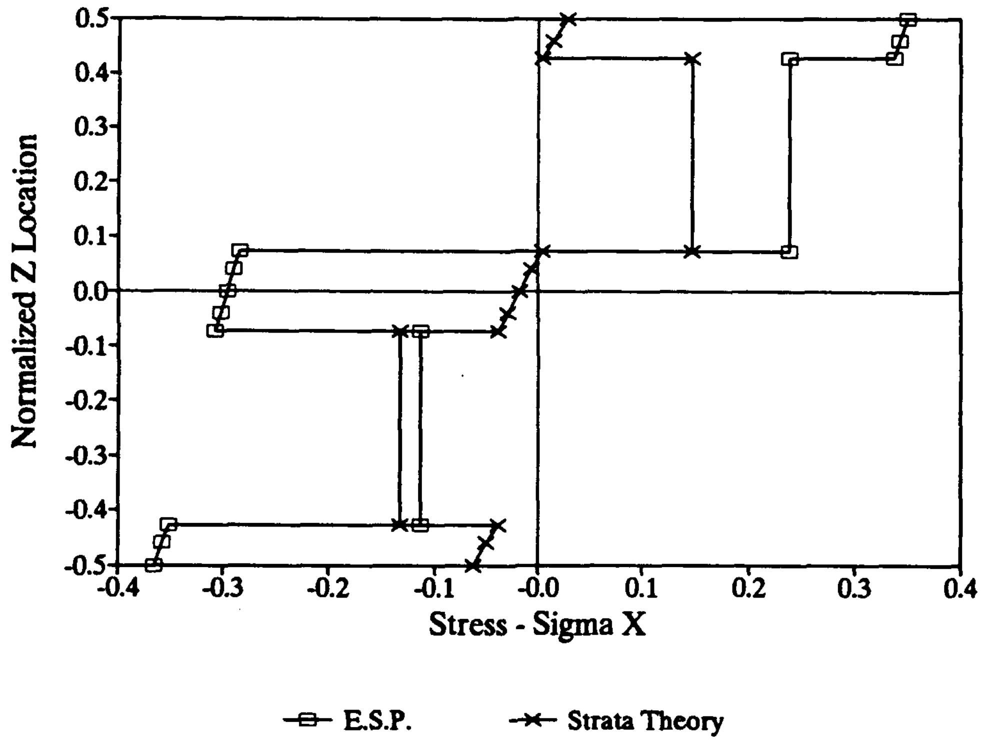

Figure 51. σ_x for cylindrical bending 2-Ply composite plate

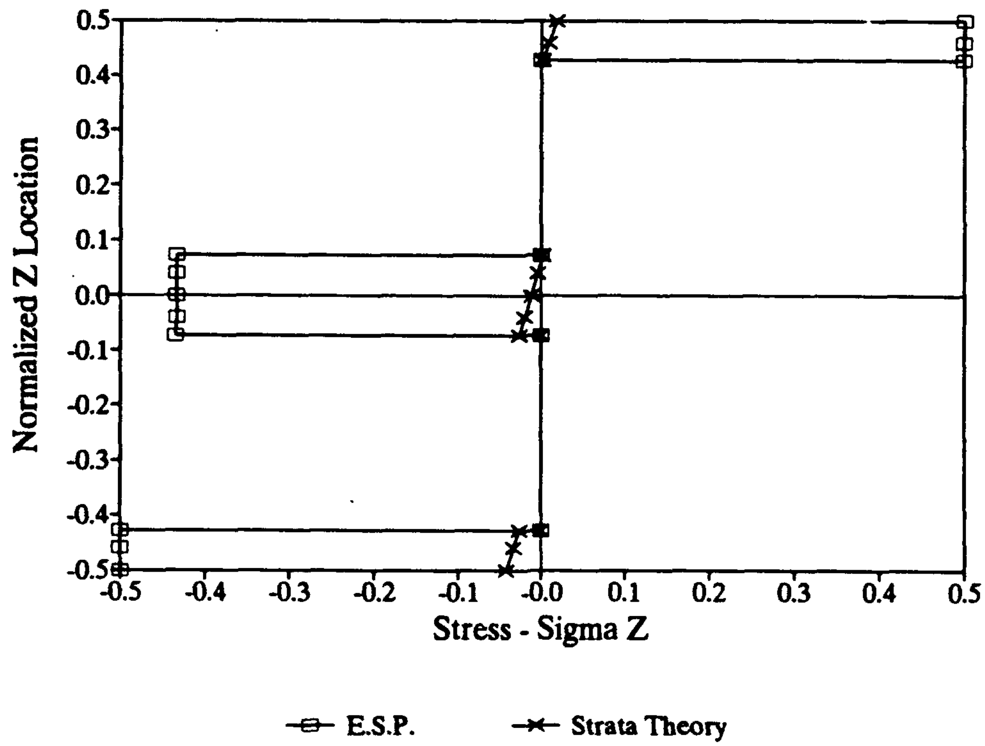

Figure 52. σ_z for cylindrical bending 2-Ply composite plate

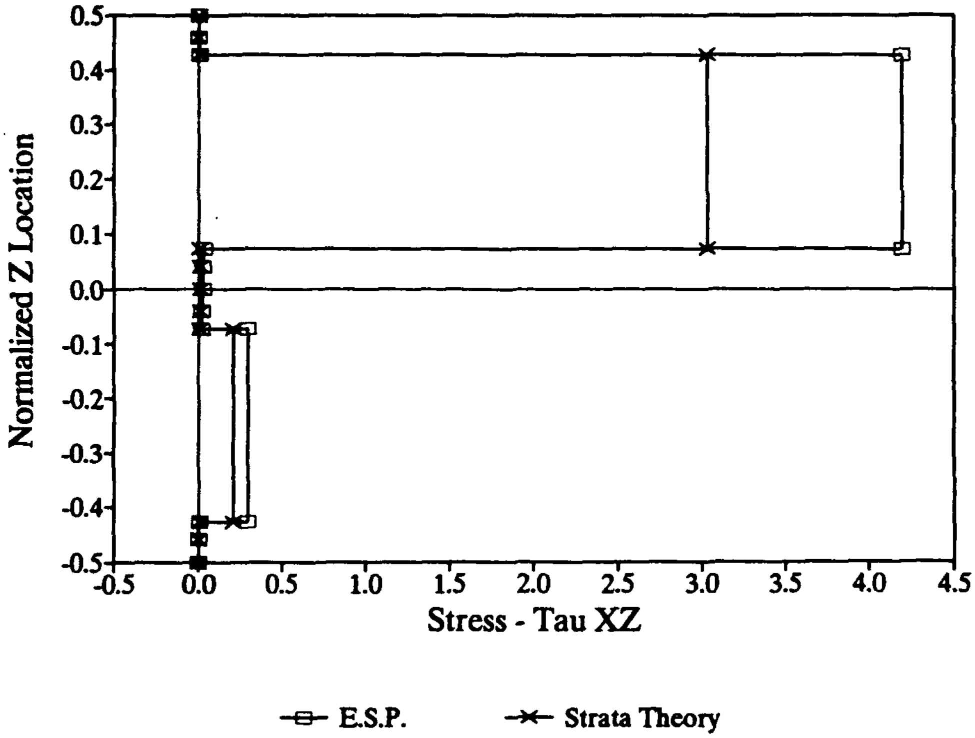

Figure 53. τ_{xz} for cylindrical bending 2-Ply composite plate

The following tables list the values of the parameters that were calculated for the preceding plots:

Table 13. Matrix strata elastic coefficients (2-ply cylindrical bending plate)

K_{1m}	6.4286E+005
K_{2m}	4.2857E+005
K_{3m}	1.0714E+005

When fibers are present, the agreement between ST and ESP for the displacements u, v, and w are very good. Also, there is very good agreement between ESP and ST in the general behavior for all the stresses and strains, with the obvious

Table 14. Fiber strata elastic coefficients (2-ply cylindrical bending plate)

	Fiber Stratum 1	Fiber Stratum 1
S_{1f}	2.2886E+007	2.1949E+006
S_{2f}	2.1949E+006	2.2886E+007
S_{3f}	1.2347E+007	1.2347E+007
S_{4f}	3.6581E+005	3.6581E+005
S_{5f}	0.0000E+000	0.0000E+000
S_{6f}	0.0000E+000	0.0000E+000
S_{7f}	5.2692E+006	3.6581E+005
S_{8f}	3.6581E+005	5.2692E+006
S_{9f}	0.0000E+000	0.0000E+000
S_{10f}	1.2347E+007	4.2857E+005
S_{11f}	4.2857E+005	1.2347E+007
S_{12f}	2.2886E+007	2.2886E+007
S_{13f}	0.0000E+000	0.0000E+000

Table 15. Strata Theory coefficients (2-ply cylindrical bending plate)

$$
\begin{aligned}
A &= 5.0817\text{E-}006 \\
B_0 &= 8.6071\text{E-}008 \\
B_1 &= -4.1525\text{E-}007 \\
B_2 &= 2.6488\text{E-}010 \\
B_3 &= -1.9537\text{E-}007 \\
B_4 &= -1.7935\text{E-}009 \\
B_5 &= 3.1359\text{E-}007
\end{aligned}
$$

exception of ϵ_z. The polynomial form of the z dependence of the assumed displacement field is clearly evident in the plots of the shear stresses and shear strains.

In comparing the results of the 2-ply composite with the isotropic plate, it is obvious that the fibers dominate the behavior of the composite. While the general behavior of the ST results echo those of ESP, there is still some disagreement in the

Table 16. ESP displacement function coefficients (2-ply cylindrical bending plate)

	Matrix Stratum 1	Matrix Stratum 2	Matrix Stratum 3
λU_0	-1.5197E-006	-1.4293E-006	-1.6406E-006
λW_0	7.7067E-007	-6.4056E-007	-7.1890E-007
U_0	-3.3253E-008	1.6444E-007	2.8112E-007
W_0	5.0720E-006	5.1687E-006	5.2223E-006

Evaluated at the z coordinate at the bottom of each matrix stratum

actual values calculated for displacements, stresses, and strains. Other test cases for a square plate were examined, and they revealed that the level of agreement between ESP and ST seems to be most highly dependent upon the Poisson's ratio for the fibers (ν_f), and the order of the polynomial (N) assumed for ST.

Just as with the isotropic plate, there is considerable disagreement in ϵ_z between ST and ESP for a composite panel, but this should be expected. In the matrix strata that comprise the top and bottom surfaces of the composite, note that ϵ_z increases slightly as z gets farther away from the free surfaces at $z=\pm h/2$. This demonstrates a coupling of normal stresses. At the free surfaces of the composite, the matrix is allowed to deform freely through Poisson effects. However, at the matrix-fiber strata interface, the matrix is forced to match the displacements of the fibers, and so it cannot deform freely.

Because ϵ_z is allowed to be nonzero for ESP, ESP predicts that the presence of fibers results in significant jumps in ϵ_z through the thickness. As indicated for the rectangular plate case, the through-the-thickness shear through the fiber strata affects how ϵ_z varies for the ESP solution. A fiber ply where the fibers span the width of the plate will absorb much more strain energy than a ply with fibers that run parallel to the simply-supported edges. This induces a larger jump in ϵ_z between the top and middle matrix strata than between the middle and bottom matrix strata [Figure 49.]. If the fiber plies were reversed (90 & 0 instead of 0 & 90), there would be a greater difference in ϵ_z across the second fiber ply than the first, and ϵ_z through the middle matrix stratum would be positive by the same average magnitude (i.e. it would be "flipped" to the positive side of the graph). The cylindrical bending case is the limiting case of a rectangular plate with a high aspect ratio.

As was indicated for a rectangular plate, both ST and ESP predict that the z displacement (w) of a composite varies as a function of the plate width squared (a^2). In other words, the limit of w/a^2 as "a" goes to infinity is a constant. When the given load distribution is used, and a linear polynomial is assumed for ST (i.e. N=1), the solution for the coefficient "A" of the ST assumed displacement field predicts this limit to be

$$\lim_{a\to\infty} \frac{A}{a^2} = \frac{\rho}{\pi^2 S^0_{7_f}} \tag{84}$$

This is a fundamentally different behavior than that of an isotropic plate or for a set of stacked orthotropic plates. CPT, the exact orthotropic plate solutions developed by Pagano [8:398-411; 9:20-34], ESP, or ST (when applied to an isotropic plate) predicts that the limit of w/a^4 as "a" goes to infinity is a constant. Note the dependence on the characteristic plate width "a". Applying Navier's solution to the cylindrical bending case for an isotropic material, the normalized displacement is

$$\lim_{a\to\infty} \frac{\bar{w}}{a^4} = \frac{12\rho(1-\nu^2)}{h^3\pi^4 E} \qquad \bar{w} = w\left(x=\frac{a}{2}, z=0\right) \tag{85}$$

The expression for an isotropic plate using ST becomes

$$\lim_{a \to \infty} \frac{A}{a^4} = \frac{12\rho(1-2\nu_m)(1+\nu_m)}{h^3\pi^4 E_m(1-\nu_m)} \tag{86}$$

Similar expressions arise for CPT or the exact orthotropic plate solutions as well.

VI. Conclusions and Recommendations

Interpretation of Results

For an isotropic plate, there was a reasonable degree of agreement between ESP and ST. There was excellent agreement for τ_{xz}, τ_{yz}, γ_{xz}, and γ_{yz} when a cubic (or higher order) polynomial was used for ST. The remaining stresses and strains as well as the displacements displayed the same general behavior for both ESP and ST, although the actual values calculated disagreed by nearly a factor of 2 in some cases. Since the normal stresses are coupled through ν, and since ϵ_z is set to zero in ST, this introduces an error that is not present in ESP. As a result, the amount by which ESP and ST disagree is heavily dependent on the value of the Poisson's ratio (ν).

When fibers are present, however, the agreement between ST and ESP for the displacements u, v, and w are very good. Also, there is very good agreement between ESP and ST in the general behavior for all the stresses and strains, with the obvious exception of ϵ_z. While it is clear that the fibers dominate the behavior of the composite, the level of agreement in the actual values calculated seems to be most highly dependent upon the Poisson's ratio for the fibers (ν_f).

Since the difference in the displacements between the ESP and ST solutions for a composite are very small, the approximation that epsilon z is zero appears to be a reasonably valid assumption for ST. Although ST was not really intended to be used for detailed stress/strain prediction, the fact that ST and ESP agree so well is another confirmation of the general validity of the ST approach. It also reinforces the potential of ST as a dynamic analysis tool.

The applied load used for the example calculations had a simple sinusoidal distribution, but Fourier analysis can be used to decompose any arbitrary load into an infinite sum of sines and cosines. The equations used for the example problems can readily be generalized for more complex loads by modifying the values of p and q. However, if many terms of the Fourier series are needed to represent the applied load, considerable computation time may be required.

Areas for Further Research

It is not clear whether the behavior of the normalized displacement w as a function of plate width represents the true behavior of a composite panel. This would seem to be an important characteristic of the Stratified Plate Model that needs to be verified, or refuted, experimentally.

The ST equation development readily lends itself to a formulation that includes a dynamic term. If the ST method is to be used for mode shape prediction, verification of the behavior of ST with respect to the plate dimensions, as suggested above, is needed as a preliminary and preparatory step to determine the suitability of ST as a vibration analysis tool.

There are indications that the ST formulation would lend itself well to the development of a new type of finite element. Indeed, some preliminary work in this area has already been done. Once the formulation of such an element is developed, validation and verification is needed to determine such aspects as convergence and compatibility within and between elements.

One of the strengths of the ST method is that the order of the polynomial (N) for the assumed z dependence in ST can be selected as necessary to obtain the desired level of accuracy. This study did not explicitly examine the effect of N on the accuracy of the results, particularly for multi-ply composites, but it is likely that the value of N that yields the best trade-off between computation time and accuracy will be a matter of experience and the specific details of the problem being studied. Nevertheless, ST should be applied to multi-ply composites to ensure convergence and determine the order of the polynomial (N) required for adequate accuracy.

Also, ST should be applied to lay-ups with angle plies to examine the coupling effects, and to determine possible forms of solutions for such cases. The x- and y-dependent portions of the assumed displacement field for a simply-supported rectangular plate with cross-plies were fairly easy to obtain, but the form of the displacement field expressions when angle-plies are present are not so obvious and should be explored further.

SOLVE

```
A=10
B=10
setparam
calparam
exactsol
stxplies
quit
```

SETPARAM

```
% SETPARAM.M routine for PC-MATLAB
%
format short e
global n m A B p q matprops fibprops zdist N alpha beta tf tm FM R RMI
diary setparam.log
%
iso=0
n=1
m=1
%
Em=0.3e6
NUm=.40
Gm=Em/(1+NUm)/2
%
k1=Em*(1-NUm)/(1-2*NUm)/(1+NUm)
k2=Em*NUm/(1-2*NUm)/(1+NUm)
k3=Gm
%
KM=[k1 k2 k2 0  0  0;
    k2 k1 k2 0  0  0;
    k2 k2 k1 0  0  0;
    0  0  0  k3 0  0;
    0  0  0  0  k3 0;
    0  0  0  0  0  k3];
%
matprops=[k1 k2 k3]
%
if iso==1
    tf=[0 0]
    tm=[.25 .5 .25]
    alpha=0
    beta=1
    %
    fibprops=[0 0 0 0 0 0 0 0 0;
              0 0 0 0 0 0 0 0 0]
else
    sqrt2=sqrt(2)
    volf=sqrt2/2
    volm=1-volf
    %
    tf=[sqrt2/4 sqrt2/4]
    tm=[.25-sqrt2/8 .5-sqrt2/4 .25-sqrt2/8]
    alpha=-(1+sqrt2)/2
    beta=2+sqrt2
    %
```

```
    Ef=20e6
    NUf=.35
    Gf=Ef/(1+NUf)/2
    %
    s1=volf*Ef*(1-NUf)/(1-2*NUf)/(1+NUf)+volm*k1
    s2=k1/volm
    s3=volf*Ef*NUf/(1-2*NUf)/(1+NUf)+volm*k2
    s4=Gm/volm
    s5=0
    s6=0
    s7=volf*Gf+volm*Gm
    s8=Gm/volm
    s9=0
    s10=k2
    %
    fibprops=[s1 s2 s3 s4 s5 s6 s7 s8 s9;
              s2 s1 s3 s4 s5 s6 s8 s7 s9]
    %
    SM1=[s1  s3  0    0    0;
         s3  s2  0    0    0;
         s3 s10  0    0    0;
          0   0 s4/2  0    0;
          0   0  0   s7/2  0;
          0   0  0    0   s8/2]
    %
    SM2=[s2  s3  0    0    0;
         s3  s1  0    0    0;
         s10 s3  0    0    0;
          0   0 s4/2  0    0;
          0   0  0   s8/2  0;
          0   0  0    0   s7/2]
end
%
zdist=[.25, -.25]
N=5
```

CALPARAM

```
% CALPARAM.M routine for PC-MATLAB
%
p=n*pi/A
q=m*pi/B
%
R2=p*p+q*q
R=sqrt(R2)
%
RM=[1  R    R2   R*R2    R2*R2   R*R2*R2;
    0  1   2*R   3*R2   4*R*R2   5*R2*R2;
    0  0    2     6*R    12*R2   20*R*R2;
    1 -R    R2  -R*R2    R2*R2  -R*R2*R2;
    0  1  -2*R   3*R2  -4*R*R2   5*R2*R2;
    0  0    2    -6*R    12*R2  -20*R*R2]
%
RMI=inv(RM)
```

EXACTSOL

```
% EXACTSOL.M routine for PC-MATLAB
%
diary exactsol.log
[field1,field2,bc]=gmatlmat
zero=zeros(6,6);
%
zv1=expmat(tm(1));
zv2=expmat(tm(2));
zv3=expmat(tm(3));
%
bc1=[zv1    zero   zero;
     eye(6) zero   zero;
     zero   zv2    zero;
     zero   eye(6) zero;
     zero   zero   zv3;
     zero   zero   eye(6)]
%
bc0=bc*bc1
CondNum=rcond(bc0)
bcinv=inv(bc0)
%
force=[0;0;.5;0;0;0;0;0;0;0;0;0;0;0;0;0;0;.5]
%
UVW=bcinv*force
uvw=zeros(7,11);
uvw(7,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,-.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
%
UVWset=[UVW(1:6)];
tmBottom=.5-tm(1);
%
for ii=(1:11)
   if ii==4
      UVWset=[UVW(7:12)];
      tmBottom=-tm(2)/2;
   elseif ii==9
      UVWset=[UVW(13:18)];
      tmBottom=-.5;
   end
   uvw(1:6,ii)=expmat(uvw(7,ii)-tmBottom)*UVWset;
end
uvw
%
diary exactsol.prn
uvw'
%
stress=zeros(7,11);
stress(7,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,
            -.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
strain=zeros(7,11);
strain(7,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,
            -.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
%
StrainMat=[0 0 0 -p  0 0;
           0 0 0  0 -q 0;
           0 0 1  0  0 0;
           0 0 0  q  p 0;
           1 0 0  0  0 p;
           0 1 0  0  0 q];
%
for ii=(1:11)
   strain(1:6,ii)=StrainMat*uvw(1:6,ii);
```

```
   stress(1:6,ii)=KM*StrainMat*uvw(1:6,ii);
end
diary exstress.prn
strain'
stress'
%
diary off
if iso==0
   uvwf=zeros(3,2);
   stress=zeros(7,2);
   stress(7,1:2)=[tf(1) tf(2)];
   strain=zeros(6,2);
   strain(6,1:2)=[tf(1) tf(2)];
   %
   StrainMat=[-p 0 0;
              0 -q 0;
              q  p 0;
              0  0 p;
              0  0 q];
   %
   for ii=(1:2)
      uvwf(1,ii)=UVW(ii*6-2);
      uvwf(2,ii)=UVW(ii*6-1);
      uvwf(3,ii)=UVW(ii*6);
      strain(1:5,ii)=StrainMat*uvwf(1:3,ii);
   end
   stress(1:6,1)=SM1*StrainMat*uvwf(1:3,1);
   stress(1:6,2)=SM2*StrainMat*uvwf(1:3,2);
   diary ex-fstr.prn
   strain'
   stress'
   %
   diary off
end
```

STXPLIES

```
% STXPLIES - The name stands for "STRATA THEORY, CROSS-PLIES". This routine
%            creates the matrix of coefficients that arise from the PDEs
%            when using the assumed form of psi, zeta, and w.
%
c1=matprops(1);
c2=matprops(2);
c3=matprops(3);
ps=p*p;
qs=q*q;
pq=p*q;
bs=beta*beta;
stheory=zeros(2*N+3,2*N+3);
stheory(1,1)=(c3*ks(0)+ss(7,0))*ps+(c3*ks(0)+ss(8,0))*qs;
%
for ii=(1:N)
   stheory(1,ii+2)=ii*beta*c3*p*ks(ii-1);
   stheory(1,ii+N+3)=ii*beta*c3*q*ks(ii-1);
end
%
for jj=(0:N)
   stheory(jj+2,1)=jj*beta*c3*p*ks(jj-1);
   for ii=(0:N)
      kstiff=ks(ii+jj);
      stheory(jj+2,ii+2)=ii*jj*bs*c3*ks(ii+jj-2)
                +(ss(1,ii+jj)+c1*kstiff)*ps+(ss(4,ii+jj)+c3*kstiff)*qs;
```

```
      stheory(jj+2,ii+N+3)=(ss(3,ii+jj)+ss(4,ii+jj)+(c2+c3)*kstiff)*pq;
   end
end
%
for jj=(0:N)
   stheory(jj+N+3,1)=jj*beta*c3*q*ks(jj-1);
   for ii=(0:N)
      kstiff=ks(ii+jj);
      stheory(jj+N+3,ii+2)=(ss(3,ii+jj)+ss(4,ii+jj)+(c2+c3)*kstiff)*pq;
      stheory(jj+N+3,ii+N+3)=ii*jj*bs*c3*ks(ii+jj-2)
                    +(ss(2,ii+jj)+c1*kstiff)*qs+(ss(4,ii+jj)+c3*kstiff)*ps;
   end
end
%
diary strtheor.log
stheory
force=zeros(2*N+3,1);
force(1)=1;
CondNum=rcond(stheory)
%
UVW=inv(stheory)*force
uvw=zeros(7,11);
uvw(7,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,-.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
%
alphafactor=[1,1,1,0,0,0,0,0,-1,-1,-1];
for ii=(1:11)
   uvw(1,ii)=0;
   uvw(2,ii)=0;
   uvw(4,ii)=0;
   uvw(5,ii)=0;
   aplusbz=alpha*alphafactor(ii)+beta*uvw(7,ii);
   for jj=(0:N)
      uvw(1,ii)=uvw(1,ii)+UVW(jj+2)*jj*beta*aplusbz^(jj-1);
      uvw(2,ii)=uvw(2,ii)+UVW(jj+N+3)*jj*beta*aplusbz^(jj-1);
      uvw(4,ii)=uvw(4,ii)+UVW(jj+2)*aplusbz^jj;
      uvw(5,ii)=uvw(5,ii)+UVW(jj+N+3)*aplusbz^jj;
   end
   uvw(3,ii)=0;
   uvw(6,ii)=UVW(1);
end
uvw
diary strtheor.prn
uvw'
%
stress=zeros(7,11);
stress(7,1:11)=[.5,.46,0.50-tm(1),tm(2)/2,.04,0.,
                    -.04,-tm(2)/2,tm(3)-0.50,-.46,-.5];
strain=zeros(7,11);
strain(7,1:11)=[.5,.46,0.50-tm(1),tm(2)/2,.04,0.,
                    -.04,-tm(2)/2,tm(3)-0.50,-.46,-.5];
%
StrainMat=[0 0 0 -p  0 0;
           0 0 0  0 -q 0;
           0 0 1  0  0 0;
           0 0 0  q  p 0;
           1 0 0  0  0 p;
           0 1 0  0  0 q];
%
for ii=(1:11)
   strain(1:6,ii)=StrainMat*uvw(1:6,ii);
   stress(1:6,ii)=KM*StrainMat*uvw(1:6,ii);
end
diary ststress.prn
```

```
strain'
stress'
%
diary off
if iso==0
   uvwf=zeros(5,2);
   stress=zeros(7,2);
   stress(7,1:2)=[tf(1) tf(2)];
   strain=zeros(6,2);
   strain(6,1:2)=[tf(1) tf(2)];
   %
   StrainMat=[0 0 -p  0 0;
              0 0  0 -q 0;
              0 0  q  p 0;
              1 0  0  0 p;
              0 1  0  0 q];
   %
   for ii=(1:2)
      uvwf(1,ii)=0;
      uvwf(2,ii)=0;
      uvwf(3,ii)=0;
      uvwf(4,ii)=0;
      zf=zdist(ii);
      for jj=(0:N)
         uvwf(1,ii)=uvwf(1,ii)+UVW(jj+2)*jj*zf^(jj-1);
         uvwf(2,ii)=uvwf(2,ii)+UVW(jj+N+3)*jj*zf^(jj-1);
         uvwf(3,ii)=uvwf(3,ii)+UVW(jj+2)*zf^jj;
         uvwf(4,ii)=uvwf(4,ii)+UVW(jj+N+3)*zf^jj;
      end
      uvwf(5,ii)=UVW(1);
      strain(1:5,ii)=StrainMat*uvwf(1:5,ii);
   end
   stress(1:6,1)=SM1*StrainMat*uvwf(1:5,1);
   stress(1:6,2)=SM2*StrainMat*uvwf(1:5,2);
   diary st-fstr.prn
   strain'
   stress'
   %
   diary off
end
```

EXPMAT

```
function em=expmat(z)
% EXPMAT - Function to generate the matrix exponential for the boundary
%          condition matrices of the "EXACT STRATIFIED PLATE SOLUTION" for
%          the analysis of composite panels.

ezp=exp(R*z);
ezm=exp(-R*z);
eVector=[ezp; z*ezp; z*z*ezp; ezm; z*ezm; z*z*ezm];
aV=RMI*eVector
em=aV(1)*eye(6)+aV(2)*FM+aV(3)*FM*FM+aV(4)*FM*FM*FM+aV(5)*FM*FM*FM*FM
   +aV(6)*FM*FM*FM*FM*FM;
```

GMATLMAT

```
function [fm1,fm2,bc]=gmatlmat
% GMATLMAT  Function to generate the material property matrices for the
%           "EXACT" solution of the cross-ply composite panel problem.
%
```

```
% Syntax:    [FM1,FM2,BC]=GMATLMAT
%
%         p  =      nπ/a
%         q  =      mπ/b
%        mp  =      Vector of material properties (1x3)
%        fp  =      Vector of fiber strata properties (1x9)
%
%        BC  =      Boundary condition matrix
%       FM1  =      Field matrix #1
%       FM2  =      Field matrix #2
%                                       u  =  Displacement vector
%     where [FM1] u = L [FM2] u         L  =  Eigenvalue (aka lambda)
%
% This function is used in conjunction with other MATLAB routines to
% verify the "STRATA THEORY" of composite panel analysis.  Written by
% Alan L. Lesmerises.

c1=matprops(1)
c2=matprops(2)
c3=matprops(3)
c4=c2+c3
z=zeros(3,3);
%
m1=[c3 0 0;
    0 c3 0;
    0 0 c1];
%
m2=c4*[ 0   0   p;
        0   0   q;
       -p  -q   0];
%
m3=-[(p*p*c1+q*q*c3)     p*q*c4            0        ;
       p*q*c4       (p*p*c3+q*q*c1)        0        ;
         0                 0          (p*p+q*q)*c3];
%
% Above matrices appear in PDE as:
%
%      [m1]u"+[m2]u'+[m3]u=0
% or
%
%     [ [m1]   [0]  ] [Lu]       [ [0]   [m1] ] [Lu]
%     [ [0]  -[m3]  ] [ u]  = L  [ [m1]  [m2] ] [ u]
%
%
fm1=[m1 z ;
     z -m3];
%
fm2=[ z m1;
     m1 m2];
%
FM=inv(fm2)*fm1
%
m2=[ 0       0    p*c3;
     0       0    q*c3;
   -p*c2  -q*c2   0   ];
%
s1=fibprops(1,1);
s2=fibprops(1,2);
s3=fibprops(1,3);
s4=fibprops(1,4);
s7=fibprops(1,7);
s8=fibprops(1,8);
m2f1=m2+tf(1)*[(p*p*s1+q*q*s4)    p*q*(s3+s4)           0         ;
```

```
                         p*q*(s3+s4)    (p*p*s4+q*q*s2)         0        ;
                              0                0         (p*p*s7+q*q*s8)];
%
s1=fibprops(2,1);
s2=fibprops(2,2);
s3=fibprops(2,3);
s4=fibprops(2,4);
s7=fibprops(2,7);
s8=fibprops(2,8);
m2f2=m2+tf(2)*[(p*p*s1+q*q*s4)    p*q*(s3+s4)           0        ;
                 p*q*(s3+s4)    (p*p*s4+q*q*s2)         0        ;
                      0                0         (p*p*s7+q*q*s8)];
%
I3=eye(3);
bc=[m1 m2  z   z  z   z   z   z  z   z   z   z;
    z  z  -m1 -m2 m1 m2f1 z   z  z   z   z   z;
    z  z   z  I3  z  -I3  z   z  z   z   z   z;
    z  z   z   z  z   z  -m1 -m2 m1 m2f2 z   z;
    z  z   z   z  z   z   z  I3  z  -I3  z   z;
    z  z   z   z  z   z   z   z  z   z  -m1 -m2]
%
% End of function GMATLMAT.M
%
```

TREND

```
setparam
extrend=zeros(4,50);
for jndex=1:50
   A=jndex
   B=A;
   extrend(1,jndex)=A;
   extrend(2,jndex)=B;
   calparam
   exactsol
   extrend(3,jndex)=uvw(6,6);
   stxplies
   extrend(4,jndex)=uvw(6,6);
end
diary trend.prn
extrend'
diary off
quit
```

KS

```
function k=ks(order)
% KS - Evaluates the integral in z that is associated with the k's that
%      appear in the P.D.E.'s for the "STRATA THEORY" for the analysis of
%      composite panels.

k=0;
if order < 0
elseif (order/2 == fix(order/2))
   k=1/(order+1)/2^order/beta;
end
```

SS

```
function sstiff=ss(sterm,order)
%
% SS - Sum the terms of Σ s(f,j) * z^n
%
% Syntax    SS(j,order)
%
% where:
%           s(f,j)  =  membrane elasticity terms
%                f  =  fiber number
%                j  =  the elasticity term of interest
%                n  =  the "order" of the z term
%
%            sterm  =  membrane properties vector
%            order  =  the order of the z term (n above)

[n,numfibers]=size(zdist);
sstiff=0;
for index=(1:numfibers)
   sstiff=sstiff+fibprops(index,sterm)*zdist(index)^order*tf(index);
end
%
% End of SS.M function
%
```

SOLVE

```
A=10
setparam
calparam
exactsol
stxplies
quit
```

SETPARAM

```
% SETPARAM.M routine for PC-MATLAB
%
format short e
global n A p matprops fibprops zdist N alpha beta tf tm FM RMI
diary setparam.log
%
iso=0
n=1
%
Em=0.3e6
NUm=.40
Gm=Em/(1+NUm)/2
%
k1=Em*(1-NUm)/(1-2*NUm)/(1+NUm)
k2=Em*NUm/(1-2*NUm)/(1+NUm)
k3=Gm
%
KM=[k1 k2  0;
    k2 k1  0;
    0  0  k3];
%
matprops=[k1 k2 k3]
%
if iso==1
    tf=[0 0]
    tm=[.25 .5 .25]
    alpha=0
    beta=1
    %
    fibprops=[0 0 0 0 0 0 0 0 0;
              0 0 0 0 0 0 0 0 0];
else
    sqrt2=sqrt(2)
    volf=sqrt2/2
    volm=1-volf
    %
    tf=[sqrt2/4 sqrt2/4]
    tm=[.25-sqrt2/8 .5-sqrt2/4 .25-sqrt2/8]
    alpha=-(1+sqrt2)/2
    beta=2+sqrt2
    %
    Ef=20e6
    NUf=.35
    Gf=Ef/(1+NUf)/2
    %
    s1=volf*Ef*(1-NUf)/(1-2*NUf)/(1+NUf)+volm*k1
```

```
    s2=k1/volm
    s3=volf*Ef*NUf/(1-2*NUf)/(1+NUf)+volm*k2
    s4=Gm/volm
    s5=0
    s6=0
    s7=volf*Gf+volm*Gm
    s8=Gm/volm
    s9=0
    s10=k2
    %
    fibprops=[s1 s2 s3 s4 s5 s6 s7 s8 s9;
              s2 s1 s3 s4 s5 s6 s8 s7 s9]
    %
    % SM1=[s1  s3  0   0   0;
    %      s3  s2  0   0   0;
    %      s3 s10  0   0   0;
    %       0   0 s4/2 0   0;
    %       0   0  0  s7/2 0;
    %       0   0  0   0  s8/2]
    %
    % SM2=[s2  s3  0   0   0;
    %      s3  s1  0   0   0;
    %      s10 s3  0   0   0;
    %       0   0 s4/2 0   0;
    %       0   0  0 -s8/2 0;
    %       0   0  0   0 -s7/2]
    %
    SM1=[s1 0;
         0 s7/2]
    %
    SM2=[s2 0;
         0 s8/2]
end
%
zdist=[.25, -.25]
N=5
```

CALPARAM

```
% CALPARAM.M routine for PC-MATLAB
%
p=n*pi/A
ps=p*p
RM=[1  p   ps  p*ps;
    0  1  2*p  3*ps;
    1 -p   ps -p*ps;
    0  1 -2*p  3*ps]
RMI=inv(RM)
```

EXACTSOL

```
% EXACTSOL.M routine for PC-MATLAB
%
diary exactsol.log
[field1,field2,bc]=gmatlmat
zero=zeros(4,4);
%
zv1=expmat(tm(1));
zv2=expmat(tm(2));
zv3=expmat(tm(3));
%
```

```
bc1=[zv1    zero   zero;
     eye(4) zero   zero;
     zero   zv2    zero;
     zero   eye(4) zero;
     zero   zero   zv3;
     zero   zero   eye(4)]
%
bc0=bc*bc1
CondNum=rcond(bc0)
bcinv=inv(bc0)
%
force=[0;.5;0;0;0;0;0;0;0;0;0;.5]
%
UW=bcinv*force
uw=zeros(5,11);
uw(5,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,-.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
%
UWset=[UW(1:4)];
tmBottom=.5-tm(1);
%
for index=(1:11)
   if index==4
      UWset=[UW(5:8)];
      tmBottom=-tm(2)/2;
   elseif index==9
      UWset=[UW(9:12)];
      tmBottom=-.5;
   end
   uw(1:4,index)=expmat(uw(5,index)-tmBottom)*UWset;
end
uw
%
diary exactsol.prn
uw'
diary off
%
stress=zeros(4,11);
stress(4,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,
                  -.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
strain=zeros(4,11);
strain(4,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,
                  -.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
%
StrainMat=[0 0 -p 0;
           0 1  0 0;
           1 0  0 p];
%
for ii=(1:11)
   strain(1:3,ii)=StrainMat*uw(1:4,ii);
   stress(1:3,ii)=KM*StrainMat*uw(1:4,ii);
end
diary exstress.prn
strain'
stress'
diary off
%
if iso==0
   uwf=zeros(2,2);
   stress=zeros(3,2);
   stress(3,1:2)=[tf(1) tf(2)];
   strain=zeros(3,2);
   strain(3,1:2)=[tf(1) tf(2)];
   %
```

```
   StrainMat=[-p 0;
               0 p];
   %
   for ii=(1:2)
      uwf(1,ii)=UW(ii*4-1);
      uwf(2,ii)=UW(ii*4);
      strain(1:2,ii)=StrainMat*uwf(1:2,ii);
   end
   stress(1:2,1)=SM1*StrainMat*uwf(1:2,1);
   stress(1:2,2)=SM2*StrainMat*uwf(1:2,2);
   diary ex-fstr.prn
   strain'
   stress'
   %
   diary off
end
```

STXPLIES

```
% STXPLIES - The name stands for "STRATA THEORY, CROSS-PLIES". This routine
%            creates the matrix of coefficients that arise from the PDEs
%            when using the assumed form of psi and w.
%
c1=matprops(1);
c2=matprops(2);
c3=matprops(3);
ps=p*p;
bs=beta*beta;
stheory=zeros(N+2,N+2);
stheory(1,1)=(c3*ks(0)+ss(7,0))*ps;
%
for ii=(1:N)
   stheory(1,ii+2)=ii*beta*c3*p*ks(ii-1);
end
%
for jj=(0:N)
   stheory(jj+2,1)=jj*beta*c3*p*ks(jj-1);
   for ii=(0:N)
      stheory(jj+2,ii+2)=ii*jj*bs*c3*ks(ii+jj-2)+(ss(1,ii+jj)+c1*ks(ii+jj))*ps;
   end
end
%
diary strtheor.log
stheory
force=zeros(N+2,1);
force(1)=1;
CondNum=rcond(stheory)
UW=inv(stheory)*force
uw=zeros(5,11);
uw(5,1:11)=[.5,.46,.5-tm(1),tm(2)/2,.04,0.,-.04,-tm(2)/2,tm(3)-.5,-.46,-.5];
alphafactor=[1,1,1,0,0,0,0,0,-1,-1,-1];
for ii=(1:11)
   uw(1,ii)=0;
   uw(3,ii)=0;
   aplusbz=alpha*alphafactor(ii)+beta*uw(5,ii);
   for jj=(0:N)
      uw(1,ii)=uw(1,ii)+UW(jj+2)*jj*beta*aplusbz^(jj-1);
      uw(3,ii)=uw(3,ii)+UW(jj+2)*aplusbz^jj;
   end
   uw(2,ii)=0;
   uw(4,ii)=UW(1);
end
```

```
uw
diary strtheor.prn
uw'
diary off
%
stress=zeros(4,11);
stress(4,1:11)=[.5,.46,0.50-tm(1),tm(2)/2,.04,0.,
                        -.04,-tm(2)/2,tm(3)-0.50,-.46,-.5];
strain=zeros(4,11);
strain(4,1:11)=[.5,.46,0.50-tm(1),tm(2)/2,.04,0.,
                        -.04,-tm(2)/2,tm(3)-0.50,-.46,-.5];
%
StrainMat=[0 0 -p 0;
           0 1  0 0;
           1 0  0 p];
%
for ii=(1:11)
   strain(1:3,ii)=StrainMat*uw(1:4,ii);
   stress(1:3,ii)=KM*StrainMat*uw(1:4,ii);
end
diary ststress.prn
strain'
stress'
%
diary off
if iso==0
   uwf=zeros(3,2);
   stress=zeros(4,2);
   stress(4,1:2)=[tf(1) tf(2)];
   strain=zeros(4,2);
   strain(4,1:2)=[tf(1) tf(2)];
   %
   StrainMat=[0 -p 0;
              1  0 p];
   %
   for ii=(1:2)
      uwf(1,ii)=0;
      uwf(2,ii)=0;
      zf=zdist(ii);
      for jj=(0:N)
         uwf(1,ii)=uwf(1,ii)+UW(jj+2)*jj*zf^(jj-1);
         uwf(2,ii)=uwf(2,ii)+UW(jj+2)*zf^jj;
      end
      uwf(3,ii)=UW(1);
      strain(1:2,ii)=StrainMat*uwf(1:3,ii);
   end
   stress(1:2,1)=SM1*StrainMat*uwf(1:3,1);
   stress(1:2,2)=SM2*StrainMat*uwf(1:3,2);
   diary st-fstr.prn
   strain'
   stress'
   %
   diary off
end
```

EXPMAT

```
function em=expmat(z)
% EXPMAT - Function to generate the matrix exponential for the boundary
%          condition matrices of the "EXACT STRATIFIED PLATE SOLUTION" for
%          the analysis of composite panels.
```

```
ezp=exp(p*z);
ezm=exp(-p*z);
eVector=[ezp; z*ezp; ezm; z*ezm];
aV=RMI*eVector
em=aV(1)*eye(4)+aV(2)*FM+aV(3)*FM*FM+aV(4)*FM*FM*FM;
```

GMATLMAT

```
function [fm1,fm2,bc]=gmatlmat
% GMATLMAT  Function to generate the material property matrices for the
%           "EXACT" solution of the cross-ply composite panel problem.
%
% Syntax:   [FM1,FM2,BC]=GMATLMAT
%
%        p  =     nπ/a
%       mp  =     Vector of material properties (1x3)
%       fp  =     Vector of fiber strata properties (1x9)
%
%       BC  =     Boundary condition matrix
%      FM1  =     Field matrix #1
%      FM2  =     Field matrix #2
%                                           u  =  Displacement vector
%      where [FM1] u = L [FM2] u            L  =  Eigenvalue (aka lambda)
%
% This function is used in conjunction with other MATLAB routines to
% verify the "STRATA THEORY" of composite panel analysis.  Written by
% Alan L. Lesmerises.

c1=matprops(1);
c2=matprops(2);
c3=matprops(3);
c4=c2+c3;
z=zeros(2,2);
%
m1=[c3 0;
    0 c1];
%
m2=c4*p*[0 1;
        -1 0];
%
m3=[-p*p*c1     0     ;
       0     -p*p*c3];
%
% Above matrices appear in PDE as:
%
%      [m1]u"+[m2]u'+[m3]u=0
% or
%
%     [ [m1]   [0]  ] [Lu]       [ [0]   [m1] ] [Lu]
%     [ [0]  -[m3]  ] [ u]  = L  [ [m1]  [m2] ] [ u]
%
%
fm1=[m1 z ;
     z -m3];
%
fm2=[ z m1;
     m1 m2];
%
FM=inv(fm2)*fm1
%
m1=[c3 0;
    0 c1];
```

```
%
m2=[ 0  p*c3;
   -p*c2  0 ];
%
s1=fibprops(1,1)*tf(1);
s2=fibprops(1,2)*tf(1);
s3=fibprops(1,3)*tf(1);
s4=fibprops(1,4)*tf(1);
s7=fibprops(1,7)*tf(1);
s8=fibprops(1,8)*tf(1);
m2f1=m2+[p*p*s1   0   ;
           0    p*p*s7];
%
s1=fibprops(2,1)*tf(2);
s2=fibprops(2,2)*tf(2);
s3=fibprops(2,3)*tf(2);
s4=fibprops(2,4)*tf(2);
s7=fibprops(2,7)*tf(2);
s8=fibprops(2,8)*tf(2);
m2f2=m2+[p*p*s1   0   ;
           0    p*p*s7];
%
I2=eye(2);
bc=[m1 m2  z   z  z   z   z   z  z   z   z   z;
    z  z  -m1 -m2 m1 m2f1 z   z  z   z   z   z;
    z  z   z   I2 z  -I2  z   z  z   z   z   z;
    z  z   z   z  z   z  -m1 -m2 m1 m2f2 z   z;
    z  z   z   z  z   z   z   I2 z  -I2  z   z;
    z  z   z   z  z   z   z   z  z   z  -m1 -m2]
%
% End of function GMATLMAT.M
%
```

TREND

```
setparam
extrend=zeros(3,50);
for jndex=1:50
   A=jndex
   extrend(1,jndex)=A;
   calparam
   exactsol
   extrend(2,jndex)=uw(4,5);
   stxplies
   extrend(3,jndex)=uw(4,5);
end
diary trend.prn
extrend'
diary off
quit
```

KS

```
function k=ks(order)
% KS - Evaluates the integral in z that is associated with the k's that
%      appear in the P.D.E.'s for the "STRATA THEORY" for the analysis of
%      composite panels.

k=0;
if order < 0
elseif (order/2 == fix(order/2))
```

```
   k=1/(order+1)/2^order/beta;
end
```

SS

```
function sstiff=ss(sterm,order)
%
% SS - Sum the terms of Σ s(f,j) * z^n
%
% Syntax    SS(j,order)
%
% where:
%           s(f,j)  =  membrane elasticity terms
%                f  =  fiber number
%                j  =  the elasticity term of interest
%                n  =  the "order" of the z term
%
%            sterm  =  membrane properties vector
%            order  =  the order of the z term (n above)

[n,numfibers]=size(zdist);
sstiff=0;
ssum=0;
for index=(1:numfibers)
   sstiff=sstiff+fibprops(index,sterm)*zdist(index)^order*tf(index);
end
%
% End of SS.M function
%
```

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Vita

Captain Alan L. Lesmerises was born on January 7, 1959 in Manchester, NH. He graduated from Trinity High School, Manchester, NH, in 1976. He then attended Daniel Webster College (1976-77), majoring in Aeronautical Engineering. He enlisted in the U.S. Air Force in December 1977, and was assigned to Plattsburg AFB, NY as a Jet Engine Technician. He continued his education at Plattsburg State University College, SUNY (1977-79). In 1983, he was assigned to the University of Oklahoma after being accepted into the Airman Education and Commissioning Program. He graduated with honors in May 1986 with a Bachelor of Science degree in Aerospace Engineering. After attending Officer Training School, he was assigned to the Fuels and Lubrication Division, Aero Propulsion Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson AFB, OH. During this assignment, he was program manager for several research and development contracts, and developed computer software. He also analyzed test data, and designed test equipment and facilities for fundamental combustion research. While there, he co-authored six technical papers based on this research.

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A new model for composites represents a composite panel as an alternating stack of isotropic matrix and orthotropic fiber-dominated layers. An energy approach results in a set of partial differential equations and boundary conditions where the mechanical properties of the matrix and fibers appear separately. Two solution methods are developed. The first, called the "Strata Theory" is developed for general applications, while the second, called the "Exact Stratified Plate", is used as a benchmark for comparison to Strata Theory. Both solutions assume zero gradients of displacement through the thickness of a fiber layer. They are compared for an isotropic material and a two-ply composite with cross-plies, using simply-supported cases of a square plate and cylindrical bending.

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