AD-A243 862

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED Final/1 Dec 88 - 31 May 91

4. TITLE AND SUBTITLE "STUDIES IN GLOBAL, BIFURCATION & SYMMETRY" (U)	5. FUNDING NUMBERS 61102F 2304/A4)
6. AUTHOR(S) Phillip Holmes	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Cornell University College of Engineering 212 Kimball Ithaca, NY 14853	8. PERFORMING ORGANIZATION REPORT NUMBER AFOSR-TR- 91 0928
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/NM Building 410 Bolling AFB DC 20332-6448	10. SPONSORING/MONITORING AGENCY REPORT NUMBER AFOSR-89-0200
11. SUPPLEMENTARY NOTES	
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for Public Release; Distribution Unlimited	12b. DISTRIBUTION CODE UL

13. ABSTRACT (*Maximum 200 words*)

Models of Turbulent Boundary Layers. With J.L. Lumley, S. Leibovich, postdoc and students I have used the proper orthogonal decomposition theorem to provide optimal bases for finite dimensional subspaces so that relatively low dimensional projections of the Navier-Stokes equations can be calculated. These dynamical systems retain key features of the turbulence production mechanisms. They inherit symmetries from physical space which lead to the existence of structurally stable heteroclinic cycles, in turn creating intermittent dynamics remarkably similar to the bursting phenomenon observed in experiments. Ours are perhaps the first rationally derived low (-10-50) dimensional models for turbulence in open flows and they offer promise of improved understanding of basic mechanisms and design of drag control strategies. Unlike other groups using Karhunen-Loeve methods, we have carrie out rather complete analyses of the projected ODE's. This work resulted in papers and also led to the mathematical questions.

14. SUBJECT TERMS	15. NUMBER OF PAGES 16	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT SAR

FINAL REPORT RESEARCH ACCOMPLISHMENTS 1989-1991, AFOSR 89-0200: STUDIES IN GLOBAL BIFURCATION AND SYMMETRY

We describe our research individually. Since we frequently collaborate, there is some overlap.

Philip Holmes

RESEARCH PROJECTS

I have worked in several areas over the past four years, not all of which were supported by AFOSR 89-0200. Since some of the work done under AFOSR 89-0200 arose from work suppoorted by other agencies, I mention it where apporopiate.

(1) Models of Turbulent Boundary Layers. With J.L.Lumley, S.Leibovich, postdocs and students I have used the proper orthogonal decomposition theorem to provide optimal bases for finite dimensional subspaces so that relatively low dimensional projections of the Navier-Stokes equations can be calculated. These dynamical systems retain key features of the turbulence production mechanisms. They inherit symmetries from physical space which lead to the existence of structurally stable heteroclinic cycles, in turn creating intermittent dynamics remarably similar to the bursting phenomenon observed in experiments. Ours are perhaps the first rationally derived low (~10-50) dimensional models for turbulence in open flows and they offer promise of improved understanding of basic mechanisms and design of drag control strategies. Unlike other groups using Karhunen-Loeve methods, we have carried out rather complete analyses of the projected ODEs. This work resulted in papers [1.67, 1.78, 1.80 and 1.85] and also led to the mathematical questions of (2) below.

(2) Global Dynamics of Symmetric Systems & Perturbation of Heteroclinic Attractors. Numerical and analytical studies in (1) showed that both discrete and continuous symmetry groups could exert striking effects on the global dyanamics of sets of PDE and ODE. This prompted several studies of O(2) equivariant vector fields in which, with D.Armbruster (MSI postdoc) and J.Guckenheimer, I was able to provide complete unfoldings and analyses of the dynamics and prove that specific heteroclinic interactions were responsible for the rich dynamics of systems such as the Kuramoto-Sivashinsky PDE. Also prompted by (1), Stone and I studied the influence of random and deterministic perturbations on systems

possessing such heteroclinic attractors. More recently, my graduate student S. Campbell has completed a study of symmetry breaking from O(2) to D4 and shown that, while certain heteroclinic cycles survive, other entirely new solutions appear (modulted travelling waves which change their direction of propagation). Both students, Emily Stone and SueAnn Campbell, have now completed their PhDs. The work under this heading was reported in [1.71, 1.72, 1.74, 1.76 1.84 and 1.87].

(3) Dynamics of Nonlinear Dissipative Evolution Equations. In 1988, as a result of MSI sponsored visits by R.D.James and J.M.Ball, we began studying model problems which have the potential to create fine structure, characteristic of certain material phase transformations, as energy is dissipated. These are essentially infinite dimensional problems, very different from PDE that possess inertial manifolds. Here arbitrarily high wave number "modes" are crucial to the long term dynamics and subtle distinctions between weak and strong convergence are important. R.L.Pego and P.J.Swart (MSI fellowship student- 1990-1) are also involved in the project. [1.83]. Pieter Swart has now copleted his PhD studies and taken a postdoc at Carnegie Mellon.

(4) Nonlinear Vibrations. O.O'Reilly (TAM grad student) and I studied the vibrations of a taut string subject to vertical harmonic excitation but free to move out of the vertical plane, both experimentally and theoretically. We showed that periodic, quasiperiodic and chaotic whirling motions could be understood in terms of a two degree of freedom averaged system and its integrable Hamiltonian limit. In connection with this problem, we worked with A.Mielke on bifurcation and chaos in unfoldings of homoclinic orbits to a Hamiltonian saddle-center [1.89, 1.90]. O'Reilly graduated in 1990 and is currently a postdoc at ETH, Zurich.

(5) Nonlinear Stability of Rotating Systems. Inspired by the semester organised by Marsden on Hamiltonian systems and stability in 1989, my student Brett Zombro and I have been using the "energy momentum" method to study the stability of discrete and continuum systems. We focus especially on situations where the method fails (the amended potential is indefinite with even index) and we have found explicit criteria for linear stability/instability in these cases [1.91].

TRAINING OF GRADUATE STUDENTS

During the period 1989-91 one M.Sc. and seven Ph.D. students graduated after writing theses under my direction: E. Stone, 1989; T. Kiemel, 1990; V. Brunsden, 1990 (M.Sc.); O. O'Reilly, 1990; C. Moore, 1991; S.A. Campbell

(1991); G. Berkooz (1991) and P.J. Swart (1991). AFOSR 89-0200 provided partial support and paid tuition for three of these students: O'Reilly, Moore and Campbell. I am currently supervising the research of three graduate students.

PUBLICATIONS 1989-91 See CV attached.

John Guckenheimer

RESEARCH PROJECTS

My research from the period from 1987-1991 has been supported by several Federal agencies, including AFOSR. A summary of my activities follows:

(1) Iterations of One Dimensional Maps. I have pursued the mathematics of iterations of maps of the interval for fifteen years, making a number of basic discoveries. In the past four years, there have been two notable contributions. The first applies to transformations that are the limit of period doublings and gives a simple geometric argument that the natural scales in these systems decrease geometrically. This argument has been generalized by Sullivan to all infinitely renormalizable unimodal mappings and used by him as a key step in his program to give geometric proofs of the Feigenbaum conjectures. The second result is joint work with Stewart Johnson that establishes distortion estimates on the iterates of unimodal maps with negative Schwarzian derivative.

(2) Dynamics and Bifurcations of Symmetric Systems. Motivated by work at Cornell on turbulent boundary layers, I rediscovered in concrete examples an important phenomenon that had been described earlier in a more abstract setting by Mike Field: dynamical systems with symmetry can have structurally stable heteroclinic orbits. Philip Holmes, Dieter Armbruster and I gave explicit examples and showed their relationship to problems originating in fluid mechanics and partial differential equations. The examples were placed within the context of multiparameter bifurcation problems. Our analyses of these bifurcation problems are among the first and most complete such analyses that have been given for systems with symmetry. Dieter Armbruster, Seunghwhan Kim and I gave an example of a two parameter bifurcation problem in symmetric systems for which one can demonstrate that there is a region of chaotic systems adjacent to the codimension two point which forms a positive angle there. This is still the only example with this property that has been described. Alex Mahalov and I have extended the work with Armbruster and Holmes to obtain results about three-wave interactions.

(3) Diffeomorphisms of the Two Torus. Seunghwhan Kim, Claude Baesens, Robert Mackay and I undertook an extensive investigation of bifurcations of families of diffeomorphisms of the two dimensional torus. These systems are the simplest mathematical models for the behavior of three coupled oscillators. With a combination of numerical study and geometric analysis, we mapped many of the main features that occur in the resonant behavior of these systems. We prove some theorems that relate the topology of the torus to the dynamics of the diffeomorphisms, and we formulated a much longer list of observations that can be stated as conjectures.

(4) Planar Vector Fields. Richard Rand, Dana Schlomiuk and I used computer algebra systems to investigate questions related to the number of limit cycles found in planar vector fields with polynomial coefficients (Hilbert's sixteenth problem). We performed perturbation calculations to analyze the number of limit cycles that can appear in a nearly divergence free vector field with quadratic coefficients.

(5) Computational Dynamics. Seunghwhan Kim and I developed a software package, entitled kaos, for the interactive investigation of bifurcation problems in dynamical systems. This package currently runs on Sun workstations and is being widely distributed. It has a number of unique features that make it considerably more efficient for determining the bifurcation diagrams associated to multiparameter systems than other programs that have been written for the numerical study of dynamical systems.

A second generation software package of this type has been developed with students Mark Myers, Patrick Worfolk and Rick Wicklin. The internal structure of the program has been redesigned in an object oriented manner with a view to making further extensions based upon new algorithms easy to incorporate. There are unpublished results on new algorithms for the computation of two (and higher) dimensional stable and unstable manifolds and for the computation of where Hopf bifurcations occur in families of dynamical systems.

(6) Analysis of Surface Waves. Dieter Armbruster, Seunghwhan Kim and I have investigated models for experiments of surface waves excited by vertical oscillation of containers with square cross-section. We have brought into clearer focus the discrepancy that still remains between calculations based upon truncation of the classical fluid equations describing surface waves and the experimental data of Simonelli and Gollub.

(7) Dynamics of Neurons. Isabel Laboriau and I have undertaken a system-

atic study of the dynamics of the Hodgkin-Huxley equations describing voltage clamped action potentials of squid giant axons. Using a combination of computer algebra and the program "kaos" described above, we are able to produce bifurcation diagrams for this system as pairs of parameters are varied. One observation of this work is that there are dynamical phenomena displayed by the Hodgkin-Huxley system that are topologically incompatible with the dynamics of planar vector fields.

Ron Harris-Warrick and I have developed a collaboration to explore the ability of extensions of the Hodgkin-Huxley equations to account for the dynamical behavior of conditionally bursting neurons in the lobster stomatogastric ganglion. Our intent is to develop models, investigate them and perform experiments to test whether models patterned after the Hodgkin-Huxley equations, but with more ion channels, can reproduce observed behavior of a single isolated cell in varying environments. This work is still in progress.

TRAINING OF GRADUATE STUDENTS:

I have been supervising several graduate students who are now in the research stage of their programs. Brad Bond has been studying the forced spherical pendulum and expects to complete his dissertation during Fall, 1991. Ramin Farzaneh has been studying questions involving diffeomorphisms of the torus and expects to finish his dissertation in Spring, 1991. Y. Y. Du is studying bifurcation problems arising in surface waves and should complete his degree in 1992 or 1993. Mark Myers, Rick Wicklin and Patrick Worfolk expect to complete their studies in 1993. In addition, I have been an active member of the special committees supervising several students who have completed degrees in the past three years, including Fred Adler, Gal Berkooz, Steve Lantz and Alex Mahalov.

PUBLICATIONS 1987-1991:

G. Dangelmayr and J. Guckenheimer, On a four parameter family of planar vector fields, Arch. Rat. Mech. Anal., 97, 321-352, 1987.

Renormalization of one dimensional mappings and strange attractors, Contemporary Mathematics 58 part III, 143–160, 1987.

Limit Sets of S-unimodal maps with zero entropy, Comm. Math. Phys. 110, 655–659, 1987.

Comments on multiple bifurcations, Proceedings of 1986 Army Research Conference.

______ and P. Holmes, Structurally stable heteroclinic cycles, Proc. Camb. Phil. Soc. 103, 189–192, 1988.

______, R. Rand and D. Schlomiuk, Degenerate homoclinic cycles in perturbations of quadratic hamiltonian systems, Nonlinearity, 2, 405–418, 1989.

Dieter Armbruster, ______ and P. Holmes, Heteroclinic cycles and modulated travelling waves in systems with O(2) symmetry, Physica D29, 257–282, 1988.

Dieter Armbruster, ______ and P. Holmes, Kuramoto-Sivashinsky Dynamics on the center-unstable manifold, SIAM J. Appl. Math, 49, 676–691, 1989.

______, R. Rand and D. Schlomiuk, Integrability of plane quadratic vector fields, Expositiones Mathematicae, 8, 3–25, 1990.

Obstacles to modelling large dynamical systems, Mathematical Approaches to Problems in Resource Management and Ecology, C. Chavez et al., ed. Lecture Notes in Mathematical Biology 81,319–327, 1989.

S. Kim, R. MacKay and ______, Resonance regions for families of torus maps, Nonlinearity, 2, 391–404, 1989.

______ and S. Johnson, Distortion of S-unimodal maps, Annals of Mathematics, 132, 71–130, 1990.

D. Armbruster, ______ and S. Kim, Chaotic Dynamics in Systems with Square Symmetry, Physics Letters A, 140, 416–420, 1989.

S. Kim, ______, FunTorus, interactive explorations of the two dimensional torus, MSI Technical Report, 1989.

S. Kim and ________, The two dimensional torus: Numerical explorations and mathematical conjectures, Chaos/XAOC Soviet-American Perspectives on Nonlinear Science, AIP,145–152, 1990.

C. Baesens, ________, S. Kim and R. MacKay, Simple Resonance Regions of Torus Diffeomorphisms, Proceedings of IMA Workshop on Chemical Reactors, to appear.

G. Berkooz, ________, P. Holmes, J. Lumley, J. Marsden, N. Aubry and E. Stone, Dynamical Systems Theory Approach to the Wall Region, to appear.

C. Baesens, ________, S. Kim and R. MacKay, Three Coupled Oscillators: Mode-locking, Global Bifurcations and Toroidal Chaos, Physica D, 49, 387–485, 1991.

__________ and S. Kim, kaos, Dynamical System Toolkit with Interactive Graphic Interface, MSI Technical report, 1990.

D. Armbruster, _________, and S. Kim, Resonant Surface Waves in a Square Container, Differential Equations and Computer Algebra, M. Singer, ed. , Academic Press, 61–76, 1991.

_________ and I. Labouriau, Bifurcation of the Hodgkin Huxley Equations: A New Twist, preprint.

Symmetry Properties of Confined Convective States, preprint.

_________ and S. Johnson, Beyond Hyperbolicity: Expansion Properties of One Dimensional Mappings, preprint.

Computational Environments for Exploring Dynamical Systems, Int,. Journal of Bifurcation and Chaos, 1, 269–276,1991.

_________ and A. Mahalov, Resonant Triad Interactions in Systems with O(2) Symmetry, Physica D, to appear

Richard Rand

RESEARCH PROJECTS

(1) Use of COMPUTER ALGEBRA to automate PERTURBATION calculations for dynamical systems. These methods consist of a wide variety of schemes for obtaining approximate analytical solutions (as opposed to numerical simulations), and were previously executed by hand. The use of computer algebra has revolutionized the field by offering greater accuracy and speed, and by permitting calculations to be completed which would never be attempted by hand. Some of my work has been reported in the book, "Perturbation Methods, Bifurcation Theory and Computer Algebra", with D.Armbruster, published by Springer in 1987. The following classical methods have been computer-algebracized: Lindstedt's method, the two variable expansion method (also known as multiple scales), averaging, Lie transforms, normal forms and center manifolds, the method of Liapunov and Schmidt, matched asymptotic expansions and the asymptotic expansion of integrals.

My most recent work in this area has involved using computer algebra to treat perturbation problems which involve ELLIPTIC FUNCTIONS. Although the use of elliptic functions in perturbation theory has been discussed in the literature for years, progress on such problems has been slow because of the greater computational complexity involved in such calculations. This work has shown that by using computer algebra, elliptic functions can be a practical tool for applied mathematicians.

(2) Dynamics of slowly-evolving Hamiltonian systems. This work has involved two separate projects. The first, with graduate student V.Coppola, involved CHAOS in a system which possessed a periodically disappearing figure-eight double homoclinic loop. Each time the separatrix is born, a given motion may be attracted to the left (L) or right (R) separatrix loop, thus resulting in a sequence of L's and R's corresponding to a given initial condition, e.g., LRRLRRRLLLLR... . We were able to show analytically (using the averaged equations) that this sequence displays sensitive dependence on initial conditions (SDIC), a criterion often used to describe chaos. This work has recently been extended by graduate student Jaquelyn Bridge who has quantified SDIC by introducing the notion of "sensitivity" of the symbol sequence to a change in initial conditions. Her current Ph.D. thesis research involves using contour maps of sensitivity to characterize this and related systems. A second project, accomplished with T.Mingori while I was on Sabbatic at UCLA in 1989-90, involves the phenomenon of RESONANT CAPTURE, i.e., the failure of a rotating mechanical system to be spun up, due to its resonant interaction with another system. The phenomenon is important in the dynamics of dual-spin spacecraft. Starting from a simple mechanical system consisting of an unbalanced rotor attached to an elastic support, we derived an abstract model of resonant capture. Using perturbation theory and elliptic functions we were able to determine which initial conditions lead to capture, and which pass through the resonant manifold and escape capture. This work has recently been extended by graduate student Chris Hall (who incidentally is a Captain in the U.S. Air Force, on leave from the Air Force Institute of Technology), who has extended the approach by obtaining approximate solutions to the averaged equations on either side of the moving separatrix, connecting these by an approximate boundary layer solution in the separatrix region.

(3) Bifurcations and chaos in piecewise-linear dynamical systems. This work involved two related projects, one with S.Shaw of Michigan State, and another with F.Moon and his associates. In the first case we modeled an inverted pendulum which impacted against rigid walls, and in the second case we investigated a model of a single pin joint consisting of two truss members joined by a smooth pin with play, the latter motivated by application to space truss structures. In both cases we were able to follow a series of bifurcations of families of simple periodic motions which led to chaos.

(4) Hilbert's 16th problem. Together with J.Guckenheimer and D.Schlomiuk we investigated a special case of this classic problem concerning the maximum

number of limit cycles possible in polynomial flows in the plane. The work involved an extremely complicated computer algebra calculation and was concerned with the bifurcation of limit cycles off of homoclinic loops.

(5) Dynamics of COUPLED OSCILLATORS. Work in this area has involved three projects:

a) Motivated by the biological oscillations in the spinal chord of a fish, we have, with P.Holmes, A.Cohen and graduate students D.Storti, W.Keith and T.Chakraborty, pursued a number of projects involving coupled oscillators. These have ranged from two coupled van der Pol oscillators to n coupled phase-only oscillators. In all cases we have been interested in the nature of the bifurcations accompanying transition from phase lock to drift.

b) Together with Professor M.Paidoussis and G.X.Li we have studied the dynamics of a pipe conveying a fluid. This involved a Galerkin projection onto 2 modes, and then a center manifold analysis of the resulting pair of o.d.e.'s. We found we could obtain estimates for the parameter value at which a limit cycle (born through a Hopf bifurcation) changed it stability, the second step in a sequence of bifurcations leading to chaos.

c) Together with Professors Alex Vakakis, C.H.Pak and Frank Moon, we have studied the bifurcation of nonlinear normal modes in two classes of conservative two degree of freedom systems. The use of computer algebra has permitted us to consider a system with general cubic forces having 6 parameters. For this class of systems we were able to obtain the points of maximum degeneracy in the 6 dimensional parameter space. Unfolding these singularities showed that there were systems in which a nonlinear normal mode could locally bifurcate into 5 related periodic motions (but no more than 5). This work showed how families of periodic motions are related.

(6) Resonance in a robot arm. Together with F.Moon and graduate student M.Golnaraghi we used computer algebra and perturbations to investigate the nature of resonances in a model of a high-speed flexible-arm robot.

(7) Model of corneal surgery. Together with biologist H.Howland and graduate student S.Lubkin, we developed an elastic shell model of the cornea. By varying the shell's thickness, we were able to model the effects of the surgical procedure called a radial keratotomy, aimed at changing the curvature of the cornea and eliminating the need for corrective lenses.

(8) Problem of Three Bodies. Together with graduate student B.Viswanadham, we used computer algebra and perturbation methods to obtain expressions for the transition curves in the elliptic restricted problem. We obtained expressions valid to order 50 in the eccentricity of the elliptical orbit of the primaries.

(9) Dynamics of Green Plants. Together with biologist J.Ellenson we investigated the dynamics of water transport in leaves, with special emphasis on the flow through stomata (pores) in the leaf surface. The model involved limit cycle oscillators coupled by diffusion. Our most recent work in this area involves circumnutation (movement in a circle) of young seedlings, with graduate student Sharon Lubkin. The motion, which has an observed period of 2 - 10 hours, has been modeled by coupled reaction-diffusion equations.

TRAINING OF GRADUATE STUDENTS

During this period I supervised the theses of the following students :J.Len (Ph.D. 1987), V.Coppola (M.S. 1988, Ph.D.1989), B.Viswanadham (M.S.1989), Howard Susskind (1991).

I currently am supervising three doctoral candidates, Jacquelyn Bridge, Chris Hall and Sharon Lubkin.

PUBLICATIONS 1989-1991

Analytical Approximation for Period-Doubling Following a Hopf Bifurcation R.H.Rand Mechanics Research Communications 16:117-123 (1989)

Computer Algebra and the Elliptic Restricted Problem of Three Bodies R.H.Rand, B.Viswanadham in Proceedings of Twelfth Canadian Congress of Applied Mechanics, May 28-June 2, 1989, Carleton University, Ottawa, pp.248-249

Degenerate Homoclinic Cycles in Perturbations of Quadratic Hamiltonian Systems J.Guckenheimer, R.Rand, D.Schlomiuk Nonlinearity 2:405-418 (1989)

Perturbation Methods and Computer Algebra R.Rand Chapter 6 in Symbolic Computation: Applications to Scientific Computing, ed. R.Grossman, SIAM, pp.139-151 (1989)

Resonance in a High-Speed Flexible-Arm Robot M.F.Golnaraghi, F.C.Moon, R.H.Rand Dynamics and Stability of Systems 4:169-188 (1989)

Integrability of Plane Quadratic Vector Fields D.Schlomiuk, J.Guckenheimer and R.Rand Expositiones Mathematicae 8:3-25 (1990)

Averaging Using Elliptic Functions: Approximation of Limit Cycles V.T.Coppola and R.H.Rand Acta Mechanica 81:125-142 (1990)

Bifurcations and Chaos in a Forced Zero-Stiffness Impact Oscillator G.X.Li, R.H.Rand and F.C.Moon International J. Nonlinear Mechanics 25:417-432 (1990)

Computer Algebra, Elliptic Functions and Chaos V.T.Coppola and R.H.Rand in Proceedings of 1990 ASME International Computers in Engineering Conference, Vol.1, pp.193-200, Amer.Soc.Mech.Eng. (1990)

Spinup Through Resonance of Rotating Unbalanced Systems with Limited Torque R.J.Kinsey, D.L.Mingori and R.H.Rand AIAA paper no. 90-2966 in Proceedings of AIAA/AAS Astrodynamics Conference, Aug.20-22,1990, Portland,OR, Part 2, pp.805-813

Using Computer Algebra to Handle Elliptic Functions in the Method of Averaging R.H.Rand in Symbolic Computations and Their Impact on Mechanics, eds. A.K.Noor, I.Elishakoff, G.Hulbert, pp.311-326, Amer.Soc.Mech.Eng., PVP-Vol.205, (1990)

Analytical Model of Corneal Surgery H.C.Howland, R.H.Rand, S.L.Lubkin J.Biomechanical Eng. 113:239-241 (1991)

MACSYMA Program to Implement Elliptic Functions in the Method of Averaging V.T.Coppola, R.H.Rand in Computer Aided Proofs in Analysis, eds. K.R.Meyer and D.S.Schmidt, pp.71-89, Springer Verlag (1991)

Chaos in a System with a Periodically Disappearing Separatrix V.T.Coppola, R.H.Rand Nonlinear Dynamics 1: 401-420 (1990)

Chaotic Motions of a Constrained Pipe Conveying Fluid: Comparison Between Simulation, Analysis, and Experiment M.P.Paidoussis, G.X.Li, R.H.Rand J.Applied Mechanics 58:559-565 (1991)

Spinup Dynamics of Axial Dual-Spin Spacecraft C.D.Hall,R.H.Rand AIAA paper no. 91-403 in Proceedings of AIS/AIAA Astrodynamics Conference, Aug.19-22,1991, Durango,CO