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Order statistics play an important role in reliability. The life time of any coherent system is the first order statistic of the, generally dependent, lives of its cut sets. For the important class of k-out-of-n systems, the lifetime of the system is the $n-k+1$th order statistic of the lives of its components, which are often assumed to be independent. Therefore the reliability of many systems can be easily stated as a probability concerning an order statistic.

A system is called a second order r-out-of-k system if it is a r-out-of-k system based on k modules, without common components, and where each module is an a_i-out-of-n_i system. Two features of such systems are of interest, namely the probability that a particular module is among the modules that failed before the failure of the system and the number of failed components at the time of the failure of the system. In this paper, we review results regarding these features for some special cases of second order r-out-of-k systems, emphasizing their applications to optimal allocation problems.

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Abstract

Order statistics play an important role in reliability. The life time of any coherent system is the first order statistic of the, generally dependent, lives of its cut sets. For the important class of k–out-of–n systems, the lifetime of the system is the $n-k+1$th order statistic of the lives of its components, which are often assumed to be independent. Therefore the reliability of many systems can be easily stated as a probability concerning an order statistic.

A system is called a second order r-out-of-k system if it is a r-out-of-k system based on k modules, without common components, and where each module is an a_i-out-of-n_i system. Two features of such systems are of interest, namely the probability that a particular module is among the modules that failed before the failure of the system and the number of failed components at the time of the failure of the system. In this paper, we review results regarding these features for some special cases of second order r-out-of-k systems, emphasizing their applications to optimal allocation problems.

1. Introduction

Order statistics play an important role in reliability. The life time of any coherent system is the first order statistic of the, generally dependent, lives of its cut sets. For the important class of k–out-of–n systems, the lifetime of the system is the $n-k+1$th order statistic of the lives of its components, which are often assumed to be independent. Therefore the reliability of many systems can be easily stated as a probability concerning an order statistic.

Consider a system S consisting of modules $P_1, P_2, \ldots, P_k$ where the modules have no common components. Such a system is called a second order r–out-of–k system if it is an r–out-of–k system based on the k modules and P_i is an a_i–out-of–n_i system, $i = 1, 2, \ldots, k$. The lifetime of the system is the $k-r+1$th order statistic of the independent lifetimes of its modules, which are themselves order statistics of the independent lives of their respective components. The rich structure of the system S allows us to investigate several interesting questions which are essentially questions regarding distributions of order statistics. The following two events are of special interest:

1. The event that the lifetime of a particular module P_1 is less than or equal to the lifetime of the system. This is the same as the event that the rank of the lifetime of P_1 is less that or equal to $k - r + 1$ among the lifetimes of the k modules. The probability of this event is defined to be the role of the module P_1 in the failure of the system S in El-Neweihi and Sethuraman (1991). This probability is useful to determine the contribution of P_1 towards the failure of the system and can be viewed as a measure of the importance of the module P_1.

2. The event that at least l of the $n_1 + \cdots + n_k$ components have failed at the time of the failure of the system. This is the same as the event that the lifetime of the system S is greater than or equal to the lth order statistic of the lives of the $n_1 + \cdots + n_k$ components. The probability of this event is useful in assessing the damage to the system (measured in terms of the number of failed components) at the time of its failure.

Several interesting properties of the probabilities of the above mentioned events were derived in El-Neweihi,Proschan and Sethuraman (1978), El-Neweihi (1980), El-Neweihi and Sethuraman (1991) and Ross, Shahshahani and Weiss (1980).

In this paper we review results from these papers accentuating their applications to optimal allocation problems.

2. Series-parallel system

Consider a system S which is a series system based on modules $C_0, C_1, \ldots, C_k$ where C_i is a parallel system based on n_i components, $i = 1, \ldots, k$. We assume that the lifetimes of $n = n_0 + n_1 + \cdots + n_k$ components are independent with a common continuous distribution. In this section we review results from Proschan, El-Neweihi and Sethuraman (1978) who studied this system. In the following $\mathbf{n}$ will stand for the vector $(n_1, n_2, \cdots, n_k)$.

Let T_{ij} be the lifetimes of the jth component in module $P_i, 1 \leq j \leq n_i, 0 \leq i \leq k$. Let $T_i = \max_{1 \leq j \leq n_i} T_{ij}$ be the lifetimes of module $P_i, 0 \leq i \leq k$. The probability that the failure of the cut set C_0 causes the failure of the system S, *i.e.* the role of C_0, will be denoted by $P(n_0; \mathbf{n})$. It is easy to see that $P(n_0; \mathbf{n}) = P(T_0 < \min_{1 \le i \le k} T_i)$. This provides a method to obtain Theorem 2.1 below which gives a compact expression for $P(n_0; \mathbf{n})$.

Theorem 2.1

$$P(n_0; \mathbf{n}) = \int_0^1 \prod_{i=1}^{n} (1 - x_i)^{n_i} n_0 x^{n_0 - 1} \, dx.$$

From this it follows that $P(n_0; \mathbf{n})$ is a Schur-concave function of $\mathbf{n}$. The implication of this satement is that C_0 is more likely to fail first if the remaining cut sets are homogeneous in size than if they are more heterogeneous.

When maintaining a system S as above one will have to repair it when it fails. When the system fails one will have to inspect the cut sets $C_0, \ldots, C_k$ to see which one has failed. It may be physically more convenient to inspect and repair C_0 than the other cut sets. In this situation one would like to maximize $P(n_0; \mathbf{n})$. Theorem 2.1 above shows that this is done by equalizing the sizes of the cut sets $C_1, \ldots, C_k$.

Let $L(\mathbf{n})$ be the number of components that have failed in all the modules at the time of the failure of the system S. The following were proved in El-Neweihi, Proschan and Sethuraman, (1978):

1 : $L(\mathbf{n}) \overset{st}{\geq} L(\mathbf{n}^*)$ if $\mathbf{n}^* \overset{m}{\geq} \mathbf{n}$.
2 : The distribution of $L(\mathbf{n})$ is NBU.

It was also conjectured in that paper that the distribution of $L(\mathbf{n})$ is IFR; this was later proved in Ross, Shahshahani and Weiss (1980).

3. A (k+1−r+1)-out-of-(k+1) system based on parallel modules

Consider a system S constructed from $k+1$ modules $P_0, P_1, \ldots, P_k$. Assume that P_i contains n_i components whose lifetimes have a common continuous distribution $F_i(x)$, $i = 0, \ldots, k$. Assume that the $n_0 + \cdots + n_k$ components are independent. Let $\mathbf{n}$ denote $(n_1, \ldots, n_k)$. Consider the following structure **(A)** for S:

A1 : The modules $P_0, P_1, \ldots, P_k$ are all parallel systems, and
A2 : the system S is a $(k+1-r+1)$-out-of-$(k+1)$ system based on the $k+1$ modules $P_0, P_1, \ldots, P_k$.

This means that the system S fails as soon as r modules fail. In this section we review results from El-Neweihi and Sethuraman (1991) who studied this system.

Denote the lifetimes of the modules P_i by T_i, $i = 0, \ldots, k$ and let $R_0, R_1, \ldots, R_k$ be the ranks of $T_0, T_1, \ldots, T_k$. Denote the probability that P_0 is among the r modules that failed first and caused the failure of the system by

$$P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = \text{Prob}\{R_0 \leq r\}.$$

A study of properties of the quantity $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is useful to determine the contribution of the module P_0 towards the failure of S. This quantity may be viewed as a measure of importance of the module P_0

The system considered in this section reduces to the series-parallel system considered in Section 2 when $r = 1$ and $F_1 = F_2 = \cdots = F_k = F$.

Let $h_{r|k}(p_1, \ldots, p_k) = P\{\sum_i^k Y_i \geq r\}$ where $Y_1, \ldots, Y_k$ are k independent Bernoulli random variables with parameters $p_1, \ldots, p_k$. The quantity $h_{r|k}(p_1, \ldots, p_k)$ represents the reliability of an r-out-of-k system with k independent components having reliabilities $p_1, \ldots, p_k$.

A compact expression for $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is given by the following theorem.

Theorem 3.1

$$P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = 1 - \int h_{r|k}((F_1(x))^{n_1}, \ldots, (F_k(x))^{n_k}) dF_{T_0}(x).$$

The following theorem can be shown by using Theorem 3.1 and a result on order statistics from heterogeneous distributions found in Pledger and Proschan (1971).

Theorem 3.2 *For each n_0, F_0 and F, $P_r(n_0, F_0; \mathbf{n}, F)$ is Schur-concave in* $\mathbf{n}$.

This theorem states that the module P_0 is more likely to be among the modules that fail before the failure of the system S when the sizes of the modules $P_1, \ldots, P_k$ are more homogeneous. This fact is intuitively more obvious when $r = 1$, the case considered in El-Neweihi, Proschan and Sethuraman (1978). Theorem 3.2 shows that this is true for all values of r.

Theorem 3.2 has an application to optimal allocation along the lines of the remark following Theorem 2.1.

Assume that $n_1 = \cdots = n_k = n$ and that the life distribution F_i of the components of the module P_i have proportional hazards, *i.e.*, $\bar{F}_i(x) = \exp(-\lambda_i R(x)), i = 1, \ldots, k$. In this case, $P_r(n_0, F_0; n, \mathbf{F})$ is a function which depends on $\mathbf{F}$ only through λ and therefore may be denoted by $P_{r+}(n_0, F_0; n, \lambda)$. Theorem 3.3 below shows that $P_{r+}(n_0, F_0; n, \lambda)$

is Schur-concave in $\boldsymbol{\lambda}$ when $r = 1$. We do not know whether this result will extend to other cases of r.

Theorem 3.3 $P_{1+}(n_0, F_0; n, \boldsymbol{\lambda})$ *is Schur-concave in* $\boldsymbol{\lambda}$.

We can give more complete results if we assume that the distributions F_i have proportional left-hazards. Assume that $F_i(x) = \exp(-\lambda_i A(x)), i = 1, \ldots, k$. In this case, $P_r(n_0, F_0; n, \mathbf{F})$ is a function which depends on $\mathbf{F}$ only through $\boldsymbol{\lambda}$ and therefore may be denoted by $P_{r-}(n_0, F_0; n, \boldsymbol{\lambda})$. In Theorem 3.4 below we show that $P_{r-}(n_0, F_0; n, \boldsymbol{\lambda})$ is Schur-concave in $\boldsymbol{\lambda}$.

Theorem 3.4 $P_{r-}(n_0, F_0; n, \boldsymbol{\lambda})$ *is Schur-concave in* $\boldsymbol{\lambda}$.

El-Neweihi (1980) studied the joint monotonicity properties of $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ in $\mathbf{n}, \mathbf{F}$. He considered the case $r = 1$ and showed that $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$ is an AI function of $(\mathbf{n}, \mathbf{F})$. Example 2.8 of El-Neweihi and Sethuraman (1991) shows that this AI property is not generally true for other values of r.

4. Series system based on a_{i+1}-out-of-n_i systems

Consider an alternate structure **(B)** for the system S.

B1 : The module P_i is an $a_i + 1$-out-of-n_i system, $i = 0, \ldots, k$, and

B2 : the system S is a series system based on $P_0, P_1, \ldots, P_k$.

The system considered in this section reduces to the series-parallel system considered in Section 2 when $a_i = 0, i = 0, 1, \ldots, k$ and $F_1 = F_2 = \cdots = F_k = F$. This system allows for more general modules than the system considered in Section 3 and requires the modules to be connected in series.

Let T_{ij} be the lifetimes of the jth component in module $P_i, 1 \leq j \leq n_i, 0 \leq i \leq k$. Then, T_i, the lifetime of the module P_i, is the $n_i - a_i$th order statistic among $T_{ij}, 1 \leq j \leq n_i, 0 \leq i \leq k$. The probability that the module P_0 causes the system to fail, $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$, will now be denoted by $P(a_0, n_0, F_0; \mathbf{a}, \mathbf{n}, \mathbf{F})$ and this is equal to $P(T_0 < \min_{1 \leq i \leq k} T_i)$. We will say that $F \leq G$ if $F(x) \leq G(x)$ for all x. The following theorem gives an AI property using this ordering on distribution functions.

Theorem 4.1 $P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F})$ *is AI in* $\mathbf{n}, \mathbf{F}$, *for each* a_0, n_0, F_0, *and* a.

Theorem 4.8 of El-Neweihi (1980) treats the special case of the above when $a = a_0 = 0$.

We now give an application of the above results to an optimal allocation problem.

Suppose that the sizes $n_1, \ldots, n_k$ of the modules $P_1, \ldots, P_k$ are in increasing order. Suppose that we have collections of components with reliabilities $p_1 \geq \cdots \geq p_k$ at a particular time t. Theorem 4.1 shows that the reliability of S at time t is maximized by allocating components of reliability p_i to the module $P_i, i = 1, \ldots, k$.

The following theorem considers the case $n_i = n, F_i = F, i = 1, 2, \ldots, k$.

Theorem 4.2 *$P(a_0, n_0, F_0; \mathbf{a}, n, F)$ is Schur-concave in* $\mathbf{a}$.

Consider a series system based on modules $P_1, \ldots, P_k$ where P_i ia an a_i-out-of-n system, $i = 1, \ldots, k$. Assume that all the components have i.i.d. lifetimes. Theorem 4.2 above shows that the reliability of the system is maximized by equalizing the a_i's.

The case when $a_i = a, F_i = F, i = 1, 2, \ldots, k$ was treated in El-Neweihi, Proschan and Sethuraman (1978) where the following theorem was established.

Theorem 4.3 *$P(a_0, n_0, F_0; a, \mathbf{n}, F)$ is Schur-concave in* $\mathbf{n}$.

Theorem 4.3 shows that the probability that module P_0 fails first is Schur-concave in $\mathbf{n}$. We can ask the question whether the probability that module P_0 is among the first r modules to fail is also Schur-concave. The following example shows that this is not so for $r = 2$.

Example 4.4 Let $k = 2, a_1 = 1, a_2 = 1, F_0 = F_1 = F_2 = F$ where F is the uniform distribution on $[0, 1]$. Then
The probability that module P_0 is among the first two modules to fail

$$= 1 - \int [t^{n_1+n_2} + (n_1 + n_2)(1-t)t^{n_1+n_2-1} + n_1 n_2 (1-t)^2 t^{n_1+n_2-2}]$$
$$\times \binom{n_0}{a_0} (n_0 - a_0) t^{n_0 - a_0 - 1} (1-t)^{a_0} dt.$$

The integrand is Schur-concave in $\mathbf{n}$ and hence this probability is Schur-convex.

Theorems 4.3 has an obvious application to optimal allocation along the lines of the remarks following Theorem 4.1 and 4.2.

5. Number of failed components at system failure

Consider a series-parallel system S based on k modules $P_1, \ldots, P_k$, which are parallel systems with sizes $n_1, \ldots, n_k$, respectively. Suppose that the common distribution of the lifetimes of components in P_i is F_i, $i = 1, \ldots, k$. Let $L(\mathbf{n}, \mathbf{F})$ be the number of failed components in all the modules at the time of failure of the system S. In section 2 we reviewed results obtained by El-Neweihi, Proschan and Sethuraman (1978) and Ross, Shahshahani and Weiss (1980) on $L(\mathbf{n}, F)$ when $F_1 = \cdots, F_k = F$. El-Neweihi and Sethuraman (1991) studies properties of $E(L(\mathbf{n}, \mathbf{F}))$ without assuming that $F_1 = \cdots, F_k = F$. In this section we review those results.

Let $T_{ij}, j = 1, \ldots, n_i$ be the lifetimes of the n_i components in $P_i, i = 1, \ldots, k$. Let $T = \min_{1 \le i \le k} \max_{1 \le j \le n_i} T_{ij}$ be the lifetime of the system S. Clearly

$$L(\mathbf{n}, \mathbf{F}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} I\{T \ge T_{ij}\},$$

where $I\{A\}$ is the indicator of the event A. The following lemma gives a useful expression for $E(L(\mathbf{n}, \mathbf{F}))$.

Lemma 5.1 $E(L(\mathbf{n}, \mathbf{F})) = \sum_{i=1}^{k} n_i \int \{\prod_{l=1, l \ne i}^{k} [1 - (F_l(x))^{n_l}]\} dF_i(x)$.

This lemma can be used to show the following theorem.

Theorem 5.2 *The expected number of failed components in the system S at the time of system failure $E(L(\mathbf{n}, \mathbf{F}))$ is AI in $\mathbf{n}, \mathbf{F}$.*

An implication of the above result to optimal allocation in a series-parallel system S is as follows. Let S be a series system consisting of modules $P_1, \ldots, P_k$ be k which are parallel systems with $n_1 \le \cdots \le n_k$ components, respectively. Suppose that we have collections of components with life distributions $F_1 \le \cdots \le F_k$. Then one should allocate components with life distributions $F_{(n-i+1)}$ to the module P_i to minimize the expected number of component failures at the time of the failure of system S.

We now consider a parallel-series system S' where the modules $P'_1, \ldots, P'_k$ are series systems with the same number of components n. Assume further that $\bar{F}'_i(x) = \exp(-\lambda_i x), i = 1, \ldots, k$. Theorem 5.3 below shows that, when $k = 2$, the expected number of component failures at system failure is Schur-convex in (λ_1, λ_2).

Theorem 5.3 *Let $B(n, \mathbf{F}')$ be the expected number of component failures at system failure in the parallel-series system S' described above. Let $k = 2$. Then $B(n, \mathbf{F}')$ is Schur-convex in (λ_1, λ_2).*

Theorem 5.3 has an application to optimal allocation along the lines of the remark following Theorem 5.2.

6. Further extensions

The results reviewed in this paper pertain to special second order r-out-k systems. These are the only results available at this time. There is a need to investigate questions similar to those reviewed in this paper for more general second order systems . This would be a first step. New kinds of questions also arise for these systems. One can study the role of groups of modules rather than that of a single module. The role of a group of modules can be defined to be the probability that at least m of the modules in the group have failed prior to the failure of the system, where m can vary from 1 to the size of the group.

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