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**ON CONSISTENCY RELATIONS IN
NONLINEAR FRACTURE MECHANICS†**

by

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and

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February, 1986

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Abstract

A simple form of consistency relations between generalized forces and displacements for systems exhibiting power-law behavior is presented. The later discussion focuses on certain details regarding applications of the relations to nonlinear fracture mechanics, emphasizing the finite element analysis of a single edge-cracked strip subjected to remote tension under plane strain conditions.

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Introduction

→ Within the last decade there has been observed a serious computational effort towards obtaining numerical solutions to problems of nonlinear fracture mechanics. The handbook formatted reference (Kumar et al., 1981), gives extensive tabulations of important parameters (J -integral, CTOD, etc.) for various configurations, loadings and material behaviors, which makes it useful for engineering analysis, and the review of prior work in the field permits its use as a good starting point for further research. The later work of Shih and Needleman (1984) exposes some quantitative discrepancies between the two results. These discrepancies cause confusion and implicitly raise the legitimate question: 'Who is right?'. The purpose of this work is to help to resolve the matter of quality of reported data.

There are three parts in this paper. The first one focuses on the derivation of consistency relations between generalized forces and displacements in fairly general types of nonlinear systems, including the class of traction prescribed boundary-value problems for power-law isotropic materials, exhibiting in pure tension stress-strain behavior

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{\sigma}{\sigma_0} \right)^n, \quad (1)$$

where ϵ_0 and σ_0 are reference strain and stress values, and n is a material exponent varying from one for the linear material to infinity for the rigid plastic material.

The second section discusses some specific features of solutions for fracture mechanics configurations, stressing both the validity of the application of the consistency relations in principle and their potential use in the detection of errors.

The paper is concluded with a numerical example to which the consistency conditions are applied – the single edge-cracked strip subjected to remote tension under

plane strain conditions. The results are compared against the data reported by Kumar et al. (1981) and Shih and Needleman (1984).

Throughout the paper we use boldface letters for vectors, matrices and tensors. Dot stands for an appropriate inner product. All computations reported herein have been performed using the *ABAQUS* finite element program on a *DATA GENERAL MV-10000* computer.

Consistency Relations

Let us consider some body loaded by a set of generalized forces $\mathbf{Q}$. The set of generalized displacements is taken as derivable from the constitutive potential F , which depends on the overall geometry, material properties and the forces:

$$\mathbf{q} = \frac{\partial F}{\partial \mathbf{Q}}. \quad (2)$$

We confine our attention to the class of potentials which are both convex and homogeneous functions of $\mathbf{Q}$ degree $(n + 1)$:

$$F(\alpha \mathbf{Q}_1 + (1 - \alpha) \mathbf{Q}_2) \leq \alpha F(\mathbf{Q}_1) + (1 - \alpha) F(\mathbf{Q}_2), \quad (3)$$

$$\frac{\partial F}{\partial \mathbf{Q}} \cdot \mathbf{Q} = (n + 1) F. \quad (4)$$

In equation (3) it is understood that $\mathbf{Q}_1, \mathbf{Q}_2$ are arbitrary sets of generalized forces and $0 \leq \alpha \leq 1$. In the relation (4), we adopt Euler's Theorem on homogeneous functions as a definition. One of the important properties of convex functions is the positive semidefiniteness of the Hessian matrix Γ (tangent compliance in the force-displacement framework):

$$\Gamma = \frac{\partial^2 F}{\partial \mathbf{Q}^2}. \quad (5)$$

The inner product of Γ with $Q \otimes Q$ leads to an important conclusion that the potential functions F defined above are nonnegative.

The alternative definition of a homogeneous function allows us to reduce the effective number of independent variables by one; the most general representation serving our purposes is:

$$F(Q) = f(x) [r(Q)]^{n+1}, \quad (6)$$

where $f(x)$ is a nonnegative function of the reduced set x , which dimension is one less than that of Q , and $r(Q)$ is an arbitrarily defined norm in the Q -space. The key point of the forthcoming derivation is that the generalized displacements in equation (2) are the partial derivatives of the same function and, therefore, should be related. We limit our demonstration to the case of two generalized forces. The extension to a multidimensional set is transparent and will not be considered here. In this case equation (6) may be reduced to

$$F = \frac{1}{n+1} f(x) Q^{n+1}, \quad Q := Q_1 > 0, \quad x := \frac{Q_2}{Q_1}. \quad (7)$$

Straightforward differentiation leads to the following expressions for q and Γ :

$$q = \left(\frac{(n+1)f - xf'}{f'} \right) \frac{Q^n}{n+1}, \quad (8)$$

$$\Gamma = \left(\begin{array}{cc} (n+1)nf - 2nxf' + x^2f'' & \text{symmetry} \\ nf' - xf'' & f'' \end{array} \right) \frac{Q^{n-1}}{n+1}, \quad (9)$$

where prime denotes differentiation with respect to x . To enforce positive semidefiniteness of Γ we require both Γ_{22} and determinant of Γ to be nonnegative, which gives

$$f'' \geq 0, \quad (10)$$

$$ff'' - \frac{n}{n+1} f'^2 \geq 0. \quad (11)$$

But since the function $f(x)$ is nonnegative, condition (11) is sufficient to guarantee positive semidefiniteness.

Introduction of an auxiliary function y , in a way a conjugate of x , and given by

$$y := \frac{q_2}{q_1} = \frac{f'}{(n+1)f - xf'}, \quad (12)$$

permits the rewriting of condition (11) in a simpler form:

$$y' \geq 0. \quad (13)$$

Now, instead of considering the related pair of functions $f(x)$ and $f'(x)$ to describe the generalized displacements, we can use the pair $f(x)$ and $y(x)$ related via (12). If values of $f(x)$ and $y(x)$ are given at some point, say $x = x_0$, $f(x_0) = f_0$, $y(x_0) = y_0$, then integration of (12) leads to

$$f(x) = f_0 \exp \left((n+1) \int_{x_0}^x \frac{y(t) dt}{1 + ty(t)} \right). \quad (14)$$

But for any value of $x \geq x_0$ with $x_0 \leq t \leq x$ we may write the following inequality

$$\frac{y_0}{1 + ty_0} \leq \frac{y(t)}{1 + ty(t)} \leq \frac{y(x)}{1 + ty(x)}, \quad (15)$$

which, after the integration, gives us the main result of this section:

$$\frac{1 + xy_0}{1 + x_0y_0} \leq \left(\frac{f}{f_0} \right)^{\frac{1}{n+1}} \leq \frac{1 + xy(x)}{1 + x_0y(x)} \quad (16)$$

It is important to mention that (11) and (13) are equivalent only if $q_1 > 0$ for all the points within the interval $[x_0, x]$, which implicitly imposes conditions on both x_0 and x .

Application to Fracture Mechanics

A broad class of problems of fracture mechanics can be described by a generic problem - one is given a configuration containing a crack, loaded by forces Q , and is asked to determine the value of the J -integral (Rice, 1968) to characterize the local fields (Hutchinson, 1968, Rice and Rosengren, 1968) and q - displacements, at the remote distances, due to the introduction of the crack,

$$q = q^t - q^h, \quad (17)$$

where q^t and q^h stand for generalized displacements in two auxiliary problems. The first one is a prescribed traction boundary-value problem for the given configuration. The second one is identical, but there is no crack.

The constitutive potential $F(Q)$ becomes equal to the difference in the complementary energies of the auxiliary problems:

$$F(Q) = \frac{1}{n+1} \int_V (\mathbf{E}^t \cdot \mathbf{T}^t - \mathbf{E}^h \cdot \mathbf{T}^h) dV, \quad (18)$$

where $\mathbf{T}$ and $\mathbf{E}$ stand for the stress and strain tensors, respectively.

The connection between this class of problems and the one described in the first section is obvious but, nevertheless, there are some important details to be considered both due to the necessity to perform numerical analysis and the special features of the fracture mechanics problems *per se*.

At first, we would like to address the matter of possible loss of homogeneous structure of $F(Q)$. In principle, the linearity of equilibrium, compatibility equations, and boundary conditions, combined with the constitutive law, requires an analytical solution to the problem to be homogeneous. But if we have to model an incompressible

material and, therefore, employ a penalty procedure in numerical analysis, we can encounter deviations from homogeneity. The simplest way to introduce a penalty is by linear relations between isotropic components of the stress and strain tensors:

$$\text{tr} \mathbf{T} = \frac{K \sigma_0}{3 \epsilon_0} \text{tr} \mathbf{E}. \quad (19)$$

We argue, hueristically at best, that as K tends to infinity, the influence of the hydrostatic stress diminishes, and material response tends toward incompressibility. An attempt to conserve homogeneity by using a power-law penalty may easily lead to numerical problems as we operate with large numbers. Therefore, we can claim that degree of compressibility and deviation from the homogeneous structure are implicitly related, and in the limiting case of a large K , material tends towards both homogeneous and incompressible response. The straightforward application of (16) may be of use in detecting errors in interpretation of penalty term for sophisticated variational formulations combining both regular displacement based and hybrid (displacement and pressure) formulations of the finite element method, as implemented, for example, in *ABAQUS*. Later on we present a numerical example of this relation.

The question of convexity is especially interesting for the fracture mechanics problems. The difficulty here is that there is no single boundary-value problem which may be directly analyzed to determine $F(\mathbf{Q})$, but rather two auxilary problems. Convexity of each problem is guaranteed by the monotonicity of the stress-strain curve or, more precisely, by the convexity of the strain energy density function (Marsden and Hughes, 1983). The function F is nonnegative (Rice, 1968), but is not necessarily convex. The straightforward mathematical example is $F(Q_1, Q_2) = |Q_1 Q_2|^{\frac{n+1}{2}}$.

In order to demonstrate the possible loss of convexity in a physical problem, we address the case of a penny-shaped crack embedded in an infinite isotropic power-law

matrix subjected to axisymmetric remote loading, characterized by the axial and radial stresses Q_1 and Q_2 , respectively. We consider three materials which can be generated from (1) as tensorial extensions. The first one is the incompressible material given by

$$\mathbf{E} = \frac{3}{2}\epsilon_0 \left(\frac{\sigma}{\sigma_0} \right)^{n-1} \frac{\mathbf{S}}{\sigma_0}, \quad (20)$$

$$\sigma = \left(\frac{3}{2} \mathbf{S} \cdot \mathbf{S} \right)^{\frac{1}{2}}, \quad (21)$$

where $\mathbf{S}$ is the stress deviator and we take $n = 3$. The second material is linear elastic ($n=1$) with Young's modulus $E = \sigma_0/\epsilon_0$ and Poisson's ratio ν . The last case is an isotropic, power-law material, *compressible* material, which constitutive equations are derived from (20),(21) by substitution of the tensor $\mathbf{T}$ itself for the stress deviator $\mathbf{S}$. The values of σ_0, ϵ_0 and ν are not important as they are appearing as constant multipliers.

The first example is treated in detail by He and Hutchinson (1981), (1983); for the case $Q_1 > 0, Q_1 > Q_2$. The suggested functional fit to the numerical solution is given by

$$F(\mathbf{Q}) = 4\sigma_0\epsilon_0 a^3 \left(1 + \frac{3}{n} \right)^{-\frac{1}{2}} \cdot \left(\frac{Q_1 - Q_2}{\sigma_0} \right)^{n-1} \cdot \left(\frac{Q_1}{\sigma_0} \right)^2, \quad (22)$$

where a is the crack radius. The above expression gives us concave $F(\mathbf{Q})$, which can be seen directly from deriving the Hessian matrix. The formula is derived from a perturbation technique, agrees well with numerical solutions providing $x < 0.6$, and for larger x it fails to give an accurate estimate (He and Hutchinson, 1983). The finite element analysis of the problem, which we have conducted, shows that at approximately the same point ($x \approx .6$) the complete numerical solution gains convexity; therefore, we may conclude that there is conditional convexity in this case.

The well-known solution (Sneddon, 1964) for the isotropic linear material coincides with (22) for $n=1$ and $\nu = \frac{1}{2}$, and is given by

$$F(Q) = \frac{8(1-\nu^2)}{3E} a^3 Q_1^2. \quad (23)$$

This expression gives only one non-zero component, Γ_{11} , of the Hessian matrix, which, of course, retains positive semidefiniteness, but relations (16) degenerate to triviality.

By performing a finite element analysis we find that the third material gives us an unconditionally convex potential, and the main result of the first section is of use.

In the above examples we have encountered three possible situations for two-dimensional Q-space; namely, relations (16) are relevant at some points in the domain, they are relevant throughout Q-space, or they can not be applied at all. The heuristic conclusion may be drawn if we consider the dimensions of the Q-space of the two auxiliary and the main problem for all three materials. In the first example the solution of the problem without the crack depends only on the applied equivalent Mises stress $\bar{\sigma}$. If we introduce a new set of generalized forces, namely

$$Q'_1 = \frac{Q_1 + 2Q_2}{3} \quad Q'_2 = Q_1 - Q_2, \quad (24)$$

where the first equation of (24) defines applied hydrostatic pressure and the second the equivalent Mises stress, then the dimension of Q'-space is one in the context of the problem without the crack. The main and the other auxiliary problem, on the other hand, exhibit two-dimensional load space. It is clear from (23) that an analogous situation occurs with the linear elastic material, except that the main problem is the one which has the reduced space. The potential due to the introduction of the crack does depend on the single force Q_1 , but, by superposition, is independent of Q_2 . Only for the last example is there a truly two-dimensional Q-space for all three problems.

It is clear that in the first two examples, there exists some set Q , which essentially forms the null-space of $F(Q)$ for the main or the auxiliary problem. Obviously, it is hydrostatic pressure for the incompressible material and stressing parallel to the crack plane for the main problem in the linear case. The loosely defined induction is, that in order to employ (16), all three problems must have the two-dimensional Q -space. A more rigorous statement would, perhaps, require the definition of F on the complement of the null-space.

The conclusion is hueristic but, nevertheless, seems to rehabilitate the 'misbehavior' of the otherwise mathematically 'loyal' equations.

It is important to note that this conclusion does not put any question marks on the substantial body of theoretical and computational effort (Budiansky et al. 1981, Rafalsky, 1985) in terms of the derivations and the implementation of a variational principle for determination of F directly, because here we deal only with the generalized force space of boundary tractions, rather than with a function space of Ritz procedure.

To conclude this section we would like to mention the possibility of including the J -integral into a gradient structure analogous to (2), which leads to the correlation between near and far fields and, of course, to another group of consistency relations. The procedure that was initially suggested by Parks et al. (1983) and later was applied by Shih and Needleman (1984), is essentially based on finite difference approximation of the gradient scheme for planar and axysimmetric problems.

Numerical Example

A single edge-cracked strip, *Figure 1*, subjected to a remote loading under plane strain conditions is considered. The dimensions are:

$$a = 10, \quad b = 20, \quad L = 60.$$

The constitutive behavior is modeled by (19)-(21) with numerical values of material constants:

$$\sigma_0 = 1, \quad \epsilon_0 = 1, \quad K = 10^6, \quad n = 5.$$

The values of σ_0 and ϵ_0 do not correspond to any real material, but as long as we are able to scale solutions, the idea of operating with more computationally convenient numbers is rather helpful. The finite element discretization is given in *Figure 2*. The plane strain eight-node hybrid element with nine integration points and a bi-linear interpolation for pressure is employed.

The remote forces per unit thickness are tensile load $N = 20$ and varying *positive* bending moment M , tending to close the crack. We are interested in the pure tension solution and apply bending only for the purpose of simulation of conditions (16). The generalized forces are identified as $Q_1 := N$, $Q_2 := M/b$ with corresponding pair of generalized displacements $q_1 := \delta$ and $q_2 := \theta b$. It is understood that the generalized displacements refer to the contribution due to the crack, as discussed in the previous section.

The potential $F(Q)$ and the J -integral are taken in convenient dimensional forms

$$F(Q) = \frac{1}{n+1} \sigma_0 \epsilon_0 b^2 f(x) \left(\frac{N}{\sigma_0 b} \right)^{n+1}, \quad (25)$$

$$J = \sigma_0 \epsilon_0 b h(x) \left(\frac{N}{\sigma_0 b} \right)^{n+1}. \quad (26)$$

The results of the computations are given in *Table 1*. We conclude that the current analysis gives results close to those of Shih and Needleman (1984) for the values of the rotation and the J -integral; the displacement is somewhere between the earlier reported data. It is worth mentioning that the main point of discrepancy between the previous data is in the near field quantities; therefore, our results rather support the data reported by Shih and Needleman (1984).

The next part of our analysis is concerned with the simulation of data for relations (16). It is obvious that in this example problem, consistency relations can be of importance. But on the other hand, if values of x and x_0 are sufficiently close we expect the interference of the numerical noise in the tabulated data to be substantial; therefore, we have to decide on a minimum value, x_{min} , such that for all $x - x_0 \geq x_{min}$, consistency relations give us reasonable conclusions.

We adopt a very simplified estimate. The difference between bounding terms in (15) at the pure tension limit $x_0 = 0$ is $(x - x_0)(y - y_0)$, and this should remain positive in the most unfavorable case. As we have four kinematical data entries in the above formula, then we require

$$y - y_0 \geq 4e_q, \quad (27)$$

where e_q is the relative error in q . From the homogeneity and dimensional considerations we can take:

$$e_q = ne_R, \quad (28)$$

where e_R is the relative error in nodal reaction force in the virtual work sense, and in the analysis with the *ABAQUS* program, the maximum value of this error has been set at 10^{-4} . In expression (28), we have neglected a dimensionless constant multiplier expected to be of order unity. Relations (27),(28) form the implicit conditions on x_{min} . The results of computations are given in Table 2 and give $x_{min} = 8.10^{-4}$, which is in a fair agreement with conditions (27),(28) which give the value of $x_{min} = 3.10^{-4}$. Therefore, we may claim that our data is *probably acceptable*.

The section is concluded by the consideration of the penalty term as a possibility for the loss of homogeneity. In this set of computations we take K varying from 1 to 10^6 , keeping the rest of the material constants to be the same.

The results of *Table 3* show that the maximum difference for various $K \geq 100$, in x_{min} , J -integral and q , only appears in the fourth digit. These results suggest that one might be able to use moderate values of the penalty K . There are clear advantages to such a procedure, because the computations converge more rapidly and there is less chance to encounter numerical difficulties. For small values of K (below 100) the most sensitive parameter turns out to be x_{min} , though the physical quantities remain within an acceptable variation.

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Reference	f	y	h
[1]	83.90	-2.393	309.9
[2]	113.1	-2.430	407.4
current	99.00	-2.677	390.2

Table 1. Pure Tension Data, $n = 5, K = 10^6, a/b = 1/2$.

x	f	y	Lower Bound	Intermediate Value	Upper Bound
.0000	99.039237	-2.677448	—	—	—
.0001	98.881223	-2.676500	1.000268	1.000266	1.000268
.0002	98.720673	-2.675612	1.000535	1.000537	1.000536
.0003	98.564139	-2.674622	1.000803	1.000802	1.000804
.0004	98.405337	-2.673701	1.001071	1.001071	1.001072
.0005	98.247815	-2.672751	1.001338	1.001338	1.001341
.0006	98.090018	-2.671816	1.001606	1.001606	1.001609
.0007	97.929625	-2.670958	1.001873	1.001880	1.001878
.0008	97.776636	-2.669905	1.002140	1.002141	1.002147
.0009	97.618779	-2.668991	1.002408	1.002411	1.002416
.0010	97.461915	-2.668057	1.002675	1.002679	1.002685

Table 2. Combined Loading Data, $K = 10^6, n = 5, a/b = 1/2$.[†]

$\log K$	h	f	y	$x_{\min}$
0	393.7	101.8	-2.644	.0060
1	390.5	99.30	-2.674	.0012
2	390.3	99.05	-2.676	.0008
4	390.2	99.05	-2.677	.0008
6	390.2	99.04	-2.677	.0008

Table 3. Pure Tension Data for Variable Penalty Term, $x = 0, n = 5, a/b = 1/2$.

[†] In the upper block of the table, relations (16) are not valid, in the next block they should be valid according to (27),(28), and in the last part relations (16) are unconditionally valid.

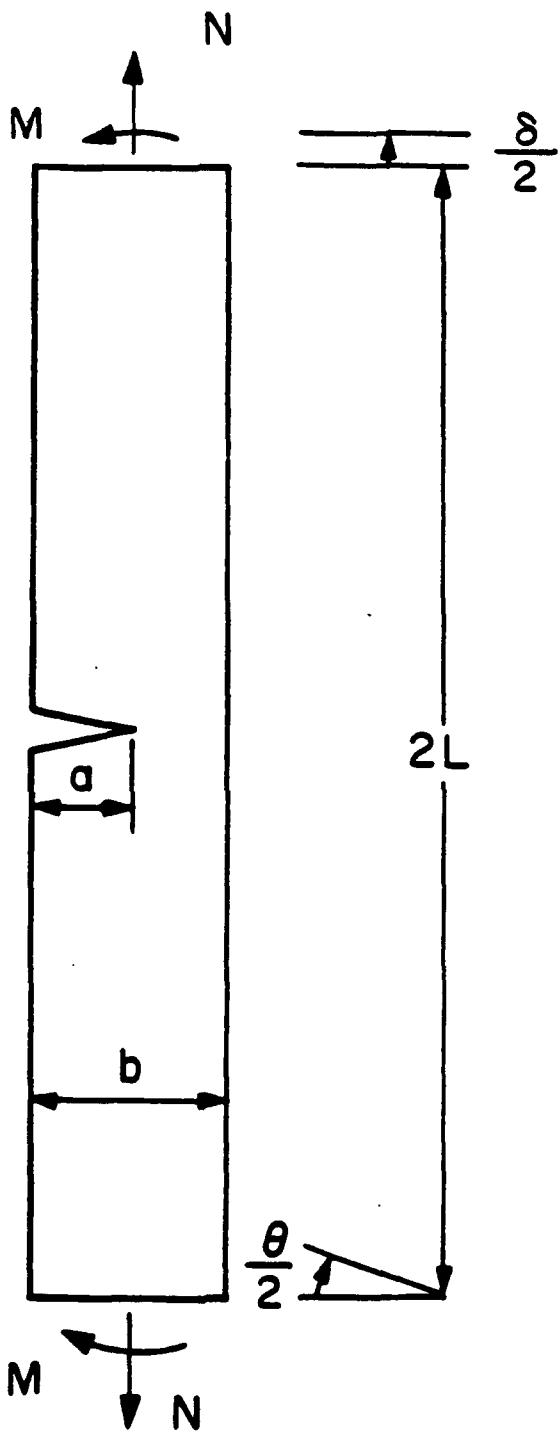


Figure 1. Single Edge-Cracked Strip

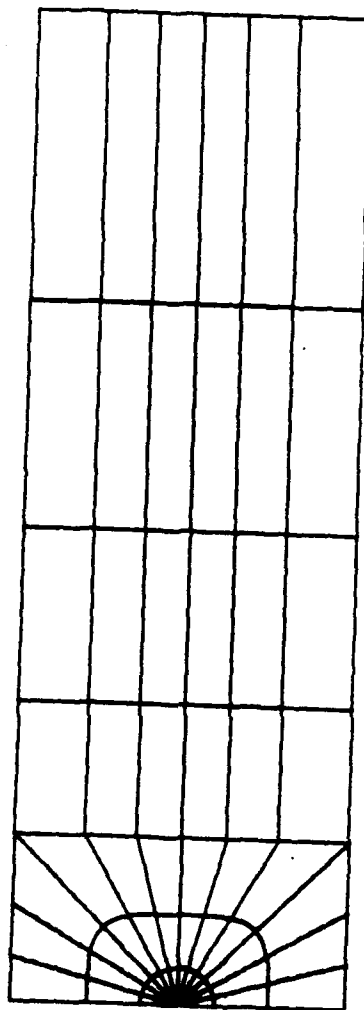


Figure 2. Finite Element Mesh