The Cartesian product of a k-extendable and an ℓ-extendable graph is $(k+\ell+1)$-extendable

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1. Introduction and Terminology

Let us start with the definition of a k-extendable graph G. Suppose k is an integer such that $1 \leq k \leq (|V(G)| - 2)/2$. A graph G is k-extendable if G is connected, has a perfect matching (a 1-factor) and any matching in G consisting of k edges can be extended to (i.e., is a subset of) a perfect matching.

The **extendability number** of G, ext$G$, is the maximum k such that G is k-extendable. A natural problem is to determine the extendability number of a graph G. In particular, we would like to know the extendability number of some special graphs, for example, the n-dimensional cube Q_n which has 2^n vertices and $n2^{n-1}$ edges. For small values of n, you can easily verify that Q_n is $(n-1)$-extendable. On the other hand, a k-extendable graph is $(k+1)$-connected (see [3]) or [4]) and so the n-cube Q_n cannot be n-extendable. Thus, it is plausible to conjecture that the extendability number of Q_n is $n-1$.

The n-cube, Q_n, is an example of a graph which is the *Cartesian product* of two smaller graphs. In general, the **Cartesian product** $G_1 \times G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) of the Cartesian product graph are adjacent if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1. In particular, the n-cube, Q_n, can be defined inductively by letting $Q_1 = K_2$ and for all $n \geq 2$, let $Q_n = Q_{n-1} \times K_2$.

Viewed from this point of view, when trying to prove that Q_n is $(n-1)$-extendable, the question naturally arises as to how highly extendable the graph $G \times K_2$ is if G is, say, k-extendable. Or even more generally, how highly extendable is the Cartesian product of a k-extendable and an ℓ-extendable graph?

2. Main Results

The main result of this paper is the following theorem.

Theorem 1. If G_1 and G_2 are k-extendable and ℓ-extendable graphs, respectively, then their Cartesian product $G_1 \times G_2$ is $(k + \ell + 1)$-extendable.

An important special case of Theorem 1 is

Theorem 2. If G is a k-extendable graph then $G \times K_2$ is $(k + 1)$-extendable.

We have recently learned that J. Liu and Q. Yu [2] have independently proved the following generalization of Theorem 2, as well as Theorem 4.

Theorem 3. Let G_1 be a k-extendable graph and G_2 be a connected graph. Then the Cartesian product $G_1 \times G_2$ is $(k + 1)$-extendable.

Each of these theorems implies that the cube Q_n is $(n-1)$-extendable and the products $Q_{k+\ell+2} = Q_{k+1} \times Q_{\ell+1}$ and $Q_{k+2} = Q_{k+1} \times K_2$ show that these results are best possible for some classes of graphs.

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Proofs of Theorems 1 and 2 are based on the following technical type theorem.

Theorem 4. Suppose that G is a k-extendable graph, $v_1, v_2, \cdots, v_r, w_1, w_2, \cdots, w_s$ are arbitrary vertices of $G, e_1, e_2, \cdots, e_t$ are independent edges of G not incident to the vertices $v_1, \cdots, v_r, w_1, \cdots, w_s$ and suppose that $r \geq 1$, $r + s + t \leq k + 1$. Then G contains a perfect matching extension of $\{e_1, \cdots, e_t\}$ not containing any edge joining the vertex sets $\{v_1, \cdots v_r\}$ and $\{w_1, \cdots, w_s\}$.

After the proof of Theorem 4, we shall first prove Theorem 2 via a single application of Theorem 4 in a relatively simple case and then we shall prove Theorem 1 by applying Theorem 4 several times in more complicated situations.

Let us make an important remark about notation before proceding. Throughout the paper, a matching M is a graph G means not just a set of independent edges in G but a subgraph of G each component of which has exactly two vertices. In accordance with this, if M is any matching in G and V_0 is a vertex set in $V(G)$, the induced subgraph $M_G(V_0)$ — or more briefly $M(V_0)$ — is a forest with components each of which has either one or two vertices.

Proof of Theorem 4. We will need the following two lemmas. (The proof of the first may be found in [4].)

Lemma 1. Every k-extendable graph is $(k + 1)$-connected. ∎

Lemma 2. Let G be a k-extendable graph ($k \geq 1$) and let $x_1 y_1$, $x_2 y_2, \cdots, x_t y_t$ ($t < k$) be independent edges in G. Then deleting the endvertices of these edges, the resulting graph $G - \{x_1, y_1, x_2, y_2, \cdots, x_t, y_t\}$ is $(k - t)$-extendable.

Proof of Lemma 2. Notice that it is sufficient to prove the statement for $t = 1$; the general statement follows by induction on t. By definition, $|V(G)| \geq 2k + 2$ and so $|V(G) - \{x_1, y_1\}| \geq 2k$. By Lemma 1, G is $(k + 1)$-connected and so $G - \{x_1, y_1\}$ is connected.

Graph G is k-extendable and so it is 1-extendable, as well (see [4]), and hence $x_1 y_1$ can be extended to a perfect matching of G. Hence the graph $G - \{x_1, y_1\}$ has a perfect matching. Finally let $e_1, e_2, \cdots, e_{k-1}$ be arbitrary independent edges in $G - \{x_1, y_1\}$. The graph G is k-extendable and so the edge set $\{e_1, e_2, \cdots, e_{k-1}, x_1 y_1\}$ extends to a perfect matching M in G and $M - \{x_1, y_1\}$ is thus a perfect matching in $G - \{x_1, y_1\}$ containing the edge set $\{e_1, \cdots, e_{k-1}\}$. ∎

Now we are ready to prove Theorem 4. If $t = k$, then $r = 1$ and $s = 0$ and the claim is obvious.

So let us suppose that $t < k$. Let G be a k-extendable graph, $V = \{v_1, \cdots, v_r\}$ and $W = \{w_1, \cdots, w_s\}$ be arbitrary disjoint vertex sets, and $e_1 = x_1 y_1, e_2 = x_2 y_2, \cdots, e_t = x_t y_t$ be independent edges of G not incident with any vertex in $V \cup W$ where $r \geq 1$, $r + s + t \leq k + 1$. By Lemmas 2 and 1, respectively, the graph $H = G - \{x_1, y_1, \cdots, x_t, y_t\}$ is $(k - t)$-extendable and hence $(k - t + 1)$-connected.

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We wish to prove that there is a matching M in the bipartite subgraph H^* of H having vertex bipartition $V \cup (V(H) - (V \cup W))$ which covers all the vertices in V. Suppose not. Then by P. Hall's bipartite matching theorem [1], there is a set $V_0 \subseteq V$ with

$$|\Gamma_{H^*}(V_0)| < |V_0|.$$

Then for the set

$$W_0 = (V - V_0) \cup \Gamma_{H^*}(V_0) \cup W,$$

we have

$$|W_0| \leq r + s - 1 \leq k - t$$

and

$$|V(H) - (V_0 \cup W_0)| \geq |V(H)| - |V_0| - |W_0| \geq 2(k-t) + 2 - r - (k-t) \geq 1,$$

and so $V(H) - (V_0 \cup W_0)$ is non-empty. But then the set W_0 separates the non-empty set V_0 from the non-empty set $V(H) - V_0 - W_0$ in H, contradicting the $(k-t+1)$-connectedness of the graph H. ∎

Proof of Theorem 2. To facilitate the proofs of Theorems 1 and 2, we now introduce the concept of a *shadow*.

Throughout this paper, it will be useful to think of graph $H = G \times K_2$ as two graphs G' and G'' both isomorphic to G with vertex sets $\{v_1', \cdots, v_n'\}$ and $\{v_1'', \cdots, v_n''\}$, respectively, joined by a perfect matching consisting of the edges $v_1' v_1'', \cdots, v_n' v_n''$. Let F be a forest in graph H each component of which consists of one or two vertices. (In the proof of Theorem 2, we will have two-vertex components only, i.e., F will be a matching.)

The **shadow** $S_{G'}(F)$ of F in G' is a forest F' in G' whose components (each of which has one or two vertices) have their vertex sets defined as in the following table:

Vertex Set of the component of F	Vertex set of the shadow of the component of F in G'
$\{v_i'\}$	$\{v_i'\}$
$\{v_i''\}$	$\{v_i'\}$, if $v_i' \notin V(F)$ $\emptyset$, if $v_i' \in V(F)$
$\{v_i', v_j'\}$	$\{v_i', v_j'\}$
$\{v_i', v_i''\}$	$\{v_i'\}$
$\{v_i'', v_j''\}$	$\{v_i', v_j'\}$ if $\{v_i', v_j'\} \cap V(F) = \emptyset$, $\{v_i'\}$, if $v_i' \notin V(F)$ and $v_j' \in V(F)$, $\{v_j'\}$, if $v_i' \in V(F)$ and $v_j' \notin V(F)$, $\emptyset$, if $\{v_i', v_j'\} \subseteq V(F)$

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Clearly the number of components of the shadow $S_{G'}(F)$ is at most the number of components of F.

Now, we are ready to prove Theorem 2. Let G be a k-extendable graph and let G' and G'' denote the subgraphs in $H = G \times K_2$ isomorphic to G. Let M be a matching of $k+1$ edges in H. We distinguish two cases according to the setting of M in H.

Case 1. $E(M) \subseteq E(G') \cup E(G'')$.

If, say, $E(M) \subseteq E(G')$ then taking the shadow $S_{G''}(M)$ and an arbitrary perfect matching in $H - (V(M) \cup V(S_{G''}(M)))$ (e.g. the edges joining $V(G') - V(M)$ and $V(G'') - V(S_{G''}(M))$), we obtain a perfect matching extension of M.

On the other hand, if $E(M) \not\subseteq E(G')$ and $E(M) \not\subseteq E(G'')$ then $|E(M) \cap E(G')| \leq k$ and $|E(M) \cap E(G'')| \leq k$. By the k-extendability of $G' \simeq G'' \simeq G$, we can take a perfect matching extension of $E(M) \cap E(G')$ and $E(M) \cap E(G'')$ in G' and G'', respectively, and their union is a perfect matching extension of M in H.

Case 2. $E(M) \not\subseteq E(G') \cup E(G'')$.

Let $v_1, v_2, \cdots, v_r$ $(r \geq 1)$ be the isolated vertices of the shadow $S_{G'}(M)$ which are the shadows of the edges of M joining $V(G')$ to $V(G'')$ and let $w_1, \cdots, w_s$ be the other isolated vertices of $S_{G'}(M)$. Furthermore, let $\{x_1, y_1\}, \cdots, \{x_t, y_t\}$ be the vertex sets of the two-vertex components of $S_{G'}(M)$ $(r + s + t \leq k + 1)$. Then G' contains a perfect matching extension M' of the edge set $\{x_1 y_1, \cdots, x_t y_t\}$ with no edge $v_i w_j$ $(1 \leq i \leq r, 1 \leq j \leq s)$ by Theorem 4. Let $e'_{t+1}, \cdots, e'_{t+q}$ $(q \leq r)$ be the edges of M' incident to the vertex set $\{v_1, \cdots, v_r\}$ and let $e''_{t+1}, \cdots, e''_{t+q}$ be the edges of the G'' shadow of the two-vertex components determined by the edges $e'_{t+1}, \cdots, e'_{t+q}$. If the edge set $(E(M) \cap E(G'')) \cup \{e''_{t+1}, \cdots, e''_{t+q}\}$ has at most k elements then it has a perfect matching extension M'' in G'' since G'' is k-extendable. And if $(E(M) \cap E(G'')) \cup \{e''_{t+1}, \cdots, e''_{t+q}\}$ has $k+1$ elements, then $E(M) \cap E(G') = \emptyset$, $s = 0$ and $M'' = S_{G''}(M')$ is a perfect matching extension of it. In both cases, $M' \cup M''$ is a perfect matching in H. Now, swap the edge pairs $\{e'_i, e''_i\}$ $(i = t+1, \cdots, t+q)$ into pairs $\{f_{i1}, f_{i2}\}$ of edges joining $V(G')$ and $V(G'')$ such that $e'_i, f_{i1}, e''_i, f_{i2}$ constitute a 4-cycle. The resulting matching

$$((M' \cup M'') - \{e'_{t+1}, e''_{t+1}, \cdots, e'_{t+q}, e''_{t+q}\}) \cup \{f_{t+1,1}, f_{t+1,2}, \cdots, f_{t+q,1}, f_{t+q,2}\}$$

is a perfect matching extension of M in H. ∎

Proof of Theorem 1. Let $V = \{v_{ij} : 1 \leq i \leq |V(G_1)|, 1 \leq j \leq |V(G_2)|\}$ be the vertex set of the Cartesian product $G_1 \times G_2$. The subgraphs of $G_1 \times G_2$ induced by the vertex sets $\{v_{ij} : 1 \leq j \leq |V(G_1)|\}$ and $\{v_{ij} : 1 \leq i \leq |V(G_2)|\}$ are called the i-th row and the j-th column of $G_1 \times G_2$ and will be denoted by R_i and T_j, respectively. Note that $R_i \simeq G_1$ $(i = 1, 2, \cdots, |V(G_2)|)$ and $T_j \simeq G_2$ $(j = 1, 2, \cdots, |V(G_1)|)$.

Let M be an arbitrary matching in $G_1 \times G_2$. For a vertex set $V_0 \subset V(G_1 \times G_2)$ the induced subgraph $M[V_0 \cap V(M)]$ will be called the trace of M in V_0 and will be denoted by $M(V_0)$. The components of $M(V_0)$ have one or two vertices and will be called trace elements of M in V_0. (Typically, V_0 will be the vertex set of one or two rows or columns, and we will then write $M(R_i)$, $M(R_{i_1} \cup R_{i_2})$, etc.)

From now on, let M be a given matching of $k + \ell + 1$ edges in $G_1 \times G_2$. We will prove that M is extendable to a perfect matching in $G_1 \times G_2$. (The other conditions are obviously fulfilled, namely that $G_1 \times G_2$ is connected and has a perfect matching.)

Lemma 3. At least one of the following two statements is true.

(i) The trace of M in any row has at most $k + 1$ components and the trace of M in the union of any two rows has at most $k + 2$ components.

(ii) The trace of M in any column has at most $\ell + 1$ components and the trace of M in the union of any two columns has at most $\ell + 2$ components.

Proof of Lemma 3. Suppose that the first half of (i) is false, i.e., suppose $M(R_i)$ has as least $k + 2$ components for some i.

Consider any column T_j. The (at least $k + 1$) components of M, whose traces in R_i are completely in $R_i - \{v_{ij}\}$, are not incident to $V(T_j)$ and so $M(T_j)$ has at most $(k + \ell + 1) - (k + 1) = \ell$ components and the first part of statement (ii) holds.

Now consider two arbitrary columns T_{j_1} and T_{j_2}. The (at least k) components of M whose traces in R_i are completely in $R_i - \{v_{ij_1}, v_{ij_2}\}$ are not incident to $V(T_{j_1} \cup T_{j_2})$ and so $M(T_{j_1} \cup T_{j_2})$ has at most $\ell + 1$ components. Thus the second part of statement (ii) holds as well.

Now, suppose that the second half of (i) is false, i.e., suppose $M(R_{i_1} \cup R_{i_2})$ has at least $k + 3$ components for some indices i_1, i_2. Consider any column T_j. The (at least $k + 1$) components of M whose traces in $R_{i_1} \cup R_{i_2}$ are completely in $(R_{i_1} \cup R_{i_2}) - \{v_{i_1 j}, v_{i_2 j}\}$ are not incident to $V(T_j)$ and so $M(T_j)$ has at most ℓ components. Thus the first part of statement (ii) is satisfied. Now, consider any two columns T_{j_1} and T_{j_2}. The (at least $k - 1$) components of M whose traces in $R_{i_1} \cup R_{i_2}$ are completely in $R_{i_1} \cup R_{i_2} - \{v_{i_1 j_1}, v_{i_1 j_2}, v_{i_2 j_1}, v_{i_2 j_2}\}$ are not incident to $V(T_{j_1} \cup T_{j_2})$ and so $M(T_{j_1} \cup T_{j_2})$ has at most $\ell + 2$ components. Thus the second part of statement (ii) holds in this case as well. ∎

Appealing to the row-column symmetry of Lemma 3, from now on, without loss of generality, we will assume that statement (i) is true. We will start by *trying* to swap all vertical edges of M for pairs of horizontal edges inducing four-cycles so that the resulting matching in each row R_i should be extendable to a perfect matching of R_i. Then swapping back the edges in the four-cycle, we will obtain a perfect matching extension of M in $G_1 \times G_2$.

Remark 1. Notice that if we swap a vertical edge $v_{i_1 j_1} v_{i_2 j_1}$ into some pair of horizontal edges $\{v_{i_1 j_1} v_{i_1 j_2}, v_{i_2 j_1} v_{i_2 j_2}\}$, then the number of matching edges in the current matching will increase by one. However, the number of trace components in any row or in the union of any two rows will not change except in the union $R_{i_1} \cup R_{i_2}$. However, we will not return to these two rows again since all the vertical edges joining them are swapped at the same time.

Keeping only the ordering rule: "start with the pairs of rows whose union contains $k + 2$ trace components", we will try to perform the following inductive *General Row Step* for all pairs of rows joined by at least one vertical edge of M. (Clearly, by symmetry, there

is a corresponding *General Column Step* for all pairs of columns joined by at least one *horizontal* edge of M.)

General Row Step. Let R_{i_1} and R_{i_2} be two rows of $G_1 \times G_2$ joined by at least one vertical edge of M_0, when M_0 is the matching obtained from M after some number of previous applications (perhaps none) of the General Row Step. Suppose that the number of components of the shadow $S_{R_{i_1}}(M_0(R_{i_1} \cup R_{i_2}))$ is not more than $k+1$. Let $v_{i_1 j_1}, v_{i_1 j_2}, \cdots, v_{i_1 j_r}$ be the shadow vertices in R_{i_1} of the vertical edges in $M_0(R_{i_1} \cup R_{i_2})$, $v_{i_1 k_1}, v_{i_1 k_2}, \cdots, v_{i_1 k_s}$ be the other shadow components consisting of one vertex and $f_1, f_2, \cdots, f_t$ be the edges in $S_{R_{i_1}}(M_0(R_{i_1} \cup R_{i_2}))$, $r + s + t \le k+1$, $r > 0$. R_{i_1} is k-extendable, so Theorem 4 implies that there are some indices $j_1^*, j_2^*, \cdots, j_r^*$ such that $\{j_1^*, \cdots, j_r^*\} \cap \{k_1, \cdots, k_s\} = \emptyset$ and the edge set $\{v_{i_1 j_1} v_{i_1 j_1^*}, \cdots, v_{i_1 j_r} v_{i_1 j_r^*}\} \cup \{f_1, \cdots, f_t\}$ is extendable to a perfect matching of R_{i_1}, and then so is $\{v_{i_1 j_1} v_{i_1 j_1^*}, \cdots, v_{i_1 j_r} v_{i_1 j_r^*}\} \cup (E(M_0) \cap E(R_{i_1}))$. Furthermore, the edge set $\{v_{i_2 j_1} v_{i_2 j_1^*}, \cdots, v_{i_2 j_r} v_{i_2 j_r^*}\} \cup (E(M_0) \cap E(R_{i_2}))$ is extendable to a perfect matching in R_{i_2} even if this is a set of $k + 1$ edges. (It cannot have *more* than $k + 1$ edges since we assumed that (i) of Lemma 3 holds.) For suppose it has $k + 1$ edges. Then the set $E(M_0) \cap E(R_{i_2})$ has $k + 1 - r$ elements, say, $e_1, e_2, \cdots, e_{k+1-r}$. But then $f_1, \cdots, f_t$ must be their shadows in R_{i_1}, (and so $t = k + 1 - r$) since $S_{R_{i_1}}(M_0(R_{i_1} \cup R_{i_2}))$ has at most $k + 1$ components. Thus, the edge set $\{v_{i_2 j_1} v_{i_2 j_1^*}, \cdots, v_{i_2 j_r} v_{i_2 j_r^*}\} \cup \{e_1, \cdots, e_{k+1-r}\}$ is extendable to a perfect matching in R_{i_2} just as their shadows were extendable to a perfect matching in R_{i_1}. Now, swap the vertical edges of $M_0(R_{i_1} \cup R_{i_2})$ into the pairs $\{v_{i_1 j_1} v_{i_1 j_1^*}, v_{i_2 j_1} v_{i_2 j_1^*}\}, \cdots, \{v_{i_1 j_r} v_{i_1 j_r^*}, v_{i_2 j_r} v_{i_2 j_r^*}\}$, leaving all other edges of M_0 unchanged to get a new updated matching M_0.

What can prevent us from applying the General Row Step as long as we have vertical matching edges between some pairs of rows? There are two possibilities.

Case 1. There is a row R_i containing $k + 1$ edges of the original matching M which are not extendable to a perfect matching in R_i. (Note that there cannot be *more* than $k + 1$ such edges of R_i by our assumption after the proof of Lemma 3. Note also that if we obtain $k + 1$ edges in R_i by means of a swap, then they are extendable, as we have seen in the General Row Step just above.)

In this case, we claim that *all horizontal* edges of M in G can be swapped for vertical edges so that those in any column can be extended to a perfect matching of this column. Thus we will then have the original vertical edges of M and a set of swapping 4-cycles each containing either one or two horizontal edges of M. Moreover, if one of these swapping 4-cycles *abcd* contains only one horizontal edge of M—say *ab*—then neither *c* nor *d* is an endpoint of any edge in M.

In other words, if Case 1 occurs, we can finish as follows: swap *all horizontal* edges of M to get swapping 4-cycles such that the vertical edges in any column of a swapping 4-cycle, together with the vertical edges of M in that column, extend to a perfect matching of that column. Taking the union of all these perfect matchings over all columns of $G_1 \times G_2$, we get a perfect matching of $G_1 \times G_2$ all the edges of which are vertical. Now swap back on all swapping 4-cycles to get a perfect matching of $G_1 \times G_2$ containing all edges of M and we are finished.

Let us proceed to justify this claim.

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As in the proof of Lemma 3, it is easy to see in this case that for every column T_j, the trace $M(T_j)$ has at most $\ell + 1$ components and *does not* contain $\ell + 1$ vertical edges because the total number of vertical edges is at most $(k + \ell + 1) - (k + 1) = \ell$ and for the union of any two columns T_{j_1} and T_{j_2}, the trace $M(T_{j_1} \cup T_{j_2})$ has at most $\ell + 2$ components.

First consider all pairs of columns T_{j_1} and T_{j_2} such that $M(T_{j_1} \cup T_{j_2})$ contains *exactly* $\ell + 2$ components. Then vertices v_{ij_1} and v_{ij_2} are necessarily incident with two different horizontal edges of M. Thus v_{ij_1} and v_{ij_2} are two different isolated vertices of $M(T_{j_1} \cup T_{j_2})$. But then each of the shadows $S_{T_{j_1}}(M(T_{j_1} \cup T_{j_2}))$ and $S_{T_{j_2}}(M(T_{j_1} \cup T_{j_2}))$ has at most $\ell + 1$ components.

Therefore, if T_{j_1} and T_{j_2} were joined by one or more horizontal edges of M, then we can apply the General Column Step to simultaneously swap each of these horizontal edges joining the two columns for a pair of vertical edges. Thus we finally arrive at a new matching in $T_{j_1} \cup T_{j_2}$ consisting of vertical edges only.

We can do this type of swapping in *all* pairs of columns T_{j_p} and T_{j_q} in which the trace of M has exactly $\ell + 2$ components because in so doing, we *do not* swap the edges of M which lie in R_i and so the number of components in the shadows $S_{T_{j_p}}(M(T_{j_p} \cup T_{j_q}))$ and $S_{T_{j_q}}(M(T_{j_p} \cup T_{j_q}))$ is at most $\ell + 1$.

Secondly, we can swap the remaining horizontal edges of M which join the pairs of columns in the union of which the trace of M has at most $\ell + 1$ components. So it is at this stage that all $k + 1$ horizontal lines of M in row R_i get swapped for vertical lines via swapping 4-cycles. Now every column contains a certain set of at most $\ell + 1$ vertical edges which extends to a perfect matching in this column. (We have used Theorem 4 repeatedly here, but applied to columns rather than rows. We repeat for emphasis here that in these pairs of columns these sets of at most $\ell + 1$ vertical edges which have arisen from swaps on horizontal edges of M joining the two columns, *do not* touch any other edges of the original matching M!)

The column-by-column union of all these perfect matchings yields a perfect matching P' for $G_1 \times G_2$ consisting of all vertical lines and among all these vertical lines are all the vertical lines of M. But then one can swap back on the swapping 4-cycles associated with perfect matching P' to get a perfect matching P for $G_1 \times G_2$ which picks up all the horizontal lines of M and retains all the vertical lines of M. So P is a perfect matching for $G_1 \times G_2$ containing M and we are done.

Case 2. At the beginning or after some number of swaps (perhaps none), there are two rows R_{i_1} and R_{i_2} joined by a vertical edge of the present matching M_0 such that $M_0(R_{i_1} \cup R_{i_2})$ has $k + 2$ components and both shadows $S_{R_{i_1}}(M_0(R_{i_1} \cup R_{i_2}))$ and $S_{R_{i_2}}(M_0(R_{i_1} \cup R_{i_2}))$ consist of $k + 2$ components, as well, i.e., the hypothesis of Theorem 4 and those of the General Row Step do not apply.

In this case, every component of $M_0(R_{i_1} \cup R_{i_2})$ has some nonempty shadow in both rows. (This means that if $M_0(R_{i_1})$ and $M_0(R_{i_2})$ contain some component in the same column then either they must be endvertices of the same vertical edge or the endvertices of two horizontal edges that are *not* shadows of each other.)

We will prove that we can apply the General Column Step repeatedly for M and its modifications and for any two columns joined by some horizontal edges of the matching

M.

Let us consider two columns T_{j_1} and T_{j_2} joined by some horizontal edges of M. If the number of components of $M((R_{i_1} \cup R_{i_2}) \cap (T_{j_1} \cup T_{j_2}))$ is at most two then the number of components of $M(T_{j_1} \cup T_{j_2})$ is at most $\ell + 1$ (this can be proved in a manner similar to Lemma 3) and we can apply the General Column Step for $T_{j_1} \cup T_{j_2}$ to swap the horizontal edges joining T_{j_1} and T_{j_2}.

So, we may assume that $M((R_{i_1} \cup R_{i_2}) \cap (T_{j_1} \cup T_{j_2}))$ has at least 3 components. Then, say, for T_{j_1}, the trace $M((R_{i_1} \cup R_{i_2}) \cap T_{j_1})$ has two components, i.e., $v_{i_1 j_1}$ and $v_{i_2 j_1}$ which are endvertices of two distinct edges e_1 and e_2 of M.

We now claim: (*) that we may assume that e_1 and e_2 are in R_{i_1} and R_{i_2}, respectively. Suppose this is not the case and one of these edges is vertical, say $e_1 = v_{i_1 j_1} v_{i_3 j_1}$. If we had not swapped the edge e_1 at some previous time, then the shadow $S_{R_{i_2}}(M_0(R_{i_1} \cup R_{i_2}))$ could not have $k + 2$ components, contradicting the hypotheses of Case 2. Thus, we swapped e_1 at some previous time and $M(R_{i_1} \cup R_{i_3})$ had $k + 2$ components just like $M(R_{i_1} \cup R_{i_2})$ by the ordering rule.

If $M((R_{i_1} \cup R_{i_3}) \cap (T_{j_1} \cup T_{j_2}))$ has at most two components then $M(T_{j_1} \cup T_{j_2})$ has at most $\ell + 1$ components and we can apply Theorem 4 and the General Column Step just like above.

But now $v_{i_1 j_1}$ and $v_{i_3 j_1}$ constitute a trace component of two vertices (i.e., it is an edge) and so $M((R_{i_1} \cup R_{i_3}) \cap (T_{j_1} \cup T_{j_2}))$ can have three components only if both $v_{i_1 j_2}$ and $v_{i_3 j_2} \in V(M)$. Then, $M(T_{j_1} \cup T_{j_2})$ may have $\ell + 2$ components (but not more). However, the component with vertex set $\{v_{i_1 j_1}, v_{i_3 j_1}\}$ does not and will not have any shadow in T_{j_2} (and this situation will not change later). So we can apply Theorem 4 and the General Column Step for $T_{j_1} \cup T_{j_2}$. (Remember: T_{j_2} has at most $\ell + 1$ shadow elements of $M(T_{j_1} \cup T_{j_2})$ because $v_{i_1 j_1} v_{i_3 j_1}$ has no image in T_{j_2}.) This completes the proof of claim (*).

From now on, we may assume that, say, $M((R_{i_1} \cup R_{i_2}) \cap T_{j_1})$ has two components and the edges of M incident to $v_{i_1 j_1}$ and $v_{i_2 j_1}$ are horizontal, and if $M((R_{i_1} \cup R_{i_2}) \cap T_{j_2})$ has two components then the edges of M incident to $v_{i_1 j_2}$ and $v_{i_2 j_2}$ are horizontal, as well. Thus, we can apply Theorem 4 and the General Column Step for each pair of columns joined by some horizontal edges of M if the trace of M in their union contains $\ell + 2$ and $\ell + 3$ components, respectively, since in this case at least one or both of the horizontal edges of M incident to $\{v_{i_1 j_1}, v_{i_2 j_1}\}$ have no shadow in T_{j_2}. Furthermore, in these applications of the General Column Step, we claim that we did not swap any horizontal edge of M in $R_{i_1} \cup R_{i_2}$. Suppose it is not the case and we swapped some edge $v_{i_1 j_1} v_{i_1 j_2}$. Then $M(T_{j_1} \cup T_{j_2})$ had at least $\ell + 2$ components and so $v_{i_2 j_1}, v_{i_2 j_2} \in V(M)$. However then $M(\{v_{i_1 j_1} v_{i_1 j_2}\})$ has no shadow in R_{i_2} which contradicts the original assumption of Case 2.

Thus one by one, we can apply Theorem 4 and the General Column Step for all pairs of columns joined by some horizontal edges of M if the trace of M in their union consists of at least $\ell + 2$ components since we did not change any horizontal edges of M in $R_{i_1} \cup R_{i_2}$. Finally, we can apply Theorem 4 and the General Column Step for the pairs of columns joined by some horizontal edges of M the union of which intersects at most $\ell + 1$ components of M. Thus, we can swap all horizontal edges into vertical pairs of edges and we can extend the resulting matching to a perfect matching of $G_1 \times G_2$ columnwise.

Swapping back the edge couples in the swapping four-cycles, we obtain a perfect matching extension of M in $G_1 \times G_2$ as desired. ∎

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