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THE ROLE OF A MODULE IN THE FAILURE OF A SYSTEM

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THE ROLE OF A MODULE IN THE FAILURE OF A SYSYTEM

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Abstract

Arrangement increasing and Schur functions play a central role in establishing stochastic inequalities in several areas of statistics and reliability. The role of a module in the failure of a system measures the importance of the module. We define the role to be the probability that this module is among the modules that failed before the failure of the system. A system is called a second order r-out-of-k system if it is a r-out-of-k system based on k modules, without common components, and where each module is an a_i-out-of-n_i system. For such systems, we show that the role of a module is an arrangement increasing or Schur function of parameters that describe the system. These results allow us to compare the role of a module under different values of the parameters of the system.

1. Introduction

In Reliability Theory, after answering questions concerning the **reliability** of a system, the **importance of a component** in a system becomes the next natural question to study. The importance of a component may be measured in many ways. It may be measured by the increment in reliability of the system per unit increase in the reliability of the component. This view is taken in the pioneering paper of Birnbaum (1969). Boland, El-Neweihi and Proschan (1988) and Natvig (1985) have built upon this concept of importance.

The probability that a component is among the components that failed before the failure of a system provides another measure of the importance of the component. This view can be found in Fussell and Vesely (1972) and Barlow and Proschan (1975).

A general summary of many different ways to measure the importance of a component may be found in the expository paper of Boland and El-Neweihi (1990).

A system generally consists of modules which themselves are subsystems of individual components. In this work we will talk about the role of a module in the failure of a system. There can be several notions of the **role** of a **module**. In this paper, we define the **role** of a module to be the probability that the the module is among the modules that caused the failure of the system.

We will compare the role of a module with the role of another module, or compare the role of several modules simultaneously, or compare the role of a module under several values of other parameters of the system. Each of these comparisons can be made by showing that the role of a module is an arrangement increasing or Schur function of the appropriate arguments. In this expository paper we describe such results without proof. The complete proofs are given in the cited references.

The theory of arrangement increasing (AI) and Schur functions play a central role in establishing stochastic inequalities in several areas of statistics and reliability. This theory is well established, for instance see Proschan and Sethuraman (1978), Hollander,Proschan and Sethuraman (1978). A comprehensive treatment of these functions is given in Marshall and Olkin (1980). We therefore do not give the definitions and known facts concerning arrangement increasing and Schur functions.

2. Series-parallel system

Consider a system S which is a series system based on modules $C_0, C_1, \ldots, C_k$ where C_i is a parallel system based on n_i components, $i = 1, \ldots, k$. We assume that the lifetimes of $n = n_0 + n_1 + \cdots + n_k$ components are independent with a common continuous distribution. This system was studied in Proschan, El-Neweihi and Sethuraman (1978). In the following $\mathbf{n}$ will stand for the vector $(n_1, n_2, \cdots, n_k)$.

The probability that the failure of the cut set C_0 causes the failure of the system S, *i.e.* the role of C_0, will be denoted by $P(n_0; \mathbf{n})$. It is easy to see that $P(n_0; \mathbf{n})$ is decreasing in n_0 and increasing in $\mathbf{n}$. Theorem 2.1 below gives a compact expression to evaluate $P(n_0; \mathbf{n})$.

Theorem 2.1

$$P(n_0;\mathbf{n}) = \int_0^1 \prod_{i=1}^{n}(1-x_i)^{n_i} n_0 x^{n_0-1}\,dx.$$

From this it follows that $P(n_0;\mathbf{n})$ is a Schur-concave function of $\mathbf{n}$. The implication of this satement is that C_0 is more likely to fail first if the remaining cut sets are homogeneous in size than if they are more heterogeneous.

Let $n_0 \leq n_1 \leq \cdots \leq n_k$. The order in which the cut sets will fail is another quantity of interest. This will compare the relative roles of all the cut sets. Let

$$Q(i_0, i_1, \ldots, i_k) = P(C_{i_0} < C_{i_1} < \cdots < C_{i_k}).$$

The following theorem can be found in El-Neweihi, Proschan and Sethuraman (1978).

Theorem 2.2

$$P(C_0 < C_1 < \cdots < C_k) = \prod_{i}^{k} n_i \prod_{i=1}^{k}(\sum_{j=0}^{i} n_j)^{-1}.$$

This shows that $Q(i_0, i_1, \ldots, i_k)$ is a AI function of $(i_0, i_1, \ldots, i_k)$ and thus the modules C_i are more likely to fail in the order of their sizes.

Let $L(\mathbf{n})$ be the number of components that have failed in all the modules at the time of the failure of the system S. The following were proved in El-Neweihi, Proschan and Sethuraman, (1978):

1 : $L(\mathbf{n}) \overset{st}{\geq} L(\mathbf{n}^*)$ if $\mathbf{n}^* \overset{m}{\geq} \mathbf{n}$.
2 : The distribution of $L(\mathbf{n})$ is NBU.

It was also conjectured in that paper that the distribution of $L(\mathbf{n})$ is IFR; this was later proved in Ross, Shahshahani and Weiss (1980).

3. A (k+1−r+1)-out-of-(k+1) system based on parallel modules

Consider a system S constructed from $k+1$ modules $P_0, P_1, \ldots, P_k$. Assume that P_i contains n_i components whose lifetimes have a common continuous distribution $F_i(x)$, $i = 0, \ldots, k$. Assume that the $n_0 + \cdots + n_k$ components are independent. Let $\mathbf{n}$ denote $(n_1, \ldots, n_k)$. Consider the following structure **(A)** for S:

A1 : The modules $P_0, P_1, \ldots, P_k$ are all parallel systems, and
A2 : the system S is a $(k+1-r+1)$-out-of-$(k+1)$ system based on the $k+1$ modules $P_0, P_1, \ldots, P_k$.

This means that the system S fails as soon as r modules fail .

Denote the lifetimes of the modules P_i by T_i, $i = 0, \ldots, k$ and let $R_0, R_1, \ldots, R_k$ be the ranks of $T_0, T_1, \ldots, T_k$. Denote the probability that P_0 is among the r modules that failed first and caused the failure of the system by

$$P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = \text{Prob}\{R_0 \leq r\}.$$

A study of properties of the quantity $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is useful to determine the contribution of the module P_0 towards the failure of S. This quantity may be viewed as a measure of importance of the module P_0

The system considered in this section reduces to the series-parallel system considered in Section 2 when $r = 1$ and $F_1 = F_2 = \cdots = F_k = F$.

Let $h_{r|k}(p_1, \ldots, p_k) = P\{\sum_i^k Y_i \geq r\}$ where $Y_1, \ldots, Y_k$ are k independent Bernoulli random variables with parameters $p_1, \ldots, p_k$. The quantity $h_{r|k}(p_1, \ldots, p_k)$ represents the reliability of an r-out-of-k system with k independent components having reliabilities $p_1, \ldots, p_k$.

A compact expression for $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is given by the following theorem.

Theorem 3.1

$$P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = 1 - \int h_{r|k}((F_1(x))^{n_1}, \ldots, (F_k(x))^{n_k}) dF_{T_0}(x).$$

The following theorem can be shown by using Theorem 3.1 and a result on order statistics from heterogeneous distributions found in Pledger and Proschan (1971).

Theorem 3.2 *For each n_0, F_0 and F, $P_r(n_0, F_0; \mathbf{n}, F)$ is Schur-concave in* $\mathbf{n}$.

This theorem states that the module P_0 is more likely to be among the modules that fail before the failure of the system S when the sizes of the modules $P_1, \ldots, P_k$ are more homogeneous. This fact is intuitively more obvious when $r = 1$, the case considered in El-Neweihi, Proschan and Sethuraman (1978). Theorem 3.2 shows that this is true for all values of r.

Let $P_{r*}(n_0, F_0; \mathbf{n}, F)$ be the probability that module P_0 is the rth module to fail among the modules $P_0, P_1, \ldots, P_k$. Clearly, $P_{r*}(n_0, F_0; \mathbf{n}, F) = P_r(n_0, F_0; \mathbf{n}, F) -$

$P_{r-1}(n_0, F_0; \mathbf{n}, F)$ and is therefore the difference of two Schur functions. It is not true that $P_{r*}(n_0, F_0; \mathbf{n}, F)$ is Schur-concave in $\mathbf{n}$. For instance when $k = 2, r = 2$ and $F_0 = F_1 = F_2 = F$, we have $P_{r*}(n_0, F_0; \mathbf{n}, F) = \int_0^1 (x^{n_1} + x^{n_2} - 2x^{n_1+n_2}) n_0 x^{n_0-1} dx$, which is Schur-convex in $\mathbf{n}$, for each n_0. This remark shows that the claim in Theorem 3.8 in El-Neweihi (1980) is false.

Assume that $n_1 = \cdots = n_k = n$ and that the life distribution F_i of the components of the module P_i have proportional hazards, *i.e.*, $\bar{F}_i(x) = \exp(-\lambda_i R(x)), i = 1, \ldots, k$. In this case, $P_r(n_0, F_0; n, \mathbf{F})$ is a function which depends on $\mathbf{F}$ only through λ and therefore may be denoted by $P_{r+}(n_0, F_0; n, \lambda)$. Theorem 3.3 below shows that $P_{r+}(n_0, F_0; n, \lambda)$ is Schur-concave in λ when $r = 1$. We do not know whether this result will extend to other cases of r.

Theorem 3.3 $P_{1+}(n_0, F_0; n, \lambda)$ *is Schur-concave in* λ.

We can give more complete results if we assume that the distributions F_i have proportional left-hazards. Assume that $F_i(x) = \exp(-\lambda_i A(x)), i = 1, \ldots, k$. In this case, $P_r(n_0, F_0; n, \mathbf{F})$ is a function which depends on $\mathbf{F}$ only through λ and therefore may be denoted by $P_{r-}(n_0, F_0; n, \lambda)$. In Theorem 3.4 below we show that $P_{r-}(n_0, F_0; n, \lambda)$ is Schur-concave in λ .

Theorem 3.4 $P_{r-}(n_0, F_0; n, \lambda)$ *is Schur-concave in* λ .

El Neweihi (1980) studied the joint monotonicity properties of $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ in $\mathbf{n}, \mathbf{F}$. He considered the case $r = 1$ and showed that $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$ is an AI function of $(\mathbf{n}, \mathbf{F})$. Example 2.8 of El-Neweihi and Sethuraman (1991) shows that this AI property is not generally true for other values of r.

4. Series system based on a_{i+1}-out-of-n_i systems

Consider an alternate structure **(B)** for the system S.

B1 : The module P_i is an $a_i + 1$-out-of-n_i system, $i = 0, \ldots, k$, and

B2 : the system S is a series system based on $P_0, P_1, \ldots, P_k$.

The system considered in this section reduces to the series-parallel system considered in Section 2 when $a_i = 0, i = 0, 1, \ldots, k$ and $F_1 = F_2 = \cdots = F_k = F$. This system allows for more general modules than the system considered in Section 3 and requires the modules to be connected in series.

The probability that the module P_0 causes the system to fail, $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$, will now be denoted by $P(a_0, n_0, F_0; \mathbf{a}, \mathbf{n}, \mathbf{F})$. We will say that $F \le G$ if $F(x) \le G(x)$ for all x.

The following theorem gives an AI property using this ordering on distribution functions.

Theorem 4.1 *$P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F})$ is AI in $\mathbf{n}, \mathbf{F}$, for each a_0, n_0, F_0, and a .*

Theorem 4.8 of El-Neweihi (1980) treats the special case of the above when $a_i = 0, i = 0, 1, 2, \ldots, k$.

We now give an application of the above results to an optimal allocation problem. Suppose that the sizes $n_1, \ldots, n_k$ of the modules $P_1, \ldots, P_k$ are in increasing order. Suppose that we have collections of components with reliabilities $p_1 \geq \cdots \geq p_k$ at a particular time t. Theorem 4.1 shows that the reliability of S at time t is maximized by allocating components of reliability p_i to the module $P_i, i = 1, \ldots, k$.

The following theorem considers the case $n_i = n, F_i = F, i = 1, 2, \ldots, k$.

Theorem 4.2 *$P(a_0, n_0, F_0; \mathbf{a}, n, F)$ is Schur-concave in $\mathbf{a}$.*

The case when $a_i = a, F_i = F, i = 1, 2, \ldots, k$ was treated in El-Neweihi, Proschan and Sethuraman (1978) where the following theorem was established.

Theorem 4.3 *$P(a_0, n_0, F_0; a, \mathbf{n}, F)$ is Schur-concave in $\mathbf{n}$.*

Theorem 4.3 shows that the probability that module P_0 fails first is Schur-concave in $\mathbf{n}$. We can ask the question whether the probability that module P_0 is among the first r modules to fail is also Schur-concave. The following example shows that this is not so for $r = 2$.

Example 4.4 Let $k = 2, a_1 = 1, a_2 = 1, F_0 = F_1 = F_2 = F$ where F is the uniform distribution on $[0, 1]$. Then
The probability that module P_0 is among the first two modules to fail

$$= 1 - \int [t^{n_1+n_2} + (n_1 + n_2)(1-t)t^{n_1+n_2-1} + n_1 n_2 (1-t)^2 t^{n_1+n_2-2}]$$
$$\times \binom{n_0}{a_0} (n_0 - a_0) t^{n_0 - a_0 - 1} (1-t)^{a_0} dt.$$

The integrand is Schur-concave in $\mathbf{n}$ and hence this probability is Schur-convex.

Theorems 4.2 and 4.3 have obvious applications to optimal allocation along the lines of the remark following Theorem 4.1.

5. Dual systems

Every coherent structure possesses a dual structure. The dual of a parallel structure is a series structure. The dual of a k-out-of-n structure is an $n-k+1$-out-of-n structure, and is a structure of the same type. Consider the system S with structure $\mathbf{A}$ based on the modules $P_0, P_1, \ldots, P_k$ as in Section 2. The dual of this is a system S' based on the modules $P'_0, P'_1, \ldots, P'_k$, consisting of $n_0, n_1, \ldots, n_k$ components, and possessing the structure $\mathbf{A}'$ as follows:

A′1 The modules $P'_0, P'_1, \ldots, P'_k$ are all series systems, and
A′2 the system S' is an r-out-of-$k+1$ system based on the $k+1$ modules $P'_0, P'_1, \ldots, P'_k$.

This means that the system S' fails as soon as $k-r+1$ modules fail. Let T_i be the lifetime of the modules P_i, $i = 0, \ldots, k$ and let $R_0, R_1, \ldots, R_k$ be the ranks of $T_0, T_1, \ldots, T_k$. Let T'_i be the lifetime of the modules P'_i, $i = 0, \ldots, k$ and let $R'_0, R'_1, \ldots, R'_k$ be the ranks of $T'_0, T'_1, \ldots, T'_k$. Suppose that $T'_i = f(T_i)$ where f is a positive, strictly decreasing and continuous function. This happens when the lifetimes of the components in S' are the same function f of the lifetimes of the corresponding components of S. Let $P'_r(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ be the probability that R'_0 is less than or equal to r, that is P'_0 is among the first r modules to fail in S'.

It is easy to see that $P'_{k-r+1}(n_0, F'_0; \mathbf{n}, \mathbf{F}') = 1 - P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$, that is, the probability that P'_0 is among the modules that caused the failure of the system S' is the complement of the probability that P_0 is among the modules that caused the failure of the system S.

Theorems 5.1 to 5.3 below follow directly from the above relationship between dual structures, see El-Neweihi and Sethuraman (1991).

Theorem 5.1 *For each n_0, F'_0, F', $P'_r(n_0, F'_0; \mathbf{n}, F')$ is Schur-convex in* $\mathbf{n}$.

Theorem 5.2 *The probability that P'_0 fails last among all the $k+1$ modules is $1 - P'_{k'}(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ and is arrangement decreasing in* $\mathbf{n}, \mathbf{F}'$.

Theorem 5.3 *Let $\bar{F}'_i(x) = \exp(-\lambda_i R(x))$, $i = 1, \ldots, k$(the proportional hazards case). Then $P'_r(n_0, F'_0; n, \mathbf{F}')$ is Schur-convex in* $\boldsymbol{\lambda}$.

We will now consider the dual of the system S with the structure $\mathbf{B}$ defined in Section 3. This is a system S' with modules $P'_0, P'_1, \ldots, P'_k$ satisfying the following structure.

B′1 The module P_i in an $(n_i - a_i)$-out-of-n_i system , $i = 0, \ldots, k$, and
B′2 the system S' is a parallel system based on the modules $P'_0, P'_1, \ldots, P'_k$.

We will denote the probability that P_0' fails last by $P'(a_0, n_0, F_0'; \mathbf{a}, \mathbf{n}, \mathbf{F}')$.

The following theorems follow by using the relation between dual structures.

Theorem 5.4 *For each* a_0, n_0, F_0' *and* a, $P'(a_0, n_0, F_0'; a, \mathbf{n}, \mathbf{F}')$ *is arrangement decreasing in* $\mathbf{n}, \mathbf{F}'$.

Theorem 5.5 *For each* a_0, n_0, F_0' *and* F', $P'(a_0, n_0, F_0'; \mathbf{a}, n, F')$ *is Schur-concave in* $\mathbf{a}$.

6. Further extensions

The structures that have been considered in this paper are special cases of second order r-out-of-k systems. A definition of such a system S is as follows. Let $P_1, \ldots, P_k$ be k modules with no common components where each module is a a_i-out-of-n_i system. The system S fails as soon as $k - r + 1$ of the modules $P_1, \ldots, P_k$ fail.

We need to investigate questions similar to those considered in this paper for such second order systems in general. This would be a first step. New kinds of questions also arise for these systems. One can study the role of groups of modules rather than that of a single module. The role of a group of modules can be defined to be the probability that at least m of the modules in the group have failed prior to the failure of the system, where m can vary from 1 to the size of the group.

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