Management Science Research Report #MSRR-575

Decomposition of Balanced Matrices. Part VII: A Polynomial Recognition Algorithm

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October 1991

This work was supported in part by NSF grants DDM-8800281, DDM-8901495 and DDM-9001705.

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1 Introduction

In this paper we present an algorithm to test whether a bipartite graph $G(V^r, V^c; E)$ is balanced. The running time of the algorithm is bounded by a polynomial function of the cardinalities m and n of the node sets V^r and V^c.

Recall from Part I that a node cutset S of G is a *double star cutset* if there exist two adjacent nodes, $u, v \in V$ such that $S = N(u) \cup N(v)$. The algorithm in this section decomposes G using double star cutsets. The lemma below shows a relation between double star cutsets and extended star cutsets for undominated bipartite graphs as defined below.

Definition 1.1 *A node u in a bipartite graph G is said to be* dominated *if there exists a node v, distinct from u, such that $N(u) \subseteq N(v)$. A bipartite graph G is said to be* undominated *if G contains no dominated nodes.*

Lemma 1.2 *If G is an undominated graph containing an extended star cutset, then G contains a double star cutset.*

Proof: Let $S = (u; T; A; N)$ be an extended star cutset of G and let $G_1, G_2, \ldots, G_l$ be the connected components of $G \setminus S$. Define $S^* = N(u) \cup N(v)$, where v is a node in A. Clearly, $S \subseteq S^*$. Suppose S^* is not a double star cutset of G. Then all the nodes in one of the connected components of $G \setminus S$, say G_k, belong to $S^* \setminus S$. Hence $V(G_k) \subset N(u) \cup N(v)$. Consider a node $x \in V(G_k)$ and assume w.l.o.g. that $x, u \in V^r$. Now $N(x) \cap V(G_k) \subset N(u)$ and $N(x) \cap V(G_j) = \emptyset$, for all $j \neq k$. Hence x is a dominated node. □

From the bipartite graph G, the algorithm creates a number of undominated induced subgraphs of G and decomposes each of them by double star cutsets. A proof of the validity of this approach is given in the last section.

A 2-join in a bipartite graph G, as defined in Part I, is a set of edges $E^* = E(K_{AB}) \cup E(K_{DF})$ belonging to two bicliques K_{AB} and K_{DF} such that $E(K_{AB})$ and $E(K_{DF})$ are not cutsets of G and no connected component of the partial graph $G \setminus E^*$ contains a node in A and a node in B or a node in D and a node in F.

Definition 1.3 *A 2-join $E^* = E(K_{AB}) \cup E(K_{DF})$ is* stable *if it satisfies the following properties:*

- *No node in A is adjacent to a node in D and no node in B is adjacent to a node in F.*
- *Sets A and B contain at least two nodes each or sets D and F contain at least two nodes each.*

Since 1-joins and strong 2-joins imply the existence of an extended star cutset, Theorem 1.1[IV] and Theorem 10.1[V] imply the following version of Theorem 4.9[I].

Theorem 1.4 *Let G be a balanced bipartite graph which is not restricted balanced. Then G has an extended star cutset or a stable 2-join.*

Now Lemma 1.2 and Theorem 1.4 above imply the following version of Theorem 4.9[I] for undominated balanced bipartite graphs.

Theorem 1.5 *Let G be an undominated balanced bipartite graph which is not restriced balanced. Then G has a double star cutset or a stable 2-join.*

The next section contains definitions and some properties of bipartite graphs that are essential for the validity of the algorithm given in Section 3. The algorithm uses four procedures which are described in Sections 4 to 7. The validity and polynomiality of each of the procedures is shown in the respective sections. The last section contains a proof of the validity of the algorithm.

2 Smallest unquad holes

Let H^* be a smallest unquad hole in a bipartite graph G which is not balanced. In this section we study properties of strongly adjacent nodes to H^*.

Definition 2.1 *A node u strongly adjacent to a hole H in G is* odd-strongly adjacent *if u has an odd number of neighbors in H. If u has an even number of neighbors in H, then u is* even-strongly adjacent. *The sets $A_r(H)$ and $A_c(H)$ contain the odd-strongly adjacent nodes to H which belong to V^r and V^c respectively.*

The following fundamental properties of the sets $A_r(H^*)$ and $A_c(H^*)$, associated with a smallest unquad hole H^* have been proven by Conforti and Rao in [15].

Property 2.2 *There exists a node $x_r \in V^r \cap V(H^*)$ which is adjacent to all the nodes in $A_c(H^*)$.*

Property 2.3 *There exists a node $x_c \in V^c \cap V(H^*)$ which is adjacent to all the nodes in $A_r(H^*)$.*

Property 2.4 *Every even strongly adjacent node to H^* is a twin of a node in H^**

The above properties have been used in [16] to design a polynomial algorithm to test whether a linear bipartite graph is balanced. To test balancedness of a bipartite graph, we need the following additional properties of strongly adjacent nodes.

Definition 2.5 *A* tent $\tau(H,u,v)$ *is a configuration of G induced by a hole H and two adjacent nodes u and v which are even strongly adjacent to H with the following property:*

The nodes of H can be partitioned into two subpaths P_u and P_v containing the nodes in $N(u) \cap H$ and $N(v) \cap H$ respectively.

A tent $\tau(H,u,v)$ is referred to as a *tent containing H*. We now study properties of a tent $\tau(H^*,u,v)$ containing a smallest unquad hole H^* and we assume throughout the paper that the first node, say u in the definition of a tent $\tau(H,u,v)$ belongs to V^r and that node v belongs to V^c. We use the notation of Figure 1, where nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order, when traversing H^* counterclockwise, starting from u_1.

Lemma 2.6 *Nodes v_0, u_1, u_2 satisfy at least one of the following properties:*

- *The set $A_r(H^*)$ is contained in $N(v_0) \cup N(u_1)$.*
- *The set $A_r(H^*)$ is contained in $N(v_0) \cup N(u_2)$.*

Nodes u_0, v_1, v_2 satisfy at least one of the following properties:

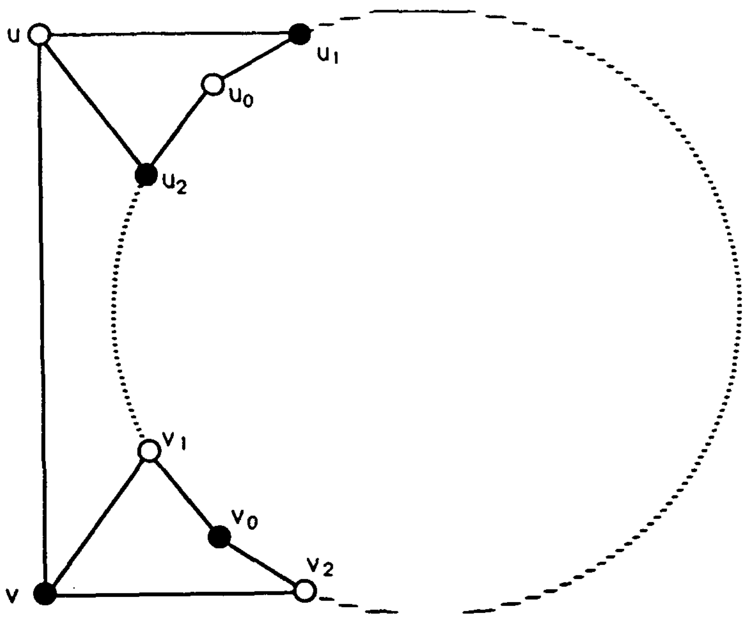

Figure 1: A tent

- *The set $A_c(H^*)$ is contained in $N(u_0) \cup N(v_1)$.*
- *The set $A_c(H^*)$ is contained in $N(u_0) \cup N(v_2)$.*

Proof: We prove the first part. Suppose $w \in A_r(H^*)$ is not adjacent to v_0. Consider the hole H_1^* obtained from H^* by replacing v_0 with node v of $\tau(H^*, u, v)$. Now w cannot be adjacent to v, for otherwise w is even strongly adjacent to H_1^*, violating Property 2.4. Node u is in $A_r(H_1^*)$ and has neighbors u_1, u_2 and v in H_1^*. Since w is in $A_r(H_1^*)$ and w is not adjacent to v, by Property 2.3 it follows that w is adjacent to u_1 or u_2. By Property 2.3, all nodes in $A_r(H_1^*)$ must have a common neighbor in H_1^*. It follows that this common neighbor must be u_1 or u_2. The proof of the second part is identical. □

Lemma 2.7 *Let $\tau(H^*, u, v)$ and $\tau(H^*, w, y)$ be two tents, where w_1, w_2 are the neighbors of w and y_1, y_2 are the neighbors of y in H^*. Let w_0 and y_0 be the common neighbors of w_1, w_2 and y_1, y_2 respectively. Then at least one of the following properties holds:*

- *Nodes u_1 and u_2 coincide with w_1 and w_2.*

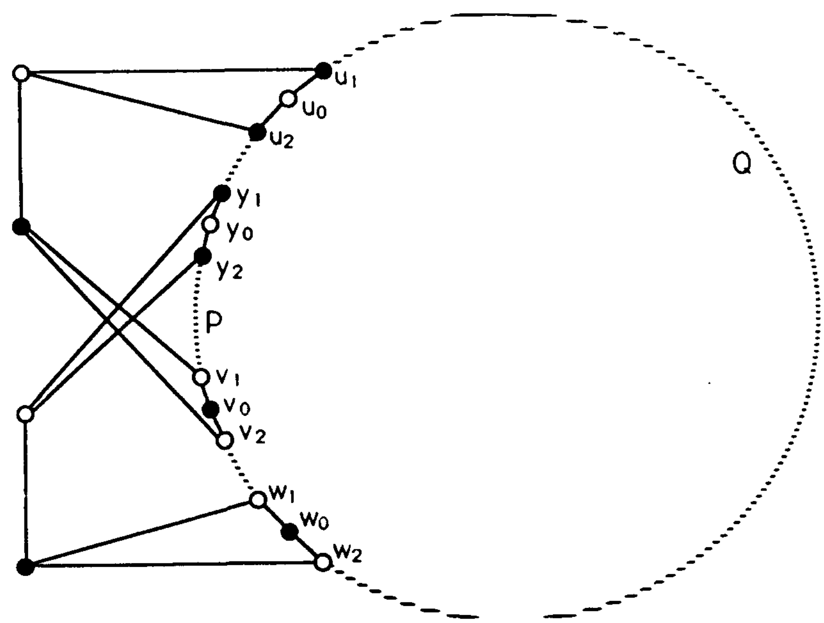

Figure 2:

- *Nodes v_1 and v_2 coincide with y_1 and y_2.*
- *Nodes u_0 and y are adjacent.*
- *Nodes v_0 and w are adjacent.*

Proof: Suppose the contrary. Then node u does not coincide with w, node v does not coincide with y, nodes u_0 and y are not adjacent and nodes v_0 and w are not adjacent. Let P denote the u_2v_1-subpath of H^* not containing any other neighbor of u or v. Similarly, let Q denote the v_2u_1-subpath of H^* not containing any other neighbor of u or v, see Figure 2.

Now it follows that y_1 and y_2 are contained in P or Q and w_1 and w_2 are contained in P or Q. Assume w.l.o.g. that y_1 and y_2 are contained in P. Since H^* is an unquad hole of smallest cardinality, it follows that both P and Q must be of length $1 \bmod 4$. We now prove the following two claims:

Claim 1 *Node y is not adjacent to u and node w is not adjacent to v.*

Proof of Claim 1: Suppose that y and u are adjacent. Since u_0 and y are not adjacent, it follows that y_0 does not coincide with u_1 or u_2. By the definition of a tent, v_0 does not coincide with u_1 or u_2. Furthermore, since

v_1 and v_2 do not coincide with y_1 and y_2, we have that v_0 does not coincide with y_0.

Consider the unquad hole H_1^*, obtained from H^* by replacing v_0 and y_0 with v and y respectively. Now u has four neighbors in a smallest odd hole H_1^*, violating Property 2.4. Hence y is not adjacent to u. By symmetry, it follows that w is not adjacent to v.

Claim 2 *Nodes w_1 and w_2 belong to Q.*

Proof of Claim 2: Suppose the claim is false. Then w_1 and w_2 belong to P. By assumption, y_1 and y_2 belong to P. Note that the w_1y_2-subpath of P is of length 1 *mod* 4 and contains nodes w_2 and y_1 and at least five edges. Now $C = v, v_1, P, u_2, u, v$ is an even hole of length at least eight. By Claim 1, y is not adjacent to u and w is not adjacent to v. Let C^* be the hole obtained from C by replacing the subpath $P_{y_2w_1}$ of P with the path y_2, y, w, w_1. Now C^* is an unquad hole of smaller cardinality than H^*. This completes the proof of Claim 2.

Now a simple counting argument shows that the hole $H = u, v, v_1, P_{v_1y_2}, y_2, y, w, w_2, Q_{w_2u_1}, u_1, u$ is unquad and is of smaller cardinality than H^*. $\square$

Definition 2.8 *A hole H is said to be* clean in G *if the following three conditions hold:*

(i) *No node is odd-strongly adjacent to H.*

(ii) *Every even-strongly adjacent node is a twin of a node in H.*

(iii) *There is no tent containing H.*

We show in Section 6 that if G is not balanced, an unquad hole of smallest cardinality which is clean belongs to one of the final blocks of a decomposition of a bipartite graph with extended star cutsets. In Section 5 we show how to obtain a clean unquad hole in a bipartite graph which is not balanced.

3 A Recognition Algorithm

In this section we give an algorithm to test whether a bipartite graph is balanced.

Definition 3.1 *A wheel with three spokes and at least two sectors having length 2 is said to be a* short 3-wheel.

ALGORITHM

Input: A bipartite graph G.
Output: G is identified as balanced or not balanced.

Step 1 Apply Procedure 1 to check whether G contains a short 3-wheel. If so, G is not balanced, otherwise go to Step 2.

Step 2 Apply Procedure 2 to create at most m^4n^4 induced subgraphs of G, say $G_1, \ldots, G_i, \ldots, G_p$ such that, if G is not balanced, at least one of the induced subgraphs created, say G_i, contains an unquad hole of smallest cardinality which is clean in G_i.

Step 3 Apply Procedure 3 to each of the induced subgraphs $G_1, \ldots, G_i, \ldots, G_p$ to decompose them into undominated induced subgraphs $F_1, \ldots, F_j, \ldots, F_q$ that do not contain a double star cutset. While decomposing a graph with a double star cutset $N(u) \cup N(v)$, Procedure 3 also checks the existence of a 3-path configuration containing nodes u and v and nodes in two distinct connected components resulting from the decomposition. If such a configuration is found, then G is not balanced, otherwise go to Step 4.

Step 4 Apply Procedure 4 to each of the induced subgraphs $F_1, \ldots, F_j, \ldots, F_q$ to decompose them using 2-joins into blocks $B_1, \ldots, B_k, \ldots, B_r$ not containing an extended star cuset or a 2-join.

Step 5 Test whether any of the blocks $B_1, \ldots, B_k, \ldots, B_r$ contains an unquad cycle. If so, G is not balanced, otherwise G is balanced.

Remark 3.2 *An algorithm to test whether a bipartite graph contains an unquad cycle can be found in [11] or [25]. Hence the details of Step 5 are omitted in this paper.*

4 Identification of Short 3-wheels

The following procedure tests whether a bipartite graph contains a short 3-wheel.

PROCEDURE 1

Input: A bipartite graph G.

Output: G contains a short 3-wheel or G does not contain such a configuration.

Step 1 Enumerate all distinct subsets of six nodes with three nodes in V^r and three nodes in V^c and declare them as unscanned. Go to Step 2.

Step 2 If all subsets are scanned, G does not contain a short 3-wheel, stop. Otherwise choose an unscanned subset U. If U induces a 6-cycle $C = a_1, a_2, a_3, a_4, a_5, a_6, a_1$, having unique chord a_2a_5, go to Step 3. Otherwise declare U as scanned and repeat Step 2.

Step 3 Remove the nodes in $N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) \setminus \{a_1, a_3\}$. If a_1 and a_3 are in the same connected component, then a short 3-wheel with spokes a_2a_1, a_2a_3, a_2a_5 is identified, stop. If not, remove the nodes in $N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) \setminus \{a_4, a_6\}$. If a_4 and a_6 are in the same connected component, then a short 3-wheel with spokes a_5a_2, a_5a_4, a_5a_6 is identified, stop. Otherwise declare U as scanned return to Step 2.

Remark 4.1 *The complexity of this procedure is of order $O(m^4n^4)$.*

5 Induced Subgraphs Containing Clean Unquad Holes

In this section, we show how to create at most m^4n^4 induced subgraphs of G such that, if G is not balanced, one of the subgraphs, say G_t, contains a smallest unquad hole which is clean in G_t.

Definition 5.1 *Given a configuration $F(V(F), E(F))$, and a node v in F, we denote with $N_F(v)$ the set $N(v) \cap V(F)$.*

We define F_{ijkl} to be the induced subgraph obtained by removing the nodes in $N_F(j) \setminus \{i, k\}$ and the nodes in $N_F(k) \setminus \{j, l\}$.

PROCEDURE 2

Input: A bipartite graph G.

Output: A family $\mathcal{L} = G_1, G_2, \ldots, G_p$, where $p \leq m^4n^4$, of induced subgraphs of G such that if G is not balanced, one of the subgraphs in $\mathcal{L}$, say G_t, contains a smallest unquad hole that is clean in G_t.

Step 1 Let $\mathcal{L}^* = \{G_{ijkl} \mid \text{nodes } i,j,k,l \text{ induce the chordless path } i,j,k,l\}$

Step 2 Let $\mathcal{L} = \{Q_{ijkl} \mid$ the graph Q is in $\mathcal{L}^*$, nodes in $\{i,j,k,l\}$ belong to Q and induce the chordless path $i,j,k,l\}$.

We now prove the validity of Procedure 2.

Lemma 5.2 *If G is not balanced, one of the graphs in $\mathcal{L}$, say G_t, contains an unquad hole H^*, smallest in G, and H^* is clean in G_t.*

Proof: Assume G is not balanced. Then G contains a smallest unquad hole H^*. Recall that the sets $A_r(H^*)$ and $A_c(H^*)$ are defined with respect to G. Consider the following two cases:

Case 1 *There is no tent in G containing H^*.*

Proof of Case 1: By Property 2.2, there exists a node $j \in V^r(G) \cap V(H^*)$ that is a common neighbor of all nodes in $A_c(H^*)$. Let i,k be the neighbors of j in H^* and let l be the other neighbor of k in H^*. Then the graph G_{ijkl} contains H^*, but does not contain any node in $A_c(H^*)$, and belongs to $\mathcal{L}^*$. By considering G_{ijkl} and applying Property 2.3, it follows that $\mathcal{L}$ contains a graph G_t and H^* is clean in G_t.

Case 2 *The graph G contains a tent $\tau(H^*,u,v)$.*

Proof of Case 2: By Lemma 2.6, the set $A_r(H^*)$ is contained in $N(v_0) \cup N(u_1)$ or in $N(v_0) \cup N(u_2)$ and the set $A_c(H^*)$ is contained in $N(u_0) \cup N(v_1)$ or in $N(u_0) \cup N(v_2)$. Assume w.l.o.g. that $A_r(H^*)$ is contained in $N(v_0) \cup N(u_1)$.

Suppose $A_c(H^*)$ is contained in $N(u_0) \cup N(v_1)$ and let u^* and v^* be the neighbors of u_1 and v_1, which are distinct from u_0 and v_0 respectively. By Lemmas 2.6 and 2.7, it follows that the graph $G_{u^* u_1 u_0 u_2}$, which belongs to $\mathcal{L}^*$, contains H^* and satisfies the following properties:

- No node in $A_c(H^*)$ that is adjacent to u_0 belongs to $G_{u^* u_1 u_0 u_2}$.
- No node in $A_r(H^*)$ that is adjacent to u_1 belongs to $G_{u^* u_1 u_0 u_2}$.
- The graph $G_{u^* u_1 u_0 u_2}$ does not contain a node w, in a tent $\tau(H^*,w,y)$, where w_1 and w_2 coincide with u_1 and u_2.
- The graph $G_{u^* u_1 u_0 u_2}$ does not contain a node y, in a tent $\tau(H^*,w,y)$, where y and u_0 are adjacent.

As a consequence of Lemmas 2.6 and 2.7, applied to $G_{u^* u_1 u_0 u_2}$, it follows that $\mathcal{L}$ contains an induced subgraph of G, say G_t, which contains H^* and H^* is clean in G_t. If $A^c(H^*)$ is contained in $N(u_0) \cup N(v_2)$, the proof is identical. □

6 Double Star Cutset Decompositions

We describe a procedure to decompose a bipartite graph into blocks which are induced subgraphs and do not contain a double star cutset. While decomposing the graph into blocks, the procedure also checks the existence of a 3-path configuration that contains nodes in at least two connected components.

PROCEDURE 3

Input: A bipartite graph F not containing a short 3-wheel.

Output: Either F contains a 3-path configuration, hence F is not balanced or a list of undominated induced subgraphs $F_1, \ldots, F_j, \ldots, F_q$ of F, where $q \leq |V^c(F)|^2 |V^r(F)|^2 \leq m^2 n^2$ with the following properties:

- The graphs $F_1, \ldots, F_j, \ldots, F_q$ do not contain a double star cutset.
- If the input graph F contains a smallest unquad hole which is clean in F, then one of the graphs in the list, say F_i, contains an unquad hole H^*, that is smallest in F and H^* is clean in F_i.

Step 1 Delete dominated nodes in F until F becomes undominated. Let $\mathcal{M} = \{F\}, \mathcal{T} = \emptyset$.

Step 2 If $\mathcal{M}$ is empty, stop. Otherwise remove a graph R from $\mathcal{M}$. If R has no double star cutset, add R to $\mathcal{T}$ and repeat Step 2. Otherwise, let $S = N_R(u) \cup N_R(v)$ be a double star cutset of R. Let $R_1, \ldots, R_l$ be the connected components of $R \setminus S$, let $R_1^*, \ldots, R_l^*$ be the corresponding blocks, i.e. R_i^* is induced by $V(R_i) \cup S$. Go to Step 3.

Step 3 Consider every pair of nonadjacent nodes u_p and v_q such that node u_p is adjacent to u and node v_q is adjacent to v. If both u_p and v_q have neighbors in two distinct connected components of $R \setminus S$, there is a $3PC(u_p, v_q)$ and F is not balanced. Otherwise go to Step 4.

Step 4 From each block R_i^*, remove dominated nodes in $(N(u) \cup N(v)) \setminus \{u, v\}$, until no such node exists. Now remove further any dominated node until the block becomes undominated.

Add to $\mathcal{M}$ all the undominated blocks that contain at least one chordless path of length 3. Go to Step 2.

Remark 6.1 *If a node $w \in (N(u) \cup N(v)) \setminus \{u, v\}$ belongs to the undominated block R_i^* at the end of Step 4, Then w is adjacent to at least one node in the connected component R_i.*

Before proving the validity of Procedure 3, we need the following definition:

Definition 6.2 *Let G be a graph containing a hole H. Then $C(H) = \{H_i \mid H_i$ is obtained from H by a sequence of holes $H = H_0, H_1, \ldots, H_i$, where H_j and H_{j-1}, for $j = 1, 2, \ldots, i$, differ in at most one node $\}$.*

Lemma 6.3 *Let G be a graph containing a smallest unquad hole H^*, but not containing a short 3-wheel. If H^* is clean in G, then every hole H_i^* in $C(H^*)$ is clean in G.*

Proof: Let H_1^* be a hole that differs from H^* in only one node. In order to prove the lemma, it is sufficient to show that H_1^* is clean in G.

Since H_1^* is an unquad hole of smallest cardinality, by Property 2.4, condition (ii) of Definition 2.8 is satisfied. Hence, if the lemma is false, condition (i) or (iii) of Definition 2.8 is not satisfied. Therefore we consider the following two cases.

Case 1 *Condition (i) of Definition 2.8 is not satisfied.*

Proof of Case 1: Now a node w must be odd-strongly adjacent to H_1^*. Since no node is odd-strongly adjacent to H^*, it follows that w has three neigbors, say w_1, w_2, w_3 in H_1^*. Two of these neighbors, say w_1 and w_2 must be in H^* and have a common neighbor, say w_0 in H^*. Since w_3 is in H_1^* but not in H^*, it follows that H_1^* is obtained from H^* by replacing some node $u \neq u_1$, u_2 in H^* with w_3. Let u_1 and u_2 be the neighbors of u in H^*. Note that w_3 is adjacent to u_1 and u_2 and u does not coincide with w_1 or w_2. Hence u_1 and u_2 do not coincide with w_0. Now $\tau(H^*, w_3, w)$ is a tent, contradicting the assumption that H^* is clean in G.

Case 2 *Condition (iii) of Definition 2.8 is not satisfied.*

Proof of Case 2: There must be a tent $\tau(H_1^*, u, v)$. We first show the following claim:

Claim *At least one of the nodes u_1, u_2, v_1, v_2 does not belong to the hole H^*.*

Proof of Claim: Assume not. Since u and v are not in H_1^*, it follows that at most one of them is in H^*. If u is in H^*, then u_0 is not in H^* and v is odd-strongly adjacent to H^*. So u is not in H^* and, by symmetry, node v is not in H^*.

Assume that neither u nor v belong to H^* and let $w \neq u_1, u_2, v_1, v_2$ be a node in H^* but not in H_1^*. Nodes w and u are not adjacent, otherwise node u is odd-stongly adjacent to H^*, contradicting the assumption that H^* is clean. By symmetry, it follows that nodes w and v are not adjacent. Now $\tau(H^*, u, v)$ is a tent, contradicting the assumption that H^* is clean and the proof of the claim is complete.

By the above claim, one of the nodes u_1, u_2, v_1, v_2 is not in H^*. Assume w.l.o.g. that u_2 is not in H^*. Clearly, node u is not in H^*. Node v is not in H^*, otherwise node v_0 is not in H^*, node u_2 coincides with v_0 and $\tau(H_1^*, u, v)$ is not a tent.

Thus the hole H_1^* is obtained from H^* by replacing a node w with u_2, where w is adjacent to u_0. Let u_3 in H^* be the other neighbor of u_2. It follows that u_3 is adjacent to w. Let Q denote the v_1u_3-subpath of H^* not containing v_2, see Figure 3. Consider the hole $C = u, v, v_1, Q, u_3, w, u_0, u_1, u$. Now the wheel (C, u_2) is a short 3-wheel, contradicting the fact that G does not contain a short 3-wheel. $\square$

Remark 6.4 *Assume that the graph F contains a smallest unquad hole H^* that is clean in F. If F does not contain a short 3-wheel, an undominated graph obtained from F in Step 1 of Procedure 3 contains a clean unquad hole in the family $C(H^*)$.*

Lemma 6.5 *Let F be a bipartite graph satisfying the following properties:*

- *The graph F does not contain a short 3-wheel.*
- *The graph F contains a smallest unquad hole H^* that is clean in F.*

Then the output of Procedure 3 is one of the following:

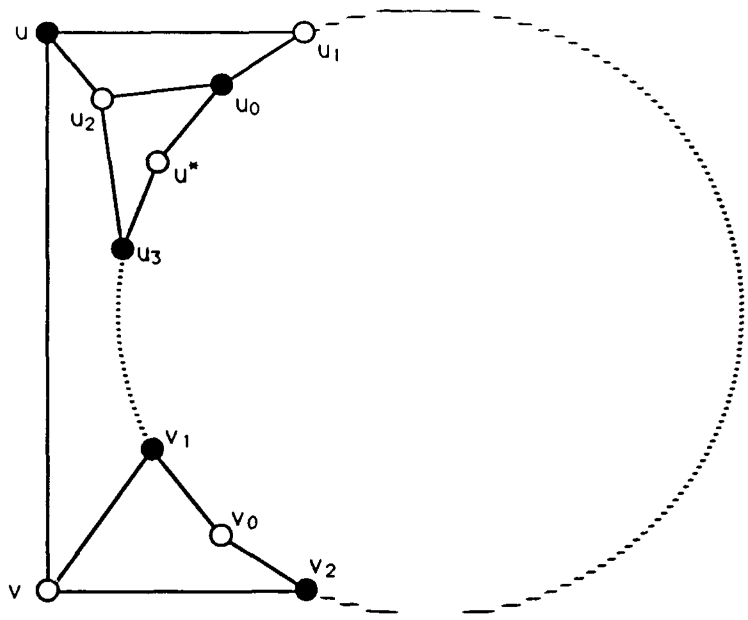

Figure 3:

- *A 3-path configuration is detected in Step 3.*
- *One of the undominated blocks, say F_i, obtained as ouput of Procedure 3, contains an unquad hole in $C(H^*)$.*

Proof: Let $S = N(u) \cup N(v)$ be a double star cutset of F. Let $F_1, \ldots, F_t$ be the connected components of $F \setminus S$ and $F_1^*, \ldots, F_t^*$ be the corresponding blocks. We first show that an unquad hole $H' \in C(H^*)$ is contained in some block F_i^* obtained at the end of Step 3. There are three cases to consider.

Case 1 *Both nodes u and v belong to H^*.*

Proof of Case 1: Let u_1 and v_1 in H^* be the other neighbors of u and v respectively. Now the nodes in $V(H^*) \setminus \{u, v, u_1, v_1\}$ are in some connected component F_i and F_i^* contains H^*.

Case 2 *Either node u or node v is in H^*.*

Proof of Case 2: Assume w.l.o.g. that u is in H^* and v is not in H^*. Let u_1 and u_2 be the neighbors of u in H^*. Note that v can have at most one neighbor distinct from u in H^*. Suppose v does not have any neighbor other than u in H^*. Then the nodes in the set $V(H^*) \setminus \{u, u_1, u_2\}$ are in some connected component F_i and F_i^* contains H^*.

Suppose v has one other neighbor, say v_1, in H^*. Now v_1 and u must have a common neighbor, say u_1, in H^*. Now the nodes in the set $V(H^*) \setminus \{u, u_1, u_2\}$ are in some connected component F_i and it follows that F_i^* contains H^*.

Case 3 *Neither u nor v belongs to H^*.*

Proof of Case 3: Assume w.l.o.g. that $|N(u) \cap V(H^*)| \leq |N(v) \cap V(H^*)|$ There are three subcases to consider:

Case 3.1 *The set $N(u) \cap V(H^*)$ is empty.*

Proof of Case 3.1: If $|N(v) \cap V(H^*)| = 0$ or 1, the unquad hole H^* is preserved in some block F_i^*. Suppose now that $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let v_0 be the common neigbor of v_1 and v_2 in H^*. Now the nodes in $V(H^*) \setminus \{v_0, v_1, v_2\}$ will be in some connected component F_i. If v_0 is in F_i, then the block F_i^* contains H^*. If v_0 is not in F_i, let H" be obtained from H^* by replacing v_0 with v. Now H" belongs to $\mathcal{C}(H^*)$ and the block F_i^* contains H".

Case 3.2 $N(u) \cap V(H^*) = \{u_1\}$.

Proof of Case 3.2: Now $|N(v) \cap V(H^*)| = 1$ or 2. Suppose $N(v) \cap V(H^*) = \{v_1\}$. If u_1 and v_1 are adjacent in H^*, then H^* is preserved in some block F_i^*. Suppse u_1 and v_1 are not adjacent. Let P and Q be the two u_1v_1-subpaths of H^*. The nodes in $V(P) \setminus \{u_1, v_1\}$ will be in some connected component F_i and the nodes in $V(Q) \setminus \{u_1, v_1\}$ will be in some connected component F_j. If the two connected components coincide, H^* is preserved in F_i^*. If the two connected components do not coincide, there is a $3PC(u_1, v_1)$ and Step 3 in Procedure 3 detects this 3-path configuration.

Suppose $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let v_0 be the common neighbor of v_1 and v_2 in H^*. Now u_1 must be adjacent to either v_1 or v_2 for otherwise we have an unquad hole of smaller cardinality than H^*. Suppose u_1 and v_1 are adjacent. Now the nodes in $V(H^*) \setminus \{u_1, v_1, v_0, v_2\}$ will be in some connected component F_i. If v_0 is in the same connected component F_i then H^* is preserved in F_i^*. Suppose v_0 is not in the same connected component F_i. Let H" be obtained from H^* by replacing v_0 with v. Now H" belongs to $\mathcal{C}(H^*)$ and the block F_i^* contains H".

Case 3.3 $N(u) \cap V(H^*) = \{u_1, u_2\}$.

Proof of Case 3.3: Now $N(v) \cap V(H^*) = \{v_1, v_2\}$.Let u_0 be the common neighbor of u_1 and u_2 in H^* and let v_0 be the common neighbor of v_1 and v_2 in H^*. If u_0 is not adjacent to v and v_0 is not adjacent to u there is a tent $\tau(H^*, u, v)$. So assume w.l.o.g. that u_0 coincides with v_1. Then v_2 is adjacent to u_2 and H^* is preserved in some block F_i^*.

Thus in all cases some block F_i^* contains the unquad hole H^* or an unquad hole H" in $\mathcal{C}(H^*)$. Now by Lemma 6.2 the unquad hole H" is clean in F and hence H" clean in F_i^*. By Remark 6.3 the undominated graph F_i^* defined in Step 4 of Procedure 3 must contain an unquad hole in $\mathcal{C}(H^*)$. Repeating the same argument for every undominated block F_i^*, which contains an unquad hole in the family $\mathcal{C}(H^*)$ and is added to the list $\mathcal{M}$, the lemma follows. □

Lemma 6.6 *The number of induced subgraphs in the list $\mathcal{T}$ produced by Procedure 3 is bounded by $|V^c(F)|^2|V^r(F)|^2$.*

Proof: Let $S = N(u) \cup N(v)$ be a double star cutset of F. Let $F_1, \ldots, F_t$ be the connected components of $F \setminus S$ and let $F_1^*, \ldots, F_t^*$ be the corresponding undominated blocks. We prove the following two claims.

Claim 1 *No two distinct undominated blocks contain the same chordless path of length 3.*

Proof of Claim 1: Suppose by contradiction that a chordless path $P = a, b, c, d$ belongs to two distinct undominated blocks F_i^* and F_j^*. Then $\{a, b, c, d\} \subseteq N_F(u) \cup N_F(v)$. There are three cases to consider.

Case 1 *Both nodes u and v belong to $\{a, b, c, d\}$.*

Proof of Case 1: Node d cannot coincide with u for otherwise a and d are adjacent and P is not a chordless path. Similarly d does not coincide with v and a does not coincide with u or v. Hence we can assume that $u = b$ and $v = c$. From Step 4 of Procedure 3 it follows that node a has at least one neighbor in each of the connected components F_i and F_j for otherwise it would have been deleted from one or both the undominated blocks F_i^* and F_j^*. Similarly node d has at least one neighbor in each of the connected components F_i and F_j. Now Step 3 of Procedure 3 detects a 3-path configuration.

Case 2 *Either u or v belongs to $\{a, b, c, d\}$.*

Proof of Case 2: The same argument used in Case 1 shows that node u coincides with b or c. Assume w.l.o.g. that u and b coincide. Now a and c are neighbors of u, d is adjacent to v and both a and d must have at least one neighbor in F_i and F_j. Again Step 3 of Procedure 3 detects a 3-path configuration.

Case 3 *Both u and v do not belong to $\{a, b, c, d\}$.*

Proof of Case 3: As in the previous cases both a and d must have at least one neighbor in F_i, at least one neighbor in F_j and Step 3 of Procedure 3 detects a 3-path configuration. This completes the proof of Claim 1.

Claim 2 *The graph F contains at least one chordless path of length 3 which is not contained in any of the undominated blocks F_i^*.*

Proof of Claim 2: Each of the connected components $F_1, \ldots, F_t$ must contain at least two nodes, since F is an undominated graph. At least one node in F_i must be adjacent to a node in $N_F(u) \cup N_F(v)$. Assume w.l.o.g. that node p_i in F_i is adjacent to a neighbor of v, say d_i. Suppose now no node in F_i is adjacent to a node in $N(u)$. Then by Step 4 of Procedure 3, the undominated block F_i^* does not contain any neighbor of u other than v. This in turn implies that in the same step node u would have been deleted from F_i^*. Now $P = p_i, d_i, v, u$ is a chordless path of length 3 in F but P is not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. So a node in F_i must be adjacent to a node, say s_i, which is a neighbor of u. Repeating the same argument for $j = 1, \ldots, t$, it follows that each connected component F_j contains a node, say w_j, which is adjacent to a node, say $s_j \in N_F(u)$. Suppose now s_j has a neighbor, say g in a connected component F_k, distinct from F_j. Let q be a neighbor of g in F_k. Then $P = q, g, s_j, w_j$ is a chordless path of length 3 which is contained in F but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$.

Suppose now that s_j does not have any neighbor in a connected component, say F_l. Then in Step 4 of Procedure 3, node s_j is deleted from the undominated block F_l^*. Now the path w_l, s_l, u, s_j is a chordless path of length 3 which is contained in F but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. This completes the proof of the claim.

Every undominated block that is added to the list $\mathcal{M}$ in Step 4 of Procedure 3 contains a chordless path of length 3. Hence every undominated block that is added to the list $\mathcal{T}$ in Step 2 contains a chordless path of length 3. By Claim 1, the same chordless path of length 3 is not in any other undominated block that is added to the list $\mathcal{T}$. By Claim 2, it follows that the number of double star cutsets used to decompose the graph F with Procedure 3 is at most $|V^c(F)|^2|V^r(F)|^2$. Hence the lemma follows. $\Box$

7 2-Join Decompositions

In this section we describe a procedure to decompose a bipartite graph into blocks that do not contain an extended star cutset or a stable 2-join, as defined in Definition 1.3. Cornuéjols and Cunningham [17] give a set of rules to construct an efficient algorithm for the identification of a 2-amalgam, of which a 2-join is a special case. The rules are applicable to any graph, not necessarily bipartite. We specialize their rules to construct an algorithm to identify a stable 2-join in a bipartite graph.

Our procedure considers every possible set of six edges to obtain a stable 2-join containing the given set of edges.

PROCEDURE 4

Input A bipartite graph G, not containing an extended star cutset.

Output A list of bipartite graphs $\mathcal{N} = B_1, B_2, \ldots, B_r$, where $r \leq m^2n^2$, satisfying the following properties:

i No graph in the list $\mathcal{N}$ contains a stable 2-join.

ii The graph G is balanced if and only if all the graphs in the list $\mathcal{N}$ are balanced.

Step 1 Let $\mathcal{L} = \{G\}$, and $\mathcal{N} = \emptyset$.

Step 2 If $\mathcal{L} = \emptyset$, stop. Otherwise remove a graph R from $\mathcal{L}$. Enumerate all distinct subsets of six nodes with three nodes in $V^r(R)$ and three nodes in $V^c(R)$ and declare them as unscanned. Go to Step 3.

Step 3 If all subsets in $V^r(R)$ are scanned, add R to $\mathcal{N}$ and return to Step 2. Otherwise choose an unscanned subset U. If the subgraph of R induced by the nodes in U does not consist of two connected components, one being a cycle of length four and the other containing a single edge, declare U as scanned and repeat Step 3.

Otherwise let c_1, r_1, c_2, r_2, c_1 be the cycle and c_3r_3 be the edge and go to Step 4.

Step 4 Define $A = \{c_1, c_2\}$, $B = \{r_1, r_2\}$, $D = \{c_3\}$, $F = \{r_3\}$, $S = \emptyset$, $T = V(R) \setminus U$. Apply Procedure 5 to check whether there exists a stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$. If no such stable 2-join exists, go to Step 5. If a stable 2-join has been identified, construct the blocks R_1^* and R_2^*, add them to the list $\mathcal{L}$ and return to Step 2.

Step 5 Define $A = \{c_1, c_2\}$, $B = \{r_1, r_2\}$, $D = \{r_3\}$, $F = \{c_3\}$, $S = \emptyset$, $T = V(R) \setminus U$. Apply Procedure 5 to check whether there exists a stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$. If no such stable 2-join exists, declare U as scanned and return to Step 3. If a stable 2-join has been identified, construct the blocks R_1^* and R_2^*, add them to the list $\mathcal{L}$ and return to Step 2.

PROCEDURE 5

Input A bipartite graph R and node disjoint bicliques K_{AB} and K_{DF} such that no node in A is adjacent to a node in D and no node in B is adjacent to a node in F.

Output Either a stable 2-join $E^* = E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$ is identified, or no such stable 2-join exists.

Step 1 Let $S = \emptyset$ and $T = V(R) \setminus (A \cup B \cup D \cup F)$. Go to Step 2.

Step 2 All nodes in T are unscanned. Go to Step 3.

Step 3 If all nodes in T are scanned, go to Step 5. Otherwise consider an unscanned node $u \in T$ and go to Step 4.

Step 4 Apply the Rules 1 to 11 in sequence. If any of the Rules 1 to 5 is applicable, there is no stable 2-join E^*, stop. If any of the rules 6 to 11 is applicable, return to Step 2. If none of the rules is applicable, declare u as scanned and return to Step 3.

Rule 1 If u is adjacent to a node in A and a node in F, there is no stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 2 If u is adjacent to a node in B and a node in D, there is no stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 3 If u is adjacent to a node in S, a node in B and a node in F, there is no stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 4 If u is adjacent to a node in S and there exist two nodes $f_1, f_2 \in F$ such that u and f_1 are adjacent but u and f_2 are nonadjacent, there is no stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 5 If u is adjacent to a node in S and there exist two nodes $b_1, b_2 \in B$ such that u and b_1 are adjacent but u and b_2 are nonadjacent, there is no stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 6 If u is adjacent to a node in A and a node in D, remove u from T and add u to S.

Rule 7 If u is not adjacent to any node in $A \cup B$ and there exist two nodes $d_1, d_2 \in D$ such that u and d_1 are adjacent but u and d_2 are nonadjacent, remove u from T and add it to S.

Rule 8 If u is not adjacent to any node in $D \cup F$ and there exist two nodes $a_1, a_2 \in A$ such that u and a_1 are adjacent but u and a_2 are nonadjacent, remove u from T and add it to S.

Rule 9 If u is adjacent to all nodes in F and to at least one node in S, but u is not adjacent to any node in $A \cup B$, remove u from T and add it to D.

Rule 10 If u is adjacent to all nodes in B and to at least one node in S, but u is not adjacent to any node in $D \cup F$, remove u from T and add it to A.

Rule 11 If u is adjacent to at least one node in S, but u is not adjacent to any node in $B \cup F$, remove u from T and add it to S.

Step 5 Remove from T every node u that is adjacent to all nodes in A and add u to B. Remove from T every node v that is adjacent to all nodes in D and add v to F. Let $A' = A$, $B' = B$, $D' = D$ and $F' = F$. Now $E(K_{A'B'}) \cup E(K_{D'F'})$ defines a stable 2-join, separating $A' \cup D' \cup S$ from $B' \cup F' \cup T$.

Remark 7.1 *The rules in Step 4 of Procedure 5 are forcing in the sense that if any of them holds, node u must be removed from T and added to one of the sets A, D or S if there is a stable 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$. Rules 1 to 5 detect a contradiction that arises as a consequence of removing u from T and adding to one of the sets A, D or S. If all the nodes in T are scanned in Step 3 of Procedure 5, it follows that the bicliques identified by Procedure 5 define a stable 2-join. Moreover the graphs in the list $\mathcal{N}$ do not contain a stable 2-join.*

Lemma 7.2 *Let G be a bipartite graph not containing an extended star cutset, and $\mathcal{N} = B_1, B_2, \ldots, B_r$ be the list of graphs produced from G by Procedure 4. Then $r \leq m^2n^2$ and the graphs in $\mathcal{N} = B_1, B_2, \ldots, B_r$ do not contain an extended star cutset or a stable 2-join. Moreover if G is balanced all the graphs in the list $\mathcal{N}$ are balanced and if G is not balanced at least one graph in the list $\mathcal{N}$ is not balanced.*

Proof: Let G be a bipartite graph, not containing an extended star cutset, that is decomposed by Procedure 4. Suppose $E^* = E(K_{AB}) \cup E(K_{DF})$ is a stable 2-join of G that separates G_1 from G_2 and let G_1^* and G_2^* be the corresponding blocks. We now show that G_1^* and G_2^* do not contain an extended star cutset. Suppose G_1^* contains an extended star cutset $S = (x; X; Y; N)$. Let the nodes in A and D belong to G_1 and let nodes b and f in G_1^* represent the nodes in B and F respectively. The nodes b and f are connected by a path P_{bf} which is of length 3, 4, 5 or 6.There are four cases to consider.

Case 1 *Node x coincides with b or f.*

Proof of Case 1: Assume w.l.o.g. that x coincides with b. Since P_{bf} is of length at least 3 and E^* defines a stable 2-join, it follows that node f and the nodes in D are not in S. Hence S separates the nodes in D from a node in $G_1 \setminus A$. If $X = \{x\}$, then S is a star cutset of G_1^* separating the nodes in D from a node in $G_1 \setminus A$. Now every node in B defines a star cutset of G separating the nodes in D from a node in $G_1 \setminus A$. Hence X must contain at least two nodes. Then at least two nodes in A are contained in Y. Let x^* be a node in B . Let $N^* = N_G(x^*) \setminus Y$ and $X^* = (X \setminus \{x\}) \cup B$. Now $S^* = (x^*, X^*, Y, N^*)$ defines an extended star cutset of G separating the nodes in D from a node in $G_1 \setminus A$.

Case 2 *Node x is an intermediate node of P_{bf}.*

Proof of Case 2: At least one of the nodes b or f is not in S since P_{bf} is of length at least 3. Assume w.l.o.g. that node f is not in S. Now S is a star cutset of G_1^* separating the nodes in D from a node in $G_1 \setminus A$. Then node d must be a star cutset of G_1^* separating the nodes in D from a node in $G_1 \setminus A$ and we are in Case 1.

Case 3 *Node x is in A or in D.*

Proof of Case 3: Assume w.l.o.g. that x is in A. Now node $f \notin X$ since E^* defines a stable 2-join. Then S is an extended star cutset of G_1^* separating f from a node in $G_1 \setminus S$. If node b is not in S, it follows that X is a star cutset of G separating the nodes in F from a node in $G_1 \setminus S$. Suppose now node b is in S. Then it follows that b is in N. Let $N^* = (N \setminus \{b\}) \cup B$. Now $S^* = (x, X, Y, N^*)$ is an extended star cutset of G separating the nodes in F from a node in G_1.

Case 4 *Node x is in G_1 but not in $A \cup D$.*

Proof of Case 4: Now node b or f is not in S. Assume w.l.o.g. that f is not in S. Then S is an extended star cutset of G_1^* separating node f from a node in $G_1 \setminus S$. If node b is not in S it follows that X is an extended star cutset of G separating the nodes in F from a node in $G_1 \setminus S$. Suppose now node b is in S. Then b must be in X and Y must contain at least two nodes in A. Let $X^* = (X \setminus \{b\}) \cup B$. Now $S^* = (x, X^*, Y, N)$ is an extended star cutset of G separating the nodes in F from a node in G_1.

Hence G_1^* does not contain an extended star cutset. By symmetry, G_2^* does not contain an extended star cutset. Now repeating the same argument for every graph that is added to the list $\mathcal{L}$, it follows that every graph in the list $\mathcal{N}$ produced by Procedure 4 does not contain an extended star cutset. By Remark 7.1, the graphs in the list $\mathcal{N}$ do not contain a stable 2-join. Now a repeated application of Theorem 2.4[I] shows that if G is balanced, all the graphs in the list $\mathcal{N}$ are balanced and if G is not balanced at least one graph in the list $\mathcal{N}$ is not balanced.

In order to complete the proof of the lemma we now show that the number of graphs in the list $\mathcal{N}$ is less than or equal to m^2n^2. Let G_1^* and G_2^* be the two blocks defined by a stable 2-join of the graph G. Now a cycle of length 4 in one of the two bicliques of the stable 2-join is in G but not in G_1^* or G_2^*. Moreover neither G_1^* nor G_2^* contains a cycle of length 4 that is not contained in G. Hence the number of stable 2-joins identified by Procedure 4 is less than or equal to m^2n^2 and consequently the number of graphs in the list $\mathcal{N}$ is less than or equal to m^2n^2.

This completes the proof of the lemma. □

8 Validity of the Algorithm

We now prove the validity of the algorithm given in Section 3.

Theorem 8.1 *The running time of the algorithm described in Section 3 is bounded from above by a polynomial function of the cardinalities m and n of the node sets V^r and V^c respectively. Moreover the algorithm correctly identifies a bipartite graph G as balanced or not.*

Proof: The algorithm described in Section 3 applies the four procedures given in Sections 4 to 7 respectively. The running time of each of these procedures has been shown in its respective section to be bounded from above by a polynomial function of m and n. The algorithms in [11] and [25], to check whether a bipartite graph contains an unquad cycle, are bounded from above by a polynomial function of m and n. Hence the running time of the algorithm described in Section 3 is bounded from above by a polynomial function of m and n.

Suppose G is balanced. Clearly G cannot contain a short 3-wheel or a 3-path configuration. All the induced subgraphs of G are balanced and the graphs produced by Procedures 2 and 3 are balanced. Consequently, by Lemma 7.2, all the graphs in the final list $\mathcal{N}$ produced by Procedure 4 are balanced and do not contain an extended star cutset or a stable 2-join. Now by Theorem 1.4 every graph in the list $\mathcal{N}$ does not contain an unquad cycle. Then Step 5 of the algorithm identifies G as balanced.

Suppose G is not balanced. If G contains a short 3-wheel, Step 1 of the algorithm identifies G as not balanced. Suppose G does not contain a short 3-wheel. Clearly G contains an unquad hole of smallest cardinality. Now by Lemma 5.2 one of the induced subgraphs of G, say G_i, in the list produced by Procedure 2 contains an unquad hole H^*, of smallest cardinality, which is clean in G_i. Now G_i is one of the graphs considered for double star cutset decompositions by Procedure 3. By Lemma 6.4, Procedure 3 either detects a 3-path configuration or one of the undominated blocks, say F, in the final list produced by Procedure 3 contains an unquad hole in the family $\mathcal{C}(H^*)$. In the former case clearly G is not balanced. In the latter case, F is one of the graphs considered for stable 2-join decompositions by Procedure 4. Now by Lemma 7.2, one of the blocks, say B_j, produced by Procedure 4 is not balanced. Clearly the block B_j contains an unquad hole and hence an unquad cycle. Hence Step 5 of the algorithm identifies G as not balanced.

This completes the proof of the theorem. □

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) DECOMPOSITION OF BALANCED MATRICES. PART VII: A POLYNOMIAL RECOGNITION ALGORITHM		5. TYPE OF REPORT & PERIOD COVERED Technical Report, Oct 1991
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(S) Michele Conforti Gerard Cornuejols M.R. Rao		8. CONTRACT OR GRANT NUMBER(S) DDM-8800281 DDM-8901495 DDM-9001705
9. PERFORMING ORGANIZATION NAME AND ADDRESS Graduate School of Industrial Administration Carnegie Mellon University Pittsburgh, PA 15213		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 434) Arlington, VA 22217		12. REPORT DATE October 1991
		13. NUMBER OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS (If different from Controlling Office)		15. SECURITY CLASS (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G:

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1, K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1, K_2 disconnects G.
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G, where S can be partitioned into three disjoint sets T, A, N, such that $T \neq \emptyset$, some node $x \in T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$, then $|A| \geq 2$ and every node of T is adjacent to every node of A.

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r, V^c; E)$ as follows: The node nets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij}=1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.