

AD-A244 171

ATION PAGE

Form Approved
OMB No. 0704-0188

Average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering the collection of information, and sending comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. (Leave blank)		2. REPORT DATE		3. REPORT TYPE AND DATES COVERED Final/1 Aug 88 - 31 Jul 91	
4. TITLE AND SUBTITLE ANALYSIS AND REGULATION OF NONLINEAR SYSTEMS (U)				5. FUNDING NUMBERS 61102F 2301/A1	
6. AUTHOR(S) Eduardo D. Sontag					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Rutgers Department of Mathematics at New Brunswick Hill Center for Mathematical Sciences New Brunswick, NJ 08903-3072				8. PERFORMING ORGANIZATION REPORT NUMBER 91 0933	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448				10. SPONSORING/MONITORING AGENCY REPORT NUMBER AFOSR-88-0235	
11. SUPPLEMENTARY NOTES					
12a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for Public Release; Distribution Unlimited					
12b. DISTRIBUTION CODE UL					
13. ABSTRACT (Maximum 200 words) The research reported deals with the control of nonlinear systems. Among the topics covered are: State-space and I/O stabilization, systems with saturated controls, universal formulas for Lyapunov-function based feedback, neural-net controllers, discrete-time controllability and sample control, input/output algebraic-differential equations, and certain types of Hamiltonian systems.					
14. SUBJECT TERMS					
15. NUMBER OF PAGES					
16. PRICE CODE					
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED		18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED		19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	
20. LIMITATION OF ABSTRACT SAR					

91-19227



ANALYSIS AND REGULATION OF NONLINEAR SYSTEMS

Final Report, AFOSR-88-0235, 8/1/88 to 7/31/91

Eduardo D. Sontag*

SYCON - Rutgers Center for Systems and Control
Department of Mathematics, Rutgers University

ABSTRACT

The research reported deals with the control of nonlinear systems. Among the topics covered are: State-space and I/O stabilization, systems with saturated controls, universal formulas for Lyapunov-function based feedback, neural-net controllers, discrete-time controllability and sampled control, input/output algebraic-differential equations, and certain types of Hamiltonian systems.

1 Introduction

A striking amount of progress was achieved during the past few years on difficult questions in nonlinear control theory. First-rate researchers have directed their efforts towards the identification of differences with the well understood linear case as well as the introduction of new techniques. The recent books by Isidori and Van der Schaft/Nijmeijer deal exclusively with the recent developments. The PI's graduate mathematics textbook [1] mentions briefly some of these topics in the context of more classical linear results. Another new book, by Slotine and Li, includes an excellent exposition of various applied aspects. As emphasized in Slotine-Li's book, the current interest in nonlinear techniques by researchers in areas as diverse as robotics, aerospace, and process control, is based on the availability of powerful new tools for global analysis and design, which provide major improvements in simplicity, robustness, and operating range over more classical linear approaches. That book shows with detailed and realistic examples the applicability of the nonlinear theory.

The main objective of control theory is to modify the behavior of a dynamical system, typically with the purpose of regulating certain variables or tracking desired signals. In most cases, either *stability* of the closed-loop system is an explicit requirement or the problem can be recast in a form that involves stabilization (e.g., of an error signal). For linear systems, the associated problems can now be treated fairly satisfactorily, but in the nonlinear case the area is very far from being settled. The emphasis of the PI's work has been mainly on questions of (global) stabilization, both in the state-space and input/output senses. *State-space stability* relates to classical dynamical systems ideas; in many practical situations, it can be achieved by means of linearization under feedback (Brockett, Hunt, Meyer, Su, Jakubczyk, Respondek), via control-Lyapunov ideas (Artstein, Jurdjevic-Quinn, Tsinias), via the promising approach of zero-dynamics (Byrnes and Isidori), or, for more local studies, through Center Manifold Techniques (Abed, Aeyels, Bacciotti, Boothby, Marino, Crouch, and many others). *Input/output stability* developed in parallel, often using techniques based on small-gain arguments (Desoer, Sandberg, Vidyasagar, Zames, and others). It is perhaps surprising that these two frameworks are not automatically related, as is the case for linear systems. (Even for feedback linearizable systems, this relation is more subtle than might appear: Linearizing a system and then stabilizing the equivalent linearization does *not* in general result in a closed-loop system that is input/output stable.) Building on previous work by others, the PI started in [3] a line of research that leads to the synthesis of i/o stabilizing laws starting with state-space stabilizers, hence providing a method for *transferring* results from one context to the other.

Lyapunov functions are often employed in practical control design and in robust control theory. A systematic technology is being developed for obtaining *explicit* feedback laws from given "control-Lyapunov" functions. This theory elaborates and refines previous work by many authors.

Another direction of research deals with saturation effects; even for linear systems, the necessary theory requires nontrivial nonlinear tools. (According to the book by Slotine and Li, "saturation is probably the most commonly encountered nonlinearity in control engineering.") Recent work coauthored by the PI has resulted in general techniques for stabilizing systems in the presence of control saturation. This line of work also motivated research of the PI and students into other related issues, such as the effect of saturation and finite precision on observability.

*Graduate students in project: Francesca Albertini, Yuandan Lin, Renée Schwarzschild, Eva Siegelmann, Yuan Wang

2 State-Space and I/O Stabilization

A great deal of effort has been directed by many towards the general problem of stabilizing systems

$$\dot{x} = f(x, u) \quad (1)$$

with respect to a given equilibrium point; see for instance the survey paper [43], which includes a rather large bibliography, as well as the respective sections of the books cited earlier. In order to talk about stabilization, which deals with closed-loop behavior, it is first necessary to understand what is meant by "stability" of an open-loop system such as (1). Technicalities aside, there are two *conceptually* very different manners of defining this notion.

The first approach to defining stability of control systems relies on the standard Lyapunov definition from dynamical systems theory, and disregards the control: Global asymptotic stability of the origin for the system $\dot{x} = f(x, 0)$ obtained by setting $u \equiv 0$ (assume $f(0, 0) = 0$). This notion is called *state-space stability* or *zero-input stability*. (One may also consider the corresponding local definitions; for simplicity, the following discussion is only for the global case.)

An alternative way of introducing stability for (1) arises from operator-theoretic approaches to systems. In essence, what is desired is not only that the system be zero-input stable as described above, but that, in addition, a "nice" input $u(\cdot)$ should produce a "nice" state trajectory $x(\cdot)$ when starting from any initial state. That is to say, one imposes a condition on the operator "(initial state, control) $\mapsto$ ensuing state trajectory." This *input/output* (or, unless only partial observations are of interest, *input-to-state*) *stability* is central to the study of robustness with respect to actuator and sensor noise, stability using partial measurements, and controller parameterization. There are many possible precise definitions of input/output stability. One possibility is to add the requirement *BIBS: bounded inputs produce bounded state* (or output) trajectories. Even here, there are variants: The state bounds could be required to depend only on input and initial state bounds—but not on the actual values of these—, and this dependence could be required to be linear, which gives rise to "finite-gain" stability ("finite incremental gain" if the operator is not just continuous but also Lipschitz). These issues are discussed in textbooks by Vidyasagar and others. For the purposes of the present discussion, i/o stability will simply mean *state-space stability plus BIBS*. With this meaning, a basic fact is as follows. (See [3], [28] for proofs of various versions of this result.)

Theorem. *I/O stability implies that controls that converge to zero produce trajectories that converge to zero, from any initial state.*

Contrary to the situation that holds for linear systems, in general state-space stability does not imply i/o stability. This is illustrated by the trivial example $\dot{x} = -x + ux^2$; the system is zero-input stable but $u \equiv 1$ makes every trajectory with $x(0) > 1$ escape to infinity in finite time. The gap between the concepts, and associated problems, have been known for a long time, and appear in slightly different form in the classical study of "total stability." In a control context, there has been work characterizing restrictive conditions under which the implication does hold. One such case is that of systems with globally Lipschitz right-hand side and *exponentially* stable dynamics for $u \equiv 0$, as done in work by Varaiya and Liu in the mid-1960s and by Sastry and Isidori (but note that this result is false if asymptotic stability is not exponential, as illustrated by the example $\dot{x} = -\tanh x + u$).

Given that the two broad notions of stability do not in general coincide, it makes sense to ask the *design* question: If one can make a system state-space stable by using *feedback*, can one also make it i/o stable?

The *stabilization* problem is that of finding a feedback law of the type $u = k(x) + \beta(x)v$, $k(0) = 0$, so that the closed-loop system $\dot{x} = F(x, v) = f(x, k(x) + \beta(x)v)$ is stable in the desired sense (with new control v). That is, one wants a system which is *feedback equivalent* to the original one and stable: as usual in nonlinear control, one requires that the square matrix $\beta(x)$ be invertible for all x , so no instantaneous controllability is lost. Note that the more restricted problem of *state-space stabilization* is that of finding just k (as β is now irrelevant) so that $\dot{x} = F(x, 0) := f(x, k(x))$ be globally asymptotically stable; for *i/o stabilization* one wants that in addition $\dot{x} = F(x, v)$ be BIBS. It is perhaps surprising, given the gap between the two properties, that if a system is stabilizable in the state-space sense then it is also stabilizable in the i/o sense. More precisely:

Theorem. *If there is a smooth k so that $\dot{x} = F(x, 0)$ is state-space stable then there are also a (generally different) smooth k and a smooth β so that $\dot{x} = F(x, v)$ is i/o stable.*

This was proved in [9]. The paper [3] had previously shown that, for systems *affine in controls*

$$\dot{x} = f(x, u) = f_0(x) + G(x)u \quad (2)$$

one may always take $\beta = \text{identity}$. Models of this type, possibly with control bounds, cover most interesting aerospace finite-dimensional control applications. (The blanket assumption is being made, when stating the above and latter results, that all systems considered have smooth, i.e. infinitely differentiable, right-hand side f , and that states $x(t)$ and control values $u(t)$ lie in Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, though far less is needed in general.)

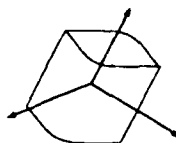
2.1 Cascades

The concept of i/o stability is particularly useful when studying cascades of systems, because series connections of i/o stable systems are again i/o stable. (In contrast, series connections of state-space stable systems are not necessarily globally state-space stable.) By a cascade one means a system with triangular structure

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z, u) \end{aligned}$$

One can easily stabilize a cascade if it is known that the first system (the z -subsystem) is stabilizable and that the second one ($\dot{x} = f(x, z)$) is i/o stable. As discussed in [24] and [28], sometimes it may be possible to use a preliminary feedback that makes the second system i/o stable, as needed for the argument to hold. The idea can be applied to systems that admit various types of partial linearization, such as the standard model for the angular momentum of a satellite controlled by two pairs of opposing gas jets. In terms of the angles between a body frame and a fixed frame, under some simplifying assumptions the equations become

$$\begin{aligned} \dot{\omega}_1 &= A_1 \omega_2 \omega_3 + \alpha_1 u_1 \\ \dot{\omega}_2 &= A_2 \omega_1 \omega_3 + \alpha_2 u_2 \\ \dot{\omega}_3 &= A_3 \omega_1 \omega_2 \end{aligned}$$



The A_i 's are related to the inertial coefficients, and A_3, α_1, α_2 are all nonzero. Then the procedure results in the simple feedback law

$$u_1 = -\omega_1 - \omega_2 - \omega_2 \omega_3, \quad u_2 = \omega_1^2 - \omega_3 + 2\omega_1 \omega_2 \omega_3.$$

which globally stabilizes the equilibrium $\omega = 0$. Indeed, under feedback and coordinate changes, one may simplify the equations to

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= u_1 \\ \dot{z} &= u_2 \end{aligned}$$

which has the form of a scalar system $\dot{x} = v_1 v_2$ cascaded with two integrators. If the system $\dot{x} = v_1 v_2$ is first made to be i/o stable (which one knows can be done, because it is clearly state-space stabilizable, and using the results cited earlier) then it may be expected that the cascade can also be stabilized. Indeed, letting $v_1 = -x + w_1, v_2 = x^2 + w_2$ the equation $\dot{x} = v_1 v_2$ becomes $\dot{x} = -x^3 + q(x, w_1, w_2)$ which is i/o stable since q has degree less than 3 in x . In terms of the new variables $\eta := x + y$ and $\zeta = z - x^2$, the closed-loop system consists of the system $\dot{\eta} = -\eta, \dot{\zeta} = -\zeta$ feeding into

$$\dot{x} = -x^3 + q(x, \eta, \zeta)$$

and is therefore globally asymptotically stable. A different control law can be obtained for this satellite model using instead zero-dynamics design (work by Byrnes and Isidori). Performance comparisons between different techniques are as yet unclear.

2.2 Coprime Factorizations

There has been considerable interest in parameterizations of controllers for nonlinear systems (for the linear case, see for instance the textbook [1], Section 6.5). This has motivated the search for coprime factorization conditions, with important contributions by Desoer, Hammer, Hunt, Krener, Verma, and others. The paper [26] constructed factorizations, giving formulas that reduce automatically in the linear case to the by-now "classical" formulas that the PI had obtained with Khargonekar in the 1980s.

More precisely, consider the system (1) and a fixed initial state x_0 , and let $P : \mathcal{D}(P) \rightarrow L_{\infty, \epsilon}^p$, where $\mathcal{D}(P) \subseteq L_{\infty, \epsilon}^m$, be the operator obtained by defining $P(u) :=$ state resulting from input $u(\cdot)$ and initial state x_0 . Here $\mathcal{D}(P)$, the domain of P , is the set of controls for which this solution is well-defined for all t , and $L_{\infty, \epsilon}^k$ denotes the set of measurable essentially bounded maps $[0, \infty) \rightarrow \mathbb{R}^k$, for any k . The operator P admits a coprime right factorization if there exist three i/o stable operators $A : L_{\infty, \epsilon}^m \times L_{\infty, \epsilon}^p \rightarrow L_{\infty, \epsilon}^m$, $N : L_{\infty, \epsilon}^m \rightarrow L_{\infty, \epsilon}^p$, and $D : L_{\infty, \epsilon}^m \rightarrow L_{\infty, \epsilon}^m$, such that D is causally invertible, $\mathcal{D}(D^{-1}) = \mathcal{D}(P)$, $P = ND^{-1}$ and, if I denotes the identity in $L_{\infty, \epsilon}^m$,

$$A \circ \begin{pmatrix} D \\ N \end{pmatrix} = I.$$

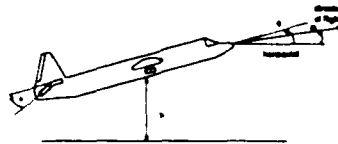
The precise definition of i/o stable operator is given in [3], but for purposes of the present discussion, it can be taken to mean that inputs converging to zero produce outputs that also converge to zero. In [26] it is shown that: *If (1) is smoothly stabilizable then its input to state mapping admits a coprime right factorization.* A version of this result in terms of "Bezout" factorizations, for which A can be split into two parts and the familiar formulas $VD + SN = I$ appear instead, was given in [3], for systems affine in controls.

If there is also given an output map $y = h(x)$ and P is now the operator $u \mapsto y$, then [26] shows how to extend the result on existence of factorizations under the additional assumption that an observer exists.

3 Systems With Saturating Controls

Nonlinear control provides tools that allow one to deal with certain questions that are on the surface of an apparently linear nature. Particularly interesting examples of such questions deal with actuator saturation in linear systems. (Saturation of actuators is a real and potentially important effect, as evidenced by the classical case of an X-15 experimental aircraft that crashed in 1966 due to ailerons reaching maximum deflection, causing loss of controllability along the roll axis.) The search for controllers of systems subject to such saturation can be seen as a problem in nonlinear control, and that is the point of view taken here. As a simple illustration, take the model of an airplane flying on a vertical plane and at a constant ground speed. The control is the rate of the elevator angle, assumed subject to magnitude constraints. Let z denote this angle; the pitch angle is denoted by ϕ , the ground speed by c , the flight path angle by α , and the altitude by h . For small angles, the above quantities are related by the following differential equations:

$$\begin{aligned} \dot{\alpha} &= a(\phi - \alpha) \\ \ddot{\phi} &= -\omega^2(\phi - \alpha - bz) \\ \dot{h} &= c\alpha \\ \dot{z} &= \sigma(u) \end{aligned}$$



where $\omega > 0$ is a constant representing a natural oscillation frequency and a, b are positive constants. The saturation nonlinearity σ is shown explicitly, for emphasis; it could be, for instance, the function $\sigma(u) = u$ for $|u| < 1$, $\sigma(u) = \text{sign } u$ otherwise. A routine application of linear control theory provides a stabilizing law if one disregards the bounds. But with saturation, global stability is of course not guaranteed (it is easy to compute examples showing this failure of stability). This remark, that linear laws when saturated can lead to instability, has motivated much past research; in tracking problems and some disturbance rejection problems, for example, the phenomenon called "integrator wind-up" often arises when using a linear integral feedback law derived by ignoring saturation limits.

Many authors have proved estimates, -using the circle criterion and other techniques related to small-gain analysis,- to guarantee stability in regions of the state-space. But rather than asking when a linear control law $u = Fx$ remains globally stabilizing when saturated, or estimating the domain where attractivity still holds if not globally stabilizing, in joint work with Sussmann we considered in [33] the system as a *nonlinear* system, taking the saturation into account in the model. This was of course not a new idea, in so far as optimal control techniques can be applied -resulting for instance in a bang-bang control law for the above airplane, when solving a time-optimal problem. But by looking for non-necessarily optimal laws, one finds more smooth controllers. The paper [33] gave a very general result on stabilization of linear systems with input saturation, resulting in *smooth feedback laws*. The hypotheses made in [33] apply in particular to the above airplane example. Work in progress has resulted in a simple and very explicit representation of the needed feedback laws. This work uses in an essential manner the theory of i/o stability discussed earlier.

The class of systems considered in [33] is given as in:

$$\dot{x} = Ax + B\theta(u), \quad (3)$$

where $\theta(u) = (\sigma(u_1), \dots, \sigma(u_m))'$ and $\sigma(u)$ is a saturation function as above. Here A and B are $n \times n$ and $n \times m$ matrices respectively, and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is assumed bounded, globally Lipschitz, continuously differentiable at the origin, and with $\sigma'(0) \neq 0$. (This includes for instance, the "squashing" function given by the standard sigmoid $\tanh(u)$, as well as the saturation function discussed earlier.) Of interest are only global problems; local stabilization can always be achieved by the use of a linear control law. One of the main results in [33] is as follows:

Theorem. *For the system (9), there is a globally (state-space) stabilizing smooth feedback if and only if the system is asymptotically null-controllable, that is, if from any state one can asymptotically reach the origin.*

In other words, subject only to the obvious necessary condition, there are smooth feedback stabilizers. As a corollary, one can also obtain a feedback guaranteeing i/o stability. Furthermore, asymptotic null-controllability for these systems is easy to characterize by a well-known criterion: The pair (A, B) must be stabilizable in the ordinary sense, and all eigenvalues of A must have nonpositive real part.

4 Control-Lyapunov Functions

Consider a system affine in controls, as in (2). If there is some feedback law k that stabilizes in the state-space sense, and if k is continuous away from the origin, then converse Lyapunov theorems guarantee the existence of a positive definite (i.e., $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$.) and proper (i.e., $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$) smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that energy can be decreased with open-loop controls. That is, V is a *control Lyapunov function (clf)*: it satisfies $\inf_u \{a(x) + B(x)u\} < 0$ for each $x \neq 0$, with the notations $a(x) := \nabla V(x) f_0(x)$ and $B(x) = (b_1(x), \dots, b_m(x)) := \nabla V(x) G(x)$. Moreover, if k is continuous at the origin, then this function has the additional *small control property (scp)* [with respect to (2)]: controls can be picked small if the state is small. (That is: For each $\delta > 0$ there is an $\epsilon > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \epsilon$, then there is some u with $\|u\| < \delta$ such that $a(x) + B(x)u < 0$.)

In a 1971 paper the PI had showed that the existence of a clf is in fact also necessary if there is just asymptotic controllability, that is, if from any state one can asymptotically reach the origin, provided that one does not require smoothness of V and the defining equation is replaced by an equation involving

Dini derivatives. Independently, Artstein showed at about the same time that if there is a smooth clf V , then there is a feedback law which globally stabilizes the system, and k is smooth on $\mathbb{R}^n - \{0\}$. If in addition V satisfies the scp, then k can be chosen to be *almost smooth* on $\mathbb{R}^n$, meaning not only smooth away from the origin, but also continuous on all of $\mathbb{R}^n$.

One goal in smooth feedback stabilization is to provide conditions under which asymptotic controllability implies smooth feedback stabilizability; the results just mentioned say that the former corresponds to existence of *continuous* clf's, and the latter to the existence of *smooth* such functions.

Since clf's are as a general rule easier to obtain than the feedback laws themselves –after all, in order to prove that a given feedback law stabilizes, one often has in addition to provide a suitable Lyapunov function,– Artstein's Theorem provides in principle a very powerful approach to nonlinear stabilization. However, until recently all proofs relied on nonconstructive (partition of unity) arguments. To make it a practical technique, one needs a more explicit construction. In [10] the PI gave one such explicit, very elementary, construction of k from V and the vector fields defining the system. A further advantage of the method in [10], in addition to its extreme simplicity and ease of implementation, is that it provides automatically an *analytic* feedback law if the original vector fields as well as the clf are analytic.

The construction in [10] is based on the following argument, which for simplicity is given here only for single-input ($m = 1$) systems. Assume that V is a clf for the system $\dot{x} = f(x) + ug(x)$. Denote $a(x) := \nabla V(x) \cdot f(x)$, $b(x) := \nabla V(x) \cdot g(x)$. The condition that V is a clf is precisely the statement that $b(x) \neq 0 \Rightarrow a(x) < 0$ for all nonzero x . In other words, for each such x , the pair $(a(x), b(x))$ is stabilizable when seen as a single-input, one-dimensional linear system. On the other hand, giving a feedback law $u = k(x)$ for the original system, with the property that the same V is a Lyapunov function for the obtained closed-loop system $\dot{x} = f(x) + k(x)g(x)$ is equivalent to asking that $\nabla V(x) \cdot (f(x) + k(x)g(x)) < 0$, that is, $a(x) + k(x)b(x) < 0$ for all nonzero x . In other words, $k(x)$, seen as a 1×1 matrix, must be a constant linear feedback stabilizer for $(a(x), b(x))$, for each fixed x .

Now interpret $(a(x), b(x))$ as a family of linear systems parameterized by x . This family depends smoothly on x . From the theory of families of systems or "systems over rings" one knows that since each such linear system is stabilizable there exists indeed a smoothly dependent k as wanted. Moreover, this k can be chosen to be analytically or rationally dependent if the original family is, that is, if the original system and clf are. The result in the smooth and analytic cases is due to Delchamps (see for instance [1], Exercise 7.5.11), but in this very simple case (the family is one-dimensional), the construction of k can be carried out directly without explicit recourse to the general result. Indeed, one can show directly that the following feedback law: $k := -\frac{a + \sqrt{a^2 + b^2}}{b}$ (with $k(x) := 0$ when $b(x) = 0$) works. This results from the solution of an LQ problem, and is analytic, in fact algebraic, on a, b . (The apparent singularity due to division by b is removable.) Along trajectories of the corresponding closed-loop system, one has that $\frac{dV}{dt} = -\sqrt{a^2 + b^2} < 0$ as desired. This feedback law may fail to be continuous at zero. However, if one modifies it slightly to $k := -\frac{a + \sqrt{a^2 + b^2}}{b}$ it does become continuous (under the scp assumption).

In connection with the discussion on i/o stability, it is worth remarking that, provided that the function $a(x)^2 + b(x)^2$ is proper ("radially unbounded" is the terminology in the engineering literature) the above law gives automatically also i/o stability, when using $\beta \equiv I$.

The clf work has recently been extended to *bounded* controls in [16], and nonnegative controls in [37].

To illustrate the latter results, take the case of single-input systems ($m = 1$) and control value set the unit interval $(-1, 1)$ (or the corresponding closed interval). Now the assumption is that in the equation defining the clf property, controls are restricted to the allowed control-value set. If V is a clf satisfying the scp, then

$$k(a, b) := \begin{cases} -\frac{a + \sqrt{a^2 + b^2}}{b(1 + \sqrt{1 + b^2})} & \text{if } b \neq 0, \\ 0, & \text{if } b = 0. \end{cases} \quad (4)$$

globally stabilizes, is real-analytic in a, b , and has values always with magnitude less than 1. In addition, this is guaranteed to be continuous at the origin (hence almost-smooth in the sense mentioned above).

5 Neural Network Approaches

There has been much activity lately in the area encompassed by the term "neural networks," and many practical successes of the associated technologies have been reported both in the scientific and in the popular press. A recent workshop (October 24-26, 1990) at the McDonnell Douglas Corp., attended by researchers from various Air Force contractors and development labs, centered on aerospace applications of neural nets. Several presentations focused on the use of network techniques in flight systems design, such as fault detection and classification (and associated problems of reconfigurable aircraft control), development of controls valid over large flight envelopes, and even precision laying of composites on aircraft structures. A recurrent topic of discussion during the workshop was the lack of theoretical foundations for the analysis and comparison of different network tools.

The PI is engaged in a program of research whose main purpose is to carry out a rigorous mathematical analysis for problems in neural nets. Such a development, besides being intrinsically of interest, will lead to a better understanding of the issues involved in the design of efficient algorithms as well as in understanding the capabilities and limitations of these models. Some of this work dealt with the fact that, among two possible variants of the so-called backpropagation training method for sigmoidal nets, both of which are used in practice, one is a better generalization of the older perceptron training algorithm than the other (see e.g. [13]). Various results on the local minima structure of the associated gradient descent problems were obtained too (see [8]). New representation results showed that nets consisting of sigmoidal neurons have at least twice the representational capabilities of nets that use classical threshold neurons, at least when this increase is quantified in terms of classification and interpolation power (see [32], [47] [48], [18]). Other recent work established a basic result quantifying the capabilities of neural nets as universal computing devices (see [19]).

One of the main lines of work in this research has dealt with nets as controllers. We showed that "two hidden layer nets" are necessary as well as sufficient for the solution of typical nonlinear control problems. Most current experimental work uses *single*-hidden-layer nets, based on an (incorrect) intuition grounded on well-known approximation theorems.

Nets with Heaviside nonlinearities can be seen as implementing a particular type of discontinuous feedback strategy. The basic obstruction has to do with the power of hidden units in allowing the existence of sections of certain projections. Topological constraints on what can be done with just one hidden layer imply that, even though one such layer is indeed sufficient when controlling linear systems, the structure is not rich enough to deal with nonlinear control problems, as illustrated in [40].

Figure 1 illustrates a net with two hidden layers and a single output neuron. The processing elements ("neurons") receive inputs from preceding layers, and evaluate a fixed nonlinear function θ of a weighted sum of these inputs. In neural net practice, one often takes θ to be the *standard sigmoid* $\theta(x) = \sigma(x) = \frac{1}{1+e^{-x}}$ or equivalently, up to translations and change of coordinates, the hyperbolic tangent $\theta(x) = \tanh(x)$. Another usual choice is the *hardlimiter*, *Heaviside*, or *threshold* function $\theta(x) = \mathcal{H}(x) = 0$ if $x \leq 0$ and $= 1$ if $x > 0$, which can be approximated well by $\sigma(\gamma x)$ when the "gain" γ is large. Most of the analysis, including studies of circuit complexity and the theory of threshold logic, has been done for $\mathcal{H}$, but in practice one often uses the standard sigmoid. Direct connections from input neurons to the output neuron are sometimes allowed; they add a linear term to the function computed by such a net. In some studies the output is passed through a last nonlinearity, but for many purposes, such as function interpolation, this is undesirable. Also, more than one output neuron may be of interest, and this can be easily incorporated in the model.

Consider nonlinear control systems

$$\dot{x}(t) = P(x(t), u(t)) \quad (5)$$

whose states evolve in $\mathbb{R}^n$, having controls $u(t)$ that take values in $\mathbb{R}^m$, and with P sufficiently smooth and satisfying $P(0,0) = 0$. We assume that the system can be *locally* stabilized with linear feedback, i.e. there is some matrix F so that the closed loop system with right-hand side $P(x(t), Fx(t))$ is locally asymptotically stable. The system (5) is *asymptotically controllable* if for each state x_0 there is some infinite control sequence $u(0), u(1), \dots$ such that the corresponding solution with $x(0) = x_0$ satisfies that

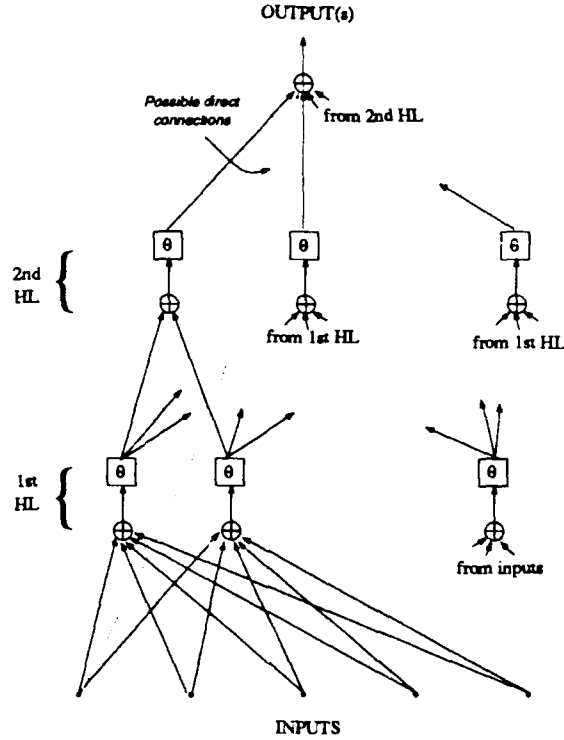


Figure 1: *Feedforward Net.*

$x(t) \rightarrow 0$ as $t \rightarrow \infty$. This condition is obviously the weakest possible one if any type of controller is to stabilize the system. We say that $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *type-1 feedback* if each coordinate f of K is a single-hidden layer net with possible connections, and that it is a *type-2 feedback* if it can be written in this manner with each f being a two-hidden layer net. These are feedback laws that can be computed by nets with one or two hidden layers respectively, and having m output neurons and possible direct connections from inputs to outputs (these connections provide the linear term). Then one will have:

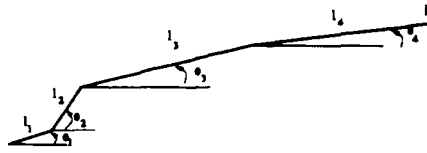
Theorem. Let $\theta = \mathcal{H}$. For each system as above, and each compact subset C of the state space, there exists some feedback law of type 2 that globally stabilizes (5) on C , that is, so that C is in the domain of asymptotic stability of $x^+ = P(x, K(x))$. On the other hand, there exist systems as above, and compact sets C , for which no possible feedback law of type 1 stabilizes C . ■

The negative result remains if $\theta = \tanh$, or if θ is one of many other functions used in neural net practice.

One explicit illustration of the obstructions due to just one hidden layer, for a tracking problem, is as follows. Assume that it is desired to control a planar four-link rotational manipulator, with torques being applied at each joint. It is assumed that unmeasured disturbances affect the first two joints, and what is desired is a control law which, given a reference tip position and the current angles of the first two joints, will produce a set of coordinates for the last two joints resulting in the wanted tip location. (See Figure 5.) In particular, choosing joint constraints $\theta_i \in [-\pi/2, \pi/2], i = 1, 2, \theta_3 \in [-7\pi/4, \pi/4], \theta_4 \in [\pi/4, 3\pi/4]$, and link lengths $l_1 = l_2 = 1, l_3 = 6, l_4 = 6\sqrt{2}$, it can be proved that the problem is solvable using two hidden layers but not using just one hidden layer.

6 Other Topics

Various other topics were treated in this project, related to nonlinear control but different from stabilization. Some of the work dealt with discrete-time control systems, in particular with the generalization of the Chow-Krener Lemma from continuous time, and the application of such concepts to the analysis



of chaotic systems. Other work dealt with the representability of systems by high order differential equations, and prefiltered versions of these, as needed for parameter identification. Another line of work imposed mechanical (energy) constraints in the form of the equations (2), and provided results that then apply to time-optimal problems for manipulators. Finally, a fair amount of effort has been directed towards the study of the effect of saturations and quantization on other systems properties such as observability and realizability.

6.1 Nonlinear Discrete-Time Systems

Modern control techniques typically result in complex regulation mechanisms, which must be implemented using digital computers. Consequently, an important area of research is that of studying the constraints imposed by computer control.

The behavior of continuous-time plants under digital control can be modeled by discrete-time systems. Starting in the mid-seventies in joint work with Rouchaleau, the PI had been interested in problems of control for nonlinear discrete time systems. During the early 1980s the focus shifted largely to questions of accessibility and controllability. This work benefitted from collaboration with B. Jakubczyk, and resulted in the major paper [2].

Consider discrete-time control systems

$$x(t+1) = f(x(t), u(t)), \quad t = 0, 1, 2, \dots, \quad (6)$$

where for simplicity it is assumed here that f is real-analytic and states and controls take values $x(t) \in X$, $u(t) \in \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ open. (More generally, one can deal with just smooth f , and states and controls taking values in manifolds.) Systems of this type but with discrete-valued states and controls have long been studied in automata and sequential machine theory, but the continuous-valued case has only recently become the subject of serious investigation in so far as controllability properties are concerned.

Much interest in (6) has been motivated by the problem of modeling physical systems under digital control via *sampling*. Recall that sampling is the process under which the state of a continuous time system is measured at discrete instants, and control actions are taken also at discrete instants (see e.g. Section 2.10 in [1]). The main point is that, as far as the control algorithm is concerned, the physical system is a discrete-time system described by an equation of the type (6), where $f(x, u)$ is the solution of the differential equation (1) at the end of an interval of length δ assuming that the initial state was x and the control was held constantly equal to u .

The study of controllability questions for continuous-time systems has been the subject of a concentrated research effort, as documented for instance in the books cited earlier. Recall that the *reachable set* or *forward-accessible set* from a state $x^0 \in X$ consists of those states to which one may steer x^0 using arbitrary —measurable essentially bounded in the continuous-time case— controls. The *orbit* or *forward-backward accessible set* from x^0 is defined as the set consisting of all states to which x^0 can be steered using both motions of the system as well as negative time motions. That is, z is in the orbit of x if there exists a sequence of states $x_0 = x, x_1, \dots, x_k = z$ such that for each $i = 1, \dots, k$ either x_i is reachable from x_{i-1} or x_{i-1} is reachable from x_i . Negative time motions are in general not physically realizable, but the orbit is an extremely useful object to study. In particular, it is always a submanifold of X whose dimension can be obtained from Lie-algebraic computations, and group-theoretic techniques are relevant. In contrast, the reachable set is not in general a manifold, though one may define a “dimension” for it in various natural and equivalent ways, for instance meaning the largest dimension of a

submanifold that it includes. (See [44] for an exposition of general theorems on manifold structures for orbits.)

In continuous-time, a fundamental fact in controllability studies is what may be called the *Chow-Krener property*: *The dimension of the orbit from each $x^0 \in X$ equals the dimension of the reachable set from x^0 .* (The property holds as stated for analytic systems; in the more general smooth case one would need an additional Lie-algebraic assumption.) That is, the reachable set contains a submanifold of the state space and it is in turn contained in a submanifold of the same dimension, and this dimension can be computed from the rank of certain matrices formed by taking iterated Jacobians of the vector fields defining the system, evaluated at the state x^0 . These Lie-theoretic characterizations are "direct" in that they do not involve integration of the differential equation, and they are closely related to more classical geometric material related to Frobenius' theorem.

Perhaps the most important difference between discrete and continuous time is that the Chow-Krener property fails for the former, even for systems obtained through the sampling of one-dimensional analytic systems; a counterexample was given in older papers by the PI.

One of the main difficulties in the general discrete-time case is due to the possible noninvertibility of the one-step transition maps $f_u := f(\cdot, u)$ which means that *semigroups* tend to appear where *groups* would appear in the continuous case, so less algebraic structure is available. An useful fact however is that discrete-time systems that arise through sampling are so that each of these maps is indeed a diffeomorphism (possibly not everywhere defined) of the state space. This is analogous to the situation in classical dynamical system theory, where one studies time-one diffeomorphisms and Poincaré maps associated to differential equations. The reference [2] deals with systems (6) that are *invertible*, that is, the maps f_u are assumed to be diffeomorphisms. For such systems, the paper derived several characterizations of accessibility and studied the geometric structure of accessible sets, and as a consequence a theorem was proved showing that the Chow-Krener property does hold provided that the state x^0 be an equilibrium state ($f(x^0, 0) = x^0$).

The nature of the results in [2] can be illustrated with an example. Denote by f_0^k the k -th power of f_0 with respect to composition, and define the following vector fields depending on u :

$$\begin{aligned} X_u^+(x) &= \left. \frac{\partial}{\partial v} \right|_{v=0} f_u^{-1} \circ f_{u+v}(x), \\ X_u^-(x) &= \left. \frac{\partial}{\partial v} \right|_{v=0} f_u \circ f_{u+v}^{-1}(x), \end{aligned}$$

and more generally for each integer k and for $\sigma = \pm$,

$$(Ad_0^k X_u^\sigma)(x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_0^{-k} \circ f_u^{-\sigma} \circ f_{u+v}^\sigma \circ f_0^k(x),$$

where $-\sigma = -, +$ if $\sigma = +, -$ respectively.

Theorem. *The following statements hold for any analytic system (6) and equilibrium state x^0 :*

- (a) *The orbit from x^0 has dimension equal to the dimension of the vector space*

$$\text{Lie} \{ Ad_0^k X_u^+ \mid k \geq 0, u \in \mathbb{R} \} (x^0).$$

- (b) *The reachable set from x^0 has dimension equal to the dimension of the vector space*

$$\text{Lie} \{ Ad_0^k X_u^\sigma \mid k \in \mathbb{Z}, u \in \mathbb{R}, \sigma = \pm \} (x^0).$$

The Chow-Krener property in the equilibrium case is an easy consequence of these characterizations. Note that the conditions in this Theorem involve iterated compositions of transitions corresponding to only *one* control -arbitrarily taken as the zero control. The "naive" conditions that one can give based on the implicit function theorem for the above accessibility properties would involve compositions of all transition mappings, as well as, for forward-backward accessibility, of their (possibly hard to

compute) inverses. Moreover, in the particular case when the system has for instance the form $x(t+1) = x(t) + g(x(t), u(t))$ with $g(x, 0) \equiv 0$, the "Ad's" become all the identity and no compositions at all need be computed.

The above Theorem is valid in the global sense that the dimensions of these spaces being equal to n (the dimension of the state space) at *every* point is equivalent to the corresponding property holding from every state x^0 . But the pointwise version is false, for nonequilibria. Recent work with our student Albertini has shown that one can obtain the pointwise version in the case of systems satisfying certain stability conditions; see [49], [20].

6.2 A Class of Hamiltonian Systems

As emphasized in much recent work, it is often very useful to impose additional structure on the system model (1) in order to exploit special properties of mechanical systems. In the paper [27] the PI described a class of Hamiltonian systems in the context of which a number of results about time-optimal control can be obtained. Using this axiomatic class (the axioms can be typically checked for mechanical systems directly from first principles), and the relations along Lie brackets that automatically hold there, one may derive a number of results for robotic manipulators which previously required a very large amount of symbolic computation. The main results characterized regions of the state space where singular trajectories cannot exist, and provided high-order conditions for optimality. This work was motivated by previous studies by the PI (with Sussmann) done specifically for robotic manipulators.

6.3 Input/Output Equations

The work in this area was carried out together with former graduate student Y. Wang. The topic is the relation between realizability and the existence of "input output equations" of the type

$$E(w(t), w'(t), w''(t), \dots, w^{(r)}(t)) = 0 \quad (7)$$

where

$$w(\cdot) = (u(\cdot), y(\cdot))$$

are the i/o pairs of the system. This relation is essential in understanding issues of identifiability, and it generalizes to nonlinear systems the equivalence between autoregressive representations and finite dimensional linear realizability. The work resulted in a far-reaching generalization of the PI's previous theory for polynomial discrete-time systems covering the full analogy to the discrete case as well as results that are stronger than in that case. A number of papers have been written (see e.g. [14], [15], [29]).

The functional relation E is usually estimated, for instance through least squares techniques, if a parametric general form is chosen, such as using polynomials of a fixed degree. For example, in linear systems theory one often deals with degree-one polynomials E :

$$y^{(k)}(t) = a_1 y(t) + \dots + a_k y^{(k-1)}(t) + b_1 u(t) + \dots + b_k u^{(k-1)}(t) \quad (8)$$

(or their frequency-domain equivalent, transfer functions; the difference equation analogue is sometimes called an "autoregressive moving average" representation). In the linear case, such representations form the basis of much of adaptive control and identification theory. State-space formalisms are more useful than i/o equations for problems in nonlinear control, however. Thus a basic question is that of deciding when a given i/o operator admits a state-space representation. The work in the project dealt mostly with those systems that are affine in controls as in Equation (2), in which the entries of f_0 and G , as well as an output function, can be expressed in terms of rational or polynomial functions of the state.

For the linear case, one knows that an equation such as (8) can be reduced, by adding state variables for enough derivatives of the output y , to a linear time-invariant system. In frequency-domain terms, rationality of the transfer function is equivalent to realizability. For nonlinear systems this reduction presents a far harder problem, one that is to a great extent unsolved. The problem is basically that of in

some sense replacing a nontrivial equation (7) by a system of first-order equations which does not involve derivatives of the inputs. (For the converse, it is easy to show, by elementary arguments involving finite transcendence degree, that any i/o operator realizable by a rational state space system satisfies some i/o equation of type (7), with E a polynomial.)

The paper [15] develops many basic analytic properties of Fliess operators used in establishing the result that such an operator satisfies an algebraic i/o equation (i.e., all pairs $(u, F[u])$ do) *if and only if* it is realizable by a singular polynomial system. The proofs are based on a careful analysis of the concept of *observation space*, introduced by Lo in the mid-1970s –and by the PI for discrete-time,– and developed further by Fliess, Bartosiewicz, and others. One of the central technical results relates two different definitions of this space, one in terms of smooth controls and another in terms of piecewise constant ones: these two definitions are seen to coincide ([11]). One of them immediately relates to i/o equations, while the other is related to realizability through the notions of observation algebras and observation fields which are the natural analogues of the corresponding discrete-time concepts studied earlier and were introduced for differential equations by Bartosiewicz.

In [15], the results from [14] relating algebraic i/o equations to internal realizability are complemented with a result relating analytic i/o equations and local internal realizability. To do this, one first constructs a “meromorphic” realization by studying the properties of meromorphically finitely generated field extensions, and then imposes similar properties on the observation fields already introduced in the former paper. Finally, by a perturbation approach, together with the Fliess Lie rank condition for realizability, one concludes that around each point there is local analytic realization. That paper combines the techniques from differential algebra used in the polynomial case with techniques from differential geometry (Lie rank finiteness).

Work in progress, to be reported at the 1991 Conference on Decision and Control, deals with the replacing of differential algebraic equations by *integral* equations, in so far as possible. The latter are far less sensitive to noise and more suitable for identification algorithms that involve prefiltering (as done in the linear case). In the linear case, differential equations are always equivalent to integral ones, but that is seen not to be so in the nonlinear case.

6.4 Sign-Linear Systems

Recent work with graduate student R. Schwarzschild has dealt with the basic systems theory (controllability, observability, realization) of systems whose underlying dynamics are linear but for which only certain limited logical observations are possible. This relates to questions of quantization in digital control, as well as the general area of the interface between classical (continuous-variable) control systems and logical devices. In the particular case in which only the sign of the output can be observed, a fairly complete theory seems to result, and a paper is in preparation on that topic. That paper deals with *sign-linear* systems:

$$x(t+1) \{ \text{or } \dot{x}(t) \} = Ax(t) + Bu(t), \quad y(t) = \text{sign}(Cx)$$

where A is an $n \times n$ matrix, B is $n \times m$, and C is $1 \times n$ (with the convention that $\text{sign}(0) = 0$.) and associated *sign-linear* i/o maps of the type $y(t) = \text{sign}(A_1 u(t-1) + \dots A_n u(0))$ or the analogous continuous-time maps (convolution followed by sign). One may also consider multivariable outputs, but the statements become slightly more complicated.

Such systems, which can be seen as an extreme case (1-bit) of the systems with quantized observations studied in much recent work by Delchamps, appear in a variety of areas. For instance, they model the situation when linear systems are used as classifiers, providing a recursive version of the Widrow-type “adelines” used in pattern analysis. They (and similar models that are based on bilinear systems) also arise in relaxation techniques for learning finite automata. Finally, such systems provide a natural class of nonsmooth nonlinear systems, a class that combines logical and switching devices together with more classical continuous variables.

The results that have been obtained parallel those known for standard linear systems, but with a few, perhaps unexpected, differences. Some of the results are as follows: (a) Minimal-dimensional realizations are unique up to a change of variables in the state space and a positive rescaling of outputs.

(b) Finite-dimensional realizability can be characterized in the usual manner using Hankel matrices. (c) If a realization of a given i/o map is controllable and observable, in the usual sense of control theory, then it is minimal. Conversely, a minimal realization is necessarily final-state observable (that is, there is a control allowing for determination of the state at the end of the interval of application) but, in the discrete-time case, minimal realizations may not be observable. (In continuous-time, final-state and [initial-state-] observability coincide.)

Because of the possible lack of observability of minimal realizations, for some discrete-time sign-linear i/o maps it is the case that the abstract "canonical" realization predicted by automata theory is not given by a sign-linear system. Some preliminary results are available in that case, which can be sometimes easily described in terms of combinations of finite automata and linear systems. A detailed paper has been just submitted summarizing results obtained to date ([21]).

7 Bibliography (publications in 1989-91)

(Coauthors: (FA): Francesca Albertini, (BJ): Bronek Jakubczyk. (YL): Yuandan Lin, (RS): Renée Schwarzschild, (ES): Eva Siegelmann, (HS): Heñtor Sussmann, (YW): Yuan Wang, (YY): Yutaka Yamamoto.)

Textbook:

1. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Springer, New York, 1990.

Journal Papers:

(Note: SCL= *Sys. Control Letters*, SICOPT= *SIAM J. Control and Opt.*, TAC= *IEEE Trans. Automatic Control*, LAA= *Linear Alg. Applcs.*, CS= *Complex Systems*, FM= *Forum Math.*, NN= *Neural Networks*, NC= *Neural Computation*, JCSS= *J. Comp. Syst. Sci.*, AML= *Appl. Math. Lett.*)

2. (BJ) "Controllability of nonlinear discrete-time systems: A Lie-algebraic approach," SICOPT28(1990): 1-33.
3. "Smooth stabilization implies coprime factorization," TAC34(1989): 435-443.
4. (YY) "On the existence of approximately coprime factorizations for retarded systems," SCL13(1989): 53-58.
5. (YW) "Pole shifting for families of linear systems depending on three parameters," LAA138(1990): 3-38.
6. (with A. Arapostathis, B. Jakubczyk, H.-G. Lee, and S.I. Marcus) "The effect of sampling on linear equivalence and feedback linearization," SCL13(1989): 373-381.
7. (HS) "Further comments on the stabilizability of the angular velocity of a rigid body," SCL12 (1989): 213-217.
8. (HS) "Backpropagation can give rise to spurious local minima even for networks without hidden layers," CS3(1989): 91-106.
9. "Further facts about input to state stabilization", TAC35(1990): 473-476.
10. "A 'universal' construction of Artstein's theorem on nonlinear stabilization," SCL13(1989): 117-123.
11. (YW) "On two definitions of observation spaces," SCL13(1989): 279-289.
12. "Sigmoids distinguish better than Heavisides," NC1(1989): 470-472.
13. (HS) "Backpropagation separates where perceptrons do," NN4(1991): 243-249.
14. (YW) "Algebraic differential equations and rational control systems," SICOPT(1991): to appear.
15. (YW) "Generating series and nonlinear systems: analytic aspects, local realizability, and i/o representations," FM(1991): to appear.
16. (YL) "A universal formula for stabilization with bounded controls," SCL16(1991): 393-397.
17. "Feedback stabilization using two-hidden-layer nets," submitted.
18. "Feedforward nets for interpolation and classification," JCSS(1992): to appear.
19. (ES) "Turing computability with neural nets," AML(1991): to appear.
20. (FA) "Discrete-time transitivity and accessibility: Analytic systems," submitted.
21. (RS) "Linear systems with sign-observations," submitted.

Conference Proceedings:

(Note: CDC= *IEEE Conf. on Decision and Control*, CISS= *Conf. Inform. Sci. and Systems*, ACC= *American Control Conf.*, ECC= *European Control Conf.*, IJCNN= *Int. Joint Conf. on Neural Networks*.)

22. (HS) "Backpropagation separates when perceptrons do," IJCNN89.
23. (YW) "Input-output equations and realizability of continuous-time systems," CISS89.
24. "Some recent results on nonlinear feedback," CISS89.
25. (HS) "Remarks on local minima in backpropagation," CISS89.
26. "Some connections between stabilization and factorization," CDC89.
27. "Remarks on the time-optimal control of a class of Hamiltonian systems," CDC89.

28. "Remarks on stabilization and input-to-state stability," CDC89.
29. (YW) "Realization and I/O relations: The analytic case," CDC89.
30. (with H. Dewan) "Extrapolatory methods for speeding up the BP algorithm," IJCNN90.
31. (with Asano, T., J. Hershberger, J. Pach, D. Souvaine, and S. Suri) "Separating bi-chromatic points by parallel lines," in *Proceedings of the Second Canadian Conf. on Computational Geometry*, Ottawa, Canada, 1990, p. 46-49.
32. "Comparing sigmoids and heavisides," CISS90.
33. (HS) "Nonlinear output feedback design for linear systems with saturating controls," CDC90.
34. "Capabilities of four- vs three-layer nets, and control applications," CISS91.
35. (RS) "Quantized systems, saturated measurements, and sign-linear systems," CISS91.
36. (FA) "Accessibility of discrete-time nonlinear systems, and some relations to chaotic dynamics," CISS91.
37. (YL) "Further universal formulas for Lyapunov approaches to nonlinear stabilization," CISS91.
38. (RS) "Algebraic theory of sign-linear systems," ACC91.
39. (with W. Maass and G. Schnitger) "On the computational power of sigmoid versus boolean threshold circuits," *Proc. of the 32nd Annual IEEE Conference on Foundations of Computer Science*, San Juan, Puerto Rico, Oct. 1991.
40. "Feedback Stabilization Using Two-Hidden-Layer Nets," ACC91.
41. (FA) "Some connections between chaotic dynamical systems and control systems," ECC91.

Book Chapters:

42. (YW) "Input/output equations and realizability," in *Realization and Modelling in System Theory* (M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, eds.), Birkhäuser, 1990, pp. 125-132.
43. "Feedback stabilization of nonlinear systems," in *Robust Control of Linear Systems and Nonlinear Control* (M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, eds.), Birkhäuser, 1990, pp. 61-81.
44. "Integrability of certain distributions associated with actions on manifolds and applications to control problems," in *Nonlinear Controllability and Optimal Control* (H. Sussmann, ed.), Dekker, 1990, pp. 81-131.
45. "Constant McMillan degree and the continuous stabilization of families of transfer matrices," in *Control of Uncertain Systems* (D. Hinrichsen and B. Martensson, eds.), Birkhäuser, 1990, pp. 289-295.
46. "Kalman's controllability rank condition: from linear to nonlinear," in *Mathematical System Theory: The Influence of R. E. Kalman* (A. C. Antoulas, ed.), Springer, 1991.
47. "Capabilities and training of feedforward nets," in *Neural Networks: Theory and Applications* (R.J. Mammone and Y.Y. Zeevi, eds.), Academic Press, 1991, pp. 303-321.
48. "Remarks on interpolation and recognition using neural nets," in *Advances in Neural Information Processing Systems 3* (R.P. Lippmann, J. Moody, and D.S. Touretzky, eds.), Morgan Kaufmann, 1991, pp. 939-945.
49. (FA) "Transitivity and forward accessibility of discrete-time nonlinear systems," in *Controlled Dynamical Systems* (B. Bonnard, M. Bride, J.P. Gauthier, and I. Kupka, eds.), Birkhäuser, 1991.
50. "Input/Output and State-Space Stability," in *New Trends in Systems Theory* (G. Conte et al, eds.), Birkhäuser, 1991.