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TESTING THE MINIMAL REPAIR ASSUMPTION IN AN IMPERFECT REPAIR MODEL

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Abstract

We propose two nonparametric tests of the assumption that imperfectly repaired systems are minimally repaired in the models of Brown and Proschan (1983) and Block, Borges, and Savits (BBS) (1985). The large sample theory for these tests is derived from the asymptotic joint distribution of the survival function estimator of Whitaker and Samaniego (1989) and the ordinary empirical survival function based on the initial failure times of new, or perfectly repaired systems. Simulation results are also provided for the null hypothesis case, and under the alternatives proposed by Kijima (1989). Models assuming minimal repair specify that upon repair, a failed system is returned to the working state, while the effective age of the system is held constant; that is, the distribution of the time until the next failure of the repaired system is the same as for a system of the same age which has not yet failed. These models are common in the literature of operations research and reliability, and probabilistic results and the recently proposed inferential procedures of Whitaker and Samaniego (1989) and Hollander, Presnell, and Sethuraman (1989) depend on the minimal repair assumption. Though tests have been proposed for goodness of fit of the model when a particular form of the distribution is assumed, we know of no previous proposal of a nonparametric method to test this assumption.

KEYWORDS: Imperfect repair, repairable system, nonhomogeneous Poisson process, martingale.

1. INTRODUCTION

Models which allow for imperfect repair are clearly needed for reliability analysis of repairable systems, and minimal repair provides a mathematically tractable approach to this modeling problem. Models involving minimal repair appear frequently in the literature of reliability and operations research (see Ascher and Feingold (1984) for references). A particular example is the age-dependent minimal repair model (BBS model) of Block, Borges, and Savits (1985), which generalizes the imperfect repair model of Brown and Proschan (1983). In the BBS model, a system with survival time distribution F is put on test at time zero. Upon failure at age t, one of two types of repair is performed: with probability $p(t)$ a perfect repair is performed and the system is returned to the "good-as-new" state; otherwise, a minimal repair is performed and the device is returned to the working state, but is only as good as a working item of age equal to the age of the device at failure. In the former case, we consider the effective age of the system to be returned to zero (a new item has effective age 0), while in the latter case the effective age of the system is unchanged from the effective age at failure. Thus, if a minimal repair is performed on an item failing at age t, then the repaired item has survival function

$$\overline{F}(s|\,t) = \frac{\overline{F}(s+t)}{\overline{F}(t)}, \quad s \geq 0,$$

where we use the notation $\overline{F}$ to indicate the survival function, $1-F$, of a distribution F. The process is continued after repair, with each subsequent failure being followed by a perfect repair with probability $p(t)$, or a minimal repair with probability $q(t) = 1 - p(t)$, where t is the effective age of the failed device.

Whitaker and Samaniego (1989) and Hollander, Presnell, and Sethuraman (HPS) (1989) propose several inferential procedures for this model. These procedures and their asymptotic distribution theory depend strongly on the assumption that imperfectly repaired systems are minimally repaired, although in practice this assumption is tenuous. Minimal repair is generally motivated as an approximation to the situation where some failing component(s) of a complex multi-component system is replaced or repaired, but we would not expect the model to fit exactly the repairs of an actual system. This is discussed by Bergman (1985), who distinguishes between *statistical* minimal repair (as defined above) and *physical* minimal repair, in which the failed system is restored to its exact physical condition just before failure. Some interesting arguments against the indiscriminate use of minimal repair models are also given by Arjas and Norros (1989) and Natvig (1990), who refer to the type of minimal repair considered here as 'black box' minimal repair, and contrast this with models which take into account available information about the condition of the system. In spite of the widespread use and criticism of minimal repair, however, we know of no previous proposal of a method for testing the assumption of minimal repair which does not assume a particular form for the underlying survival distribution of the system.

In this paper we propose two nonparametric tests of the minimal repair assumption in the BBS model. These tests are based on the notion of contrasting two available estimators of the system survival function. We assume that n systems are observed under the BBS model, each until the time of its first perfect repair (or equivalently, a single system is observed until the time of its n^{th} perfect repair). Regardless of the mode of repair, a consistent estimator of the system survival function is available, namely the empirical survival function, $\widehat{\overline{F}}_e$, based on the initial failure times of the systems under observation. Under the BBS model, the estimator of Whitaker and Samaniego (1989), $\widehat{\overline{F}}$, also provides a consistent estimator of the system survival function, while if the minimal repair assumption of the model fails to hold, this estimator might diverge from the true system survival function. In Section 2. we propose a Kolmogorov-Smirnov type test based on the maximum absolute difference between $\widehat{\overline{F}}$ and $\widehat{\overline{F}}_e$, and a test based on a Mann-Whitney-Wilcoxon type statistic of the form $\int \widehat{F}_e \, d\widehat{F}$. Theoretical justification of these procedures is given and they are applied to the data of Proschan (1963). In Section 3. the results of a simulation study of these procedures are given. Finally, in Appendix A. we give proofs of the theoretical results presented in Section 2..

2. TWO TESTS OF MINIMAL REPAIR

2.1 The Model

The following model for sampling from an age-dependent minimal repair process is given in (HPS) and is repeated here for convenience. To simplify the exposition, we assume that F is continuous and that $F(t) < 1$ for all $t < \infty$, although we generally give expressions applicable to the more general case.

A sequence of failure ages obtained under a model of perennial minimal repair may be defined as follows. Let F be a life distribution, and let $\{X_0 \equiv 0, X_1, X_2, \dots\}$ be a record-value sequence based on F; that is, $(X_k)_{k=1}^{\infty}$ is a Markov process with the conditional distribution of X_k given $X_0, \dots, X_{k-1}$ being given by $\overline{F}(t \mid X_{k-1}) = \overline{F}(t)/\overline{F}(X_{k-1})$, for $t \geq X_{k-1}$ and $k \geq 1$. Note that the counting process

$$N^*(t) = \#\{\, k : X_k \leq t \,\}$$

is just a nonhomogeneous Poisson process with mean function equal to the cumulative hazard function of F,

$$\Lambda(t) = \int_0^t \frac{dF(s)}{\overline{F}(s-)} \, .$$

Perfect repair is now introduced through the use of independent uniform $(0,1)$ random variables, $\{U_1, U_2, \dots\}$, also independent of $(X_k)_{k=1}^{\infty}$. Defining $\delta_k = I(U_k < p(X_k))$ and $\nu = \inf\{k : \delta_k = 1\}$, where we take $\inf \emptyset = \infty$, we see that $P(\delta_k = 1 | X_1, \dots, X_k, \delta_1, \dots, \delta_{k-1}) = p(X_k)$. Clearly, observation of $\{(X_1, \delta_1), \dots, (X_\nu, \delta_\nu)\}$ is equivalent to observation of a system under the BBS model until the time of the first perfect repair, although ν may in general be infinite.

Let H be the subdistribution function defined by

$$H(t) = P(X_\nu \le t, \nu < \infty).$$

Then H has cumulative hazard function

$$\Lambda_H(t) = \int_0^t p(s)\, d\Lambda(s), \tag{2.1}$$

and H is a proper distribution function if and only if

$$\int_0^\infty p(s)\, d\Lambda(s) = +\infty. \tag{2.2}$$

In this case, X_ν is almost surely finite with distribution function H. We will assume that (2.2) holds for the remainder of the paper.

2.2 Theoretical Results

Now suppose that we observe n such systems, so that we have independent record value sequences $\{(X_{jk})_{k=1}^{\infty} : 1 \le j \le n\}$ from F, independent uniform random variables $\{U_{jk} : k \ge 1, 1 \le j \le n\}$, and we observe $\{(X_{jk}, \delta_{jk}) : 1 \le k \le \nu_j, 1 \le j \le n\}$. Let $\widehat{F}_e$ be the empirical cumulative distribution function formed in the usual way from the initial failure ages of the system, $X_{11}, \dots, X_{n1}$, let $\widehat{H}$ be the empirical c.d.f. of $X_{1\nu_1}, \dots, X_{n\nu_n}$, and let

$$N_j^*(t) = \#\{k : X_{jk} \le t\}, \tag{2.3}$$

$$N_j(t) = N_j^*(t \wedge X_{j\nu_j}), \tag{2.4}$$

and

$$N(t) = \sum_{j=1}^{n} N_j(t). \tag{2.5}$$

Note that $N(t)$ is the total number of *observed* failures through time t, and $n\widehat{\bar{H}}(t-)$ is the number of systems still under observation, or "at risk," just before t. Let T be the first failure age at which only one item is at risk:

$$T = \sup\{t : \widehat{\bar{H}}(t) = \frac{1}{n}\} = \min\{X_{(i)} : \widehat{\bar{H}}(X_{(i)}-) = \frac{1}{n}\},$$

where the $X_{(i)}$ are the ordered values of the combined observed failure ages of the n BBS processes. A natural estimator of the cumulative hazard function is then

$$\widehat{\Lambda}(t) = \int_0^t \frac{J(s)}{n\widehat{\overline{H}}(s-)}\, dN(s) = \sum_{X_{(i)} \le t \wedge T} \frac{1}{n\widehat{\overline{H}}(X_{(i)}-)}. \tag{2.6}$$

where $J(s) = I(s \le T)$, and we take $\frac{0}{0} = 0$. With this notation, the survival function estimator of Whitaker and Samaniego (1989) is

$$\widehat{\overline{F}}(t) = \prod_{s \le t}(1 - \Delta\widehat{\Lambda}(s)) = \prod_{X_{(i)} \le t \wedge T}\Big(1 - \frac{1}{n\widehat{\overline{H}}(X_{(i)}-)}\Big). \tag{2.7}$$

We define

$$C(t) = \int_0^t \frac{dF(s)}{\overline{H}(s-)\overline{F}(s)}, \qquad \widehat{C}(t) = \int_0^t \frac{d\widehat{F}(s)}{\widehat{\overline{H}}(s-)\widehat{\overline{F}}(s)}, \tag{2.8}$$

$$L(t) = \frac{1}{\overline{F}(t)} - 1 - C(t), \qquad \widehat{L}(t) = \frac{1}{\widehat{\overline{F}}(t)} - 1 - \widehat{C}(t), \tag{2.9}$$

and

$$G = L/(1+L), \qquad \widehat{G} = \widehat{L}/(1+\widehat{L}). \tag{2.10}$$

The following results are proven in Appendix A.:

Theorem 2.1. As $n \to \infty$,

$$\frac{\sqrt{n}(\widehat{F} - \widehat{F}_e)}{\overline{F}} \xrightarrow{\mathcal{D}} B(L) \qquad \text{in } D[0,\infty),$$

where B is Brownian motion on $[0,\infty)$.

Corollary 2.2.

$$\sqrt{n}\frac{\overline{G}}{\overline{F}}\big(\widehat{F} - \widehat{F}_e\big) \xrightarrow{\mathcal{D}} B^0(G) \quad \text{in } D[0,\infty),$$

and

$$\sqrt{n}\frac{\widehat{\overline{G}}}{\widehat{\overline{F}}}\big(\widehat{F} - \widehat{F}_e\big) \xrightarrow{\mathcal{D}} B^0(G) \quad \text{in } D[0,\infty),$$

where B^0 is the Brownian bridge on $[0,1]$.

Corollary 2.3.

$$\sqrt{n}(\widehat{F} - \widehat{F}_e) \xrightarrow{\mathcal{D}} \overline{F} \times B(L) \qquad \text{in } D[0,\infty].$$

Corollary 2.4. Let $V = \int_0^\infty \widehat{F}_e \, d\widehat{F}$. Then

$$\sqrt{n}\left(V - \frac{1}{2}\right) \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

where

$$\sigma^2 = \frac{1}{12} - \frac{1}{4}\int_0^\infty \frac{\overline{F}^3(s)}{\overline{H}(s-)} \, dF(s). \tag{2.11}$$

The result of Corollary 2.2 leads to a Kolmogorov-Smirnov type test of the minimal repair assumption, to be carried out by referring the statistic

$$S_\tau = \sup_{0 \le t \le \tau} \sqrt{n} \frac{\widehat{G}(t)}{\widehat{\overline{F}}(t)} \left|\widehat{F}(t) - \widehat{F}_e(t)\right|$$

to a table of the distribution of the supremum of the absolute value of the Brownian bridge over the interval $[0, \widehat{G}(\tau)]$ (Koziol and Byar 1975; Hall and Wellner 1980). Corollary 2.4, on the other hand, suggests that we refer the statistic

$$V^* = \sqrt{n}\left(V - \frac{1}{2}\right) / \hat{\sigma}_V$$

to a standard normal distribution, where

$$\hat{\sigma}_V^2 = \frac{1}{12} - \frac{1}{4}\int_0^\infty \frac{\widehat{\overline{F}}^3(s)}{\widehat{\overline{H}}(s-)} \, d\widehat{F}(s).$$

(The consistency of $\hat{\sigma}$ is demonstrated in HPS, pp. 27–28.) Other tests based on the simple idea of searching for a statistically significant discrepancy between $\widehat{F}_e$ and $\widehat{F}$ are also possible, but we will consider only these two.

Remark 2.5. A strictly monotone transformation of the time axis does not alter the value of V^* or the value of S_τ, if τ is also transformed. But if $p(t) \equiv p$ is a constant function, then the transformation $X'_{jk} = \Lambda(X_{jk})$ reduces us to the situation of observing the BBS model with the same constant $p(\cdot)$ and with exponential F. Thus, when $p(\cdot)$ is a constant function, the distribution of V^* under the null hypothesis does not depend on F. If u is fixed and $\tau = F^{-1}(u)$, then the null distribution of S_τ is also independent of F. Of course the distributions in both cases depend on p and in the latter case on u.

2.3 An Example

As an example, we have applied the procedures above to the Boeing air conditioner data of Proschan (1963). The data are reproduced in Table 1. For this analysis, we have treated the intervals between failures as inter-failure times

Plane Number												
7907	7908	7909	7910	7911	7912	7913	7914	7915	7916	7917	8044	8045
194	413	90	74	55	23	97	50	359	50	130	487	102
15	14	10	57	320	261	51	44	9	254	493	18	209
41	58	60	48	56	87	11	102	12	5		100	14
29	37	186	29	104	7	4	72	270	283		7	57
33	100	61	502	220	120	141	22	603	35		98	54
181	65	49	12	239	14	18	39	3	12		5	32
	9	14	70	47	62	142	3	104			85	67
	169	24	21	246	47	68	15	2			91	59
	447	56	29	176	225	77	197	438			43	134
	184	20	386	182	71	80	188				230	152
	36	79	59	33	246	1	79				3	27
	201	84	27	**	21	16	88				130	14
	118	44	**	15	42	106	46					230
	**	59	153	104	20	206	5					66
	34	29	26	35	5	82	5					61
	31	118	326		12	54	36					34
	18	25			120	31	22					
	18	156			11	216	139					
	67	310			3	46	210					
	57	76			14	111	97					
	62	26			71	39	30					
	7	44			11	63	23					
	22	23			14	18	13					
	34	62			11	191	14					
		**			16	18						
		130			90	163						
		208			1	24						
		70			16							
		101			52							
		208			95							

Table 1. Intervals Between Failures of Boeing Air Conditioner Systems.

between minimal repairs. Proschan omits any failure interval immediately following a major overhaul (indicated by ** in Table 1), and we have omitted the intervals following such an overhaul from our analysis, since it is impossible to determine the age of the unit after the overhaul. Thus the values below the **'s in Table 1 are not used. We have treated the age at which a major overhaul occurs as the time of the first perfect repair for that airplane. This affects planes 7908, 7909, 7910, and 7911. For purposes of this example, we treat the last observed failure ages of the remaining planes as the times of their first perfect repair.

We have arbitrarily decided to compute the S statistic over the interval from 0 to 500 hours. The value of S in this case is 0.7705, and $\widehat{G}(500) =$ .9902. This value of S is less than the 50th percentile of its asymptotic null distribution, so that the test yields no evidence against the minimal repair assumption. For the Wilcoxon-like test statistic, $V = 0.4984$, $\hat{\sigma}_V = 0.1753$, and $V^* = -0.03323$, again yielding no evidence against the minimal repair assumption. In Figure 1, we have plotted $\widehat{\widehat{F}}$ and $\widehat{\widehat{F}}_e$ on the interval $[0, 500]$. In agreement with the results of the test procedures, there is little visual evidence to suggest a meaningful discrepancy between the estimators. As we shall see in Section 3., however, these tests appear to have very low power for a sample

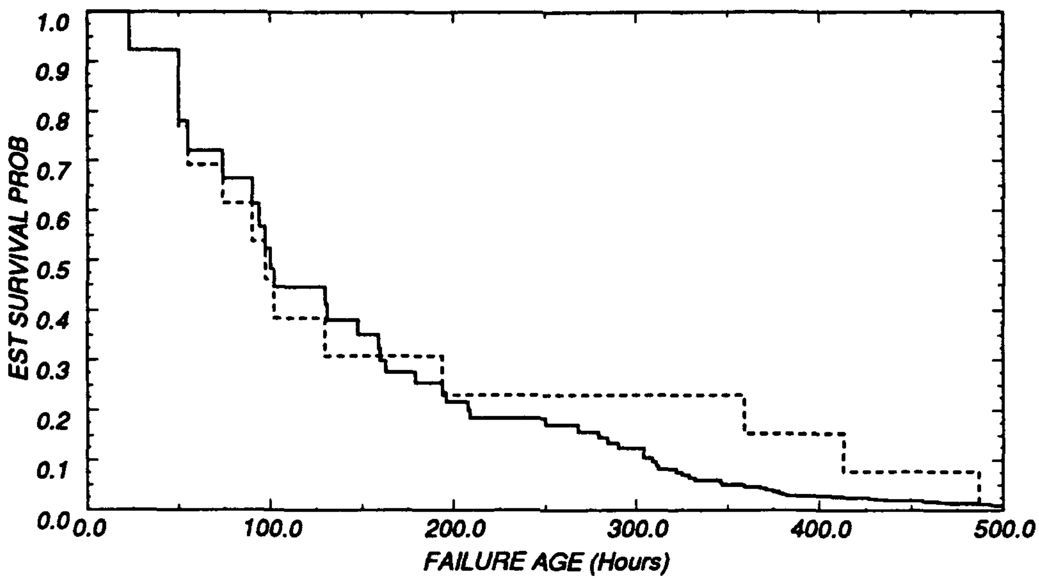

Figure 1. Plot of $\hat{\hat{F}}$ (solid) and $\hat{\hat{F}}_e$ (dashed) for Boeing Data.

size as small as 13, so that a larger sample size is needed to effectively test the minimal repair assumption in this example.

3. SIMULATION RESULTS

We have carried out simulation studies of both the size and power of the tests proposed. Computations were carried out on the University of Florida Statistics Department's system of DEC workstations. Simulation programs were written in FORTRAN. The uniform generator used as a basis for all other random numbers was the universal random number generator described in Marsaglia, Tsang, and Zaman (1990). Exponential random variables were generated using the function REXP given by Marsaglia and Tsang (1984). The memoryless property of the exponential distribution makes it easy to generate record values by simply adding an independent exponential to the current record value. For the general gamma(a) distribution, the initial record values were generated in the obvious way: generate gamma variates until the current record is exceeded. These gamma random variables were generated using the squeeze method of Marsaglia (1977). Once this sequence produced a value greater than a (arbitrary, but it is clear that at some point the starting algorithm must be abandoned), an algorithm described by Marsaglia and Tsang (1984) for generating random variables from the tail of a distribution was employed.

The sample sizes examined were 10, 20, 30, 50, 100, and 200. For each sample generated, both S and V^* were calculated. Since the interval $[0, \tau]$

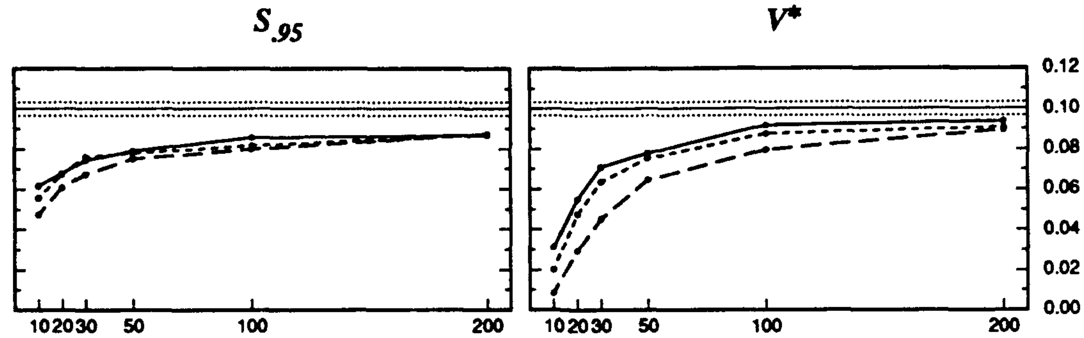

Figure 2. Size of S and V^* with nominal size 0.10, with p held constant at 0.50 (—), 0.25 (-), and 0.10 (solid). Estimates based on 30,000 iterations.

on which the statistic is computed may affect the behavior of the S test, we collected results of computing the statistic to the 90^{th}, 95^{th}, and 99^{th} percentile of the underlying distribution By a slight abuse of our earlier notation, we will denote these statistics by $S_{.90}$, etc. The tables of Koziol and Byar (1975) were used to find the appropriate value of $\lambda_\alpha(\widehat{K}(\tau))$, after first rounding $\widehat{K}(\tau)$ to the nearest tenth. The BBS process was simulated with constant p taking values .10, .25, and .50.

The estimated sizes of the tests based on $S_{.95}$ and V^* with nominal size $\alpha = 0.10$ are presented graphically in Figure 2. (The results for $S_{.90}$ and $S_{.99}$ are similar to those for $S_{.95}$.) These estimates are based on 30,000 iterations of the simulation. Recall that under the null hypothesis with constant p, the null distributions of the statistics considered here do not depend on the underlying F. The light dotted lines at $\alpha = 0.10 \pm 1.96 \times 0.3/\sqrt{30,000}$ are provided for reference. Both the $S_{.95}$ and the V^* tests are conservative at the sample sizes examined, with the discrepancy between the nominal and the true sizes decreasing monotonically with the sample size. The value of p also has an effect on the size, particularly in the case of V^*, with larger p yielding a smaller size for the test. Since the tests are at least not anticonservative, we continue by examining their power.

To study the power of the test procedures, we consider two alternatives to minimal repair, Models 1 and 2 of Kijima (1989). Let X_k represent the actual age (as measured by the clock) and let Y_k represent the effective age (Kijima calls this the *virtual* age) of the system at the k^{th}failure. Thus the conditional survival function of $\Delta X_k = X_k - X_{k-1}$ given $\{X_1, \ldots, X_{k-1}, Y_1, \ldots, Y_{k-1}\}$ is $\overline{F}(s | Y_{k-1}) = \overline{F}(s+Y_{k-1})/\overline{F}(Y_{k-1})$, for $s \geq 0$. In Kijima's Model 1, the effective age of the system after the k^{th}repair is

$$Y_k = Y_{k-1} + A_k \Delta X_k \,,$$

where A_k is a random variable taking values in the unit interval independently of $Y_1, \ldots, Y_{k-1}$ and $X_1, \ldots, X_k$. The idea here is that the repair affects only aging of the system accumulated since the last repair. In Kijima's Model 2 on the other hand, the effective age of the system after the k^{th}repair is given by

$$Y_k = A_k(Y_{k-1} + \Delta X_k)\,,$$

where A_k is as in Model 1.

In our simulation study, the A_k's in these models were taken to be constant (variously, 0.75, 0.50, 0.25, and 0.10). The underlying distributions considered were gamma with shape parameters 2, 4, and 6. In lieu of tables of the simulation results, we provide Figures 3–8. Again we provide results only for $S_{.95}$, as the results for $S_{.90}$ and $S_{.99}$ were similar. These figures support the following conclusions:

- The power of both tests increases as A_k decreases. This is to be expected since values of A_k close to one in either model correspond to models which are "close" to the minimal repair model.

- With all other factors being constant, the power of both tests tends to be higher for Model 2 than for Model 1. This is not surprising, since for the same value of A_k, repair under Model 2 has a greater effect on the effective age than repair under Model 1.

- The power of both tests increases as p decreases. Again, this is expected, since smaller values of p correspond to the observation of more repairs and hence more information about the mode of repair.

- The power of both tests increases as the gamma shape parameter a increases away from 1. This again is not surprising, since values of a close to 1 correspond to an underlying distribution closer to the exponential, for which both Models 1 and 2 are equivalent to the minimal repair model.

- The test based on V^* tends to have greater power for large sample sizes (> 30), while the test based on $S_{.95}$ has greater power for small sample sizes (< 30). We have no obvious explanation for this behavior.

Since we know of no competing procedures which do not assume a particular form for F, it is difficult to make an overall qualitative judgement of the power of these tests. In either of the models considered, a procedure based on estimation of a parameter of the distribution of A_k (such as the mean when the A_k's are random and i. i. d.) might have better power, but might not perform as well against other types of alternatives as our more general approach.

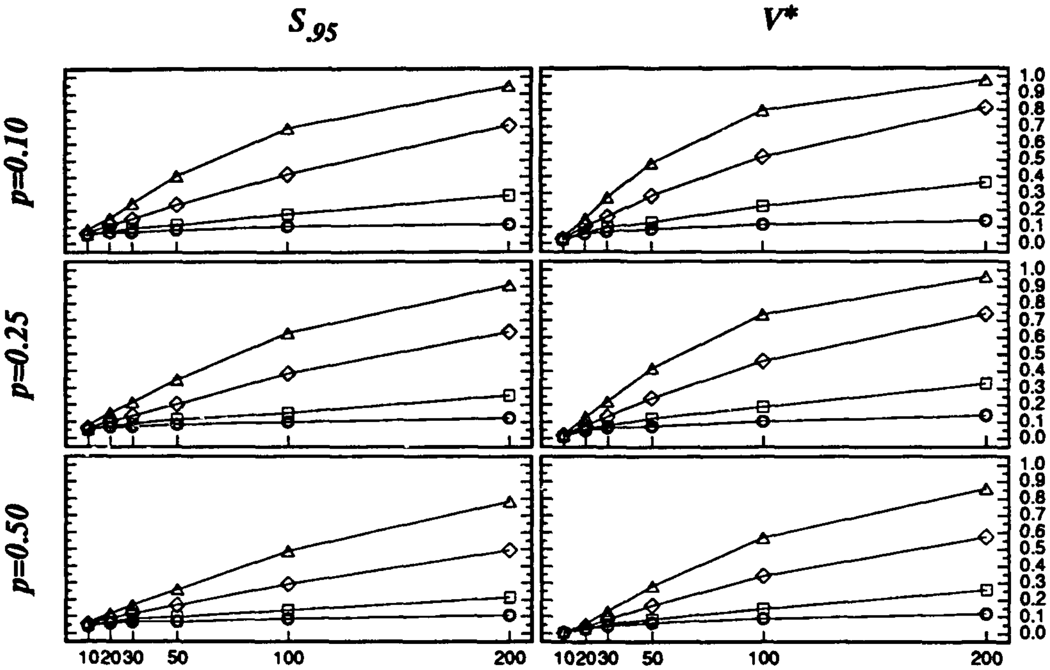

Figure 3. Estimated power (5000 simulations) of $S_{.95}$ and V^* against Kijima's Model 1 with F = Gamma(2), and $A_n = 0.75(\circ), 0.50(\square), 0.25(\diamond), 0.10(\triangle)$.

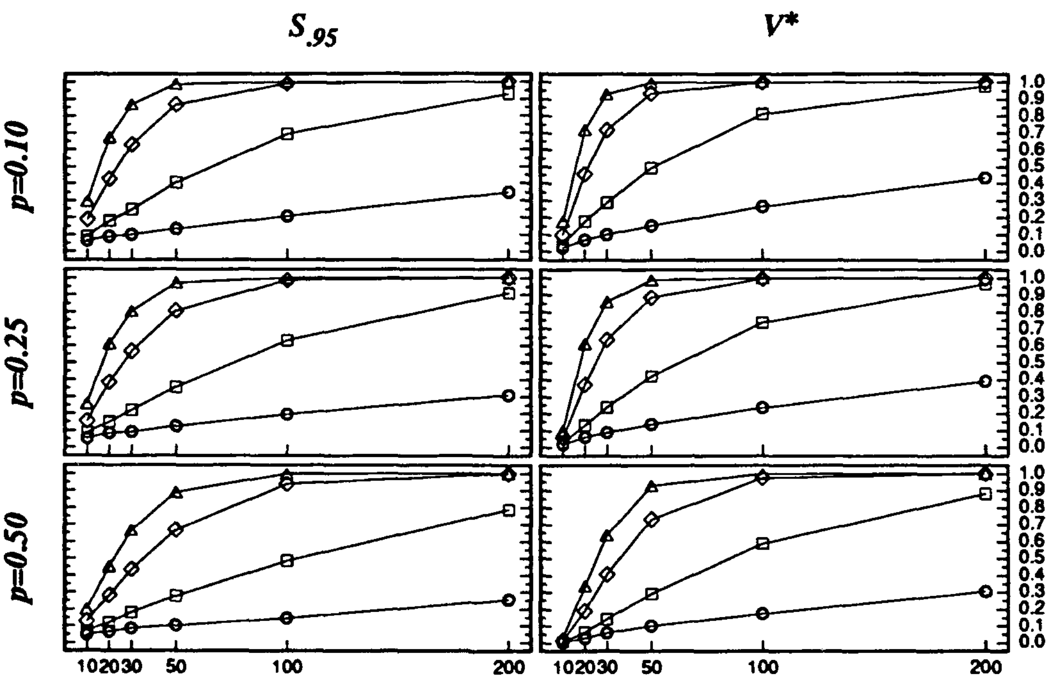

Figure 4. Estimated power (5000 simulations) of $S_{.95}$ and V^* against Kijima's Model 1 with F = Gamma(4), and $A_n = 0.75(\circ), 0.50(\square), 0.25(\diamond), 0.10(\triangle)$.

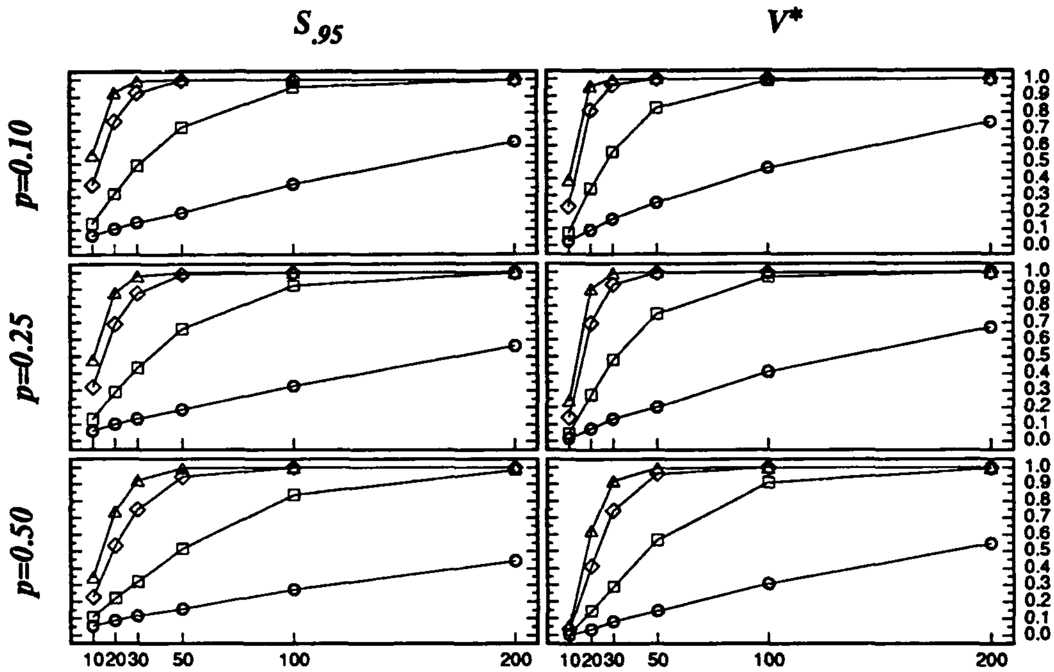

Figure 5. Estimated power (5000 simulations) of $S_{.95}$ and V^* against Kijima's Model 1 with F = Gamma(6), and $A_n = 0.75(\circ), 0.50(\square), 0.25(\diamond), 0.10(\triangle)$.

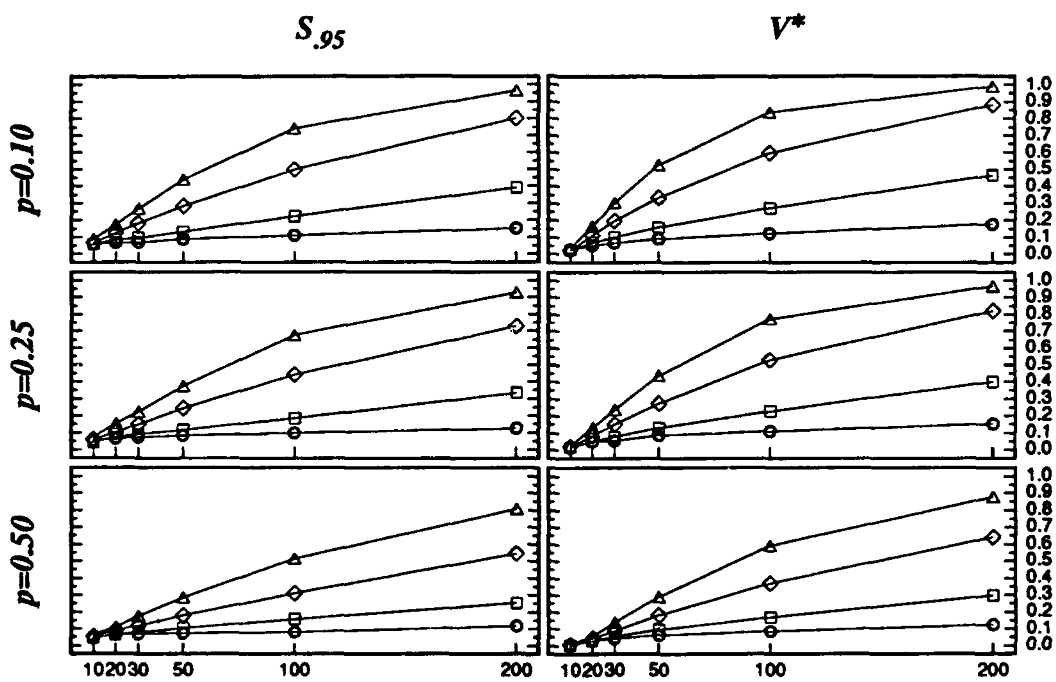

Figure 6. Estimated power (5000 simulations) of $S_{.95}$ and V^* against Kijima's Model 2 with F = Gamma(2), and $A_n = 0.75(\circ), 0.50(\square), 0.25(\diamond), 0.10(\triangle)$.

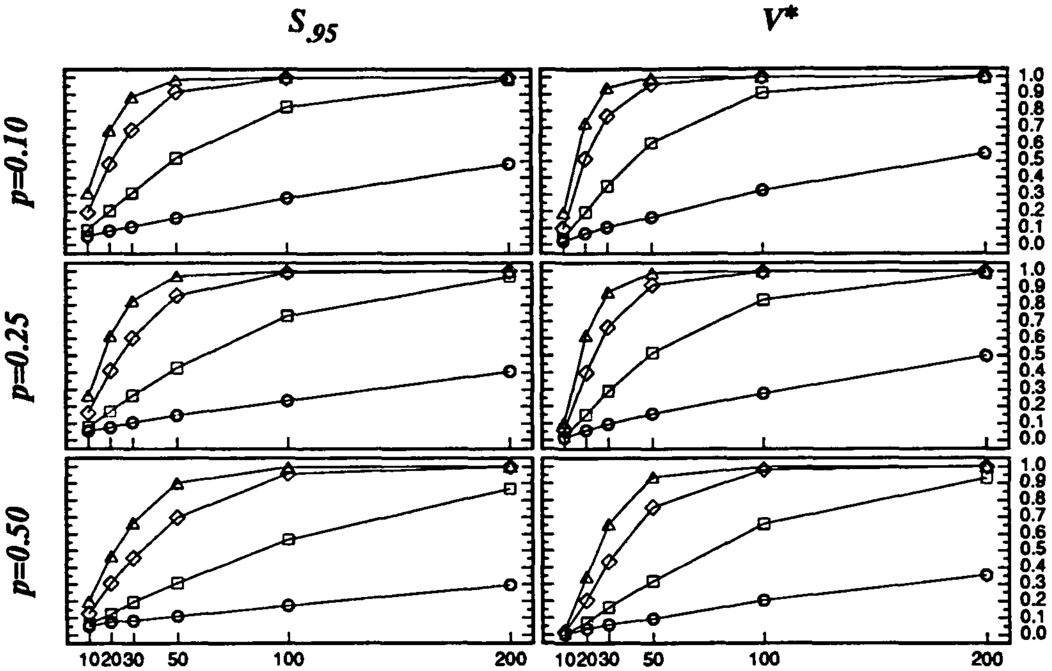

Figure 7. Estimated power (5000 simulations) of $S_{.95}$ and V^* against Kijima's Model 2 with F = Gamma(4), and $A_n = 0.75(\circ), 0.50(\square), 0.25(\diamond), 0.10(\triangle)$.

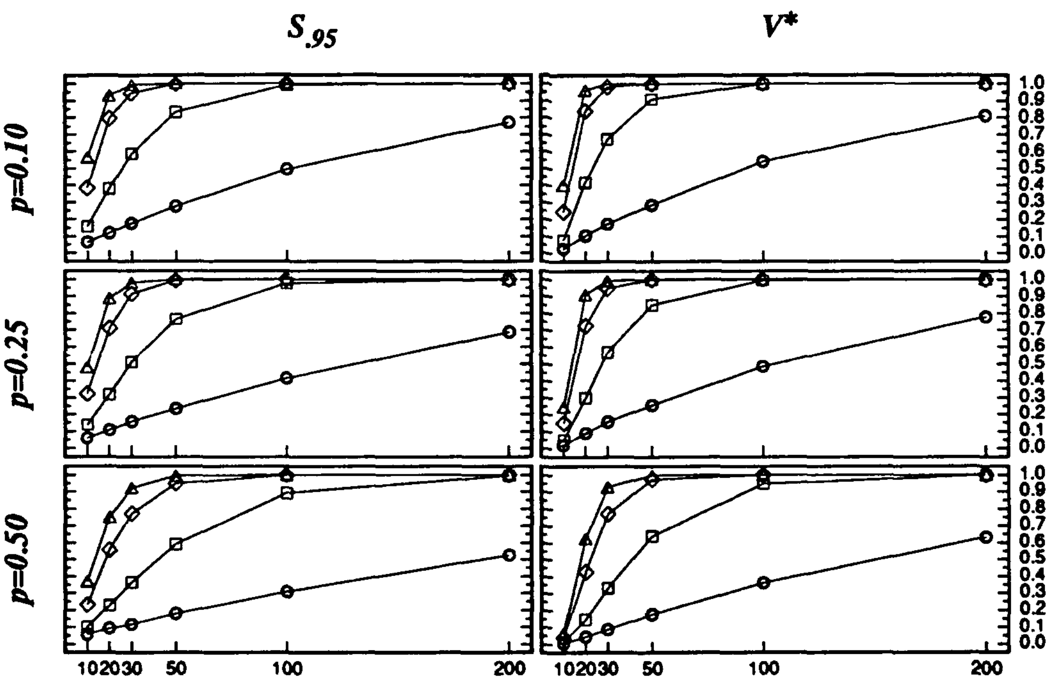

Figure 8. Estimated power (5000 simulations) of $S_{.95}$ and V^* against Kijima's Model 2 with F = Gamma(6), and $A_n = 0.75(\circ), 0.50(\square), 0.25(\diamond), 0.10(\triangle)$.

A. PROOFS

In this section, we follow the usual convention of taking all σ-fields to contain the P-null sets of $\mathcal{F}$, where $(\Omega, \mathcal{F}, P)$ is our basic probability space. Most of the results on stochastic integration which we need here are conveniently collected, with references, in Appendix B of Shorack and Wellner (1986): in particular, we will generally use Theorem B.3.1 without further citation. In this regard, we note here that for all the *local* martingales below, the localizing sequence of of stopping times may be taken to be any sequence of constants diverging to positive infinity (Hollander et al. 1989, Remark 2.2). Also, integrands are readily verfied to be predictable, and generally are bounded on finite intervals by some constant, so that, e.g., condition (b) of this theorem is easily verified. Additional results which we will need are the related Proposition 18.13 of Métivier (1982) and Lemmas A.1 and A.2 of Doss and Chiang (1991). Since the localizing sequences for the processes considered may be taken to be a common sequence of constants, we can use the latter two lemmas (which are stated for martingales) without modification.

A.1 The Basic Martingale Dependence Structure

Returning to the basic model of Section 2., let N_j^* be defined as in (2.3) and let

$$\mathcal{F}_t^{(j)} = \sigma(\{ N_j^*(s) : s \le t\} \cup \{ U_{jk} : k \ge 1 \}).$$

Then

$$M_j^*(t) = N_j^*(t) - \Lambda(t)$$

is a locally square-integrable martingale with respect to $\mathcal{F}_t^{(j)}$ with predictable quadratic variation

$$\langle M_j^* \rangle(t) = \int_0^t (1 - \Delta\Lambda(s))\, d\Lambda(s),$$

and with N_j as in (2.4),

$$M_j(t) = N_j(t) - \Lambda(t \wedge X_{j\nu_j}) = \int_0^t I(X_{j\nu_j} \ge s)\, dM_j^*(s)$$

is a locally square-integrable $\mathcal{F}_t^{(j)}$-martingale with

$$\langle M_j \rangle(t) = \int_0^t I(X_{j\nu_j} \ge s)(1 - \Delta\Lambda(s))\, d\Lambda(s)$$

(Hollander et al. 1989). Similarly, we note that $I(X_{j1} \le t) = N_j^*(t \wedge X_{j1})$, and that (note that both $I(X_{j1} \ge t)$ and $I(X_{j\nu_j} \ge t)$ are predictable),

$$M_j^e(t) = I(X_{j1} \le t) - \Lambda(t \wedge X_{j1}) = \int_0^t I(X_{j1} \ge s)\, dM_j^*(s)$$

is a square-integrable $\mathcal{F}_t^{(j)}$-martingale with

$$\langle M_j^e \rangle(t) = \int_0^t I(X_{j1} \geq s)(1 - \Delta\Lambda(s))\, d\Lambda(s)\,.$$

Finally, by Proposition 18.13 of Métivier (1982),

$$\begin{aligned}\langle M_j, M_j^e \rangle(t) &= \int_0^t I(X_{j\nu_j} \geq s) I(X_{j1} \geq s)\, d\langle M_j^e \rangle(s) \\ &= \int_0^t I(X_{j1} \geq s)(1 - \Delta\Lambda(s))\, d\Lambda(s). \end{aligned} \qquad \text{(A.1)}$$

Now let

$$\mathcal{F}_t = \bigvee_{j=1}^{n} \mathcal{F}_t^{(j)}\,.$$

Then, noting that $n\widehat{F}_e(t) = \sum_{j=1}^n I(X_{j1} \leq t)$, $n\widehat{\overline{F}}_e(t-) = \sum_{j=1}^n I(X_{j1} \geq t)$, and $n\widehat{\overline{H}}(t-) = \sum_{j=1}^n I(X_{j\nu_j} \geq t)$, and recalling the definition of N in (2.5), it follows from the independence of the $\mathcal{F}_t^{(j)}$ (Doss and Chiang 1991, Lemmas A.1 and A.2) that

$$M_e(t) = \sum_{j=1}^n M_j^e(t) = n\widehat{F}_e(t) - n\int_0^t \widehat{\overline{F}}_e(s-)\, d\Lambda(s) \qquad \text{(A.2)}$$

and

$$M(t) = \sum_{j=1}^n M(t) = N(t) - n\int_0^t \widehat{\overline{H}}(s-)\, d\Lambda(s) \qquad \text{(A.3)}$$

are locally square-integrable $\mathcal{F}_t$-martingales with

$$\langle M_e \rangle(t) = \sum_{j=1}^n \langle M_j^e \rangle(t) = n\int_0^t \widehat{\overline{F}}_e(s-)(1 - \Delta\Lambda(s))\, d\Lambda(s)$$

and

$$\langle M \rangle(t) = \sum_{j=1}^n \langle M_j \rangle(t) = n\int_0^t \widehat{\overline{H}}(s-)(1 - \Delta\Lambda(s))\, d\Lambda(s)\,.$$

Since M_j and $M_{j'}^e$ are independent when $j \neq j'$, we may again apply Lemma A.2 of Doss and Chiang (1991) to get $\langle M_j, M_{j'}^e \rangle = 0$. Thus, by (A.1) and the bilinearity of the $\langle \cdot, \cdot \rangle$-operator

$$\begin{aligned}\langle M, M_e \rangle(t) &= \sum_{j=1}^n \sum_{j'=1}^n \langle M_j, M_{j'}^e \rangle(t) \; = \; \sum_{j=1}^n \langle M_j, M_j^e \rangle(t) \\ &= \sum_{j=1}^n \int_0^t I(X_{j1} \geq s)(1 - \Delta\Lambda(s))\, d\Lambda(s) \\ &= n\int_0^t \widehat{\overline{F}}_e(s-)(1 - \Delta\Lambda(s))\, d\Lambda(s). \end{aligned} \qquad \text{(A.4)}$$

These results provide us with the basic martingale structure needed to prove the results of Section 2.. The first step is to establish a martingale representation for $(\widehat{F} - \widehat{F}_e)/\overline{F}$.

A.2 Martingale Representations

Recalling (2.6) and (2.7), we define

$$\widehat{\Lambda}_e(t) = \int_0^t \frac{1}{\widehat{\overline{F}}_e(s-)}\, d\widehat{F}_e(s),$$

and note that

$$\widehat{\overline{F}}_e(t) = \prod_{s \le t}(1 - \Delta\widehat{\Lambda}_e(s)).$$

For brevity, let $Z = \sqrt{n}(\widehat{F} - F)/\overline{F}$, $Z_e = \sqrt{n}(\widehat{F}_e - F)/\overline{F}$, and $\tilde{Z} = Z - Z_e = \sqrt{n}(\widehat{F} - \widehat{F}_e)/\overline{F}$. Using Duhammel's equation (Gill and Johansen 1990; or see Shorack and Wellner 1986, Lemma 7.2.1) and (A.2), we find that

$$\begin{aligned} Z_e(t) &= \sqrt{n}\int_0^t \frac{\widehat{\overline{F}}_e(s-)}{\overline{F}(s)}\, d[\widehat{\Lambda}_e(s) - \Lambda(s)] \\ &= \frac{1}{\sqrt{n}}\int_0^t \frac{1}{\overline{F}(s)}\, dM_e(s), \end{aligned} \tag{A.5}$$

and similarly (Hollander et al. 1989),

$$\begin{aligned} Z(t) &= \sqrt{n}\int_0^t \frac{\widehat{\overline{F}}(s-)}{\overline{F}(s)}\, d[\widehat{\Lambda}(s) - \Lambda(s)] \\ &= \frac{1}{\sqrt{n}}\int_0^t \frac{\widehat{\overline{F}}(s-)}{\widehat{\overline{H}}(s-)\overline{F}(s)}\, dM(s). \end{aligned} \tag{A.6}$$

In both (A.5) and (A.6), the integrand on the right-hand side is predictable and bounded on any finite interval $(0,\tau]$ by a constand $(1/\overline{F}(\tau))$. Thus Z, Z_e, and $\tilde{Z}$ are locally square-integrable martingales with

$$\begin{aligned} \langle Z_e\rangle(t) &= \frac{1}{n}\int_0^t \frac{1}{\overline{F}^2(s)}\, d\langle M_e\rangle(s) \\ &= \int_0^t \widehat{\overline{F}}_e(s-)\frac{dF(s)}{\overline{F}^2(s-)\overline{F}(s)} \end{aligned}$$

and

$$\begin{aligned} \langle Z\rangle(t) &= \frac{1}{n}\int_0^t \left(\frac{\widehat{\overline{F}}(s-)}{\widehat{\overline{H}}(s-)\overline{F}(s)}\right)^2 d\langle M\rangle(s) \\ &= \int_0^t \frac{\widehat{\overline{F}}^2(s-)}{\widehat{\overline{H}}(s-)}\,\frac{dF(s)}{\overline{F}^2(s-)\overline{F}(s)}. \end{aligned}$$

Also, applying Proposition 18.13 of Métivier (1982) to (A.4), (A.5), and (A.6), we see that

$$\begin{aligned}\langle Z, Z_e\rangle(t) &= \frac{1}{n}\int_0^t \left(\frac{\widehat{\overline{F}}(s-)}{\widehat{\overline{H}}(s-)\overline{F}(s)}\right)\left(\frac{1}{\overline{F}^2(s)}\right) d\langle M, M_e\rangle(s) \\ &= \int_0^t \frac{\widehat{\overline{F}}(s-)\widehat{\overline{F}}_e(s-)}{\widehat{\overline{H}}(s-)} \frac{dF(s)}{\overline{F}^2(s-)\overline{F}(s)}.\end{aligned}$$

These results combine to give

$$\begin{aligned}\langle \tilde{Z}\rangle(t) &= \langle Z\rangle(t) + \langle Z_e\rangle(t) - 2\langle Z, Z_e\rangle(t) \\ &= \int_0^t \left(\widehat{\overline{F}}_e(s-) - \left(2\frac{\widehat{\overline{F}}_e(s-)}{\widehat{\overline{F}}(s-)} - 1\right)\frac{\widehat{\overline{F}}^2(s-)}{\widehat{\overline{H}}(s-)}\right) \frac{dF(s)}{\overline{F}^2(s-)\overline{F}(s)}. \qquad \text{(A.7)}\end{aligned}$$

The martingale representation for $\tilde{Z}$ will be used below with Rebolledo's Martingale Central Limit Theorem to prove Theorem 2.1. In particular, (A.7) will be used to determine the variance-covariance function of the of the limiting Gaussian process found in the theorem.

A.3 Proof of Theorem 2.1

By the Glivenko-Cantelli Theorem, $\widehat{F}_e$ and $\widehat{H}$ are uniformly strongly consistent for F and H respectively. Similarly, $\widehat{F}$ is uniformly (strongly) consistent for $\overline{F}$ (Whitaker and Samaniego 1989; Hollander et al. 1989). Thus it follows from (A.7) that for $t < \infty$,

$$\langle \tilde{Z}\rangle(t) \xrightarrow{P} \int_0^t \left(\frac{1}{\overline{F}(s-)} - \frac{1}{\overline{H}(s-)}\right)\frac{dF(s)}{\overline{F}(s)} = L(t).$$

The result of Theorem 2.1 will follow upon verification of Rebolledo's strong ARJ(2) condition for $\tilde{Z}$. For an arbitrary local martingale m and $\epsilon > 0$, let

$$\sigma^\epsilon[m](t) = \sum_{s\le t} |\Delta m(s)|^2 I(|\Delta m(s)| > \epsilon),$$

and let $\tilde{\sigma}^\epsilon[m]$ be the compensator of $\sigma^\epsilon[m]$. Then we must show that $\tilde{\sigma}^\epsilon[\tilde{Z}](t)$ converges in probability to 0 for each $t < \infty$. For this, we first note that by (A.2), (A.3), (A.5), and (A.6),

$$\Delta Z_e(s) = \frac{1}{\sqrt{n}}\frac{1}{\overline{F}(s)}\Delta M_e(s) = \frac{1}{\sqrt{n}}\frac{1}{\overline{F}(s)}\Delta(n\widehat{F}_e(s))$$

and

$$\Delta Z(s) = \frac{1}{\sqrt{n}}\frac{\widehat{\overline{F}}(s-)}{\overline{F}(s)\widehat{\overline{H}}(s-)}\Delta M(s) = \frac{1}{\sqrt{n}}\frac{\widehat{\overline{F}}(s-)}{\overline{F}(s)\widehat{\overline{H}}(s-)}\Delta N(s).$$

Thus the jumps of Z_e and Z are postitive, and since $|\Delta\tilde{Z}(s)| = |\Delta Z(s) - \Delta Z_e(s)|$, we have

$$|\Delta\tilde{Z}(s)|^2 I(|\Delta\tilde{Z}(s)| > \epsilon) \\ \leq \; |\Delta Z(s)|^2 I(|\Delta Z(s)| > \epsilon) + |\Delta Z_e(s)|^2 I(|\Delta Z_e(s)| > \epsilon).$$

From this inequality we see that

$$\sigma^\epsilon[Z](t) + \sigma^\epsilon[Z_e](t) - \sigma^\epsilon[\tilde{Z}](t) \\ = \sum_{s\leq t}\Big(|\Delta Z(s)|^2 I(|\Delta Z(s)| > \epsilon) + |\Delta Z_e(s)|^2 I(|\Delta Z_e(s)| > \epsilon) \\ -|\Delta\tilde{Z}(s)|^2 I(|\Delta\tilde{Z}(s)| > \epsilon)\Big)$$

is an increasing, nonnegative process. This implies that $\tilde{\sigma}^\epsilon[\tilde{Z}]$, the compensator of $\sigma^\epsilon[\tilde{Z}]$, is bounded by the sum of the compensators of $\sigma^\epsilon[Z]$ and $\sigma^\epsilon[Z_e]$. (To see this, note that more generally, if A and B are increasing processes, and $A-B$ is increasing, then $A-B$ has compensator equal to $\tilde{A}-\tilde{B}$, the difference of the compensators of A and B, respectively. Since the compensator of an increasing process is nonnegative, this implies that $\tilde{A} \geq \tilde{B}$.) Noting now that (Hollander et al. 1989)

$$\tilde{\sigma}^\epsilon[Z](t) = \int_0^t \left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)}\right)^2 \frac{1}{\hat{\bar{H}}(s-)} I\left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)\hat{\bar{H}}(s-)} > \sqrt{n}\epsilon\right) d\Lambda(s),$$

and similarly,

$$\tilde{\sigma}^\epsilon[Z_e](t) = \int_0^t \frac{\hat{\bar{F}}_e(s-)}{\bar{F}^2(s)} I\left(\frac{1}{\bar{F}(s)} > \sqrt{n}\epsilon\right) d\Lambda(s),$$

it follows easily that $\tilde{\sigma}^\epsilon[Z](t) \to 0$ and $\tilde{\sigma}^\epsilon[Z_e](t) \to 0$ in probability for all $t < \infty$. □

A.4 Proofs of Corollaries

A.4.1 Proof of Corollary 2.2

The first part of Corollary 2.2 follows directly from Theorem 2.1 by a check of covariances. This is a standard transformation, sometimes called Doob's transformation. The second part of the corollary follows from the first, and the uniform consistency (on finite intervals) of $\hat{\bar{G}}/\hat{\bar{F}}$. This consistency follows easily from the definition of $\hat{G}$ in (2.10) and the consistency of $\hat{F}_e$, $\hat{F}$, and $\hat{H}$.

A.4.2 Proof of Corollary 2.3

Weak convergence in $D[0,\infty]$ of $\sqrt{n}(\widehat{F}-F)$ is established in Hollander et al. (1989), and of course convergence of $\sqrt{n}(\widehat{F}_e-F)$ in $D[0,\infty]$ is well known. This is enough to establish tightness of the joint distribution of $\sqrt{n}(\widehat{F}-F)$ and $\sqrt{n}(\widehat{F}_e-F)$ in $(D[0,\infty])^2$. Since the limiting distributions are concentrated on $C[0,\infty]$, and since addition and subtraction are continuous at continuous points in $D[0,\infty]$, it follows that $\sqrt{n}(\widehat{F}-\widehat{F}_e)$ is tight in $D[0,\infty]$. Theorem 2.1 indentifies the finite-dimensional distributions of the limiting process, and the corollary follows.

A.4.3 Proof of Corollary 2.4

Corollary 2.4 can be proved directly in the same fashion as the proof of Theorem 5.1 of Hollander et al. (1989), or by using the Wilcoxon example of Gill (1989). We pursue the latter method here. In what follows, convergence in distribution indicates weak convergence is in $D[0,\infty]$ with the usual Skorohod topology.

By Corollary 3.1 of Hollander et al. (1989), $\sqrt{n}(\widehat{F}-F)$ converges in distribution to a Gaussian process with covariance function $\overline{F}(s)\overline{F}(t)C(s\wedge t)$, while it is well known that $\sqrt{n}(\widehat{F}_e-F)$ converges in distribution to a Gaussian process with covariance function $F(s\wedge t)\overline{F}(s\vee t)$. These results together with Corollary 2.3 identify the limiting joint distibution of $\sqrt{n}(\widehat{F}-F)$ and $\sqrt{n}(\widehat{F}_e-F)$ in $(D[0,\infty])^2$ as that of (W_1, W_1+W_2), where $W_1=\overline{F}B_1(C)$ and $W_2=\overline{F}B_2(L)$. Here C and L are as defined in (2.8) and (2.9), and B_1 and B_2 are independent Brownian motion processes.

Now, by Lemma 3 and Theorem 3 of Gill (1989),

$$\begin{aligned}\sqrt{n}\left(V-\frac{1}{2}\right) &\xrightarrow{\mathcal{D}} \int_0^\infty F\,d(W_1+W_2)+\int_0^\infty W_1\,dF \\ &\overset{\mathcal{D}}{=} \int_0^\infty W_2\,dF\,.\end{aligned}$$

This implies that the limiting distribution of $\sqrt{n}(V-1/2)$ is normal with mean 0 and variance

$$\sigma^2=\int_0^\infty\int_0^\infty \overline{F}(s)\overline{F}(t)L(s\wedge t)\,dF(s)dF(t)$$

which simplifies to (2.11).

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