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Analysis of Electromagnetic Propagation Over Variable Terrain Using The Parabolic Wave Equation

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ADMINISTRATIVE INFORMATION

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EXECUTIVE SUMMARY

This report describes the basic physics and numerical techniques used in implementing a range-dependent, tropospheric microwave propagation model. This propagation model allows for both finite surface conductivity and variable (i.e., highly irregular) surface terrain. The model is based upon the electromagnetic parabolic wave equation (PE) and employs a novel implementation of the split-step Fourier PE (SSFPE) algorithm to efficiently compute the electromagnetic radiation fields. This physics report reviews the fundamentals of the PE method and provides a detailed error analysis of the SSFPE algorithm. The generalization of the SSFPE algorithm to handle nonflat boundaries is also discussed.

The propagation model is implemented as the VTRPE (variable terrain radio parabolic equation) computer code and is used in the prediction of radar system performance. The VTRPE code has the following characteristics:

(1) full-wave propagation physics (i.e., field amplitude and phase are computed);
(2) direct solution of electromagnetic fields;
(3) exact treatment of refraction and diffraction phenomena;
(4) exact treatment of multipath phenomena;
(5) range-dependent atmospheric refractivity inputs, $N(z,r)$;
(6) infinite or finite conductivity surface boundary conditions;
(7) linear transmitter field polarization (vertical or horizontal);
(8) variable surface terrain elevation and surface dielectric properties;
(9) frequency dependent atmospheric attenuation;
(10) frequency range: $\approx 0.1 \rightarrow 30$ GHz;
(11) generalized transmitter radiation patterns;
(12) arbitrary transmitter/receiver geometry;
(13) automatic selection of the SSFPE range step-size and FFT transform size; and
(14) automatic monitoring of SSFPE solution global error.

The VTRPE model properly accounts for the dominant mechanisms governing tropospheric radio wave propagation, including the effects of anomalous propagation arising from spatial changes in atmospheric refractivity and variable terrain features.

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CONTENTS

§1.0 INTRODUCTION

This report describes the physics and numerical techniques used to model microwave propagation in range-dependent environments by means of the parabolic wave equation (PE).

The method of parabolic wave equations was first proposed in 1944 by Leontovich[1] as a means of solving elliptic partial differential wave equations. He used the technique to solve the problem of electromagnetic wave propagation above a plane earth. In 1946 Leontovich and Fock[2] applied the PE method to the problem of transhorizon radio wave propagation above a spherical earth, thereby making a breakthrough in electromagnetic wave propagation modeling. Approximately 30 years passed before a practical algorithm for solving the Leontovich–Fock parabolic wave equation was developed. In 1973, Hardin and Tappert[3] developed the split-step Fourier parabolic equaton (SSFPE) algorithm and applied it to the problem of modeling ionospheric radar propagation.[4] The split-step Fourier PE algorithm exploited advances in computer hardware and the development of the fast Fourier transform algorithm to yield an efficient numerical solution to the Leontovich–Fock parabolic wave equation. In 1977 Tappert[5] introduced SSFPE methods to the underwater acoustics community, where it rapidly became a valuable tool for predicting range-dependent underwater sound propagation. In 1983 the split-step Fourier PE method was reapplied to radar propagation by Ko, Sari, and Skura[6] to study anomalous tropospheric microwave propagation. Subsequently, Dockery and Konstanzer[7] applied the SSFPE method to analyze phased array radar performance. Since then several others, including Ryan[8] and Craig,[9] have developed electromagnetic PE models.

The scope of this report is restricted to a description of the electromagnetic PE model physics and the split-step Fourier PE algorithm. A detailed description of the numerical and computational issues associated with implementing specific algorithms is left for a later document. By intent, only the salient and most important ideas are described; no attempt is made to cover classical electromagnetism or the mathematical theory of partial differential equations. The reader is directed to standard texts for the necessary background material.

This report focuses on the propagation of radio waves in an inhomogeneous atmosphere with the transmitter and receiver located near the surface of the earth, in the troposphere. The emphasis will be on short wavelengths, the so-called microwaves, which are characteristic of radars and short-range microwave communication links. The wavelength range of interest is from about 3 meters down to about 1 cm, which corresponds to a frequency range of approximately 100 MHz to 30 GHz. This encompasses, to a large extent, the bulk of the search and acquisition radars in current use. It will be assumed that the wavelength is sufficiently small that ionospheric reflections are absent but yet long enough that large numbers of atomic or molecular resonances in the gaseous components of the atmosphere do not occur in a small wavelength interval. In other words, it will be assumed that the frequency is well above the ionospheric plasma frequency but low enough that dispersion effects are not important.

A further assumption is that the atmosphere can be modeled as a simple material medium, viz., a linear, isotropic, nonionized medium. This implies a neglect of the effects

of the earth's magnetic field on wave propagation, and a restriction of propagation paths to the lower portion of the atmosphere — the troposphere. The limitations imposed by these assumptions are flexible and depend, for instance, upon whether the transmission path under consideration is long and near the earth's surface or short and high in the atmosphere. In any case, the above restrictions are commonly accepted as being appropriate for the problem of tropspheric microwave propagation and lead to no significant errors in electromagnetic field calculations.

The methods described in this report form the basis of the VTRPE (variable terrain radio parabolic equation) computer program.[10] The primary use of the VTRPE model is in the analysis and performance prediction of radar and communication systems operating in the microwave region. In practice, the actual performance of such systems in the atmosphere is often quite different from the characteristics predicted based upon free-space propagation. Free-space ranges are often several orders of magnitude different from observed detection ranges in the atmosphere. The reason for this discrepancy is threefold:

1. The earth's surface is a finite conductor and scatters (reflects) incident energy in various directions, leading to complicated spatial interference patterns.
2. The curved earth casts a shadow giving rise to diffraction phenomena.
3. Inhomogeneities in the atmospheric index of refraction cause significant refraction or bending of radio wave energy.

To adequately represent the above types of "anomalous" propagation effects, a propagation model must incorporate full-wave propagation physics and range-dependent environmental inputs. The electromagnetic parabolic wave equation does just this.

The remainder of this report is divided into two major sections. First, the fundamental physics and mathematical relations describing tropospheric microwave propagation over a spherical earth are developed. Starting with the basic Maxwell's equations that govern the electromagnetic fields, a parabolic wave equation is derived for the nonzero field components. Second, this parabolic wave equation is numerically solved by using the split-step Fourier PE algorithm, and the resulting error terms are discussed.

§2.0 PROPAGATION PHYSICS

The scope of this section is to develop, from the fundamental equations of electrodynamics, a parabolic wave equation (PE) that governs the propagation of electromagnetic waves in the troposphere. In later sections, this parabolic wave equation will be solved numerically for the electric and magnetic radiation fields by means of the split-step Fourier PE (SSFPE) algorithm. The model problem in mind is the propagation of linearly polarized microwave radiation through an inhomogeneous troposphere over a spherical earth. The analytical development will proceed as follows: First, Maxwell's equations and some basic concepts of classical electromagnetism are reviewed in order to derive vector wave equations for the electromagnetic radiation fields. By design, the exposition is terse, and the reader is referred to standard texts on classical electromagnetic theory for the details.[11–13] Next, from the vector wave equations an equivalent scalar Helmholtz equation is derived for the nonzero field components. This Helmholtz equation is then transformed from spherical coordinates to an equivalent "earth-flattened" coordinate system. Finally, the elliptic Helmholtz equation is approximated by the Leontovich–Fock parabolic wave equation.

The discussion is restricted to the propagation of electromagnetic radiation in the troposphere, which is assumed to be a linear, isotropic nonionized medium. The electrical properties of the troposphere are modeled by a lossy dielectric. The surface of the earth will be modeled as a nonferrous dielectric sphere with finite conductivity. A source of electromagnetic radiation (i.e., a transmitter) is assumed to be located at $\vec{R}_0 = (r_0, 0, 0)$ on the polar axis of an earth-based spherical coordinate system (r, θ, ϕ). The source is assumed to emit linearly polarized, monochromatic radiation having a simple harmonic time dependence t given by $\exp(-i\omega t)$, where $\omega = 2\pi f$ is the radian frequency. (No generality is lost by this last restriction, since by Fourier's theorem any linear field of arbitrary time dependence can be synthesized from a knowledge of its spectral components.) Furthermore, it is also assumed that the frequencies of interest are all well above the ionospheric plasma frequency and that effects of the earth's magnetic field can be ignored.

2.1 Maxwell's Equations

With the above restrictions, the source-free monochromatic Maxwell's equations in rationalized mks units are (bold faced symbols denote vector fields)

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r},\omega) &= +i\omega\mathbf{B}(\mathbf{r},\omega), \\ \nabla \times \mathbf{H}(\mathbf{r},\omega) &= -i\omega\mathbf{D}(\mathbf{r},\omega), \\ \nabla \cdot \mathbf{B}(\mathbf{r},\omega) &= 0, \\ \nabla \cdot \mathbf{D}(\mathbf{r},\omega) &= 0, \end{aligned} \tag{1}$$

where

$$\mathbf{E}(\mathbf{r},\omega) = \text{electric field intensity vector (V/m)},$$
$$\mathbf{H}(\mathbf{r},\omega) = \text{magnetic field intensity vector (A/m)},$$
$$\mathbf{B}(\mathbf{r},\omega) = \text{magnetic field induction vector (Wb/m)},$$
$$\mathbf{D}(\mathbf{r},\omega) = \text{electric field displacement vector (C/m)},$$

and the implicit time-dependence $\exp(-i\omega t)$ has been suppressed. Henceforth, for notational simplicity, the explicit frequency dependence of the field quantities is suppressed.

The macroscopic fields $\mathbf{D}$ and $\mathbf{H}$ are related, respectively, to $\mathbf{E}$ and $\mathbf{B}$ by constitutive relations that characterize the electromagnetic properties of the material medium involved. For isotropic materials the constitutive relations are

$$\mathbf{D}(\mathbf{r}) = \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}),$$
$$\mathbf{B}(\mathbf{r}) = \mu(\mathbf{r})\mathbf{H}(\mathbf{r}),$$

where ϵ is the absolute (complex) permittivity and μ is the absolute magnetic permeability. For most nonferrous materials, the permeability μ is very close to that of free space μ_0.[14] Accordingly, the medium permeability will be set equal to its vacuum value: $\mu(\mathbf{r}) \Rightarrow \mu_0$. It also proves useful to define the the vacuum wave number k_0 and the dimensionless relative (complex) permittivity ε by

$$\varepsilon(\mathbf{r}) \equiv \frac{\epsilon(\mathbf{r})}{\epsilon_0} = \varepsilon_1 + i\varepsilon_2 = \varepsilon_1(\mathbf{r}) + i\frac{\sigma(\mathbf{r})}{\omega\epsilon_0},$$
$$k_0 \equiv \omega\sqrt{\mu_0\epsilon_0},$$

where ϵ_0 is the vacuum dielectric constant, σ is the medium conductivity and ε_1 is the usual relative permittivity of the medium.

Maxwell's equations (1) then assume the form:

$$\nabla \times \mathbf{E}(\mathbf{r}) = +i\omega\mu_0\mathbf{H}(\mathbf{r}), \tag{2}$$
$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon_0\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{3}$$
$$\mu_0\nabla \cdot \mathbf{H}(\mathbf{r}) = 0, \tag{4}$$
$$\epsilon_0\nabla \cdot \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0. \tag{5}$$

2.2 Vector Wave Equations

Maxwell's equations, Eqs. (2)-(5), are a set of first order partial differential equations in which the $\mathbf{E}$ and $\mathbf{H}$ fields are coupled. An equivalent set of second-order uncoupled vector wave equations may be derived from them, in which the electric and magnetic fields occur separately. For example, by taking the curl of Eq. (3), using standard vector identities,

and employing Eq. (2), one can eliminate the electric field and obtain an equation for the magnetic field **H** alone:

$$\begin{aligned}\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) &= -i\omega\epsilon_0 \nabla \times [\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})], \\ &= \frac{\nabla\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})} \times \nabla \times \mathbf{H}(\mathbf{r}) + k_0^2 \varepsilon(\mathbf{r})\mathbf{H}(\mathbf{r}).\end{aligned} \tag{6}$$

By using the vector identity $\nabla \times \nabla \times \mathbf{C} = \nabla(\nabla \cdot \mathbf{C}) - \nabla^2\mathbf{C}$, and the fact that the magnetic field is solenoidal, Eq. (6) may be cast into the form of a vector wave equation:

$$\nabla^2\mathbf{H}(\mathbf{r}) + \frac{\nabla\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})} \times \nabla \times \mathbf{H}(\mathbf{r}) + k_0^2\varepsilon(\mathbf{r})\mathbf{H}(\mathbf{r}) = 0. \tag{7}$$

Once the magnetic field is found, by solving Eq. (7), the electric field is then obtained from the Maxwell curl relation of Eq. (3):

$$\mathbf{E}(\mathbf{r}) = \frac{i}{\omega\epsilon_0\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}).$$

In a similar fashion, the magnetic field **H** may be eliminated by taking the curl of Eq. (2) and using Eq. (3) to get a vector wave equation involving the electric field alone:

$$\nabla^2\mathbf{E}(\mathbf{r}) + \nabla\left[\mathbf{E}(\mathbf{r}) \cdot \frac{\nabla\varepsilon(\mathbf{r})}{\varepsilon(\mathbf{r})}\right] + k_0^2\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0. \tag{8}$$

The Maxwell curl relation of Eq. (2) may then be used to obtain the **H**-field:

$$\mathbf{H}(\mathbf{r}) = \frac{-i}{\omega\mu_0} \nabla \times \mathbf{E}(\mathbf{r}).$$

The vector wave equations (7) and (8) are no more tractable than Maxwell's equations unless an assumption is made regarding the field polarization. In many cases, the transmitter emits radiation that is linearly polarized in the meridian plane containing the receiver. For a linearly polarized transmitter, the resulting electric field vector may be assumed to have components lying either wholly within (vertical polarization) or perpendicular to (horizontal polarization) the meridian plane containing the source and observation point. Thus, for the vertically polarized case, the nonzero magnetic field component is $\mathbf{H} = H_\phi\hat{e}_\phi$, which may be computed from Eq. (7), while for the horizontally polarized case, the only nonzero electric field component is $\mathbf{E} = E_\phi\hat{e}_\phi$, which may be obtained from Eq. (8).

2.3 Vertical Polarization

If the source emits vertically polarized radiation, then the electric field vector will have components in the meridian plane containing the source and receiver, $\mathbf{E} = E_r\hat{e}_r + E_\theta\hat{e}_\theta$, while the magnetic field vector has a single nonzero component perpendicular to this plane:

$\mathbf{H} = H_\phi \hat{\epsilon}_\phi$. It thus proves simpler to solve for the single magnetic field component by using Eq. (7) and then compute the remaining electric field components from the Maxwell curl relation of Eq. (3):

$$\mathbf{E}(\mathbf{r}) = \frac{i}{\omega\epsilon_0\varepsilon}\nabla\times\mathbf{H} = E_r(\mathbf{r})\hat{\epsilon}_r + E_\theta(\mathbf{r})\hat{\epsilon}_\theta,$$
$$E_r = \frac{i}{\omega\epsilon_0\varepsilon r\sin\theta}\frac{\partial(H_\phi\sin\theta)}{\partial\theta},$$
$$E_\theta = \frac{-i}{\omega\epsilon_0\varepsilon r}\frac{\partial(rH_\phi)}{\partial r}.$$

Working in a spherical coordinate system (r,θ,ϕ), with respective unit vectors $(\hat{\epsilon}_r,\hat{\epsilon}_\theta,\hat{\epsilon}_\phi)$, the first two terms in Eq. (7) become[15]

$$\nabla^2\mathbf{H} = \frac{-2}{r^2\sin\theta}\frac{\partial H_\phi}{\partial\phi}(\hat{\epsilon}_r + \hat{\epsilon}_\theta\cot\theta) + \hat{\epsilon}_\phi\left[\frac{1}{r}\frac{\partial^2(rH_\phi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial H_\phi}{\partial\theta})\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial^2 H_\phi}{\partial\phi^2} - \frac{H_\phi}{r^2\sin^2\theta} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial H_\phi}{\partial\phi}\right], \quad (9)$$

and

$$\frac{\nabla\varepsilon}{\varepsilon}\times\nabla\times\mathbf{H} = \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\phi}\frac{1}{r^2\sin\theta}\left[\hat{\epsilon}_r\frac{\partial(rH_\phi)}{\partial r} + \hat{\epsilon}_\theta\frac{1}{\sin\theta}\frac{\partial(\sin\theta H_\phi)}{\partial\theta}\right]$$
$$- \frac{\hat{\epsilon}_\phi}{r}\left[\frac{1}{r\sin\theta}\frac{\partial(\sin\theta H_\phi)}{\partial\theta}\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\theta} + \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial r}\frac{\partial(rH_\phi)}{\partial r}\right]. \quad (10)$$

Equations (9) and (10) reveal that, in general, the individual vector components of the magnetic field will be coupled due to the presence of the $\frac{\partial\varepsilon}{\partial\phi}$ term. This coupling, or depolarization, is analogous to the Faraday rotation observed in optically active materials. A significant simplification arises if the propagation medium is assumed to have no components of $\nabla\varepsilon(\mathbf{r})$ parallel to the source magnetic field (i.e., $\hat{\epsilon}_\phi\cdot\nabla\varepsilon = 0$). In this case, Eqs. (9) and (10) will have only a single nonzero component:

$$\hat{\epsilon}_\phi\cdot\nabla^2\mathbf{H} = \frac{1}{r}\frac{\partial^2(rH_\phi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial H_\phi}{\partial\theta}) - \frac{H_\phi}{r^2\sin^2\theta}, \quad (11)$$

$$\hat{\epsilon}_\phi\cdot\frac{\nabla\varepsilon}{\varepsilon}\times\nabla\times\mathbf{H} = \frac{-1}{r}\left[\frac{1}{r\sin\theta}\frac{\partial(\sin\theta H_\phi)}{\partial\theta}\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\theta} + \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial r}\frac{\partial(rH_\phi)}{\partial r}\right]. \quad (12)$$

The r and θ-derivatives appearing in Eqs. (11) and (12) may be combined to give the following compact form for the vector wave equation (7):

$$\frac{\varepsilon}{r}\frac{\partial}{\partial r}\frac{1}{\varepsilon}\frac{\partial(rH_\phi)}{\partial r} + \frac{\varepsilon}{r^2\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\varepsilon}\frac{\partial H_\phi}{\partial\theta} + \left(k_0^2\varepsilon - \frac{1}{r^2\sin^2\theta} - \frac{\cot\theta}{r^2\varepsilon}\frac{\partial\varepsilon}{\partial\theta}\right)H_\phi = 0. \quad (13)$$

As it stands, Eq. (13) is not in the desired Helmholtz form but can be transformed into it by a change of the dependent variable:

$$u_v(\mathbf{r}) = \frac{\sqrt{r\sin\theta}}{n} H_\phi(\mathbf{r}), \qquad n(\mathbf{r}) = \sqrt{\varepsilon(\mathbf{r})}. \tag{14}$$

where n is the complex index of refraction referenced to the vacuum dielectric constant ϵ_0. In terms of u, the partial derivative terms in Eq. (13) can be shown to have the form

$$\begin{aligned}
\frac{\partial}{\partial r}\frac{1}{\varepsilon}\frac{\partial(rH_\phi)}{\partial r} &= \frac{1}{\sqrt{\sin\theta}}\frac{\partial}{\partial r}\frac{1}{\varepsilon}\frac{\partial(u\sqrt{r\varepsilon})}{\partial r}, \\
&= \sqrt{\frac{r}{\varepsilon\sin\theta}}\left\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} - \left[\frac{1}{4r^2} + \frac{3}{4}\left(\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial r}\right)^2 - \frac{1}{2\varepsilon}\frac{\partial^2\varepsilon}{\partial r^2}\right]u\right\}, \\
&= \sqrt{\frac{r}{\varepsilon\sin\theta}}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} - \left(\frac{1}{4r^2} + \varepsilon^{\frac{1}{2}}\frac{\partial^2\varepsilon^{-\frac{1}{2}}}{\partial r^2}\right)u\right],
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\theta}\frac{\sin\theta}{\varepsilon}\frac{\partial H_\phi}{\partial\theta} &= \frac{\varepsilon}{\sqrt{r\sin\theta}}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\varepsilon}\frac{\partial(u\sqrt{\varepsilon\csc\theta})}{\partial\theta}, \\
&= \sqrt{\frac{\sin\theta}{r\varepsilon}}\left[\frac{\partial^2 u}{\partial\theta^2} + \left(\frac{1}{4} + \frac{\csc^2\theta}{4} + \frac{\cot\theta}{2\varepsilon}\frac{\partial\varepsilon}{\partial\theta} - \varepsilon^{\frac{1}{2}}\frac{\partial^2\varepsilon^{-\frac{1}{2}}}{\partial\theta^2}\right)u\right].
\end{aligned} \tag{16}$$

Now, with the new field variable u_v, Eq. (13) takes on the desired Helmholtz equation form:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_v}{\partial\theta^2} + [k_0^2\varepsilon(\mathbf{r}) + \delta\varepsilon_v(\mathbf{r})]u_v = 0, \tag{17}$$

with

$$\begin{aligned}
\delta\varepsilon_v(\mathbf{r}) &= -\frac{3}{4}\frac{\csc^2\theta}{r^2} - \frac{\cot\theta}{2r^2\varepsilon}\frac{\partial\varepsilon}{\partial\theta} - \varepsilon^{\frac{1}{2}}\frac{\partial^2\varepsilon^{-\frac{1}{2}}}{\partial r^2} - \frac{\varepsilon^{\frac{1}{2}}}{r^2}\frac{\partial^2\varepsilon^{-\frac{1}{2}}}{\partial\theta^2}, \\
&= -\frac{3}{4}\frac{\csc^2\theta}{r^2} - \frac{\cot\theta}{r^2 n}\frac{\partial n}{\partial\theta} - \frac{n}{r^2}\frac{\partial^2 n^{-1}}{\partial\theta^2} - n\frac{\partial^2 n^{-1}}{\partial r^2}.
\end{aligned} \tag{18}$$

In Eq. (17) the full ϕ-dependence is retained in the ε-dependent terms in the coefficient of u_v, even though $\frac{\partial\varepsilon}{\partial\phi} = 0$ in the derivation. This appears to be a contradiction, which will now be justified, albeit heuristically.

The terms involving $\frac{\partial\varepsilon}{\partial\phi}$ occur in the coefficients of the partial derivative operators in Eq. (11) and will be neglected. Basically this amounts to neglecting small variations in the field amplitude, but retaining the ϕ-dependence in the term $k_0^2\varepsilon$, which affects the phase of the field. This is most easily understood in the context of geometrical optics, where the large k_0 solution to

$$\left[\nabla^2 + k_0^2\varepsilon(\mathbf{r})\right]H_\phi = 0,$$

is expressed in the form[13]

$$H_\phi = Qe^{-ik_0 S},$$

where the phase function S satisfies the eikonal equation

$$(\nabla S)^2 = \varepsilon(\mathbf{r}).$$

Thus, small changes in ε can lead to large overall phase errors in the eikonal $k_0 S$ and cause appreciable errors in the field. This is not the case for the amplitude function Q, which is only weakly dependent on variations in ε.

Neglecting the azimuthal variation in Eq. (10) but including it in the $k_0^2\varepsilon$ factor is termed a $2\frac{1}{2}$-D approximation to the full 3-D Helmholtz equation. It has the advantage of retaining, to lowest order, the effects of azimuthal environmental variability while reducing computational complexity.

2.4 Horizontal Polarization

If the source emits horizontally polarized radiation, then the electric field vector has the single nonzero component $\mathbf{E} = E_\phi(\mathbf{r})\hat{\epsilon}_\phi$. Analogous with the case of vertical polarization, the single electric field component E_ϕ may be obtained from Eq. (8) which, in spherical coordinates, becomes

$$\frac{1}{r}\frac{\partial^2(rE_\phi(\mathbf{r}))}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial E_\phi(\mathbf{r})}{\partial\theta} + \left[k_0^2\varepsilon(\mathbf{r}) - \frac{1}{r^2\sin^2\theta}\right]E_\phi(\mathbf{r}) = 0. \qquad (19)$$

Again paralleling the vertical polarization case, Eq. (19) can be converted into a scalar Helmholtz equation by introducing a new dependent field variable u_h, defined by

$$u_h \equiv u_h(\mathbf{r}) = \sqrt{r\sin\theta}E_\phi(\mathbf{r}), \qquad (20)$$

and dropping the ϕ-derivative term in Eq. (19). With the new field variable, Eq. (19) then becomes

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_h}{\partial\theta^2} + \left[k_0^2\varepsilon(\mathbf{r}) + \delta\varepsilon_h(\mathbf{r})\right]u_h = 0, \qquad (21)$$

where

$$\delta\varepsilon_h(r,\theta) = \frac{-3}{4r^2\sin^2\theta}. \qquad (22)$$

Thus both the horizontal and vertical polarization cases can be expressed in terms of a scalar Helmholtz equation of the form

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + k_0^2\varepsilon(\mathbf{r}) + \delta\varepsilon(\mathbf{r})\right]u(\mathbf{r}) = 0, \qquad (23)$$

where u and $\delta\varepsilon$ are defined by Eqs. (14) and (18) for vertical polarization, and by Eqs. (20) and (22) for horizontal polarization.

The additional term $\delta\varepsilon$ is generally small, except in two cases: The first occurs for vertical polarization with propagation near the boundary separating dissimilar dielectrics with finite conductivity, in which case the spatial derivatives of the index of refraction n may become large. The second occurs for long propagation paths at low frequencies where the wave number k_0 is small and then the spherical correction terms become important. Inclusion of the $\delta\varepsilon$ correction allows one to deal in a unified manner with propagation over idealized boundaries as well as treat propagation in the presence of subsurface overburden.

2.5 Earth-Flattening Transform

The coupling of r and θ variables in Eq. (23) is not desirable for numerical purposes and can be eliminated by means of a simple transformation from the spherical (r, θ, ϕ) coordinate system to an equivalent cartesian (ξ, η, ζ) coordinate system defined by

$$\begin{aligned} \zeta &= a \ln(r/a), \\ \xi &= a\theta \cos\phi, \\ \eta &= a\theta \sin\phi, \end{aligned} \tag{24}$$

where a is the earth's radius, and $\zeta = 0$ corresponds to the surface of the earth. This transformation, suggested by Pryce,[16] is known as an earth-flattening transformation, and with the approximation $\zeta \approx h$ was also studied by Pekeris.[17]

In terms of the new coordinates defined by Eq. (24), the metric

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

becomes

$$ds^2 = e^{2\zeta/a}\left[d\xi^2 + d\eta^2 + d\zeta^2 - (1 - \frac{\sin^2\theta}{\theta^2})\frac{(\xi d\eta - \eta d\xi)^2}{\xi^2 + \eta^2}\right]. \tag{25}$$

In most problems of interest, $\theta \ll 1$, in which case Eq. (25) can be expanded in powers of θ to yield

$$ds^2 = e^{2\zeta/a}\left[d\xi^2 + d\eta^2 + d\zeta^2 - \frac{(\xi d\eta - \eta d\xi)^2}{3a^2} + O(a^{-4})\right]. \tag{26}$$

It is a fairly simple matter to determine the geodesics of the non-Euclidean (i.e., non-flat) space whose metric is given by Eq. (26). Let the source point be located at $(\xi = 0, \eta = 0, \zeta = \zeta_0)$ and limit attention to geodesics which pass through the source. From symmetry, it is obvious that the geodesics lie in a plane containing the ζ-axis. We may without any loss of generality confine ourselves to geodesics in the $\xi\zeta$-plane. These geodesics will then coincide with straight lines through the source, the equations of which are given parametrically by

$$\begin{aligned} \xi &= a \tan^{-1}\left(\tan\psi + \frac{s}{a}\sec\psi e^{-\zeta_0/a}\right) - a\psi, \\ \zeta &= \zeta_0 + \frac{1}{2}a \ln\left(1 + \frac{2s}{a}\sin\psi e^{-\zeta_0/a} + \frac{s^2}{a^2}e^{-2\zeta_0/a}\right), \end{aligned} \tag{27}$$

where s is the geodesic distance (arc length) from the source. The parameter ψ is a constant along the geodesic, being the angle the geodesic makes with the horizontal at the source. In terms of the original spherical coordinates (r, θ, ϕ), Eq. (27) reduces to the equation for a straight line:

$$\begin{aligned} r \sin\theta &= s \cos\psi, \\ r \cos\theta &= s \sin\psi + a e^{\zeta_0/a}. \end{aligned}$$

Also, the distance along the geodesic between the points $(0, 0, \zeta_0)$ and $(\xi, 0, \zeta)$ is simply

$$s = a \left[e^{2\zeta_0/a} - 2e^{(\zeta_0+\zeta)/a} \cos(\xi/a) + e^{2\zeta/a} \right]^{1/2}.$$

In what follows, the meridian plane containing the source/receiver will be taken to be the $r\theta$-plane. In this case, the earth-flattening transformation, Eq. (24), is simply defined in terms of new "cartesian" coordinates (x, z) by

$$\begin{aligned} x &= a\theta, \\ z &= a \ln(1 + h/a) \approx h(1 - \frac{h}{2a}), \quad \text{if } h/a \ll 1 \end{aligned} \tag{28}$$

where $h = r - a$ is the local altitude referenced to the mean earth radius a. This leads to the following Helmholtz separated form for Eq. (23) :

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + K^2(x, z) \right] w(x, z) = 0. \tag{29}$$

The new dependent variable w and the "effective" wave number K are defined for horizontal polarization by

$$\begin{aligned} w_h &= \sqrt{r \sin\theta}\, E_\phi(\mathbf{r}), \\ &= e^{+z/(2a)} \sqrt{a \sin(x/a)}\, E_\phi(\mathbf{x}), \\ &\approx \sqrt{x}\, E_\phi(\mathbf{x}), \quad \text{if } x/a \ll 1 \text{ and } z/a \ll 1 \end{aligned} \tag{30}$$

and

$$\begin{aligned} K_h^2 &= k_0^2 m^2 - \frac{3 \csc^2(x/a)}{4a^2}, \\ &\approx k_0^2 m^2 - \frac{3}{4x^2}, \quad \text{if } x/a \ll 1 \end{aligned} \tag{31}$$

with m the modified index of refraction, defined by

$$m \equiv n(r, \theta, \phi) \frac{r}{a} = n(\mathbf{r})(1 + \frac{h}{a}). \tag{32}$$

For vertical polarization, the new variables are

$$\begin{aligned} w_v &= \frac{\sqrt{r\sin\theta}}{n} H_\phi(\mathbf{r}), \\ &= e^{+3z/(2a)} \frac{\sqrt{a\sin(x/a)}}{m(x,z)} H_\phi(\mathbf{x}), \\ &\approx \frac{\sqrt{x}}{m} H_\phi(\mathbf{x}), \quad \text{if } x/a \ll 1 \text{ and } z/a \ll 1 \end{aligned} \tag{33}$$

and

$$\begin{aligned} K_v^2(x,z) = k_0^2 m^2 &- \frac{3\csc^2(x/a)}{4a^2} - \frac{\cot(x/a)}{am}\frac{\partial m}{\partial x} \\ &- m\frac{\partial^2 m^{-1}}{\partial x^2} - m\frac{\partial^2 m^{-1}}{\partial z^2} - \frac{m}{a}\frac{\partial m^{-1}}{\partial z}. \end{aligned} \tag{34}$$

If the medium is spherically stratified and has no range dependence, then the effective wave number takes the form

$$\begin{aligned} K_v^2(x,z) &= k_0^2 m_{eff}^2(z) - \frac{3}{4x^2}, \qquad \text{if} \quad x/a \ll 1 \\ m_{eff}^2(z) &= m^2(z) + \frac{1}{k_0^2}\left[\frac{1}{m}\frac{\partial^2 m}{\partial z^2} - 2\left(\frac{1}{m}\frac{\partial m}{\partial z}\right)^2 + \frac{1}{am}\frac{\partial m}{\partial z}\right], \\ &\approx n^2(z)(1+2z/a), \qquad \text{if} \quad z/a \ll 1 \end{aligned}$$

which is the same form as the earth-flattening approximation discussed by Pekeris. The Pekeris transform, however, was based upon $z = r - a$ and is only valid for small z/a in contrast to that given by Eq. (28), which is exact.

2.6 Boundary Conditions

The solutions to Maxwell's equations are not unique until boundary conditions on the fields are prescribed. These boundary conditions are (1) a Sommerfeld radiation-type boundary condition at infinity[18]

$$\lim_{r\to\infty} r\left(\frac{\partial A}{\partial r} - ik_0 A\right), \tag{35}$$

where A denotes the field component (i.e., $A \equiv E_\phi$ for horizontal polarization, $A \equiv H_\phi$ for vertical polarization), and (2) continuity of the tangential electric and magnetic fields at the earth's surface, $r = a$. If the propagation medium is lossy — i.e., the dielectric constant is complex with a positive imaginary part — then the Sommerfeld radiation condition may be replaced by the simpler condition that the fields vanish as $r \to \infty$.

The continuity of tangential field components at $r = a$ is implemented by modeling the earth as a locally homogeneous dielectric with finite conductivity. This is a reasonable

approximation at microwave frequencies, since the penetration depth of the fields into the earth's surface is small compared to spatial variations in the surface dielectric properties.

For example, in the case of microwave propagation over seawater, it can be shown by standard techniques that the electric and magnetic fields within the ocean decay exponentially with distance from the air–sea boundary.[12] The vertical scale length over which the field components in the water decay to $1/e$ of their value at the interface is termed the skin depth δ. The high-frequency form of δ is[19]

$$\delta \simeq \sqrt{2/\mu_0 \sigma_s \omega},$$

where σ_s is the surface conductivity. A typical value for the conductivity of seawater is $\sigma \approx 4$ S/m, which means the order of magnitude of δ is about $250/\sqrt{f}$, with f, the frequency, in hertz. For typical marine radar frequencies ($f \approx 10^{10}$ Hz), $\delta \approx 0.025$ m .

The small skin depth means that, as an approximation, the boundary condition of tangential field continuity can be replaced by the Leontovich surface impedance boundary condition[20]

$$\left.\frac{\partial(rA)}{\partial r}\right|_{r=a} = -Z(rA)\Big|_{r=a}. \tag{36}$$

The quantity Z is the local, flat surface impedance given by Fresnel formulas:

$$Z = ik_0\sqrt{\varepsilon_s - \cos^2\vartheta_g} \quad \text{for horizontal polarization,}$$

$$Z = \frac{ik_0}{\varepsilon_s}\sqrt{\varepsilon_s - \cos^2\vartheta_g} \quad \text{for vertical polarization,}$$

with ε_s the complex dielectric constant of the earth's surface and ϑ_g the grazing angle of the field at the earth's surface. In most applications, the surface grazing angle is very small, and it is appropriate to use the limiting forms

$$Z_h = ik_0\sqrt{\varepsilon_s - 1}\,, \tag{37}$$

$$Z_v = \frac{ik_0}{\varepsilon_s}\sqrt{\varepsilon_s - 1}\,. \tag{38}$$

The Leontovich impedance boundary conditions, Eq. (37) and Eq. (38), may also be transformed to earth-flattened cartesian coordinates where they become

$$\left.\frac{\partial w_h(x,z)}{\partial z}\right|_{z=0} = -\left(\frac{1}{2a} + ik_0\sqrt{\varepsilon_s - 1}\right) w_h(x,0) \tag{39}$$

for horizontal polarization and

$$\left.\frac{\partial w_v(x,z)}{\partial z}\right|_{z=0} = -\left(-\frac{1}{2a} + \frac{1}{m(x,0)}\left.\frac{\partial m(x,z)}{\partial z}\right|_{z=0} + ik_0\frac{\sqrt{\varepsilon_s - 1}}{\varepsilon_s}\right) w_v(x,0) \tag{40}$$

for vertical polarization.

2.7 Propagation Factor

In analyzing the various propagation phenomena, it is useful to separate those system parameters not influenced by the environment, antenna radiation characteristics for example, from propagation effects that are environmentally influenced, such as ducting caused by spatial variations in the refractivity. Following Kerr,[21] define the one-way generalized transmission equation, which relates the power received by an omnidirectional receiver at a point in space, $P_r(\mathbf{r})$, to the power emitted from a transmitting antenna, P_t, by

$$\frac{P_r(\mathbf{r})}{P_t} = G_t \left[\frac{F}{(2k_0R)}\right]^2, \tag{41}$$

where

$$\begin{aligned} G_t &= \text{transmitting antenna power gain,} \\ k_0 &= \text{vacuum wavenumber} = \frac{2\pi}{\lambda}, \\ R &= \text{distance between transmitter and receiving point,} \\ F &= \text{pattern propagation factor.} \end{aligned}$$

The radiating characteristics of an antenna are specified in terms of the antenna radiation pattern function $f(\theta, \phi)$ where (θ, ϕ) are the zenith and azimuthal angles, respectively, of a spherical coordinate system centered at the antenna, with polar axis pointed in the direction of maximum transmission. The antenna radiation pattern function f is defined to be the ratio of electric (or magnetic) field strength $\mathbf{E}(\theta, \phi)$ radiated in the direction (θ, ϕ), to the peak transmitted field strength E_0:

$$f(\theta, \phi) \equiv \frac{\mathbf{E}(\theta, \phi)}{E_0}.$$

The radiation pattern function is related to the time-averaged Poynting vector S of the radiated wave field by

$$S(\theta, \phi) = |f(\theta, \phi)|^2 S_0,$$

where S_0 is the energy flow per unit area corresponding to the peak field E_0. In general, the antenna radiation pattern function f is a complex valued quantity.

The antenna gain G is expressed in terms of the pattern function as

$$G = \frac{4\pi}{\int_{(4\pi)} |f(\theta, \phi)|^2 \, d\Omega}.$$

Note that the radiation pattern function f and the corresponding antenna gain G are defined with respect to free-space propagation conditions, and therefore do not include any environmental effects.

Environmental effects are included in the generalized transmission equation via the pattern propagation factor F. The pattern propagation factor is defined as the ratio of the field magnitude at a point in space, $\mathbf{E}(\mathbf{r})$, to the magnitude of the field at the same point but under free-space conditions, $\mathbf{E}_0(\mathbf{r})$:

$$F = \left| \frac{\mathbf{E}(\mathbf{r})}{\mathbf{E}_0(\mathbf{r})} \right|. \tag{42}$$

This definition of the pattern propagation factor assumes that the transmitting antenna is aligned with its maximum response axis pointed directly at the observation point. The pattern propagation factor is the key part of Eq. (41), and is the fundamental quantity to be computed by the PE model. Because of the large dynamic ranges of the various terms in Eq. (41), it is customary to work in dB-space by defining the dimensionless variables PF and PL by

$$PF \equiv 20 \log |F|, \tag{43}$$

$$PL \equiv 20 \log(2k_0 R) - 20 \log |F|. \tag{44}$$

The quantity PF is often called the propagation factor, while PL is denoted the path loss.

The remaining task is to compute the pattern propagation factor F defined by Eq. (42), which, for linearly polarized radiation, is just the ratio of the ϕ-components of the electric or magnetic field vectors:

$$F \equiv F_h(\mathbf{r}) = \frac{|E_\phi(\mathbf{r})|}{|E_\phi^{fs}(\mathbf{r})|}, \quad \text{horizontal polarization} \tag{45}$$

and

$$F \equiv F_v(\mathbf{r}) = \frac{|H_\phi(\mathbf{r})|}{|H_\phi^{fs}(\mathbf{r})|}, \quad \text{vertical polarization} \tag{46}$$

where E_ϕ^{fs} and H_ϕ^{fs} are the free-space electric and magnetic field components. By convention, the pattern propagation factor is defined with respect to the free-space field of a unit-strength point dipole.

Following Papas,[22] the free-space dipole fields are computed by using the dyadic Green's function $\Gamma(\mathbf{r}, \mathbf{r}_0)$, which is the solution of

$$\nabla \times \nabla \times \Gamma(\mathbf{r}, \mathbf{r}_0) - k^2 \Gamma(\mathbf{r}, \mathbf{r}_0) = \mathbf{u}\delta(\mathbf{r} - \mathbf{r}_0),$$

where $\mathbf{u}$ is the unit dyadic. The dyadic Green's function Γ can be expressed in terms of a scalar Green's function G_0 as

$$\Gamma = (\mathbf{u} + \frac{1}{k^2} \nabla \nabla) G_0,$$

where the scalar Green's function G_0 satisfies

$$(\nabla^2 + k^2) G_0(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0). \tag{47}$$

The appropriate outgoing wave solution of Eq. (47) is

$$G_0(\mathbf{r},\mathbf{r}_0) = \frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}. \tag{48}$$

Two types of dipole fields are required: (1) a vertically polarized field arising from a point vertical electric dipole (VED) with electric dipole moment oriented along the z-axis, $\mathbf{p} = p_z\hat{e}_z$; and (2) a horizontally polarized field arising from a point vertical magnetic dipole (VMD) with magnetic dipole moment oriented along the z-axis, $\mathbf{m} = m_z\hat{e}_z$. Employing the dyadic Green's function Γ, the electric and magnetic fields from a VED are given by

$$\mathbf{E}_{ed}(\mathbf{r}) = -\frac{1}{\epsilon}\nabla\times(\mathbf{p}\times\nabla G_0), \tag{49}$$

$$\mathbf{H}_{ed}(\mathbf{r}) = i\omega\mathbf{p}\times\nabla G_0, \tag{50}$$

while the fields from a VMD are given by

$$\mathbf{H}_{md}(\mathbf{r}) = -\nabla\times(\mathbf{m}\times\nabla G_0), \tag{51}$$

$$\mathbf{E}_{md}(\mathbf{r}) = -i\omega\mu_0\mathbf{m}\times\nabla G_0. \tag{52}$$

For both the VED and VMD cases, let a unit-strength point dipole be located at $\mathbf{r}_0 = (r_0,0,0)$ in a spherical coordinate system (r,θ,ϕ), with the dipole moment oriented along the polar axis. Using the scalar free-space Green's function G_0, Eq. (48), and equations (50) and (52), the ϕ-components of the dipole fields are given by

$$H_{ed}(\mathbf{r}) = \frac{-\omega}{4\pi R}e^{ik_0R}\sin\theta\left(\frac{r}{R}\right)\left(k_0+\frac{i}{R}\right), \quad \text{VED} \tag{53}$$

$$E_{md}(\mathbf{r}) = \frac{\omega\mu_0}{4\pi R}e^{ik_0R}\sin\theta\left(\frac{r}{R}\right)\left(k_0+\frac{i}{R}\right), \quad \text{VMD} \tag{54}$$

where $R = |\mathbf{r}-\mathbf{r}_0| = \sqrt{r^2+r_0^2-2rr_0\cos\theta}$, and θ is the polar angle.

The propagation factor for horizontal polarization $F \equiv F_h$ then becomes

$$\begin{aligned} F_h(\mathbf{r}) &= \left|\frac{E_\phi(\mathbf{r})}{E_{md}(\mathbf{r})}\right|, \\ &= \frac{4\pi}{k_0\omega\mu_0}\frac{|u(\mathbf{r})|R^2}{(r\sin\theta)^{\frac{3}{2}}}\left[1+(k_0R)^{-2}\right]^{-\frac{1}{2}}, \end{aligned} \tag{55}$$

while that for vertically polarized radiation $F \equiv F_v$ is

$$\begin{aligned} F_v(\mathbf{r}) &= \left|\frac{H_\phi(\mathbf{r})}{H_{ed}(\mathbf{r})}\right|, \\ &= \frac{4\pi}{k_0\omega}\frac{|n(\mathbf{r})u(\mathbf{r})|R^2}{(r\sin\theta)^{\frac{3}{2}}}\left[1+(k_0R)^{-2}\right]^{-\frac{1}{2}}. \end{aligned} \tag{56}$$

To express the propagation factor F in earth-flattened (x, z) coordinates, the dipole fields defined in Eq. (53) and Eq. (54) need to be converted. First, the source-receiver separation distance R becomes

$$\begin{aligned} R = |\mathbf{r} - \mathbf{r}_0| &= \sqrt{(r - r_0)^2 + 4rr_0 \sin^2(\theta/2)}, \\ &= 2ae^{(z+z_0)/(2a)} \sqrt{\sinh^2(\frac{z - z_0}{2a}) + \sin^2(\frac{x}{2a})}, \\ &\approx (1 + \frac{z + z_0}{2a}) \sqrt{(z - z_0)^2 + x^2} + O(x^4/a^2), \\ &\rightarrow \sqrt{(z - z_0)^2 + x^2}, \qquad \text{if } x/a \ll 1 \text{ and } (z + z_0)/a \ll 1 \, . \end{aligned} \tag{57}$$

The last relation in Eq. (57) has a relative error of $< 2.5\text{x}10^{-3}$ for horizontal ranges $x < 200$ km and altitudes $z < 30$ km. In like manner, the angular dipole coefficient $(r/R)\sin\theta$ has the limiting forms:

$$\begin{aligned} \frac{r}{R} \sin\theta &= (a/R) e^{z/a} \sin(x/a), \\ &\rightarrow \frac{x}{R}, \quad \text{if } x/a \ll 1 \text{ and } (z + z_0)/a \ll 1 \\ &\rightarrow 1, \quad \text{if } (z - z_0)/x \ll 1 \, . \end{aligned} \tag{58}$$

Finally, the propagation factor takes the following limiting form for altitudes and ranges typical of tropospheric propagation:

$$\begin{aligned} F_h(\mathbf{x}) &\approx \frac{4\pi R^2}{k_0 \omega \mu_0} x^{-\frac{3}{2}} \frac{|w(\mathbf{x})|}{\sqrt{1 + (k_0 R)^{-2}}}, \\ &\rightarrow \frac{4\pi\sqrt{x}}{k_0 \omega \mu_0} \frac{|w(\mathbf{x})|}{\sqrt{1 + (k_0 x)^{-2}}}, \quad \text{if } (z - z_0)/x \ll 1 \end{aligned} \tag{59}$$

and

$$\begin{aligned} F_v(\mathbf{x}) &= \frac{4\pi R^2}{k_0 \omega} x^{-\frac{3}{2}} \frac{|m(\mathbf{x}) w(\mathbf{x})|}{\sqrt{1 + (k_0 R)^{-2}}}, \\ &= \frac{4\pi}{k_0 \omega} \frac{\sqrt{x} |m(\mathbf{x}) w(\mathbf{x})|}{\sqrt{1 + (k_0 x)^{-2}}}, \quad \text{if } (z - z_0)/x \ll 1 \, . \end{aligned} \tag{60}$$

2.8 Parabolic Wave Equation

Although Eq. (29) follows exactly from Maxwell's equations, analytic solutions are only possible if K is independent of x. Nor is direct numerical solution an easy alternative. This follows from the fact that Eq. (29) is an elliptic partial differential equation; hence solving for a solution at one point requires that it must be numerically solved over the entire propagation domain simultaneously. Standard finite difference approaches to solving

elliptic boundary value problems also tend to require griding schemes on the order of a wavelength. For the long ranges involved in tropospheric propagation, this would lead to unacceptably large problem sizes.

To deal with these issues, Leontovich and Fock introduced the concept of a parabolic wave equation to treat the problem of transhorizon radio wave propagation. Their approach may be summarized as follows: First, an envelope transformation is performed to remove the anticipated rapid phase variation:

$$w(\mathbf{x}) = e^{+i\bar{k}x}\psi(\mathbf{x}), \tag{61}$$

where $\bar{k}$ is a reference wave number. (For most tropospheric problems, the natural choice is $\bar{k} = k_0$.) Next, Eq. (61) is substituted into Eq. (29) to get

$$2i\bar{k}\frac{\partial\psi(\mathbf{x})}{\partial x} + \frac{\partial^2\psi(\mathbf{x})}{\partial z^2} + \left(K^2(\mathbf{x}) - \bar{k}^2\right)\psi(\mathbf{x}) = -\frac{\partial^2\psi(\mathbf{x})}{\partial x^2}. \tag{62}$$

The essence of the Leontovich–Fock PE approximation is to assume that variations of the envelope function with range are small relative to other terms, and therefore to drop the $\frac{\partial^2\psi}{\partial x^2}$ term to get a parabolic equation in the longitudinal coordinate x:

$$2i\bar{k}\frac{\partial\psi(\mathbf{x})}{\partial x} + \frac{\partial^2\psi(\mathbf{x})}{\partial z^2} + \left(K^2(\mathbf{x}) - \bar{k}^2\right)\psi(\mathbf{x}) = 0. \tag{63}$$

The advantage of Eq. (63) over Eq. (62) is that it is parabolic rather than elliptic in range. This means that it is an initial value problem and can be solved efficiently by marching algorithms. Also, the surface boundary conditions retain the same form in the PE approximation as in the original Helmholtz equation.

The justification for dropping the second derivative term is usually made on the grounds that the envelope function ψ is a slowly varying function of the range coordinate x. While pedagogically correct, such a derivation of PE does not yield any insight into the approximation nor the errors that are incurred in using it. An alternative approach to deriving the PE is via factorization of the elliptic wave equation. If we define an operator $Q(\mathbf{x})$ by

$$Q(\mathbf{x}) = \sqrt{\frac{\partial^2}{\partial z^2} + K^2(\mathbf{x})}, \tag{64}$$

then Eq. (29) can be expressed in the equivalent factored form

$$\left(\frac{\partial}{\partial x} + iQ\right)\left(\frac{\partial}{\partial x} - iQ\right)w(\mathbf{x}) + i\left[\frac{\partial}{\partial x}, Q\right]w(\mathbf{x}) = 0. \tag{65}$$

(The notation $\left[\frac{\partial}{\partial x}, Q\right] = \frac{\partial Q}{\partial x} - Q\frac{\partial}{\partial x}$ is the commutator of the operators $\frac{\partial}{\partial x}$ and Q.)

For range-independent propagation, the commutator $\left[\frac{\partial}{\partial x}, Q\right] = 0$ and the equation satisfied by outwardly propagating waves is just

$$\frac{\partial w(\mathbf{x})}{\partial x} = iQ(\mathbf{x})w(\mathbf{x}). \tag{66}$$

Equation (66) is an exact formal solution to the range- independent Helmholtz equation, and is the most general PE that is exact for range independent media. Following Tappert,[5] a parabolic wave equation with the Q operator defined by Eq. (64) will be denoted as the general PE (GPE) and the Q operator will be denoted as the GPE propagator. GPE is the most complete PE that is evolutionary in range and neglects backscattering. For range-independent environments, it is exact within the limits of the far-field approximation, and is the starting point for all numerical PE algorithms. In following sections, the computational techniques used to solve the PE will be derived.

2.9 Variable Terrain Physics

The preceding sections have dealt with the problem of radio wave propagation over a smooth spherical earth. This enabled the surface boundary conditions on the electromagnetic field components to be satisfied on the conformal spherical shell $r = a$, where a is the average earth radius. In reality, the surface of the earth is not smooth but has a nonuniform topography. What effect does this varying terrain have on the propagation of radio waves? Intuitively, one expects that the Leontovich surface impedance boundary condition

$$\left.\frac{\partial(rA)}{\partial r}\right|_{r=a} = -Z(rA)\Big|_{r=a}, \tag{67}$$

will be modified. This, in fact, is the case.

If the terrain relief $\zeta(\theta)$ is a function of the meridian angle θ, such that $r = a + \zeta(\theta)$ is the radial distance corresponding to the local surface, the boundary condition in Eq. (67) will become

$$\left.\frac{\partial(rA)}{\partial \mathbf{n}}\right|_{r=a+\zeta(\theta)} = -Z(rA)\Big|_{r=a+\zeta(\theta)}, \tag{68}$$

where A denotes the electromagnetic field ϕ-component (i.e., $A \equiv E_\phi$ for horizontal polarization; $A = H_\phi$ for vertical polarization), and $\frac{\partial}{\mathbf{n}}$ is the normal derivative to the surface defined by ζ. This surface normal derivative is specified by

$$\frac{\partial}{\partial \mathbf{n}} \equiv \frac{\frac{\partial}{\partial r} - \frac{\tau}{r}\frac{\partial}{\partial \theta}}{\sqrt{1+\tau^2}}, \quad \tau = \frac{1}{r}\frac{\partial \zeta}{\partial \theta}. \tag{69}$$

Using Eq. (69), the Leontovich boundary condition may be written as

$$\left.\frac{\partial(rA)}{\partial r}\right|_{r=a+\zeta} - \tau \left.\frac{\partial A}{\partial \theta}\right|_{r=a+\zeta} = -Z\sqrt{1+\tau^2}(rA)\Big|_{r=a+\zeta}. \tag{70}$$

The new boundary condition, Eq. (70), is seen to be more complicated than the smooth surface condition in Eq. (67) due to the presence of the term involving $\frac{\partial A}{\partial \theta}$. This complication is difficult to deal with numerically, and it seems logical to investigate whether a modification to the standard earth-flattening transformation would, in effect, "smooth" out the variable terrain. Accordingly, let us define the surface-locked cartesian coordinate system (x, h), where x is the local horizontal (range) coordinate and h is a new vertical coordinate measured with respect to the local terrain elevation. The new flattening transform is defined by

$$\begin{aligned} x &= a\theta, \\ h &= a\ln(r/a) - \zeta(\theta). \end{aligned} \tag{71}$$

In terms of the (x, h) coordinates, the normal derivative, Eq. (69), takes on the simple form

$$\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial h},$$

and the boundary condition Eq. (70) becomes

$$\left.\frac{\partial(rA)}{\partial h}\right|_{h=0} = -Z(rA)\Big|_{h=0}.$$

Thus, the new transformation defined by Eq. (71) has indeed led to a simplification of the surface boundary condition—but at what price?

If the transform defined by Eq. (71) is applied to the scalar Helmholtz equation in spherical coordinates

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + k_0^2 n^2(r,\theta) + \delta n(\mathbf{r})\right] u(r,\theta) = 0, \tag{72}$$

then the result is

$$\left[1 + (\frac{\alpha}{a})^2\right]\frac{\partial^2 u}{\partial h^2} - \frac{2a}{a}\frac{\partial^2 u}{\partial x \partial h} - \frac{\dot{\alpha}}{a^2}\frac{\partial u}{\partial h} + \frac{\partial^2 u}{\partial x^2} + \left[k_0^2 m^2 + \delta m\right] u(h,x) = 0 \tag{73}$$

where

$$\alpha \equiv \frac{\partial \zeta}{\partial \theta}, \quad \dot{\alpha} \equiv \frac{\partial^2 \zeta}{\partial \theta^2}.$$

Equation (73) is no longer in Helmholtz format—the flattening transformation has destroyed its separability by admixing the xh-derivatives.

The complications in Eq. (73) can be shown to arise from the $\frac{\partial^2}{\partial \theta^2}$ term in Eq. (72). Ultimately the Leontovich–Fock parabolic approximation is to be applied, which amounts to keeping terms to first order in $\frac{\partial}{\partial \theta}$. This suggests an approach that first makes a PE approximation in spherical coordinates to Eq. (72) and then applies the earth-flattening transformation Eq. (71). Accordingly, let us define the envelope function ψ by

$$u(r,\theta) = e^{i\bar{k}a\theta}\psi(r,\theta),$$

and then write Eq. (72) in parabolic form as

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \psi}{\partial r} + \frac{i2\bar{k}a}{r}\frac{\partial \psi}{\partial \theta} + \left(k_0^2 n^2 + \delta n - \frac{\bar{k}^2 a^2}{r^2}\right)\psi = 0. \tag{74}$$

Next, let us apply the transform, Eq. (71), to get

$$\frac{\partial^2 \psi}{\partial h^2} + i2\bar{k}\left(\frac{\partial \psi}{\partial x} - \frac{\alpha}{a}\frac{\partial \psi}{\partial h}\right) + \left[k_0^2 m^2 + \delta m - \bar{k}^2\right]\psi = 0. \tag{75}$$

The term involving $\frac{\partial \psi}{\partial h}$ can be removed by a simple change of variable:

$$\psi \Rightarrow e^{i\bar{k}\alpha h}\psi,$$

in which case Eq. (75) takes on the final form:

$$\frac{\partial^2 \psi}{\partial h^2} + i2\bar{k}\frac{\partial \psi}{\partial x} + \left[k_0^2 m^2 + \delta m + \bar{k}^2\left(\frac{\alpha^2}{a^2} - 2h\dot{\alpha} - 1\right)\right]\psi = 0. \tag{76}$$

Comparing with the form of the PE for a smooth surface, Eq. (63), the effects of variable terrain are seen to be incorporated into a modified wavenumber K_{mod} defined by

$$K_{mod}^2(h,x) = k_0^2 m^2 + \delta m + \bar{k}^2\left(\frac{\alpha^2}{a^2} - 2h\dot{\alpha} - 1\right). \tag{77}$$

The inclusion of variable terrain in the PE approximation is thus modeled by modifying the medium index of refraction via Eq. (77) and then proceeding with numerical solutions as in the smooth-surface PE. Having solved Eq. (76) for the field in xh-coordinates, it is a simple matter to transform them back to physical space and then compute the propagation factor F.

§3.0 NUMERICAL SOLUTIONS

This section describes methods for the numerical solution of the generalized one-way parabolic wave equation

$$\frac{\partial \psi(\mathbf{x})}{\partial x} = iQ(\mathbf{x})\psi(\mathbf{x}), \tag{78}$$

where the generalized PE (GPE) operator Q is defined to be

$$Q(\mathbf{x}) = \sqrt{\frac{\partial^2}{\partial z^2} + K^2(\mathbf{x})} - \bar{k}, \tag{79}$$

with $\bar{k}$ a reference wave number. Because of the square root appearing in Eq. (79), the usual techniques for solving parabolic partial differential equations are not possible. This is because Q belongs to the class of pseudodifferential operators[23] since it contains both multiplicative, K^2, and differential, $\frac{\partial^2}{\partial z^2}$, operators under the radical. Hence, $Q(\mathbf{x})\psi(\mathbf{x})$ cannot be expressed as a finite Taylor series in local operators about the point $\mathbf{x}$. The Q operator may be expressed in terms of a dimensionless "kinetic energy" operator T and a "potential energy" operator V as

$$Q(\mathbf{x}) = k_0\sqrt{1 - T(z) - V(x,z)} - \bar{k}, \tag{80}$$

where the kinetic and potential energy operators are

$$\begin{aligned} T(z) &\equiv -\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}, \\ V(x,z) &\equiv 1 - K^2(x,z)/k_0^2. \end{aligned} \tag{81}$$

If $\|T + V\| \ll 1$, then a formal Taylor series expansion of Eq. (80) gives

$$Q(\mathbf{x}) \approx Q_1(\mathbf{x}) = k_0 - \bar{k} - \frac{k_0}{2}(T + V), \tag{82}$$

which is the standard parabolic equation (SPE) operator first proposed by Tappert.[5] The SPE approximation to Q basically assumes that the propagation occurs within a small cone of angles centered about $\bar{k}$.

An alternative approximation to Q was developed by Feit and Fleck[24] for propagation in optical fibers and used by Thompson and Chapman[25] for underwater acoustic propagation. In this form, the GPE operator is approximated as:

$$Q(\mathbf{x}) \approx Q_2(\mathbf{x}) = k_0(\sqrt{1 - T} - 1) + K(\mathbf{x}) - \bar{k}. \tag{83}$$

This approximation is known as a wide-angle parabolic equation (WAPE) operator since it is valid for much wider propagation angles than the SPE.

Suppose, for the moment, that the GPE Q-operator is not a function of the range coordinate x: $Q \to Q(z)$. Then, the formal solution of Eq. (78) is just

$$\psi(x,z) = e^{i(x-x_0)Q(z)}\psi(x_0,z),$$

where the exponential of the operator Q is defined by its power series expansion. The conventional numerical approach to solving Eq. (78) is to use finite-difference methods and approximate the exponential operator $\exp[iQ(x-x_0)]$ by using Cayley's method:[26]

$$e^{iQ(x-x_0)} \simeq \frac{1+\frac{1}{2}iQ\Delta x}{1-\frac{1}{2}iQ\Delta x}.$$

If the Q operator is now discretized by a finite-difference approximation in z, a complex tridiagonal system of equations results. This is simply the standard Crank–Nicholson method for a parabolic partial differential equation.[27]

There are two fundamental problems with this approach however. First, the truncation error of the Crank-Nicholson scheme is only second order in Δz and Δx, and thus requires small mesh intervals to achieve high accuracy. This in turn leads to very large matrix systems and many range steps. The second problem with the finite-difference approach is a lack of rigorous error bounds on the solution that can be monitored during the computations to assure a fixed, preset global error.

To deal with both these issues, an alternative solution method is used that is based upon spectral operator techniques. This spectral technique is known as the split-step Fourier PE algorithm (SSFPE) and was developed by Hardin and Tappert.[3] The remainder of this section deals with development of the SSFPE algorithm, associated error bounds, and numerical implementation using the fast Fourier transform algorithm.

3.1 Magnus Expansion

Given the parabolic equation Eq. (78), there still exists a problem of developing a numerical solution. The difficulty arises from the nonlocal nature of the Q operator. To solve the GPE, techniques from time-dependent quantum scattering theory are used to express the solution in terms of an evolution operator.[28] Let us define the wave evolution operator $U(\mathbf{x},\mathbf{x}_0)$ that determines the PE wave field $\psi(\mathbf{x})$ at the point $\mathbf{x} = (x,z)$ in terms of the wave field $\psi(\mathbf{x}_0)$ at the point $\mathbf{x}_0 = (x_0,z)$ by the equation

$$\psi(\mathbf{x}) = U(\mathbf{x},\mathbf{x}_0)\psi(\mathbf{x}_0).$$

It follows by substitution into Eq. (78) that U is an operator satisfying the partial differential equation

$$\frac{\partial U(\mathbf{x},\mathbf{x}_0)}{\partial x} = iQ(\mathbf{x})U(\mathbf{x},\mathbf{x}_0). \tag{84}$$

Clearly, U must also satisfy the initial condition

$$U(\mathbf{x}_0,\mathbf{x}_0) = 1. \tag{85}$$

and possess the group property

$$U(\mathbf{x},\mathbf{x}_0) = U(\mathbf{x},\mathbf{x}_1)U(\mathbf{x}_1,\mathbf{x}_0). \tag{86}$$

Furthermore, if there are no dissipative processes present, it follows from the complex Poynting theorem that the z-integrated field energy density must be constant with range

$$\int |\psi(x,z)|^2\,dz = \int |\psi(x_0,z)|^2\,dz,$$

and this leads to the important result that the evolution operator U must be unitary: $U^\dagger(\mathbf{x},\mathbf{x}_0) = U^{-1}(\mathbf{x},\mathbf{x}_0)$, where $U^\dagger$ is the (Hermitian) adjoint of U. The importance of having a unitary operator for numerical work should not be underestimated, since this form for U leads to numerical methods that can be shown to be absolutely convergent for all range step sizes.

Integrating Eq. (84) and using Eq. (85) yields an integral equation for the evolution operator

$$U(\mathbf{x},\mathbf{x}_0) = 1 + i\int_{x_0}^{x} Q(t,z)U(t,x_0,z)\,dt\,.$$

Solving the above integral equation iteratively yields Dyson's expansion for U:

$$\begin{aligned} U(x,x_0,z) = 1 + i\int_{x_0}^{x} Q(t,z)\,dt - \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2\, Q(x_1,z)Q(x_2,z) \\ - i\int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \int_{x_0}^{x_2} dx_3\, Q(x_1,z)Q(x_2,z)Q(x_3,z) + \cdots \end{aligned} \tag{87}$$

The x-ordering of the integrands in Eq. (87) is very important because, in general, the Q operators at different x-positions do not commute. Thus the ordering of the operators is determined by the range to which they refer, with operators referring to earlier ranges always appearing to the right. Though the Dyson expansion is a formally exact solution for U, it is not very useful for calculations, since if the series in Eq. (87) is truncated after a finite number of terms, the result is no longer a unitary operator.

A formal solution to Eq. (84) that does preserve the unitary nature of U can be found by using a technique developed by Magnus.[29] The Magnus result is based upon an exponential operator expansion which effectively includes all terms in the Dyson expansion. The details are found in Ryan,[8] and yield

$$\begin{aligned} U(x,x_0,z) = \exp\Bigg(i\int_{x_0}^{x} Q(x_1,z)\,dx_1 - \frac{1}{2}\int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2\,[Q(x_1,z),Q(x_2,z)] \\ - \frac{i}{6}\int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \int_{x_0}^{x_2} dx_3 \Big\{ [[Q(x_1,z),Q(x_2,z)],Q(x_3,z)] \\ + [[Q(x_3,z),Q(x_2,z)],Q(x_1,z)] \Big\} + \cdots \Bigg). \end{aligned} \tag{88}$$

While the Magnus expansion provides an exact formal solution to the GPE wave field, three approximations must be made before it is useful for numerical computations. First, the infinite series in the Magnus exponent must be truncated. Keeping the first term in the operator exponential in Eq. (88), yields the first-order Magnus approximation U_1:

$$U(x, x_0, z) \simeq U_1(x, x_0, z),$$
$$U_1(x, x_0, z) \equiv e^{i \int_{x_0}^{x} Q(t,z)\, dt} = e^{i(x-x_0)H}, \tag{89}$$

where H is the range-averaged PE "Hamiltonian"

$$H = \frac{1}{\Delta} \int_{x_0}^{x_0+\Delta} Q(t, z)\, dt\,, \quad \Delta = x - x_0. \tag{90}$$

Second, the pseudodifferential GPE operator, Q, is factored into the sum of two ordinary operators, one of which is independent of range:

$$Q(x, z) \approx A(z) + B(x, z). \tag{91}$$

This factorization of Q is not unique; in general, the A and B operators do not commute with each other. For example, if the SPE operator, Eq. (82), is used to approximate $Q(x)$, then

$$A(z) = \frac{1}{2k_0} \frac{\partial^2}{\partial z^2}\,, \tag{92}$$
$$B(x, z) = (k_0 - \bar{k}) - \frac{k_0}{2} V(x, z), \tag{93}$$

while if the WAPE operator, Eq. (83), is used

$$A(z) = \sqrt{k_0^2 + \tfrac{\partial^2}{\partial z^2}} - k_0 \tag{94}$$
$$B(x, z) = K(x, z) - \bar{k}. \tag{95}$$

Finally, the Hamiltonian H is evaluated by using the midpoint rule as

$$H(x, z) = \frac{1}{\Delta} \int_{x_0}^{x_0+\Delta} Q(t, z)\, dt\,,$$
$$\approx Q(x_0 + \Delta/2, z) + \frac{\Delta^2}{24} \frac{\partial^2 Q(t, z)}{\partial x^2}, \quad t \in (x_0, x_0 + \Delta),$$
$$\approx A(z) + B(\bar{x}, z) \quad \bar{x} \equiv x_0 + \Delta/2\,. \tag{96}$$

If this is done, a formal solution of Eq. (78) can be expressed in exponential operator form as

$$\psi(x + \Delta, z) \simeq U_1(x + \Delta, x, z)\, \psi(x, z),$$
$$= e^{+i\Delta H(x,z)}\, \psi(x, z),$$
$$= \exp\left\{ +i[\Delta A(z) + \int_{x}^{x+\Delta} B(t, z)\, dt] \right\} \psi(x, z),$$
$$\approx \exp\left\{ +i\Delta[A(z) + B(\bar{x}. z)] \right\} \psi(x, z). \tag{97}$$

3.2 Split-Step PE Algorithm

The first-order Magnus expansion, Eq. (97), is not suitable for numerical work since the A and B operators appearing in the exponent do not in general commute. To resolve this, the Trotter product formula[30] is used to symmetrically factor the Magnus expansion into the product of simpler operators, U_A and U_B

$$U_A = U_A(\Delta) = e^{+i\Delta A(z)},$$
$$U_B = U_B(\Delta) = e^{+i\Delta B(x+\Delta/2,z)},$$

to yield the split-step PE algorithm:

$$\begin{aligned}\psi(x+\Delta,z) &= U_A(\tfrac{\Delta}{2})\,U_B(\Delta)\,U_A(\tfrac{\Delta}{2})\,\psi(x,z),\\ &= e^{+i\Delta A(z)/2}\,e^{+i\Delta B(x+\Delta/2,z)}\,e^{+i\Delta A(z)/2}\,\psi(x,z).\end{aligned} \tag{98}$$

The operator U_A is known as the free-space propagator and is the exact formal solution to the parabolic equation in the absence of refraction. Physically, the split-step PE algorithm amounts to

(1) a half-step of free-space propagation, $U_A(\frac{\Delta}{2})$;
(2) a phase correction, $U_B(\Delta)$, to account for refractive effects; and
(3) finally, another half-step of free-space propagation, $U_A(\frac{\Delta}{2})$.

In actual use, the split-step algorithm is used to advance the PE field multiple range steps:

$$\psi_n = e^{i\Delta_n A/2}\,e^{i\Delta_n B_n}\,e^{i\Delta_n A/2}\,\psi_{n-1}\,,$$

where

$$\begin{aligned}\Delta_n &= x_n - x_{n-1},\\ \psi_n &= \psi(x_n,z),\\ B_n &= B[(x_{n-1}+x_n)/2,z]\,.\end{aligned}$$

By using the group property of the evolution operator, Eq. (86), the split-step algorithm may be iterated to give

$$\begin{aligned}\psi_n &= e^{i\Delta_n A/2}e^{i\Delta B_n}e^{i\Delta_n A/2}e^{i\Delta_{n-1}A/2}e^{i\Delta B_{n-1}}e^{i\Delta_{n-1}A/2}\,\psi_{n-2},\\ &= e^{i\Delta_n A/2}e^{i\Delta B_n}e^{i(\Delta_n+\Delta_{n-1})A/2}e^{i\Delta B_{n-1}}e^{i\Delta_{n-1}A/2}\,\psi_{n-2},\\ &= e^{i\Delta_n A/2}e^{i\Delta_n B_n}e^{i(\Delta_n+\Delta_{n-1})A/2}\cdots e^{i\Delta_2 B_2}e^{i(\Delta_2+\Delta_1)A/2}e^{i\Delta_1 B_1}e^{i\Delta_1 A/2}\,\psi_0\,.\end{aligned}$$

3.3 Split-Step PE Truncation Error

At each step of the PE calculation, there will be a certain amount of error induced in the solution due to round-off error and the intrinsic error of the split-step PE algorithm.

The round-off error is caused by performing calculations using floating-point arithmetic on finite-word-length computers. This type of error is highly dependent upon the hardware characteristics and actual numerical algorithms used, and will not be discussed further. The intrinsic error is termed truncation error and arises from the various approximations in going from the exact Magnus solution in Eq. (88) to the split-step algorithm in Eq. (98).

The split-step PE is thus predicated on four approximations:

(1) approximating the exact square root GPE operator, Eq. (79), by a finite sum of ordinary operators;
(2) truncating the formally exact Magnus solution for the evolution operator and keeping only the first term in the exponent, Eq. (89);
(3) evaluating the PE Hamiltonian by the midpoint rule, Eq. (96); and
(4) approximating the exponential Magnus operator by a symmetrized factorization of individual operators, Eq. (98).

Each of these steps introduces an intrinsic truncation error in the split-step PE solution—the ability to quantify these errors is crucial to the successful implementation of a numerical PE code.

Although the Magnus expansion provides an exact formal solution to the PE wave field, three approximations must be made to it before numerical computations are feasible. First, the infinite series in the Magnus exponent is truncated after the first term

$$U(x_0+\Delta, x_0, z) \approx e^{i\Delta H}, \qquad H = \frac{1}{\Delta}\int_{x_0}^{x_0+\Delta} Q(t,z)\,dt\,, \tag{99}$$

where H is the range-averaged PE Hamiltonian.

The truncation errors incurred by using Eq. (99) are a function of the range dependence of the Q operator. If Q, or equivalently K, is independent of x, then Eq. (99) is exact. For range-dependent K, the truncation error in U may be quantified by examining the next term in the Magnus expansion and gives

$$\begin{aligned} U(x_0+\Delta, x_0, z) &\approx \exp\left\{ i\Delta H - \frac{1}{2}\int_{x_0}^{x_0+\Delta} dx_1 \int_{x_0}^{x_1} dx_2 \left[Q(x_1,z), Q(x_2,z)\right] \right\}, \\ &= \exp\left\{ i\Delta H - \frac{\Delta^3}{6}\left[\frac{\partial Q}{\partial x}, Q\right] + O(\Delta^4) \right\}. \end{aligned} \tag{100}$$

Thus, when Q is range-dependent, truncation errors proportional to $O(\Delta^3)$ are incurred no matter what form of the PE operator is used. Furthermore, how the range-averaged PE Hamiltonian H is evaluated critically affects the error budget. For example, if Q is expanded in a Taylor series about the midpoint, $\bar{x} = (x+x_0)/2$, and the integral in Eq. (98) is evaluated by the midpoint rule, then

$$U(x_0+\Delta, x_0, z) = \exp\left(i\Delta Q(\bar{x},z) + \frac{\Delta^3}{24}\left\{ iQ''(\bar{x},z) - 4\left[Q'(\bar{x},z), Q(\bar{x},z)\right]\right\} + O(\Delta^4) \right),$$

where $Q'(\bar{x},z) = \partial Q(\bar{x},z)/\partial x$ and $Q''(\bar{x},z) = \partial^2 Q(\bar{x},z)/\partial x^2$.

The local truncation error T_1 incurred in the PE field solution by using Eq. (99), with Q evaluated at the midpoint $\bar{x}$, is then

$$\begin{aligned} T_1 &\equiv \left[U(x_0+\Delta,x_0,z) - e^{i\Delta Q(\bar{x},z)}\right]\psi(x_0,z), \\ &\simeq \frac{\Delta^3}{24}\left\{iQ''(\bar{x},z) - 4\left[Q'(\bar{x},z)Q(\bar{x},z) - Q(\bar{x},z)Q'(\bar{x},z)\right]\right\}\psi(x_0,z), \\ &= \frac{\Delta^3}{24}\left\{i\frac{\partial^2 B(\bar{x},z)}{\partial x^2} - 4\left[\frac{\partial B(\bar{x},z)}{\partial x}A(z) - A(z)\frac{\partial B(\bar{x},z)}{\partial x}\right]\right\}\psi(x_0,z). \end{aligned} \tag{101}$$

A second type of truncation error occurs if the symmetrized operator splitting, defined by Eq. (98), is employed to compute the propagator. This splitting error can be evaluated by using the Baker–Campbell–Hausdorff[31–33] (BCH) expansion of two noncommuting operators:[34]

$$e^A e^B = \exp\left(A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}[A,[A,B]] + \tfrac{1}{12}[[A,B],B] + \cdots\right).$$

Applying the BCH expansion to Eq. (98) gives

$$e^{i\Delta A/2}\, e^{i\Delta \bar{B}}\, e^{i\Delta A/2} = \exp i\left\{\Delta A + \Delta\bar{B} - \frac{\Delta^3}{12}[[A,\bar{B}],\bar{B}] + \frac{\Delta^3}{24}[[\bar{B},A],A] + O(\Delta^4)\right\},$$

and the local truncation error T_2 caused by using the Trotter product formula, with B evaluated at the midpoint is thus

$$\begin{aligned} T_2 &\equiv \left\{e^{i\Delta[A+B(\bar{x})]} - e^{i\Delta A/2}\, e^{i\Delta\bar{B}}\, e^{i\Delta A/2}\right\}\psi(x_0,z), \\ &\approx \frac{i\Delta^3}{24}\left\{2\left[[A,B(\bar{x})],B(\bar{x})\right] - \left[[B(\bar{x}),A],A\right]\right\}\psi(x_0,z), \\ &= \frac{i\Delta^3}{24}\left[2(AB^2 - 2BAB + B^2A) - (BA^2 - 2ABA + A^2B)\right]\psi(x_0,z). \end{aligned} \tag{102}$$

If the SPE operator is used to approximate Q, then the third-order local truncation error terms are given explicitly by

$$T_1 = -\frac{\Delta^3}{48}\left(ik_0\frac{\partial^2\bar{V}}{\partial x^2} + 2\frac{\partial^3\bar{V}}{\partial z^2\partial x}\right)\psi(x_0,z), \tag{103}$$

and

$$\begin{aligned} T_2 = \frac{i\Delta^3}{48k_0}\Bigg\{&\left[\frac{1}{4}\frac{\partial^4 V}{\partial z^4} + ik_0\frac{\partial^3 V}{\partial z^2\partial x} - k_0^2\left(\frac{\partial\bar{V}}{\partial z}\right)^2\right]\psi(x_0,z) \\ &+ \left[i2k_0\frac{\partial^2\bar{V}}{\partial z\partial x} - \frac{\partial^3\bar{V}}{\partial z^3}\right]\frac{\partial\psi(x_0,z)}{\partial z} + \frac{\partial^2\bar{V}}{\partial z^2}\frac{\partial^2\psi(x_0,z)}{\partial z^2}\Bigg\}, \end{aligned} \tag{104}$$

where $\bar{V} = V(\bar{x}, z)$. Equations (103) and (104) show that the local truncation error for the symmetrized operator splitting is linear in frequency (via the k_0 terms) and cubic in the range step-size, Δ.

To simplify the analysis, let us assume that the potential energy function V is independent of the range x. (In most applications, the horizontal x-derivatives of V are much smaller than corresponding vertical z-derivatives.) In this case, the third-order local absolute error in the PE field, $\delta^{(3)}\psi$, caused by advancing with the range step Δ is given by

$$\delta^{(3)}\psi(x_0+\Delta,z) = \frac{i\Delta^3}{48k_0}\left\{\left[\frac{1}{4}\frac{\partial^4\bar{V}}{\partial z^4} - k_0^2\left(\frac{\partial\bar{V}}{\partial z}\right)^2\right]\psi(x_0,z) - \frac{\partial^3\bar{V}}{\partial z^3}\frac{\partial\psi(x_0,z)}{\partial z} + \frac{\partial^2\bar{V}}{\partial z^2}\frac{\partial^2\psi(x_0,z)}{\partial z^2}\right\}. \quad (105)$$

Instead of working with the pointwise error estimate, $\delta^{(3)}\psi$ defined above, it proves more useful to consider the total relative error per range step $E^{(3)}$, which is normalized by the vertical energy density of the PE field:

$$E^{(3)} = \frac{\|\delta^{(3)}\psi\|}{\|\psi\|}, \equiv \frac{\int \psi^*\delta^{(3)}\psi\,dz}{\int|\psi|^2\,dz}. \quad (106)$$

The term $\|\delta^{(3)}\psi\|$ may be evaluated by integrating by parts and takes the form

$$\begin{aligned}\|\delta^{(3)}\psi\| &= \frac{i\Delta^3}{48k_0}\int\psi^*\left[\frac{1}{4}\frac{\partial^4\bar{V}}{\partial z^4}\psi - k_0^2\left(\frac{\partial\bar{V}}{\partial z}\right)^2\psi - \frac{\partial^3\bar{V}}{\partial z^3}\frac{\partial\psi}{\partial z} + \frac{\partial^2\bar{V}}{\partial z^2}\frac{\partial^2\psi}{\partial z^2}\right]dz, \\ &= \frac{i\Delta^3}{48k_0}\left\{\int\left[k_0^2\left(\frac{\partial\bar{V}}{\partial z}\right)^2|\psi|^2 + \frac{\partial^2\bar{V}}{\partial z^2}\left|\frac{\partial\psi}{\partial z}\right|^2 - \tfrac{1}{4}\frac{\partial^2\bar{V}}{\partial z^2}\frac{\partial^2|\psi|^2}{\partial z^2}\right]dz \right. \\ &\quad \left. + \tfrac{1}{4}\left.\frac{\partial^2\bar{V}}{\partial z^2}\right|_{z=0}\frac{\partial|\psi(x_0,0)|^2}{\partial z}\right\}. \end{aligned} \quad (107)$$

To ensure that the total error in the computed PE solution is bounded, the PE step-size Δ is chosen so that the third-order error term $E^{(3)}$ is small relative to the first-order term $E^{(1)}$:

$$\left|\frac{E^{(3)}}{E^{(1)}}\right| \le \epsilon_{rel} \ll 1,$$

where ϵ_{rel} is a specified error tolerance, and

$$E^{(1)} = \frac{\|\delta^{(1)}\psi\|}{\|\psi\|}, \qquad \|\delta^{(1)}\psi\| \equiv \frac{\Delta}{2k_0}\int\psi^*\left[\frac{\partial^2}{\partial z^2} - k_0^2\bar{V}\right]\psi\,dz. \quad (108)$$

This implies that the PE range step-size should be chosen so that

$$\Delta^2 \leq 24\epsilon_{rel}\frac{\|\delta^{(1)}\psi\|}{\|\delta^{(3)}\psi\|},$$

and determines the PE range step, Δ, by

$$\Delta^2 = \frac{24\epsilon_{rel}\int \psi^*\left[\frac{\partial^2}{\partial z^2} - k_0^2\bar{V}\right]\psi\, dz}{\int\left[k_0^2\left(\frac{\partial\bar{V}}{\partial z}\right)^2|\psi|^2 + \frac{\partial^2\bar{V}}{\partial z^2}\left|\frac{\partial\psi}{\partial z}\right|^2 - \frac{1}{4}\frac{\partial^2\bar{V}}{\partial z^2}\frac{\partial^2|\psi|^2}{\partial z^2}\right]dz + \frac{1}{4}\frac{\partial^2\bar{V}}{\partial z^2}\frac{\partial|\psi|^2}{\partial z}\Big|_{z=0}}. \quad (109)$$

In numerical implementation, the vertical derivatives appearing in Eq. (109) are evaluated by numerical finite-difference approximations. As the PE code advances the field, the local error budget is monitored and the range step-size is dynamically adjusted to keep the local error below a preset threshold. The question of a more detailed error analysis will not be covered in this report. The interested reader is referred to Ryan[8] for analysis of the split-step PE truncation errors.

3.4 Split-Step Fourier PE Algorithm

The PE solution, $\psi(x+\Delta, z)$, at range $x+\Delta$ is obtained from the known field, $\psi(x, z)$, at range x by means of the split-step PE algorithm, Eq. (98). The presence of the differential operator, $A(z)$, in the exponent can be dealt with by transforming to a basis in which $A(z)$ is diagonal. One such basis is the Fourier basis, with the z-space Fourier transform $\mathcal{F}$ being defined as

$$\Psi(x,p) = \mathcal{F}[\psi(x,z)] \equiv \int_{-\infty}^{\infty}\psi(x,z)e^{-ipz}\,dz,$$

and the inverse transform $\mathcal{F}^{-1}$ being defined by

$$\psi(x,z) = \mathcal{F}^{-1}[\Psi(x,p)] \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(x,p)e^{+ipz}\,dp.$$

The conjugate transform variable p can be associated with the vertical wave number, via $p = k_0\sin\theta$, where θ is the propagation angle with respect to the horizontal.

As is well known, the vertical derivative operator in z-space is related to the conjugate p-space operator by

$$\left(\frac{\partial}{\partial z}\right)^n \Rightarrow (-ip)^n,$$

in which case the SPE operator Eq. (92) is implemented via

$$\begin{aligned} e^{i\Delta A(z)/2}\psi(x,z) &= \mathcal{F}^{-1}\left[e^{-i\Delta p^2/4k_0}\Psi(x,p)\right], \\ &= \mathcal{F}^{-1}\left\{e^{-i\Delta p^2/4k_0}\mathcal{F}\left[\psi(x,z)\right]\right\}. \end{aligned} \quad (110)$$

Similarly, the WAPE operator Eq. (94) is implemented via

$$\begin{aligned} e^{i\frac{\Delta}{2}A(z)}\psi(x,z) &= \mathcal{F}^{-1}\left[e^{-i\frac{\Delta}{2}(k_0-\sqrt{k_0^2-p^2})}\Psi(x,p)\right], \\ &= \mathcal{F}^{-1}\left[e^{-i\frac{\Delta}{2}(k_0-\sqrt{k_0^2-p^2})}\mathcal{F}\left[\psi(x,z)\right]\right]. \end{aligned} \tag{111}$$

Now any function may be expressed in terms of even and odd components, so the PE field can also be written as

$$\psi(x,z) = \psi_e(x,z) + \psi_o(x,z),$$

where

$$\begin{aligned} \psi_e(x,z) &\equiv \tfrac{1}{2}[\psi(x,z) + \psi(x,-z)], \\ \psi_o(x,z) &\equiv \tfrac{1}{2}[\psi(x,z) - \psi(x,-z)]. \end{aligned}$$

Being a linear operator, the Fourier transform may also be expressed in terms of even and odd components as

$$\Psi(x,p) = \mathcal{F}[\psi(x,z)] = \Psi_e(x,p) + \Psi_o(x,p),$$

where Ψ_e and Ψ_o are represented in terms of Fourier sine and cosine transforms as

$$\Psi_e(x,p) = \mathcal{F}_c[\psi_e(x,z)] \equiv 2\int_0^\infty \psi_e(x,z)\cos pz\, dz,$$

and

$$\Psi_o(x,p) = \mathcal{F}_s[\psi_o(x,z)] \equiv -i2\int_0^\infty \psi_o(x,z)\sin pz\, dz.$$

The corresponding inverse sine and cosine transforms are defined by

$$\psi_e(x,z) = \mathcal{F}_c^{-1}[\Psi_e(x,p)] \equiv \frac{1}{\pi}\int_0^\infty \Psi_e(x,p)\cos pz\, dp,$$

and

$$\psi_o(x,z) = i\mathcal{F}_s^{-1}[\Psi_o(x,p)] \equiv \frac{i}{\pi}\int_0^\infty \Psi_o(x,p)\sin pz\, dp.$$

In practice, the infinite Fourier transforms are replaced by finite discrete Fourier transforms (DFTs) over the domain $(0, Z_{max})$, which in turn are evaluated numerically using the fast Fourier transform algorithm.[35] The actual FFT algorithms implement fast real-valued sine and cosine transforms to reduce core storage and improve computational speed.[35,36]

When using DFTs, it is important to satisfy the Nyquist sampling theorem to avoid transform aliasing problems.[37] If P_{max} is the maximum vertical wave number and N is the discrete cosine/sine transform size, then the Nyquist condition is just

$$Z_{max}P_{max} = \pi N\,. \tag{112}$$

3.5 Reflectionless Absorber Boundary

The appropriate boundary condition to be satisfied as $z \to \infty$ by the PE field $\psi(x,z)$ is the Sommerfeld outgoing wave radiation condition, Eq. (35). Since the split-step Fourier algorithm employs finite Fourier transforms, the implementation of a radiation-type boundary condition is quite complicated. This follows from the fact that truncation of the infinite z-domain down to a finite interval in the Fourier transform leads to the introduction of spurious discrete standing wave solutions in the vertical. In effect, the terminal impedance at the end of the transform grid is not properly "matched" to the radiation boundary condition.

To circumvent this problem and attenuate the spurious standing wave solutions introduced by the finite Fourier transforms, a complex absorber potential $V_{abs}(z)$ is added to the split-step B operator:

$$B(x,z) \Rightarrow B(x,z) + \frac{ik_0}{2} V_{abs}(z).$$

The particular form of the complex absorber or "sponge" is found by recourse to a nuclear optical model[38] analog based upon the theory of reflectionless potentials.[39] The specific form chosen is

$$V_{abs}(z) = V_0 \operatorname{sech}^2[\alpha(z - Z_{max})] \tag{113}$$

where the parameters $\{V_0, \alpha\}$ are determined parametrically by minimizing transmission and reflection coefficients from the sponge region.[40]

3.6 PE Starting Fields

The split-step PE method must be initialized with a starting field distribution $\psi(x_0, z)$ at some distance x_0 from the source, since the parabolic wave equation is not valid in the region of the source. Two options are available to compute the starting fields. First, the initial z-space field may be obtained by analytic methods, assuming free-space conditions and treating the source as a point radiator. This option, though, is not useful for highly directional antennas.

The second approach is to use the duality of the antenna aperture field distribution and antenna radiation pattern function.[41] In free space, the antenna radiation pattern function $f(p)$ and the antenna aperture field distribution $A(z)$ are a Fourier transform pair:

$$f(p - p_0) = \int_{-\infty}^{\infty} A(z) e^{-i(p-p_0)z}\, dz\,, \tag{114}$$

where

$$p = k_0 \sin\theta, \tag{115}$$

$$p_0 = k_0 \sin\theta_0. \tag{116}$$

In Eq. (115), θ is the elevation angle measured with respect to the horizontal ($\theta > 0$ is up), and in Eq. (116), θ_0 is the antenna mainlobe vertical pointing direction. The antenna pattern function is used to construct even and odd symmetry p-space fields in the form

$$\Psi_e(0,p) = f(p)e^{-ipz_0} + f(-p)e^{+ipz_0}, \tag{117}$$

and

$$\Psi_o(0,p) = f(p)e^{-ipz_0} - f(-p)e^{+ipz_0}, \tag{118}$$

where the Fourier shift theorem has been used to properly account for a nonzero source altitude, z_0. Given the above even/odd symmetry p-space fields, the corresponding z-space PE field is obtained by taking the inverse cosine or sine transform of Eq. (117) or Eq. (118), respectively:

$$\psi_e(0,z) = \mathcal{F}_c^{-1}[\Psi_e(0,p)],$$

and

$$\psi_o(0,z) = \mathcal{F}_s^{-1}[\Psi_o(0,p)].$$

3.7 Antenna Patterns

The PE method is capable of modeling the radiation emitted by directional antennas, provided that the complex antenna radiation pattern, Eq. (114), is known. Often this information is not available for specific radar systems, so simplified generic antenna patterns are used. These generic patterns display some of the features of real antennas while retaining fairly simple analytical forms. Each of the analytic antenna patterns is specified in terms of a normalized p-space steering parameter, t, defined by

$$t = \frac{\sin\theta - \sin\theta_0}{\sin(\frac{1}{2}\theta_{b\,p})} = \frac{p - p_0}{p_{\frac{1}{2}}},$$

where

$$p_{\frac{1}{2}} = k_0 \sin(\frac{1}{2}\theta_{bw}),$$

and θ_{bw} is the half-power (i.e., 3-dB down) beamwidth. The following analytic antenna radiation patterns are useful:

3.7.1 sin(x)/x

The uniformly illuminated aperture corresponds to a radiation pattern having the functional form

$$f(t) = \frac{\sin(at)}{at}, \quad a = 1.3916. \tag{119}$$

The scale parameter a is determined by solving the nonlinear equation

$$\frac{\sin x}{x} = \frac{1}{\sqrt{2}}.$$

This pattern has the narrowest mainlobe width, at the expense of high sidelobe levels.

3.7.2 Gaussian

The Gaussian antenna radiation pattern has the functional form

$$f(t) = e^{-a^2 t^2}, \quad \text{where} \quad a^2 = \frac{\ln 2}{2}. \tag{120}$$

This pattern is the optimal compromise between sidelobe level and mainlobe width

3.7.3 Compound

The compound radiation pattern is a one-parameter pattern that is a blend of the uniformly illuminated aperture and the cosine-squared aperture distribution, having the functional form

$$f(t) = \frac{2}{1+c}\frac{\sin(at)}{at}\left[c + \frac{1-c}{2}\frac{1}{1-(at/\pi)^2}\right], \quad 0 \le c \le 1.$$

The parameter a controls the mainlobe width and is determined by solving the nonlinear equation

$$\frac{\sin a}{a}\left[2c + \frac{1-c}{1-(a/\pi)^2}\right] = \frac{1+c}{\sqrt{2}}.$$

The uniform aperture corresponds to $c = 1$, while the cosine-squared aperture corresponds to $c = 0$.

3.7.4 Hansen

Another single parameter pattern is obtained from the circular aperture distributions analyzed by Hansen.[42] This pattern has the functional form

$$f(t) = \frac{H}{i_1(H)}\begin{cases} \frac{i_1(\sqrt{x})}{\sqrt{x}}, & \text{for } x \ge 0 \\ \frac{j_1(\sqrt{|x|})}{\sqrt{|x|}}, & \text{for } x < 0, \end{cases} \tag{121}$$

where

$$x = H^2 - (at)^2,$$

and j_1 and i_1 are the first-order spherical and modified Bessel functions defined by

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z},$$
$$i_1(z) = -\frac{\sinh z}{z^2} + \frac{\cosh z}{z}.$$

The quantity a, which determines the 3-dB point in the pattern, is found by solving the transcendental equation

$$C \equiv \frac{i_1(H)}{\sqrt{2}H} = \begin{cases} \frac{i_1(\sqrt{H^2-a^2})}{\sqrt{H^2-a^2}}, & \text{for } C \ge \frac{1}{3} \\ \frac{j_1(\sqrt{a^2-H^2})}{\sqrt{a^2-H^2}}, & \text{for } C < \frac{1}{3}. \end{cases}$$

The first sidelobe level, SLL, in the radiation pattern can be shown to have a value of

$$SLL = 30.84 + 20 \log \left[\frac{3i_1(H)}{H} \right] \quad \text{dB},$$

down from the peak, and is located at

$$a\frac{p - p_0}{p_{\frac{1}{2}}} = \sqrt{1 + \left(\frac{5.763}{H} \right)^2}.$$

The parameter H allows a trade-off between low sidelobe level and mainlobe beamwidth.

§4.0 CONCLUSION

This report describes the basic physics and analytical techniques used to model microwave propagation in range-dependent environments. These techniques have been implemented in a range-dependent propagation computer model — the VTRPE code. The VTRPE computer model is used to predict and analyze the performance of radar and communication systems operating in spatially varying environments. The methods described herein account for spatial variations in the atmospheric refractivity as well as for variations in surface dielectric properties and topography.

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This report describes the fundamental physics and basic numerical algorithms that form the basis of a range-dependent microwave propagation model. The model accounts for the effects of nonhomogeneous atmospheric refractivity fields, finite surface conductivity and variable surface terrain elevation. Both vertical and horizontal field polarization and general antenna radiation patterns are included in the analysis. The electromagnetic field propagation is modeled using the split-step Fourier parabolic wave equation (SSFPE) algorithm, modified to account for an impedance-type boundary condition. The various approximations used in developing the SSFPE algorithm are discussed, along with local and global error estimates.

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