AFIT/GE/ENG/91D-28

WAVE PROPAGATION IN A RANDOMLY
INHOMOGENEOUS MEDIUM - A STUDY
OF THE PROBLEM

THESIS

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AFIT/GE/ENG/91D-28

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Wave Propagation in a Randomly Inhomogeneous Medium - A Study of the Problem

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The problem of calculating the scintillation index of an atmospherically propagating spherical wave is examined. The fourth statistical moment of the complex field is obtained by using Feynman (path) integral techniques applied to the stochastic parabolic equation. The general trajectory of each Feynman integral is approximated by a truncated Fourier-sine series and the infinite-fold integration of the Feynman integral is reduced to a three-fold Riemann integral which is shown to match results derived under different assumptions. This thesis is highly tutorial.

Path Integration, Feynman Integration, Functional Integration, Scintillation Index 106

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WAVE PROPAGATION IN A RANDOMLY INHOMOGENEOUS MEDIUM - A STUDY OF THE PROBLEM

THESIS

Presented to the Faculty of the School of Engineering

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In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Electrical Engineering

Kyle Hunter, B.S.E.E., B.S. MATH, M.A. MATH

Captain, USAF

December 1991

Acknowledgments

When I began work on this thesis, I began to possess a mind set which can best be summarized by a quote from an american icon:

> A man has to know his limitations. *Clint Eastwood as Dirty Harry in Magnum Force*

However, after working with Dr. Joseph Gozani, whom I admire and to whom am greatly indebted, I changed my previous mind set to one which can best summarized by an american cult icon

> Don't dream it, be it ! *Tim Curry as Doctor Frank-n-furter in The Rocky Horror Picture Show*

But, what kind of acknowledgment would this be if I didn't show the appropriate gratitude to all the others to whom I am indebted. I am indebted to my AFIT advisor Dr. Byron Welsh for his excitement and encouragement at my evolving (though still miniscule) understanding of path integration. Additionally, I am indebted to my thesis committee, Dr. Steve Rogers and Lt. Col. Dave Norman for the probing questions which I could answer coherently and accurately. Of course, I have parents. Their encouragement at my progress can't be overlooked. My thanks to you both and to all the others I've mentioned. Lastly, I thank me. Afterall, I actually wrote it.

Kyle Hunter

Preface

This thesis examines the problem of wave propagation in a randomly inhomogeneous medium from the standpoint of Feynman integration. Also known as path integration or functional integration, Feynman integration oftentimes presents the uninitiated reader with conceptual difficulties; therefore, this thesis was written with a tutorial style in mind. Moreover, there are numerous appendices and far greater detail and verboseness in the main chapters than would normally be considered necessary in a thesis; however, it is hoped that excessive verbosity will be an asset in this case.

Kyle Hunter

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Abstract

The problem of calculating the scintillation index of an atmospherically propagating spherical wave is examined. The fourth statistical moment of the complex field is obtained by using Feynman (path) integral techniques applied to the stochastic parabolic equation. The general trajectory of each Feynman integral is approximated by a truncated Fourier-sine series and the infinite-fold integration of the Feynman integral is reduced to a three-fold Riemann integral which is shown to match results derived under different assumptions.

WAVE PROPAGATION IN A RANDOMLY INHOMOGENEOUS MEDIUM - A STUDY OF THE PROBLEM

I. Introduction to the Problem

The Overview

Wave propagation in a randomly inhomogeneous medium, for example monochromatic light propagating in the atmosphere results in changes in both the phase and the amplitude of the propagating wave. If the fluctuations of the amplitude and phase are weak, then it is possible to describe the characteristics of the amplitude and phase using perturbation techniques. In this case, the observed changes of the phase of the wave in an observation plane, some distance from the wave's origin, will be small. The region over which an observation plane may be placed, and still facilitate the measurement of small phase fluctuations, is known variously as the weak fluctuation regime, the perturbational regime, or the asymptotic regime. Moreover, as the propagation distance increases or the strength of the random inhomogeneities increases, the intensity changes in the observation plane of the propagating wave will increase. For sufficiently strong random inhomogeneities in the atmosphere, the intensity variations will attain an extremum and the scintillation index, the normalized intensity variance, may exceed unity. If the scintillation index exceeds unity, then the propagation distance or region over which scintillation index does so is known as the strong fluctuation regime. Finally, if the propagation distance is increased further still or the random inhomogeneities become stronger yet, then the intensity fluctuations will begin to decrease and asymptotically saturate at a fixed level. This region of asymptotic saturation is known as the saturation regime. Wave propagation in random media are characterized by the medium's statistical characteristics. This implies that various statistics, and particular moments must be obtained. These can be found by averaging over stochastic complex wave functions obtained by Feynman (path) integration.

The Problem

Functional integration, by its vary nature, requires multiple integrations over a continuum of elements of a function space. There is presently a method to approximate a functional integral, and therefore the integration continuum, by using a finite subset of an infinite orthogonal function expansion to approximate the functional integrand. The resulting approximation can then be numerically integrated to obtain the necessary complex amplitude. The accuracy of some of the initial subsets, a finite number of terms from a Fourier series, looks ver, promising. The application of these techniqes to the proplem of wave propagation in a randomly inhomogeneous medium is a fairly recent development and not all of the ramifications are at present fully known or understood.

The Research Questions

During the course of research for this thesis, it desired to obtain the answers to some relevant questions.

1. Using only one sinusoid, the first term in a Fourier-sine series, what accuracy does it provide when used to approximate functional integration?
2. What is the computational complexity of this approximation?
3. Does the characteristics of this approximation dictate the use of a particular numerical integration scheme?

These questions are obviously not all inclusive. In fact, it will be necessary to drastically reduce the potential research extent in order to obtain any meaningful results. This will be done by making certain assumptions.

The Research Assumptions

As it will be explained in greater depth in Chapter 2, all propagating electromagnetic waves must satisfy the wave equation which is derived from Maxwell's equations. However, this derived equation is a second order in space, second order in time, partial differential equation. If the situation is further compounded by requiring propagation in a randomly inhomogeneous medium, then the wave equation becomes a second order stochastic partial differential equation. By making several prudent, realistic assumptions, it is possible to simplify the solution to the wave equation, a Feynman integral, into a solution that is less complicated. In order to make this simplification, it will be necessary to make the following assumptions:

1. The parabolic approximation to the stochastic Helmholtz equation holds.
2. The propagating wave is monochromatic.
3. The wavelength of the wave's carrier is in the optical regime.
4. The characteristic size of the random inhomogeneities of the propagation medium greatly exceeds the wavelength of the carrier.
5. The velocity of light will greatly exceed the maximum velocity of the fluctuations which implies nonrelativistic motion.
6. Near field effects are not relevant as they are small and we will consider propagation distances which exceed the characteristic length of the near field.
7. Although the Feynman integral methods was borrowed from quantum mechanics, quantum mechanical effects are assumed to be inconsequential in all limiting cases of the functional integrals.

Assumption 1, the parabolic approximation, will require that back scattering effects as well as large scattering angles from the mean direction of propagation, be ignored. As it is shown in Appendix A, this necessitates that the complex amplitude of a propagating wave vary slowly as compared to the mean wavelength of the carrier.

Assumption 2 does not excessively limit the applicability of the results of this thesis; it is possible to decompose a general propagating wave into its spatial Fourier components and then consider each spatial frequency separately; however, this thesis will not cover such an analysis.

Assumption 3 will allow effects of wave depolarization due to the random inhomogeneities to be ignored because the depolarization will be small.

Assumption 4 is realistic as the size of the inhomogeneities in Earth's atmosphere ranges from several millimeters, the inner scale, to about a hundred meters, the outer scale.

Assumption 5 requires a physically realizable propagation medium such as the earth's atmosphere. Notice this assumption might not be valid for optical wave propagation in a turbulent plasma.

Assumption 6 states that the near field effects may be ignored. The Huygens-Fresnel principle provides a rigorous solution to the parabolic wave equation for positive propagation distances which exceed the near field regime. This will be covered in more detail in Chapter 3.

Assumption 7 requires the consideration of only classical, that is, non-quantum mechanical phenomena.

The Scope

This thesis will investigate the problem of wave propagation in random media and some of the equations that will be generated by this investigation. Additionally, an accleration technique which can be used to solve the final resulting equation will be provided.

The Research Methodology

The numerical integration of continuous functions is a field of study in itself. The integrand functions, which will be considered in this thesis, are highly oscillatory; therefore, many of the more favored numerical integration schemes, such as Gaussian quadrature, will not achieve acceptable accuracy. Thus, in general, one finds that for many currently available numerical integration routines that:

1. The proposed numerical integration scheme may not converge within the prescribed number of iterations.
2. The proposed numerical integration scheme may not converge at all. Some integration schemes recursively subdivide every interval and compare the results of each subdivision. If the solution series, which is generated by comparing the integral over an interval with all the recursive integrals of that interval, is bounded, but fails to converge, or worse yet, diverges to infinity, then the solution and the associated method of solution are useless.
3. The proposed numerical integration scheme rapidly converges to an incorrect solution. This is possible if each subdivision matches some pathological characteristic of the integrand.

The Research Benefits

As this thesis is directed primarily towards a study of the problem of analyzing wave propagation in a randomly inhomogeneous medium, the benefits of this research will primarily be to enable the reader of this thesis to better understand the problem and its characteristics. That is, by casting a physical problem into an equivalent mathematical form, it is possible to obtain a deeper understanding of said problem. The Feynman integral method will provide an intuitively pleasing way to analyze the problem; therefore, it will allow the reader to obtain greater insight into the problem of wave propagation in a randomly inhomogeneous medium. This basic understanding of a difficult problem is, and should always be, a goal of basic theoretical research into said problem.

The next chapter contains a literature review of wave propagation analysis using Feynman integration, as well as other currently favored analysis methods.

II. Literature Review

Justification

Imagine a shinny penny on the bottom of a clear, dark-bottomed swimming pool on a calm day. Standing on the edge of the pool, one can easily see that penny. If you toss in a rock, the penny's image will dance about, and it may even disappear entirely. The atmosphere is obviously not that swimming pool, however both the hypothetical pool and the atmosphere randomly distort optical images by distorting propagating optical waves. That is, the propagating optical wave is distorted randomly in both phase and in amplitude by the medium in which it is propagating. It is natural to inquire as to characteristics of the propagating optical wave. It is the characteristics of an atmospherically contaminated propagating optical wave with which this thesis will be concerned.

The atmosphere is, as a first approximation, a chunk of air with varying densities generated by thermal heating from the sun. Large sections of air warm, rise, and eventually break into smaller sections. The smaller sections disperse into even smaller air pockets. These air pockets can range in size from a few millimeters, the inner scale, to several hundred meters, the outer scale (17:388,390). It is the motion of the air pockets, along with their varying relative indices of refraction, that cause the randomization of the amplitudes and phases of atmospherically propagating optical waves. Because the atmosphere introduces these randomizations into propagating optical waves, the characteristics of the optical waves, their amplitude and phase, are unpredictable. Therefore, it is the statistical characteristics of the propagating optical waves which must be considered. Moreover, the method by which the optical wave's statistics are obtained will be dictated by the fluctuating statistical characteristics of the medium, and the distance through which the optical wave has propagated. There are several mathematical methods which can be used to either calculate or approximate the statistical characteristics of an optical wave propagating through a randomly inhomogeneous medium, such as the atmosphere. These mathematical methods will, in turn, have conditions under which they are applicable.

The choice of the particular mathematical analysis method is often complicated by the fact that the characteristics of the optical wave can influence the choice of the mathematical method used to analyze them. In order to gain a more thorough understanding of the especially powerful approach of functional integration, it is necessary to be familiar with some of the other methods which are currently mentioned in the literature along with their regions of applicability and any constraints to their use. To do this, it will be necessary to start at the beginning with the wave equation.

In free space, a propagating monochromatic optical wave must obey the wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(r,t) = 0 \tag{2.1}$$

where$U(r,t)$ is the optical field, a function of space and time, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, r is the position vector (x, y, z), c is the speed of light, t is time. A closed solution of Eq. 2.1 is, except for trivial or highly symmetric cases, extremely difficult, if not impossible, to obtain. Therefore, numerical solutions to Eq. 2.1 must be obtained. However, the solution of Eq. 2.1 requires a global approach. That is, the solution must be obtained in all volumes of interest, subject to all the boundary conditions simultaneously (23:1). It is possible to reduce the computational complexity of Eq. 2.1 by considering a reduced class of problems. By considering cases with negligible backscatter and small propagation angles about the mean direction of propagation, Eq 2.1 will be simplified. This simplification, which results in a new equation, is often called the parabolic approximation to the wave equation. These characteristics of this simplified equation will be explored in the next section.

Parabolic Equation

The derivation of the parabolic approximation to the wave equation for an optical wave propagating in the z direction can be found in Appendix A. It is shown in Appendix A that Eq. 2.1 may be reduced to

$$(2jk\frac{\partial}{\partial z} + \nabla_\perp^2 + k^2 \tilde{\epsilon}\,(\rho, z))u(\rho, z) = 0 \tag{2.2}$$

where $u(\rho, z)$ is the complex amplitude of the optical wave, $\nabla_\perp^2 = \frac{\delta^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, j is $\sqrt{-1}$, k is $\frac{2\pi}{\lambda}$ and λ is the carrier wavelength, $\tilde{\epsilon}\,(\rho, z)$ is the fluctuating component of the medium's dielectric permeability, $\rho = (x, y)$, namely ρ is in the plane transverse to the mean direction of propagation, z.

The simplification of Eq. 2.1 to Eq. 2.2 is a direct result of considering the propagation of the optical wave in small angles about the mean direction of propagation. The small angle simplification which results in Eq. 2.2 has been known under various names. Some of these names include: "small angle approximation", "paraxial approximation", "the standard parabolic approximation" (14:838), and the randomly occurring implied "usual approximation" , where the article's author assumes a certain level of reader sophistication (2:516). The reason for this interest in small angles about the mean propagation direction is that some random inhomogeneous media, under certain conditions, exhibit only small wave field scattering angles. Therefore, the wave equation may be reduced to the parabolic wave equation. However, this small angle restriction is more fundamentally a necessary condition to maintain the validity of the approximation of Eq. 2.1 by Eq. 2.2. Specifically, the backscatter of the optical wave, or acoustical wave for that matter, is considered to be negligible (10:895). The interested reader will find Appendix A has further details.

The parabolic approximation to the wave equation is an active area of research for numerous authors including (29:212), (10:894), (8:171), (7:1224), (6:38-3), (12:297), and (13:1355). Furthermore, authors in other fields including acoustics and quantum mechanics are also extremely interested in this equation because of the existence of equations which are functionally similar to Eq 2.2. In acoustics, the propagation of sound in the ocean is modeled as an inhomogeneous second order partial differential equation (23:6). Therefore, the general solution to Eq 2.2 and the acoustics problem will be similar. In quantum mechanics, the motion of elementary particles in a conservative field is modeled by an inhomogeneous partial differential equation known as the Schrödinger wave equation (29:225), (16:48). This equation is equivalent to the parabolic approximation to the wave equation and therefore general solutions will be similar to both equations. It is indeed fortunate that these diverse fields possesses similar equations, because the solution obtained in one field of study may be used in the other research fields, subject to the appropriate boundary conditions (29:225,226), (23:7,8,18,19). There is a plethora of different names for Eq. 2.2 in the current literature. Eq. 2.2 is also known as the parabolic approximation to the stochastic Helmholtz equation, the parabolic approximation to the wave equation (14:838), reduced scalar Helmholtz equation (12:297), complex heat equation (7:1224), diffusion equation (16:49).

As it was stated earlier, Eq. 2.1 usually doesn't posses a closed-form solution, and the random effects of the atmosphere necessitate the study of statistical characteristics, rather than, the deterministic characteristics of the optical wave. However, by considering Eq. 2.2 and the statistics of the atmosphere (more generally the medium in which propagation occurs), it is possible to apply different mathematical techniques to the solution of Eq. 2.2. The next sections will consider the various mathematical methods.

Method of Geometrical Optics

The method of geometrical optics is the lowest approximation to the wave field amplitude in a random inhomogeneous medium. In this multiple scattering method, the accuracy of the solution increases with decreasing wavelength (10:894). Furthermore, the range of validity for geometric optics can be large (25:2). Additionally, the method facilitates the analysis of multiple effects, including refraction, and is the basis for the method of smooth perturbations and the Markov approximation method (25:2).

Geometrical optics, the method with which most individuals are familiar, is the method whereby optical fields are modeled by straight line or ray constructions. This method is the simplest of the analytical methods used to obtain a solution to the problem of wave propagation in a randomly inhomogeneous medium. The simplicity is achieved by ignoring diffractional effects. The simplicity is attained when the umbra and penumbra of shadows are determined by the geometry of transparent, translucent, and opaque objects in the propagation medium. Because of this simplicity, the geometric optics method can't predict the effects of diffraction nor can it take them into account. Thus, simplicity is traded for accuracy; "The method is simple in that, unlike the method of smooth perturbations and Markov approximation, it does not take into consideration diffraction effects, and therefore is not as versatile as the other two methods" (25:1). However, the method has certain advantages over other methods. Specifically, geometrical optics permits multiple effect modeling and the modeling of effects that are more difficult to describe using other methods. Additionally, certain propagation effects will continue to maintain validity beyond the geometrical optics (validity) regime (25:2). The elimination of diffractional effects may be justified under certain geometries; however, when diffraction must be considered, the method of smooth perturbations provides a possibly better solution framework.

Method of Smooth Perturbations

This single scattering method is one of the older methods still being used to solve Eq. 2.2. This method, also known as either Rytov's method, or the asymptotic method in the current literature, uses first order perturbation methods on the logarithm of the optical wave's complex amplitude to compute an approximate solution to Eq. 2.2. This method may be used whenever the scintillation index, the normalized intensity variance of the optical wave, is small, and diffractional effects must be included (10:894), (14:2111). Thus, whenever the scintillation index,

$$\beta^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} \tag{2.3}$$

where $\langle \cdot \rangle$ denotes ensemble averaging, is less than 0.3, it is generally agreed that the method of smooth perturbations is applicable. Appendix B contains a more complete discussion of the method for the interested reader. The accuracy of the smooth perturbation method has been called into question by some authors (2:516). However, an asymptotic solution approach, a variation of the method of smooth perturbations, used to solve Eq. 2.2 in the case of strong intensity fluctuations has become available recently (15:2111). Additionally, other methods including functional integration methods have shown the Rytov method to be quite accurate under certain constraints. These constraints are complex and depend upon both the size of the root-mean-square phase fluctuations and the ratio of the longitudinal scale size of the inhomogeneities to the extent of the Fresnel zone of the optical wave (10:895,896). The solution of Eq. 2.2 by the method of smooth perturbations is still a stochastic quantity as the solution depends upon the statistical characteristics of the random medium. It is from this solution that the statistical quantities of interest are calculated. It is possible to bypass the solution of Eq. 2.2 entirely and calculate the statistical quantities directly. This method will be explored in the following section.

Method of Markov Approximations

The method of Markov approximations facilitates the determination of closed partial differential equations for the statistical quantities of the optical wave field. The derivation of these statistical moment equations eliminates the necessity of explicitly solving Eq. 2.2. As shown in Appendix C, the statistical equations are obtained by approximating the optical wave propagation in the inhomogeneous random medium as a Markov process (29:213), (25:79). The method of Markov approximation is according to (7:1234) and (8:171) a more accurate method to solve Eq. 2.2 than the method of smooth perturbations. In order for the method of Markov approximations to be applicable to a propagation problem, it is necessary that,

1. The characteristics of the wave field are dependent only upon the section of the medium through which the wave has already propagated.
2. The longitudinal correlation radius of $\tilde{\epsilon}$ is small compared to the characteristic length of the changes in $u(\rho, z)$ (the complex amplitude) in the z direction (25:79).

The first condition is actually impossible for the general situation of optical wave propagation in a random medium such as the atmosphere. Atmospherically propagating waves will be backscattered to a small degree. These backscattered waves will interact later with forward propagating waves and therefore, the forward propagating wave will be dependent upon the characteristics of the medium through which it has not yet encountered. Therefore, a direct implication of the first requirement necessitates the consideration of solutions for Eq. 2.2 as opposed to Eq. 2.1 (28:5.11).

The second requirement simply means that the correlation function $B_\epsilon(z, z') = \langle \tilde{\epsilon}\ (\rho, z)\ \tilde{\epsilon}\ (\rho, z') \rangle$ is non-zero over an interval which is small compared to the characteristic length of changes in the complex amplitude, $u(\rho, z)$, of the optical wave in the mean direction of propagation. This in turn requires that "longitudinal correlation radius of ϵ be the smallest dimension in the problem" (29:215). When this holds, $B_\epsilon(z, z') = \langle \tilde{\epsilon}\ (\rho, z)\ \tilde{\epsilon}\ (\rho, z') \rangle$ can be approximated by a scaled Dirac delta distribution, in which case $\tilde{\epsilon}$ is said to be delta correlated. Clearly one advantage, which is often exploited in the literature, is the ability to extend the integration range of a function with small support, such as a Dirac delta function, from a finite to an infinite integration range. This expansion in the integration range allows the approximation of certain correlation functions as Fourier Transforms of power spectral densities (29:218).

If the previous two necessary conditions are approximately satisfied, then the Markov approximation method may be applied to situations where the scintillation index exceeds unity. That is, when the method of smooth perturbations fails to hold, namely, strong intensity fluctuations (10:894). The accuracy of the resulting equations of the statistical moments will be dependent to the degree that the above two assumptions are satisfied. Finally, just as the method of Markov approximations eliminated the need to explicitly solve for the solution to Eq. 2.2, the application of numerical modeling methods and modern computer hardware has, in some circumstances, eliminated the need to even consider the statistics of the solution.

Computer Modeling

The application of Fast Fourier Transform algorithms, finite element methods and multi-gigaflop performance levels in modern computing hardware has allowed the computer modeling of many random medium propagation problems. It is possible to numerically solve Eq. 2.1 over all locations in a given volume simultaneously using modern hardware and software. However, even with today's most powerful computing systems, it is Eq. 2.2 rather than Eq. 2.1 which is modeled. In one method, the statistical equations for the optical wave are obtained in integral form and the method of Monte Carlo integration is applied to the equations (2:517). A different method obtains solutions which are allowed to propagate. The solutions to Eq. 2.2 can be obtained relatively easily for arbitrarily thin volumes transverse to the mean propagation direction. These solutions are then propagated to adjacent thin parallel volumes sequentially just as the actual optical wave would propagate through the medium using known laws for optical propagation (15:2111). However, computer simulation, though a powerful tool, has disadvantages over the previously mentioned analytical methods. Simulations, as accurate as they can be, often conceal fundamentally important mechanisms in the propagation of optical waves. Additionally, aliasing, the contamination of lower frequency components by higher frequency components, is often an ignored problem when modeling continuous phenomena using discrete data on computers. Presently there is great interest in one especially powerful method from the field of quantum mechanics which can be used to model and solve Eq. 2.2 in a form suitable for numerical or asymptotic analysis methods (6:38-1). This method, known as functional integration, is investigated in the next section.

Functional Integration

Richard Feynman, in his doctoral dissertation in 1942, proposed a novel solution to the Schrödinger wave equation, as shown in Appendix D. His solution allowed the number of integrations in a finite-fold integral to diverge to infinity (16:48), (12:298), (29:227), (23:4). Feynman's technique spurred the investigations of others into the area of research which later came to be known as "integration in a function space" (16:48). Feynman's original integral solution of the Schrödinger wave equation came to be known as a Feynman integral, although it was determined that this integral was simply a special case of a Wiener integral (5:127). The study of Wiener integrals, also known today as path integrals, began with the work of N. Wiener in the 1920's (16:48). Because of the functional similarity of the parabolic equation to the Schrödinger wave equation, the functional integration approach can be used to solve the parabolic equation Eq. 2.2. Moreover, it has been shown that the functional integration approach can be applied to the wave equation Eq. 2.1 itself (6:38-3). It is the method of functional integration applied to the parabolic equation with which this thesis will be primarily concerned.

A Justification from Fourier Optics This section will give an intuitive, nonrigorous justification of the functional integral technique to solve the parabolic equation, Eq. 2.4 in a homogeneous medium ($\tilde{\epsilon}\ (\rho, z) = 0$). The solution for the more general case of an inhomogeneous medium will be found in chapter 3. The solution to Eq. 2.4

$$(2jk\frac{\partial}{\partial z} + \nabla_{\perp}^{2})u(\rho, z) = 0 \tag{2.4}$$

is given by

$$u(\rho, z) = \frac{k \exp(jkz)}{2\pi jz} \iint \exp\left(\frac{jk|\rho - \rho'|^{2}}{2z}\right) u(\rho', 0)d^{2}\rho' \tag{2.5}$$

where $u(\rho, z)$ is the complex amplitude of the optical wave at the $z = z$ plane, $u(\rho, 0)$ is the complex amplitude of the incident optical wave at the $z = 0$ plane, $\rho = \rho(x, y)$ is a two-dimensional position vector in the $z = z$ plane, $\rho' = \rho'(x', y')$ is a two-dimensional position vector in the $z = 0$ plane, $j = \sqrt{-1}$, $k = \frac{2\pi}{\lambda}$ and λ is the optical carrier wavelength.

Eq. 2.5 is a solution which is valid for all positive values of z. If one considers the geometry in Fig. 2.1, then it is easy to see that if the complex amplitude of a propagating optical wave is known at the $z = 0$ plane, then it can be obtained at the $z = z$ plane using Eq. 2.5. Furthermore, Eq. 2.5 must hold for all values of z' where $0 \leq z' \leq z$.

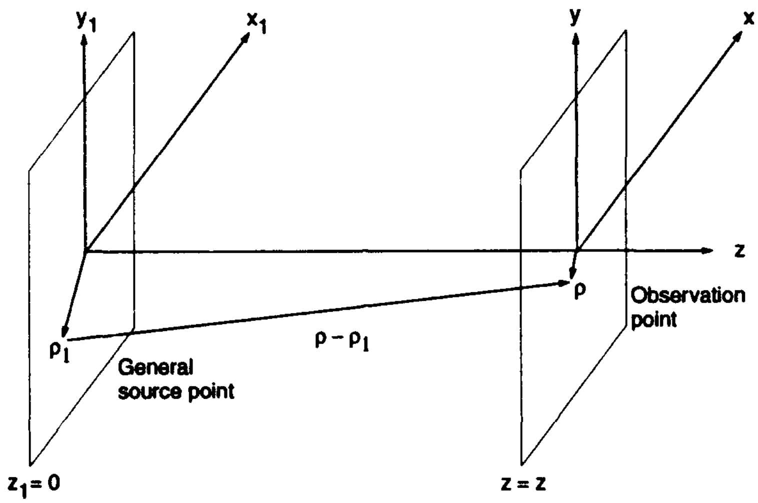


Figure 2.1. Single interval propagation

Because Eq. 2.5 holds for all values of z' such that $0 \leq z' \leq z$, it holds for $z' = \frac{z}{2}$. This simply means that in order to be able to observe an optical wave at the $z = z$ observation plane, it was necessary to be able to observe it at some intermediate location along the path of propagation. So, the optical wave will propagate from the image plane to the observation plane at $z' = \frac{z}{2}$. Then, the optical wave will propagate from this object plane to the observation plane at $z' = z$. Thus, referring to Fig. 2.2 equation Eq. 2.5 can be applied to the separate domains: $0 \leq z' \leq \frac{z}{2}$ and $\frac{z}{2} \leq z' \leq z$.

Therefore, Eq. 2.5 can be applied to each of the half-length sections and a four fold integral is

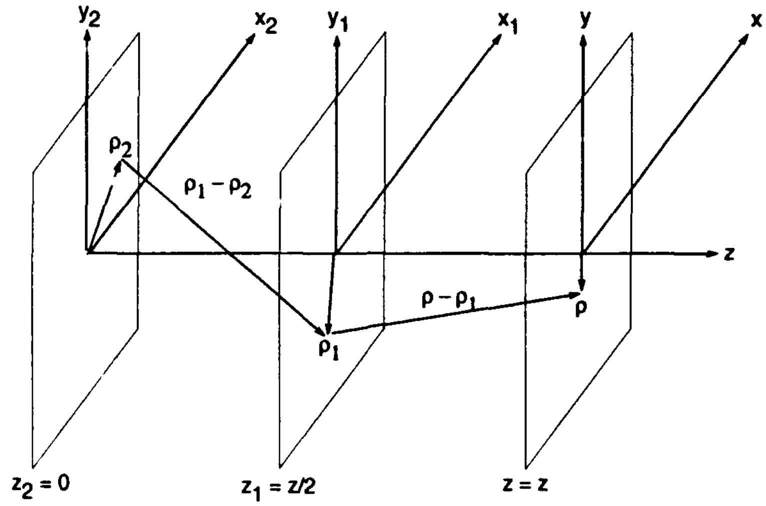


Figure 2.2. Double interval propagation

obtained

$$u(\rho, z) = \exp(jkz)\left(\frac{2k}{2\pi jz}\right)^2 \iint \iint \exp\left(\frac{2jk(|\rho_1 - \rho|^2 + |\rho_2 - \rho_1|^2)}{2z}\right) u(\rho_2, 0) d^2\rho_1 d^2\rho_2 \tag{2.6}$$

where the region of integration is $\mathbf{R}^2 \times \mathbf{R}^2$. It is possible to see the emergence of a path. The complex amplitude at the point (x, y) is determined by the concatenated rays or path $(\rho_1 - \rho), (\rho_2 - \rho_1)$. It is possible to continue to divide the original interval into three sections, four sections and so on. For the case of an arbitrary number of sections, the geometry is shown in Fig. 2.3.

As the number of intervals becomes large, the length of each interval tends toward zero. Thus, the finite-fold integral becomes a functional integral or path integral in the limit as the size of each interval vanishes. The path in this case being any possible ray trajectory $\rho_n, \rho_{n-1}, ..., \rho_1$ from the furthest observation plane backward to the original object plane. The intermediate planes each alternately act as object and observation planes and in the limit, the set of polygonal paths so generated will almost span the set of all possible continuous paths from the original object plane to the final observation plane uniformly. Formally, the solution to the parabolic

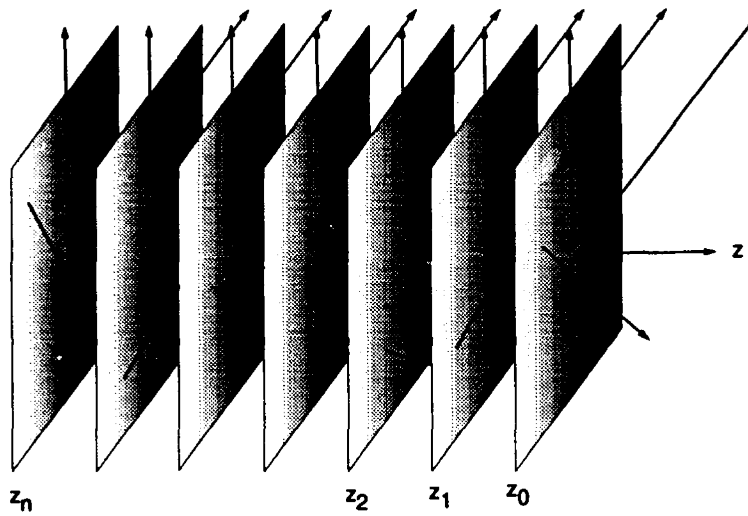


Figure 2.3. Multiple interval propagation

equation is given by Eq. 2.7

$$u(\rho, z) = \lim_{n\to\infty} \exp(jkz)\left(\frac{nk}{2\pi jz}\right)^n \iint \cdots \iint \exp\left(\frac{jnk}{2z}\sum_{m=1}^{n}|\rho_m - \rho_{m-1}|^2\right) u(\rho_0, 0)\prod_{m=0}^{n-1} d^2\rho_m \tag{2.7}$$

where the integration region is $\lim_{n\to\infty}\mathbf{R}^{2n}$. Of course for free space, its easy to show that the preceding path integral is fully equivalent to the Green's function from the Huygens-Fresnel integral. As it was stated earlier, the previous path integral derivation showed, in a non-rigorous way, the applicability of the functional integral approach. It could be argued that the Huygens-Fresnel formula would give the correct results over a single propagation interval without the need to resort to the recursive subdivisions that the functional integration method encompassed. This is true; however, the previous example was for a deterministic, homogeneous medium. In the case of a randomly inhomogeneous medium, the functional integral approach to solving the parabolic equation, parallels the heuristic method of considering thin slabs of the medium to be approximated as phase screens. This idea will be expanded further in the following chapter.

Up until now, the term complex amplitude has been used extensively to describe the strength of the propagating optical wave. However, neither intensity detectors, photographic film nor most optically dependent biological organisms can see complex amplitudes. All three "see" intensities. Therefore, it is necessary to complicate matters by considering not only the complex amplitude of a propagating wave and all of its statistical characteristics, but also the product of the complex amplitude and its complex conjugate and the associated statistical characteristics of the intensity. Because the statistical characteristics of the randomly inhomogeneous medium will further complicate matters, it is therefore necessary to consider the strength of the fluctuations in the randomly inhomogeneous medium.

Weak, Strong and Saturated Regimes It has been shown that for a Gaussian random phase screen β^2 has the characteristics as shown in Fig. 2.4 for an incident unity amplitude plane wave (15:2111), (25:74). From this figure there are three regions which are of interest. The first is the region in which the value of β^2 increases from zero to one, the weak fluctuation regime. The second region, if it exists, is the region in which β^2 is greater than one, the strong fluctuation regime. The final region, if it exists is where β^2 is asymptotically decreasing to one, the saturation regime.

The functional integration approach to solving the parabolic equation in a randomly inhomogeneous medium is an especially powerful one as the method can be applied to all three regions of intensity fluctuations, the weak, the strong and the saturated. This is an especially useful characteristic as it is possible to compare the results from the other mathematical methods to the results from the Feynman integration method.

Regions of Applicability

This section will simply provide a tabular listing of the applicability constraints of the various methods previously listed.

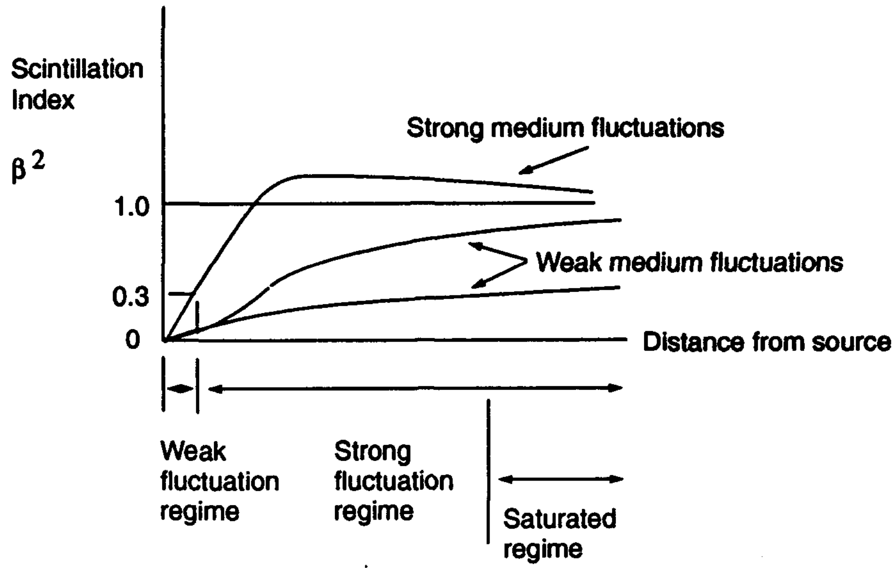


Figure 2.4. Scintillation Index

Method	Regimes solution is valid	Advantages	Disadvantages
Geometrical Optics	Non-diffraction	Very simple method	Diffractional effects ignored
Method of Smooth perturbations	Weak	Relatively simple method	Limited to weak fluctuations
Markov Approximation	Weak and Strong	Eliminates the need to solve the parabolic equation	Unrealistic assumptions possible
Computer Modeling	All	Eliminates the need to calculate statistics	Underlying mechanisms are obscured Possible aliasing
Feynman Integration	All	Accurate and intuitive	Infinite-fold integrals difficult to analyze

Table 2.1. Characteristics between various solution methods.

Summary

This chapter has provided a current literature review. This review included several of the techniques used to solve the problem of wave propagation in a randomly inhomogeneous medium. The next chapter will explore the characteristics of path integration as applied the this problem.

III. Feynman Integration applied to Optical Wave Propagation in a Randomly Inhomogeneous Medium

Introduction

This chapter follows an unpublished paper by Dr. J. Gozani, Dr. V. Tatarskii and Dr. V. Zavarontnyi (19:1). Therefore, in order to help the reader, who might become hopelessly lost in the succeeding sections, a brief overview of the theoretical development will be provided. This chapter will present a development of path integration applied to wave propagation in a randomly inhomogeneous medium so that the scintillation index, the normalized intensity variance, for a spherical wave may be calculated. This will be accomplished in three parts.

First, the existence of the path integral solution will be briefly mentioned by demonstrating the similarity between the Schrödinger wave equation, which has a path integral solution, and the parabolic equation, with which this thesis is concerned.

Second, in order to understand the development of the Green's function path integral for the fourth statistical moment of the field, a two-fold path integral solution to the two-fold stochastic parabolic wave equation will be developed. This development will be accomplished by applying the Huygens-Fresnel integral solution for wave propagation in a deterministic homogeneous medium to thin sections of the randomly inhomogeneous propagation medium. It will be shown that as the number of sections over which the Huygens-Fresnel integral is applied increases, the resulting solution will become more accurate. In the limit as the number of sections becomes unbounded, the resulting solution will become a path integral with stochastic components. A similar development will yield a Green's function path integral with stochastic components. It will be possible to separate the resulting solution into two factors. The first factor will represent the effects of a completely deterministic homogeneous propagation medium and the second factor will represent the effects of the stochastic deviations of the propagation medium about its mean characteristics. Then, each path in the path integral will be approximated by a truncated Fourier-sine series using a Taylor-functional series expansion.

Third, using the preceding development of the two-fold path integral for the field as a guide, the fourth statistical moment of the field will be calculated. The stochastic components of the resulting eight-fold Green's function path integral will be averaged over the ensemble of all paths from the source point to the observation point. After the averaging is performed, the resulting equation will be manipulated in a manner similar to the two-fold path integral up to and including the approximation of every trajectory by a truncated Fourier-sine series. Then the truncated Fourier-sine series will next be approximated by the first term in the series. This approximation will allow the eight-fold path integral to be approximated by an ordinary three-fold Riemann integral which has appeared in a similar form in the current literature.

Lastly, as it is the scintillation index, namely the normalized intensity variance, which is ultimately desired, it behooves the reader to keep this ultimate goal in mind during the succeeding derivations.

Parabolic Equation and Schrödinger Wave Equation Similarity

As presented in Appendix D, the solution of the Schrödinger wave equation may be cast into a "sequential Wiener integral with an imaginary variance term, namely a Feynman integral (16:55)." The one-dimensional Schrödinger wave equation has the form,

$$\left(\frac{\partial}{\partial t} - \frac{j\hbar}{2m}\frac{\partial^2}{\partial x^2} + \frac{j}{\hbar}V(x,t)\right)\psi(x,t) = 0 \tag{3.1}$$

where $\psi(x,t)$ is a wave function, $V(x,t)$ is a potential energy function, and m, j and $\hbar$ are constants (see Appendix D). Following a derivation similar to that used to derive the one-dimensional Schrödinger wave equation solution, it is possible to show that there is a corresponding two-dimensional Feynman integral solution to the two-dimensional Schrödinger wave equation. The two-dimensional Schrödinger wave equation has the form,

$$\left(\frac{\partial}{\partial t} - \frac{j\hbar}{2m}\nabla_\perp^2 + \frac{j}{\hbar}V(\rho,t)\right)\psi(\rho,t) = 0 \tag{3.2}$$

where $\psi(\rho,t)$ is a two-dimensional wave function, $V(\rho,t)$ is a two-dimensional potential energy function, $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $\rho = (x,y)$. Recall the form of the parabolic equation,

$$\left(2jk\frac{\partial}{\partial z} + \nabla_\perp^2 + k^2\,\tilde{\epsilon}\,(\rho,z)\right)u(\rho,z) = 0 \tag{3.3}$$

where $\tilde{\epsilon}\,(\rho, z)$ is the fluctuating part of the medium's dielectric permeability and $u(\rho, z)$ is the complex amplitude of the propagating wave. Dividing both sides by $2jk$ results in,

$$\left(\frac{\partial}{\partial z} - \frac{j}{2k}\,\nabla_{\perp}^{2} + \frac{k}{2j}\,\tilde{\epsilon}\,(\rho, z)\right) u(\rho, z) = 0. \tag{3.4}$$

It is easy to see the similarity between Eq. 3.2 and Eq. 3.4. Because of this similarity, it is possible to cast the solution of Eq. 3.4 into a Feynman integral (6:38-3). Interestingly enough, it is possible to obtain the solution to the stochastic Helmholtz equation, without the parabolic approximation constraints, as a multi-dimensional Feynman integral (6:38-3); however, this thesis will not investigate that topic other than to mention it so that the interested reader may pursue it further.

Following a derivation similar to the one in the chapter 2 subsection "A justification from Fourier optics", it is possible to solve the stochastic parabolic equation using Feynman integration directly. Therefore, it is possible to obtain a solution to the problem of wave propagation in a randomly inhomogeneous medium by either noting the similarity between the Schrödinger wave equation and the parabolic equation or by a direct derivation using the notion of an infinite number of phase screen equivalent propagation regions. The heuristic derivation of such a direct solution follows in the next section.

Derivation of the Feynman Integral Solution of the Stochastic Parabolic Equation

This section will present a non-rigorous derivation of the solution to the stochastic ($\tilde{\epsilon} \neq 0$) parabolic equation using Feynman integration. This derivation will establish the earlier claim of an intuitively pleasing solution. The interested reader can find a more mathematically rigorous derivation of the relationship between the stochastic parabolic equation and its Feynman integral solution in (29:227,228), or (7:1225-1227). Moreover, there is a derivation and solution of a related acoustical stochastic parabolic equation available in the literature (23:6-12). Lastly, there exists an extensive reference table which relates quantum mechanical functionals to acoustical functionals which appear in Feynman integral solutions to the Schrödinger wave equation and acoustical parabolic equation respectively (23:19). As it was stated in chapter 2, the functional similarities of the equations in acoustics, quantum mechanics and optical wave propagation has lead to similar general solutions. Therefore, it is important to consider the techniques and results from other research fields for novel methods to obtain solutions to wave propagation in a random medium. This is why the work in other research fields has been brought to the reader's attention.

Beginning with Eq. 3.5 which is derived in Appendix E, the complex amplitude, $u(\rho, z)$, of an initially unit amplitude plane wave propagating in the z direction, for a relatively short distance through the randomly inhomogeneous medium, is given approximately by,

$$u(\rho'', z) \simeq \frac{k}{j2\pi z} \iint \exp\left(\frac{jk}{2z}|\rho'' - \rho'|^2\right) \exp\left(\frac{jkz}{2} \tilde{\epsilon}\,(\rho', z)\right) d^2\rho' \tag{3.5}$$

where ρ'' is a vector representing the observation or measurement location in the observation plane, ρ' is a vector representing a point-source location in the image plane, z is the propagation distance. Implicit in the previous equation was the assumption that $\tilde{\epsilon}\,(\rho, z)$ was changing slowly across a small propagation distance z. As the distance between the image plane and observation plane decreases, the above approximation will become more accurate. A closer examination of the above equation will yield an interesting interpretation. The effects of the inhomogeneous medium on the propagating unit amplitude plane wave may be duplicated by an equivalent idealized random phase screen placed before a homogeneous medium, so long as the propagation distance is sufficiently small (29:228). The actual dimensions of "sufficiently small" will be explained later. An idealized phase screen is a mathematical construction which has the physical properties of vanishingly small (differential) thickness, and a transmittance characteristic which changes only the phase, and not the amplitude, of an incident optical wave. Thus, all the effects of the randomly inhomogeneous medium may be modeled as having been induced by an equivalent random phase screen. Therefore, by recognizing this interpretation, Eq. 3.5 may be rewritten with the phase screen interpretation as,

$$u(\rho'', z) \simeq \frac{k}{j2\pi z} \iint \underbrace{\exp\left(\frac{jkz}{2} \tilde{\epsilon}\,(\rho', z)\right)}_{\substack{\text{Phase screen or} \\ \text{random phase plane wave}}} \exp\left(\frac{jk}{2z}|\rho'' - \rho'|^2\right) d^2\rho' \tag{3.6}$$

We can rewrite this equation in another form by first noting that for a thin slab of the propagation medium we have

$$\tilde{\epsilon}\,(\rho', z) \simeq \tilde{\epsilon}\,(\rho', 0) \tag{3.7}$$

as we've assumed that $\tilde{\epsilon}$ varies slowly in the z direction. This being the case, we can write

$$u(\rho'', z) = \frac{k}{j2\pi z} \iint u(\rho', 0) \exp\left(\frac{jk}{2z}|\rho'' - \rho'|^2\right) d^2\rho' \tag{3.8}$$

where $\exp\left(\frac{jkz}{2}\,\tilde{\epsilon}\,(\rho', z)\right) = u(\rho', 0)$. Furthermore, if a subdivision process, similar to the one applied in chapter 2 to the problem of propagation in a homogeneous medium, is applied to this randomly inhomogeneous medium (Eq. 3.5), we will obtain a "sequential Wiener integral with imaginary variance parameter" (5:127). This is another way of saying a Feynman integral will result as a solution. This solution is written formally as,

$$\begin{aligned} u(\rho'', z) = \lim_{n\to\infty} u(\rho_n, z) = \\ \lim_{n\to\infty}\left(\frac{nk}{j2\pi z}\right)^n \iint \cdots \iint \exp\left(\frac{jnk}{2z}\sum_{m=1}^{n}|\rho_m - \rho_{m-1}|^2\right) \\ \exp\left(\frac{jnk}{2z}\sum_{m=0}^{n-1}\tilde{\epsilon}\left(\rho_m, \frac{mz}{n}\right)\right) u(\rho_0, 0)\prod_{m=0}^{n-1} d^2\rho_m \end{aligned} \tag{3.9}$$

where ρ'' has been defined as $\lim_{n\to\infty}\rho_n$ and a more general boundary condition $u(\rho_0, 0)$ has been included.

By interpreting the previous infinite-fold integral as it was done for Eq. 3.5, then it becomes clear that the effects of a randomly inhomogeneous medium on a propagating plane wave will be indistinguishable from the effects of an equivalent (though infinite) set of ideal random phase screens. Additionally, it was assumed that $\tilde{\epsilon}$ could be approximated as being nearly constant across the propagation distance and as the propagation subdivision width decreases and eventually becomes a differential in thickness, the approximate solution, which corresponds to an infinite number of small propagation regions, becomes an exact solution. Therefore, unlike the case of an homogeneous medium, it is necessary to let the number of subdivisions and therefore the number of equivalent phase screens become unbounded to obtain an exact solution. Thus, it is seen that the complex amplitude of an optical wave propagating in an inhomogeneous medium is represented as a two-dimensional Feynman integral. This integral is, as we have seen, is the limiting case of replacing the random inhomogeneous medium by one containing a set of uniformly spaced random phase screens and allowing the number of phase screens to become unbounded.

The Feynman Integral Solution in the Limit We can make an additional interpretation concerning Eq. 3.9. This interpretation will result in a functional integral form which has appeared in the literature (5:135), (22:1264), (1:37), (21:8), (29:228), (23:11). By considering Eq. 3.9 it follows that the functional integrand,

$$\exp\left(\frac{jnk}{2z}\sum_{m=1}^{n}|\rho_m - \rho_{m-1}|^2\right)\exp\left(\frac{jkz}{2n}\sum_{m=0}^{n}\tilde{\epsilon}\,(\rho_m, \frac{mz}{n})\right) \tag{3.10}$$

may be rearranged to yield,

$$\exp\left(\frac{jk}{2}\sum_{m=1}^{n}\left|\frac{|\rho_m - \rho_{m-1}|}{\frac{z}{n}}\right|^2\frac{z}{n}\right)\exp\left(\frac{jk}{2}\sum_{m=0}^{n}\tilde{\epsilon}\,(\rho_m, m\frac{z}{n})\frac{z}{n}\right) \tag{3.11}$$

Now, consider the following interpretation of the above expression. Eq. 3.11 is a functional of the n + 1 variables $\rho_0, \rho_1 \cdots \rho_n$. Equivalently, it is possible to consider Eq. 3.11 as a functional of a differentiable parametric function $\rho(\tau)$ where τ is a continuous parameter evaluated at discrete locations,

$$\{\rho_0,\ \rho_1,\ \rho_2,\ \cdots,\ \rho_n\} = \left\{\rho(\tau)|_{\tau=0,\ \tau=\frac{z}{n},\ \tau=2\frac{z}{n}\cdots,\ \tau=(n-1)\frac{z}{n},\ \tau=z}\right\} \tag{3.12}$$

That is, $\rho(\tau)$ is a continuous variable of a continuous parameter τ sampled at the evenly spaced points $\tau = 0, \tau = \frac{z}{n}, \tau = 2\frac{z}{n}\cdots$. In the limit as n becomes unbounded in Eq. 3.11, the finite, discrete sampling of the functional $\rho(\tau)$ will become effectively continuous. The astute reader will immediately disagree as to the sampling being continuous as, for example, there will be no sampling at any of the irrational numbers due to the limiting process described above. That is true; however, because any irrational number may be approximated as closely as desired by a convergent sequence of rational numbers, $\rho(\tau)$ will be sampled effectively everywhere.

Now consider a simple substitution of $\delta\tau$ for $\frac{z}{n}$ into Eq. 3.11. This will yield a more apparent result from elementary calculus,

$$\exp\left(\frac{jk}{2}\sum_{m=1}^{n}\left\{\frac{|\rho_m - \rho_{m-1}|}{\delta\tau}\right\}^2\delta\tau\right)\exp\left(\frac{jk}{2}\sum_{m=0}^{n}\tilde{\epsilon}\,(\rho_m, m\delta\tau)\delta\tau\right) \tag{3.13}$$

To begin, consider the first exponential factor in the previous equation. It follows that

$$\left\{\frac{|\rho_m - \rho_{m-1}|}{\delta\tau}\right\}^2 \approx \left|\frac{\rho(\tau)|_{\tau=\frac{mz}{n}} - \rho(\tau)|_{\tau=\frac{(m-1)z}{n}}}{\tau|_{\tau=\frac{mz}{n}} - \tau|_{\tau=\frac{(m-1)z}{n}}}\right|^2 \simeq \left|\frac{d\rho(\tau)}{d\tau}\right|^2\Bigg|_{\frac{mz}{n}} \tag{3.14}$$

Therefore, in the limit as n becomes unbounded it follows that,

$$\exp\left(\frac{jk}{2}\sum_{m=1}^{n}\left\{\frac{|\rho_m - \rho_{m-1}|}{\frac{z}{n}}\right\}^2 \frac{z}{n}\right) \Longrightarrow \exp\left(\frac{jk}{2}\int_0^z \left|\frac{d\rho(\tau)}{d\tau}\right|^2 d\tau\right) \tag{3.15}$$

Moreover, the second exponential factor in Eq. 3.13, may be manipulated in a similar fashion. Therefore, in the limit as n becomes unbounded it follows that,

$$\exp\left(\frac{jk}{2}\sum_{m=0}^{n} \tilde{\epsilon}\,(\rho_m, m\frac{z}{n})\frac{z}{n}\right) \Longrightarrow \exp\left(\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(\rho(\tau),\tau)d\tau\right) \tag{3.16}$$

Therefore, the entire functional integrand may be written as,

$$\exp\left(\frac{jk}{2}\int_0^z \left|\frac{d\rho(\tau)}{d\tau}\right|^2 d\tau\right)\exp\left(\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(\rho(\tau),\tau)d\tau\right) \tag{3.17}$$

where τ is the limiting continuous variable ρ. Therefore, it is possible to write Eq. 3.9 formally as,

$$u(\rho, z) = \iint D^2[\rho(\tau)]\exp\left(\frac{jk}{2}\int_0^z \left|\frac{d\rho(\tau)}{d\tau}\right|^2 d\tau\right)\exp\left(\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(\rho(\tau),\tau)d\tau\right) \tag{3.18}$$

where $D^2[\rho(\tau)] = \lim_{n\to\infty}\left(\frac{nk}{j2\pi z}\right)^n \prod_{m=1}^{n} d^2\rho_m$. This is one of the formal forms for the Feynman integral solution that is often found in the literature. It should be noted that when we have a deterministic medium, $\tilde{\epsilon} = 0$ and the preceding integral will reduce to Eq. 2.7. The reader should be aware that similar to the Riemann integral where the integration variable is simply a dummy variable, $\rho(\tau)$ is similarly a dummy integration variable. Therefore, Eq. 3.18 is fully equivalent to

$$u(\rho, z) = \iint D^2[x]\exp\left(\frac{jk}{2}\int_0^z \left|\frac{dx}{d\tau}\right|^2 d\tau\right)\exp\left(\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(x,\tau)d\tau\right) \tag{3.19}$$

where $x = x(\tau)$ and τ is the limiting continuous variable.

The preceding formal equation, it must be understood is only that, formal. It has been found that the Feynman integral does not possess a countably additive measure (26:8). Therefore, it would be more accurate if the preceding equation was rewritten as

$$u(\rho, z) = \sum_{\text{paths}} \exp\left(\frac{jk}{2}\int_0^z \left|\frac{dx}{d\tau}\right|^2 d\tau\right) \exp\left(\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(x, \tau) d\tau\right) \tag{3.20}$$

where "paths" means add all the contributions from all paths.

Green's Function Path Integral Following the derivation of the preceding equation, it is possible to show that the Green's function for propagation in a randomly inhomogeneous medium is given by

$$G(\rho'', z; \rho', 0) = \iint_{x(0)=\rho'}^{x(z)=\rho''} D^2[x] \exp\left\{\frac{jk}{2}\int_0^z |\dot{x}(\tau)|^2 d\tau\right\} \exp\left\{\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(x(\tau), \tau) d\tau\right\} \tag{3.21}$$

where each trajectory $[x]$ begins at ρ' when $\tau = 0$ and ends at ρ'' when $\tau = z$, the overdot indicates differentiation with respect to τ and each trajectory was the limiting case of a polygonal curve. The total complex field amplitude at an observation point due to several point sources or to a continuum of point sources in a source plane some distance away is then given by

$$u(\rho'', z) = \iint G(\rho'', z; \rho', 0) u(\rho', 0) d^2\rho' \tag{3.22}$$

which is simply a superposition integral with a Green's function integrand. Let the trajectory $x(\tau)$ be decomposed into two components, $\overline{x}(\tau)$ and $\tilde{x}\,(\tau)$. The first component, $\overline{x}(\tau)$ is the straight line connecting a given source point, ρ' with the observation point ρ''. The equation of this line is simply

$$\overline{x}(\tau) = \left(\frac{\rho'' - \rho'}{z}\right)\tau + \rho' \tag{3.23}$$

where $\tau \in [0, z]$. The second component. $\tilde{x}\,(\tau)$ is simply the difference between $x(\tau)$ and $\overline{x}(\tau)$, that is, the fluctuating component. Therefore, a given trajectory $x(\tau)$ may be written as

$$x(\tau) = \overline{x}(\tau) + \widetilde{x}(\tau) \tag{3.24}$$

It is important to notice that the fluctuating component $\widetilde{x}(\tau)$ vanishes at the boundaries $\tau = 0$ and $\tau = z$. It is important to notice that the strait line component corresponds to a completely deterministic propagation medium and the fluctuating component corresponds to the stochastic deviations of the propagation medium about its mean characteristics. Applying this trajectory decomposition to every trajectory in the Green's function in Eq. 3.21, we obtain

$$\begin{aligned} G(\rho'', z; \rho', 0) = & \iint_{\widetilde{x}(0)=0}^{\widetilde{x}(z)=0} D^2[\widetilde{x}] \\ & \exp\left\{\frac{jk}{2}\int_0^z \left[\left|\dot{\widetilde{x}}(\tau)\right|^2 + \left(\frac{|\rho''-\rho'|}{z}\right)^2 + 2\dot{\widetilde{x}}(\tau)\cdot\left(\frac{\rho''-\rho'}{z}\right)\right] d\tau\right\} \\ & \exp\left\{\frac{jk}{2}\int_0^z \widetilde{\epsilon}\,(\overline{x}(\tau) + \widetilde{x}(\tau), \tau) d\tau\right\} \end{aligned} \tag{3.25}$$

where we have used

$$\begin{aligned} |\dot{x}(\tau)|^2 &= \dot{x}(\tau)\cdot\dot{x}(\tau) \\ &= \left(\dot{\overline{x}}(\tau) + \dot{\widetilde{x}}(\tau)\right)\cdot\left(\dot{\overline{x}}(\tau) + \dot{\widetilde{x}}(\tau)\right) \\ &= |\dot{\overline{x}}(\tau)|^2 + |\dot{\widetilde{x}}(\tau)|^2 + 2\dot{\overline{x}}(\tau)\cdot\dot{\widetilde{x}}(\tau) \\ &= \left(\frac{|\rho''-\rho'|}{z}\right)^2 + |\dot{\widetilde{x}}(\tau)|^2 + 2\left(\frac{\rho''-\rho'}{z}\right)\cdot\dot{\widetilde{x}}(\tau) \end{aligned} \tag{3.26}$$

Let's consider the first exponential term in the path integral above.

$$\exp\left(\frac{jk}{2}\int_0^z \left[\left|\dot{\widetilde{x}}(\tau)\right|^2 + \left(\frac{|\rho''-\rho'|}{z}\right)^2 + 2\dot{\widetilde{x}}(\tau)\cdot\left(\frac{\rho''-\rho'}{z}\right)\right] d\tau\right) \tag{3.27}$$

We can simplify this equation by first noting

$$\exp\left(\frac{jk}{2}\int_0^z \left(\frac{|\rho''-\rho'|}{z}\right)^2 d\tau\right) = \exp\left(\frac{jk|\rho''-\rho'|^2}{2z}\right) \tag{3.28}$$

and second noting that

$$\exp\left(\int_0^z \dot{\tilde{x}}(\tau)\cdot\left(\frac{\rho''-\rho'}{z}\right)d\tau\right) = \exp\left(\left(\frac{\rho''-\rho'}{z}\right)\cdot \tilde{x}\,(\tau)\right)\Big|_0^z = \exp(0) = 1 \qquad (3.29)$$

The integral in the preceding exponential vanishes due to the boundary conditions on $\tilde{x}\,(\tau)$. We can now write the Green's function path integral as

$$G(\rho'', z; \rho', 0) = \exp\left(\frac{jk}{2z}|\rho''-\rho'|^2\right)\iint_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[\tilde{x}] \qquad (3.30)$$
$$\exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2 + \tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}\,(\tau),\tau)d\tau\right\}$$

For free space ($\tilde{\epsilon}=0$), the Green's function is known to be

$$G_0(\rho'', z; \rho', 0) = \frac{k}{j2\pi z}\exp\left(\frac{jk}{2z}|\rho''-\rho'|^2\right) \qquad (3.31)$$

where the subscript 0 is used to denote free space. Therefore, we can write the Green's function for a general medium as

$$G(\rho'', z; \rho', 0) = G_0(\rho'', z; \rho', 0)F(z) \qquad (3.32)$$

where

$$F(z) = \frac{j2\pi z}{k}\iint_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[\tilde{x}]\exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2 + \tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}\,(\tau),\tau)d\tau\right\} \qquad (3.33)$$

We will be interested in the statistical characteristics of $F(z)$ because $G_0(\rho'', z; \rho', 0)$ is completely deterministic.

As an aside, the astute reader will immediately realize that the free space Greens function will supply a normalizing condition on the path integral of Eq. 3.30. Specifically, when the propagation medium is free space, we have $\tilde{\epsilon}=0$. Therefore, it follows that for free space

$$G_0(\rho'', z; \rho', 0) = \exp\left(\frac{jk}{2z}|\rho''-\rho'|^2\right)\iint_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[x]\exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2\right\} \qquad (3.34)$$

which further implies that

$$\int\!\!\int_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[x] \exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2\right\} = \frac{1}{j\lambda z} \tag{3.35}$$

Before we begin the next section, it will prove enlightening to verify that the Green's function path integral will, at least for the case of free space, reduce to the regular Green's function which is derivable from the Huygens-Fresnel integral. Additionally, it is hoped that this derivation will help convince the reader that the paraxial approximation upon which the Huygens-Fresnel integral is based and the paraxial approximation upon which the Feynman integral is based are equivalent. To begin, we know that the Green's function for the Huygens-Fresnel integral is given by

$$G_{HF}(\rho'', z, \rho', 0) = \frac{k}{j2\pi z}\exp\left(\frac{jk}{2z}|\rho'' - \rho'|^2\right) \tag{3.36}$$

where the HF subscript will be used to denote the "Huygens-Fresnel" Green's function. Now, we will form the sequence of approximations to the Green's function and then show that each approximation is in fact equal to every other approximation. Specifically, consider the case where there is one subdivision for the trajectories. In this case, let ρ_0 be a general location vector in the source plane and ρ_1 be a general location vector in the measurement plane. As shown in chapter 2, we have for the complex field amplitude,

$$u(\rho_1, z) = \frac{k}{j2\pi z}\int\!\!\int u(\rho_0, 0)\exp\left(\frac{jk}{2z}|\rho_1 - \rho_0|^2\right) d^2\rho_0 \tag{3.37}$$

By using the property of the Green's function,

$$u(\rho'', z) = \int\!\!\int G(\rho'', z, \rho', 0) u(\rho', 0) d^2\rho' \tag{3.38}$$

we see that the resulting Green's function is equal to G_{HF}. Now consider a propagation region with two subdivisions. In this case let ρ_0 be as before a location vector in the source plane. Additionally, let ρ_2 replace ρ_1 as a location vector in the measurement plane. Lastly, let ρ_1 now represent a general location point which is midway along the propagation medium between ρ_0

and ρ_2. In this case, the complex field amplitude in the measurement plane which will be dependent upon the complex field amplitude at the intermediate (ρ_1) plane will be

$$u(\rho_2, z) = \left(\frac{k}{j2\pi z/2}\right)^2 \iint \iint u(\rho_0, 0) \exp\left(\frac{jk}{2z/2} \sum_{n=1}^{2} |\rho_n - \rho_{n-1}|^2\right) d^2\rho_1 d^2\rho_0 \tag{3.39}$$

it follows that the Green's function may be written as

$$G(\rho_2, z, \rho_0, 0) = \left(\frac{k}{j2\pi z/2}\right)^2 \iint \exp\left(\frac{jk}{2z/2}(|\rho_2 - \rho_1|^2 + |\rho_1 - \rho_0|^2)\right) d^2\rho_1 \tag{3.40}$$

Now, consider the factor

$$|\rho_2 - \rho_1|^2 + |\rho_1 - \rho_0|^2 \tag{3.41}$$

which may be rewritten as

$$2\left|\rho_1 - \left(\frac{\rho_2 + \rho_0}{2}\right)\right|^2 + \frac{1}{2}|\rho_2 - \rho_0|^2 \tag{3.42}$$

When the preceding factor is substituted into Eq. 3.40, we obtain

$$\begin{aligned} G(\rho_2, z, \rho_0, 0) = \left(\frac{k}{j2\pi z/2}\right)^2 \exp\left(\frac{jk}{2z}|\rho_2 - \rho_0|^2\right) \\ \iint \exp\left(\frac{j2k}{2z/2}\left|\rho_1 - \left(\frac{\rho_2 + \rho_0}{2}\right)\right|^2\right) d^2\rho_1 \end{aligned} \tag{3.43}$$

which is a gaussian integral. Such an integral is easily evaluated and yields

$$G(\rho_2, z, \rho_0, 0) = \frac{k}{j2\pi z} \exp\left(\frac{jk}{2z}|\rho_2 - \rho_0|^2\right) = G_{HF}(\rho_2, z, \rho_0, 0) \tag{3.44}$$

the free space Green's function. Lastly, it is relatively simple, though extremely tedious, to show that for N subdivisions, the Green's function

$$G(\rho_N, z, \rho_0, 0) = \left(\frac{k}{j2\pi z/N}\right)^N \iint \cdots \iint \exp\left(\frac{jk}{2z/N}\sum_{n=1}^{N}|\rho_n - \rho_{n-1}|^2\right)\prod_{m=1}^{N-1} d^2\rho_m \tag{3.45}$$

can also be simplified so that

$$G(\rho_N, z, \rho_0, 0) = G_{HF}(\rho_N, z, \rho_0, 0) \tag{3.46}$$

Therefore, because the choice of N was arbitrary, we can take the limit as N becomes unbounded and we conclude that

$$G(\rho'', z; \rho', 0) = \exp\left(\frac{jk}{2z}|\rho'' - \rho'|^2\right) \iint_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[x] \exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2 d\tau\right\} \tag{3.47}$$

therefore,

$$\iint_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[x] \exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2 d\tau\right\} = \frac{1}{j\lambda z} \tag{3.48}$$

and so we conclude that the Green's function from the Huygens-Fresnel integral and the Green's function from the Feynman integral are equal for free space.

Orthogonal Function Expansion It is known that if a function satisfies certain requirements, then it possible to expand it over an infinite orthogonal function set. The choice of the orthogonal functions and therefore, whether the set is countable or not, is usually chosen to simplify the problem at hand. For our problem, we will choose a Fourier-sine series as the orthogonal function set. This set will be used to perform an expansion of the fluctuating part of each trajectory in the Green's function path integral.

Let the fluctuating part of the trajectory $\tilde{x}\ (\tau)$ be expanded in a Fourier-sine series as

$$\tilde{x}\ (\tau) = \sum_{n=1}^{\infty} x_n \phi_n(\tau) \tag{3.49}$$

where

$$\phi_n(\tau) = \frac{1}{n\pi}\sqrt{\frac{2z}{k}}\sin\left(\frac{n\pi\tau}{z}\right) \tag{3.50}$$

and $x_n \in \mathbf{R}^2$. The use of the above Fourier-sine series results in the simplification

$$\exp\left\{\frac{jk}{2}\int_0^z \left|\dot{\tilde{x}}(\tau)\right|^2 d\tau\right\} = \exp\left\{\frac{j}{2}\sum_{n=1}^{\infty}|x_n|^2\right\} \tag{3.51}$$

as the cross terms, all being orthogonal, will integrate to zero.

Approximation of the Green's Function Path Integral In this section, we will use a finite truncation of the Fourier-sine series to obtain an approximate value of $G(\rho'', z; \rho', 0)$. Consider the path integral

$$I(z) = \iint_{\tilde{x}(0)=0}^{\tilde{x}(z)=0} D^2[x]\exp\left\{\frac{j}{2}\sum_{n=1}^{\infty}|x_n|^2 + \frac{jk}{2}\int_0^z \tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}_\infty(\tau),\tau)d\tau\right\} \tag{3.52}$$

where

$$\tilde{x}_N(\tau) = \sum_{n=1}^{N} x_n\phi_n(\tau) \tag{3.53}$$

We seek to find a finite value of N such that $\tilde{x}_\infty(\tau)$ can be replaced by $\tilde{x}_N(\tau)$ in Eq. 3.52. This is not a simple problem as the choice of N will be dictated by several factors including the statistics of $\tilde{\epsilon}$. This problem not withstanding, we may make some simplifications to Eq. 3.52.

Let us begin with the integral term in the exponent of the path integrand in Eq. 3.52

$$\int_0^z \tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}_\infty(\tau),\tau)d\tau \tag{3.54}$$

It is seen that $\tilde{\epsilon}$ is a functional of the functions $\overline{x}$ and $\tilde{x}_N$. Therefore, by following the method given in Appendix F for expanding an integral of a functional in a variational Taylor series we get for the zeroth variational term

$$\int_0^z \tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}_N(\tau),\tau)d\tau \tag{3.55}$$

while the first variational term is

$$\frac{1}{1!}\int_0^z \tilde{x}_{\overline{N}}(\tau)\cdot\frac{\partial\,\tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}_N\,(\tau),\tau)}{\partial\,\tilde{x}_N\,(\tau)}d\tau \tag{3.56}$$

and the second variational term is

$$\frac{1}{2!}\int_0^z \tilde{x}_{\overline{N}}(\tau)\cdot\tilde{x}_{\overline{N}}(\tau)\cdot\frac{\partial^2\,\tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}_N\,(\tau),\tau)}{\partial(\tilde{x}_N\,(\tau))^2}d\tau \tag{3.57}$$

where

$$\tilde{x}_{\overline{N}}(\tau)=\sum_{n=N+1}^{\infty}x_n\phi_n(\tau) \tag{3.58}$$

Higher order terms are similarly defined as the interested reader will find in Appendix F. Similar to the more familiar Taylor series in which the value of a function at a fixed point is approximated by the first term of the series, it is possible to approximate Eq. 3.54 by the first term of its variational series given by Eq. 3.55. Additionally the other exponential factor in the path integrand of Eq. 3.52 may be simply truncated after N terms. Doing so results in

$$I_N(z)=\iint_{\tilde{x}_N(0)=0}^{\tilde{x}_N(z)=0}D^2[\tilde{x}_N]\exp\left\{\frac{j}{2}\sum_{n=1}^{N}|x_n|^2+\frac{jk}{2}\int_0^z\tilde{\epsilon}\,(\overline{x}(\tau)+\tilde{x}_N\,(\tau),\tau)d\tau\right\} \tag{3.59}$$

This result will be used in the next section where this thesis will be concerned with the fourth statistical moment of the complex amplitude of a propagating optical wave. The ultimate accuracy of these approximations will be determined by comparing the predictions of these approximations for scintillation index with reality. Before we investigate the fourth statistical moment of the field, it should be noted that the previous equation is the result of making two different, though related types of approximations. The first approximation was made by assuming every trajectory of the Feynman integral could be approximated by a truncated Fourier-sine series. The second approximation was made by assuming that the difference between the $\tilde{\epsilon}$ functional of the original trajectory and the $\tilde{\epsilon}$ functional of the approximate trajectory was sufficiently small so that the former could be approximated by the later. In other words, the approximation by the first term in the variational Taylor series was sufficiently accurate.

Fourth Statistical Moment

The fourth statistical moment of the field, which gives the intensity variance of the propagating optical wave, will be investigated in this section. The other moments of major interest, the first and second, can be obtained in a manner similar to that which will be used to obtain the fourth moments. Therefore, this thesis will consider only the derivation of the fourth moment of the field. The derivation of the fourth statistical moment of the field will culminate in the calculation of the scintillation index for a point source. Thus, the general fourth statistical moment will be obtained and then the four independent points in the general fourth moment equation will be merged into a single point. This is the ultimate goal of which the reader must be mindful during the reading of the succeeding sections.

It is reasonable to wonder why the general fourth moment equation is obtained and then the four independent points replaced with a single point rather than begin with a single point in the first place. After all, the point source scintillation index or normalized intensity variance, is ultimately desired. The necessity of doing so for the extended medium parallels the necessity of doing so for a *phase screen*. The closed differential equation which describes the fourth statistical moment for the phase screen case is dependent upon the second partial derivatives of the four independent points. Is is only after the appropriate equations have been solved that in the limit, the four points are allowed approach each other and ultimately be replaced by a single point (25:22).

As we have already seen, the field at an observation point may be written in terms of all point source contributions in the object plane. Specifically this was a superposition integral of the Green's function of the problem. Therefore, the fourth moment of the field may be written as

$$\Gamma_4 = \langle U(\rho_1'', z)U^*(\rho_2'', z)U(\rho_3'', z)U^*(\rho_4'', z)\rangle \tag{3.60}$$

where $\langle\cdot\rangle$ denotes ensemble averaging. Assuming the mean direction of propagation is the z direction, we have for the field

$$U(\rho, z) = u(\rho, z)\exp(jkz) \tag{3.61}$$

Using this substitution we obtain

$$\Gamma_4(\rho_1'', \rho_2'', \rho_3'', \rho_4'', z) = \langle u(\rho_1'', z) u^*(\rho_2'', z) u(\rho_3'', z) u^*(\rho_4'', z) \rangle \tag{3.62}$$

This may be written in terms of the appropriate Green's functions and superposition integrals resulting in

$$\begin{aligned} &\Gamma_4(\rho_1'', \rho_2'', \rho_3'', \rho_4'', z) = \\ &\quad \iint\iint\iint\iint \langle G(\rho_1'', z; \rho_1', 0) G^*(\rho_2'', z; \rho_2', 0) G(\rho_3'', z; \rho_3', 0) G^*(\rho_4'', z; \rho_4', 0) \rangle \\ &\quad u(\rho_1', 0) u^*(\rho_2', 0) u(\rho_3', 0) u^*(\rho_4', 0) d^2\rho_1' d^2\rho_2' d^2\rho_3' d^2\rho_4' \end{aligned} \tag{3.63}$$

Let the statistical moment of the Green's functions be denoted as

$$\begin{aligned} G_{<4>} &= G_{<4>}(\rho_1'', \rho_2'', \rho_3'', \rho_4'', z; \rho_1', \rho_2', \rho_3', \rho_4', 0) = \\ &\langle G(\rho_1'', z; \rho_1', 0) G^*(\rho_2'', z; \rho_2', 0) G(\rho_3'', z; \rho_3', 0) G^*(\rho_4'', z; \rho_4', 0) \rangle \end{aligned} \tag{3.64}$$

Each of the Green's functions in the above ensemble average may be written in terms of the appropriate path integral which results in the fourth-order path integral

$$\begin{aligned} G_{<4>} = &\iint_{x_1(0)=\rho_1'}^{x_1(z)=\rho_1''} \iint_{x_2(0)=\rho_2'}^{x_2(z)=\rho_2''} \iint_{x_3(0)=\rho_3'}^{x_3(z)=\rho_3''} \iint_{x_4(0)=\rho_4'}^{x_4(z)=\rho_4''} D^2[x_1] D^2[x_2] D^2[x_3] D^2[x_4] \\ &\exp\left\{\frac{jk}{2}\int_0^z \left(|\dot{x}_1(\tau)|^2 - |\dot{x}_2(\tau)|^2 + |\dot{x}_3(\tau)|^2 - |\dot{x}_4(\tau)|^2\right) d\tau\right\} \\ &\left\langle \exp\left\{\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(x_1(\tau), \tau) - \tilde{\epsilon}\,(x_2(\tau), \tau) + \tilde{\epsilon}\,(x_3(\tau), \tau) - \tilde{\epsilon}\,(x_4(\tau), \tau) d\tau\right\}\right\rangle \end{aligned} \tag{3.65}$$

If for the moment, let us consider the stochastic quantity E

$$E = \left\langle \exp\frac{jk}{2}\left\{\int_0^z \tilde{\epsilon}\,(x_1(\tau), \tau) - \tilde{\epsilon}\,(x_2(\tau), \tau) + \tilde{\epsilon}\,(x_3(\tau), \tau) - \tilde{\epsilon}\,(x_4(\tau), \tau) d\tau\right\}\right\rangle \tag{3.66}$$

Let us assume that the Markov approximation, as described more fully in Appendix C holds for $\tilde{\epsilon}$. Let us also assume that $\tilde{\epsilon}$ is a zero mean gaussian random field. Then the above ensemble average may be simplified. Specifically, we assume a correlation function of the form

$$B_\epsilon(\rho - \rho', z - z') = \left\langle \tilde{\epsilon}\,(\rho, z)\,\tilde{\epsilon}\,(\rho', z') \right\rangle = A(\rho - \rho')\delta(z - z') \tag{3.67}$$

where $\delta(\cdot)$ is the Dirac delta distribution. Therefore, we assume that $\tilde{\epsilon}$ is a gaussian delta-correlated random field as in (29:214). The gaussian assumption is a good assumption as the effects of each small section of the randomly inhomogeneous medium can be modeled as an independent identically distributed random variable. The assumption of delta correlation is a reasonable first approximation as it implicitly assumes no backscattering effects. These assumptions allow the simplification of the statistical quantity E

$$E = \exp\left\{ \frac{-k^2}{8} \left\langle \left[\int_0^z \left(\tilde{\epsilon}\,(x_1(\tau), \tau) - \tilde{\epsilon}\,(x_2(\tau), \tau) + \tilde{\epsilon}\,(x_3(\tau), \tau) - \tilde{\epsilon}\,(x_4(\tau), \tau) \right) \right]^2 \right\rangle \right\} \tag{3.68}$$

Expanding terms, integrating and taking averages, results in

$$E = \exp\left\{ \frac{-k^2}{4} \int_0^z F[x_1(\tau), x_2(\tau), x_3(\tau), x_4(\tau), \tau] d\tau \right\} \tag{3.69}$$

where

$$\begin{aligned} F[x_1(\tau), x_2(\tau), x_3(\tau), x_4(\tau), \tau] = 4A(0) - \\ A(x_1(\tau) - x_2(\tau)) + A(x_1(\tau) - x_3(\tau)) - \\ A(x_1(\tau) - x_4(\tau)) - A(x_2(\tau) - x_3(\tau)) + \\ A(x_2(\tau) - x_4(\tau)) - A(x_3(\tau) - x_4(\tau)) \end{aligned} \tag{3.70}$$

and

$$A(0) - A(\rho) = 2 \iint (1 - \cos(k \cdot \rho)) \Phi_\epsilon(k, 0) d^2k \tag{3.71}$$

where $\Phi_\epsilon(k, 0)$ is the two-dimensional power spectral density of the the fluctuations in ϵ. Furthermore, for a pure Kolmogorov law

$$A(0) - A(\rho) = 1.46 C_\epsilon^2 |\rho|^{\frac{5}{3}} \tag{3.72}$$

under the Markov approximation (29:219). Returning to the fourth order path integral we have,

$$G_{<4>} = \iint_{x_1(0)=\rho_1'}^{x_1(z)=\rho_1''} \iint_{x_2(0)=\rho_2'}^{x_2(z)=\rho_2''} \iint_{x_3(0)=\rho_3'}^{x_3(z)=\rho_3''} \iint_{x_4(0)=\rho_4'}^{x_4(z)=\rho_4''} D^2[x_1]D^2[x_2]D^2[x_3]D^2[x_4] \exp\left\{\frac{jk}{2}\int_0^z |\dot{x}_1(\tau)|^2 - |\dot{x}_2(\tau)|^2 + |\dot{x}_3(\tau)|^2 - |\dot{x}_4(\tau)|^2 d\tau\right\} \exp\left\{\frac{-\pi k^2}{4}\int_0^z F[x_1(\tau), x_2(\tau), x_3(\tau), x_4(\tau), \tau]d\tau\right\} \tag{3.73}$$

Examination of the previous equation will reveal that the first exponential term which is independent of the fluctuations of the medium represents a pure diffraction factor. This would be the only factor if propagation were occurring in free space. The second exponential factor represents combined effects of both diffraction and refraction. This term is the scattering function of the fourth moment along trajectories influenced by refraction and diffraction. Furthermore, for short distances, such as the differential width of a phase screen, the diffractional effects of the pure diffractional factor dominate the diffraction-refraction factor. For longer propagation distances, this will in general no longer be true; however, it is not correct to state that at long propagation distances the combination diffraction-refraction (scattering) term will dominate the pure diffraction term, but rather that the pure diffractional factor will not necessarily dominate (18:1).

The previous equation which represents the statistical effects of a propagation medium on four beams, may be simplified by the following variable substitution

$$\begin{aligned} 4R(\tau) &= x_1(\tau) + x_2(\tau) + x_3(\tau) + x_4(\tau) \\ \rho(\tau) &= x_1(\tau) - x_2(\tau) + x_3(\tau) - x_4(\tau) \\ 2r_1(\tau) &= x_1(\tau) + x_2(\tau) - x_3(\tau) - x_4(\tau) \\ 2r_2(\tau) &= x_1(\tau) - x_2(\tau) - x_3(\tau) + x_4(\tau) \end{aligned} \tag{3.74}$$

These 4 substitutions result in

$$|\dot{x}_1(\tau)|^2 - |\dot{x}_2(\tau)|^2 + |\dot{x}_3(\tau)|^2 - |\dot{x}_4(\tau)|^2 = 2\dot{R}(\tau)\cdot\dot{\rho}(\tau) + 2\dot{r}_1(\tau)\cdot\dot{r}_2(\tau) \tag{3.75}$$

and

$$F[x_1(\tau), x_2(\tau), x_3(\tau), x_4(\tau), \tau] = F[r_1(\tau), r_2(\tau), \rho(\tau), \tau] = \tag{3.76}$$
$$A(r_1(\tau) + \frac{1}{2}\rho(\tau)) + A(r_1(\tau) - \frac{1}{2}\rho(\tau)) +$$
$$A(r_2(\tau) + \frac{1}{2}\rho(\tau)) + A(r_2(\tau) - \frac{1}{2}\rho(\tau)) +$$
$$A(r_1(\tau) + r_2(\tau)) - A(r_1(\tau) - r_2(\tau))$$

It is seen that the scattering factor will be independent of $R(\tau)$ under the preceding variable transformation. At the present stage of development, the Green's function is

$$G_{<4>} = \iint_{R(0)=R'}^{R(z)=R''} \iint_{\rho(0)=\rho'}^{\rho(z)=\rho''} \iint_{r_1(0)=r_1'}^{r_1(z)=r_1''} \iint_{r_2(0)=r_2'}^{r_2(z)=r_2''} D^2[R]D^2[\rho]D^2[r_1]D^2[r_2] \tag{3.77}$$
$$\exp\left\{jk\int_0^z \dot{R}(\tau)\cdot\dot{\rho}(\tau) + \dot{r_1}(\tau)\cdot\dot{r_2}(\tau) - \frac{\pi k^2}{4}F[r_1(\tau), r_2(\tau), \rho(\tau), \tau]d\tau\right\}$$

As it was done previously, consider the trajectories $R(\tau)$ and $\rho(\tau)$ as strait lines connecting source and observation points with some deviation superimposed upon them. Then we can make the following substitutions

$$\begin{aligned} R(\tau) &= \overline{R}(\tau) + \tilde{R}(\tau) \\ \overline{R}(\tau) &= \left(\frac{R'' - R'}{z}\right)\tau + R' \\ \rho(\tau) &= \overline{\rho}(\tau) + \tilde{\rho}(\tau) \\ \overline{\rho}(\tau) &= \left(\frac{\rho'' - \rho'}{z}\right)\tau + \rho' \end{aligned} \tag{3.78}$$

where $R'' = R(\tau)|_{\tau=z}$, $R' = R(\tau)|_{\tau=0}$, and so fourth. Using these substitutions, it follows directly that

$$\exp\left\{jk\int_0^z \dot{R}(\tau)\cdot\dot{\rho}(\tau)d\tau\right\} = \tag{3.79}$$
$$\exp\left\{\frac{jk}{z}(R''-R')\cdot(\rho''-\rho')\right\}\exp\left\{jk\int_0^z \dot{\tilde{R}}(\tau)\cdot\dot{\tilde{\rho}}(\tau)d\tau\right\}$$

as the cross terms will integrate to zero. Furthermore, the remaining integral may be integrated by parts which results in

$$\exp\left\{jk\int_0^z \dot{\tilde{R}}(\tau)\cdot\dot{\tilde{\rho}}(\tau)d\tau\right\} = \exp\left\{-jk\int_0^z \tilde{R}(\tau)\cdot\ddot{\tilde{\rho}}(\tau)d\tau\right\} \tag{3.80}$$

The Green's function now takes the form

$$G_{<4>} = \exp\left\{\frac{jk}{z}(R''-R')\cdot(\rho''-\rho')\right\} \tag{3.81}$$
$$\int_{\tilde{R}(0)=R'}^{\tilde{R}(z)=R''}\int_{\tilde{\rho}(0)=\rho'}^{\tilde{\rho}(z)=\rho''}\iint_{r_1(0)=r_1'}^{r_1(z)=r_1''}\iint_{r_2(0)=r_2'}^{r_2(z)=r_2''} D^2[\tilde{R}]D^2[\tilde{\rho}]D^2[r_1]D^2[r_2]$$
$$\exp\left\{-jk\int_0^z \tilde{R}(\tau)\cdot\ddot{\tilde{\rho}}(\tau) - \dot{r}_1(\tau)\cdot\dot{r}_2(\tau)d\tau\right\}\exp\left\{\frac{-\pi k^2}{4}\int_0^z F[r_1(\tau),r_2(\tau),\tilde{\rho}(\tau)+\overline{\rho}(\tau)]d\tau\right\}$$

A path integral simplifying formula referred to as the "delta functional" property of path integrals exists which can be used to further simplify the preceding path integral (9:22). This path integral relation is given by

$$\iint D[x(z)]D[y(z)]\exp\left\{-jk\int_0^R x(z)\cdot(\ddot{y}(z)-f(z))dz\right\}F[y(z)] = \left(\frac{k}{2\pi R}\right)^2 F[g(z)] \tag{3.82}$$

where $x(z)$, $y(z)$, $f(z)$, and $g(z)$ are vector valued functions and

$$\ddot{g}(z) = f(z) \tag{3.83}$$

and where $g(z)$ satisfies the boundary conditions for $y(z)$. Using this simplification formula, the

Green's function for the fourth moment becomes

$$G_{<4>} = \left(\frac{k}{2\pi z}\right)^2 \exp\left\{\frac{jk}{z}(R'' - R') \cdot (\rho'' - \rho')\right\} \tag{3.84}$$
$$\iint_{r_1(0)=r_1'}^{r_1(z)=r_1''} \iint_{r_2(0)=r_2'}^{r_2(z)=r_2''} D^2[r_1] D^2[r_2] \exp\left\{jk \int_0^z \dot{r}_1(\tau) \cdot \dot{r}_2(\tau) d\tau\right\}$$
$$\exp\left\{\frac{-\pi k^2}{4} \int_0^z F[r_1(\tau), r_2(\tau), \overline{\rho}(\tau)] d\tau\right\}$$

$G_{<4>}$ is seen to be independent of the value of $\overline{\rho}(\tau)$. Because of this independence, the final value of $G_{<4>}$ will not change irrespective of the value of $\overline{\rho}(\tau)$. Therefore it makes sense to assign it a value which will simplify our calculations. For reasons which will soon become apparent, we will set $\overline{\rho}(\tau)$ equal to zero. This being the case, it follows that the Green's function becomes

$$G_{<4>} = \left(\frac{k}{2\pi z}\right)^2 \exp\left\{\frac{jk}{z}(R'' - R') \cdot (\rho'' - \rho')\right\} \tag{3.85}$$
$$\iint_{r_1(0)=r_1'}^{r_1(z)=r_1''} \iint_{r_2(0)=r_2'}^{r_2(z)=r_2''} D^2[r_1] D^2[r_2] \exp\left\{jk \int_0^z \dot{r}_1(\tau) \cdot \dot{r}_2(\tau) d\tau\right\}$$
$$\exp\left\{\frac{-\pi k^2}{4} \int_0^z F[r_1(\tau), r_2(\tau)] d\tau\right\}$$

where the abbreviation $F[r_1(\tau), r_2(\tau)] = F[r_1(\tau), r_2(\tau), 0]$ has been used. At this point, we recall that the Green's function was composed of the effects from 4 independent point sources or beams. Now, we see that the Green's function has been manipulated into a form that is equivalent to a Green's function with two beams. Therefore, the Green's function now represents two effective point sources or beams.

Now consider the decomposition of the two trajectories $r_1(\tau)$ and $r_2(\tau)$. The decomposition will parallel the previous decompositions for $R(\tau)$ and $\rho(\tau)$. Specifically, let

$$r_1(\tau) = \overline{r_1}(\tau) + \tilde{r}_1(\tau) \tag{3.86}$$
$$r_2(\tau) = \overline{r_2}(\tau) + \tilde{r}_2(\tau)$$

where

$$\overline{r_1}(\tau) = \left(\frac{r_1'' - r_1'}{z}\right)\tau + r_1' \tag{3.87}$$
$$\overline{r_2}(\tau) = \left(\frac{r_2'' - r_2'}{z}\right)\tau + r_2'$$

and $r' = r(\tau)|_{\tau=0}$, $r'' = r(\tau)|_{\tau=z}$, and so fourth. Additionally, let

$$\tilde{r_1}\ (\tau) = \sum_{n=1}^{\infty} x_n \phi_n(\tau) \tag{3.88}$$
$$\tilde{r_2}\ (\tau) = \sum_{n=1}^{\infty} y_n \phi_n(\tau)$$

where x_n and y_n are the ordered pair Fourier-sine series coefficients and

$$\phi_n(\tau) = \frac{1}{n\pi}\sqrt{\frac{2z}{k}}\sin\left(\frac{n\pi}{z}\tau\right) \tag{3.89}$$

are the orthogonal functions on $[0, z]$. The previously mentioned substitutions for $r_1(\tau)$ and $r_2(\tau)$ will result in

$$\dot{r_1}(\tau)\cdot\dot{r_2}(\tau) = \tag{3.90}$$
$$\left\{\left(\frac{r_1'' - r_1'}{z}\right) + \sqrt{\frac{2}{kz}}\sum_{n=1}^{\infty} x_n \cos\left(\frac{n\pi}{z}\tau\right)\right\}\cdot\left\{\left(\frac{r_2'' - r_2'}{z}\right) + \sqrt{\frac{2}{kz}}\sum_{n=1}^{\infty} y_n \cos\left(\frac{n\pi}{z}\tau\right)\right\}$$

By taking the dot product and integrating from 0 to z in the preceding equation, one will obtain a simplification of Eq. 3.85 resulting in

$$G_{<4>} = \left(\frac{k}{2\pi z}\right)^2 \exp\left\{\frac{jk}{z}(R'' - R')\cdot(\rho'' - \rho')\right\} \tag{3.91}$$
$$\exp\left\{\frac{jk}{z}(r_1'' - r_1')\cdot(r_2'' - r_2')\right\}\iint_{\tilde{r_1}(0)=r_1'}^{\tilde{r_1}(z)=r_1''}\iint_{\tilde{r_2}(0)=r_2'}^{\tilde{r_2}(z)=r_2''} D^2[\tilde{r_1}]D^2[\tilde{r_2}]$$
$$\exp\left\{j\sum_{n=1}^{\infty} x_n\cdot y_n - \frac{\pi k^2}{4}\int_0^z F[\overline{r_1}(\tau)+\tilde{x}_\infty(\tau), \overline{r_2}(\tau)+\tilde{y}_\infty(\tau), \overline{\rho}(\tau)]d\tau\right\}$$

where

$$\widetilde{x}_N\ (\tau) = \sum_{n=1}^{N} x_n \phi_n(\tau) \tag{3.92}$$
$$\widetilde{y}_N\ (\tau) = \sum_{n=1}^{N} y_n \phi_n(\tau)$$

and where $x_n, y_n \in \mathbf{R}^2$.

Previously we noted that $G_{<4>}$ is independent of the value of $\overline{\rho}(\tau)$, and we choose a convenient value for it, namely zero. We now wish to examine the consequence of such a decision. Recall

$$\overline{\rho}(\tau) = \left(\frac{\rho'' - \rho'}{z}\right)\tau + \rho' \tag{3.93}$$

where

$$\rho' = \rho_1' - \rho_2' + \rho_3' - \rho_4' \tag{3.94}$$
$$\rho'' = \rho_1'' - \rho_2'' + \rho_3'' - \rho_4''$$

Let's consider the explicit evaluation of the the preceding two equations for two specific values of τ. When $\tau = 0$ we have

$$\rho_1' - \rho_2' = \rho_4' - \rho_3' \tag{3.95}$$

and when $\tau = z$ it follows that

$$\rho_1'' - \rho_2'' = \rho_4'' - \rho_3'' \tag{3.96}$$

Recall from elementary vector analysis, two vectors (directed line segments) are equal if and only if they are parallel. The location of the two vectors is irrelevant. Therefore, the choice of $\overline{\rho}(\tau) = 0$ implies that the vector $\rho_1' - \rho_2'$ must be parallel to $\rho_4' - \rho_3'$ just as $\rho_1'' - \rho_2''$ must be parallel to $\rho_4'' - \rho_3''$. The first equality states that a quadrilateral with the source points ρ_1', ρ_2', ρ_3', ρ_4' at each of its vertices is in fact a parallelogram. The same holds for the four observation points ρ_1'',

ρ_2'', ρ_3'', ρ_4''. Fig. 3.1 illustrates this more clearly. This geometry of source and observation points will not create any problems for our derivations as in the limit, the sides of the parallelogram will become infinitesimally small and the four points will be merged into a single one (24:56).

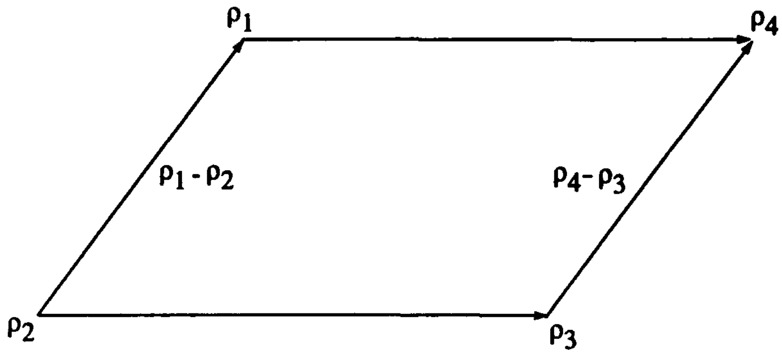


Figure 3.1. The four source or measurement points form a parallelogram.

It would appear that the equation for the fourth statistical moment is hopelessly complicated with the infinite number of integrations contained in the Feynman integral. However, as in the case of the single path integral, we will approximate the infinite number of integrations inherent in the four-fold two-dimensional Feynman integral with a finite number (in reality 3) of ordinary Riemann integrals. The validity of this approximation may be verified either by mathematically rigorous means (which is the ideal method) or by comparing the predictions of the modified equations with reality (which is what this thesis will do). As our derivations now exist, the path integral $\int\int D^2[x]$ is the integration over all sinusoids of the form given by Eq. 3.89. The approximation of the path integral by a finite number of functions will result in a finite number of ordinary Riemann integrations.

As the interested reader will find in Appendix F, it is possible to expand the functional

$$\int_0^z F[\overline{r_1}(\tau) + \widetilde{x}_\infty(\tau) + \overline{r_2}(\tau), \widetilde{y}_\infty(\tau)]d\tau \tag{3.97}$$

in a two-dimensional variational Taylor series. It therefore follows from Appendix F that the zeroth variation of the above equation for N terms is given by

$$\int_0^z F[\overline{r_1}(\tau)+\tilde{x}_N(\tau),\overline{r_2}(\tau)+\tilde{y}_N(\tau)]d\tau \tag{3.98}$$

higher order variational terms will be defined similar to the one-dimensional case. As already stated, the accuracy of our approximation will be validated by the accuracy with which the results predict reality. Moreover, the task of actually evaluating the resulting integrations must be addressed. As it was stated previously and as it will be repeated here, we have made two approximations. The first approximation was made by assuming every trajectory of the Feynman integral could be approximated by a truncated Fourier-sine series. The second approximation was made by assuming that the difference between the $\tilde{\epsilon}$ functional of the original trajectory and the $\tilde{\epsilon}$ functional of the approximate trajectory was sufficiently small so that the former could be approximated by the later. In other words, the approximation by the first term in the variational Taylor series was sufficiently accurate. The case of $N = 1$ will be investigated in the next section.

The Fourth Moment for a Point Source In this section we are now ready to consider the case of a point source and a single observation point. In order to simplify the derivations, we will assume that both the source point and observation point lie along the z axis ($\overline{r_1}(\tau) = \overline{r_2}(\tau) = 0$). Additionally, we will consider the case of only one sinusoid for the approximation and therefore the symbol ˆ will be placed over a variable indicate its two-dimensional nature. Finally, the turbulence will be modeled as a pure Kolmogorov law. Therefore, the fourth statistical moment apart from a constant becomes

$$G_{<4>} = K(z)\iint_{\mathbf{R}^2}\iint_{\mathbf{R}^2} d^2\hat{x}d^2\hat{y}\exp\left\{j\hat{x}\cdot\hat{y} - \frac{\pi k^2}{4}\int_0^z F[\hat{x}\phi(\tau),\hat{y}\phi(\tau)]d\tau\right\} \tag{3.99}$$

where $\hat{x} = (x_1, x_2)$, $\hat{y} = (y_1, y_2)$, $\phi(\tau) = \phi_1(\tau)$ and

$$F[\hat{x}\phi(\tau),\hat{y}\phi(\tau)] = 1.46C_\epsilon^2\left(2|\hat{x}|^{5/3} + 2|\hat{y}|^{5/3} + |\hat{x}-\hat{y}|^{5/3} + |\hat{x}+\hat{y}|^{5/3}\right)\phi^{5/3}(\tau) \tag{3.100}$$

and

$$K(z) = \left(\frac{k}{2\pi z}\right) \exp\left\{\frac{jk}{z}[(R'' - R') \cdot (\rho'' - \rho') + (r_1'' - r_1') \cdot (r_2'' - r_2')]\right\} \tag{3.101}$$

Performing the integration in τ yields

$$G_{<4>} = K(z) \iint_{\mathbf{R}^2} \iint_{\mathbf{R}^2} dx_1 dy_1 dx_2 dy_2 \exp\{j(x_1 y_1 + x_2 y_2)\} \tag{3.102}$$
$$\exp\left\{\frac{-1.46\sqrt{2}k^{7/6}\Gamma^2(4/3)}{\Gamma(8/3)} C_\epsilon^2 z^{11/6} \left(2(|\hat{x}|^2)^{5/6} + 2(|\hat{y}|^2)^{5/6} - (|\hat{x} - \hat{y}|^2)^{5/6} + (|\hat{x} + \hat{y}|^2)^{5/6}\right)\right\}$$

where $\Gamma(\cdot)$ is the Gamma function. The previous equation was written in a form to facilitate the substitution of polar coordinates. By making the substitutions

$$\begin{aligned} \hat{x} &= (x_1, x_2) = (r_1 \cos(\theta_1), r_1 \sin(\theta_1)) \\ \hat{y} &= (y_1, y_2) = (r_2 \cos(\theta_2), r_2 \sin(\theta_2)) \end{aligned} \tag{3.103}$$

Eq. 3.102 becomes

$$K(z)G_{<4>} = \iint_0^{2\pi} d\theta_1 \iint_0^{2\pi} d\theta_2 \iint_0^{\infty} r_1 dr_1 \iint_0^{\infty} r_2 dr_2 \exp\{j r_1 r_2 \cos(\theta_1 - \theta_2)\} \tag{3.104}$$
$$\exp\left\{C(z)\left(2r_1^{5/3} + 2r_2^{5/3}\right)\right\}$$
$$\exp\left\{C(z)\left((r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) - (r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2))\right)\right\}$$

where

$$C(z) = \frac{-1.46\sqrt{2}k^{7/6}\Gamma^2(4/3)}{\Gamma(8/3)} C_\epsilon^2 z^{11/6} \tag{3.105}$$

The substitution of

$$\begin{aligned} \phi &= \theta_1 - \theta_2 \\ \phi' &= \theta_2 \end{aligned} \tag{3.106}$$

will permit ϕ' to be integrated by inspection, resulting in

$$G_{<4>} = \left(\frac{k}{2\pi z}\right)^2 2\pi \int_0^{2\pi} d\phi \int_0^{\infty} r_1 dr_1 \int_0^{\infty} r_2 dr_2 \exp\{jr_1r_2\cos(\phi)\} \tag{3.107}$$
$$\exp\left\{C(z)\left(2r_1^{5/3} + 2r_2^{5/3}\right)\right\}$$
$$\exp\left\{C(z)\left((r_1^2 + r_2^2 - 2r_1r_2\cos(\phi))^{5/6} - (r_1^2 + r_2^2 + 2r_1r_2\cos(\phi))^{5/6}\right)\right\}$$

Further simplifications are possible by considering the integration of ϕ on $[0, \pi/2]$ instead of $[0, 2\pi]$ and recognizing the symmetry of $G_{<4>}$ along the line $r_1 = r_2$. These substitutions will now be performed one at a time. First, the integration range will be divided in half. Specifically, the integrand of the previous triple integral may be considered as a function of $\cos(\phi)$ only. Additionally, the previously mentioned integrand is an even function in $\cos(\phi)$. Then we may write

$$\begin{aligned} \int_0^{2\pi} f(\cos(\phi))d\phi &= \int_0^{\pi} f(\cos(\phi))d\phi + \int_{\pi}^{2\pi} f(\cos(\phi))d\phi \\ &= \int_0^{\pi} f(\cos(\phi))d\phi + \int_0^{\pi} f(\cos(\phi + \pi))d\phi \\ &= \int_0^{\pi} f(\cos(\phi))d\phi + \int_0^{\pi} f(-\cos(\phi))d\phi \\ &= 2\int_0^{\pi} f(\cos(\phi))d\phi \end{aligned} \tag{3.108}$$

A similar series of simplifications may be made which will result in the integration range of ϕ being reduced by half again. This second reduction in the integration region will cause the factor $\exp(jr_1r_2\cos(\phi))$ to be replaced with $2\cos(r_1r_2\cos(\phi))$. Therefore, it follows that

$$G_{<4>} = 8\pi K(z) \int_0^{\pi/2} d\phi \int_0^{\infty} r_1 dr_1 \int_0^{r1} r_2 dr_2 \cos\left(r_1r_2\cos(\phi)\right) \tag{3.109}$$
$$\exp\left\{C(z)\left(2r_1^{5/3} + 2r_2^{5/3} - (r_1^2 + r_2^2 - 2r_1r_2\cos(\phi))^{5/6} - (r_1^2 + r_2^2 + 2r_1r_2\cos(\phi))^{5/6}\right)\right\}$$

The final simplification will be to reduce the integration region $0 \leq r_1 < \infty, 0 \leq r_1 < \infty$ with $0 \leq r_2 \leq r_1 < \infty$. Thus, we have arrived at the final integral form

$$G_{<4>} = 16\pi K(z) \int_0^{\pi/2} d\phi \int_0^{\infty} r_1 dr_1 \int_0^{r1} r_2 dr_2 \cos\left(r_1 r_2 \cos(\phi)\right) \tag{3.110}$$
$$\exp\left\{C(z)\left(2r_1^{5/3} + 2r_2^{5/3} - (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi))^{5/6} - (r_1^2 + r_2^2 + 2r_1 r_2 \cos(\phi))^{5/6}\right)\right\}$$

One of the final substitutions which may be made concerns the function $C(z)$. The Rytov solution for the relative intensity fluctuations is given by (25:103)

$$\beta_0^2 = 0.307 C_\epsilon^2 k^{7/6} z^{11/6} \tag{3.111}$$

We notice that the units of C_ϵ are meters$^{-1/3}$, the units of k is meters^{-1} and z is simply meters.

Therefore, if we make the substitution

$$C(z) = -0.619\beta_0^2 \tag{3.112}$$

the analysis of Eq. 3.110 may be performed under dimensionless conditions.

Eq. 3.110 is a highly oscillatory integral and because of this property, special techniques must be used to try to evaluate it. The naive method, also known as the brute force method, is to simply perform the triple integral indicated above. The final integral over r_1 may be evaluated by considering a sequence of partial integrations for larger and larger values of r_1. In the limit as r_1 tends to infinity, it is hoped that the partial contributions to the total integral tend to zero. However, in this case, hope does not mirror reality and for a finite precision machine convergence is nearly impossible to attain. In order to accelerate the convergence we can replace the integrand with a function which converges to zero much more rapidly (18:1). Specifically, we can subtract the value of the integrand for very large values of r_1 and then add it back again later. With an abuse of the $\lim_{r_1 \to \infty}$ notation, we will want to consider

$$\left(1 - \lim_{r_1\to\infty} + \lim_{r_1\to\infty}\right) 16\pi K(z) \int_0^{\pi/2} d\phi \tag{3.113}$$
$$\int_0^{\infty} r_1 dr_1 \int_0^{r1} r_2 dr_2 \cos\left(r_1 r_2 \cos(\phi)\right)$$
$$\exp\left\{-0.619\beta_0^2\left(2r_1^{5/3} + 2r_2^{5/3} - (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi))^{\frac{5}{6}} - (r_1^2 + r_2^2 + 2r_1 r_2 \cos(\phi))^{\frac{5}{6}}\right)\right\}$$

Therefore, consider the case where $\lim_{r_1 \to \infty}$. This equivalent to the case where $|x| \to \infty$ in Eq. 3.103. Therefore, as r_1 tends to infinity, Eq. 3.110 is fully equivalent to Eq. 3.102 which is shown below for convenience.

$$G_{<4>} = K(z) \iint_{\mathbf{R}^2} \iint_{\mathbf{R}^2} d^2\hat{x} d^2\hat{y} \exp\{j(x_1y_1 + x_2y_2)\} \tag{3.114}$$
$$\exp\left\{-0.619\beta_0^2 \left(2(|\hat{x}|^2)^{5/6} + 2(|\hat{y}|^2)^{5/6} - (|\hat{x} - \hat{y}|^2)^{5/6} + (|\hat{x} + \hat{y}|^2)^{5/6}\right)\right\}$$

In the limit as x tends to infinity and y remains unchanged, the integrand of Eq. 3.114

$$\exp\left\{-0.619\beta_0^2 \left(2(|\hat{x}|^2)^{5/6} + 2(|\hat{y}|^2)^{5/6} - (|\hat{x} - \hat{y}|^2)^{5/6} + (|\hat{x} + \hat{y}|^2)^{5/6}\right)\right\} \tag{3.115}$$

may be rewritten as

$$\exp\left\{j(x_1y_1 + x_2y_2) - 0.619\beta_0^2 \left(2(|\hat{x}|^2)^{5/6} + 2(|\hat{y}|^2)^{5/6} - (|\hat{x}|^2)^{5/6} + (|\hat{x}|^2)^{5/6}\right)\right\} \tag{3.116}$$

which simplifies to

$$\exp\left\{j(x_1y_1 + x_2y_2) - 0.619\beta_0^2 \left(2|\hat{y}|^{5/3}\right)\right\} \tag{3.117}$$

Therefore, we will be interested in the following integration

$$L(\beta_0^2) = \iint_{\mathbf{R}^2} \iint_{\mathbf{R}^2} dx_1dy_1dx_2dy_2 \exp\left\{j(x_1y_1 + x_2y_2) - 0.619\beta_0^2 \left(2|y|^{5/3}\right)\right\} \tag{3.118}$$

Which is equivalent to the integration

$$L(\beta_0^2) = \iint_{\mathbf{R}^2} \iint_{\mathbf{R}^2} dx_1dy_1dx_2dy_2 \exp\left\{j(x_1y_1 + x_2y_2) - 1.238\beta_0^2 \left(|y_1^2 + y_2^2|^{5/6}\right)\right\} \tag{3.119}$$

The integrations with respect to x_1 and y_1 may be performed first where

$$\int_{-\infty}^{\infty} \exp(jx_1y_1)dx_1 = 2\pi\delta(y_1) \tag{3.120}$$

resulting in

$$L(\beta_0^2) = 2\pi \iint_{\mathbf{R}^2} dx_2 dy_2 \int_{-\infty}^{\infty} dy_1 \delta(y_1) \exp\{jx_2y_2\} \tag{3.121}$$
$$\exp\left\{-1.238\beta_0^2 \left(y_1^2 + y_2^2\right)^{(5/6)}\right\} =$$
$$2\pi \iint_{\mathbf{R}^2} dx_2 dy_2 \exp\{jx_2y_2\} \exp\left\{-1.238\beta_0^2 \left(y_2^2\right)^{(5/6)}\right\}$$

The integrations on x_2 and y_2 may be similarly performed yielding

$$L(\beta_0^2) = 4\pi^2 \tag{3.122}$$

Therefore, it follows that we will be interested in the evaluation of

$$16\pi K(z) \int_0^{\pi/2} d\phi \int_0^{\infty} r_1 dr_1 \int_0^{r1} r_2 dr_2 \cos\left(r_1 r_2 \cos(\phi)\right) \tag{3.123}$$
$$\left[\exp\left\{-0.619\beta_0^2 \left(2r_1^{5/3} + 2r_2^{5/3} - (r_1^2 + r_2^2 - 2r_1r_2\cos(\phi))^{5/6} - (r_1^2 + r_2^2 + 2r_1r_2\cos(\phi))^{5/6}\right)\right\} -\right.$$
$$\left.\exp\left\{-1.238\beta_0^2 r_2^{5/3}\right\}\right] + 4\pi^2 K(z)$$

The preceding integral is the final form which may be numerically integrated, although we will not explicitly do so as we could not obtain a convergent solution for all positive values of β^2. In particular, rapid convergence for values of $\beta^2 > 10.0$ was obtained using Gauss-Kronrod integration. The corresponding value of z the propagation distance is given by simply solving

$$0.307 C_\epsilon^2 k^{7/6} z^{11/6} = \beta^2 > 10.0 \tag{3.124}$$

which results in

$$z > (32.57 C_\epsilon^{-2} k^{-7/6})^{6/11} \tag{3.125}$$

Therefore, given a particular optical wavelength, and a value of $C_\epsilon = C_n$ which is altitude dependent, we can obtain a minimum propagation range for which a numerical solution to Eq. 3.110 will rapidly converge.

An equation for the scintillation index very similar to Eq. 3.110 which was derived under a different set of assumptions has been numerically integrated by Beran and Whitman (3:2140). Therefore, the result of performing the operations of Eq. 3.123 will yield a function of β_0^2 which will closely match the scintillation index derived by Beran and Whitman. Their calculated scintillation index was shown to be accurate to the theoretical scintillation index within ten percent. Therefore, we expect a similar magnitude in error for Eq. 3.123 in its predictions of the scintillation index. The scintillation index from Eq. 3.123 is qualitatively very similar to the scintillation index for strong medium fluctuations which appears in Fig. 2.4.

This chapter has examined a small aspect of the problem of wave propagation in a randomly inhomogeneous medium. The next chapter will examine some of the consequences of this examination.

IV. Conclusions, Contributions and Recommendations

Research Questions Answered

During the course of research for this thesis, it was desired to obtain the answers to some relevant questions which are repeated from Chapter 1 for convenience.

1. Of the presently known subsets which have any practical usefulness, what accuracy do they provide when used to help approximate functional integration?
2. What is the computational complexity of these subsets?
3. Do the characteristics of the function subsets used to approximate the functional integrand dictate the use of a particular numerical integration scheme?

This thesis investigated the approximation of an arbitrary continuous trajectory with a truncated Fourier-sine series of only one sinusoid; therefore, the answers to the above three questions will have to be accepted with such an understanding. Specifically,

1. The accuracy of a single sinusoid is estimated to be on the order of ten percent error between the predicted scintillation index and the theoretical scintillation index for a point source based upon the results of Beran and Whitman (3:2140). This estimation of the error comes from the comparison of the three-fold integral obtained from the Feynman integral used in this thesis to obtain the fourth moment of the field and the similar three-fold integral examined by Beran and Whitman. They obtained an error of about ten percent in their research and its reasonable to assume the error in this thesis is on the same order of magnitude.
2. The computational complexity of this particular approximating function (a single sinusoid) was extremely high. Initial computational runs on a Cray -YMP using Gauss-Kronrod adaptive integration schemes often did not converge in under one hour. Higher values of β_0^2 did not converge at all. The results of Beran and Whitman were obtained by heuristically choosing particular integration regions and examining the limit of the resulting sequence of partial sums. They reported acceptable convergence times which we could not duplicate. However, their results were accepted as valid.

3. Based upon the experience gained in writing this thesis and the results reported by Beran and Whitman, it would seem that the analysis of the integration scheme is extremely important. Apparently, the integration region must be subdivided so that the available integration schemes, for example Gauss-Kronrod, will achieve acceptable convergence in a reasonable time. The exact nature of the subdivided integration regions is presently very poorly understood.

Contributions

Using the present sinusoidal approximation to the path integrand and the presently implemented numerical integration scheme to calculate Eq. 3.110, we see that are some limitations as to what size of a propagation distance will yield a convergent numerical solution. Specifically, as of the time of the writing of this thesis, the limitation is given by

$$z > 6.68 C_{\epsilon}^{-12/11} k^{-7/11} \tag{4.1}$$

meaning that values of z larger than this will yield a convergent solution. As an example, using a measured C_{ϵ}^2 at 12497 meters (above sea level) of 5.0 10^{-14} (meter$^{-2/3}$) and an optical wavelength of 7000 angstroms (red light), we find a convergent solution for $z > 4550$ meters (27:71).

Recommendations

In order to obtain a better understanding of the method of path integration to find the scintillation index, it is recommended that the nature of the highly oscillatory integrand used in the Eq. 3.123 be better understood. Specifically, the necessary integration scheme and region over which such a scheme is applied must be rigorously understood. Additionally, a single sinusoid as an approximation to a general trajectory is clearly not a good approximation. It is possible that there exist other orthonormal function sets which could be used to make the necessary approximations (perhaps wavelets). Lastly, though this work was directed at gaining a better understanding of the problem from the path integral standpoint, there may be important future applications which can be derived from the work contained herein. Therefore, this

direction of research should, if possible, continue to receive support.

Appendix A. *Parabolic Wave Equation*

As a convenience to the reader, this appendix has been provided. This appendix will follow the method used in (29:211-212) for the development of the parabolic wave equation. Starting from Maxwells' equations and considering a non-conducting, charge-free medium with a unity magnetic permeability, and a slowly temporally varying dielectric permeability $\epsilon(r,t) = \epsilon(\rho, z)$, Maxwell's equations can be written

$$\nabla \times E = \frac{-1}{c}\frac{\partial H}{\partial t} \tag{A.1}$$

$$\nabla \times H = \frac{1}{c}\frac{\partial(\epsilon E)}{\partial t} \tag{A.2}$$

$$\nabla \cdot (\epsilon E) = 0 \tag{A.3}$$

where E is the electric field strength, H is the magnetic field strength, $r = (x, y, z)$ is a position vector, ∇ is the gradient operator, ϵ is the dielectric permeability of the medium, and c is the speed of light.

In the atmosphere, the dielectric permeability is very slowly changing, therefore, $\epsilon(r,t) \simeq \epsilon(\rho, z)$. Thus, letting the varying portion of ϵ be expressed as,

$$\tilde{\epsilon}\,(\rho, z) = \frac{\epsilon(\rho, z) - \langle\epsilon(\rho, z)\rangle}{\langle\epsilon(\rho, z)\rangle} \tag{A.4}$$

or by neglecting the explicit (ρ, z) dependence

$$\tilde{\epsilon} = \frac{\epsilon - \langle\epsilon\rangle}{\langle\epsilon\rangle} \tag{A.5}$$

where $\langle\cdot\rangle$ denotes ensemble averaging. If the reasonable requirement that the atmosphere exhibits non-relativistic characteristics,

$$\frac{\langle|v|\rangle}{c} \ll \langle|\,\tilde{\epsilon}\,|\rangle, \tag{A.6}$$

where v is the characteristic velocity of atmospheric turbulent inhomogeneities, and c is the velocity of light is made then it follows that,

$$(\nabla^2 + k^2)E = -k^2 \tilde{\epsilon}\, E - \nabla(E\nabla \tilde{\epsilon}) \tag{A.7}$$

It can be shown that the last term in the above equation can be neglected for large-scale inhomogeneities: $l_0 \gg \lambda$. This being the case, a stochastic wave equation which holds for any component of the electric field intensity E results,

$$(\nabla^2 + k^2)E = -k^2 \tilde{\epsilon}\, E \tag{A.8}$$

Thus,

$$\nabla^2 E + k^2(1+ \tilde{\epsilon})E = 0 \tag{A.9}$$

for any component x, y, or z. The above equation is known as the stochastic Helmholtz equation. A rigorous derivation of the parabolic approximation to the wave equation can be found in (25:38-44). However, for this synopsis, a more intuitive and simplistic derivation will be used. This derivation will parallel the method presented in (28:5.11). An optical wave propagating in the z direction, may be written as,

$$U(x, y, z, t) = u(x, y, z)\cos(kz - \omega t) \tag{A.10}$$

where $U(x, y, z, t)$ is the optical field at the point (x, y, z) and the time t, $u(x, y, z)$ is the real-valued amplitude of the wave at the point (x, y, z), $k = \frac{2\pi}{\lambda}$ and λ is the carrier wavelength, and ω is the angular frequency of the carrier. By considering U to be the real part of an analytic signal, U_a, it follows that,

$$U(x, y, z) = \Re\{U_a(x, y, z)\exp(jkz)\exp(-jwt)\} \tag{A.11}$$

where $\Re(\cdot) = \frac{1}{2}((\cdot) + (\cdot)^*)$, and $(\cdot)^*$ means complex conjugation. By considering a monochromatic optical wave and the variation in time to be implicit, it is possible to write the last equation as,

$$U(x, y, z) = \Re\{U_a(x, y, z) \exp(jkz)\} = \Re\{u(\rho, z) \exp(jkz)\} \tag{A.12}$$

where $u(\rho, z)$ now represents the complex amplitude of the optical wave, and $\rho = (x, y)$. Henceforth, without loss of generality, this thesis will consider monochromatic disturbances (waves) which propagate in approximately the z direction. Suppose $u(\rho, z)$ changes little on the scale of λ. Then it follows that the change in $u(\rho, z)$, which is represented by $\delta u(\rho, z)$, will be much smaller than the magnitude of $u(\rho, z)$. In other words

$$|\delta u(\rho, z)| \ll |u(\rho, z)| \tag{A.13}$$

Next, consider the incremental change in $u(\rho, z)$ along an incremental change in z,

$$\delta u(\rho, z) \simeq \frac{\partial u(\rho, z)}{\partial z} \times \delta z \tag{A.14}$$

Now, because it has been assumed that $u(\rho, z)$ varies slowly with respect to the carrier frequency in the z direction, consider $\delta z \approx \lambda$ in which case,

$$\frac{\partial u(\rho, z)}{\partial z} \times \delta z \simeq \frac{\partial u(\rho, z)}{\partial z} \times \lambda \tag{A.15}$$

Thus, it follows that,

$$\left|\frac{\partial u(\rho, z)}{\partial z}\right| \lambda \ll |u(\rho, z)| \tag{A.16}$$

This results in,

$$\left|\frac{\partial u(\rho, z)}{\partial z}\right| \ll \frac{1}{\lambda}|u(\rho, z)| < k|u(\rho, z)| \tag{A.17}$$

Following a similar line of reasoning, a similar result for $\frac{\partial^2 u(\rho, z)}{\partial z^2}$ follows,

$$\left|\frac{\partial^2 u(\rho, z)}{\partial z^2}\right| \ll k\left|\frac{\partial u(\rho, z)}{\partial z}\right| \tag{A.18}$$

The substitution of equations A.16, A.17 and A.18 into equation A.9 results in the simplified scalar-stochastic parabolic wave equation,

$$\left(2jk\frac{\partial}{\partial z} + \nabla_{\perp}^{2} + k^{2}\,\tilde{\epsilon}\,(\rho, z)\right) u(\rho, z) = 0. \tag{A.19}$$

where $\nabla_{\perp}^{2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. It is important to notice that equation A.19 is a first order partial differential equation in z, so there will be only one boundary condition in z.

Appendix B. *Method of Smooth Perturbations*

Asymptotic Expansion

This appendix is incuded as a convenience to the reader who may not be fully aware of the method of smooth perturbations, also known as Rytov's method. The derivations in this appendix will closely follow those found in (25:46-50).

A rigorous derivation of the parabolic equation would exclude the effects of backscattering for a wave propagating in an inhomogeneous medium. Therefore, Eq. A.19 holds for small angles about the mean propagation path. The problem with equation A.19 is that, except in the most trivial of cases, it is not solvable in closed form. Therefore, the method of smooth perturbations, also known as Rytov's method is applied. This method will approximate the actual solution by using the first term of an asymptotic series expansion. Consider an equation of the form,

$$f(x,\xi) = 0. \tag{B.1}$$

where ξ is a small-valued parameter. One method to solve the above equation is to assume a solution of the form,

$$f(x,\xi) = \sum_{i=0}^{\infty} \xi^i f_i(x) \tag{B.2}$$

where $\{f_i(x)\}_{i=0}^{\infty}$ is a sequence of (possibly unbounded) functions. The solution to B.2 is obtained by equating similar magnitudes or powers of ξ. For example, suppose it is desired to solve the equation

$$\exp(x^2\xi) = \cos(x+\xi) \tag{B.3}$$

for small values of ξ. By inspection, when ξ is zero then the solution is simply, $x = 0$; however, we might also be interested in solutions in x for small non-zero values of ξ. By expanding the exponential and sinusoidal terms into their respective Maclurian series in the above equation we have

$$\left(1 + x^2\xi + \frac{x^4\xi^2}{2!} + \frac{x^6\xi^3}{3!}\cdots\right) = \tag{B.4}$$
$$\left(1 - \frac{x^2 + 2x\xi + \xi^2}{2!} + \frac{x^4 + 4x^3\xi + 6x^2\xi^2 + 4x\xi^3 + \xi^4}{4!}\cdots\right)$$

We can obtain a solution by equating similar powers in ξ. For example, the first few simultaneous equations will be obtained by equating terms in ξ^0, ξ^1 and ξ^2

$$\begin{aligned} 1 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cdots \\ x^2 &= -\frac{2x}{2!} + \frac{4x^3}{4!}\cdots \\ \frac{x^4}{2!} &= \frac{-1}{2!} + \frac{6x^2}{2!}\cdots \end{aligned}$$

It is easy to see that the first of the preceding three equations will result in the equality

$$1 = \cos(x) \tag{B.5}$$

which immediately implies $x = 2n\pi$ for $n = \cdots -2, -1, 0, 1, 2\cdots$. Each equality based upon the next higher power in ξ will further reduce the allowed set of possible solutions. It is entirely possible that there will not be a solution. This can be seen to occur if every possible solution is eliminated by a succeding equation. It is this type of solution technique upon which the Rytov method is based.

The Rytov Method

The complex amplitude, $u(\rho, z)$ of a propagating wave may be rewriten in polar form,

$$u(\rho, z) = A_0 \exp\left[ln\frac{A(\rho, z)}{A_0} + jS(\rho, z)\right] = A_0 \exp(\mathcal{X} + jS) = A_0 \exp[\Phi(\rho, z)] \tag{B.6}$$

where A_0 is the amplitude of an initially uniform plane wave propagating into the random inhomogeneous medium (25:48). If the substitution $A_0 \exp[\Phi(\rho, z)] = u(\rho, z)$ is made into equation A.19 one will obtain the equation:

$$2jk\frac{\partial \Phi}{\partial z} + \nabla_{\perp}^{2}\Phi + (\nabla_{\perp}\Phi)^2 + k^2 \tilde{\epsilon}\,(\rho, z) = 0 \tag{B.7}$$

This equation is less difficult to solve than equation A.19 as the fluctuating term $\tilde{\epsilon}$ as an additive term and not a multiplicitive factor. As stated in (25:48), equation B.7 is solved by assuming an asymptotic solution of the form $\Phi = \Phi_1 + \Phi_2 + \ldots$. Additionally, the scale size of Φ_p is assumed to be on the order of $(\sigma_\epsilon)^p$. Thus, $|\Phi_1| > |\Phi_2| > |\Phi_3| \ldots$ and equation B.7 can be recast into a collection of recursive, though linear equations. The set of equations was generated by grouping terms with the same relative magnitude. Terms which are on the size of σ_ϵ, $(\sigma_\epsilon)^2$, $(\sigma_\epsilon)^3)$ and so on, are collected into related equations, where σ_ϵ is the standard deviation of the medium's dielectric permeability ϵ.

$$2jk\frac{\partial \Phi_1}{\partial z} \quad + \quad \nabla_{\perp}^{2}\Phi_1 = -k^2 \tilde{\epsilon}\,(\rho, z) \tag{B.8}$$

$$2jk\frac{\partial \Phi_2}{\partial z} \quad + \quad \nabla_{\perp}^{2}\Phi_2 = -(\nabla_{\perp}\Phi_1)^2 \tag{B.9}$$

$$2jk\frac{\partial \Phi_3}{\partial z} \quad + \quad \nabla_{\perp}^{2}\Phi_3 = -2\,\nabla_{\perp}\,\Phi_1 \times \nabla_{\perp}\Phi_2 \tag{B.10}$$

Thus, equation B.8 is of size σ_ϵ, equation B.9 is of size $(\sigma_\epsilon)^2$ and equation B.10 is of size $(\sigma_\epsilon)^3$ (25:48). What usually happens in the method of smooth perturbations is that Φ is approximated by only the first term. Thus, $\Phi \simeq \Phi_1$. In order for this approximation to be reasonable, it is necessary for $|\Phi_2|$ to be small. This will happen if $(\nabla_{\perp}(\Phi_1)^2$ is small. Equation B.9 will have a solution in Φ_2 of the form:

$$\Phi_2(\rho, z) = \int_0^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(\rho - \rho', z - z')(\nabla_{\perp}\Phi_1)^2 d^2\rho' dz' \tag{B.11}$$

where the Green's function for the problem, K is given by

$$K(\rho - \rho', z - z') = \frac{-1}{4\pi(z - z')} \exp\left[\frac{jk|\rho - \rho'|^2}{2(z - z')}\right] \tag{B.12}$$

If $(\nabla_{\perp}(\Phi_1)^2$ is small, then Φ_2 in equation B.11 will be small and the approximation $\Phi \simeq \Phi_1$ will be accurate. The magnitude of $(\nabla_\perp(\Phi_1)^2$ necessary for the above approximation to hold has been found to be $(\nabla_\perp(\Phi_1)^2 \ll k^2\sigma_\epsilon$. If this holds then $(\nabla_\perp(\Phi_1)^2$ being small implies transverse variations in Φ must be relatively small and therefore Φ must be relatively smooth in the transverse plane. This is why this method is known as the method of smooth perturbations.

If Rytov's method is applied to the case of an initially unit amplitude plane wave striking a phase screen, then it has been shown that the phase variance of the resulting wave is given by

$$\beta_0^2 = 0.307 C_\epsilon^2 k^{\frac{7}{6}} z^{\frac{11}{6}} \tag{B.13}$$

where C_ϵ^2 is the structure constant of the medium's dielectric permeability, $k = \frac{2\pi}{\lambda}$ and λ is the carrier wavelength, z is the propagation distance from the phase screen to the measurement plane. One should immediately notice the nearly linear dependence in k and nearly quadratic dependence in z for the parameter β_0^2. The parameter β_0^2 will prove to be very useful in allowing the path integrals to be expressed in a dimensionless form.

Appendix C. *Method of Markov Approximations*

This appendix has been provided as a convenience to the reader and draws heavily from (29:215-217). It will provide a more detailed examination of the Markov approximation method. Beginning with the parabolic wave equation and following the method in (25:85-86) one has

$$(2jk\frac{\partial}{\partial z} + \nabla_\perp^2 + k^2 \tilde{\epsilon}\,(\rho, z))u(\rho, z) = 0 \tag{C.1}$$

Now we must recall the two conditions which must be satisfied.

First, $u(\rho, z)$ must be statistically independent of the effects of the random inhomogenities for any location z' > z in the direction of propagation (no backscattering effects).

Second, the longitudional correlation radius of $\tilde{\epsilon}\,(\rho, z))$ must be less than the characteristic time of changes in $u(\rho, z)$. This means that $u(\rho, z)$ can't resolve the individual effects of the fluctuations in $\tilde{\epsilon}\,(\rho, z)$. With these assumptions, the procedure is to consider two terms of equation C.1 and apply the identity:

$$\left(\frac{\partial}{\partial z} + f(\rho, z)\right) v(\rho. z) = \exp\left(-\int_0^z f(\rho, \xi)d\xi\right) \frac{\partial}{\partial z}\left[\exp\left(\int_0^z f(\rho, \xi)d\xi\right) v(\rho, z)\right] \tag{C.2}$$

Therefore,

$$\left(2jk\frac{\partial}{\partial z} + k^2 \tilde{\epsilon}\,(\rho, z)\right) v(\rho, z) = \\ 2jk\left[\exp\left(\frac{k^2}{2jk}\int_0^z \tilde{\epsilon}\,(\rho, \xi)d\xi\right)\right] \frac{\partial}{\partial z}\left[\exp\left(\frac{k^2}{2jk}\int_0^z \tilde{\epsilon}\,(\rho, \xi)d\xi\right) v(\rho, z)\right] \tag{C.3}$$

Thus, we have:

$$2jk\left[\exp\left(\frac{-jk}{2}\int_0^z \tilde{\epsilon}\,(\rho, \xi)d\xi\right)\right] \frac{\partial}{\partial z}\left[\exp\left(\int_0^z \frac{-jk}{2}\tilde{\epsilon}\,(\rho, \xi)d\xi\right) v(\rho, z)\right] = -\nabla_\perp^2 v(\rho, z) \tag{C.4}$$

Rearranging the factors yields

$$-2jk\frac{\partial}{\partial z}\left[\exp\left(\int_0^z \frac{-jk}{2}\tilde{\epsilon}\,(\rho, \xi)d\xi\right) v(\rho, z)\right] = \left[\exp\left(\frac{jk}{2}\int_0^z \tilde{\epsilon}\,(\rho, \xi)d\xi\right)\right] \nabla_\perp^2 v(\rho, z) \tag{C.5}$$

Next, integrating from 0 to z yields

$$-2jk\left[\exp\left(\int_0^z \frac{-jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)v(\rho,z)\right]-2jkv(\rho,0)= \int_0^z\left[\exp\left(\frac{-jk}{2}\int_0^{z'}\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\nabla_\perp^2\,v(\rho,z')\right]dz' \tag{C.6}$$

Continuing,

$$\exp\left(\int_0^z \frac{-jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)v(\rho,z)= -v(\rho,0)+\frac{-1}{2jk}\int_0^z\left[\exp\left(\frac{-jk}{2}\int_0^{z'}\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\nabla_\perp^2\,v(\rho,z')\right]dz' \tag{C.7}$$

Solving for $v(\rho,z)$ yields,

$$v(\rho,z) = \left[-\exp\left(\int_0^z \frac{jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\right]v(\rho,0)+\left[\frac{-1}{2jk}\exp\left(\int_0^z \frac{jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\right] \left[\int_0^z\exp\left(\frac{-jk}{2}\int_0^{z'}\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\nabla_\perp^2\,v(\rho,z')dz'\right] \tag{C.8}$$

Now, the last two integrals in the above equation may be combined to yield,

$$v(\rho,z)=\left[-\exp\left(\int_0^z \frac{k^2}{2jk}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\right]v(\rho,0)+ \frac{-1}{2jk}\int_0^z\exp\left(\int_{z'}^z \frac{jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\nabla_\perp^2\,v(\rho,z')dz' \tag{C.9}$$

Now, the expectation of both sides is taken,

$$\langle v(\rho,z)\rangle = \left\langle\left[-\exp\left(\int_0^z \frac{k^2}{2jk}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\right]v(\rho,0)\right\rangle+ \left\langle\frac{-1}{2jk}\int_0^z\exp\left(\int_{z'}^z \frac{jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi\right)\nabla_\perp^2\,v(\rho,z')dz'\right\rangle \tag{C.10}$$

Consider the second term in the last equation. Its expectation may be simplified by noting that the factor $\int_{z'}^z \frac{jk}{2}\,\tilde{\epsilon}\,(\rho,\xi)d\xi$ is dependant only on values of z for z ≥ z', while the factor $\nabla_\perp^2 v(\rho,z')$ is dependent only on values of z for z ≤ z'. Therefore, the two factors are statistically independent except on a set of measure zero. So, the expectation operation may be simplified,

$$\langle v(\rho,z)\rangle = \left\langle \left[-\exp\left(\int_0^z \frac{k^2}{2jk} \tilde{\epsilon}\,(\rho,\xi)d\xi \right)\right] v(\rho,0) \right\rangle + \tag{C.11}$$
$$\frac{-1}{2jk}\int_0^z \left\langle \exp\left(\int_{z'}^z \frac{jk}{2} \tilde{\epsilon}\,(\rho,\xi)d\xi \right)\right\rangle \left\langle \nabla_\perp^2 v(\rho,z')\right\rangle dz'$$

If the statistics of $\tilde{\epsilon}\,(\rho,\xi)$ are known then, equation C.11 is a closed equation for the mean of the field amplitude. For example, the statistics of $\tilde{\epsilon}\,(\rho,\xi)$ might be gaussian. If this is the case, a closed partial differential equation for the mean of $v(\rho,z)$ can be obtained (25:87). Higher order moments of $v(\rho,z)$ may be obtained by generalizating the above method (25:88) and this will be done shortly.

By performing steps similar to those already demonstrated it is possible to derive a closed equation for an arbitrary statistical moment. In particular if

$$\Gamma_{n,m}(z) = \Gamma_{n,m}(\rho_1,\rho_2,\cdots,\rho_n,\rho_1',\rho_2',\cdots,\rho_m',z) = \tag{C.12}$$
$$\langle u(\rho_1,z)u(\rho_2,z)\cdots u(\rho_n,z)u^*(\rho_1',z)u^*(\rho_2',z)\cdots u^*(\rho_m',z)\rangle$$

and

$$Q_{n,m}(z) = Q_{n,m}(\rho_1,\rho_2,\cdots,\rho_n,\rho_1',\rho_2',\cdots,\rho_m',z) = \tag{C.13}$$
$$\tilde{\epsilon}\,(\rho_1,z) + \tilde{\epsilon}\,(\rho_2,z) + \cdots + \tilde{\epsilon}\,(\rho_n,z) - \tilde{\epsilon}^*\,(\rho_1',z) - \tilde{\epsilon}^*\,(\rho_2',z) - \cdots - \tilde{\epsilon}^*\,(\rho_m',z)$$

then as shown in (29:216), it follows that

$$2jk\Gamma_{n,m}(z) - 2jk\Gamma(0)\left\langle \exp\left\{ \frac{jk}{2}\int_0^z Q(x)dx \right\}\right\rangle + \tag{C.14}$$
$$\int_0^z \left\langle \exp\left\{ \frac{jk}{2}\int_0^z Q(x)dx \right\}\right\rangle \left(\nabla_1^2 + \cdots + \nabla_n^2 - \nabla_1'^2 - \cdots - \nabla_m'^2 \right) \Gamma_{n,m}(x)dx = 0$$

where $\nabla_i^2 = \frac{\partial^2}{\partial \rho_i^2}$. This is a closed partial differential equation for an arbitrary statistical moment of the complex amplitude. If a further assumption is made about $\tilde{\epsilon}$, namely that $\tilde{\epsilon}$ has a Gaussian distribution and is delta correlated, specifically

$$\langle \tilde{\epsilon}\,(\rho_1, z_1)\, \tilde{\epsilon}^*\,(\rho_2, z_2)\rangle = A(\rho_1 - \rho_2)\delta(z_1 - z_2) \tag{C.15}$$

then as shown in (29:217) it follows that

$$2jk\frac{\partial}{\partial z}\Gamma_{n,m}(z) + \left(\nabla_1^2 + \cdots + \nabla_n^2 - \nabla_1'^2 - \cdots - \nabla_m'^2 \cdots\right)\Gamma_{n,m}(z) + \frac{jk^3}{4}F_{n,m}\Gamma_{n,m}(z) = 0 \tag{C.16}$$

where

$$F_{n,m} = F_{n,m}(\rho_1, \rho_2, \cdots, \rho_n, \rho_1', \rho_2', \cdots, \rho_m') = \sum_{a=1}^{n}\sum_{b=1}^{n} A(\rho_a - \rho_b) - 2\sum_{a=1}^{n}\sum_{b=1}^{m} A(\rho_a - \rho_b') + \sum_{a=1}^{m}\sum_{b=1}^{m} A(\rho_a' - \rho_b') \tag{C.17}$$

The assumption that $\tilde{\epsilon}$ is Gaussian distributed follows from the law of large numbers which states that the sum of independent and identically distributed random variables tends to a Gaussian distribution. The assumption of delta correlation is a good approximation if the correlation radius of $\tilde{\epsilon}$ is small. The solution for $\Gamma_4(z, \rho_1, \rho_2, \rho_3, \rho_4)$ can be obtained for a phase screen of thickness (δz) and is found to be

$$\begin{aligned}\Gamma_4(\rho_1, \rho_2, \rho_3, \rho_4, z) = \left(\frac{k}{j2\pi z}\right)^4 \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \Gamma_4(\rho_1', \rho_2', \rho_3', \rho_4', 0)\\ \exp\left\{\frac{jk}{2z}|(\rho_1 - \rho_1'|^2 - |\rho_2 - \rho_2'|^2 + |\rho_3 - \rho_3'|^2 - |\rho_4 - \rho_4'|^2\right\}\\ \exp\left\{\frac{\delta z k^2}{8}\left(2D(\rho_1' - \rho_2') - 2D(\rho_1' - \rho_3') + 2D(\rho_1' - \rho_4') + 2D(\rho_2' - \rho_3')\right)\right\}\\ \exp\left\{\frac{\delta z k^2}{8}\left(2D(\rho_2' - \rho_4') + 2D(\rho_3' - \rho_4')\right)\right\} d^2\rho_1' d^2\rho_2' d^2\rho_3' d^2\rho_4'\end{aligned} \tag{C.18}$$

where $D(x) = A(0) - A(x)$. For an initially totally coherent plane wave, $\Gamma_4(\rho_1', \rho_2', \rho_3', \rho_4', 0) = 1$. A final simplification may be perfomed by using the substitutions

$$\begin{aligned}\rho_1' - \rho_2' &= r_1\\ \rho_1' - \rho_4' &= r_2\end{aligned} \tag{C.19}$$

$$\rho_1' - \rho_3' = r_3$$
$$\rho_1' + \rho_2' = 2r$$
$$\rho_1 - \rho_2 = p_1$$
$$\rho_1 - \rho_4 = p_2$$

and integrating over r and r_3 results in

$$\Gamma_4(z, p_1, p_2) = \left(\frac{k}{2\pi z}\right)^2 \int\int \exp\left\{\frac{jk}{z}(r_1 - p_1)\cdot(r_2 - p_2) - \frac{k^2\delta z}{4}F(r_1, r_2)\right\} d^2r_1 d^2r_2 \tag{C.20}$$

where $F(r_1, r_2) = 2D(r_1) + 2D(r_2) - 2D(r_1 - r_2) - 2D(r_1 + r_2)$. The preceding result gives the fourth statistical moment at a distance z behind a phase screen of thickness δz for an initially totally coherent plane wave. It is possible to consider Eq. C.20 another way. Examination of Eq. C.20 reveals the integration of a function over the entire plane. The integrand of Eq. C.20 may therefore be considered to be, apart from a constant, a Green's function which describes the fourth statistical moment of a point source. This Green's function has the form

$$G_{<4>}(\rho_1, \rho_2, \rho_3, \rho_4) = \exp\left\{\frac{jk}{z}(\rho_1 - \rho_3)\cdot(\rho_2 - \rho_4) - \frac{k^2\delta z}{4}\left[2D(\rho_1) + 2D(\rho_2) - D(\rho_1 - \rho_2) - D(\rho_1 + \rho_2)\right]\right\} \tag{C.21}$$

It will be found to be useful to compare the preceding Green's function with a functionally similar path integral. This is done in chapter 3.

Appendix D. *The Feynman Integral Schrödinger wave Equation Relationship*

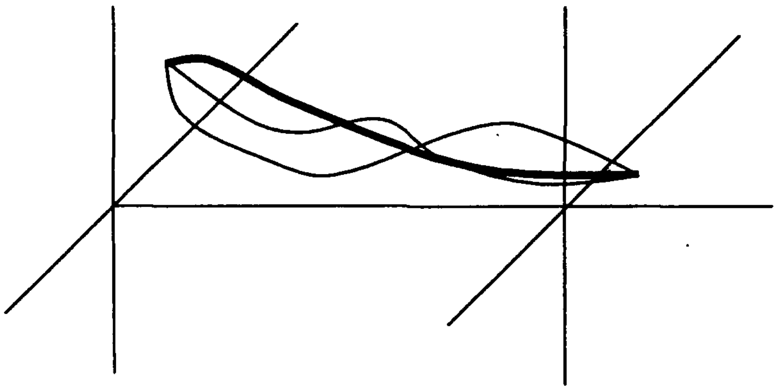
Introduction

This appendix, which draws directly from (11:26-78) is included to help the reader understand the origin of the Feynman integral. This appendix will closely follow the original method of deriving the Schrödinger wave equation from its Feynman integral solution. There are several reasons for including this appendix, namely:

1. To provide the reader of this thesis with the methodology used in some of the quantum mechanics related literature. This is justifiable because as previously noted, many of the methods and solutions of quantum mechanics have been used to solve wave propagation problems.
2. To provide an alternative example of the relationship between a parabolic equation, the Schrödinger wave equation, (a first order in time, second order in space partial differential equation) and a Feynman integral.
3. To provide some additional justification to use the functional integral approach to solve related problems.

The Classical Trajectory

In the everyday experiences of people, the effects of the Heisenberg uncertainty principle can largely be ignored. That is, the kinematics of common objects, cars, horses, books, all obey Newton's laws of motion. The expected path or trajectory that a moving object will follow is called the classical path (11:26). In the quantum mechanical world, the Heisenberg uncertainty principle states that possibilities other than the classical or expected trajectory are possible. Fig. shows some of these paths. So, according to the Heisenberg uncertainty principle, it is possible for a car spontaneously decay or for a horse to fly to the moon. Of course this is obviously not how we have come to expect cars and horses to behave and therefore the probability of such occurances is vanishingly small. Thus, the classical trajectory is in the quantum mechanical world, simply the trajectory with the greatest probability of occuring. It turns out that there is a functional defined on all the possible trajectories which exhibits an extremum for the classical trajectory. This is the Action functional.



— Possible trajectory

▬ Classical trajectory

Figure D.1. The subdivision of a trajectory

The Principle of Least Action

Consider a particle initially at (x_a, y_a, z_a) at the time t_a which travels, in a continuous fashion, to (x_b, y_b, z_b) at the time t_b and obeys Newton's laws of motion. The continuous path is therefore be described as the continuous parametric function $(x(t), y(t), z(t))$. The action functional of the particle along the path $(x(t), y(t), z(t))$ is defined as,

$$S[x,y,z] = \int_{t_a}^{t_b} (E_k - E_p)\, dt = \int_{t_a}^{t_b} L\left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}, x(t), y(t), z(t), t\right) dt \qquad \text{(D.1)}$$

where $S[x,y,z]$ is the action functional along the path (x(t), y(t), z(t)), $E_k = \frac{m}{2}\left[\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2\right]$ is the kinetic energy, $E_p = V(x(t), y(t), z(t), t)$ is the potential energy, $L(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}, x(t), y(t), z(t), t)$ is the difference between the Newtonian kinetic and potential energies. The action functional for different trajectories can either be similar or different. In the case of the classical trajectory, the action functional will be an extremum. This means that for a slight perturbation, δ of the classical path, there should be no difference in the action to the first order in δ. Thus, if $(\delta x, \delta y, \delta z)$ is a slight perturbation of the classical path which matches the classical path at the end points, as shown in Figthen this implies,

$$S[x(t) + \delta x(t), y(t) + \delta y(t), z(t) + \delta z(t)] - S[x(t), y(t), z(t)] = 0 \tag{D.2}$$

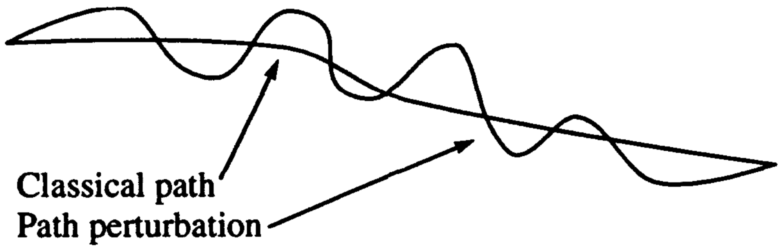


Figure D.2. A path with and without a slight perturbation

It is possible to expand $S[x(t) + \delta x(t), y(t) + \delta y(t), z(t) + \delta z(t)]$ in a Taylor series about $(x(t), y(t), z(t))$ which means that L can be expanded to first order as,

$$\begin{aligned}
L(&\frac{dx(t)}{dt} + \delta\frac{dx(t)}{dt}, \frac{dy(t)}{dt} + \delta\frac{dy(t)}{dt}, \frac{dz(t)}{dt} + \delta\frac{dz(t)}{dt},\\
&x(t) + \delta x(t), y(t) + \delta y(t), z(t) + \delta z(t), t)\\
&= L(\dot{x} + \delta\dot{x}, \dot{y} + \delta\dot{y}, \dot{z} + \delta\dot{z}, x + \delta x, y + \delta y, z + \delta z, t)\\
&\simeq L(\dot{x}, \dot{y}, \dot{z}, x, y, z) + \\
&\delta\dot{x}\frac{\partial L}{\partial\dot{x}} + \delta x\frac{\partial L}{\partial x} + \delta\dot{y}\frac{\partial L}{\partial\dot{y}} + \delta y\frac{\partial L}{\partial y} + \delta\dot{z}\frac{\partial L}{\partial\dot{z}} + \delta z\frac{\partial L}{\partial z}
\end{aligned} \tag{D.3}$$

where the over-dot represents partial differentiation with respect to t,second and higher order terms in $\dot{\delta}(\cdot)$ and $\ddot{\delta}(\cdot)$ have been dropped. Now, the action functional of the perturbed path is written,

$$\begin{aligned}
&S[x + \delta x, y + \delta y, z + \delta z] = \\
&\int_{t_a}^{t_b} L(\dot{x}, \dot{y}, \dot{z}, x, y, z, t)dt + \int_{t_a}^{t_b}\left(\delta\dot{x}\frac{\partial L}{\partial\dot{x}} + \delta x\frac{\partial L}{\partial x}\right)dt + \\
&\int_{t_a}^{t_b}\left(\delta\dot{y}\frac{\partial L}{\partial\dot{y}} + \delta y\frac{\partial L}{\partial y}\right)dt + \int_{t_a}^{t_b}\left(\delta\dot{z}\frac{\partial L}{\partial\dot{z}} + \delta z\frac{\partial L}{\partial z}\right)dt
\end{aligned} \tag{D.4}$$

it then follows that,

$$\delta S = S[x+\delta x, y+\delta y, z+\delta z] - S[x,y,z] = \tag{D.5}$$
$$\int_{t_a}^{t_b} \left\{ \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \right) \delta x + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} \right) \delta y + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} \right) \delta z \right\} dt$$

Additionally, it is necessary that $\delta x = \delta y = \delta z = 0$ at the trajectory endpoints t_a and t_b. Thus, in order that $\delta S = 0$ to first order in $\delta x, \delta y, \delta z$, it is sufficient that,

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \right) = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} \right) = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} \right) = 0 \tag{D.6}$$

which is seen to be the classical lagrangian of the Newtonian particle. The notion of a trajectory or path can be more generalized as it will be shown in a later section.

Wave Functions

Nothing in the quantum mechanical world, as already noted, is certain. Therefore highly probable, probable, possible, improbable and highly improbable events all have associated probabilities. A wave function is a function which describes these probabilities. Written as $\psi(x,t)$, its magnitude squared gives the probability of a certain event at the location x at the time t. That is, if $\psi(x,t)$ is the wave function associated with an event A, then the probability of event A is

$$|\psi(x,t)|^2 \tag{D.7}$$

here x is the one-dimensional space variable. It is easy to extend x into three dimensions (x, y, z). Given an event, there are several possible paths or trajectories. Each possible path or trajectory has an associated probability of occurance which in turn is described by a wave function. If there are two sequential events, event A followed by event B, then there is composite wave function to describe the composite event AB. Furthermore, there is an important relation between the individual wave functions for the individual events A and B and given by,

$$\psi(x_b, t_b) = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a)\psi(x_a, t_a)dx_a \tag{D.8}$$

where $K(x_b, t_b, x_a, t_a)$ is another wave function which will be described in the next section.

The Wave Function K

The wave function $K(x_2, t_2, x_1, t_1)$, which appeared in the previous section is an integral over paths. It is given by,

$$K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A} \int\int \cdots \int \exp\left(\frac{j}{\hbar} S[b,a]\right) \frac{dx_1}{A} \frac{dx_1}{A} \frac{dx_2}{A} \cdots \frac{dx_{n-1}}{A} \tag{D.9}$$

where $A = \left(\frac{j2\pi\hbar\epsilon}{m}\right)$ and ϵ and x_i are shown in the following figure, and the dependence of K on t_a and t_b is understood.

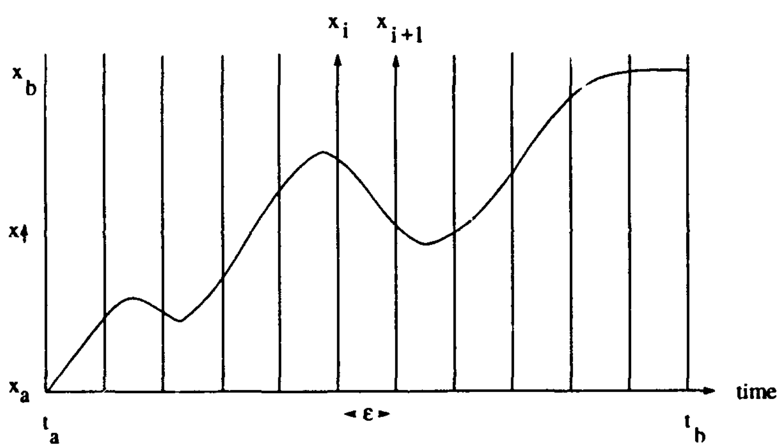


Figure D.3. The subdivision of a trajectory

Often one finds in the current literature the approximation of continuous trajectories or paths by piecewise linear approximations as shown in Fig(29:226). These approximating functions, which are also called polygonal functions, are used to simplify the limiting process in Eq. D.9. It is important to note that given any continuous curve in $\mathbf{R}^2$, it is possible to approximate it as closely as desired with a polygonal function. That is, given a continuous curve in $\mathbf{R}^2$, there is a sequence of polygonal functions which converges to the given continuous curve and infact, the convergence is uniform. We now have all the necessary tools to derive the Schrödinger wave equation. This will be done in the following section.

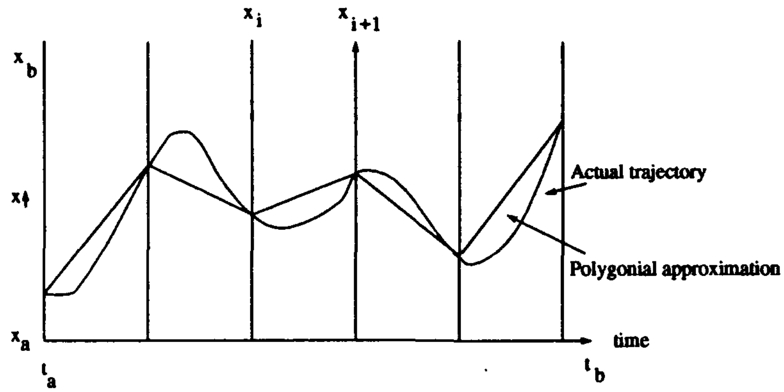


Figure D.4. The approximation of a trajectory by a polygonal function

Derivation of the Schrödinger Wave Equation

The one dimensional version of the Schrödinger wave equation will now be derived. As previously discussed, the derivation of this equation will help reinforce the readers understanding of the relationship between a partial differential equation and its path integral solution. Beginning with Eq. D.8, for a Newtonian particle at position x_a at time t_a and position x_b at time t_a, where x_a and x_b are points on the continuous trajectory x. Let $|t_b - t_a| = \epsilon'$ which is very small then,

$$\psi(x_b, t_b) = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_b)\psi(x_a, t)dx_a \tag{D.10}$$

For values of ϵ' small enough, that is, smaller than ϵ, the path integral in Eq. D.9 will reduce to a regular integral,

$$K(x_b, x_a) \simeq \frac{1}{A}\int \exp\left(\epsilon'\frac{j}{\hbar}S[x_b, x_a]\right) dx_a \tag{D.11}$$

For small ϵ' the time derivitive of the continuous trajectory x with respect to t may be written,

$$\dot{x} = \lim_{\epsilon'\to 0}\left(\frac{x_b - x_a}{t_b - t_a}\right) = \lim_{\epsilon'\to 0}\left(\frac{x_b - x_a}{\epsilon'}\right) \simeq \left(\frac{x_b - x_a}{\epsilon'}\right) \tag{D.12}$$

Additionally, for small values of ϵ', x_b is close to x_a, and therefore, with error on the order of ϵ'

that is O(ϵ'), $x_b \simeq \frac{x_b+x_a}{2}$, $L(\dot{x},x,t) \simeq L(\frac{x_b-x_a}{\epsilon'}, \frac{x_b+x_a}{2}, t)$. Therefore,

$$S[x_b, x_a] \simeq L\left(\frac{x_b - x_a}{\epsilon'}, \frac{x_b + x_a}{2}, t\right) \tag{D.13}$$

Therefore, the substitution of Eq. D.13, Eq. D.11 and $|x_b - x_a| = \tau$ into Eq. D.10, will result in Eq. D.14,

$$\psi(x_b, t+\epsilon') = \int_{-\infty}^{\infty} \frac{1}{A} \exp\left[\epsilon' \frac{j}{\hbar} L\left(\frac{\tau}{\epsilon'}, \frac{x_a + (x_a + \tau)}{2}, \epsilon'\right)\right] \psi(x_a + t) dx_a \tag{D.14}$$

The Lagrangian for a Newtonian particle traveling on a one dimensional trajectory is given by Eq. D.15

$$L(\dot{x}, x, t) = \frac{m\dot{x}^2}{2} + V(x,t) \tag{D.15}$$

Thus, making this substitution into Eq. D.14 results in Eq. D.16,

$$\psi(x_b, t+\epsilon') = \int_{-\infty}^{\infty} \frac{1}{A} \exp\left[\epsilon' \frac{j}{\hbar}\left(\frac{m\tau^2}{2} + V(x_a + \frac{\tau}{2}, t)\right)\right] \psi(x_a, t) dx_a \tag{D.16}$$

This may be simplified to,

$$\psi(x_b, t+\epsilon') = \int_{-\infty}^{\infty} \frac{1}{A} \exp\left(\frac{jm\epsilon'\tau^2}{2\hbar}\right) \exp\left(\frac{j\epsilon'}{\hbar} V(x_a + \frac{\tau}{2}, t)\right) \psi(x_a, t) dx_a \tag{D.17}$$

The first exponential term in the integrand in Eq. D.17 is highly oscillatory. Moreover, the contrabutions will be vanishingly small for large values of τ. Consider values of τ such that $|\frac{jm\tau^2}{2\hbar\epsilon'}|$ varies by at most one radian then,

$$\left|\frac{jm\tau^2}{2\hbar\epsilon'}\right| \leq 1 \Longrightarrow \tau \leq \sqrt{\frac{2\hbar\epsilon'}{m}} \tag{D.18}$$

Therefore, values of τ such that Eq. D.18 holds will give a non-zero contribution to Eq. D.17.

A function $f(x,t)$ which is continuous in all differential orders may be expanded in a

Taylor series about the temporal point t' as,

$$f(x,t') = \sum_{n=0}^{\infty} \frac{\left[\left(\frac{d}{dt}\right)^n f(x,t)\right](t-t')^n}{n!}\Bigg|_{t=t'} \tag{D.19}$$

By considering the Taylor expansion of the wave function $\psi(x_b, t+\epsilon')$ about the point t, it may be approximated to first order in ϵ' as,

$$\psi(x_b,t) \simeq \psi(x_b,t) + \frac{\partial}{\partial t}(\psi(x_t, t+\epsilon')\epsilon' \tag{D.20}$$

Similarly, $\exp\left(\frac{j\epsilon'}{\hbar}V(x_a+\frac{\tau}{2},t)\right)$ may be expanded in a Taylor series to first order in ϵ' as,

$$\exp\left(\frac{j\epsilon'}{\hbar}V(x_a+\frac{\tau}{2},t)\right) \simeq 1 + \frac{-j\epsilon'}{\hbar}V(x_a+\frac{\tau}{2},t) \tag{D.21}$$

Lastly, $\psi(x_b,t)$ may be expanded in a Taylor series about x_a to first order in ϵ' which by Eq. D.18 means to second order in τ as,

$$\psi(x_b,t) \simeq \psi(x_a,t) + \tau\frac{\partial\psi(x_b,t)}{\partial x_b} + \tau_2^2\frac{\partial^2\psi(x_a,t)}{\partial x_b^2}\Bigg|_{x_b=x_a} \tag{D.22}$$

The previous approximations may now be used in Eq. D.17 to yield,

$$\begin{aligned}\left(1+\frac{\partial}{\partial t}\right)\psi(x_b,t) = \\ \int_{-\infty}^{\infty}\frac{1}{A}\exp\left(\frac{jm\tau^2}{2\hbar\epsilon'}\right)\left[1-\frac{j\epsilon'}{h}V(x_a,t)\right]\left[1+\tau\frac{\partial}{\partial x_b}+\frac{\tau^2}{2}\frac{\partial^2}{\partial x_b^2}\right]\psi(x_b,t)d\tau\end{aligned} \tag{D.23}$$

This equation can be simplified by noting that,

$$\int_{-\infty}^{\infty}\frac{1}{A}\exp\left(\frac{jm\tau^2}{2\hbar\epsilon'}\right)\tau d\tau = 0 \tag{D.24}$$

and that,

$$\int_{-\infty}^{\infty}\frac{1}{A}\exp\left(\frac{jm\tau^2}{2\hbar\epsilon'}\right)\tau^2 d\tau = \frac{j\hbar\epsilon'}{m} \tag{D.25}$$

By equating terms in Eq. D.23 of order ϵ'^0 Eq. D.26 results,

$$\int_{-\infty}^{\infty} \frac{1}{A} \exp\left(\frac{jm\tau^2}{2\hbar\epsilon'}\right) \psi(x_b, t) d\tau \tag{D.26}$$

This will hold if $A = \sqrt{\frac{j2\pi\hbar\epsilon'}{m}}$. By using the simplifications affored by Eq. D.24 and Eq. D.25 with the value for A, the differential equation of Eq. D.27 results,

$$\left(1 + \frac{\partial}{\partial t}\right) \psi(x_b, t) = \left(1 - \frac{j\epsilon'}{\hbar} V(x_b, t)\right) \left(1 + \frac{j\hbar\epsilon'}{2m} \frac{\partial^2}{\partial x_b^2}\right) \psi(x_b, t) \tag{D.27}$$

By using some simple algebra and remembering that $\epsilon'\tau^2$ is of order ϵ'^2 and can be dropped, Schrödingers wave equation results,

$$\frac{\partial \psi(x, t)}{\partial t} = \left(\frac{j\hbar}{2m} \frac{\partial^2}{\partial x^2} - \frac{j}{\hbar} V(x, t)\right) \psi(x, t) \tag{D.28}$$

where x_b has been replaced with x. Examination of the above partial differential equation reveals that it is a first order in time, second order in space equation. Although it is not elementary, it is possible to follow the above steps by considering the space variable to be two dimensional. In that case, it will be possible to demonstrate a relationship between the two dimensional Schrödinger wave equation and its two dimensional Feynman integral solution. Lastly, by allowing the temporal variable to be replaced with a one dimensional space variable which is transverse to the other two space variables, it is possible to convert the two dimensional Schrödinger wave equation into an equivalent parabolic wave equation which will have an associated Feynman integral solution.

Appendix E. *Solution of the Helmholtz Equation*

This appendix is included as an aide to the reader who may wish to briefly review the Helmholtz equation and some of its characteristics.

The Solution to the Deterministic Helmholtz Equation

This section will give a solution of the deterministic Helmholtz equation under the condition of a slowly varying complex amplitude in the z direction. That is, the assumptions for the parabolic approximation hold. In this case, the solution to the Helmholtz equation Eq. E.1 with a unity boundary condition,

$$(\nabla^2 + k^2)u(\rho, z) = 0 \tag{E.1}$$

where $u(\rho, z)$ is the complex amplitude of the optical wave after a propagation distance of z, k is $\frac{2\pi}{\lambda}$ and λ is the carrier wavelength, $\rho = (x, y)$, namely ρ is in the plane transverse to the mean direction of propagation, z. A solution is given by a superposition integral of the form,

$$u(\rho, z) = \int\int G(\rho, \rho', z)u(\rho', 0)d^2\rho' \tag{E.2}$$

where $G(\rho, \rho', z)$ is a Greens function which satisfies the equation

$$(\nabla^2 + k^2)G(\rho, \rho', z) = \delta(\rho - \rho') \tag{E.3}$$

where $\delta(\rho - \rho')$ is the Dirac delta distribution is given by (4:378), (25:56),

$$u(\rho, z) = \frac{-1}{2\pi} \int\int_{\Sigma} \frac{\partial}{\partial z} \left(\frac{\exp(jk\sqrt{|\rho - \rho'|^2 + z^2})}{\sqrt{|\rho - \rho'|^2 + z^2}} \right) d^2\rho' \tag{E.4}$$

where $u(\rho, z)$ is the complex amplitude of the optical wave after a propagation distance of z, k is $\frac{2\pi}{\lambda}$ and λ is the carrier wavelength, $\rho = (x, y)$, namely ρ is in the measurement or observation plane transverse to the mean direction of propagation, z, $\rho' = (x', y')$, namely ρ is in the object plane, z = 0, $\frac{\exp(jk\sqrt{|\rho-\rho'|^2+z^2}}{\sqrt{|\rho-\rho'|^2+z^2}}$ is the Greens function, Σ is the region where $u(\rho', 0)$ is non-vanishing.

If Eq. E.4 is considered for values of z such that z >> λ and z >> $|\rho - \rho'|^2$, then the solution to the Helmholtz equation Eq. E.1 may be approximated by,

$$u(\rho, z) = \frac{\exp(jkz)}{j\lambda z} \int\int_{\Sigma} \exp\left(\frac{jk}{2z}|\rho - \rho'|^2\right) d^2\rho' \tag{E.5}$$

Eq. E.5 is known as the Huygens-Fresnel integral.

Interestingly enough, Eq. E.5 is the solution to the parabolic approximation to the deterministic Helmholtz equation Eq. E.6,

$$(2jk\frac{\partial}{\partial z} + \nabla_{\perp}^{2})u(\rho, z) = 0 \tag{E.6}$$

where $u(\rho, z)$ is the complex amplitude of the optical wave after a propagation distance of z, $\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $j = \sqrt{-1}$. Thus, the solution to the parabolic equation requires the consideration of small angles about the mean direction of propagation which is the same as requiring that $u(\rho, z)$ vary slowly in the mean propagation direction, z. But, this simply means that $u(\rho, z)$ has a narrow *spatial spectrum* for the z spatial frequency. The effects of the medium have been implicitly ignored. However, the inclusion of the random process $\epsilon(\rho, z)$ will not appreciably complicate the solution to the Helmholtz equation previously derive.

The Solution to the Stochastic Helmholtz Equation

The stochastic Helmholtz equation has already been derived in Appendix A from Maxwell's electromagnetic field equations. It was found to be,

$$(\nabla^2 + k^2(1 + \tilde{\epsilon}\,(\rho, z)))u(\rho, z) = 0 \tag{E.7}$$

If the solution to Eq. E.7 is considered for the region such that the Fresnel approximation holds then, the solution will take the form,

$$u(\rho, z) = \frac{k}{j2\pi z} \int\int u(\rho, z) \exp\left(\frac{jk}{2z}|\rho - \rho'|^2\right) \exp\left(\frac{jkz}{2}\,\tilde{\epsilon}\,(\rho', z)\right) d^2\rho' \tag{E.8}$$

where the approximation,

$$\sqrt{1+\tilde{\epsilon}}\sqrt{|\rho-\rho'|^2+z^2} \simeq (1+\frac{\tilde{\epsilon}}{2})(1+\frac{|\rho-\rho'|^2}{2z^2})z \simeq \frac{\tilde{\epsilon}\, z}{2}+z+\frac{|\rho-\rho'|^2}{2z} \qquad \text{(E.9)}$$

has been made. Additionally, in this approximation is the implicit assumption that $\tilde{\epsilon}\ (\rho, z)$ is relatively constant over a small increment in the z direction. Thus, the above approximation will become more accurate as the incremental change in z is small.

Appendix F. *Variational methods*

Functions and Functionals

This appendix which closely follows (24) is included as an aide to the reader who may not be fully conversant in functional or variational differentiation. The terms rule, function, mapping are all synonymous in this appendix, as are the terms variational differentiation and functional differentiation. In addition, a great amount of mathematical rigor has been sacrificed for both increased readability and decreased length. If this loss of rigor causes excessive concern, then you are advised to refer to a good book on functional and variational principles.

Recall that the simplistic idea of a function is a mapping or rule from one set to another. For example, the function given by $y = 2x$ looks innocent enough. However, if we assume x is only defined on the set of integers $\{1, 2, 33, 44\}$, then y may only take on the values $\{2, 4, 66, 88\}$. Therefore the domain of the function (the allowed values of x) and the range of the function (the allowed values of y) are also extremely important in the total idea of the function. Taking the simplification of the idea of a function still further, we could consider a function to be a machine. We send numbers into the machine and other numbers come out. The types of numbers we can send into the machine is constrained by the function's domain. The numbers the machine will return are constrained by both the domain and the range. Similar to the idea a function is the idea of a functional. A functional is a rule or mapping that is defined on some set of functions (the domain) and which returns real numbers (the range). A functional might be defined by the rule which returns 1 for all continuous functions and 0 for all discontinuous functions. Another example of a functional is the area under a continuous curve. This similarity between functions and functionals has some interesting implications. Therefore, a functional is also a machine; however, the functional machine has functions as its input and real numbers as its output.

Function Differentiation

The derivative of a function f at a point x is given by

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{F.1}$$

whenever the limit exists. For example the derivative of x^3 at the point 1 is given by

$$\frac{dx^3(1)}{dx} = \lim_{\Delta x \to 0} \left. \frac{(x + \Delta x)^3 - x^3}{\Delta x} \right|_{x=1} \tag{F.2}$$

Expanding, cancelling, and evaluating the resulting terms results in the expected result 3. An example of a function without a derivative is the mapping is the general polygonal function which is used to define a trajectory in its limit (see appendix D). On the trajectory where two straight lines join, the function describing the trajectory will not be differentiable. The single exception is the case when both lines have the same slope. Thus, we can see from this example that in the limit as the number of subdivisions of the propagation interval tends to infinity, the trajectories, will for the most part be everywhere continuous; however, except for *the strait line* connecting source and receiver, not one of the trajectories will be differentiable at every point in the propagation interval. By the very construction of the trajectories as the limiting case of all polygonal curves connecting source point to observation point, one finds the set of points where the trajectory function to be nondifferentiable to be countably infinite.

Functional or Variational Differentiation

There are several definitions of functional or variational differentiation including the Gâteaux variation and the Fréchet variation; however, we shall use the following definition (24:22)

$$\frac{\delta f[u]}{\delta u[x_0]} = \lim_{|\Delta x| \to 0, max|\delta u| \to 0} \frac{f[u + \delta u] - f[u]}{\int_{\Delta x} \delta u(x) dx} \tag{F.3}$$

where δu, a small variation or deviation in u, is nonzero only over a small region Δx about the point x_0 and provided that the limit exists for all possible small variations δu of u and contractions of Δx. Although this definition appears to be imposing, it is not appreciably any more difficult to use than the definition function differentiation with which we are more familiar. Furthermore, an example of functional differentiation would follow the first example of function differentiation which appeared in the previous section and therefore will be omitted.

Combined Function and Variational Differentiation

This section contains some rules which will be used later in this thesis. These are rules for the variational differentiation of functionals and functions of functionals. They are given without proof nor derivation. Again, the interested reader is urged to consult rigorous treatment elsewhere.

Rule 1

$$\frac{\delta}{\delta u(x)}\left(af[u]+bg[u]\right)=a\frac{\delta f[u]}{\delta u(x)}+b\frac{\delta g[u]}{\delta u(x)} \tag{F.4}$$

where a and b are independent of $u(x)$.

Rule 2

$$\frac{\delta}{\delta u(x)}f(G[u])=\frac{df}{dG[u]}\times\frac{\delta G[u]}{\delta u(x)} \tag{F.5}$$

where f is a function and G is a functional.

Rule 3

$$\frac{\delta}{\delta u(x)}u(y)=\delta(x-y) \tag{F.6}$$

where $u(x)$ is a function, $\delta(x-y)$ is the Dirac delta distribution with argument $(x-y)$.

Rule 4

$$\frac{\delta}{\delta u(x)}\frac{d}{dx}u(y)=\frac{d}{dx}\frac{\delta}{\delta u(x)}u(y)=\delta'(x-y) \tag{F.7}$$

where $\delta'(x-y)$ is the first derivative of the Dirac delta distribution with argument x-y.

Rule 5

$$\frac{\delta}{\delta u(x)}\int f(y)u(y)dy=f(x) \tag{F.8}$$

where $f(x)$ is a function.

Function and Functional Expansions

A function at a fixed point may be expanded in a power series about another fixed point. That is $f(x)$ may be expanded about the point y as

$$f(x) = f(y) + \sum_{n=1}^{N} \frac{(x-y)^n f^{(n)}(y)}{n!} + R_N \tag{F.9}$$

where $f^{(n)}$ denotes the nth derivative of the function f, R_N is the remainder after N terms. If the value of R_N tends to 0 as N tends to infinity, the above power series expansion is known as a Taylor series. It is implicit that the function f and its derivatives are defined at y. In operator form, the Taylor series would appear formally as

$$f(x) = \exp\left\{(x-y)\frac{d}{dy}\right\} f(y) \tag{F.10}$$

The equivalence of Eq. F.9 and Eq. F.10 can be seen by applying the usual expansion for $\exp(x)$.

Functional or Variational Taylor Series

A functional may be similarly expanded in a variational series. The variational series analog to a Taylor series is more complicated as one would expect. Specifically, a variational Taylor series expansion of the functional F with argument which is a function u when expanded about the function v takes the form

$$\begin{aligned} F[u] &= F[v] + \\ &\frac{1}{1!}\int (u(x') - v(x'))\frac{\delta F[v]}{\delta v(x')}dx' + \\ &\frac{1}{2!}\int\int (u(x') - v(x'))(u(x'') - v(x''))\frac{\delta^2 F[v]}{\delta v(x')\delta v(x'')}dx'dx'' + \\ &\frac{1}{3!}\int\int\int (u(x') - v(x'))(u(x'') - v(x''))(u(x''') - v(x''')) \\ &\qquad \frac{\delta^3 F[v]}{\delta v(x')\delta v(x'')\delta v(x''')}dx'dx''dx''' + \cdots + R_N \end{aligned} \tag{F.11}$$

As in the case of the function Taylor series, if the size of the term R_N tends to zero as N tends to infinity, the above functional series is known as a variational Taylor series. Size, in the case of this thesis, will be defined as the maximum of the magnitude of R_N. For example, suppose we have the functional $F[x] = x^3$ where x is any function from the collection $C_{[0,1]}$, where $C_{[0,1]}$ is the collection of all continuous functions defined on the closed interval [0, 1]. Clearly, the functions sin(x) and cos(x) are members of $C_{[0,1]}$. Therefore, the expansion of F[sin(x)] about F[cos(x)] is given formally by

$$
\begin{aligned}
&F[\sin(x)] = F[\cos(x)]+ \\
&\quad \frac{1}{1!}\int(\sin(x') - \cos(x'))\frac{\delta F[\cos(x)]}{\delta\cos(x')]}dx' + \\
&\quad \frac{1}{2!}\int\int(\sin(x') - \cos(x'))(\sin(x'') - \cos(x''))\frac{\delta^2 F[\cos(x)]}{\delta\cos(x')\delta\cos(x'')}dx'dx'' + \\
&\quad \frac{1}{3!}\int\int\int(\sin(x') - \cos(x'))(\sin(x'') - \cos(x''))(\sin(x''') - \cos(x''')) \\
&\qquad \frac{\delta^3 F[\cos(x)]}{\delta\cos(x')\delta\cos(x'')\delta\cos(x''')}dx'dx''dx''' + \ldots
\end{aligned}
\tag{F.12}
$$

The application of rules 1 through 5 will result in

$$
\begin{aligned}
&F[\sin(x)] = F[\cos(x)]+ \\
&\quad \frac{1}{1!}\int 3(\sin(x') - \cos(x'))\cos^2(x')\delta(x - x')dx' + \\
&\quad \frac{1}{2!}\int\int 6(\sin(x') - \cos(x'))(\sin(x'') - \cos(x''))\cos(x') \\
&\qquad \delta(x - x')\delta(x' - x")dx'dx'' + \\
&\quad \frac{1}{3!}\int\int 6(\sin(x') - \cos(x'))(\sin(x'') - \cos(x''))(\sin(x''') - \cos(x''')) \\
&\qquad \delta(x - x')\delta(x' - x")\delta(x'' - x''')dx'dx''dx''' + \ldots
\end{aligned}
\tag{F.13}
$$

Simplifying this equation results in

$$F[\sin(x)] = \cos^3(x) + 3\cos^2(x)(\sin(x) - \cos(x)) + 3\cos(x)(\sin(x) - \cos(x))^2 + (\sin(x) - \cos(x))^3 = \sin^3(x) \tag{F.14}$$

a tautology, which should not be surprising.

Appendix G. *Important terms*

Action functional along a path - Given the trajectory (also called the path) of a Newtonian particle, the action functional $S[x_2, x_1]$ is defined as,

$$S[x_2, x_1] = \int_{t_1}^{t_2} L(\dot{x}, x, t)dt \tag{G.1}$$

where x_1 is the initial location of the Newtonian particle at time t_1, x_2 is the final location of the Newtonian particle at time t_2, $L(\dot{x}, x, t)$ is the classical Lagrangian of the particle, The notion of an Action functional can be generalized beyond that of a Newtonian particle, by appropriately generalizing the Lagrangian of the particle.

Feynman integral - An integral over a function space. The measure of the integral is implicitly that of Richard Feynman's original construction. It is formally written as,

$$K[x_2, x_1] = \lim_{\epsilon \to 0} \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1\ \ terms} A^{-n} exp\left(\frac{j}{\hbar} S[x_2, x_1]\right) \prod_{m=1}^{n-1} dx_m \tag{G.2}$$

where x_1 is the initial location along the continuous trajectory, $S[x_2, x_1]$ is the action functional, A is defined as $\sqrt{\frac{j2\pi\hbar\epsilon}{m}}$, $\hbar$ = Planks constant divided by 2π $\epsilon = \frac{|t_1 - t_2|}{n}$ is a time subdivision.

It should be noted that the Feynman integral, unlike the sequential Wiener integral with a real variance parameter, does not possess a countable measure and therefore is not rigorously defined. However, this lack of rigor has not created any difficulties with the application of Feynman integrals to practical problems, it simply means that the mathematically rigorous rules governing integration in a function space are incomplete at present.

Fraunhofer approximation - A more restrictive approximation than the Fresnel approximation such that the Fresnel approximation may be approximated as a Fourier transform of the optical disturbance.

Fresnel approximation - An approximation of the Huygens-Fresnel integral for small angles about the mean direction of propagation and sufficient distance from the optical disturbance such that the following approximation holds,

$$\frac{\exp(jkz)}{j\lambda z}\int\int\left(\frac{exp(|\rho-\rho',z|)}{|\rho-\rho',z|}\right)d\rho' \simeq \frac{\exp(jkz)}{j\lambda z}\int\int exp\left(\frac{jk}{2z}|\rho-\rho'|^2\right)d\rho' \quad \text{(G.3)}$$

where $|\rho-\rho',z| = \sqrt{(\rho-\rho')^2+z^2}$, $|\rho-\rho'| = \sqrt{(\rho-\rho')^2}$, z is the distance form the source of the optical disturbance to the observation point, the integration region is implicitly $\mathbf{R}^2$.

Fresnel zone - A circular region, in a plane transverse to a point source along the mean direction of propagation. The radius of the (first) Fresnel zone is approximated as $\sqrt{\lambda z}$, where z is the distance from the point source to the transverse plane and λ is the carrier wavelength. It originated from the location of the first destructive interference zone and from analysis of weak propagation medium fluctuations. See the figure below.

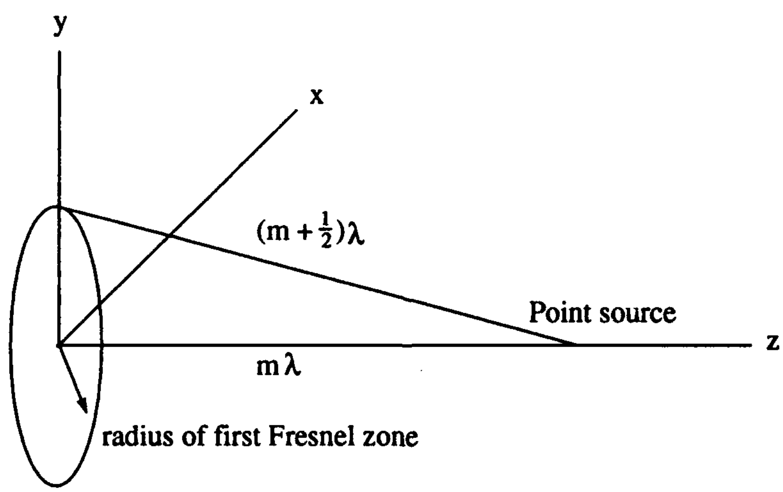


Figure G.1. First Fresnel zone

Function space - A collection of functions which is closed under (usually) addition and multiplication and has the appropriate identity elements and element inverses.

Functional - A mapping from a function space into the real numbers. An example of a functional is the the mapping,

$$X[f] = \begin{cases} \frac{\partial f}{\partial x}\Big|_{x=0} & \text{if } f(x) \text{ is differentiable at } 0 \\ 0 & \text{otherwise} \end{cases} \tag{G.4}$$

Another example is the mapping of the area under all real-valued bounded continuous functions defined on [0, 1] into $\mathbb{R}$. In this case, the functional A[f] could be defined as,

$$A[f] = \int_0^1 f(\tau)d\tau \tag{G.5}$$

Functional integral - Integration of a functional over a given function space. See Feynman integral and Wiener integral.

Green's function - A point source solution to an associated operator equation. For example, given $L[u(x)] = f(x)$, the associated Green's function, $G(x;\xi)$ when substituted for $u(x)$ will yield: $L[G(x;\xi)] = \delta(x-\xi)$ where, $\delta(x-\xi)$ is the dirac delta distribution. For wave propagation, the Green's function is the spherical wave given by $\frac{\exp(jk|r|)}{|r|}$.

Helmholtz equation - A reduced form of the wave equation. This equation takes the form,

$$(\nabla^2 + k^2 n^2(r,\omega))u(r,\omega) = f(r) \tag{G.6}$$

where ∇^2 is the three-dimensional Laplacian operator, $u(r,\omega)$ is a (possibly complex) wave amplitude, $n(r,\omega)$ is the index of refraction, r is a 3-dimensional position vector, k is $\frac{2\pi}{\lambda}$ and λ is the carrier wavelength, ω = ck is a fixed carrier angular frequency, c is the speed of light, $f(r)$ represents wave sources in the medium.

Inhomogeneity - A region of a medium, such as the atmosphere, which exhibits uniform conditions locally. These conditions may be slightly to vastly different from the local conditions only a short distance away.

Markov approximation - An approximation where a stochastic differential equation may be approximated as a markov process. This approximation enables the derivation of closed differential equations for the statistical moments of the independent variable of the original differential equation.

Markov process - An independent increment random or stochastic process which is conditionally dependent only on the most recent interval. That is, the conditional distribution of all past behavior intervals of the markov process is dependent only on the most recent interval.

Maxwell's equations - A set of electromagnetic field equations which describes the interaction and propagation of static and dynamic electric and magnetic fields as well as electromagnetic fields.

Measure - Rule by which differential contributions to an integral are weighted and added. It must be explicitly given, or in the case of Riemann integration implicitly given.

Method of smooth perturbations - A method used to obtain the complex wave amplitude in the parabolic equation. It is restricted to weak intensity fluctuations with the normalized intensity less than 0.3. See also Rytov's method.

Parabolic approximation of the wave equation - Also known as the parabolic equation - An approximation to the Helmholtz equation which results from restricting the class of admissible solutions to functions which vary slowly (with respect to the carrier frequency) in the mean direction of propagation. This simplification replaces the 3-dimensional Laplacian operator in the Helmholtz equation with a simpler 2-dimensional Laplacian operator in the transverse plane and a first derivative in the original mean direction of propagation.

Path integral - A functional integral. This integral is taken as the limit of a multidimensional integral which "samples" functional integrand at a finite number of locations. The limiting process allows the number of samples to become unbounded.

Perturbation methods - A mathematical method of solving equations which equates similar orders of magnitude in a small perturbation or deviation in the original equation. The equating of similar orders of magnitude in the perturbation term, allows the investigation of zeroth, first, second, and so on, order changes in the perturbation.

Polygonal curve - A piecewise linear continuous curve. Such a curve, in general, is not differentiable at each location where two consecutive line segments intersect. For example, in Fig. G the derivative fails to exist for the top most polygonal curve at locations a, b, c and d. Let $\{S_n\}_{n=1}^{\infty}$ denote the sequence composed of sets of points where the polygonal curve is recursively subdivided. Referring again to Fig. G let $S_1 = \{a, b, c, d\}$, $S_2 = \{e, f, g, h, i, j, k\}$, $S_3 = \{l, m, n, o, p, q, r, s, t, u, v, w, x\}$, and $S_4 = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y\}$. We see that by construction $S_1 \subset S_2 \subset S_3 \subset S_4$. We see that in the limit as n becomes unbounded, S_n will in general be a continuous curve which is not differentiable at any rational number. In other words, the curve represented by $\lim_{n\to\infty} S_n$ will differentiable everywhere except on a set of measure zero. Such a limiting curve represents the general differential contribution to the Feynman (path) integral just as the differential contribution $f(x + dx)dx$ does to the Riemann integral $\int f(x)dx$.

Quantum mechanics - Statistical study of kinematics of objects at the Fermi size (10^{-13} cm). Objects which greatly exceed this distance exhibit classical kinematics.

Rytov's approximation - Asymptotic expansion of the complex amplitude of the optical wave that maintains only the first term.

Rytov's Method - Also known as the method of smooth perturbations.

Scale size - Also known as: characteristic size, most commonly occurring size, representative size, expected size. In inhomogeneous mediums, the scale size refers to the size of the random inhomogeneities.

Scintillation - Observed intensity variations in an optical wave due to random constructive and destructive interference effects.

Scintillation Index - A normalization of the intensity variations. The scintillation index is denoted by β^2 and is defined as the ratio of the intensity variance to the mean intensity squared.

Strong intensity fluctuations - A condition in which the scintillation index exceeds 1.

Saturated Regime - Sub-region of the strong fluctuation regime where the scintillation index is asymptotically decreasing to 1. This region is characterized by the almost totally incoherent intensity contributions from incoming optical waves.

Scalar wave equation - A second order in time second order in space partial differential equation

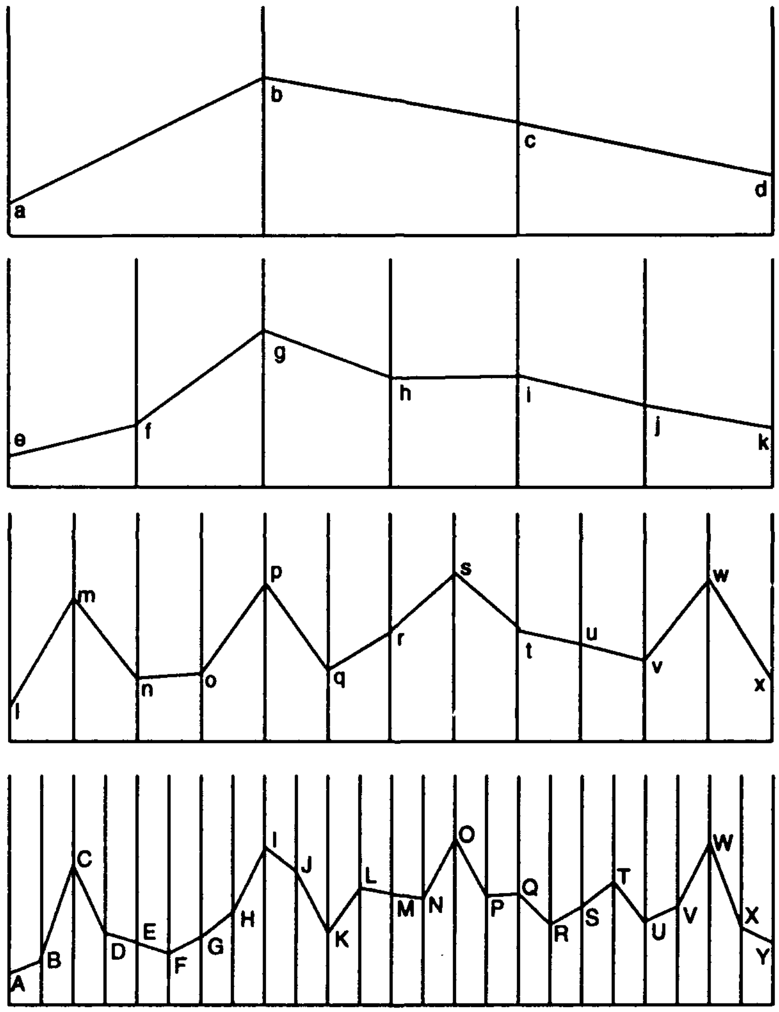


Figure G.2. Sequential evolution of a general path of the form,

$$\frac{\partial^2}{\partial x^2}T(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}T(x,t) \tag{G.7}$$

where $T(x,t)$ is the wave amplitude (possibly complex), c is a constant, x is the space variable, t is the time variable.

Stochastic scalar wave equation - A scalar valued wave equation augmented with non-deterministic parameters.

Stochastic wave equation - A vector valued wave equation augmented with non-deterministic parameters.

Taylor series - A power series expansion of a continuously differentiable function.

Weak intensity fluctuations - No firm definition exhists; however, the generally accepted defination is when the intensity variance normalized by the mean intensity squared is less than 1.

Wiener integral or sequential Wiener integral - A functional integral defined over Wiener measurc.

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SUPPLEMENTARY

INFORMATION

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