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WELL-COVERED GRAPHS: A SURVEY

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ABSTRACT

A graph G is well-covered (or w-c) if every maximal independent set of points in G is also maximum. Clearly, this is equivalent to the property that the greedy algorithm for constructing a maximal independent set always results in a maximum independent set. Although the problem of independence number is well-known to be NP-complete, it is trivially polynomial for well-covered graphs.

The concept of well-coveredness was introduced by the author in [Pl] and was first discussed therein with respect to its relationship to a number of other properties involving the independence number.

Since then, a number of results about well-covered graphs have been obtained. It is our purpose in this paper to survey these results for the first time. As the reader will see, many of the results we will discuss are quite recent and have not as yet appeared in print.

1. Background, Terminology and an Updated Lattice of Implications

In this paper, all graphs will be assumed to be connected. A set of points is independent if no two of its members are joined by a line. The size of a largest (i.e., maximum) independent set will be denoted by $\alpha(G)$ and this number will be called the independence number of G . A point cover for a graph G is a set $S \subseteq V(G)$ such that every line of G has at least one endpoint in S . Let us denote the size of a smallest point cover in a graph G by $\tau(G)$. It is clear that the complement of a point cover is an independent set (and vice-versa) and that in fact, the complement of a minimum point cover is a maximum independent set (and vice-versa). Thus we always have that $\alpha(G) + \tau(G) = |V(G)|$. We thus have the option of adopting the point of view of independent sets or of point covers. Historically, well-covered graphs (the subject of this paper) were defined in terms of point covers (hence the name), but more recently, most researchers in the area have converted to the independent set point of view. We too will follow this more recent trend.

In 1972, Karp [K] showed that the independence number problem for graphs in general is NP-complete. Thus it is considered most unlikely that a polynomial algorithm will ever be found to compute $\alpha(G)$.

It is natural, therefore, to look for interesting families of graphs for which this parameter is polynomially computable. A number of such families have been found. For example, it has long been known that bipartite graphs have polynomially computable independence

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* work supported by ONR Contract #N00014-91-J-1142

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number. The reason is that in a bipartite graph G , we always have $\nu(G) = \tau(G)$ by the well-known minimax theorem of König (Cf. Theorem 1.1.1 of [LP].) Here $\nu(G)$ denotes the size of a largest matching in G . But matching is well-known to be polynomial in all graphs by a famous result of Edmonds (cf. Theorem 9.1.8 of [LP].) Hence it immediately follows that the independence number is polynomially computable as well.

Graphs in which $\nu(G) = \tau(G)$ are said to have the König Property. We have just observed that every bipartite graph has the König Property. However, so do a number of non-bipartite graphs. It has been shown independently by Deming [De], Sterboul [Ste] and Lovász [Lo] that graphs with the König Property are polynomially recognizable.

For a more comprehensive list of other graph families with polynomially computable independence number, we refer the reader to [GJ] and [LP].

Well-covered graphs are, in a sense, the most trivial family of graphs having polynomial independence number. Consider the so-called *greedy* algorithm for constructing a maximal independent set in a graph. Namely, begin by choosing any point v_1 in graph G , put it in a set I and delete it and its neighbors from G to obtain a graph G_{v_1} . Now select any point v_2 in G_{v_1} , put it into set I also, delete it and its neighbors from G_{v_1} to obtain a yet smaller graph $(G_{v_1})_{v_2}$. Continue this procedure until all points have either been chosen for membership in I or deleted. Clearly, the set I which we have obtained in polynomial time is a maximal independent set in G . However, it may be far from a maximum independent set! But let us call a graph G well-covered if no matter which point in G we start with, the above greedy procedure always results in an independent set which is maximum. Note that this property is clearly equivalent to demanding that every maximal independent set in G is also maximum.

Intuitively speaking, the property of being well-covered would appear to be quite strong and hence perhaps it is not unreasonable to hope that a good characterization and/or a polynomial recognition algorithm can be found for this class of graphs. Alas, as we shall see in Section 2, it is unlikely that this will ever be the case.

We now proceed to several other definitions with which we shall deal in this paper.

A line in a graph G is said to be α -critical (or simply critical) if $\alpha(G - e) > \alpha(G)$. If every line of G is critical, G is called α -critical. A point v of graph G is called essential if v lies in every minimum point cover of G . Denote the set of essential points of G by $V_e(G)$. Note that if $V_e(G) = \emptyset$ then each point of G lies in some maximum independent set, and vice-versa. Graphs exhibiting this property have been christened B -graphs by Parthasarathy and Ravindra [PR]. Clearly, every well-covered graph is a B -graph, as is every α -critical graph. (It is an easy exercise for the reader to show that the converse assertions are false in both cases.) A point $v \in G$ is said to be τ -critical if $\tau(G - v) < \tau(G)$. If every point $v \in G$ is τ -critical, we say that G is τ -point-critical. This notion seems to have first been studied by Erdős and Gallai in their 1961 paper [EG]. It is easily seen that a graph G is τ -point-critical if and only if for all points $v \in G$, there is a maximum independent set which does not contain v . Again it is immediate that every B -graph is τ -point-critical, but again the interested reader can easily show that the converse implication fails.

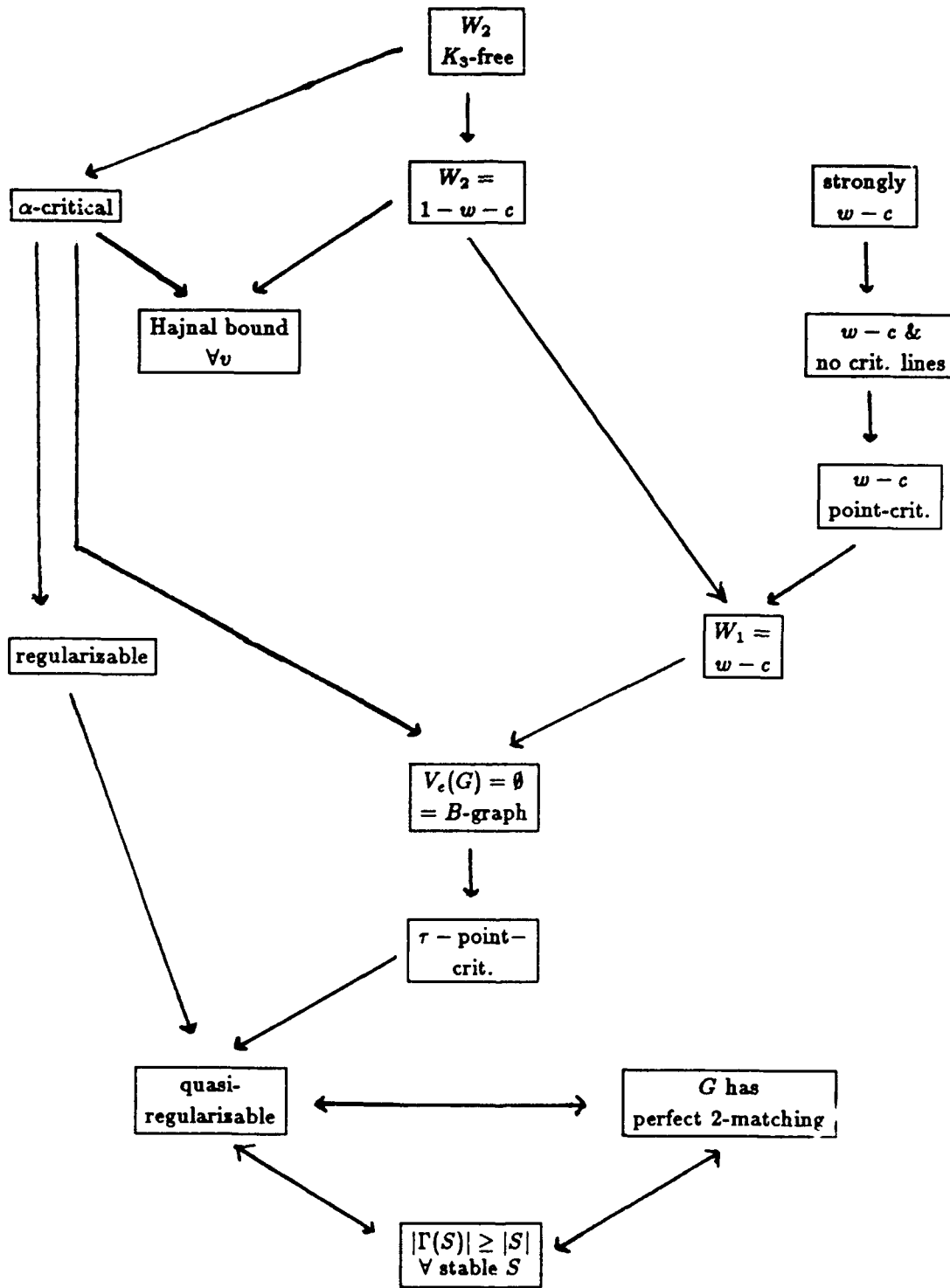
A 2-matching in graph G is an assignment of integer weights 0, 1 or 2 to the lines of G such that the sum of the weights of the lines incident with each point is at most

us say that a graph G is in class A_v if it is well-covered and for all points $v \in V(G)$, the graph $G_v = G - (\{v\} \cup \Gamma(v))$ has no isolates. Campbell showed that the class A_v properly contains the class W_2 and that the graphs in the larger class A_v also satisfy the Hajnal degree bound.

It is interesting to note at this point that we have established that the properties of being α -critical and belonging to class W_2 separately imply a number of weaker properties concerning independence number. However, it is easy to see that in general, the α -critical and W_2 properties are independent. On the other hand, Staples obtained the only non-trivial condition known to date which connects the two properties when she proved that if G is a *triangle-free* member of class W_2 , then G is also α -critical.

Next, let us define a well-covered graph G to be **strongly well-covered** if $G - e$ is also well-covered for all lines $e \in E(G)$. Pinter [Pi] has shown that *no* line in a strongly well-covered graph is α -critical. Earlier, Staples [Stal] showed that if a well-covered graph has no α -critical lines, then it has the property that $G - v$ is *not* well-covered, for all points $v \in V(G)$. (Graphs with this latter property are called well-covered point-critical graphs.)

We close this section with the diagram on the following page which summarizes most of the implications we have discussed in this section.



A Diagram of Implications

2. Complexity Issues for Well-Covered Graphs

Can we recognize a well-covered graph in polynomial time? Although this basic algorithmic question remains unanswered, it seems likely that the answer will turn out to be "no". But more about that below.

First let us point out that the property of being *not* well-covered is easily seen to be in NP. In order to see this, one need only exhibit two maximal independent sets of the graph G under consideration which have different cardinalities.

What is not so apparent is that the property of being not well-covered is NP-hard. Proofs of this result have recently reached the writer from three different sources. Chvátal and Slater independently proved it and have now co-authored a joint paper [CS] containing their proof. Independently, Sankaranarayana and Stewart [SS] have produced a virtually identical proof. The only difference is that Chvátal and Slater use reduction from 3SAT while Sankaranarayana and Stewart reduce from SAT. We will outline the proof as given by Sankaranarayana and Stewart.

Theorem 2.1. ([CS;SS]) The property of being well-covered is co-NP-complete.

Outline of Proof. Let ISAT be any instance of SAT. That is to say, ISAT is a conjunction of m clauses c_1, \dots, c_m where each clause is itself a disjunction of a subset of boolean variables u_1, \dots, u_n (and their complements). Given ISAT, we now build a graph G with point set $V(G) = \{c_1, \dots, c_m\} \cup \{u_1, \bar{u}_1, \dots, u_n, \bar{u}_n\}$. The lines of graph G are defined as follows. Every pair of distinct clause points are joined. Each literal u_i is joined to its negation \bar{u}_i . Finally, each clause point c_i is joined to literal u_j (resp. \bar{u}_j) if u_j (resp. \bar{u}_j) is contained in clause c_i . (Note that it is assumed here that no clause contains both a literal and its negation, for such a clause is always satisfiable and can hence be eliminated.)

It is now straightforward to prove that ISAT is satisfiable if and only if graph G is not well-covered.

First suppose that ISAT is satisfiable. Suppose an assignment of values to the literals is made which makes ISAT true. Then the set of those literals which are true in this assignment clearly form an independent set of size n which is maximal. Moreover, this set is not maximum, for one can easily find an independent set in G of size $n+1$; namely, choose one point corresponding to a clause and then add n additional literal points obtained by choosing from each literal pair $\{u_j, \bar{u}_j\}$ that one literal which is *not* in the chosen clause.

Thus G is not well-covered.

Conversely, suppose G is not well-covered. It is easy to see that all maximal independent sets in G of size $n+1$ must contain a clause point. Now construct an independent set S_1 of size n by choosing exactly one of the two literals u_i, \bar{u}_i in such a way that S_1 is maximal. But we have maximality only if each clause point c_i is adjacent to (at least) one of the literals chosen for set S_1 . Thus if one assigns the value *true* to each literal in S_1 each clause will be satisfied and hence so will ISAT.

Thus any polynomial recognition algorithm for non-well-covered graphs could be used to find an assignment of truth values which makes ISAT true. It thus follows that the property of not being well-covered is at least as hard as SAT and hence is NP-hard. ■

Now let us return to the question of whether the property of being well-covered is also in NP. It is NP-hard, since its negation is NP-hard. Thus if it were in NP, it would follow that $NP = co-NP$! Although this could conceivably be true without having $P = NP$, it too is considered highly unlikely by the vast majority of complexity theorists.

In a different direction, Sankaranarayana and Stewart [SS] have investigated a number of other known NP-complete properties, but limited to the class of well-covered graphs. To mention only several of their results, they have shown that the Hamilton cycle problem, the dominating cycle problem, the chromatic number problem, the max cut problem and the dominating set problem all remain NP-complete even when restricted to the class of well-covered graphs. Moreover, the graph isomorphism problem for well-covered graphs is equivalent to the general graph isomorphism problem—a problem the complexity of which remains unknown.

Let us close this section with a remark about the matching-analogue of the well-covered property. A graph G is said to be **equi-matchable** if every maximal matching is maximum. The problem of characterizing this family of graphs was first posed by Grünbaum in 1974 [G]. Lesk, Plummer and Pulleyblank [LPP] solved this problem in 1983 when they obtained a characterization of such graphs which leads to a polynomial recognition algorithm.

3. Well-Covered Bipartite Graphs and Very Well-Covered Graphs

We now begin our survey of the well-covered property with respect to various special classes of graphs. Let us start with bipartite graphs.

If G is a bipartite graph and $e = uv$ is a line in G , let G_e denote the subgraph of G induced by $\Gamma(u) \cup \Gamma(v)$. The following was proved by Ravindra in 1977 [Ra1].

Theorem 3.1. Let G be a connected bipartite graph. Then G is well-covered if and only if G contains a perfect matching F such that for every line $e = uv$ in F , G_e is a complete bipartite graph.

A characterization of well-covered trees follows as an easy corollary of this result.

Corollary 3.2. A tree T is well-covered if and only if it is K_1 or its pendant lines form a perfect matching.

Notice that if G is any well-covered graph, then $\alpha(G) \leq |V(G)|/2$. This is easy to see, for if G contained a maximum independent set of size greater than $|V(G)|/2$, then no point in the complement of the maximum independent set could itself lie in a maximum independent set. The family of well-covered graphs in which equality is attained in the above inequality, namely, those for which $\alpha(G) = |V(G)|/2$, were characterized by Nelson and Staples [Sta1] in 1975 and independently by Favaron in 1982 [Fa].

Let us define a graph G to be **very well-covered** if it is well-covered and has $\alpha(G) = |V(G)|/2$. For the sake of conciseness, let us use the following terminology. Suppose a graph G has a perfect matching F . Then the matching $F = \{a_1b_1, \dots, a_nb_n\}$ has **property P** if (a) no point $w \in G$ satisfies $w \sim a_i$ and $w \sim b_i$ where $a_ib_i \in F$ and (b) no set of two independent points $\{u, v\} \subseteq V(G)$ satisfies $u \sim a_i$ and $v \sim b_i$ where $a_ib_i \in F$.

Parts (i) and (ii) of the following theorem are proved in [Sta1] while the equivalence of all three statements is proved in [Fa].

Theorem 3.3. For any graph G the following are equivalent:

- (i) G is very well-covered,
- (ii) there exists a perfect matching in G which satisfies property P,
- (iii) G contains a perfect matching and every perfect matching in G satisfies property P.

Of course, if G is well-covered and, in addition, *bipartite* with point bipartition $A \cup B$ it follows immediately that both A and B are maximal independent sets. (Remember: we are not allowing our graphs to have isolated points.) Thus $|A| = |B|$ and $\alpha(G) = |A| = |V(G)|/2$. But by the well-known minimax theorem of König for bipartite graphs, graph G must then have a perfect matching. In fact, it is easy to characterize bipartite well-covered graphs and, in particular, trees, as corollaries of the preceding theorem. These corollaries were first proved by Ravindra in 1977 [Ra1]. To the best of the author's knowledge, Staples and Ravindra were unaware of each other's work at the time.

From part (iii) of Theorem 3.3 it follows that since matching is polynomial, there is thus a polynomial algorithm for recognizing very well-covered graphs and hence also well-covered bipartite graphs.

We now turn to the two areas of well-covered graph research which seem to have produced the bulk of results to date.

4. The Well-Covered Property versus Girth

The *girth* of a graph G is the length of any shortest cycle in G . Substantial contributions to the theory of well-covered graphs dealing with their girth are due to Finbow, Hartnell and Nowakowski. In particular, Finbow and Hartnell [FiH] introduced the idea of studying well-covered graphs from the point of view of girth in the context of analysis of a certain two-person game. In particular, they obtained the following result. Note that a *pendant line* in a graph is any line incident with a point of degree 1.

Theorem 4.1. Let G be a graph with girth at least 8. Then G is well-covered if and only if its pendant lines form a perfect matching in G .

In cooperation with Nowakowski, in 1987 these same two authors [FHN2] generalized the above result to obtain a characterization of all well-covered graphs having girth at least 5. (This paper has not yet appeared at the time of this writing.) In order to state their result, we must first introduce some new terminology.

A 5-cycle C_5 of a graph G is called *basic* if cycle C_5 does not contain two adjacent points of degree 3 or more. In a given graph G let $C(G)$ denote the set of all points which belong to basic 5-cycles. Denote by $P(G)$ the set of points in G which are incident with pendant lines in G . Then G is said to belong to the class **PC** if $V(G)$ can be expressed as $P(G) \cup C(G)$, where $P(G) \cap C(G) = \emptyset$ and the subgraph of G induced by $P(G)$ is a matching.

The main result of [FHN2] can now be stated.

Theorem 4.2. Let G be a graph with girth at least 5. Then G is well-covered if and only if G belongs to the class PC or is one of the six exceptions shown in Figure 1.

Outline of Proof. It is straightforward to show that all graphs in class PC are well-covered. In fact, every maximal independent set in a graph G in class PC has size $|P(G)|/2 + 2|C(G)|/5$. That the six exceptional graphs are well-covered can be easily checked.

The hard work lies in showing that the converse holds; namely, that the PC graphs and the six exceptions are the *only* well-covered graphs of girth at least 5.

The key to the rest of the proof lies in the concept of an *extendable* point. A point v in a well-covered graph G is *extendable* if $G - v$ is also well-covered and $\alpha(G - v) = \alpha(G)$.

Assume, now, that G is a well-covered graph with girth at least 5. It is shown next that this is equivalent to the following condition: if u and v are adjacent points with $\deg(u) \geq 3$, then if v is extendable, so is u . Next it is shown that if v is an extendable point in such a graph, then either v is incident with exactly one pendant line or else v lies on a basic 5-cycle C and the neighbors of v on C are of degree 2 in G . The authors then prove that if v is an extendable point and lies on a basic 5-cycle then all neighbors of v not on this basic 5-cycle are also extendable. Moreover, if v is incident with a pendant line then any neighbor of v which is not of degree 1 is also extendable. Finally, these results are used to show that no two basic 5-cycles can have a point in common.

The authors now use these preliminary results to show that a well-covered graph of girth at least 5 belongs to class PC if and only if G contains at least one extendable point.

The most difficult—and most lengthy—part of the characterization is now to prove that if G is well-covered and has girth at least 5 and has *no* extendable point, then it must be one of the exceptional graphs shown in Figure 1. The proof proceeds by considering the maximum degree. It is easy to see that K_1 (an exceptional graph) and K_2 (a member of class PC) are the unique well-covered girth at least 5 graphs with $\maxdeg(G) \leq 1$. The authors then proceed to show that C_7 is the only exceptional graph with $\maxdeg(G) = 2$, that P_{10} , P_{13} and P_{14} are the only exceptions having $\maxdeg(G) = 3$, that Q_{13} is the sole exception with $\maxdeg(G) = 4$ and finally, that there are *no* exceptional graphs having maximum degree greater than 4.

■

—PLACE FIGURE 1 ABOUT HERE—

By inspecting the exceptional graphs, the following corollary is immediate.

Corollary 4.3. Let $G \neq K_1, C_7$ have girth at least 6. Then G is well-covered if and only if its pendant lines form a perfect matching of G .

We observe that the characterization given in Theorem 4.2 immediately gives an obvious polynomial algorithm for recognizing well-covered graphs with girth at least 5.

So what about well-covered graphs with girth 3 or 4? It seems safe to say that those of us who have worked in the field of well-covered graphs have long felt that triangles are the real heart of the difficulty. Finbow, Hartnell and Nowakowski, however, in a second paper in 1989 [FHN3] characterized well-covered graphs in which triangles are allowed, but which have no 4-cycles nor 5-cycles. (Note that in particular, in such graphs no cliques of size greater than 3 can exist.)

In order to state this characterization, the concept of a *simplicial* point must be introduced. Let $\Gamma[v]$ denote the "closed" neighborhood of point v ; that is, let $\Gamma[v] = \Gamma(v) \cup \{v\}$. A point $v \in G$ is said to be *simplicial* if $\Gamma[v]$ is a complete graph. Now, following [FHN3], let us define a family \mathcal{F} to be that class of graphs G in which there exists a point set $\{v_1, v_2, \dots, v_k\} \subseteq V(G)$ where for each $i = 1, \dots, k$, point v_i is simplicial, $|\Gamma[v_i]| \leq 3$ and $\Gamma[v_1] \cup \Gamma[v_2] \cup \dots \cup \Gamma[v_k]$ is a partition of $V(G)$.

The characterization can now be stated as follows.

Theorem 4.4. G is a well-covered graph containing no 4-cycles nor 5-cycles if and only if $G \in \mathcal{F}$ or G is one of the two exceptional graphs shown in Figure 2.

It is easy to see that in any well-covered graph if u and v are two simplicial points, then either $\Gamma[u] \cap \Gamma[v] = \emptyset$ or $\Gamma[u] = \Gamma[v]$. Clearly, then, the graphs characterized in Theorem 4.4 can also be recognized in polynomial time.

—FIGURE 2 GOES ABOUT HERE—

5. The Well-Covered Property versus Degrees

The second of the two primary directions of research on well-covered graphs to date is the study of well-coveredness versus degree properties. One such result has already been mentioned in Section 2. We refer to the theorem which states that all graphs in W_2 satisfy the Hajnal degree bound.

Staples also realized that a 1972 result of Meyer [Me] has, as a corollary, a result relating well-covered graphs to their degree sequence. Let us state his theorem and then its implication for well-covered graphs as a corollary.

Theorem 5.1. Let G be a graph with point set $V(G) = \{v_1, \dots, v_n\}$ where $1 \leq \deg(v_1) \leq \deg(v_2) \leq \dots \leq \deg(v_n)$. If k is any integer, $2 \leq k \leq n$ such that $\deg(v_n) + \dots + \deg(v_{n-k+2}) \leq n-k$, then every independent set with fewer than k points is contained in an independent set with k points.

If we set $k = \alpha(G)$ in the above theorem we get the following.

Corollary 5.2. If the sum of the degrees of the $\alpha(G) - 1$ points of G of largest degree does not exceed $|V(G)| - \alpha(G)$, then G is well-covered.

Before proceeding further, we present a result which, although easily obtained, turns out to be extremely useful in the proofs of most of the theorems of this section.

Theorem 5.3. If G is a well-covered graph which is not complete, then for each $v \in V(G)$, the graph $G_v = G - \Gamma[v]$ is also well-covered.

Note that the graph G_v may be disconnected. However, it is trivial from the definition that a disconnected graph is well-covered if and only if each of its components are well-covered.

Campbell [Ca] and Campbell and Plummer [CP] took yet another approach when they began the study of well-covered *cubic* (i.e., 3-regular) graphs. The three graph "fragments" A , B and C shown in Figure 3 below will be important in this context. Our studies proceeded via increasing values of point-connectivity.

—PUT FIGURE 3 ABOUT HERE—

Theorem 5.4. Let G be a cubic well-covered graph with $\kappa(G) = 1$. Then G is obtained by joining a finite number of the three fragments A , B and C in a "path" containing at least one A .

Let us denote the class of well-covered graphs characterized in Theorem 5.4 as \mathcal{G}_1 . Clearly, the smallest member of class \mathcal{G}_1 has 18 points. It is also clear, but perhaps something of a surprise, that all the graphs in class \mathcal{G}_1 are planar.

In characterizing the cubic well-covered graphs with $\kappa = 2$, it is useful to subdivide this class into two subclasses. Denote by \mathcal{G}_{2I} the subclass of these graphs in which each cutset of size 2 is independent and let \mathcal{G}_{2A} denote the rest; i.e., those cubic well-covered graphs having $\kappa = 2$ and having at least one cutset of two adjacent points.

Theorem 5.5. Let G be a cubic well-covered graph having $\kappa(G) = 2$. Then if G belongs to class \mathcal{G}_{2I} , G arises by joining a sequence of fragments of type A and B in a cyclic configuration. Moreover, such a configuration must contain at least two type A fragments.

The planar members of \mathcal{G}_{2I} are precisely those cyclic configurations obtained using only fragments of type A .

On the other hand, if G belongs to class \mathcal{G}_{2A} , G arises by joining a sequence of fragments of type B or C in a path-like configuration of the form $CBB \cdots BC$, in which there may be no fragments at all of type B .

All members of class \mathcal{G}_{2A} are planar.

Now we deal with the remaining class of cubic well-covered graphs; namely those with connectivity 3. The sequence of results leading up to the (very recent) final resolution of this case is quite interesting, and we shall therefore describe them.

After a notable lack of success on attacking the family in its complete generality, it was decided to try to investigate those which are *planar*. The motivation to do so lay in

the realization that we might make some progress on what appeared to be an impossible case-by-case analysis through the use of what has come to be called *the Theory of Euler Contributions*.

This technique applicable to planar graphs was first introduced by Lebesgue [Leb] in 1940. It was then further developed by Ore [O] in his work on the 4-color problem, by Ore and Plummer [OP] and by Plummer and Toft [PT] to treat another type of graph coloring (called *cyclic coloration*).

Suppose connected graph G is embedded in the plane. Let v be any point in G . We define the **Euler contribution** of point v to be

$$\Phi(v) = 1 - \frac{\deg(v)}{2} + \sum \frac{1}{x_i}$$

where the sum is taken over all faces F_i incident with v and x_i denotes the number of lines in the boundary of face F_i . Using Euler's Formula, it is easy to see that $\sum \Phi(v) = 2$. But it then follows that for at least one point v in graph G , we must have $\Phi(v) > 0$. (We have christened such points **control points**.) Now suppose graph G is cubic. It then follows that for any control point v ,

$$\frac{-1}{2} + \sum \frac{1}{x_i} > 0.$$

This diophantine inequality has infinitely many solutions, but they can be classified into a number of finite families and dealt with a family at a time. For example, one family can be denoted by $(3, 3, x)$, where $x = 3, 4, \dots$. In other words, the three faces at the control point are two triangles and a third one of arbitrarily large size.

This, then, was the approach taken to characterize all cubic 3-connected planar well-covered graphs. Whereas there are infinitely many well-covered cubic graphs with connectivity 1 and 2, somewhat surprisingly, in the 3-connected planar case, there are only four!

Theorem 5.6. There are precisely four cubic 3-connected planar well-covered graphs. They are shown in Figure 4.

—FIGURE 4 GOES ABOUT HERE—

The proof of Theorem 5.6 required over 60 pages of computation and made repeated use of Theorem 5.3 applied to the many solution classes for control points referred to above.

But what about those cubic 3-connected well-covered graphs which are *not* planar? A bit of history is again in order. In 1989-91, M. Ellingham and G. Royle were both participants in the graph theory seminar at Vanderbilt University and during that time, several talks were given on well-covered graphs, including most of the results mentioned above. In view of the paucity of *planar* 3-connected cubic well-covered graphs as shown in

Theorem 5.6 above, Royle decided to conduct a computer search on all cubic graphs on a maximum of 18 points, seeking all those which were *not* planar.

So far in this paper, the cubic well-covered graphs mentioned are those in classes \mathcal{G}_1 , \mathcal{G}_{2I} , \mathcal{G}_{2A} , the one exceptional 14-point (non-planar) graph found by Finbow, Hartnell and Nowakowski and called by them P_{14} and the four planar graphs K_4 , R_3 , $C_5 \times K_2$ and E_{12} in Theorem 5.6.

In his computer search, Royle found *precisely two* additional graphs. They are $K_{3,3}$ and $K_{3,3}^*$ where the latter is the 8-point cubic graph obtained from $K_{3,3}$ by replacing one point with a triangle. (It should be remarked that both these graphs were known by Campbell to be well-covered and indeed appear in his thesis.)

This motivated Ellingham and Royle to attempt to prove that indeed *all* cubic well-covered graphs were accounted for in the above results. This they succeeded in doing and we close with the theorem which completely settles the problem of identifying all cubic well-covered graphs. These results have just been written up [CER1; CER2] and submitted while the present paper was being written.

Theorem 5.7. The cubic well-covered graphs are precisely those graphs belonging to graph families \mathcal{G}_1 , \mathcal{G}_{2I} , \mathcal{G}_{2A} , together with the exceptional graphs K_4 , R_3 , $C_5 \times K_2$, E_{12} , P_{14} , $K_{3,3}$ and $K_{3,3}^*$.

Outline of Proof. This proof, too, is quite long, but the main steps are the following. By the results of Finbow, Hartnell and Nowakowski, we can reduce the problem to graphs having girth 3 or 4. Then by the results of Campbell, we can restrict our considerations to the 3-connected case. Finally—and herein lies the crucial new insight of the authors—one can reduce the 3-connected girth 3 and 4 cases to the connectivity 1 and 2 cases and then, once again, use Campbell's results. ■

In closing this section, it is important to mention that it now follows easily that all cubic well-covered graphs are recognizable in polynomial time.

6. Results for Graphs in Class W_2

In this section, we present some known results about the subclass of well-covered graphs defined earlier in this paper and denoted by W_2 . Recall from Section 2 that the class W_2 can also be defined as the class of 1-well-covered graphs and these graphs are precisely those with the property that they are well-covered and if any point is deleted the resulting graph remains well-covered.

We remark that the W_2 property is in co-NP. For suppose G is not in W_2 . Then either it is not well-covered, and—as remarked in our section on complexity—this is easily exhibited by giving two maximal independent sets which have differing cardinality; or for some point $v \in V(G)$, graph $G - v$ is not well-covered. But the latter situation can again be certified by giving two independent sets (this time maximal in $G - v$) of differing sizes.

There is an additional point of concern to be dealt with here, however. If we again consider the definition of n -well-covered graphs given above—and in particular, the definition of 1-well-covered graphs—it seems, at first thought, that a graph G could fail to be

1-well-covered in the following way: it is well-covered and for each point $v \in V(G)$, $G - v$ is also well-covered, but $\alpha(G - v) < \alpha(G)$, for some point $v \in V(G)$. However, this is impossible, for all well-covered graphs are τ -point-critical and hence $\alpha(G - v) = \alpha(G)$ for all $v \in V(G)$.

The relationship of the property of being in class W_2 to a number of other properties was studied in Section 2. In this section we proceed to investigate certain special classes of W_2 graphs and the structure of their members.

First we remark that Staples [Sta1] was the first to construct infinite families of graphs in W_2 . Although the number of both well-covered and W_2 graphs on a given number of points is not known in general, Royle [Ro] gives us some feeling for their frequency by means of his computer search on all graphs with no more than 9 points. He found that among the 273,191 possible graphs, 9968 are well-covered and 558 are in W_2 . (The reader is referred to open question 11 at the end of this paper.)

Next we state a result due to Staples [Sta1] which relates the W_2 property to that of having *critical* lines. (Recall that we defined a line e in a graph G to be critical if $\alpha(G - e) > \alpha(G)$.)

Theorem 6.1. If graph G is in class W_2 , then $V(G)$ is spanned by critical lines.

The rest of the results in this section, unless otherwise specified, are due to Pinter in his 1991 thesis [Pi]. To begin with, we have a theorem for W_2 graphs which is analogous to Campbell's result (Theorem 5.3 of the present paper). This theorem turns out to be extremely useful in the study of W_2 graphs, much as Theorem 5.3 is for well-covered graphs in general.

Theorem 6.2. If G is a W_2 graph which is not complete, then for each $v \in V(G)$, the graph $G - \Gamma[v]$ is also in W_2 .

As was the case with well-covered graphs in general, we will group the W_2 results in this section in two main categories. First, we will address girth and then deal with degree results.

It is reasonably straightforward to show that with two small exceptions, graphs in W_2 have girth no more than 4. The two exceptions are K_2 and C_5 .

Theorem 6.3. If G is a graph in W_2 and $G \neq K_2$ or C_5 , then $\text{girth}(G) \leq 4$.

Outline of Proof. Recall the family PC of Finbow, Hartnell and Nowakowski introduced earlier. First it is shown that if G is in PC and has girth at least 5 ($G \neq K_2$ or C_5), then G is not in W_2 .

It is then routine to check that the six exceptional graphs of Figure 1 are not in class W_2 . ■

In view of Theorem 6.3, it is natural to group W_2 graphs into two categories: those containing triangles and those having girth exactly four. It is the girth 4 graphs which will occupy our attention.

It is reasonably easy to prove that no W_2 graph of girth 4 can have a cutpoint. Since we are considering only connected graphs in this paper, it then follows that, with the singular exception of K_2 , the minimum degree in any girth 4 graph in class W_2 is at least 2.

Although cutpoints are forbidden, infinite families of girth 4 graphs in W_2 having connectivity 2 can be constructed. It is known, however, that these families do not account for all such graphs. At the moment, it is also an open question as to whether or not there exist such graphs having connectivity *greater than 4*.

Pinter's main result on girth 4 graphs in class W_2 is a characterization of those which are *planar*. We can begin by showing that every such graph must have points of degree 2. This is proved by once again invoking the Theory of Euler Contributions and is rather lengthy. We can then show that points of degree 2 cannot exist "in isolation"; namely, at least one of their neighbors also has degree 2.

Now consider a recursive construction procedure described as follows. Begin with the 8-point graph shown in Figure 5. Call it G_3 .

—FIGURE 5 GOES ABOUT HERE—

Now given any graph G in the construction, let x and y be two adjacent points of degree 2 in G . Let u be the neighbor of x such that $u \neq y$. Then construct a new graph G' with precisely three more points than G as follows. Let the three new points be a , b and c . Now join a to x and b , b to a and c and c to b , u and y . Let us denote this infinite family of graphs by \mathcal{G} . Pinter proved the following.

Theorem 6.4. A graph G is a girth 4 planar member of class W_2 if and only if G is a member of the family \mathcal{G} .

Now let us turn our attention to relationships between membership in class W_2 and degree. In particular, following the line of enquiry pursued in the case of general well-covered graphs in Section 5, we will deal with *regular* graphs.

Of course, the complete graph K_2 is a member of class W_2 . On the other hand, Staples [Sta1] proved the following result.

Theorem 6.5. If $G \in W_n$ and G is not complete, then $\text{mindeg}(G) \leq n$.

Thus we may begin our study of regular graphs versus membership in W_2 with degree 2; that is, with cycles. But it is trivial to see that the only cycles which are in W_2 are C_3 and C_5 .

Hence we may move on to cubic W_2 graphs. But recall that by the very recent theorem of Campbell, Ellingham and Royle (Theorem 5.7 above), we have a nice characterization of the cubic well-covered graphs. Armed with this characterization, Pinter proved the next result.

Theorem 6.6. The only cubic graphs in class W_2 are K_4 and R_3 , the triangular prism.

Thus we may now move up to 4-regular W_2 graphs. But here, the story seems far from complete. It is straightforward to show that no such graph can contain a cutpoint. It is known that there exist such graphs with $\kappa = 2$, but knowledge is meager here; for example, no infinite families have even been constructed. An infinite family of 4-regular W_2 graphs having $\kappa = 4$ is known. Beyond that, the principal result known is the following.

Theorem 6.7. The only graph in W_2 which is 3-connected, 4-regular and planar is the 8-point graph shown in Figure 6.

The proof is yet another long struggle using the Theory of Euler Contributions.

—PUT FIGURE 6 ABOUT HERE—

7. Strongly Well-Covered Graphs

Recall that a well-covered graph is said to be **strongly well-covered** if $G - e$ is also well-covered for each line $e \in E(G)$. To the best of our knowledge, Pinter is the first to undertake the study of this subfamily of well-covered graphs. Staples [Sta1], however, initiated the study of a related subfamily—the so-called *well-covered point-critical* graphs. A well-covered graph G is said to be **well-covered point-critical** if $G - v$ is *not* well-covered, for all points $v \in V(G)$. She constructed infinite families of such graphs and showed that no such graph can have a cutpoint. She also showed that if G is a *bipartite* well-covered graph, other than K_2 , then G is well-covered point-critical if and only if it is 2-connected. Finally, she showed that if G is well-covered and has no critical lines, then G is well-covered point-critical.

What have these graphs to do with strongly well-covered graphs? Except for two trivial exceptions, every strongly well-covered is well-covered point-critical.

Theorem 7.1. If G is strongly well-covered and $G \neq K_1, K_2$, then G is well-covered point-critical.

We immediately have the following somewhat surprising result.

Corollary 7.2. The only strongly well-covered graphs which are also in class W_2 are K_1 and K_2 .

Of course it now also follows from Staples' result that all strongly well-covered graphs, except K_1 and K_2 , are 2-connected. We will see in a moment that this result can be sharpened.

Let us once again recall the Hajnal bound encountered in Sections 1 and 2. In particular, it was mentioned there that both α -critical and W_2 graphs satisfy this upper bound on the maximum degree. It is known, however, that the same cannot be said for strongly well-covered graphs. However, Pinter showed that the latter class indeed "comes close".

Theorem 7.3. If G is a connected strongly well-covered graph, then $\deg(v) \leq |V(G)| - 2\alpha(G) + 2$, for all $v \in V(G)$.

The bound in Theorem 7.3 is known to be sharp.

On the other hand, we have a non-trivial *lower* bound on the degree for strongly well-covered graphs.

Theorem 7.4. If G is strongly well-covered and $G \neq K_1, K_2, C_4$, then $\text{mindeg}(G) \geq 4$.

It is not known whether or not the bound in Theorem 7.4 is sharp. (See problem 7 in Section 9.)

Appealing to the main result of Finbow, Hartnell and Nowakowski (See Theorem 4.2), we can inspect the well-covered graphs with girth at least 5 and see that each has a point of degree at most 3. This observation, combined with Theorem 7.4, yields the next result.

Corollary 7.5. If G is strongly well-covered and $G \neq K_1, K_2, C_4$, then $\text{girth}(G) \leq 4$.

We remarked above that, with the exception of K_1 and K_2 , all strongly well-covered graphs are 2-connected. In fact, we can say more.

Theorem 7.6. If G is strongly well-covered and $G \neq K_1, K_2, C_4$, then G is 3-connected.

The proof of this result is interesting for its own sake, as it is one of the few known to the author which proceeds by induction on the independence number α . The basis step for the induction—namely, the case when $\alpha(G) = 2$ —follows from a characterization of such strongly well-covered graphs obtained by Pinter. Namely, the class of strongly well-covered graphs G with $\alpha(G) = 2$ is precisely the class of all graphs H_{2n} obtained from the complete graphs K_{2n} on an even number of points by deleting a perfect matching. Since we know the minimum degree of any strongly well-covered graph (with the exception of the three graphs listed in the hypothesis of this theorem) is at least 4, it follows that $n \geq 6$ and hence H_{2n} is 3-connected. The basis step of the induction is thus verified.

Is 3-connectivity the best that we can hope for here? The answer is unknown. (See Section 9.)

The final result we will mention for strongly well-covered graphs is consistent with the method of attack upon well-covered graphs in general and W_2 graphs in particular as previously outlined in this paper. Namely, what can be said in the *planar* case?

Here, fortunately, we have a complete answer.

Theorem 7.7. The only planar strongly well-covered graphs are K_1, K_2, C_4 and the octahedron.

The octahedron is shown in Figure 7. It is not surprising that in the proof of Theorem 7.7, once again use is made of the Theory of Euler Contributions.

—Figure 7 GOES ABOUT HERE—

Let us close this section with a comment on the abundance of strongly well-covered graphs. Recall that in his computer search through the class of 273,191 graphs having at most nine points, Royle [Ro] found that 9968 of them were well-covered. Upon further checking, he found that only 4 of these are strongly well-covered! This leads one to be more hopeful, perhaps, that a characterization of this class will be found in the not too distant future. On the other side of the coin, however, using the idea of lexicographic product, Pinter has constructed infinite families of strongly well-covered graphs.

8. Other Results on Well-Covered Graphs

Staples [Sta1; Sta2] discovered a number of constructions for families of well-covered graphs. Further constructions have been provided by Campbell [Ca] and Pinter [Pi]. One of the more general construction procedures developed to date involves the concept of *lexicographic product*. Suppose H is a graph with n points and let $\{G_i\}_{i=1}^n$ be a family of n graphs. Suppose $V(H) = \{v_1, \dots, v_n\}$ and let us implicitly associate with point v_i the graph G_i for each $i = 1, \dots, n$. Define the *lexicographic product* of H and $\{G_1, \dots, G_n\}$, written $H \circ (G_1, \dots, G_n)$, as follows: $V(H \circ (G_1, \dots, G_n)) = V(G_1) \cup \dots \cup V(G_n)$ and $E(H \circ (G_1, \dots, G_n)) = E(G_1) \cup \dots \cup E(G_n) \cup \{xy \mid x \in V(G_i), y \in V(G_j) \text{ and } v_i \sim v_j \text{ in graph } H\}$.

The following result is essentially due to Topp and Volkman [TV2].

Theorem 8.1. If H is well-covered and each G_i , for $i = 1, \dots, |V(H)|$ is a family of well-covered graphs with $\alpha(G_i) = \alpha(G_j)$ for all i and j , then the lexicographic product $H \circ (G_1, \dots, G_{|V(H)|})$ is well-covered and has independence number $\alpha(H)\alpha(G_1)$.

Pinter was able to employ the lexicographic product in a similar way to obtain infinite classes of W_2 graphs and also strongly well-covered graphs. We mention one of his results in this direction.

Theorem 8.2. If H is well-covered and each G_i , for $i = 1, \dots, |V(H)|$ is a family of W_2 graphs with $\alpha(G_i) = \alpha(G_j)$ for all i and j , then the lexicographic product $H \circ (G_1, \dots, G_{|V(H)|})$ is a W_2 graph with independence number $\alpha(H)\alpha(G_1)$.

Topp and Volkman have also studied various other types of graph products with respect to the well-covered and the very well-covered properties. The interested reader is again referred to [TV2].

We will now give the briefest of summaries of several other papers which deal directly or indirectly with the well-covered property. In [To] and [TV1], Topp and Volkman study the well-covered property for some very special classes of graphs: the so-called k -trees, $C_{(n)}$ -trees, block graphs and unicyclic graphs.

Hartnell [Har] has introduced a more restricted variation of the well-covered property with his *radius 2 independent set* concept. A set of points S in a graph G is said to be **radius 2 independent** if for every two points in S have disjoint closed neighborhoods. That is, for every pair of points u and v in S , $\Gamma[x] \cap \Gamma[y] = \emptyset$. He then characterizes those

trees which have the property that every maximal radius 2 independent set is maximum. These graphs arise in considering a certain 2-person game.

Dean and Zito [DZ] have begun to study yet another variant of the well-covered property. They define a graph G to be k -extendable if every independent set of size k is contained in a maximum independent set. A graph is thus well-covered if and only if it is k -extendable for all k , $1 \leq k \leq \alpha(G)$. At the other extreme, the 1-extendable graphs are just those graphs for which $V_e(G) = \emptyset$; i.e., the so-called B -graphs. (See Section 1.) The authors use the concept of k -extendability to show that two types of perfect graphs—namely, those with bounded clique size and those which contain no induced 4-cycle—can be checked for well-coveredness in polynomial time.

Note that a graph may be 2-extendable without being 1-extendable. However, Dean and Zito are able to show that every 2-extendable graph which is not also 1-extendable must be the join of a complete graph and another graph which is both 1-extendable and 2-extendable.

(The choice of terminology “ k -extendable” is somewhat unfortunate since there are a number of papers in print in which this term is applied to matchings rather than independent sets of points.)

Finbow, Hartnell and Nowakowski [FHN1] and Finbow and Hartnell [FiH2] have also introduced several new properties related to, but different from, well-coveredness.

In [FHN1] a graph G is defined to be **well-dominated** if every minimal dominating set is of the same cardinality. It is very easy to see that a well-dominated graph must be well-covered, but not conversely. Motivated by their girth 5 theorem for well-covered graphs (cf. Theorem 4.2 above), the three authors observe that among the six exceptional graphs of Theorem 4.2, precisely four are also well-dominated. They then determine which graphs in class PC are well-dominated in terms of how their basic 5-cycles are joined. As a corollary of this work, the authors then are able to show that for graphs of girth at least 6, the concepts of well-dominated and well-covered coincide.

In [FiH2], the authors define two additional subclasses of well-covered graphs, the so-called *EIDL* graphs and a subclass of these called *EDL* graphs. The reader is referred to the paper for the details.

Now let us define a set of points S in a graph G to be k -independent if each point in S is adjacent to no more than $k - 1$ other points in S . (Thus a set of points is 1-independent if and only if it is independent.) Favaron and Hartnell [FaH] define a graph G to be **well- k -covered** if every maximal k -independent set is maximum. They then characterize those trees which are well- k -covered (not to be confused with the well-covered k trees of Topp [To]!) as well as those well-2-covered graphs of girth at least 8.

One can view well-covered graphs as those having precisely one size of maximal independent set. With this perspective in mind, Finbow, Hartnell and Whitehead [FHW] relax the definition of well-covered graphs and say that a graph is in class M_2 if it has precisely two sizes of maximal independent sets. They then obtain a very complicated characterization of such graphs having girth at least 8.

Finally, we close with a class of well-covered graphs of quite a different nature. Call a function w which associates with each point of a graph G a non-negative real number a **weighting** of $V(G)$. For any set $S \subseteq V(G)$, denote the sum of the weights of the points

in G by $W(S; w)$. In particular, $W(G; w)$ is called the **graph weight**. Call a weighting w a **fractional cover** if $W(\Gamma[v]; w) \geq 1$ holds for all points in G . A fractional cover is **minimal** if no point can have its weight reduced and have the corresponding weighting remain a fractional cover. A graph G is then said to be **fractionally well-covered** if $W(G; w)$ is the same for all minimal fractional covers.

It is easy to show that fractionally well-covered graphs are also well-dominated and hence, well-covered by Lemma 1 of [FHN1]. On the other hand, there are well-covered graphs which are not fractionally well-covered. As an example, the authors offer a 5-cycle with a weight of $1/3$ on each point. Clearly, this is a minimal fractional cover having weight $5/3$. On the other hand, if one assigns a weight of 1 to two non-adjacent points of the 5-cycle and 0 to the remaining three points, one has another minimal fractional cover the weight of which is 2.

Recall that we defined a point $v \in G$ to be **simplicial** if the closed neighborhood $\Gamma[v]$ is a complete graph. A graph G will then be called **simplicial** if it can be partitioned into point-disjoint maximal complete subgraphs H_1, H_2, \dots, H_k where each H_i contains at least one simplicial point. The authors then obtain the following tidy characterization of fractionally well-covered graphs.

Theorem 8.4. A graph is fractionally well-covered if and only if it is simplicial.

It is now obvious via this characterization that the property of being fractionally well-covered is in NP.

9. Some Open Questions

As we have previously remarked, it is considered unlikely that a polynomial verification algorithm will be found for graphs which are well-covered. As we stated before, that would mean that the well-covered property is an NP-property and that, in turn, would imply that $NP = co-NP$, an equation thought highly unlikely by the vast majority of those working in complexity theory.

In light of the girth results of Finbow, Hartnell and Nowakowski [FHN2], it is the presence of short cycles—that is, triangles and quadrilaterals—which is causing the difficulty. We feel that the first two unsolved problems in the following list are likely the most difficult. Problems 3 through 12 may prove to be more tractable.

1. Characterize w-c graphs which contain C_3 's, but no C_4 's.
2. Characterize w-c graphs of girth 4.
3. Characterize w-c graphs with $\maxdeg(G) \leq 3$.
4. Characterize w-c graphs which are r -regular, for any $r \geq 4$.
5. Characterize W_2 graphs which are 3-connected, 5-regular and planar.

6. All known examples of girth 4 W_2 graphs have $\kappa(G) \leq 4$. Is there an upper bound for $\kappa(G)$ for girth 4 W_2 graphs?

7. It is known that G strongly w-c and $|V(G)| \geq 4$ implies G is 3-connected. Can we strengthen the conclusion to 4-connected?

8. Characterize girth 4 strongly w-c graphs.

9. Characterize those graphs which are (a) both α -critical and w-c; or (b) both α -critical and W_2 .

10. What about graphs G which are w-c, but $G - e$ is *not* w-c, for all $e \in E(G)$?

11. (a) Does $\# \text{ w-c } G\text{'s} / \# G\text{'s} \rightarrow 0$ as $|V(G)| \rightarrow \infty$?

(b) Does $\# W_2 G\text{'s} / \# G\text{'s} \rightarrow 0$ as $|V(G)| \rightarrow \infty$?

And finally, although the last problem we pose does not directly deal with well-covered graphs, we remind the reader of the common implications discussed in Section 1 for the properties of being α -critical and belonging to class W_2 .

12. Does $\# \alpha\text{-critical } G\text{'s} / \# G\text{'s} \rightarrow 0$ as $|V(G)| \rightarrow \infty$?

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 K_1

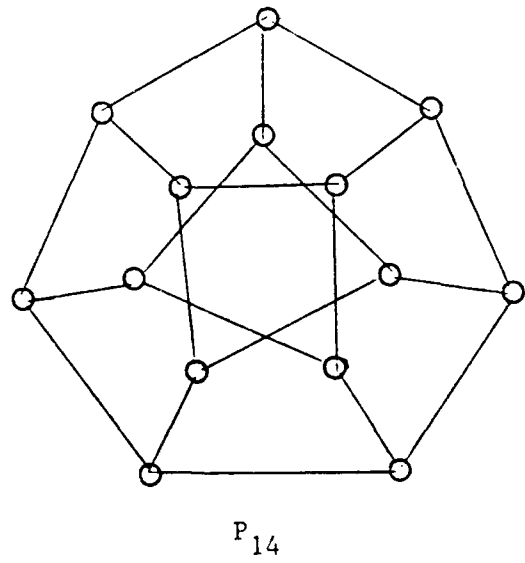
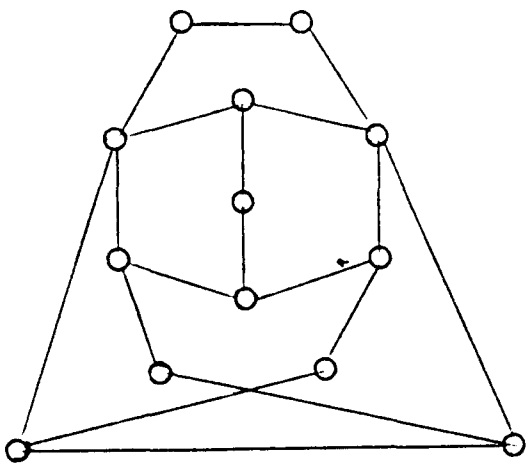
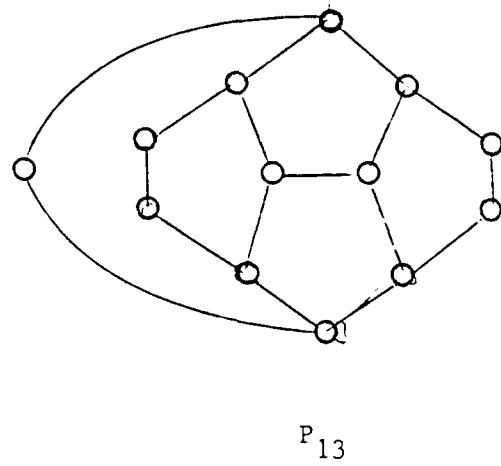
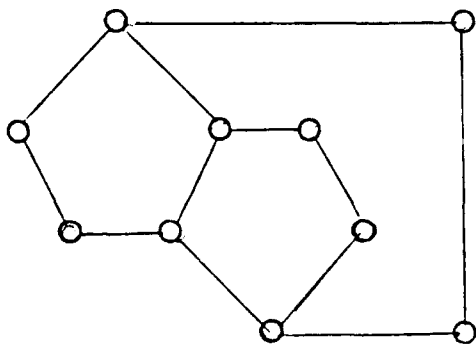
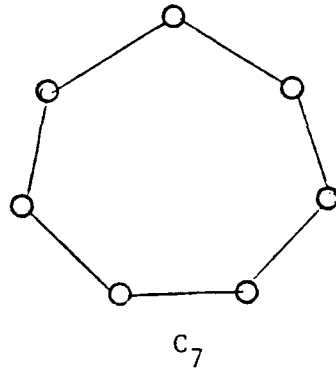
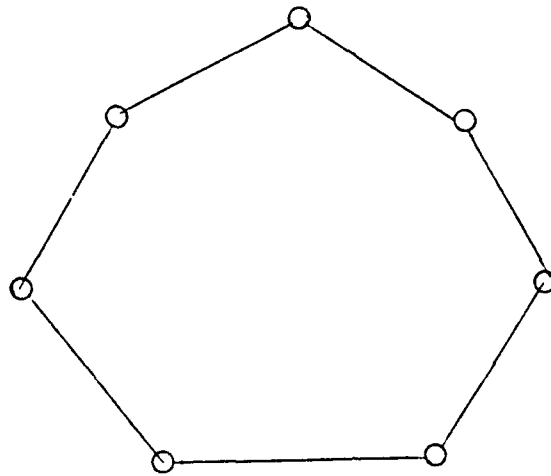
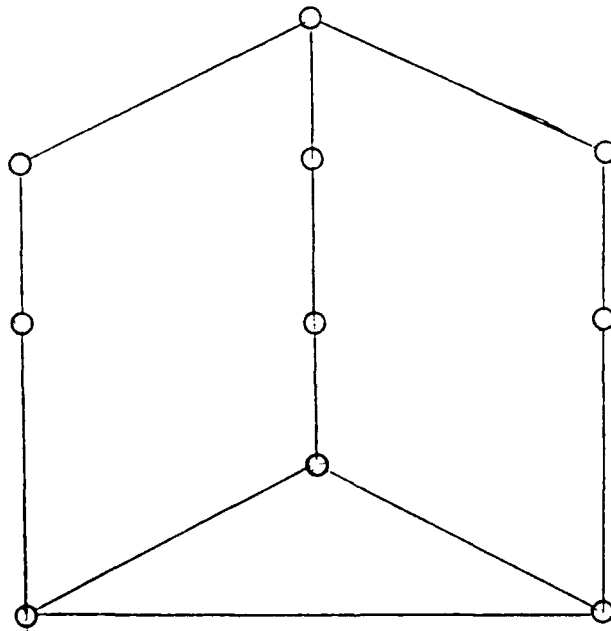


Figure 1

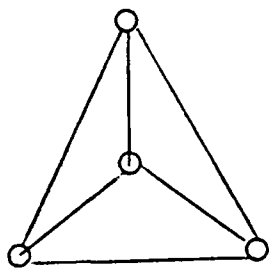


C_7

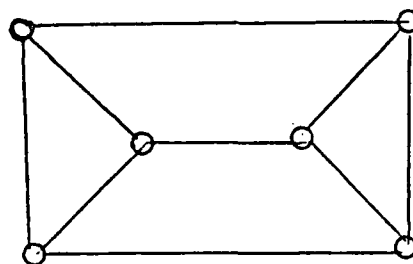


T_{10}

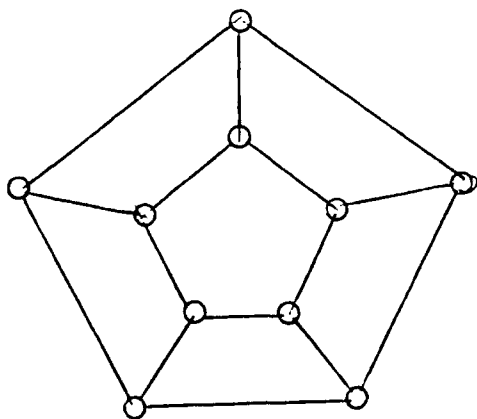
Figure 2



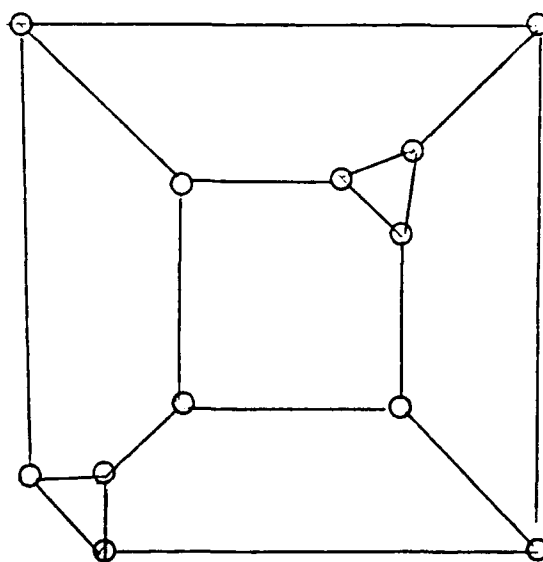
K_4



R_3



$C_5 \times K_2$



E_{12}

Figure 4

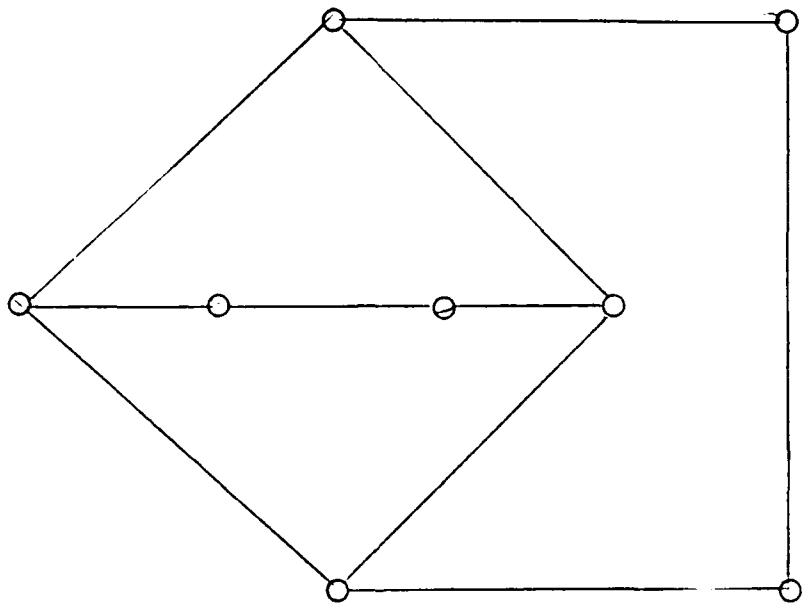


Figure 5

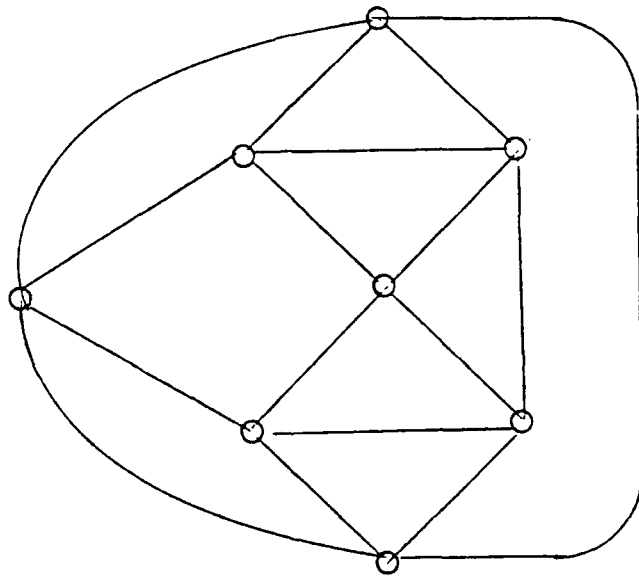


Figure 6

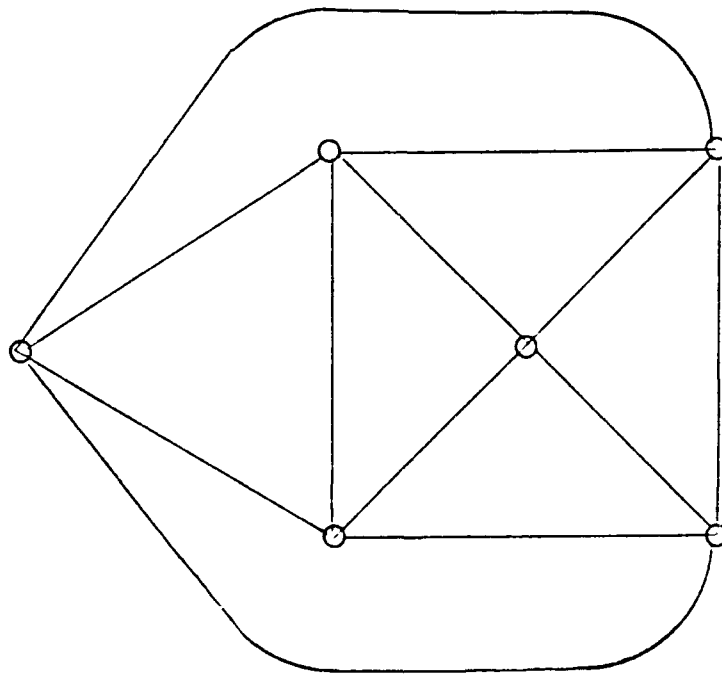


Figure 7