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TECHNICAL REPORT NO. 27

OCTOBER 1991

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Stanford, California
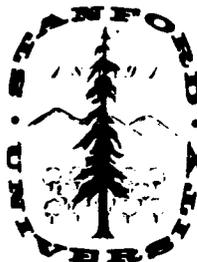
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1. Introduction

A model used frequently for time series analysis is a stationary stochastic process. If the process is Gaussian, it is completely determined by the mean of the process (a location parameter), the variance of the process (a scale parameter), and the sequence of autocorrelations (also known as lag correlations and as serial correlations). The analysis of times series differs from many other statistical analyses because of the possible dependence among observations; that dependence may be characterized by the autocorrelation sequence. For any time series analysis it is essential to make inferences about the autocorrelations.

The Fourier transform of the autocorrelation sequence provides an alternative view of the pattern of dependence. For many purposes it may be more enlightening. In this paper we consider the standardized spectral distribution function as an appropriate description of the pattern of dependence and study problems of inference concerning it. "Standardized" means that the spectral distribution is defined in terms of correlations, rather than covariances. The same information is contained in the autocorrelation sequence, the standardized spectral density, and the standardized spectral distribution, but the three forms differ in presentation.

The first inference problem treated here is the testing of a null hypothesis that completely specifies the pattern of dependence; for example, the null hypothesis might be that all of the autocorrelations are zero or equivalently that the spectral density is constant. To test this hypothesis we compare the sample standardized spectral distribution with the process standardized spectral distribution by means of a goodness of fit criterion, such as the Cramér-von Mises criterion and the Kolmogorov-Smirnov criterion. Asymptotic and other approximate distributions are obtained. The mathematics is similar to that of goodness of fit tests of probability distributions, but differs

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in an essential way. A goodness of fit test usually is consistent against all alternatives, in this case against all correlation structures different from the null hypothesis.

The Kolmogorov-Smirnov criterion can be inverted to give a confidence region for an unspecified standardized spectral distribution. Such a confidence region can be used to infer the increase in the distribution over various intervals of frequency.

Grenander and Rosenblatt (1952),(1957) studied the asymptotic distribution of the differences between the conventional sample spectral distribution function and the conventional process spectral distribution function. They argued that as a process it converges to Brownian motion with a transformed time parameter under the condition that the eighth-order moments of the innovations in the stationary linear process are finite. They proved that the Kolmogorov-Smirnov criterion for the conventional spectral distribution converges in distribution to the supremum of the limit process. This unstandardized spectral distribution, however, is not suited to questions of dependence (that is, patterns of correlation), and the limiting distributions depend on fourth-order cumulants. MacNeil (1971),(1975) considered further goodness of fit tests based on these unstandardized spectral distributions.

Bartlett (1954),(1966) proposed the sample standardized spectral distribution for testing hypotheses about correlations and asserted that the asymptotic distribution would not depend on fourth-order cumulants, but he did not find any of these distributions. Bartlett treated in more detail an analogue, namely, the integral (or sum) of the sample spectral density (periodogram) divided by the hypothetical process spectral density. This definition leads to the Brownian bridge, and the maximum of the difference between this function of the frequency and the frequency (over $[0, \pi]$) has the asymptotic distribution of the Kolmogorov-Smirnov statistic for goodness of fit of probability distributions. Priestley (1981), Section 6.2.6, summarizes these developments. See also Dzhaparidze and Osidze (1980).

Dahlhaus (1985a) showed that the difference between the sample and process standardized spectral distributions multiplied by the square root of the sample size converges weakly to a Gaussian process under several alternative conditions, but always assuming finite eighth-order moments. He obtained the covariance function, but expressed it differently from the form used in this paper. He showed that the

supremum of the absolute value of the limiting process does not have the Kolmogorov-Smirnov distribution in general and expressed the probability in terms of a boundary crossing probability involving the Brownian motion process. Dahlhaus (1988) gave a brief formal treatment of the problem with estimated parameters.

The thrust of this paper is to develop the treatment of tests of goodness of fit and confidence regions based on the knowledge of the limiting Gaussian distribution to actual applications. This study includes methods of computing the goodness of fit statistics, finding their limiting distributions, providing probability inequalities, and developing asymptotic confidence regions. As noted above, in general the process with transformed time parameter is different from the Brownian bridge. The limiting distributions are valid under weak conditions, not requiring fourth-order moments or stationarity.

2. The Empirical Process

Consider a stationary stochastic process $\{y_t\}$, $t = \cdots, -1, 0, 1, \cdots$, with $\mathcal{E}y_t = 0$, autocovariance function

$$(2.1) \qquad \mathcal{E}y_t y_{t+h} = \sigma(h), \qquad h = \cdots, -1, 0, 1, \cdots,$$

and autocorrelation function

$$(2.2) \qquad \rho_h = \sigma(h)/\sigma(0), \qquad h = \cdots, -1, 0, 1, \cdots.$$

We define the normalized spectral density as

$$(2.3) \qquad f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho_h \cos \lambda h, \qquad -\pi \le \lambda \le \pi.$$

Note that the coefficients of the trigonometric functions are the autocorrelations, not the autocovariances. The Fourier transform of the standardized spectral density is

$$(2.4) \qquad \rho_g = \int_{-\pi}^{\pi} f(\lambda) \cos \lambda g \, d\lambda, \qquad g = \cdots, -1, 0, 1, \cdots.$$

Knowledge of the standardized spectral density is equivalent to knowledge of the autocorrelations. The pattern of correlation can be described equivalently in terms of the autocorrelations or the standardized spectral density.

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Since $f(\lambda) = f(-\lambda)$, we define the standardized spectral distribution as

$$(2.5) \qquad \begin{aligned} F(\lambda) &= 2 \int_0^\lambda f(\nu) \, d\nu \\ &= \frac{1}{\pi} \left(\lambda + 2 \sum_{h=1}^\infty \rho_h \frac{\sin \lambda h}{h} \right). \end{aligned}$$

Note that $F(\pi) = 1$; the standardized spectral distribution has the properties (non-negative increments) of a probability distribution on $[0, \pi]$. In this paper we shall be concerned with inference about the standardized spectral density or distribution.

Inference is based on a sample $y_1, \cdots, y_T$. We define the sample autocovariance sequence

$$(2.6) \qquad c_h = c_{-h} = \frac{1}{T} \sum_{t=1}^{T-h} y_t y_{t+h}, \qquad h = 0, 1, \cdots .$$

The sample autocovariance is a biased estimator of the process autocovariance ($h > 0$), but it is asymptotically unbiased. We define the sample autocorrelation sequence

$$(2.7) \qquad r_h = r_{-h} = \frac{c_h}{c_0}, \qquad h = 0, 1, \cdots ,$$

the standardized sample spectral density (popularly mislabelled as the periodogram)

$$(2.8) \qquad \begin{aligned} I_T(\lambda) &= \frac{1}{2\pi c_0} \left| \sum_{t=1}^T y_t e^{i\lambda t} \right|^2 \\ &= \frac{1}{2\pi} \sum_{h=-(T-1)}^{T-1} r_h \cos \lambda h, \qquad -\pi \le \lambda \le \pi, \end{aligned}$$

and the standardized sample spectral distribution function

$$(2.9) \qquad \begin{aligned} \hat{F}_T(\lambda) &= 2 \int_0^\lambda I_T(\nu) \, d\nu \\ &= \frac{1}{\pi} \left(\lambda + 2 \sum_{h=1}^{T-1} r_h \frac{\sin \lambda h}{h} \right). \end{aligned}$$

We shall study inference based on $\sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)]$, $0 \le \lambda \le \pi$; the limiting distribution will be obtained as $T \to \infty$.

Because patterns of dependence can be described in terms of the autocorrelations, the standardized spectral density and distribution are relevant to questions of dependence, rather than the usual functions defined in terms of autocovariances; the scale

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parameter of the process is irrelevant. An additional advantage of the standardization is that the asymptotic distributions are valid under much more general conditions than without the standardization, but general conditions will be stated later in the paper. Another advantage is that $F(\lambda)$ and $\hat{F}_T(\lambda)$ have properties of theoretical and empirical probability distribution functions, respectively.

The asymptotic theory is developed for linear processes

$$(2.10) \qquad y_t = \sum_{s=0}^{\infty} \gamma_s u_{t-s}, \qquad t = \cdots, -1, 0, 1, \cdots,$$

where $\sum_{s=0}^{\infty} \gamma_s^2 < \infty$, $\mathcal{E}u_t = 0$, $\mathcal{E}u_t^2 = \sigma^2$, and $\mathcal{E}u_t u_s = 0$, $t \neq s$. In particular, if the u_t's are independently identically distributed,

$$(2.11) \qquad \sum_{s=0}^{\infty} |\gamma_s| < \infty,$$

and

$$(2.12) \qquad \sum_{s=0}^{\infty} \sqrt{s} \gamma_s^2 < \infty,$$

then for any integer H

$$(2.13) \qquad [\sqrt{T}(r_1 - \rho_1), \cdots, \sqrt{T}(r_H - \rho_H)] \xrightarrow{d} N(0, W),$$

where the (g, h) element of W is

$$(2.14) \quad w_{gh} = \sum_{r=-\infty}^{\infty} (\rho_{r+g}\rho_{r+h} + \rho_{r-g}\rho_{r+h} - 2\rho_h \rho_r \rho_{r+g} - 2\rho_g \rho_r \rho_{r+h} + 2\rho_g \rho_h \rho_r^2).$$

Note that

$$(2.15) \qquad \sigma(h) \;\; = \;\; \sigma^2 \sum_{s=0}^{\infty} \gamma_s \gamma_{s+h}, \qquad h = 0, 1, \cdots,$$

$$(2.16) \qquad \rho_h \;\; = \;\; \frac{\sum_{s=0}^{\infty} \gamma_s \gamma_{s+h}}{\sum_{s=0}^{\infty} \gamma_s^2}, \qquad h = 0, 1, \cdots,$$

and

$$(2.17) \qquad f(\lambda) = \frac{1}{2\pi \sum_{s=0}^{\infty} \gamma_s^2} \left| \sum_{s=0}^{\infty} \gamma_s e^{i\lambda s} \right|^2 .$$

In a sense the γ's are the Fourier coefficients of the square root of $f(\lambda)$. [If y_t is defined by (2.10), $\{\rho_h\}$, $f(\lambda)$, and $F(\lambda)$ are defined even if the process is not stationary.]

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The limiting distribution (2.13) was given by Bartlett (1946) under the (implicit) assumption that $\mathcal{E}u_t^4 < \infty$. That the limiting distribution is valid under the assumption of only $\mathcal{E}u_t^2 < \infty$ was shown by Anderson (1959) for autoregressive processes and by Anderson and Walker (1964) for $y_t = \sum_{s=-\infty}^{\infty} \gamma_s u_{t-s}$ and $\sum_{s=-\infty}^{\infty} |s|\gamma_s^2 < \infty$. Hannan and Heyde (1972) relaxed the condition on $\{\gamma_s\}$ to $\sum_{s=0}^{\infty} \sqrt{s}\gamma_s^2 < \infty$ when the sum was over $0, 1, 2, \ldots$ and the condition of iid u_t to martingale differences. Anderson (1991) has further relaxed the conditions on the martingle differences.

Consider

$$(2.18) \quad \sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)] = \frac{2}{\pi} \sum_{h=1}^{T-1} \frac{\sin \lambda h}{h} \sqrt{T}(r_h - \rho_h) - \frac{2}{\pi} \sum_{h=T}^{\infty} \frac{\sin \lambda h}{h} \sqrt{T}\rho_h.$$

We treat $\sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)]$ as a stochastic process on $[0, \pi]$ As $T \to \infty$, this process converges weakly to a Gaussian process with covariance function

$$(2.19) \quad 4\pi\{G[\min(\lambda, \nu)] \quad - \quad G(\lambda)F(\nu) - F(\lambda)G(\nu) + G(\pi)F(\lambda)F(\nu)\}$$
$$= 4\pi G(\pi) \left\{ \frac{G[\min(\lambda, \nu)]}{G(\pi)} - \frac{G(\lambda)}{G(\pi)}\frac{G(\nu)}{G(\pi)} \right.$$
$$\left. + \left[\frac{G(\lambda)}{G(\pi)} - F(\lambda)\right]\left[\frac{G(\nu)}{G(\pi)} - F(\nu)\right] \right\},$$

where

$$(2.20) \qquad\qquad G(\lambda) = 2 \int_0^\lambda f^2(\nu) \, d\nu.$$

The proof of this statement is given in the appendix. The first term in (2.19) was given by Grenander and Rosenblatt (1957). Durlauf (1989) derived the special case of (2.19) when $f(\lambda) = 1/(2\pi)$. Dahlhaus (1985) gave the first form of the covariance function.

We can simplify the covariance of function of the process by making the monotonic transformation

$$(2.21) \qquad\qquad u = \frac{G(\lambda)}{G(\pi)}, \qquad 0 \le \lambda \le \pi,$$

to $0 \le u \le 1$. The inverse transformation [defined properly if $f(\lambda) > 0$, $0 \le \lambda \le \pi$] is

$$(2.22) \qquad\qquad \lambda = G^{-1}[G(\pi)u], \qquad 0 \le u \le 1.$$

Now let

$$(2.23) \qquad Y_T(u) = \sqrt{T}\left(\hat{F}_T\{G^{-1}[G(\pi)u]\} - F\{G^{-1}[G(\pi)u]\}\right).$$

The covariance function of the limiting distribution of $Y_T(u)$ is

$$(2.24) \qquad 4\pi G(\pi)\{\min(u,v) - uv + q(u)q(v)\},$$

where

$$(2.25) \qquad q(u) = u - F\{G^{-1}[G(\pi)u]\}.$$

Note that $q(0) = q(1) = 0$. It is of interest that

$$(2.26) \qquad q(u) \equiv 0, \qquad 0 \le u \le 1,$$

is equivalent to

$$(2.27) \qquad \frac{G(\lambda)}{G(\pi)} - F(\lambda) \equiv 2\int_0^\lambda \left[\frac{f^2(\nu)}{G(\pi)} - f(\nu)\right] d\nu \equiv 0, \qquad 0 \le \lambda \le 1,$$

which in turn is equivalent to

$$(2.28) \qquad f(\nu)\left[\frac{f(\nu)}{G(\pi)} - 1\right] = 0 \qquad \text{a. e.}$$

In particular, $q(u) \equiv 0$ for

$$(2.29) \qquad f(\nu) = \frac{1}{2\pi} \qquad \text{or} \qquad \rho_1 = \rho_2 = \cdots = 0.$$

Durlauf (1989) has studied tests of lack of correlation.

Let $B(u)$ be the Brownian bridge; that is, $\mathcal{E}B(u) = 0$,

$$(2.30) \qquad \mathcal{E}B(u)B(v) = \min(u,v) - uv,$$

$B(u)$ is Gaussian, and sample paths are continuous with probability 1. Then

$$(2.31) \qquad \frac{1}{2\sqrt{\pi G(\pi)}}Y_T(u)\overset{\pi}{\to}B(u) + q(u)X,$$

where X has the standard normal distribution $N(0,1)$, and the covariance matrix of $B(u) + q(u)X$ is

$$(2.32) \qquad k(u,v) = \min(u,v) - uv + q(u)q(v).$$

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This covariance function is larger than the covariance function of $B(u)$, $\min(u, v) - uv$, in the Loewner sense; that is,

$$(2.33) \qquad \int_0^1 \int_0^1 k(u,v)l(u)l(v)\, du\, dv \geq \int_0^1 \int_0^1 [\min(u,v) - uv]l(u)l(v)\, du\, dv$$

for any $l(\cdot)$ for which the integrals are defined. Thus

$$(2.34) \qquad \Pr\{B(u) + Xq(u) \in \mathcal{C}\} \leq \Pr\{B(u) \in \mathcal{C}\}$$

for any convex symmetric $\mathcal{C}$ [Anderson (1955)].

3. Test of a Specific Hypothesis

3.1. Test Criteria

Consider testing the null hypothesis

$$(3.1) \qquad H : f(\lambda) = f_0(\lambda),$$

where $f_0(\lambda)$ is completely specified. Among the criteria available to test this hypothesis are the Cramér-von Mises criterion

$$(3.2) \qquad \frac{1}{4\pi G(\pi)} \int_0^1 Y_T^2(u)du = \frac{T}{2\pi G^2(\pi)} \int_0^\pi \left[\hat{F}_T(\lambda) - F_0(\lambda)\right]^2 f_0^2(\lambda)d\lambda,$$

the Kolmogorov-Smirnov criterion

$$(3.3) \qquad \frac{1}{2\sqrt{\pi G(\pi)}} \sup_{0 \leq u \leq 1} |Y_T(u)| = \sup_{0 \leq \lambda \leq \pi} \frac{\sqrt{T}}{2\sqrt{\pi G(\pi)}} |\hat{F}_T(\lambda) - F_0(\lambda)|,$$

and the Anderson-Darling statistic

$$(3.4) \qquad \frac{1}{4\pi G(\pi)} \int_0^1 Y_T^2(u)\psi(u)du,$$

where $\psi(u) = 1/[u(1-u)]$. If the null hypothesis is $f_0(\lambda) = 1/(2\pi)$, that is, complete lack of correlation, the asymptotic tests are exactly those of goodness of fit of probability distributions. See, for example, Shorack and Wellner (1986) for a review of such tests.

To carry out a test procedure, we would like to know the limiting distribution of the criterion under the null hypothesis. This is the distribution of the functional when the limiting distribution of $Y_T(u)$ is Gaussian with covariance function (2.23). Under the null hypothesis $q(u)$ is specified. The justification is the continuous mapping theorem [Theorem 5.1, Billingsley (1968), for example].

3.2. The Cramér-von Mises Criterion

The Brownian bridge has the covariance function

$$\min(u, v) - uv = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} f_j(u) f_j(v), \tag{3.5}$$

where

$$f_j(u) = \sqrt{2} \sin j\pi u \tag{3.6}$$

and $\lambda_j = (\pi j)^2$. These eigenvalues and eigenfunctions satisfy the integral equation

$$f(u) = \lambda \int_0^1 [\min(uv) - uv] f(v) dv \tag{3.7}$$

with boundary conditions $f(0) = f(1) = 0$. The eigenfunctions are normalized by $\int_0^1 f^2(u) du = 1$; they are orthogonal in the sense that $\int_0^1 f_i(u) f_j(u) du = 0$, $i \neq j$. The process has the representation.

$$B(u) \stackrel{d}{=} \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} X_j f_j(u), \tag{3.8}$$

where $X_1, X_2, \cdots$ are independent $N(0,1)$ variables. The integral can be represented as

$$\int_0^1 B^2(u) du = \int_0^1 \sum_{i,j=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} X_i f_i(u) \frac{1}{\sqrt{\lambda_j}} X_j f_j(u) du \tag{3.9}$$

$$= \sum_{j=1}^{\infty} \frac{1}{\lambda_j} X_j^2.$$

The characteristic function of (3.9) is

$$(3.10) \qquad \mathcal{E}e^{it\sum_{j=1}^{\infty} X_j^2/\lambda_j} = \prod_{j=1}^{\infty}\left(1 - \frac{2it}{\lambda_j}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{\sin\sqrt{2it}}{\sqrt{2it}}\right)^{-\frac{1}{2}}.$$

The function $D(\lambda) = \prod_{j=1}^{\infty}(1 - \lambda/\lambda_j) = \sin\sqrt{\lambda}/\sqrt{\lambda}$ is known as the Fredholm determinant of the integral equation (3.7).

Let

$$(3.11) \qquad \alpha_j = \int_0^1 q(u)f_j(u)du$$

$$= \frac{2\sqrt{2}}{G(\pi)}\int_0^{\pi} \sin\left[j\pi\frac{G(\lambda)}{G(\pi)}\right]\left[\frac{G(\lambda)}{G(\pi)} - F(\lambda)\right]f^2(\lambda)d\lambda$$

for $q(u)$ square-integrable. Then

$$(3.12) \qquad q(u) = \sum_{j=1}^{\infty}\alpha_j f_j(u).$$

The process $B(u) + Xq(u)$ has the representation

$$(3.13) \qquad B(u) + Xq(u) = \sum_{j=1}^{\infty}\left(\frac{X_i}{\sqrt{\lambda_j}} + \alpha_j X\right)f_j(u),$$

and the Cramér-von Mises criterion has the representation

$$(3.14) \qquad S = \int_0^1 [B(u) + Xq(u)]^2 du = \int_0^1\left[\sum_{j=1}^{\infty}\left(\frac{X_j}{\sqrt{\lambda_j}} + \alpha_j X\right)f_j(u)\right]^2 du$$

$$= \sum_{j=1}^{\infty}\left(\frac{X_j}{\sqrt{\lambda_j}} + \alpha_j X\right)^2$$

$$= \sum_{j=1}^{\infty}Y_j^2,$$

where $Y_j = X_j/\sqrt{\lambda_j} + \alpha_j X$. The (infinite) covariance matrix of Y_j is

$$
(3.15) \qquad
\begin{bmatrix}
\frac{1}{\lambda_1} + \alpha_1^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 & \cdots \\
\alpha_2\alpha_1 & \frac{1}{\lambda_2} + \alpha_2^2 & \alpha_2\alpha_3 & \cdots \\
\alpha_3\alpha_1 & \alpha_3\alpha_2 & \frac{1}{\lambda_3} + \alpha_3^2 & \cdots \\
\vdots & \vdots & \vdots &
\end{bmatrix}.
$$

The statistic (3.14) can be approximated by a finite sum

$$
(3.16) \qquad S_N = \sum_{j=1}^{N} Y_j^2.
$$

The difference between (3.14) and (3.16) has expectation

$$
(3.17) \qquad \mathcal{E} \sum_{j=N+1}^{\infty} Y_j^2 = \sum_{j=N+1}^{\infty} \left(\frac{1}{\lambda_j} + \alpha_j^2 \right).
$$

The expectation can be made arbitrarily small by taking N sufficiently large. Hence, as $N \to \infty$, the distribution of S_N converges to the distribution of S and the characteristic function of S_N approaches the characteristic function of S.

Let Y_N be the N-vector with Y_j as the j − th component. The covariance matrix of Y_N is

$$
(3.18) \qquad \mathcal{E} Y_N Y_N' = \Lambda_N + \alpha_N \alpha_N',
$$

where Λ_N is the diagonal matrix with $1/\lambda_j$ as the j − th diagonal element and α_N as the vector with α_j as the j − th component. Then the characteristic function of S_N is

$$
(3.19) \qquad
\begin{aligned}
\mathcal{E} \, e^{itY_N'Y_N} &= |I_N - 2it(\Lambda_N + \alpha_N\alpha_N')|^{-\frac{1}{2}} \\
&= \prod_{j=1}^{N} (1 - 2it\phi_{jN})^{-\frac{1}{2}},
\end{aligned}
$$

where ϕ_{jN} is the j − th characteristic root of $\Lambda_N + \alpha_N\alpha_N'$, that is, the j − th zero of

$$(3.20) \quad |\Lambda_N + \alpha_N \alpha'_N - \phi I_N| = \begin{vmatrix} 1 & 0 \\ -\alpha_N & \Lambda_N - \phi I_N + \alpha_N \alpha'_N \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha'_N \\ -\alpha_N & \Lambda_N - \phi I_N \end{vmatrix}$$

$$= |\Lambda_N - \phi I_N|\{1 + \alpha'_N(\Lambda_N - \phi I_N)^{-1}\alpha_N\}$$

$$= |\Lambda_N - \phi I_N|\left\{1 + \sum_{j=1}^{N} \frac{\alpha_j^2}{\frac{1}{\lambda_j} - \phi}\right\}$$

for $\phi \neq 1/\lambda_j, j = 1, \cdots, N$. We shall write

$$(3.21) \quad D_N^*(\nu) = |I_N - \nu(\Lambda_N + \alpha_N \alpha'_N)|$$

$$= \prod_{i=1}^{N}\left(1 - \frac{\nu}{\lambda_i}\right)\left\{1 + \sum_{j=1}^{N} \frac{\alpha_j^2}{\frac{1}{\lambda_j} - \frac{1}{\nu}}\right\}$$

$$= \prod_{i=1}^{N}\left(1 - \frac{\nu}{\lambda_i}\right)\left\{1 - \nu^2 \sum_{j=1}^{N} \frac{\alpha_j^2}{\lambda_j - \nu} - \nu \sum_{j=1}^{N} \alpha_j^2\right\}.$$

The first term in $D_N^*(\nu)$ is

$$(3.22) \qquad \prod_{i=1}^{N}\left(1 - \frac{\nu}{\lambda_j}\right) \longrightarrow \prod_{i=1}^{\infty}\left(1 - \frac{\nu}{\lambda_j}\right)$$

as $N \longrightarrow \infty$. The second term is

$$(3.23) \qquad 1 - \nu^2 \sum_{j=1}^{N} \frac{\alpha_j^2}{\lambda_j - \nu} - \nu \sum_{j=1}^{N} \alpha_j^2 \longrightarrow 1 - \nu^2 \sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j - \nu} - \nu \sum_{j=1}^{\infty} \alpha_j^2$$

since

$$(3.24) \qquad \sum_{j=1}^{\infty} \alpha_j^2 = \int_0^1 q^2(u) du.$$

Hence, $D_N^*(\nu)$ converges to

$$(3.25) \qquad D^*(\nu) = \prod_{i=1}^{\infty} \left(1 - \frac{\nu}{\lambda_i}\right) \left\{1 - \nu^2 \sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j - \nu} - \nu \sum_{j=1}^{\infty} \alpha_j^2\right\}.$$

The characteristic function of S is $1/\sqrt{D^*(2it)}$.

Alternatively consider the covariance function

$$(3.26) \qquad k(u,v) = \min(u,v) - uv + q(u)q(v).$$

If $q(u)$ is continuous, $k(u,v)$ is continuous and has the representation

$$(3.27) \qquad k(u,v) = \sum_{j=1}^{\infty} \frac{1}{\nu_j} g_j(u)g_j(v),$$

where ν_j and $g_j(v)$ satisfy the integral equation

$$(3.28) \qquad g(u) = \nu \int_0^1 k(u,v)g(v)dv.$$

Since $k(0,0) = k(1,1) = 0$, the functions $g_i(\cdot)$ should satisfy the boundary conditions

$$(3.29) \qquad g(0) = g(1) = 0.$$

If (3.28) is differentiated twice with respect to u and $q(u)$ is twice differentiable, we obtain

$$(3.30) \qquad g''(u) + \nu g(u) = \nu C q''(u),$$

where

$$(3.31) \qquad C = \int_0^1 q(u)g(u)du.$$

J. B. Keller has pointed out that the solution of (3.30) satisfying (3.29) and depending on ν is for $\nu \neq \pi^2 j^2, j = 1, 2, \cdots,$

$$(3.32) \qquad g(u;\nu) = -\frac{C\nu^{3/2}}{\sin\sqrt{\nu}} \left\{\sin\sqrt{\nu}(u-1) \int_0^u \sin\sqrt{\nu}t\, q(t)dt \right.$$
$$\left. + \sin\sqrt{\nu}u \int_u^1 \sin\sqrt{\nu}(t-1)q(t)dt\right\} + \nu C q(u).$$

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When we multiply (3.32) by $q(u)$, integrate from 0 to 1, and use (3.31), we obtain

$$(3.33) \qquad C = C\nu^2 \int_0^1 \int_0^1 c(u,t;\nu)q(u)q(t)du\,dt + C\nu \int_0^1 q^2(u)\,du,$$

where

$$(3.34) \qquad c(u,t;\nu) = -\frac{1}{\sqrt{\nu}\sin\sqrt{\nu}} \sin\sqrt{\nu}u \sin\sqrt{\nu}(t-1), \quad u \le t,$$
$$= -\frac{1}{\sqrt{\nu}\sin\sqrt{\nu}} \sin\sqrt{\nu}(u-1)\sin\sqrt{\nu}t, \quad u \ge t.$$

It follows from (3.33) that if $C \ne 0$ ν must satisfy

$$(3.35) \qquad 0 = 1 - \nu^2 \int_0^1 \int_0^1 c(u,t;\nu)q(u)q(t)dvdt - \nu \int_0^1 q^2(u)du.$$

If $C = 0$, then $\nu = \pi^2 j^2$ for some j: The function $c(u,t;\nu)$ is the resolvent or resolving kernel of the kernel $\min(u,t) - ut$; that is, it satisfies

$$(3.36) \qquad c(u,v;\nu) = \min(u,v) - uv + \nu \int_0^1 c(u,t;\nu)[\min(t,v) - tv]dt.$$

See Goursat (1964), for example.

Integration shows that

$$(3.37) \qquad \int_0^1 \int_0^1 c(s,t;\nu)f_i(s)f_j(t)ds\,dt = \frac{1}{\lambda_j - \nu}, \quad i = j,$$
$$= 0, \quad i \ne j.$$

Hence, the resolvent has the representation

$$(3.38) \qquad c(s,t;\nu) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \nu} f_j(s)f_j(t).$$

and

$$(3.39) \qquad \int_0^1 \int_0^1 c(s,t;\nu)q(s)q(t)ds\,dt = \sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j - \nu}.$$

Thus the right-hand side of (3.25) is the right-hand side of (3.35). Hence the characteristic function of S is $1/\sqrt{D^*(2it)}$, where $D^*(\nu)$ is given by (3.25) or by

$$(3.40) \qquad D^*(\nu) = \frac{\sin\sqrt{\nu}}{\sqrt{\nu}}\left\{1 - \nu^2 \int_0^1 \int_0^1 c(u,t;\nu)q(u)q(t)dudt - \nu \int_0^1 q^2(u)du\right\}.$$

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Let the zeros of $D^*(\nu)$ be $\nu_j, j = 1, 2, \cdots$; if $\alpha_j \neq 0, j = 1, 2, \cdots$, these zeros are distinct. (They may be distinct even though this condition is not satisfied.) Then for $\nu = \nu_j$ there is a solution $g_j(u)$ satisfying (3.28), the boundary conditions (3.29), and $\int_0^1 g^2(u) = 1$. The process has the representation

$$(3.41) \qquad B(u) + Xq(u) = \sum_{j=0}^{\infty} \frac{1}{\sqrt{\nu_j}} Z_j g_j(u),$$

where $Z_1, Z_2, \cdots$ are independent $N(0, 1)$ variables, and the integral has the representation

$$(3.42) \qquad \int_0^1 [B(u) + Xq(u)]^2 \, du = \sum_{j=1}^{\infty} \frac{1}{\nu_j} Z_j^2.$$

When the explicit form (3.40) of the Fredholm determinant is intractable or cannot be inverted, it can be approximated by $D_N^*(\nu)$ given by (3.21). For this we turn to the numerical evaluation of the characteristic roots of $\Lambda_N + \alpha_N \alpha_N'$, which we call $\phi_{1N} \geq \phi_{2N} \geq \cdots \geq \phi_{NN}$. These approximate the reciprocals of the first N eigenvalues of $k(u, v)$. They are the zeros of (3.20). A value $1/\lambda_j$ is such a zero if and only if $\alpha_j = 0$. Let $1/\lambda_1^* > 1/\lambda_2^* > \cdots > 1/\lambda_{N^*}^*$ be the values of $1/\lambda_j$ that correspond to $\alpha_j \neq 0$. Then (3.20) is

$$(3.43) \qquad |\Lambda_N - \phi I_N| \psi(\phi),$$

where

$$(3.44) \qquad \psi(\phi) = 1 + \sum_{j=1}^{N^*} \frac{\alpha_j^{*2}}{\frac{1}{\lambda_j^*} - \phi},$$

and α_j^* is the α_k corresponding to $1/\lambda_j^*$. The first derivative of $\psi(\phi)$ is

$$(3.45) \qquad \psi'(\phi) = \sum_{j=1}^{N^*} \frac{\alpha_j^{*2}}{(\frac{1}{\lambda_j^*} - \phi)^2} > 0.$$

As $\phi \to 1/\lambda_j^*$ from above, $\psi(\phi) \to -\infty$, and as $\phi \to 1/\lambda_j^*$ from below, $\psi(\phi) \to \infty$. Since $\psi(\phi)$ is continuous except at $\phi = 1/\lambda_j^*$, $j = 1, \cdots, N^*$, there is a root in the interval $(1/\lambda_j^*, 1/\lambda_{j-1}^*)$, $j = 1, \cdots, N^*$, with

$$(3.46) \qquad 1/\lambda_0^* = \sum_{j=1}^{N^*} \alpha_j^{*2} = \sum_{j=1}^{N} \alpha_j^2.$$

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The N zeros of (3.43) can now be numbered so that $1/\lambda_j \le \phi_j \le 1/\lambda_{j-1}$.

Bunch, Nielsen, and Sorenson (1978) have given a algorithm for solving the equation $\psi(\phi) = 0$. To find the root ϕ in the interval $(1/\lambda_j^*, 1/\lambda_{j-1}^*)$ for $j > 1$ define

$$(3.47) \qquad \psi_-(\phi) = \sum_{k=j}^{N^*} \frac{\alpha_k^{*2}}{\frac{1}{\lambda_k^*} - \phi}, \qquad \frac{1}{\lambda_j^*} < \phi < \frac{1}{\lambda_{j-1}^*},$$

$$(3.48) \qquad \psi_+(\phi) = \sum_{k=1}^{j-1} \frac{\alpha_k^{*2}}{\frac{1}{\lambda_k^*} - \phi}, \qquad \frac{1}{\lambda_j^*} < \phi < \frac{1}{\lambda_{j-1}^*}.$$

Then $\psi_-(\phi) < 0$ and $\psi_+(\phi) > 0$. Let ϕ_0 $(1/\lambda_j^* < \phi_0 < 1/\lambda_{j-1}^*)$ be an initial value of ϕ. For $i = 1, 2, \cdots$ define p_i, q_i, r_i, s_i to satisfy

$$(3.49) \qquad \frac{p_i}{q_i - \phi_{(i-1)}} = \psi_-(\phi_{(i-1)}), \; r_i + \frac{s_i}{1/\lambda_j^* - \phi_{(i-1)}} = \psi_+(\phi_{(i-1)}),$$

$$(3.50) \qquad \frac{p_i}{(q_i - \phi_{(i-1)})^2} = \psi_-'(\phi_{(i-1)}), \; \frac{s_i}{(1/\lambda_j^* - \phi_{(i-1)})^2} = \psi_+'(\phi_{(i-1)}).$$

Next define $\phi_{(i)}$ as the solution to

$$(3.51) \qquad -\frac{p_i}{q_i - \phi} = 1 + r_i + \frac{s_i}{1/\lambda_j^* - \phi}$$

that lies in the interval $(1/\lambda_j^*, 1/\lambda_{j-1}^*)$. If $\phi_0 \in (1/\lambda_j^*, \phi_j^*,)$ then

$$(3.52) \qquad \phi_{(i)} < \phi_{(i+1)} < \phi_j^*, \qquad i = 1, 2, \cdots,$$

and $\phi_{(i)}$ converges quadratically to ϕ_j^*, the zero of $\psi(\phi)$ that lies in the interval $(1/\lambda_j^*, 1/\lambda_{j-1}^*)$.

The cumulants of S are given by

$$(3.53) \qquad \kappa_j = 2^{j-1}(j-1)! \sum_{i=1}^{\infty} \left(\frac{1}{\nu_i} \right)^j.$$

They can also be calculated from the kernel (3.26). Let $k_1(s, t) = k(s, t)$ and $k_{j+1}(s, t) = \int_0^1 k_j(s, u) k(u, t) du$. Then

$$(3.54) \qquad \kappa_j = 2^{j-1}(j-1)! \int_0^1 k_j(s, s) ds.$$

See Anderson d Darling (1952). The expression (3.53) is obtained from the expansion of the logarithm of the characteristic function of S in powers of it; the coefficient of $(it)^j/j!$ is the j − th cumulant. kappatly, it is the coefficient of $\nu^j/j!$ in the expansion of $-\frac{1}{2}\log D^*(\nu)$, given by (3.25) or (3.40). We have [from (3.25)]

$$(3.55) \qquad -\frac{1}{2}\log D^*(\nu) = -\frac{1}{2}\prod_{j=1}^{\infty}\left(1 - \frac{\nu}{\lambda_j}\right) - \frac{1}{2}\log\left(1 - \nu\sum_{j=1}^{\infty}\frac{\alpha_j^2\lambda_j}{\lambda_j - \nu}\right)$$

The second term on the right-hand side of (3.55) is

$$(3.56) \qquad -\frac{1}{2}\log\left[1 - \nu\sum_{j=1}^{\infty}\frac{\alpha_j^2}{1 - \frac{\nu}{\lambda_j}}\right] = -\frac{1}{2}\log\left[1 - \nu\sum_{j=1}^{\infty}\alpha_j^2\sum_{i=0}^{\infty}\left(\frac{\nu}{\lambda_j}\right)^i\right].$$

The first two cumulants are

$$(3.57) \qquad \kappa_1 = \mathcal{E}S = \sum_{j=1}^{\infty}\frac{1}{\lambda_j} + \sum_{j=1}^{\infty}\alpha_j^2 = \frac{1}{6} + \int_0^1 q^2(u)du,$$

$$
\begin{aligned}
(3.58)\quad \kappa_2 &= \text{Var}S \\
&= \sum_{j=1}^{\infty}\frac{2}{\lambda_j^2} + 4\sum_{j=1}^{\infty}\frac{\alpha_j^2}{\lambda_j} + 2\left(\sum_{j=1}^{\infty}\alpha_j^2\right)^2 \\
&= \frac{2}{45} + 2\int_0^1\int_0^1[\min(u,v) - uv]q(u)q(v)dudv + 2\left(\int_0^1 q^2(u)du\right)^2.
\end{aligned}
$$

3.3.   Calculation of the Cramér-von Mises criterion

If we omit $(2\sqrt{T}/\pi)\sum_{h=T}^{\infty}\rho_h\sin\lambda h/h$, the Cramér-von Mises criterion (3.2) can be written as $T/[2\pi G^2(\pi)]$ times

$$(3.59) \qquad \int_0^{\pi}\left[\frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}(r_h - \rho_h)\right]^2\left[\frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\rho_r e^{i\lambda r}\right]^2 d\lambda$$

$$= \frac{1}{2\pi^4}\sum_{g,h=1}^{T-1}\frac{(r_g - \rho_g)(r_h - \rho_h)}{gh}\sum_{r,s=-\infty}^{\infty}\rho_r\rho_s\int_{-\pi}^{\pi}\sin\lambda g\sin\lambda h\,e^{i\lambda(r-s)}d\lambda.$$

The integral on the right-hand side of (3.53) is

$$(3.60) \qquad \frac{1}{4i^2}\int_{-\pi}^{\pi}(e^{i\lambda g} - e^{-i\lambda g})(e^{i\lambda h} - e^{-i\lambda h})e^{i\lambda(r-s)}d\lambda$$

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$$= -\frac{1}{4} \int_{-\pi}^{\pi} \left[e^{i\lambda(g+h+r-s)} - e^{i\lambda(g-h+r-s)} - e^{i\lambda(-g+h+r-s)} + e^{i\lambda(-g-h+r-s)} \right] d\lambda.$$

Since $\int_{-\pi}^{\pi} e^{i\lambda k} d\lambda = 0$ unless $k = 0$, we find that (3.53) is

$$(3.61) \quad \frac{1}{4\pi^3} \sum_{g,h=1}^{T-1} \frac{(r_g - \rho_g)(r_h - \rho_h)}{gh} \sum_{r=-\infty}^{\infty} (\rho_r \rho_{r+g-h} - \rho_r \rho_{r+g+h} - \rho_r \rho_{r-g-h} + \rho_r \rho_{r-g+h})$$

$$= \frac{1}{4\pi^3} \sum_{g,h=1}^{T-1} \frac{(r_g - \rho_g)(r_h - \rho_h)}{gh} \sum_{r=-\infty}^{\infty} (\rho_{r+g} - \rho_{r-g})(\rho_{r+h} - \rho_{r-h}).$$

Thus the Cramér-von Mises statistic can be written

$$(3.62) \quad \frac{T}{8\pi^4 G^2(\pi)} \sum_{r=-\infty}^{\infty} \left[\sum_{g=1}^{T-1} \frac{(r_g - \rho_g)(\rho_{r+g} - \rho_{r-g})}{g} \right]^2.$$

In the special case of $f_0(\lambda) = 1/(2\pi)$ (that is $\rho_1 = \rho_2 = \cdots = 0$) the Cramér-von Mises criterion (except in the part of the sum not depending in the sample) is

$$(3.63) \quad \frac{T}{\pi^2} \sum_{g=1}^{T-1} \frac{r_g^2}{g^2}.$$

In this case any finite set of $\sqrt{T} r_g$ has a limiting normal distribution in which the variables are independent standard normal variables. On this basis the limiting distribution of (3.63) is consistent with the limiting distribution of the Cramér-von Mises statistic as indicated in Section 3.2. It may be of interest to compare (3.63) with the Box-Pierce statistic $T \sum_{g=1}^{K} r_g^2$ in some fixed $K < T$.

3.4.   The Kolmogorov-Smirnov Criterion

To test $H_0 : F(\lambda) = f_0(\lambda)$ on a large-sample basis we want to find a constant c such that

$$(3.64) \quad \Pr \left\{ \frac{1}{2\sqrt{\pi G(\pi)}} \sup_{0 \le u \le 1} |Y_T(u)| \le c \right\} \to 1 - \alpha$$

for a specified α $(0 < \alpha < 1)$ as $T \to \infty$. We want to evaluate

$$(3.65) \quad \Pr \left\{ \sup_{1 \le u \le 1} |B(u) + q_0(u) X| \le c \right\}.$$

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First we derive some inequalities. Let

$$(3.66) \qquad d = \sup_{0 \le u \le 1} |q_0(u)|.$$

Then

$$(3.67) \qquad \sup_{0 \le u \le 1} |B(u) + q_0(u)X| \le \sup_{0 \le u \le 1} |B(u)| + |X|d.$$

Thus

$$(3.68) \quad \Pr\left\{ \sup_{0 \le u \le 1} |B(u)| + |X|d \le c \right\} \le \Pr\left\{ \sup_{0 \le u \le 1} |B(u) + Xq_0(u)| \le c \right\}$$

$$\le \Pr\left\{ \sup_{0 \le u \le 1} |B(u)| \le c \right\}.$$

The right-hand inequality follows from (2.34).

The distribution of $\sup_{0 \le u \le 1} |B(u)|$ is

$$(3.69) \qquad \Pr\left\{ \sup_{0 \le u \le 1} |B(u)| \le c \right\} = 1 + 2 \sum_{j=1}^{\infty} (-1)^j e^{-2j^2 c^2}.$$

Let $V = \sup_{0 \le u \le 1} |B(u)|$ and $Z = d|X|$. The density of Z is $[2/(d\sqrt{2\pi})]e^{-z^2/(2d^2)}$, $z \ge 0$. For $w > 0$

$$(3.70) \qquad \Pr\left\{ \sup_{0 \le u \le 1} |B(u)| + d|X| \le w \right\} = \Pr\{V + Z \le w\}$$

$$= \int_0^w \Pr\{V \le w - z | Z = z\} \frac{2}{d\sqrt{2\pi}} e^{-z^2/(2d^2)} dz$$

$$= \frac{2}{d\sqrt{2\pi}} \int_0^w e^{-z^2/(2d^2)} dz + \int_0^w \frac{4}{d\sqrt{2\pi}} \sum_{j=1}^{\infty} (-1)^j e^{-2j^2(w-z)^2 - z^2/(2d^2)} dz$$

$$= 2\Phi\left(\frac{w}{d}\right) - 1$$

$$+ 4 \sum_{j=1}^{\infty} (-1)^j \frac{e^{-2j^2 w^2/(1+4d^2 j^2)}}{\sqrt{1 + 4d^2 j^2}} \left[\Phi\left(\frac{w}{d\sqrt{1 + 4d^2 j^2}}\right) - \Phi\left(\frac{-4dj^2 w}{\sqrt{1 + 4d^2 j^2}}\right) \right].$$

Values of this distribution are given in Table 1. If d is small, (3.70) is an approximation to (3.65).

If (3.65) is $1 - \alpha$ and α is small, then α is approximately 2 times

$$(3.71) \qquad \Pr\left\{ \sup_{0 \le u \le 1} [B(u) + Xq_0(u)] \ge c \right\}.$$

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Suppose $0 \leq q_0(u) \leq d$. Then for $X < 0$

$$(3.72) \qquad \sup_{0 \leq u \leq 1} B(u) \leq \sup_{0 \leq u \leq 1} [B(u) + Xq_0(u)] \leq \sup_{0 \leq u \leq 1} B(u) + Xd,$$

and for $X < 0$

$$(3.73) \qquad \sup_{0 \leq u \leq 1} B(u) + Xd \leq \sup_{0 \leq u \leq 1} [B(u) + Xq(u)] \leq \sup_{0 \leq u \leq 1} B(u).$$

Then

$$(3.74) \qquad \frac{1}{2} \Pr\left\{ \sup_{0 \leq u \leq 1} B(u) \geq c \right\} + \frac{1}{2} \Pr\left\{ \sup_{0 \leq u \leq 1} B(u) - |X|d \geq c \right\}$$

$$\leq \Pr\left\{ \sup_{0 \leq u \leq 1} [B(u) + Xq_0(u)] \geq c \right\}$$

$$\leq \frac{1}{2} \Pr\left\{ \sup_{0 \leq u \leq 1} B(u) + |X|d \geq c \right\} + \frac{1}{2} \Pr\left\{ \sup_{0 \leq u \leq 1} B(u) \geq c \right\}.$$

Let $Y = \sup_{0 \leq u \leq 1} B(u)$ and $Z = d|X|$. Since $\Pr\{Y \leq y\} = 1 - e^{-2y^2}, y \geq 0$, the density of Y is $4ye^{-2y^2}$. Thus for $w \geq 0$

$$(3.75) \quad \Pr\left\{ \sup_{0 \leq u \leq 1} B(u) + d|X| \leq w \right\}$$

$$= \Pr\{Y + Z \leq w\}$$

$$= \frac{8}{d\sqrt{2\pi}} \int_0^w \int_0^{w-z} y e^{-2y^2 - z^2/(2d^2)} dy \, dz$$

$$= \frac{2}{d\sqrt{2\pi}} \int_0^w \left[1 - e^{-2(w-z)^2} \right] e^{-z^2/(2d^2)} dz$$

$$= 2\Phi\left(\frac{w}{d}\right) - 1 - \frac{2}{\sqrt{4d^2 + 1}} e^{-2w^2/(4d^2+1)} \left[\Phi\left(\frac{w}{d\sqrt{4d^2+1}}\right) - \Phi\left(\frac{-4dw}{\sqrt{4d^2+1}}\right) \right].$$

Similarly

$$(3.76) \qquad \Pr\left\{ \sup_{0 \leq u \leq 1} B(u) - d|X| \leq w \right\}$$

$$= \begin{cases} 2\Phi\left(\frac{w}{d}\right) - \frac{2}{\sqrt{4d^2+1}} e^{-2w^2/(4d^2+1)} \; \Phi\left(\frac{w}{d\sqrt{4d^2+1}}\right), & w \leq 0, \\[2em] 1 - \frac{2}{\sqrt{4d^2+1}} e^{-2w^2/(4d^2+1)} \; \Phi\left(\frac{-4dw}{\sqrt{4d^2+1}}\right), & w \geq 0. \end{cases}$$

Values of (3.75) and (3.76) are given in Tables 2 and 3, respectively.

The extremes of $q_0(u)$ can be found by setting to 0 the derivative

$$(3.77) \qquad \frac{d}{du}q_0(u) \;=\; \frac{d}{d\lambda}\left[\frac{G(\lambda)}{G(\pi)} - f_0(\lambda)\right]$$
$$= \; 2f_0(\lambda)\left[\frac{f_0(\lambda)}{G_0(\pi)} - 1\right],$$

that is, at $f_0(\lambda) = 0$ or

$$(3.78) \qquad f_0(\lambda) = G_0(\pi) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\rho_h^2.$$

We now consider the supremum of $|q_0(u)|$ over $0 \le u \le 1$ and all $f_0(\cdot)$; this is the supremum of $|[G(\lambda)/G(\pi)] - F(\lambda)|$. Since $G(\lambda)/G(\pi)$ and $F(\lambda)$ are monotonically nondecreasing in $[0,\pi]$ with $G(0)/G(\pi) = F(0) = 0$ and $G(\pi)/G(\pi) = F(\pi) = 1$,

$$(3.79) \qquad \sup_{0\le u\le 1,\, f(\cdot)} |q(u)| \le 1.$$

We shall now find an $f(\cdot)$ such that the upper bound of 1 is approached. Consider

$$(3.80) \qquad f(\lambda) = \begin{cases} c, & 0 \le \lambda \le v, \\ d, & v < \lambda \le \pi, \end{cases}$$

for some c, d, and $v(0 < v < \pi)$. Then

$$(3.81) \qquad 2\int_0^\pi f(\lambda)d\lambda = 1$$

implies

$$(3.82) \qquad d = \frac{1 - 2vc}{2(\pi - v)}.$$

Then as $v \to \pi$,

$$(3.83) \qquad \frac{G(v)}{G(\pi)} - F(v) \to \;=\; 2\frac{2\pi c + 2\pi c - 1 - 4\pi^2 c}{1 - 4\pi c + 4\pi^2 c^2}c\pi$$
$$= \; 2\pi c.$$

Hence as $c \to 1/(2\pi)$,

$$(3.84) \qquad \sup_{0\le\lambda\le\pi}\left[\frac{G(\lambda)}{G(\pi)} - F(\lambda)\right] \to 1.$$

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The probability (3.65) is a boundary value problem for the process $B(u) + Xq(u)$. It is

$$(3.85) \qquad \Pr\{-c \le B(u) + Xq(u) \le c, \ \forall u, \ 0 \le u \le 1\}.$$

Note that the sample paths are continuous with probability 1. Let

$$(3.86) \qquad W(t) = (1+t)B\left(\frac{t}{1+t}\right), 0 \le t \le \infty.$$

Then $W(t)$ is the Wiener-Brownian motion process with $\mathcal{E}W(t) = 0$ and $\mathcal{E}W(t)W(s) = \min(t, s)$; sample paths are continuous with probability 1. The probability (3.85) is

$$(3.87) \ \Pr\left\{-c(1+t) \le W(t) + X(1+t)q\left(\frac{t}{1+t}\right) \le c(1+t), \forall t, \ 0 \le t \le \infty\right\}.$$

Dahlhaus (1985) has also given this result.

Durbin (1985) has studied the first passage density of a continuous Gaussian process to a general boundary. The probability (3.71) is then the integral of this density from 0 to 1. An approximation to this probability is

$$(3.88) \qquad \frac{c}{\sqrt{2\pi}} \int_0^1 \frac{1 - t + q'(t)q(t)}{k(t,t)} e^{-c^2/[2k(t,t)]} dt,$$

Where $k(t, t) = t - t^2 + q^2(t)$. In practice (3.88) could be evaluated by numerical integration. Durbin also gives an exact expression as well as two other approximations to the first passage density. As an example he applies his formulas to $\Pr\{B(u) \ge c\}$ for several values of c. For c yielding an exact probability of .1 this approximation has an error of only .0021; for larger values of c (smaller values of α) the error is proportionally smaller.

4.   Confidence Region for the Spectral Distribution Function

An asymptotic confidence region with confidence coefficient $1 - \alpha$ for an unknown spectral distribution consists of all monotonic functions $F(\cdot)$ [$F(0) = 0, F(1) = 1$] such that

$$(4.1) \qquad \sqrt{T}\left|\hat{F}_T(\lambda) - F(\lambda)\right| \le c \qquad \forall \lambda \in [0, \pi],$$

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where c is chosen so that (3.59) is $1 - \alpha$ and $q_0(u)$ refers to the unknown distribution. Since c depends on $q_0(u)$, we need a consistent estimator of $q_0(u)$.

From the fact that $\sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)]$ has a limiting normal distribution it follows that

$$(4.2) \qquad \hat{F}_T(\lambda) \xrightarrow{P} F(\lambda).$$

Let $\bar{f}(\lambda)$ be a consistent estimator of $f(\lambda)$ in the sense that

$$(4.3) \qquad \sup_{0 \leq \lambda \leq \pi} \left| \bar{f}_T(\lambda) - f(\lambda) \right| \xrightarrow{P} 0.$$

For such an estimator the class of admissible $f(\lambda)$ must be limited. Define

$$(4.4) \qquad \bar{G}_T(\lambda) = 2 \int_0^\lambda \bar{f}_T^2(\nu) \, d\nu,$$

$$(4.5) \qquad \bar{q}_T(u) = u - \hat{F}_T\{\bar{G}_T^{-1}[\bar{G}_T(\pi)u]\}.$$

Then $\bar{q}_T(u)$ is a consistent estimator of $q_0(u)$. If c_T is the value of c for which (3.59) holds with $q_0(u)$ replaced by $\bar{q}_T(u)$, then $c_T \xrightarrow{P} c$.

The inequality (4.1) can be written

$$(4.6) \qquad \hat{F}_T(\lambda) - \frac{c}{\sqrt{T}} \leq F(\lambda) \leq \hat{F}_T(\lambda) + \frac{c}{\sqrt{T}} \qquad \forall \lambda \in [0, \pi].$$

Another problem of interest is testing that the spectral densities of two independent processes are the same; the null hypothesis is $F_1(\lambda) = F_2(\lambda)$. Suppose $\hat{F}_{T_1}(\lambda)$ and $\hat{F}_{T_2}(\lambda)$ are the corresponding two empirical processes. Under the null hypothesis.

$$(4.7) \qquad \sqrt{\frac{T_1 T_2}{T_1 + T_2}} \left[\hat{F}_{T_1}(\lambda) - \hat{F}_{T_2}(\lambda) \right]$$

converges weakly to the Gaussian process with covariance function (2.19), where $\Gamma(\lambda)$ and $G(\lambda)$ refer to the common spectral distribution and the integral of the common spectral density squared, respectively.

The Kolmogorov-Smirinov criterion

$$(4.8) \qquad \sup_{0 \leq \lambda \leq \pi} |\hat{F}_{T_1}(\lambda) - \hat{F}_{T_2}\lambda)|$$

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can be uses to test the null hypothesis. If $\bar{f}_{T_1}(\lambda)$ and $\bar{f}_{T_2}(\lambda)$ are consistent estimators of the common spectral density [in the sense of (4.3)], then

(4.9)
$$\frac{T_1}{T_1 + T_2}\bar{f}_{T_1}(\lambda) + \frac{T_2}{T_1 + T_2}\bar{f}_{T_2}(\lambda)$$

can be used to estimate $G(\lambda)$ and $q(u)$.

5. Examples

5.1. Moving Average of Order 1

Let

(5.1)
$$y_t = u_t + \alpha u_{t-1},$$

where u_t are iid $N(0, \sigma)$. Then

(5.2)
$$\rho_1 = \rho = \frac{\alpha}{1 + \alpha^2}, \qquad -\frac{1}{2} \le \rho \le \frac{1}{2},$$

(5.3)
$$f(\lambda) = \frac{1}{2\pi}(1 + 2\rho \cos \lambda),$$

(5.4)
$$\begin{aligned}
F(\lambda) &= \frac{1}{\pi} \int_0^\lambda (1 + 2\rho \cos \nu) \, d\nu \\
&= \frac{1}{\pi}(\lambda + 2\rho \sin \lambda),
\end{aligned}$$

(5.5)
$$\begin{aligned}
G(\lambda) &= \frac{1}{2\pi^2} \int_0^\lambda (1 + 4\rho \cos \nu + 4\rho^2 \cos^2 \nu) d\nu \\
&= \frac{1}{2\pi^2}\left[\lambda + 4\rho \sin \lambda + 2\rho^2(\lambda + \frac{1}{2}\sin 2\lambda)\right] \\
&= \frac{1}{2\pi^2}\left[(1 + 2\rho^2)\lambda + 4\rho \sin \lambda) + \rho^2 \sin 2\lambda\right],
\end{aligned}$$

(5.6)
$$G(\pi) = \frac{1 + 2\rho^2}{2\pi},$$

(5.7)
$$\frac{G(\lambda)}{G(\pi)} = \frac{(1 + 2\rho^2)\lambda + 4\rho \sin \lambda + \rho^2 \sin 2\lambda}{\pi(1 + 2\rho^2)}.$$

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The difference is

$$(5.8) \qquad \frac{G(\lambda)}{G(\pi)} - F(\lambda) = \frac{2\rho \sin \lambda (1 - 2\rho^2 + \rho \cos \lambda)}{\pi(1 + 2\rho^2)}.$$

This is nonnegative for $0 \le \rho \le \frac{1}{2}$ and $0 \le \lambda \le \pi$ and nonpositive for $-\frac{1}{2} \le \rho \le 0$.

The maximum of $|q(u)|$ occurs at $f(\lambda) = G(\pi)$ (by setting the derivative of $[G(\lambda)/G(\pi)] - F(\lambda)$ equal to 0), that is,

$$(5.9) \qquad 1 + 2\rho \cos \lambda = 1 + 2\rho^2;$$

that is, $\cos \lambda_0 = \rho$ or $\lambda_0 = \cos^{-1} \rho$. Then

$$(5.10) \qquad \frac{G(\lambda_0)}{G(\pi)} - F(\lambda_0) = \frac{1}{\pi}\left(\frac{4\rho \sin \lambda_0 + \rho^2 \sin 2\lambda_0}{1 + 2\rho^2} - 2\rho \sin \lambda_0\right)$$
$$= \frac{2 \cos \lambda_0 \sin^3 \lambda_0}{\pi(1 + 2 \cos^2 \lambda_0)}.$$

The maximum of $|q(u)|$ is at a zero of

$$(5.11) \qquad \frac{d}{d\lambda_0}\left[\frac{G(\lambda_0)}{G(\pi)} - F(\lambda_0)\right] = -\frac{2}{\pi} \cdot \frac{\sin^2 \lambda_0 (1 - 6 \cos^2 \lambda_0 - 4 \cos^4 \lambda_0)}{(1 + 2 \cos^2 \lambda_0)^2}.$$

The zeros of (5.11) are $\lambda_0 = 0$, π, and

$$(5.12) \qquad \cos^2 \lambda_0 = \frac{-3 \pm \sqrt{13}}{4},$$

or

$$(5.13) \qquad \rho_0^2 = \cos^2 \lambda_0 = .15138.$$

The extremum of (5.10) is

$$(5.14) \qquad \frac{2\sqrt{.1514}(.8486)^{\frac{3}{2}}}{\pi(1 + 2 \cdot .1514)} = .1483 .$$

The value of .1483 is considerably less than 1.

5.2. Autoregression of Order 1

Consider the process

$$(5.15) \qquad\qquad y_t = \rho y_{t-1} + u_t,$$

where the u_t's are uncorrelated with mean $\mathcal{E}u_t = 0$ and variance $\mathcal{E}u_t^2 = \sigma^2$. The conventional (unnormalized) spectral density is

$$(5.16) \qquad\qquad \frac{\sigma^2}{2\pi \, |1 - \rho e^{i\lambda}|^2} = \frac{\sigma^2}{2\pi(1 + \rho^2 - 2\rho \, \cos \lambda)}.$$

The variance of y_t is

$$(5.17) \qquad\qquad \mathcal{E}{y_t}^2 = \frac{\sigma^2}{1 - \rho^2}.$$

The normalized spectral density and its square are

$$(5.18) \qquad\qquad f(\lambda) = \frac{1 - \rho^2}{2\pi(1 + \rho^2 - 2\rho \, \cos \lambda)}$$

and

$$(5.19) \qquad\qquad
\begin{aligned}
f^2(\lambda) &= \frac{(1 - \rho^2)^2}{4\pi^2(1 + \rho^2 - 2\rho \, \cos \lambda)^2} \\
&= \frac{(1 - \rho^2)^2}{4\pi^2 \, |(1 - \rho e^{i\lambda})^2|^2} \\
&= \frac{(1 - \rho^2)^2}{4\pi^2 \, |1 - 2\rho e^{i\lambda} + \rho^2 e^{i2\lambda}|^2}.
\end{aligned}$$

This is $(1 - \rho^2)^2/(2\pi)$ times the spectral density of x_t satisfying

$$(5.20) \qquad\qquad x_t - 2\rho x_{t-1} + \rho^2 x_{t-2} = v_t,$$

where $\mathcal{E}v_t = 0$, $\mathcal{E}v_t^2 = 1$, and $\mathcal{E}v_t v_s = 0$, $t \neq s$. The variance of x_t is $(1+\rho^2)/(1-\rho^2)^3$. Thus

$$(5.21) \qquad\qquad
\begin{aligned}
G(\pi) &= 2 \int_0^\pi f^2(\lambda) \, d\lambda \\
&= \frac{1}{2\pi} \cdot \frac{1 + \rho^2}{1 - \rho^2}.
\end{aligned}$$

The equation $G(\pi) = f(\lambda)$ is

(5.22)
$$\frac{1+\rho^2}{1-\rho^2} = \frac{1-\rho^2}{1+\rho^2 - 2\rho \cos \lambda}$$

or

(5.23)
$$1 + \rho^2 - 2\rho \cos \lambda = \frac{(1-\rho^2)^2}{1+\rho^2}$$

or

(5.24)
$$\cos \lambda_0 = \frac{2\rho}{1+\rho^2}.$$

For example, $\cos \lambda_0 = \sqrt{2}/2 = .707$ for $\rho = \sqrt{2}-1 = .414$, that is, $\lambda_0 = \pi/4$; $\cos \lambda_0 = 0$ for $\rho = 0$, that is, $\lambda_0 = \frac{\pi}{2}$; $\cos \lambda_0 = 1$ for $\rho = 1$, that is, $\lambda_0 = 0$. The normalized spectral distribution of $\{y_t\}$ is

(5.25)
$$\begin{aligned}
F(\lambda) &= \frac{(1-\rho^2)}{\pi} \int_0^\lambda \frac{d\lambda}{1+\rho^2 - 2\rho \cos \lambda} \\
&= \frac{2(1-\rho^2)}{\pi\sqrt{(1+\rho^2)^2 - (2\rho)^2}} \tan^{-1}\left[\frac{\sqrt{(1+\rho^2)^2 - (2\rho)^2}}{1+\rho^2 - 2\rho} \tan \frac{\lambda}{2}\right] \\
&= \frac{2}{\pi}\tan^{-1}\left[\frac{1+\rho}{1-\rho}\tan\frac{\lambda}{2}\right].
\end{aligned}$$

The integral of the square of the density is

(5.26)
$$\begin{aligned}
G(\lambda) &= \frac{(1-\rho^2)^2}{2\pi^2}\int_0^\lambda \frac{d\lambda}{(1+\rho^2 - 2\rho \cos \lambda)^2} \\
&= \frac{(1-\rho^2)^2}{2\pi^2}\left[\frac{2\rho}{(1-\rho^2)^2}\cdot\frac{\sin\lambda}{1+\rho^2 - 2\rho\cos\lambda}\right. \\
&\quad \left. + \frac{1+\rho^2}{(1-\rho^2)^2}\cdot\frac{2}{(1-\rho^2)}\tan^{-1}\left(\frac{1+\rho}{1-\rho}\tan\frac{\lambda}{2}\right)\right] \\
&= \frac{\rho}{\pi^2}\cdot\frac{\sin\lambda}{1+\rho^2 - 2\rho\cos\lambda} + \frac{1+\rho^2}{\pi^2(1-\rho^2)}\tan^{-1}\left(\frac{1+\rho}{1-\rho}\tan\frac{\lambda}{2}\right) \\
&= \frac{2\rho\,\sin\lambda}{\pi(1-\rho^2)}f(\lambda) + \frac{1+\rho^2}{2\pi(1-\rho^2)}F(\lambda).
\end{aligned}$$

Then

(5.27)
$$\frac{G(\lambda)}{G(\pi)} - F(\lambda) = \frac{2\pi(1-\rho^2)}{1+\rho^2}\left[\frac{\rho\,\sin\lambda}{\pi^2(1+\rho^2 - 2\rho\cos\lambda)}\right.$$

$$+\frac{1+\rho^2}{\pi^2(1-\rho^2)}\tan^{-1}\left(\frac{1+\rho}{1-\rho}\tan\frac{\lambda}{2}\right)\Big]$$

$$-\frac{2}{\pi}\tan^{-1}\left(\frac{1+\rho}{1-\rho}\tan\lambda\right)$$

$$=\frac{2\rho(1-\rho^2)}{\pi(1+\rho^2)}\cdot\frac{\sin\lambda}{1+\rho^2-2\rho\cos\lambda}$$

$$=\frac{4\rho}{1+\rho^2}\sin\lambda f(\lambda).$$

Note that $q(u) > 0$ for $\rho > 0$ and $q(u) < 0$ for $\rho < 0$. For $\cos\lambda_0 = \frac{2\rho}{1+\rho^2}$ $\left(\sin\lambda_0 = \frac{1-\rho^2}{1+\rho^2}\right)$

$$(5.28) \qquad \frac{G(\lambda_0)}{G(\pi)} - F(\lambda_0) = \frac{2\rho(1-\rho^2)}{\pi(1+\rho^2)}\cdot\frac{1-\rho^2}{(1+\rho^2)}\cdot\frac{1+\rho^2}{(1-\rho^2)^2}$$

$$= \frac{2\rho}{\pi(1+\rho^2)}$$

$$= \frac{1}{\pi}\cos\lambda_0.$$

The difference (5.28) approaches $1/\pi = .3184$ as $\lambda_0 \to 0$ (that is, as $\rho \to 1$) and approaches $-1/\pi$ as $\lambda_0 \to \pi$ (that is, as $\rho \to -1$). This is the maximum of $|q(u)|$. Note that .3184 is less than the maximum of 1 over processes but greater than the maximum of .1486 for moving average processes of order 1.

5.3. Another Example

Suppose $f(\lambda) = (\alpha+1)|\lambda|^\alpha/(2\pi^{\alpha+1})$, $\alpha \geq 0$. Then

$$(5.29) \qquad F(\lambda) = \frac{\lambda^{\alpha+1}}{\pi^{\alpha+1}}, \qquad 0 \leq \lambda \leq \pi,$$

$$(5.30) \qquad G(\lambda) = \frac{(\alpha+1)^2}{2(2\alpha+1)\pi^{2\alpha+2}}\lambda^{2\alpha+1}, \qquad 0 \leq \lambda \leq \pi,$$

$$(5.31) \qquad G(\pi) = \frac{(\alpha+1)^2}{2(2\alpha+1)\pi},$$

$$(5.32) \qquad u = \frac{G(\lambda)}{G(\pi)} = \frac{\lambda^{2\alpha+1}}{\pi^{2\alpha+1}}, \qquad 0 \leq \lambda \leq \pi,$$

$$(5.33) \qquad \lambda = \pi u^{1/(2\alpha+1)}, \qquad 0 \le u \le 1,$$

$$(5.34) \qquad \begin{aligned} q(u) &= u - u^{(\alpha+1)/(2\alpha+1)} \\ &= u^{(\alpha+1)/(2\alpha+1)} \left(u^{\alpha/(2\alpha+1)} - 1 \right), \end{aligned}$$

$$(5.35) \qquad \frac{d}{du} q(u) = 1 - \frac{\alpha+1}{2\alpha+1} u^{-\alpha/(2\alpha+1)}.$$

The maximum of $|q(u)|$ in $[0,1]$ is at

$$(5.36) \qquad u^{\alpha/(2\alpha+1)} = \frac{\alpha}{2\alpha+1},$$

and the maximum is the absolute value of

$$(5.37) \qquad \begin{aligned} \left(\frac{\alpha}{2\alpha+1} \right)^{\frac{2\alpha+1}{\alpha}} - \left(\frac{\alpha}{2\alpha+1} \right)^{\frac{\alpha+1}{\alpha}} &= \left(\frac{1}{2+\frac{1}{\alpha}} \right)^{2+\frac{1}{\alpha}} - \left(\frac{1}{2+\frac{1}{\alpha}} \right)^{1+\frac{1}{\alpha}} \\ &= x^{-x} - x^{1-x} \\ &= x^{-x}(1-x), \end{aligned}$$

where $x = 2 + \frac{1}{\alpha}$ (≥ 2). The derivative of (5.37) with respect to x is

$$(5.38) \qquad -x^{-x}[(1-x)\log x + 2 - x],$$

which is positive for $x \ge 2$. Thus (5.37) is an increasing function of x and a decreasing function of α. As $\alpha \to \infty$, the limit of (5.37) is -.25.

6. Conditions for Weak Convergence

Grenander and Rosenblatt (1953) showed that if $\gamma_j = O(j^{-\beta})$ for some $\beta > 3/2$ and if the u_t's are independently identically distributed with $\mathcal{E}u_t = 0$, $\mathcal{E}u_t^2 = \sigma^2$, $\mathcal{E}u_t^4 = \kappa_4 + 3\sigma^4$, and $\mathcal{E}u_t^8 < \infty$, then

$$(6.1) \qquad \sqrt{T} \max_{0 \le \lambda \le \pi} |\hat{H}_T(\lambda) - H(\lambda)| \overset{d}{\to} \max_{0 \le \lambda \le \pi} |U(\lambda)|,$$

where $H(\lambda) = \sigma(0)F(\lambda)$, $\hat{H}_T(\lambda) = c_0\hat{F}_T(\lambda)$, and $U(\lambda)$ is a Gaussian process with $\mathcal{E}U(\lambda) = 0$ and covariance function

$$(6.2) \qquad \mathcal{E}U(\lambda)U(\nu) = 4\pi G[\min(\lambda, \nu)] + \frac{k_4}{\sigma^4}H(\lambda)H(\nu).$$

A numuber of authors have shown that

$$(6.3) \qquad \sqrt{T}[\hat{H}_T(\lambda) - H(\lambda)] \xrightarrow{w} U(\lambda)$$

under various alternative sets of conditions. Ibragimov and Tovstik (1964) demonstrated the limiting covariance function (6.2) under the conditions $\sum_{i=0}^{\infty} \gamma_i^2 < \infty$, $G(\pi) < \infty$, u_t independently identically distributed, and $\mathcal{E}u_t^4 < \infty$. Shaman (1971) also proved this result. See Dahlhaus (1985) for a review of the literature.

We can write

$$
\begin{aligned}
(6.4) \qquad \hat{F}_T(\lambda) - F(\lambda) &= \frac{\hat{H}_T(\lambda)}{c_0} - \frac{H(\lambda)}{\sigma(0)} \\
&= \frac{\hat{H}_T(\lambda) - H(\lambda) - [c_0 - \sigma(0)]H(\lambda)/\sigma(0)}{c_0} \\
&= \frac{\hat{H}_T(\lambda) - H(\lambda) - [\hat{H}_T(\pi) - H(\pi)]F(\lambda)}{c_0}.
\end{aligned}
$$

If $c_0 \xrightarrow{P} \sigma(0)$ as $T \to \infty$, then $\sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)]$ converges weakly to $Z(\lambda) = U(\lambda) - U(\pi)f(\lambda)$, which is Gaussian with expected value 0 and covariance function

$$
\begin{aligned}
(6.5) \quad \mathcal{E}Z(\lambda)Z(\nu) &= \mathcal{E}[U(\lambda) - U(\pi)F(\lambda)][U(\lambda) - U(\pi)F(\nu)] \\
&= 4\pi\{G[\min(\lambda, \nu) - f(\lambda)G(\nu) - G(\lambda)F(\nu) + G(\pi)F(\lambda)F(\nu)\},
\end{aligned}
$$

which is exactly (2.19).

Note that there is no term in (6.5) involving κ_4. This fact suggests that the condition $\mathcal{E}u_x^4 < \infty$ may be unnecessary. Anderson (1991b) has shown that $\sqrt{T}[\hat{F}_T - F(\lambda)] \xrightarrow{w} Z(\lambda)$ if $\{y_t\}$ satisfies (2.10) with (2.12) holding,

$$(6.6) \qquad \sum_{s=0}^{\infty} \sqrt{s}|\gamma_s| < \infty,$$

and the $\{u_t\}$ being independently identically distributed with $\mathcal{E}u_t = 0$ and $\mathcal{E}u_t^2 = \sigma^2$. This result is generalized to the following theorem:

Theorem. Suppose $\{y_t\}$ is defined by (2.10), where $\{\gamma_s\}$ satisfies (2.12)

$$(6.7) \qquad \sum_{s=0}^{\infty} \sqrt{s}\,|\gamma_s| < \infty.$$

Let $\mathcal{F}_t$ be the σ-algebra generated by $\cdots, u_{t-1}, u_t$, and let $I(A) = 1$ if A occurs and $I(A) = 0$ otherwise. Suppose $\{u_t\}$ satisfies

$$(6.8) \qquad \mathcal{E}(u_t|\mathcal{F}_{t-1}) = 0 \quad \text{a.e.,}$$

$$(6.9) \qquad \mathcal{E}(u_t^2|\mathcal{F}_{t-1}) = \sigma_t^2 \quad \text{a.e.,}$$

$$(6.10) \qquad \frac{1}{T}\sum_{t=1}^{T}\sigma_t^2 \xrightarrow{\text{P}} \sigma^2,$$

where σ^2 is a constant,

$$(6.11) \qquad \sup_{t=1,2,\cdots} \mathcal{E}[u_t^2 I(u_t^2 > a)|\mathcal{F}_{t-1}] \xrightarrow{\text{P}} 0$$

as $a \to \infty$,

$$(6.12) \qquad \frac{1}{T}\sum_{t=1}^{T}\sigma_t^2 u_{t-r} u_{t-s} \xrightarrow{\text{P}} \delta_{rs}\sigma^4, \quad r,s = 1,2,\cdots,$$

where $\delta_{rr} = 1$ and $\delta_{rs} = 0, r \neq s$,

$$(6.13) \qquad \mathcal{E}u_t^2 \leq K, \quad t = \cdots, -1, 0, 1, \cdots,$$

$$(6.14) \qquad \mathcal{E}u_t^2 u_s^2 \leq K, \quad t \neq s,$$

and

$$(6.15) \qquad \mathcal{E}u_t^2 u_{t-s} u_{t-s} = 0, \quad r \neq s, \quad r,s = 1,2,\cdots.$$

Then

$$(6.16) \qquad \sqrt{T}[\hat{F}_T(\lambda) - F(\lambda)] \xrightarrow{w} Z(\lambda).$$

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Table 1

$$\Pr\left\{\sup_{0\leq u\leq 1} |B(u)| + d\,|X| \leq w\right\}$$

w \ d	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
0.4	0.0028	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	0.0361	0.0113	0.0059	0.0037	0.0025	0.0019	0.0015	0.0012	0.0010	0.0008
0.6	0.1357	0.0625	0.0359	0.0245	0.0184	0.0146	0.0121	0.0104	0.0090	0.0072
0.7	0.2888	0.1730	0.1088	0.0768	0.0588	0.0474	0.0396	0.0340	0.0298	0.0238
0.8	0.4559	0.3251	0.2249	0.1647	0.1281	0.1042	0.0876	0.0755	0.0663	0.0532
0.9	0.6073	0.4856	0.3662	0.2800	0.2224	0.1828	0.1547	0.1339	0.1178	0.0950
1.0	0.7300	0.6294	0.5113	0.4083	0.3319	0.2770	0.2361	0.2053	0.1813	0.1467
1.1	0.8223	0.7455	0.6428	0.5358	0.4474	0.3783	0.3262	0.2851	0.2529	0.2056
1.2	0.8878	0.8325	0.7504	0.6514	0.5588	0.4806	0.4188	0.3691	0.3287	0.2689
1.3	0.9319	0.8943	0.8321	0.7487	0.6597	0.5776	0.5094	0.4526	0.4057	0.3340
1.4	0.9603	0.9359	0.8923	0.8264	0.7460	0.6654	0.5944	0.5328	0.4802	0.3994
1.5	0.9778	0.9626	0.9334	0.8839	0.8157	0.7423	0.6713	0.6075	0.5514	0.4627
1.6	0.9880	0.9790	0.9603	0.9245	0.8706	0.8053	0.7388	0.6754	0.6182	0.5233
1.7	0.9938	0.9887	0.9772	0.9530	0.9127	0.8567	0.7955	0.7347	0.6777	0.5804
1.8	0.9969	0.9941	0.9873	0.9717	0.9421	0.8970	0.8433	0.7875	0.7323	0.6329
1.9	0.9985	0.9971	0.9932	0.9835	0.9621	0.9275	0.8830	0.8319	0.7797	0.6824
2.0	0.9993	0.9986	0.9965	0.9906	0.9764	0.9512	0.9129	0.8680	0.8208	0.7260
2.1	0.9997	1.0000	0.9983	0.9948	0.9856	0.9672	0.9366	0.8983	0.8551	0.7667
2.2	0.9999	1.0000	1.0000	0.9973	0.9915	0.9777	0.9556	0.9237	0.8846	0.8015
2.3	0.9999	1.0000	1.0000	0.9986	0.9951	0.9858	0.9683	0.9431	0.9092	0.8338
2.4	1.0000	1.0000	1.0000	1.0000	0.9972	0.9911	0.9788	0.9572	0.9302	0.8614
2.5	1.0000	1.0000	1.0000	1.0000	0.9985	0.9945	0.9848	0.9697	0.9456	0.8853
3.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9984	0.9947	0.9883	0.9606
3.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9978	0.9882
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9971

Values of w for which $\Pr\left\{\sup_{0\leq u\leq 1} |B(u)| + d\,|X| \leq w\right\}$

w \ d	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
0.001	0.3742	0.4171	0.4416	0.4616	0.4742	0.4829	0.4894	0.4946	0.4991	0.5067
0.01	0.4410	0.4947	0.5238	0.5452	0.5620	0.5758	0.5874	0.5977	0.6069	0.6232
0.05	0.5196	0.5834	0.6253	0.6572	0.6833	0.7058	0.7259	0.7439	0.7606	0.7908
0.10	0.5712	0.6402	0.6906	0.7305	0.7642	0.7939	0.8206	0.8452	0.8682	0.9106
0.25	0.6765	0.7527	0.8188	0.8756	0.9262	0.9727	1.0159	1.0569	1.0961	1.1706
0.50	0.8276	0.9094	0.9921	1.0712	1.1465	1.2190	1.2894	1.3589	1.4265	1.5616
0.75	1.0192	1.1045	1.1995	1.3015	1.4050	1.5122	1.6183	1.7265	1.8358	2.0583
0.90	1.2238	1.3114	1.4158	1.5349	1.6657	1.8086	1.9552	2.1039	2.2603	2.5746
0.95	1.3581	1.4470	1.5562	1.6871	1.8340	1.9936	2.1651	2.3488	2.5247	2.8967
0.99	1.6276	1.7192	1.8385	1.9890	2.1696	2.3757	2.6007	2.8387	3.0469	3.5591
0.999	1.9495	2.0443	2.1762	2.3516	2.5687	2.8192	3.0942	3.3866	3.6906	4.3193

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Table 2

$$\Pr\left\{\sup_{0\leq u\leq 1} |B(u)| + |d|\,|X| \leq w\right\}$$

w \ d	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0198	0.0050	0.0026	0.0018	0.0013	0.0011	0.0009	0.0008	0.0007	0.0005
0.2	0.0769	0.0350	0.0198	0.0135	0.0103	0.0082	0.0069	0.0059	0.0052	0.0041
0.3	0.1647	0.0980	0.0613	0.0432	0.0331	0.0267	0.0224	0.0193	0.0169	0.0136
0.4	0.2739	0.1893	0.1299	0.0948	0.0737	0.0600	0.0505	0.0436	0.0383	0.0308
0.5	0.3935	0.2994	0.2217	0.1682	0.1331	0.1094	0.0926	0.0801	0.0705	0.0568
0.6	0.5132	0.4180	0.3291	0.2592	0.2094	0.1741	0.1483	0.1288	0.1137	0.0920
0.7	0.6247	0.5352	0.4428	0.3616	0.2985	0.2513	0.2157	0.1883	0.1669	0.1355
0.8	0.7220	0.6431	0.5541	0.4679	0.3949	0.3370	0.2919	0.2564	0.2280	0.1861
0.9	0.8021	0.7365	0.6563	0.5712	0.4928	0.4268	0.3732	0.3300	0.2949	0.2420
1.0	0.8647	0.8130	0.7446	0.6657	0.5868	0.5161	0.4561	0.4062	0.3649	0.3015
1.1	0.9111	0.8723	0.8172	0.7479	0.6729	0.6010	0.5371	0.4822	0.4356	0.3626
1.2	0.9439	0.9162	0.8738	0.8160	0.7482	0.6786	0.6134	0.5554	0.5049	0.4238
1.3	0.9660	0.9472	0.9161	0.8701	0.8115	0.7469	0.6831	0.6240	0.5711	0.4836
1.4	0.9802	0.9679	0.9462	0.9111	0.8627	0.8052	0.7450	0.6867	0.6329	0.5411
1.5	0.9889	0.9813	0.9667	0.9411	0.9026	0.8533	0.7983	0.7426	0.6894	0.5954
1.6	0.9940	0.9895	0.9802	0.9622	0.9327	0.8919	0.8433	0.7915	0.7401	0.6460
1.7	0.9969	0.9943	0.9886	0.9765	0.9548	0.9220	0.8803	0.8334	0.7850	0.6926
1.8	0.9985	0.9971	0.9937	0.9858	0.9703	0.9449	0.9101	0.8687	0.8240	0.7351
1.9	0.9993	0.9985	0.9966	0.9917	0.9810	0.9619	0.9336	0.8979	0.8576	0.7733
2.0	0.9997	0.9993	0.9983	0.9953	0.9882	0.9742	0.9517	0.9216	0.8860	0.8075
2.1	0.9999	0.9997	0.9991	0.9974	0.9928	0.9828	0.9655	0.9407	0.9097	0.8377
2.2	0.9999	0.9999	0.9996	0.9986	0.9957	0.9888	0.9758	0.9557	0.9292	0.8641
2.3	1.0000	0.9999	0.9998	0.9993	0.9975	0.9929	0.9832	0.9673	0.9451	0.8871
2.4	1.0000	1.0000	0.9999	0.9996	0.9986	0.9955	0.9886	0.9762	0.9579	0.9069
2.5	1.0000	1.0000	1.0000	0.9998	0.9992	0.9973	0.9924	0.9829	0.9681	0.9238
3.0	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9992	0.9973	0.9932	0.9751
3.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9933
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9985

Table 3

$$\Pr\left\{\sup_{0\le u\le 1}|B(u)| - |d|\,|X| \le w\right\}$$

w \ d	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0
-4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005
-2.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0020	0.0120
-1.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0013	0.0051	0.0128	0.0423
-1.25	0.0000	0.0000	0.0000	0.0000	0.0001	0.0010	0.0048	0.0136	0.0284	0.0734
-1.0	0.0000	0.0000	0.0000	0.0000	0.0007	0.0046	0.0149	0.0330	0.0579	0.1210
-0.9	0.0000	0.0000	0.0000	0.0001	0.0015	0.0080	0.0226	0.0456	0.0753	0.1458
-0.8	0.0000	0.0000	0.0000	0.0002	0.0032	0.0134	0.0334	0.0620	0.0968	0.1743
-0.7	0.0000	0.0000	0.0000	0.0007	0.0063	0.0219	0.0484	0.0831	0.1229	0.2067
-0.6	0.0000	0.0000	0.0000	0.0019	0.0121	0.0347	0.0685	0.1096	0.1541	0.2432
-0.5	0.0000	0.0000	0.0002	0.0048	0.0219	0.0533	0.0950	0.1422	0.1910	0.2840
-0.4	0.0000	0.0000	0.0009	0.0110	0.0379	0.0793	0.1290	0.1818	0.2339	0.3292
-0.3	0.0000	0.0000	0.0035	0.0235	0.0627	0.1146	0.1717	0.2289	0.2832	0.3788
-0.2	0.0000	0.0002	0.0113	0.0461	0.0993	0.1611	0.2240	0.2840	0.3390	0.4327
-0.1	0.0000	0.0030	0.0307	0.0840	0.1506	0.2201	0.2866	0.3473	0.4013	0.4908
0	0.0000	0.0194	0.0715	0.1425	0.2191	0.2929	0.3598	0.4188	0.4700	0.5528
0.1	0.0198	0.0682	0.1414	0.2243	0.3053	0.3787	0.4426	0.4973	0.5439	0.6178
0.2	0.0769	0.1488	0.2357	0.3233	0.4031	0.4719	0.5300	0.5786	0.6194	0.6829
0.3	0.1647	0.2525	0.3452	0.4309	0.5048	0.5661	0.6165	0.6580	0.6922	0.7448
0.4	0.2739	0.3690	0.4600	0.5387	0.6035	0.6556	0.6974	0.7313	0.7590	0.8010
0.5	0.3935	0.4880	0.5714	0.6397	0.6936	0.7359	0.7693	0.7959	0.8174	0.8498
0.6	0.5132	0.6006	0.6727	0.7288	0.7717	0.8046	0.8300	0.8501	0.8663	0.8904
0.7	0.6247	0.7005	0.7594	0.8034	0.8360	0.8604	0.8791	0.8937	0.9054	0.9226
0.8	0.7220	0.7841	0.8299	0.8627	0.8864	0.9038	0.9170	0.9272	0.9353	0.9472
0.9	0.8021	0.8505	0.8842	0.9077	0.9242	0.9361	0.9451	0.9519	0.9573	0.9653
1.0	0.8647	0.9004	0.9242	0.9402	0.9512	0.9591	0.9649	0.9693	0.9728	0.9779
1.25	0.9561	0.9697	0.9778	0.9829	0.9863	0.9886	0.9903	0.9915	0.9925	0.9939
1.5	0.9889	0.9928	0.9949	0.9962	0.9969	0.9975	0.9979	0.9981	0.9984	0.9987
2.0	0.9997	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000
2.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE October 1991	3. REPORT TYPE AND DATES COVERED Technical Report

4. TITLE AND SUBTITLE	5. FUNDING NUMBERS
Goodness of Fit Tests for Spectral Distributions	DAAL03-89-K-0033

6. AUTHOR(S)

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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
Department of Statistics Sequoia Hall Stanford University Stanford, CA 94305	27

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER
U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709-2211	ARO 26394.6-MA

11. SUPPLEMENTARY NOTES

The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
Approved for public release; distribution unlimited.	

13. ABSTRACT (Maximum 200 words)

The spectral distribution function of a stationary stochastic process standardized by dividing by the variance of the process is a linear function of the autocorrelations. The integral of the sample standardized spectral density (periodogram) is a similar linear function of the autocorrelations. As the sample size increases, the difference of these two functions multiplied by the square root of the sample size converges is weakly to a Gaussian stochastic process with a continuous time parameter. A monotonic transformation of this parameter yields a Brownian bridge plus an independent random term. The distributions of functionals of this process are the limiting distributions of goodness of fit criteria that are used for testing hypothese about the process autocorrelations. An application is to tests of independence (flat spectrum). The characteristic function of the Cramér-von Miese statistic is obtained; inequalities for the Kolmogorov-Smirnov criterion are given. Confidence regions for unspecified process distributions are found.

14. SUBJECT TERMS	15. NUMBER OF PAGES
Goodness of fit tests, spectral distributions, Cramér-von Mises test, Kolmogorov-Smirov test.	
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL