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13. ABSTRACT *(Maximum 200 words)*

Recently, a new class of nonlinear oscillatory equations have arisen. They have the property that the nonlinear terms can become unbounded for finite values of the variable and/or its derivative. For such systems the usual method of analysis do not apply. This report summarizes our investigations on such systems. In particular, we have carried out a detailed investigation of the mathematical properties of such systems using phase-space methods, perturbation theory based on the Hopf bifurcation theorem, and the method of harmonic balance. Properties of coupled singular oscillators were also examined.

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Nonlinear, Singular Oscillatory Systems

FINAL REPORT

Dr. Ronald E. Mickens

August 31, 1991

GRANT NUMBER: DAAL03-88-G-0002
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A. Problems Studied*

The major problem considered in this grant was the investigation of the mathematical properties of a new class of oscillators. These oscillators have the property that their defining differential equation can become unbounded for finite values of the dependent variable or its derivative. We have called such systems, "singular, nonlinear oscillators."[1] Examples of such systems occur in electronic circuits,[2] plasma physics,[3] and nonlinear vibrations.[4,5]

A model differential equation that has the required mathematical properties is the so-called WCM equation[6]

$$\ddot{x} + x = \epsilon\left[\frac{\mu - x^2}{1 - x^2}\right]\dot{x}, \tag{1}$$

where ϵ and μ are positive parameters. Of interest is the fact that the usual methods (perturbation techniques, multi-time expansions, method of slowly varying amplitude and phase), that depend on ϵ being small, cannot be applied to determine analytic approximations to the solutions of Eq. (1).[1] Hence, our investigations have centered on the following questions:

(i) What are the general solution behaviors of Eq. (1)? In particular, what can be stated about the possible existence of limit-cycles, their number, uniqueness, location, etc.?

(ii) For the construction of approximate analytic solutions, what should be the expansion parameter? Its clearly not ϵ! Is it a function of μ? If so, what function of μ?

(iii) Equation (1) can be considered a large amplitude[1,2] version of the van der Pol equation[7]

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}. \tag{2}$$

*See Bibliography for references.

What is the relationship between their solutions? How do their phase-spaces relate to each other? What is the proper limit for Eq. (1) to reduce to Eq. (2)? It is clear that the "ϵ" in the two equations are not the same.

(iv) The limit-cycle of the van der Pol equation can be investigated using the Hopf bifurcation theorem. Can the same be done for Eq. (1)?

(v) Simple harmonic balance allows the calculation of an excellent analytic approximation to the solution of Eq. (1).[1] What is the best procedure for obtaining higher-order approximations using generalized harmonic balance techniques?

(vi) What new phenomena occur when two or more singular oscillators are coupled? Do special types of periodic solutions exist? If so, how can approximate analytic solutions be obtained and their stability determined?

The following section B provides a summary of the most important results on the above problems. Section C gives a list of all abstracts and papers that have been or will be published related to the research objectives of this grant. Finally, section D is a listing of all senior personnel and graduate students who have been involved in various aspects of this research project.

B. Summary of Results

The following is a concise summary of the research accomplished under this grant. These results have been published or accepted for publication in refereed scientific journals.

References in this section refer to abstracts or papers given in section C of this report. For example, A2 and P3 are, respectively, the second abstract and the third paper listed in section C.

1. Failure of the Method of Slowly Varying Amplitude and Phase [P8]

In paper P8, Mickens and Ramadhani examined the reasons why the method of slowly varying amplitude and phase cannot be applied to singular, nonlinear oscillator equations. The breakdown occurs because the nonlinear term does not have a uniform bound on its magnitude in arbitrary finite regions of its associated phase-space. Consequently, the defining equation for the amplitude has singularities that do not correspond to the actual dynamics of the system.

2. Harmonic Balance [P1]

Early in this project, we showed that the method of simple harmonic balance provides an easy and direct procedure for calculating the fundamental properties of our model singular, nonlinear oscillator. First, relations giving the period and the amplitude of the unique, stable limit-cycle were found. Secondly, this calculation indicated that the small parameter ϵ was not the relevant expansion parameter for constructing a perturbation type solution.

3. A Perturbation Procedure [A1, P2, P3]

Since ϵ is not the parameter that can be used for the basis of a perturbation expansion, what parameter should be used? We approached this problem by considering the van der Pol oscillator differential equation. The reason for doing so is a consequence of the fact that our model singular, nonlinear equation can be considered a large amplitude form of the van der Pol equation.

The applications of the Hopf bifurcation theorem to the van der Pol equation [P2] lead us to the conclusion that the expansion parameter was μ, or more correctly the square-root of μ and its various powers. This procedure was then applied to our singular equation and a uniformly valid perturbation series solution was found [P3]. The properties of our equation, namely, that it only contained "odd" number of expressions in its rationalized form, allowed for a greatly simplified final expression for the perturbation series solution.

4. Phase-Space Techniques [P4]

An advantage of examining the geometric properties of the solutions of differential equations in phase-space is that often important information can be obtained on possible solution behavior without much initial input or work. In reference P4, we showed that just based upon phase-space considerations, our model singular, nonlinear equation had to have at least one stable limit-cycle. In more detail, for $\epsilon > 0$ and, $\mu > 0$ and small, an odd number of limit-cycles could exist. The application of the Hopf bifurcation theorem then showed that in fact only one, stable limit-cycle existed.

5. Transient Behavior [P5, P7]

The procedures of harmonic balance and perturbation discussed above allow only the determination of accurate approximations to the steady-state of our model singular oscillator. This means that the amplitude and angular frequency can be calculated, but, not the transient behavior of the solution for initial conditions not on the limit-cycle. In paper P5, we showed that such behavior could be obtained using a variant of the method of averaging. The stability of the limit-cycle was, as a by-product, also determined.

There are many dynamic systems in engineering, chemistry, biology and mathematics that can be modeled in terms of two coupled first-order ordinary differential qeuations. In paper P7, we extended the methods of paper P5 to cover this more generalized situation. [To date, I have received more than 80 requests for reprints of this paper!]

6. Coupled Singular Oscillators [A2, A5, P9, P11]

Coupled nonlinear oscillations can have solutions that are periodic, quasi-periodic or chaotic. The possible solution behaviors not only depend on the specific values of the system parameters and initial conditions, but, also on how the oscillators are coupled to each other.

Our investigations are presented in two abstracts, A2 and A5, and two papers, P9 and P11. The particular case that we studied was for two identical singular oscillators interacting by means of both linear and nonlinear couplings. Our mathematical analysis used the concept of "nonlinear normal modes." For the systems studied, we found that there were always periodic solutions that satisfied one of the symmetry conditions

$$x(t) = y(t) \quad \text{or} \quad x(t) = -y(t).$$

These solutions correspond, respectively, to symmetric and anti-symmetric nonlinear normal modes. For the case of a particular nonlinear coupling, the symmetric solution was stable and the anti-symmetric solution unstable. However, for the linear coupling situation, both types of solutions were stable. Approximate analytic expressions were obtained for these solutions. In addition, all of these results were verified by numerical integration of the equations of motion.

It should also be stated that for two linearly coupled singular oscillators, the particular "final" periodic state, that the system approached as $t \to \infty$, was a function of the choice of the values of the initial conditions. In other words, there are basins of attraction for the symmetric and anti-symmetric periodic states.

Finally, evidence was found for so-called "transition" states, i.e., states for which (numerically) the long-term behavior appears to be chaotic. The initial conditions for such states appear to be localized in phase-space.

7. Miscellaneous Problems [P6, P10]

A number of important physical systems can be modeled by van der Pol type differential equations, i.e.,

$$\ddot{x} + g(x) = \epsilon f(x)\dot{x},$$

where

$$g(x) = -g(-x), \quad f(x) = f(-x), \quad \epsilon > 0.$$

In paper P6, we examined certain mathematical properties of the periodic solutions to this class of equations as a function of the parameter ϵ.

A very interesting nonlinear oscillator is the anti-symmetric quadratic oscillator, i.e.,

$$\ddot{x} + x + \epsilon|x|x = 0.$$

We showed that all solutions of this differential equation are bounded and periodic. This follows directly from the fact that the system corresponds to a conservative oscillator having the potential energy

$$V(x) = \frac{x^2}{2} + \frac{|x|x^2}{3} .$$

Analytical approximations to the solutions have been calculated using both the method of harmonic balance and the Krylov-Bogoliubov-Mitropolsky procedure.

C. Publications

ABSTRACTS

1. R. E. Mickens and K. B. Bota, "A nonlinear singular oscillator: A perturbation procedure," Bulletin of the American Physical Society 34, 1144 (1989).

2. R. E. Mickens and K. B. Bota, "Coupled, nonlinear, singular oscillators, AAPT Announcer 19, 115 (1989).

3. O. Oyedeji and R. E. Mickens, "Poincaré plots for a nonlinear, singular oscillator differential equation," AAPT Announcer 19, 115 (1989).

14. R. E. Mickens, "Novel finite-difference schemes for differential equations, Abstracts of the Third Conference on Nonlinear Vibrations, Stability, and Dynamics of Structures and Mechanisms, A. H. Nayfeh and D. T. Mook, editors (Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University; Blacksburg, VA; June 25-27, 1990).

5. O. Oyedeji and R. E. Mickens, "Nonlinear normal modes for two coupled singular oscillators," Bulletin of the American Physical Society 36, 1303 (1991).

PAPERS

1. R. E. Mickens, "Singular, nonlinear oscillator equations," in Nonlinear Analysis and Applications, V. Lakshmikantham, editor (Marcel Dekker, New York, 1987); pps. 339-344.

2. R. E. Mickens, "Perturbation procedure for the van der Pol oscillator based on the Hopf bifurcation theorem." Journal of Sound and Vibration 127, 187 (1988).

3. R. E. Mickens, "Construction of a perturbation solution for a nonlinear, singular oscillator equation," Journal of Sound and Vibration 130, 513 (1989).

4. R. E. Mickens, "Investigation of the mathematical properties of a new nonlinear negative resistance oscillator," Circuits, Systems, and Signal Processing 8, 187 (1989).

5. R. E. Mickens, "Calculation of transient behavior for a nonlinear, singular oscillator equation," Journal of Sound and Vibration 134, 187 (1989).

6. W. E. Collins and R. E. Mickens, "Symmetry properties of van der Pol type differential equations and implications," Journal of Sound and Vibration 136, 352 (1990).

7. R. E. Mickens, "Calculation of oscillatory properties of the solutions to two coupled, first-order, nonlinear differential equations," Journal of Sound and Vibration 137, 331 (1990).

8. R. E. Mickens and I. Ramadhani, "Failure of the method of slowly varying amplitude and phase for nonlinear, singular oscillators, accepted for publication in Journal of Sound and Vibration.

9. R. E. Mickens and K. B. Bota, "Periodic symmetry modes of two coupled identical singular oscillators," Journal of Sound and Vibration 143, 180 (1990).

10. R. E. Mickens, "Oscillations of an anti-symmetric quadratic spring system," to appear in the Henry C. McBay Festschrift Volume (MIT Press; Cambridge, MA; 1992).

11. R. E. Mickens and O. Oyedeji, "Dual periodic modes for two linearly coupled identical singular oscillators," accepted for publication in Journal of Sound and Vibration.

BOOK

R. E. Mickens, Difference Equations: Theory and Applications (Van Nostrand Reinhold. New York, 1990).

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E. Report of Inventions

None

F. Bibliogrpahy

1. K. B. Bota and R. E. Mickens, "Approximate analytic solutions for singular non-linear oscillators," Journal of Sound and Vibration 96, 277 (1984).

2. S. S. Walker and J. A. Connelly, "A new negative resistance oscillator model," Circuits, Systems, and Signal Processing 2, 213 (1983).

3. P. K. Kaw, A. Sen and E. J. Valeo, "Coupled nonlinear stationary waves in a plasma," Physica 9D, 96 (1983).

4. S. Sharma and R. R. Jettappa, "Nonlinear character of motion of a bifilar pendulum," to be published.

5. Y.-T. Wang and R. Singh, "Frequency response of a nonlinear pneumatic system," Journal of Applied Mechanics 54, 209 (1987).

6. R. E. Mickens, "Investigation of the mathematical properties of a new nonlinear negative resistance oscillator," Circuits, Systems, and Signal Processing 8, 187 (1989).

7. R. E. Mickens, Nonlinear Oscillators (Cambridge University Press, New York, 1981).