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MELBOURNE, VICTORIA

Aircraft Structures Technical Memorandum 539

DETERMINATION OF ANTISYMMETRIC END LOADS FROM BULK STRESSES ON A RECTANGULAR PRISM

by

J. BENNETT

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SUMMARY

A mathematical method is presented for the determination of the normal antisymmetric end loads from known values of the bulk stress on a lateral surface of a three dimensional rectangular prism. This extends previous work which considered only the symmetric loading case. A computer program has been developed to implement this method, and results for an arbitrarily chosen case are presented.

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CONTENTS

SYMBOLS

x, y, z	Cartesian co-ordinate axes
$2a, 2b, 2c$	Length, depth, breadth of the prism
ν	Poisson's ratio
$\sigma_x, \sigma_y, \sigma_z$	Normal stress components in the x, y, z directions
$2k_1b, 2k_2c$	Lengths of loaded portion in the y and z directions
H_{mN}, K_{Nl}, L_{lm}	Fourier coefficients (Loading Antisymmetric in y)
l, m, N	Indicies of the Fourier series (Loading Antisymmetric in y)
H_{Mn}, K_{nl}, L_{lM}	Fourier coefficients (Loading Antisymmetric in z)
l, M, n	Indicies of the Fourier series (Loading Antisymmetric in z)
$f(y, z)$	Stress distribution function on the faces $x = \pm a$

1. INTRODUCTION

The stress distribution in rectangular prisms subjected to certain normal end forces on its two opposite faces is a fundamental problem in Applied Mechanics. Reference [1] supplies a method for calculating a symmetric end loading, assuming that the bulk stress $(\sigma_x + \sigma_y + \sigma_z)$ on a lateral surface of the rectangular prism is known (i.e. measurable). Such a measurement could be obtained using advanced thermal emission techniques. The aim of this paper is to extend this concept to the case where an antisymmetric end loading is applied.

A method for calculating the bulk stresses for a given antisymmetric end loading is presented in [2]. This will be referred to as the Direct solution. In the Inverse method the end loading will be determined from a known distribution of bulk stress values on a lateral surface. In this case, bulk stresses on the lateral surface will be simulated using the Direct solution.

Figure 1 shows the orientation of the rectangular prism under consideration. The equations supplied in [2] allow calculation of the stress distribution on a lateral surface for an applied loading which is antisymmetric in y and symmetric in x and z. This paper extends this solution to calculate the stress distribution on the same lateral surface for a loading antisymmetric in z and symmetric in x and y. Inverse solutions for both loading cases have been derived. A single Inverse solution could be used to determine a loading antisymmetric in y from the stress distribution on the face $y = b$, or to determine a loading antisymmetric in z from the stress distribution on the face $z = c$. However, two separate solutions are neccessary to determine the applied antisymmetric loading if only one lateral surface is available (or has been prepared) for bulk stress measurement by thermal emission.

2. DIRECT SOLUTION

The bulk stresses on the surface $y = +b$ are required for the case of a rectangular prism subjected to certain end forces. The two end loading cases considered are:

(1) Loading antisymmetric in y, symmetric in x and z

(2) Loading antisymmetric in z, symmetric in x and y

2.1 Loading Antisymmetric in Y

First define the Galerkin vector $\mathbf{F}$.

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

where

$$F_x = \sum_M \sum_n \frac{H_{Mn}a}{\alpha_{Mn}^3 \cosh \alpha_{Mn}a}[\alpha_{Mn}x \cosh \alpha_{Mn}x - (2\nu + \alpha_{Mn}a \coth \alpha_{Mn}a) \sinh \alpha_{Mn}x] \times$$
$$\times \cos \frac{n\pi z}{c} \sin \frac{M\pi y}{2b}$$

$$F_y = \sum_n \sum_l \frac{K_{nl}b}{\beta_{nl}^3 \sinh \beta_{nl}b}[\beta_{nl}y \sinh \beta_{nl}y - (2\nu + \beta_{nl}b \tanh \beta_{nl}b) \cosh \beta_{nl}y] \times$$
$$\times \cos \frac{n\pi z}{c} \cos \frac{l\pi x}{a}$$

$$F_z = \sum_l \sum_M \frac{L_{lM}c}{\gamma_{lM}^3 \cosh \gamma_{lM}c}[\gamma_{lM}z \cosh \gamma_{lM}z - (2\nu + \gamma_{lM}c \coth \gamma_{lM}c) \sinh \gamma_{lM}z] \times$$
$$\times \cos \frac{l\pi x}{a} \sin \frac{M\pi y}{2b}$$

and

$$\begin{aligned} \alpha_{Mn}^2 &= \left(\frac{M\pi}{2b}\right)^2 + \left(\frac{n\pi}{c}\right)^2 \\ \beta_{nl}^2 &= \left(\frac{n\pi}{c}\right)^2 + \left(\frac{l\pi}{a}\right)^2 \\ \gamma_{lM}^2 &= \left(\frac{l\pi}{a}\right)^2 + \left(\frac{M\pi}{2b}\right)^2 \end{aligned} \tag{1}$$

where

$$l, n = 0, 1, 2, 3... \qquad \text{and} \qquad M = 1, 3, 5, 7...$$

The stress components σ_x and τ_{yz} are then related to $\mathbf{F}$ by the following equations.

$$\sigma_x = 2(1-\nu)\frac{\partial}{\partial x}\nabla^2 F_x + \left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right) \operatorname{div} \mathbf{F}, \tag{2}$$

$$\tau_{yz} = (1-\nu)\left(\frac{\partial}{\partial z}\nabla^2 F_y + \frac{\partial}{\partial y}\nabla^2 F_z\right) - \frac{\partial^2}{\partial y \partial z}\operatorname{div} \mathbf{F}. \tag{3}$$

This gives:

$$\begin{aligned}
\sigma_x = &\sum_M \sum_n \frac{H_{Mn} a}{\cosh \alpha_{Mn} a}[(1+\alpha_{Mn} a \coth \alpha_{Mn} a)\cosh \alpha_{Mn} x - \alpha_{Mn} x \sinh \alpha_{Mn} x] \\
&\times \sin\frac{M\pi y}{2b}\cos\frac{n\pi z}{c} + \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 c^2 \sinh \beta_{nl} b}\left[2\nu n^2\pi^2 \sinh \beta_{nl} y + \left(\frac{l\pi c}{a}\right)^2\right. \\
&\left.\times [(1-\beta_{nl} b \tanh \beta_{nl} b)\sinh \beta_{nl} y + \beta_{nl} y \cosh \beta_{nl} y]\right] \cos\frac{n\pi z}{c}\cos\frac{l\pi x}{a} \\
&+ \sum_l \sum_M \frac{L_{lM} c}{\gamma_{lM}^2 b^2 \cosh \gamma_{lM} c}\left[2\nu\frac{M^2\pi^2}{4}\cosh \gamma_{lM} z + \left(\frac{l\pi b}{a}\right)^2 [(1-\gamma_{lM} c \coth \gamma_{lM} c)\right. \\
&\left.\times \cosh \gamma_{lM} z + \gamma_{lM} z \sinh \gamma_{lM} z]\right] \cos\frac{l\pi x}{a}\sin\frac{M\pi y}{2b}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\tau_{yz} = &-\sum_M \sum_n \frac{H_{Mn} a M n \pi^2}{2\alpha_{Mn}^2 bc \cosh \alpha_{Mn} a}\left[[1-(2\nu + \alpha_{Mn} a \coth \alpha_{Mn} a)]\cosh \alpha_{Mn} x\right. \\
&\left.+ \alpha_{Mn} x \sinh \alpha_{Mn} x\right] \cos\frac{M\pi y}{2b}\sin\frac{n\pi z}{c} + \sum_n \sum_l \frac{K_{nl} n\pi b}{\beta_{nl} c \sinh \beta_{nl} b}\left[\beta_{nl} y \sinh \beta_{nl} y\right. \\
&\left.- \beta_{nl} b \tanh \beta_{nl} b \cosh \beta_{nl} y\right] \sin\frac{n\pi z}{c}\cos\frac{l\pi x}{a} + \sum_l \sum_M \frac{L_{lM} M\pi c}{2\gamma_{lM} b \cosh \gamma_{lM} c} \\
&\times \left[\gamma_{lM} c \coth \gamma_{lM} c \sinh \gamma_{lM} z - \gamma_{lM} z \cosh \gamma_{lM} z\right] \sin\frac{M\pi y}{2b}\cos\frac{l\pi x}{a}
\end{aligned} \tag{5}$$

Expressions for $\sigma_y, \sigma_z, \tau_{xy}$ and τ_{xz} can be written in a similar manner.

With reference to Figure 1, the following boundary conditions apply.

$$\begin{array}{lllll}
\text{on} & y = \pm b\,; & \sigma_y = 0, & \tau_{xy} = 0 & \text{and} \quad \tau_{yz} = 0, \\
\text{on} & z = \pm c\,; & \sigma_z = 0, & \tau_{xz} = 0 & \text{and} \quad \tau_{yz} = 0, \\
\text{on} & x = \pm a\,; & \sigma_x = f_2(y,z), & \tau_{xy} = 0 & \text{and} \quad \tau_{xz} = 0.
\end{array} \tag{6}$$

All the shear stress boundary conditions are exactly satisfied by (5) and similar expressions for τ_{xy} and τ_{xz}. The *normal stress* boundary conditions are approximately satisfied by equating low order Fourier terms in equation (4), and in similar expressions for σ_y and σ_z. Taking the double Fourier transforms of these three equations (with their approximated normal stress boundary conditions) gives the expressions (7),(8) and (9).

$$
\begin{aligned}
&\sum_M \frac{4H_{Mn}\delta_1 \tanh \alpha_{Mn}a(-1)^{l+(M-1)/2}}{\alpha_{Mn}b(\alpha_{Mn}^2 a^2 + l^2\pi^2)^2}\left[\nu\left(\frac{n\pi a}{c}\right)^2(\alpha_{Mn}^2 a^2 + l^2\pi^2)\right.\\
&\left.+\left(\frac{M\pi a}{2b}\right)^2 l^2\pi^2\right] + K_{nl}\delta_2\left(1-\frac{2\beta_{nl}b}{\sinh 2\beta_{nl}b}\right)\\
&+\sum_M \frac{4L_{lM}\delta_3 \tanh \gamma_{lM}c(-1)^{n+(M-1)/2}}{\gamma_{lM}b(\gamma_{lM}^2 c^2 + n^2\pi^2)^2}\left[\nu\left(\frac{l\pi c}{a}\right)^2(\gamma_{lM}^2 c^2 + n^2\pi^2)\right.\\
&\left.+\left(\frac{M\pi c}{2b}\right)^2 n^2\pi^2\right] = 0
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
&\delta_1 = 2 \quad \text{when} \quad n = 0, \quad \delta_1 = 1 \quad \text{when} \quad n \neq 0\\
&\delta_2 = 2 \quad \text{when} \quad n = 0 \quad \text{or} \quad l = 0, \quad \delta_2 = 1 \quad \text{when} \quad n \neq 0, \quad l \neq 0\\
&\delta_3 = 2 \quad \text{when} \quad l = 0, \quad \delta_3 = 1 \quad \text{when} \quad l \neq 0
\end{aligned}
$$

$$
\begin{aligned}
&\sum_n \frac{4H_{Mn} \tanh \alpha_{Mn}a(-1)^{l+n}}{\alpha_{Mn}c(\alpha_{Mn}^2 a^2 + l^2\pi^2)^2}\left[\nu\left(\frac{M\pi a}{2b}\right)^2(\alpha_{Mn}^2 a^2 + l^2\pi^2)\right.\\
&\left.+\left(\frac{n\pi a}{c}\right)^2 l^2\pi^2\right] + \sum_n \frac{4K_{nl}\delta_1 \coth \beta_{nl}b(-1)^{n+(M-1)/2}}{\beta_{nl}c\left[\beta_{nl}^2 b^2 + \frac{M^2\pi^2}{4}\right]^2}\\
&\times\left[\nu\left(\frac{l\pi b}{a}\right)^2\left(\beta_{nl}^2 b^2 + \frac{M^2\pi^2}{4}\right) + \left(\frac{n\pi b}{c}\right)^2\frac{M^2\pi^2}{4}\right]\\
&+ L_{lM}\delta_1\left(1+\frac{2\gamma_{lM}c}{\sinh 2\gamma_{lM}c}\right) = 0
\end{aligned}
\tag{8}
$$

where

$$
\delta_1 = 2 \quad \text{when} \quad l = 0, \quad \delta_1 = 1 \quad \text{when} \quad l \neq 0
$$

$$
\begin{aligned}
&H_{Mn}\delta_1\left(1+\frac{2\alpha_{Mn}a}{\sinh 2\alpha_{Mn}a}\right)+\sum_l \frac{4K_{nl}\delta_1 \coth\beta_{nl}b(-1)^{l+(M-1)/2}}{\beta_{nl}a\left[\beta_{nl}^2b^2+\frac{M^2\pi^2}{4}\right]^2} \\
&\times\left[\nu\left(\frac{n\pi b}{c}\right)^2\left(\beta_{nl}^2b^2+\frac{M^2\pi^2}{4}\right)+\left(\frac{l\pi b}{a}\right)^2\frac{M^2\pi^2}{4}\right] \\
&+\sum_l \frac{4L_{lM}\tanh\gamma_{lM}c(-1)^{l+n}}{\gamma_{lM}a(\gamma_{lM}^2c^2+n^2\pi^2)^2}\left[\nu\left(\frac{M\pi c}{2b}\right)^2(\gamma_{lM}^2c^2+n^2\pi^2)\right. \\
&\left.+\left(\frac{l\pi c}{a}\right)^2 n^2\pi^2\right]=J_{Mn}
\end{aligned}
\tag{9}
$$

where

$$\delta_1 = 2 \quad \text{when} \quad n = 0, \quad \delta_1 = 1 \quad \text{when} \quad n \neq 0$$

and

$$J_{Mn} = \frac{1}{abc}\int_{-b}^{+b}\int_{-c}^{+c} f_2(y,z)\sin\frac{M\pi y}{2b}\cos\frac{n\pi z}{c}\,dy\,dz$$

Equations (7),(8) and (9) may then be solved simultaneously for the unknown Fourier coefficients. The stress at any point in the prism may then be calculated by substituting the Fourier coefficients into equations (4),(5) and similar.

Since the given equations are valid only for self equilibrating end loads, it is necessary to modify the required end loading $f(y,z)$ so that the net force and moment acting on each end face is zero. This produces a modified loading distribution $f_2(y,z)$.

$$f_2(y,z) = \left[f(y,z) - \frac{3My}{4cb^3}\right] \tag{10}$$

where M is the total moment due to $f(y,z)$, i.e.

$$M = \int_{-b}^{+b}\int_{-c}^{+c} f(y,z)y\,dy\,dz \tag{11}$$

The previous equations calculate the stresses due to the modified loading $f_2(y,z)$. The stress component σ_x, calculated from equation (4), is a Fourier series approximation to $f_2(y,z)$.

2.2 Loading Antisymmetric in Z

Again define the Galerkin vector $\mathbf{F}$.

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

where

$$F_x = \sum_m \sum_N \frac{H_{mN}a}{\alpha_{mN}^3 \cosh \alpha_{mN}a}[\alpha_{mN}x \cosh \alpha_{mN}x - (2\nu + \alpha_{mN}a \coth \alpha_{mN}a) \sinh \alpha_{mN}x] \times$$
$$\times \cos \frac{m\pi y}{b} \sin \frac{N\pi z}{2c}$$

$$F_y = \sum_N \sum_l \frac{K_{Nl}b}{\beta_{Nl}^3 \cosh \beta_{Nl}b}[\beta_{Nl}y \cosh \beta_{Nl}y - (2\nu + \beta_{Nl}b \coth \beta_{Nl}b) \sinh \beta_{Nl}y] \times$$
$$\times \cos \frac{l\pi x}{a} \sin \frac{N\pi z}{2c}$$

$$F_z = \sum_l \sum_m \frac{L_{lm}c}{\gamma_{lm}^3 \sinh \gamma_{lm}c}[\gamma_{lm}z \sinh \gamma_{lm}z - (2\nu + \gamma_{lm}c \tanh \gamma_{lm}c) \cosh \gamma_{lm}z] \times$$
$$\times \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b}$$

and

$$\begin{aligned} \alpha_{mN}^2 &= \left(\frac{m\pi}{b}\right)^2 + \left(\frac{N\pi}{2c}\right)^2 \\ \beta_{Nl}^2 &= \left(\frac{N\pi}{2c}\right)^2 + \left(\frac{l\pi}{a}\right)^2 \\ \gamma_{lm}^2 &= \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \end{aligned} \tag{12}$$

where

$$l, m = 0, 1, 2, 3... \qquad \text{and} \qquad N = 1, 3, 5, 7...$$

Substituting these equations into equations (2) and (3) gives:

$$
\begin{aligned}
\sigma_x = & \sum_m \sum_N \frac{H_{mN} a}{\cosh \alpha_{mN} a} [(1 + \alpha_{mN} a \coth \alpha_{mN} a) \cosh \alpha_{mN} x - \alpha_{mN} x \sinh \alpha_{mN} x] \\
& \times \cos \frac{m\pi y}{b} \sin \frac{N\pi z}{2c} + \sum_N \sum_l \frac{K_{Nl} b}{\beta_{Nl}^2 c^2 \cosh \beta_{Nl} b} \left[2\nu \frac{N^2 \pi^2}{4} \cosh \beta_{Nl} y + \left(\frac{l\pi c}{a} \right)^2 \right. \\
& \left. \times [(1 - \beta_{Nl} b \coth \beta_{Nl} b) \cosh \beta_{Nl} y + \beta_{Nl} y \sinh \beta_{Nl} y] \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} \\
& + \sum_l \sum_m \frac{L_{lm} c}{\gamma_{lm}^2 b^2 \sinh \gamma_{lm} c} \left[2\nu m^2 \pi^2 \sinh \gamma_{lm} z + \left(\frac{l\pi b}{a} \right)^2 [(1 - \gamma_{lm} c \tanh \gamma_{lm} c) \right. \\
& \left. \times \sinh \gamma_{lm} z + \gamma_{lm} z \cosh \gamma_{lm} z] \right] \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\tau_{yz} = & - \sum_m \sum_N \frac{H_{mN} a m N \pi^2}{2\alpha_{mN}^2 bc \cosh \alpha_{mN} a} \left[[1 - (2\nu + \alpha_{mN} a \coth \alpha_{mN} a)] \cosh \alpha_{mN} x \right. \\
& \left. + \alpha_{mN} x \sinh \alpha_{mN} x \right] \cos \frac{N\pi z}{2c} \sin \frac{m\pi y}{b} + \sum_N \sum_l \frac{K_{Nl} N \pi b}{2\beta_{Nl} c \cosh \beta_{Nl} b} \left[\beta_{Nl} y \cosh \beta_{Nl} y \right. \\
& \left. - \beta_{Nl} b \coth \beta_{Nl} b \sinh \beta_{Nl} y \right] \cos \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} + \sum_l \sum_m \frac{L_{lm} m \pi c}{\gamma_{lm} b \sinh \gamma_{lm} c} \\
& \times \left[\gamma_{lm} z \sinh \gamma_{lm} z - \gamma_{lm} c \tanh \gamma_{lm} c \cosh \gamma_{lm} z \right] \sin \frac{m\pi y}{b} \cos \frac{l\pi x}{a}
\end{aligned} \tag{14}
$$

Again, expressions for $\sigma_y, \sigma_z, \tau_{xy}$ and τ_{xz} can be written in a similar manner.

The boundary conditions of (6) still apply, with the exception that $f_3(y,z)$ is now used to denote the modified applied loading, i.e.

$$
\begin{array}{llll}
\text{on } y = \pm b\,; & \sigma_y = 0, & \tau_{xy} = 0 \text{ and } & \tau_{yz} = 0, \\
\text{on } z = \pm c\,; & \sigma_z = 0, & \tau_{xz} = 0 \text{ and } & \tau_{yz} = 0, \\
\text{on } x = \pm a\,; & \sigma_x = f_3(y,z), & \tau_{xy} = 0 \text{ and } & \tau_{xz} = 0.
\end{array} \tag{15}
$$

Again, all the shear stress boundary conditions are automatically satisfied by (14) and similar expressions for τ_{xy} and τ_{xz}. The normal stress boundary conditions are approximately satisfied by equating low order Fourier terms in equation (13), and in similar expressions for σ_y and σ_z. Taking the double Fourier transforms of these three equations (with their approximated normal stress boundary conditions) gives the expressions (16),(17) and (18).

$$
\begin{aligned}
&\sum_m \frac{4H_{mN}\tanh\alpha_{mN}a(-1)^{l+m}}{\alpha_{mN}b(\alpha_{mN}^2a^2+l^2\pi^2)^2}\left[\nu\left(\frac{N\pi a}{2c}\right)^2(\alpha_{mN}^2a^2+l^2\pi^2)\right.\\
&\left.+\left(\frac{m\pi a}{b}\right)^2 l^2\pi^2\right]+K_{Nl}\delta_1\left(1+\frac{2\beta_{Nl}b}{\sinh 2\beta_{Nl}b}\right)\\
&+\sum_m \frac{4L_{lm}\delta_1\coth\gamma_{lm}c(-1)^{m+(N-1)/2}}{\gamma_{lm}b\left[\gamma_{lm}^2c^2+\frac{N^2\pi^2}{4}\right]^2}\left[\nu\left(\frac{l\pi c}{a}\right)^2\left(\gamma_{lm}^2c^2+\frac{N^2\pi^2}{4}\right)\right.\\
&\left.+\left(\frac{m\pi c}{b}\right)^2\frac{N^2\pi^2}{4}\right]=0
\end{aligned}
\tag{16}
$$

where

$$\delta_1=2 \quad \text{when} \quad l=0, \quad \delta_1=1 \quad \text{when} \quad l\neq 0$$

$$
\begin{aligned}
&\sum_N \frac{4H_{mN}\delta_1\tanh\alpha_{mN}a(-1)^{l+(N-1)/2}}{\alpha_{mN}c(\alpha_{mN}^2a^2+l^2\pi^2)^2}\left[\nu\left(\frac{m\pi a}{b}\right)^2(\alpha_{mN}^2a^2+l^2\pi^2)\right.\\
&\left.+\left(\frac{N\pi a}{2c}\right)^2 l^2\pi^2\right]+\sum_N \frac{4K_{Nl}\delta_2\tanh\beta_{Nl}b(-1)^{m+(N-1)/2}}{\beta_{Nl}c(\beta_{Nl}^2b^2+m^2\pi^2)^2}\\
&\times\left[\nu\left(\frac{l\pi b}{a}\right)^2(\beta_{Nl}^2b^2+m^2\pi^2)+\left(\frac{N\pi b}{2c}\right)^2 m^2\pi^2\right]\\
&+L_{lm}\delta_3\left(1-\frac{2\gamma_{lm}c}{\sinh 2\gamma_{lm}c}\right)=0
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
&\delta_1=2 \quad \text{when} \quad m=0, \quad \delta_1=1 \quad \text{when} \quad m\neq 0\\
&\delta_2=2 \quad \text{when} \quad l=0, \quad \delta_2=1 \quad \text{when} \quad l\neq 0\\
&\delta_3=2 \quad \text{when} \quad m=0 \quad \text{or} \quad l=0, \quad \delta_3=1 \quad \text{when} \quad m\neq 0, \quad l\neq 0
\end{aligned}
$$

$$
\begin{aligned}
&H_{mN}\delta_1\left(1+\frac{2\alpha_{mN}a}{\sinh 2\alpha_{mN}a}\right)+\sum_l \frac{4K_{Nl}\tanh\beta_{Nl}b(-1)^{l+m}}{\beta_{Nl}a(\beta_{Nl}^2b^2+m^2\pi^2)^2}\\
&\times\left[\nu\left(\frac{N\pi b}{2c}\right)^2(\beta_{Nl}^2b^2+m^2\pi^2)+\left(\frac{l\pi b}{a}\right)^2 m^2\pi^2\right]\\
&+\sum_l \frac{4L_{lm}\delta_1\coth\gamma_{lm}c(-1)^{l+(N-1)/2}}{\gamma_{lm}a\left[\gamma_{lm}^2c^2+\frac{N^2\pi^2}{4}\right]^2}\left[\nu\left(\frac{m\pi c}{b}\right)^2\left(\gamma_{lm}^2c^2+\frac{N^2\pi^2}{4}\right)\right.\\
&\left.+\left(\frac{l\pi c}{a}\right)^2\frac{N^2\pi^2}{4}\right]=J_{mN} \qquad (18)
\end{aligned}
$$

where

$$\delta_1 = 2 \quad \text{when} \quad l = 0, \quad \delta_1 = 1 \quad \text{when} \quad l \neq 0$$

and

$$J_{mN} = \frac{1}{abc}\int_{-b}^{+b}\int_{-c}^{+c} f_3(y,z)\cos\frac{m\pi y}{b}\sin\frac{N\pi z}{2c}\,dy\,dz$$

Equations (16),(17) and (18) may then be solved simultaneously for the unknown Fourier coefficients. The stress at any point in the prism may then be calculated by substituting the Fourier coefficients into equations (13),(14) and similar.

The required end loading $f(y,z)$ must again be modified so that the net force and moment acting on each end face is zero. This produces a modified loading distribution $f_3(y,z)$.

$$f_3(y,z) = \left[f(y,z) - \frac{3Mz}{4bc^3}\right] \qquad (19)$$

where M is the total moment due to $f(y,z)$, i.e.

$$M = \int_{-b}^{+b}\int_{-c}^{+c} f(y,z)z\,dy\,dz \qquad (20)$$

The previous equations calculate the stresses due to the modified loading $f_3(y,z)$. The stress component σ_x, calculated from equation (13), is a Fourier series approximation to $f_3(y,z)$.

3. INVERSE SOLUTION

The main aim of this section is to provide an Inverse solution to the problem, which involves calculation of the end stress σ_x from the known bulk stress distribution ($\sigma_x + \sigma_y + \sigma_z$) on one lateral surface of the prism. Consider the case when we know the bulk stress distribution on the surface $y = +b$. On this surface, $\sigma_y = 0$, and so the bulk stress is $\sigma_x + \sigma_z$. As for the Direct solution, two loading cases are considered, i.e.

(1) Loading antisymmetric in y, symmetric in x and z

(2) Loading antisymmetric in z, symmetric in x and y

3.1 Loading Antisymmetric in Y

By summing equation (4), and a similar equation for σ_z, the following expression for the bulk stress is obtained.

$$
\begin{aligned}
\sigma_b = &\sum_M \sum_n \frac{H_{Mn} a}{\cosh \alpha_{Mn} a} \Big\{ (1 + \alpha_{Mn} a \coth \alpha_{Mn} a) \cosh \alpha_{Mn} x - \alpha_{Mn} x \sinh \alpha_{Mn} x \Big\} \\
&\times \sin \frac{M\pi y}{2b} \cos \frac{n\pi z}{c} + \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 c^2 \sinh \beta_{nl} b} \left[2\nu n^2 \pi^2 \sinh \beta_{nl} y + \left(\frac{l\pi c}{a} \right)^2 \right. \\
&\times \left. \Big\{ (1 - \beta_{nl} b \tanh \beta_{nl} b) \sinh \beta_{nl} y + \beta_{nl} y \cosh \beta_{nl} y \Big\} \right] \cos \frac{n\pi z}{c} \cos \frac{l\pi x}{a} \\
&+ \sum_l \sum_M \frac{L_{lM} c}{\gamma_{lM}^2 b^2 \cosh \gamma_{lM} c} \left[2\nu \frac{M^2 \pi^2}{4} \cosh \gamma_{lM} z + \left(\frac{l\pi b}{a} \right)^2 \Big\{ (1 - \gamma_{lM} c \coth \gamma_{lM} c) \right. \\
&\times \left. \cosh \gamma_{lM} z + \gamma_{lM} z \sinh \gamma_{lM} z \Big\} \right] \cos \frac{l\pi x}{a} \sin \frac{M\pi y}{2b} \\
&+ \sum_l \sum_M \frac{L_{lM} c}{\cosh \gamma_{lM} c} \Big\{ (1 + \gamma_{lM} c \coth \gamma_{lM} c) \cosh \gamma_{lM} z - \gamma_{lM} z \sinh \gamma_{lM} z \Big\} \\
&\times \cos \frac{l\pi x}{a} \sin \frac{M\pi y}{2b} + \sum_M \sum_n \frac{H_{Mn} a}{\alpha_{Mn}^2 b^2 \cosh \alpha_{Mn} a} \left[2\nu \frac{M^2 \pi^2}{4} \cosh \alpha_{Mn} x + \left(\frac{n\pi b}{c} \right)^2 \right. \\
&\times \left. \Big\{ (1 - \alpha_{Mn} a \coth \alpha_{Mn} a) \cosh \alpha_{Mn} x + \alpha_{Mn} x \sinh \alpha_{Mn} x \Big\} \right] \sin \frac{M\pi y}{2b} \cos \frac{n\pi z}{c} \\
&+ \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 a^2 \sinh \beta_{nl} b} \left[2\nu l^2 \pi^2 \sinh \beta_{nl} y + \left(\frac{n\pi a}{c} \right)^2 \Big\{ (1 - \beta_{nl} b \tanh \beta_{nl} b) \right. \\
&\times \left. \sinh \beta_{nl} y + \beta_{nl} y \cosh \beta_{nl} y \Big\} \right] \cos \frac{n\pi z}{c} \cos \frac{l\pi x}{a} \qquad (21)
\end{aligned}
$$

By taking the Fourier transform of this equation, as shown in Appendix A, the following results are obtained.

$$\frac{1}{abc}\int_{-c}^{+c}\int_{-a}^{+a}(\sigma_x+\sigma_z)\Big|_{y=b}\cos\frac{l\pi x}{a}\cos\frac{n\pi z}{2c}\,dx\,dz$$

$$= \sum_M \frac{4H_{Mn}\delta_1(-1)^{l+(m-1)/2}}{b\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\left[\alpha_{Mn}^3+\frac{1}{\alpha_{Mn}b^2}\left\{\nu\frac{M^2\pi^2}{4}\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]\right.\right.$$

$$\left.\left.+\left(\frac{n\pi b}{c}\right)^2\left(\frac{l\pi}{a}\right)^2\right\}\right]\tanh\alpha_{Mn}a$$

$$+\sum_M \frac{4\delta_3 L_{lM}(-1)^{n+(M-1)/2}}{b\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]^2}\left[\gamma_{lM}^3+\frac{1}{\gamma_{lM}b^2}\left\{\nu\frac{M^2\pi^2}{4}\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]\right.\right.$$

$$\left.\left.+\left(\frac{l\pi b}{a}\right)^2\left(\frac{n\pi}{c}\right)^2\right\}\right]\tanh\gamma_{lM}c$$

$$+\frac{K_{nl}\delta_2\pi^2}{\beta_{nl}^2a^2c^2}\left(a^2n^2+c^2l^2\right)\left(2\nu+1-\beta_{nl}b\tanh\beta_{nl}b+\beta_{nl}b\coth\beta_{nl}b\right) \tag{22}$$

where

$\delta_1 = 2$ when $n = 0$, $\delta_1 = 1$ when $n \neq 0$
$\delta_2 = 2$ when $n = 0$ or $l = 0$, $\delta_2 = 1$ when $n \neq 0$, $l \neq 0$
$\delta_3 = 2$ when $l = 0$, $\delta_3 = 1$ when $l \neq 0$

Equation (22) may then be applied for each value of n and l to give a set of linear simultaneous equations with the Fourier coefficients H_{Mn}, K_{nl} and L_{lM} as the unknowns. Since the boundary conditions of equation (6) still apply, equations (7) and (8) are also used in conjunction with (22). In effect, equation (22) replaces equation (9) for the Inverse solution. This imposes the restriction that $l = (M-1)/2$, so that the number of linear equations produced by (7) (8) and (22) equals the number of unknown Fourier coefficients. The resulting set of equations is then solved to produce a set of Fourier coefficients H_{Mn}, K_{nl} and L_{lM} for the Inverse problem. Equation (4) is then used to obtain the applied stress σ_x.

3.2 Loading Antisymmetric in Z

By summing equation (13), and a similar equation for σ_z, the following expression for the bulk stress is obtained.

$$
\begin{aligned}
\sigma_b = {} & \sum_m \sum_N \frac{H_{mN}a}{\cosh \alpha_{mN}a} \left\{ (1 + \alpha_{mN}a \coth \alpha_{mN}a) \cosh \alpha_{mN}x - \alpha_{mN}x \sinh \alpha_{mN}x \right\} \\
& \times \cos \frac{m\pi y}{b} \sin \frac{N\pi z}{2c} + \sum_N \sum_l \frac{K_{Nl}b}{\beta_{Nl}^2 c^2 \cosh \beta_{Nl}b} \left[2\nu \frac{N^2\pi^2}{4} \cosh \beta_{Nl}y + \left(\frac{l\pi c}{a}\right)^2 \right. \\
& \left. \times \left\{ (1 - \beta_{Nl}b \coth \beta_{Nl}b) \cosh \beta_{Nl}y + \beta_{Nl}y \sinh \beta_{Nl}y \right\} \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} \\
& + \sum_l \sum_m \frac{L_{lm}c}{\gamma_{lm}^2 b^2 \sinh \gamma_{lm}c} \left[2\nu m^2\pi^2 \sinh \gamma_{lm}z + \left(\frac{l\pi b}{a}\right)^2 \left\{ (1 - \gamma_{lm}c \tanh \gamma_{lm}c) \right. \right. \\
& \left. \left. \times \sinh \gamma_{lm}z + \gamma_{lm}z \cosh \gamma_{lm}z \right\} \right] \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} \\
& + \sum_l \sum_m \frac{L_{lm}c}{\sinh \gamma_{lm}c} \left\{ (1 + \gamma_{lm}c \tanh \gamma_{lm}c) \sinh \gamma_{lm}z - \gamma_{lm}z \cosh \gamma_{lm}z \right\} \\
& \times \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} + \sum_m \sum_N \frac{H_{mN}a}{\alpha_{mN}^2 b^2 \cosh \alpha_{mN}a} \left[2\nu m^2\pi^2 \cosh \alpha_{mN}x + \left(\frac{N\pi b}{2c}\right)^2 \right. \\
& \left. \times \left\{ (1 - \alpha_{mN}a \coth \alpha_{mN}a) \cosh \alpha_{mN}x + \alpha_{mN}x \sinh \alpha_{mN}x \right\} \right] \cos \frac{m\pi y}{b} \sin \frac{N\pi z}{2c} \\
& + \sum_N \sum_l \frac{K_{Nl}b}{\beta_{Nl}^2 a^2 \cosh \beta_{Nl}b} \left[2\nu l^2\pi^2 \cosh \beta_{Nl}y + \left(\frac{N\pi a}{2c}\right)^2 \left\{ (1 - \beta_{Nl}b \coth \beta_{Nl}b) \right. \right. \\
& \left. \left. \times \cosh \beta_{Nl}y + \beta_{Nl}y \sinh \beta_{Nl}y \right\} \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} \qquad (23)
\end{aligned}
$$

By taking the Fourier transform of this equation, as shown in Appendix B, the following results are obtained.

$$\frac{1}{abc}\int_{-c}^{+c}\int_{-a}^{+a}(\sigma_x+\sigma_z)\big|_{y=b}\cos\frac{l\pi x}{a}\sin\frac{N\pi z}{2c}\,dx\,dz$$

$$=\sum_m \frac{4H_{mN}(-1)^{m+l}}{b\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\left[\alpha_{mN}^3+\frac{1}{\alpha_{mN}b^2}\left\{\nu m^2\pi^2\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]\right.\right.$$

$$\left.\left.+\left(\frac{N\pi b}{2c}\right)^2\left(\frac{l\pi}{a}\right)^2\right\}\right]\tanh\alpha_{mN}a$$

$$+\sum_m \frac{4\delta_1 L_{lm}(-1)^{m+(N-1)/2}}{b\left[\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2\right]^2}\left[\gamma_{lm}^3+\frac{1}{\gamma_{lm}b^2}\left\{\nu m^2\pi^2\left[\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2\right]\right.\right.$$

$$\left.\left.+\left(\frac{l\pi b}{a}\right)^2\left(\frac{N\pi}{2c}\right)^2\right\}\right]\coth\gamma_{lm}c$$

$$+\frac{K_{Nl}\delta_1\pi^2}{\beta_{Nl}^2a^2c^2}\left[\frac{N^2a^2}{4}+c^2l^2\right]\Big(2\nu+1-\beta_{Nl}b\coth\beta_{Nl}b+\beta_{Nl}b\tanh\beta_{Nl}b\Big) \quad (24)$$

where

$$\delta_1=2 \quad \text{when} \quad l=0, \quad \delta_1=1 \quad \text{when} \quad l\neq 0$$

As for the case of an applied loading antisymmetric in y, a set of Fourier coefficients H_{mN}, K_{Nl} and L_{lm} may be found by solving the set of simultaneous equations produced by (16),(17) and (24). Equation (13) is then used to obtain the applied stress σ_x.

4. RESULTS AND DISCUSSION

The Direct and Inverse programs were implemented using FORTRAN 77 on an Apollo DN10000 computer. The Gaussian Quadrature method was used for the integrations required by equations (9),(18),(22) and (24). The resulting sets of linear simultaneous equations were solved using Gaussian elimination with full pivoting. Double precision was used for all calculations.

4.1 Direct Results

The equations given in the previous sections are applicable to a prism with an *arbitrary* rectangular cross section. However, for illustrative purposes, a square cross section with a square loading area was considered, where $b = c = 2$ and $k_1 = k_2 = 0.5$. First consider a loading symmetric in x and y, and antisymmetric in z. With reference to Figure 3, a stress of $f(y,z) = \pm 16$ was applied inside the loading area, with $f(y,z) = 0$ outside the loaded area. Equations (19) and (20) must first be used to produce a self-equilibrating loading $f_3(y,z)$ on the end face. From (20) the total moment is:

$$
\begin{aligned}
M &= \int_{-c}^{+c}\int_{-b}^{+b} f(y,z)z\,dy\,dz \\
&= \int_{-k_1 b}^{+k_1 b}\left[\int_{-k_2 c}^{0} -16z\,dz + \int_{0}^{+k_2 c} 16z\,dz\right]dy + 0 \\
&= 16k_2^2 c^2 2k_1 b \\
&= 32k_2^2 k_1 b c^2
\end{aligned}
$$

Applying (19) gives the modified loading function.

$$
\begin{aligned}
f_3(y,z) &= \left[f(y,z) - \frac{3Mz}{4bc^3} \right] \\
&= f(y,z) - \frac{24k_2^2 k_1 z}{c} \\
&= f(y,z) - 1.5z
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_3(y,z) &= -1.5z \text{ outside the loading area} \\
&= +16 - 1.5z \text{ inside the loading area, } 0 \le z \le k_2 c \\
&= -16 - 1.5z \text{ inside the loading area, } -k_2 c \le z \le 0
\end{aligned}
$$

Sample results from the Direct method for an applied loading antisymmetric in z are shown in Figure 4. Plots of the bulk stress distribution on the $y = +b$ surface and the end stress σ_x are presented. These results were obtained using seven terms of the Fourier series (i.e. $l, m = 0, 1, ...7$ and $N = 1, 3, ...15$), which showed good convergence. Shown in Figure 4a is the surface plot of σ_x, with its corresponding values indicated in the contour plot, Figure 4b. The plots show the stress values ranging from -18.0 to $+18.0$, in good agreement with the applied stress field $f_3(y,z)$. The bulk stress distribution σ_b on the $y = +b$ surface is shown in Figures 4c and 4d.

Similar results are shown in Figure 5 for an applied loading which is antisymmetric in y, but symmetric in x and z. The modified loading function $f_2(y,z)$, produced by equations (10) and (11) is shown in Figure 2 as:

$$
\begin{aligned}
f_2(y,z) &= -1.5y \text{ outside the loading area} \\
&= +16 - 1.5y \text{ inside the loading area, } 0 \le y \le k_1 b \\
&= -16 - 1.5y \text{ inside the loading area, } -k_1 b \le y \le 0
\end{aligned}
$$

4.2 Inverse Results

In the Inverse problem, the bulk stress field, generated by the Direct method, is used to calculate the applied end stress. For comparison, the results of the stress distribution σ_x obtained from both the Direct and Inverse methods, at different Fourier indicies, are presented. Figure 6 shows the results obtained for an applied loading antisymmetric in z, symmetric in x and y. Figure 7 shows similar results for an applied loading antisymmetric in y, symmetric in x and z.

Reference [1] presented similar results for the stress distribution for a prism with symmetric end loads. It also demonstrated that more Gauss points are needed for numeric integration to obtain a reasonable Inverse solution for a high number of Fourier coefficients. The antisymmetric solution showed a similar trend. The results presented in Figures 6 and 7 use the minimum number of Gauss points to obtain a reasonable solution. Further increases in the number of Fourier coefficients and Gauss points were not warranted for preliminary work due to the large increase in computer resources required.

5. CONCLUSION

This work has successfully extended previous work on the stress distribution in rectangular prisms to include the case of antisymmetric end loading. This distribution may be calculated from a set of bulk stress measurements on one lateral surface of the prism. Analysis has been performed for the case where the bulk stress distribution used in the Inverse method was produced by the Direct method.

Although the concept has been successfully proven, a considerable amount of work will be required to extend this methodology to a procedure which can be routinely used for the analysis of thermoelastic measurements from real structures.

REFERENCES

[1] Truong, T. and Bennett, J. "Determination of Symmetric End Loads from Bulk Stresses on a Rectangular Prism", ARL-STRUC-TM-536, Aeronautical Research Laboratory, Melbourne, Australia (1991).

[2] Sundara Raja Iyengar, K.T. and Prabhakara, M.K. "A Three Dimensional Elasticity Solution for Rectangular Prism Under End Loads", Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 49, June, 1969, pp. 321-332.

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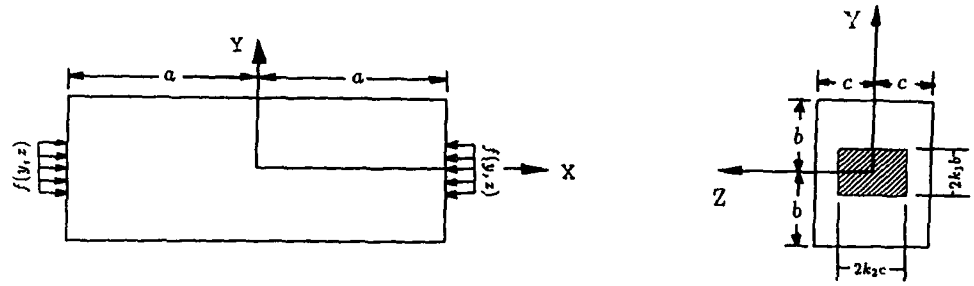


FIG. 1 RECTANGULAR PRISM WITH CENTRAL LOADING

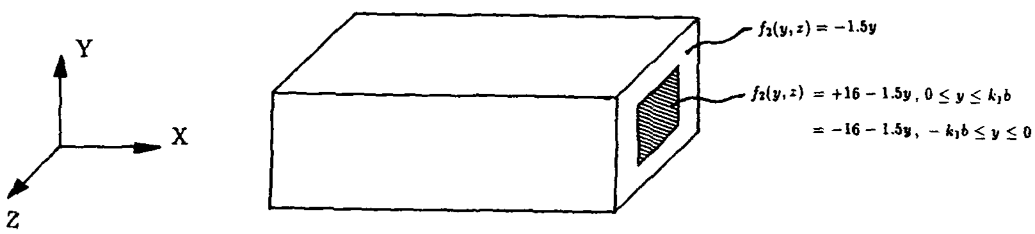


FIG. 2 APPLIED LOADING ANTISYMMETRIC IN Y

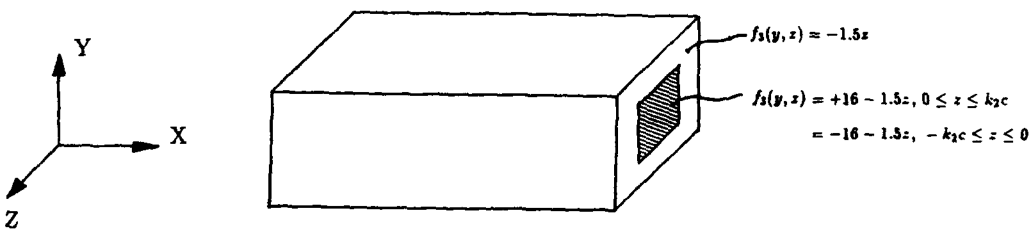


FIG. 3 APPLIED LOADING ANTISYMMETRIC IN Z

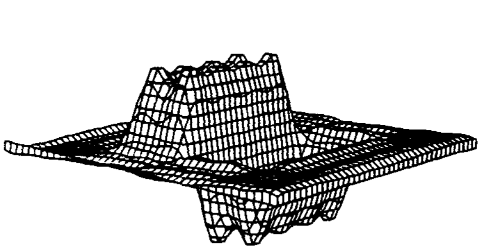


Fig. 4a Applied End Loading

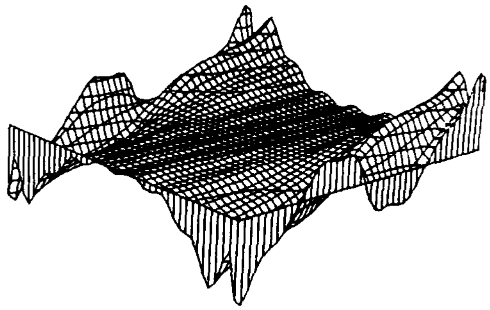


Fig. 4c Bulk Stress on $y = b$

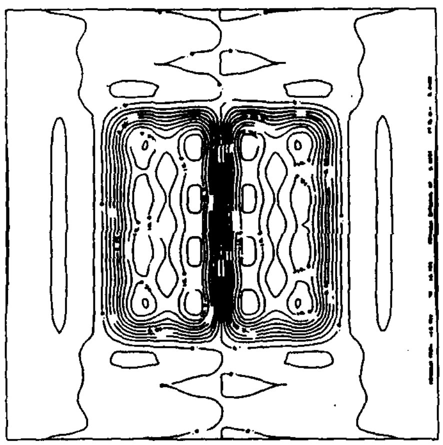


Fig. 4b Applied End Loading

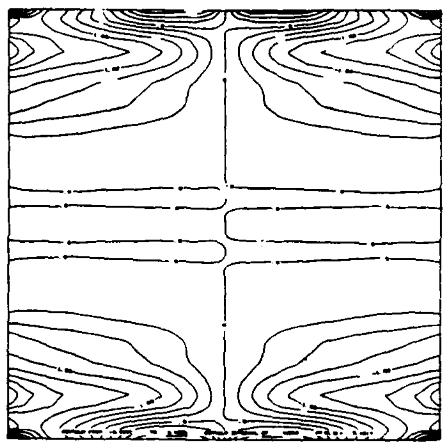


Fig. 4d Bulk Stress on $y = b$

FIG. 4 DIRECT SOLUTION FOR APPLIED LOADING ANTISYMMETRIC IN Z

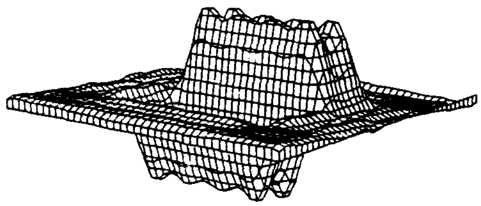


Fig. 5a Applied End Loading

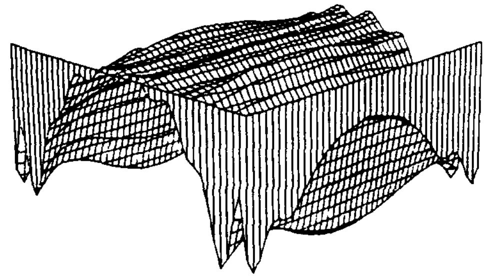


Fig. 5c Bulk Stress on $y = b$

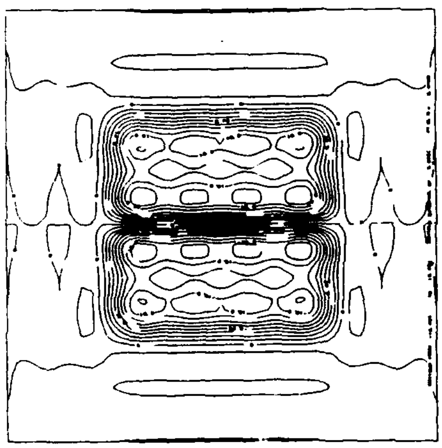


Fig. 5b Applied End Loading

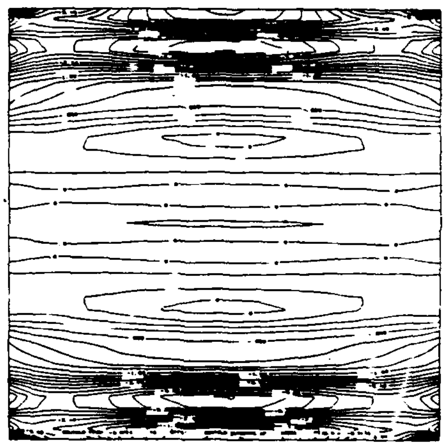
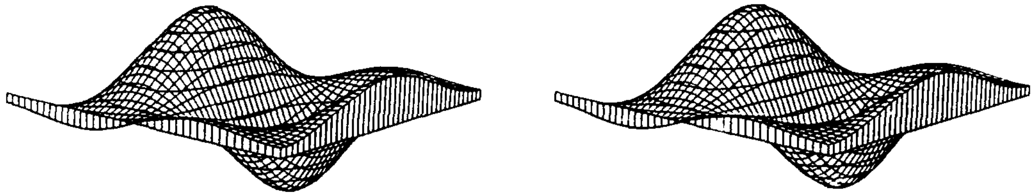


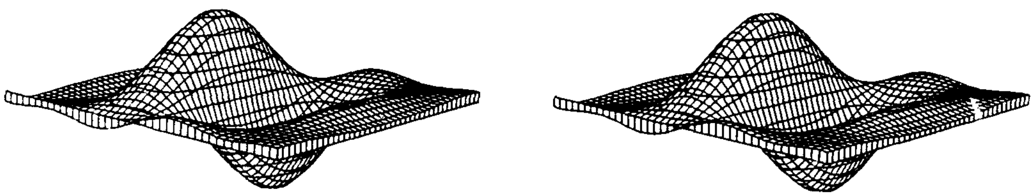
Fig. 5d Bulk Stress on $y = b$

FIG. 5 DIRECT SOLUTION FOR APPLIED LOADING ANTISYMMETRIC IN Y



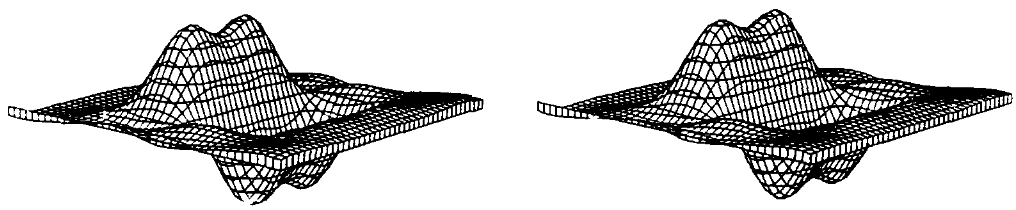
Direct Solution Inverse Solution

Fig 6a. $l, m = 0, 1$; $N = 1, 3$. **Two point integration.**



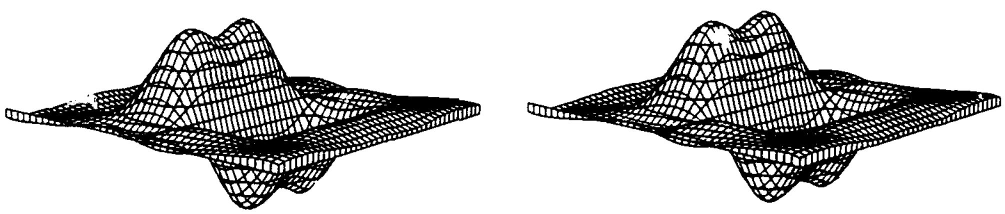
Direct Solution Inverse Solution

Fig 6b. $l, m = 0, 1, 2$; $N = 1, 3, 5$. **Two point integration.**



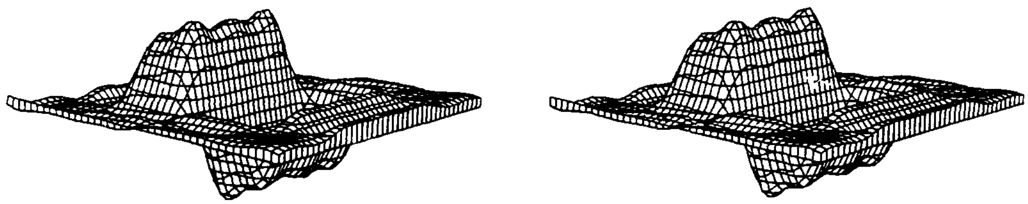
Direct Solution Inverse Solution

Fig 6c. $l, m = 0, 1...3$; $N = 1, 3...7$. **Three point integration.**



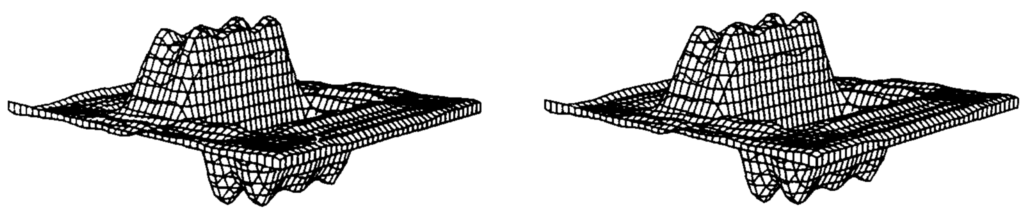
Direct Solution Inverse Solution

Fig 6d. $l, m = 0, 1...4$; $N = 1, 3...9$. **Four point integration.**



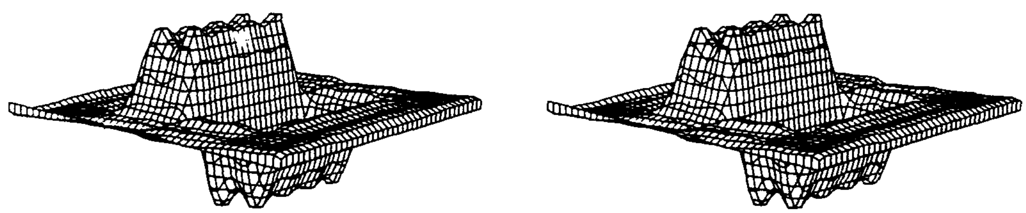
Direct Solution Inverse Solution

Fig 6e. $l, m = 0, 1 \ldots 5$; $N = 1, 3 \ldots 11$. **Four point integration.**



Direct Solution Inverse Solution

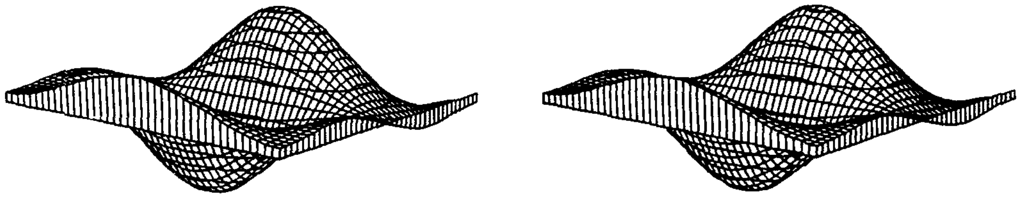
Fig 6f. $l, m = 0, 1 \ldots 6$; $N = 1, 3 \ldots 13$. **Five point integration.**



Direct Solution Inverse Solution

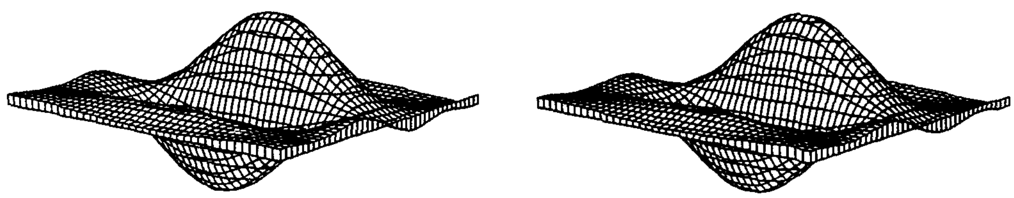
Fig 6g. $l, m = 0, 1 \ldots 7$; $N = 1, 3 \ldots 15$. **Five point integration.**

FIG. 6 DIRECT AND INVERSE SOLUTIONS FOR APPLIED LOADING ANTISYMMETRIC IN Z



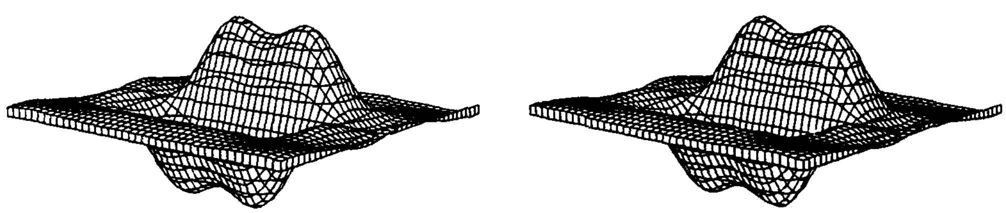
Direct Solution Inverse Solution

Fig 7a. $l, n = 0, 1$; $M = 1, 3$. Two point integration.



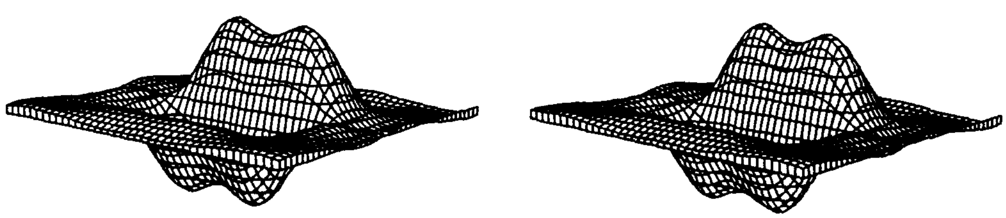
Direct Solution Inverse Solution

Fig 7b. $l, n = 0, 1, 2$; $M = 1, 3, 5$. Two point integration.



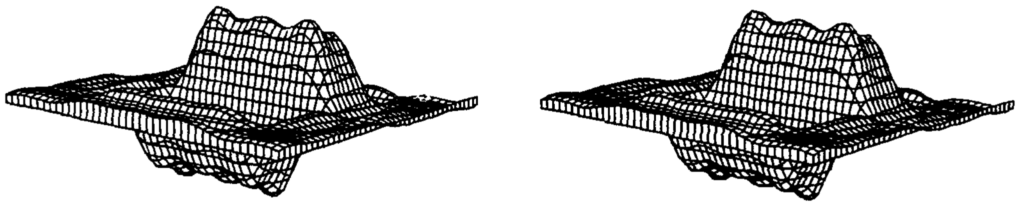
Direct Solution Inverse Solution

Fig 7c. $l, n = 0, 1 \ldots 3$; $M = 1, 3 \ldots 7$. Three point integration.



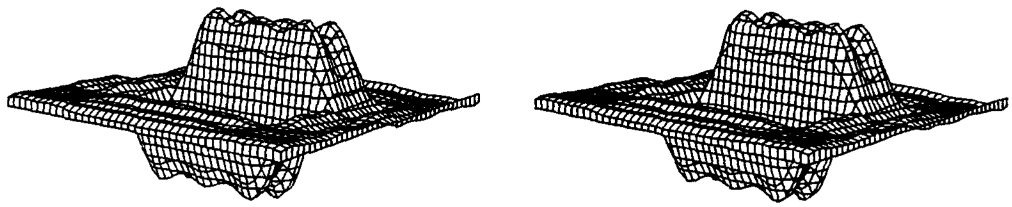
Direct Solution Inverse Solution

Fig 7d. $l, n = 0, 1 \ldots 4$; $M = 1, 3 \ldots 9$. Four point integration.



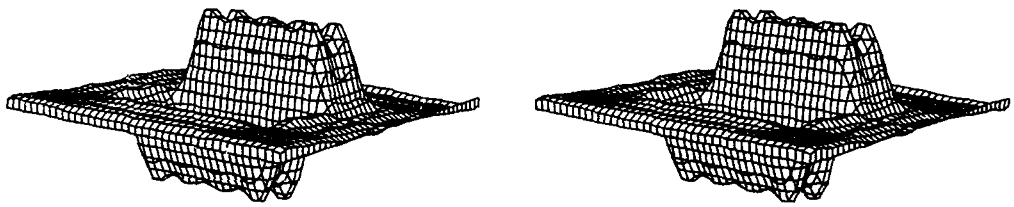
Direct Solution Inverse Solution

Fig 7e. $l, n = 0, 1...5$; $M = 1, 3...11$. Four point integration.



Direct Solution Inverse Solution

Fig 7f. $l, n = 0, 1...6$; $M = 1, 3...13$. Five point integration.



Direct Solution Inverse Solution

Fig 7g. $l, n = 0, 1...7$; $M = 1, 3...15$. Five point integration.

FIG. 7 DIRECT AND INVERSE SOLUTIONS FOR APPLIED LOADING ANTISYMMETRIC IN Y

DERIVATION OF EQUATION (22)

From Equation (21)

$$
\begin{aligned}
\sigma_b = {} & \sum_M \sum_n \frac{H_{Mn} a}{\cosh \alpha_{Mn} a} \left\{ (1 + \alpha_{Mn} a \coth \alpha_{Mn} a) \cosh \alpha_{Mn} x - \alpha_{Mn} x \sinh \alpha_{Mn} x \right\} \\
& \times \sin \frac{M \pi y}{2b} \cos \frac{n \pi z}{c} + \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 c^2 \sinh \beta_{nl} b} \left[2 \nu n^2 \pi^2 \sinh \beta_{nl} y + \left(\frac{l \pi c}{a} \right)^2 \right. \\
& \left. \times \left\{ (1 - \beta_{nl} b \tanh \beta_{nl} b) \sinh \beta_{nl} y + \beta_{nl} y \cosh \beta_{nl} y \right\} \right] \cos \frac{n \pi z}{c} \cos \frac{l \pi x}{a} \\
& + \sum_l \sum_M \frac{L_{lM} c}{\gamma_{lM}^2 b^2 \cosh \gamma_{lM} c} \left[2 \nu \frac{M^2 \pi^2}{4} \cosh \gamma_{lM} z + \left(\frac{l \pi b}{a} \right)^2 \left\{ (1 - \gamma_{lM} c \coth \gamma_{lM} c) \right. \right. \\
& \left. \left. \times \cosh \gamma_{lM} z + \gamma_{lM} z \sinh \gamma_{lM} z \right\} \right] \cos \frac{l \pi x}{a} \sin \frac{M \pi y}{2b} \\
& + \sum_l \sum_M \frac{L_{lM} c}{\cosh \gamma_{lM} c} \left\{ (1 + \gamma_{lM} c \coth \gamma_{lM} c) \cosh \gamma_{lM} z - \gamma_{lM} z \sinh \gamma_{lM} z \right\} \\
& \times \cos \frac{l \pi x}{a} \sin \frac{M \pi y}{2b} + \sum_M \sum_n \frac{H_{Mn} a}{\alpha_{Mn}^2 b^2 \cosh \alpha_{Mn} a} \left[2 \nu \frac{M^2 \pi^2}{4} \cosh \alpha_{Mn} x + \left(\frac{n \pi b}{c} \right)^2 \right. \\
& \left. \times \left\{ (1 - \alpha_{Mn} a \coth \alpha_{Mn} a) \cosh \alpha_{Mn} x + \alpha_{Mn} x \sinh \alpha_{Mn} x \right\} \right] \sin \frac{M \pi y}{2b} \cos \frac{n \pi z}{c} \\
& + \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 a^2 \sinh \beta_{nl} b} \left[2 \nu l^2 \pi^2 \sinh \beta_{nl} y + \left(\frac{n \pi a}{c} \right)^2 \left\{ (1 - \beta_{nl} b \tanh \beta_{nl} b) \right. \right. \\
& \left. \left. \times \sinh \beta_{nl} y + \beta_{nl} y \cosh \beta_{nl} y \right\} \right] \cos \frac{n \pi z}{c} \cos \frac{l \pi x}{a}
\end{aligned}
$$

On a lateral surface of $y = b$, $\sigma_y = 0$ and so the bulk stress on this surface becomes $\sigma_b = \sigma_x + \sigma_z$. Putting $y = b$ into the above equation, and noting that $\sin \frac{M\pi}{2} = (-1)^{(M-1)/2}$ for $M = 1, 3, 5 \ldots$, yields the following result.

$$
\begin{aligned}
\sigma_b = {} & \sum_M \sum_n \frac{H_{Mn} a (-1)^{(M-1)/2}}{\cosh \alpha_{Mn} a} \left\{ (1 + \alpha_{Mn} a \coth \alpha_{Mn} a) \cosh \alpha_{Mn} x - \alpha_{Mn} x \sinh \alpha_{Mn} x \right\} \cos \frac{n\pi z}{c} && \text{(I)} \\
& + \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 c^2 \sinh \beta_{nl} b} \left[2\nu n^2 \pi^2 \sinh \beta_{nl} b + \left(\frac{l\pi c}{a} \right)^2 \left\{ (1 - \beta_{nl} b \tanh \beta_{nl} b) \right. \right. && \\
& \left. \left. \times \sinh \beta_{nl} b + \beta_{nl} b \cosh \beta_{nl} b \right\} \right] \cos \frac{n\pi z}{c} \cos \frac{l\pi x}{a} && \text{(II)} \\
& + \sum_l \sum_M \frac{L_{lM} c (-1)^{(M-1)/2}}{\gamma_{lM}^2 b^2 \cosh \gamma_{lM} c} \left[2\nu \frac{M^2 \pi^2}{4} \cosh \gamma_{lM} z + \left(\frac{l\pi b}{a} \right)^2 \left\{ (1 - \gamma_{lM} c \coth \gamma_{lM} c) \right. \right. && \\
& \left. \left. \times \cosh \gamma_{lM} z + \gamma_{lM} z \sinh \gamma_{lM} z \right\} \right] \cos \frac{l\pi x}{a} && \text{(III)} \\
& + \sum_l \sum_M \frac{L_{lM} c (-1)^{(M-1)/2}}{\cosh \gamma_{lM} c} \left\{ (1 + \gamma_{lM} c \coth \gamma_{lM} c) \cosh \gamma_{lM} z - \gamma_{lM} z \sinh \gamma_{lM} z \right\} \cos \frac{l\pi x}{a} && \text{(IV)} \\
& + \sum_M \sum_n \frac{H_{Mn} a (-1)^{(M-1)/2}}{\alpha_{Mn}^2 b^2 \cosh \alpha_{Mn} a} \left[2\nu \frac{M^2 \pi^2}{4} \cosh \alpha_{Mn} x + \left(\frac{n\pi b}{c} \right)^2 \left\{ (1 - \alpha_{Mn} a \coth \alpha_{Mn} a) \right. \right. && \\
& \left. \left. \times \cosh \alpha_{Mn} x + \alpha_{Mn} x \sinh \alpha_{Mn} x \right\} \right] \cos \frac{n\pi z}{c} && \text{(V)} \\
& + \sum_n \sum_l \frac{K_{nl} b}{\beta_{nl}^2 a^2 \sinh \beta_{nl} b} \left[2\nu l^2 \pi^2 \sinh \beta_{nl} b + \left(\frac{n\pi a}{c} \right)^2 \left\{ (1 - \beta_{nl} b \tanh \beta_{nl} b) \right. \right. && \\
& \left. \left. \times \sinh \beta_{nl} b + \beta_{nl} b \cosh \beta_{nl} b \right\} \right] \cos \frac{n\pi z}{c} \cos \frac{l\pi x}{a} && \text{(VI)}
\end{aligned}
$$

To determine the Fourier coefficients H_{Mn}, K_{nl} and L_{lM} it is necessary to take the double Fourier transform of the above equation.

$$
\begin{aligned}
& \int_{-c}^{+c} \int_{-a}^{+a} (\sigma_x + \sigma_z) \cos \frac{p\pi x}{a} \cos \frac{q\pi z}{c} \, dx \, dz \\
& \qquad = \int_{-c}^{+c} \int_{-a}^{+a} \Big[\text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)} + \text{(VI)} \Big] \cos \frac{p\pi x}{a} \cos \frac{q\pi z}{c} \, dx \, dz
\end{aligned}
$$

Note that p and q are now fixed *positive* integers.

Integrating the RHS, taking term (I)

$$\int_{-c}^{+c}\int_{-a}^{+a}(\mathrm{I})\cos\frac{p\pi x}{a}\cos\frac{q\pi z}{c}\,dx\,dz$$

$$=\sum_M\sum_n \frac{H_{Mn}a(-1)^{(M-1)/2}}{\cosh\alpha_{Mn}a}\left[(1+\alpha_{Mn}a\coth\alpha_{Mn}a)\int_{-a}^{+a}\cosh\alpha_{Mn}x\cos\frac{p\pi x}{a}\,dx\right.$$
$$\left.-\alpha_{Mn}\int_{-a}^{+a}x\sinh\alpha_{Mn}x\cos\frac{p\pi x}{a}\,dx\right]\int_{-c}^{+c}\cos\frac{n\pi z}{c}\cos\frac{q\pi z}{c}\,dz$$

$$=\sum_M \frac{H_{Mn}ac\delta_1(-1)^{(M-1)/2}}{\cosh\alpha_{Mn}a}\left[(1+\alpha_{Mn}a\coth\alpha_{Mn}a)\frac{2\alpha_{Mn}(-1)^p}{\alpha_{Mn}^2+\left(\frac{p\pi}{a}\right)^2}\sinh\alpha_{Mn}a\right.$$
$$\left.-\frac{2\alpha_{Mn}^2a(-1)^p}{\alpha_{Mn}^2+\left(\frac{p\pi}{a}\right)^2}\cosh\alpha_{Mn}a+\frac{2\alpha_{Mn}(-1)^p\left[\alpha_{Mn}^2-\left(\frac{p\pi}{a}\right)^2\right]}{\left[\alpha_{Mn}^2+\left(\frac{p\pi}{a}\right)^2\right]^2}\sinh\alpha_{Mn}a\right]$$

$$=\sum_M \frac{2H_{Mn}\alpha_{Mn}\delta_1ac(-1)^{p+(M-1)/2}}{\alpha_{Mn}^2+\left(\frac{p\pi}{a}\right)^2}\left[(1+\alpha_{Mn}a\coth\alpha_{Mn}a)\tanh\alpha_{Mn}a\right.$$
$$\left.-\alpha_{Mn}a+\frac{\alpha_{Mn}^2-\left(\frac{p\pi}{a}\right)^2}{\alpha_{Mn}^2+\left(\frac{p\pi}{a}\right)^2}\tanh\alpha_{Mn}a\right]$$

Put $l=p$ and simplify to give :

$$=\sum_M \frac{4H_{Mn}\alpha_{Mn}^3ac\delta_1(-1)^{l+(M-1)/2}}{\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\tanh\alpha_{Mn}a$$

where

$$\delta_1=2\quad\text{when}\quad n=0,\quad \delta_1=1\quad\text{when}\quad n\neq 0$$

Taking term (II)

$$\int_{-c}^{+c}\int_{-a}^{+a}(II)\cos\frac{p\pi x}{a}\cos\frac{q\pi z}{c}\,dx\,dz$$

$$=\sum_n\sum_l \frac{K_{nl}b}{\beta_{nl}^2c^2\sinh\beta_{nl}b}\left[2\nu n^2\pi^2\sinh\beta_{nl}b+\left(\frac{l\pi c}{a}\right)^2\left\{(1-\beta_{nl}b\tanh\beta_{nl}b)\right.\right.$$
$$\left.\left.\times\sinh\beta_{nl}b+\beta_{nl}b\cosh\beta_{nl}b\right\}\right]\int_{-a}^{+a}\cos\frac{l\pi x}{a}\cos\frac{p\pi x}{a}\,dx\int_{-c}^{+c}\cos\frac{n\pi z}{c}\cos\frac{q\pi z}{c}\,dz$$

$$=\frac{K_{nl}ab\delta_2}{\beta_{nl}^2c\sinh\beta_{nl}b}\left[2\nu n^2\pi^2\sinh\beta_{nl}b+\left(\frac{l\pi c}{a}\right)^2\left\{(1-\beta_{nl}b\tanh\beta_{nl}b)\sinh\beta_{nl}b+\beta_{nl}b\cosh\beta_{nl}b\right\}\right]$$

$$=\frac{K_{nl}ab\delta_2}{\beta_{nl}^2c}\left[2\nu n^2\pi^2+\left(\frac{l\pi c}{a}\right)^2\left\{(1-\beta_{nl}b\tanh\beta_{nl}b)+\beta_{nl}b\coth\beta_{nl}b\right\}\right]$$

where

$$\delta_2 = 2 \quad \text{when} \quad n = 0 \quad \text{or} \quad l = 0, \quad \delta_2 = 1 \quad \text{when} \quad n \neq 0, \quad l \neq 0$$

Taking term (III)

$$\begin{aligned}
&\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{III}) \cos\frac{p\pi x}{a}\cos\frac{q\pi z}{c}\,dx\,dz \\
&= \sum_{l}\sum_{M} \frac{L_{lM}c(-1)^{(M-1)/2}}{\gamma_{lM}^2 b^2 \cosh\gamma_{lM}c}\Bigg[\left\{2\nu\frac{M^2\pi^2}{4} + \left(\frac{l\pi b}{a}\right)^2 (1-\gamma_{lM}c\coth\gamma_{lM}c)\right\} \\
&\qquad \times \int_{-c}^{+c}\cosh\gamma_{lM}z\cos\frac{q\pi z}{c}\,dz + \left(\frac{l\pi b}{a}\right)^2\gamma_{lM}\int_{-c}^{+c} z\sinh\gamma_{lM}z\cos\frac{q\pi z}{c}\,dz\Bigg] \\
&\qquad \times \int_{-a}^{+a}\cos\frac{l\pi x}{a}\cos\frac{p\pi x}{a}\,dx
\end{aligned}$$

$$\begin{aligned}
&= \sum_{M} \frac{L_{lM}\delta_3 ac(-1)^{(M-1)/2}}{\gamma_{lM}^2 b^2\cosh\gamma_{lM}c}\Bigg[\left\{2\nu\frac{M^2\pi^2}{4} + \left(\frac{l\pi b}{a}\right)^2 (1-\gamma_{lM}c\coth\gamma_{lM}c)\right\} \\
&\qquad \times \frac{2\gamma_{lM}(-1)^q}{\gamma_{lM}^2 + \left(\frac{q\pi}{c}\right)^2}\sinh\gamma_{lM}c + \left(\frac{l\pi b}{a}\right)^2\gamma_{lM}\frac{2\gamma_{lM}c(-1)^q}{\gamma_{lM}^2+\left(\frac{q\pi}{c}\right)^2}\cosh\gamma_{lM}c \\
&\qquad - \left(\frac{l\pi b}{a}\right)^2\gamma_{lM}\frac{2(-1)^q\left[\gamma_{lM}^2 - \left(\frac{q\pi}{c}\right)^2\right]}{\left[\gamma_{lM}^2+\left(\frac{q\pi}{c}\right)^2\right]^2}\sinh\gamma_{lM}c\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{M} \frac{2L_{lM}\delta_3 ac(-1)^{q+(M-1)/2}}{\gamma_{lM}b^2\left[\gamma_{lM}^2+\left(\frac{q\pi}{c}\right)^2\right]}\Bigg[\left\{2\nu\frac{M^2\pi^2}{4} + \left(\frac{l\pi b}{a}\right)^2 (1-\gamma_{lM}c\coth\gamma_{lM}c)\right\} \\
&\qquad \times \tanh\gamma_{lM}c + \left(\frac{l\pi b}{a}\right)^2\gamma_{lM}c - \left(\frac{l\pi b}{a}\right)^2\frac{\gamma_{lM}^2-\left(\frac{q\pi}{c}\right)^2}{\gamma_{lM}^2+\left(\frac{q\pi}{c}\right)^2}\tanh\gamma_{lM}c\Bigg]
\end{aligned}$$

Put $n = q$, re-arrange and simplify to give:

$$= \sum_{M} \frac{4L_{lM}\delta_3 ac(-1)^{n+(M-1)/2}}{\gamma_{lM}b^2\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]}\left[\nu\frac{M^2\pi^2}{4} + \left(\frac{l\pi b}{a}\right)^2\frac{\left(\frac{n\pi}{c}\right)^2}{\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2}\right]\tanh\gamma_{lM}c$$

where

$$\delta_3 = 2 \quad \text{when} \quad l = 0, \quad \delta_1 = 1 \quad \text{when} \quad l \neq 0$$

The integrals of parts (IV) (V) and (VI) were derived using the same approach as shown in (I) (II) and (III).

$$\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{IV}) \cos\frac{p\pi x}{a} \cos\frac{q\pi z}{c}\, dx\, dz$$

$$= \sum_{M} \frac{4L_{lM}\delta_3 \gamma_{lM}^3 ac(-1)^{n+(M-1)/2}}{\left[\gamma_{lM}^2 + \left(\frac{n\pi}{c}\right)^2\right]^2} \tanh \gamma_{lM} c$$

$$\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{V}) \cos\frac{p\pi x}{a} \cos\frac{q\pi z}{c}\, dx\, dz$$

$$= \sum_{M} \frac{4H_{Mn}\delta_1 ac(-1)^{l+(M-1)/2}}{\alpha_{Mn} b^2 \left[\alpha_{Mn}^2 + \left(\frac{l\pi}{a}\right)^2\right]} \left[\nu \frac{M^2\pi^2}{4} + \left(\frac{n\pi b}{c}\right)^2 \frac{\left(\frac{l\pi}{a}\right)^2}{\alpha_{Mn}^2 + \left(\frac{l\pi}{a}\right)^2}\right] \tanh \alpha_{Mn} a$$

$$\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{VI}) \cos\frac{p\pi x}{a} \cos\frac{q\pi z}{c}\, dx\, dz$$

$$= \frac{K_{nl} bc\delta_2}{\beta_{nl}^2 a} \left[2\nu l^2\pi^2 + \left(\frac{n\pi a}{c}\right)^2 \left\{(1 - \beta_{nl} b \tanh \beta_{nl} b) + \beta_{nl} b \coth \beta_{nl} b\right\}\right]$$

where

$$\delta_1 = 2 \quad \text{when} \quad n = 0, \quad \delta_1 = 1 \quad \text{when} \quad n \neq 0$$

$$\delta_2 = 2 \quad \text{when} \quad n = 0 \quad \text{or} \quad l = 0, \quad \delta_2 = 1 \quad \text{when} \quad n \neq 0, \quad l \neq 0$$

$$\delta_3 = 2 \quad \text{when} \quad l = 0, \quad \delta_3 = 1 \quad \text{when} \quad l \neq 0$$

The individual integral components may now be summed to give the following.

$$
\begin{aligned}
&\int_{-c}^{+c}\int_{-a}^{+c}\Big[(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})+(\mathrm{VI})\Big]\cos\frac{p\pi x}{a}\cos\frac{q\pi z}{c}\,dx\,dz \\
&=\sum_{M}\frac{4H_{Mn}\delta_1\alpha_{Mn}^3 ac(-1)^{l+(M-1)/2}}{\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\tanh\alpha_{Mn}a \\
&+\sum_{M}\frac{4H_{Mn}\delta_1 ac(-1)^{l+(M-1)/2}}{\alpha_{Mn}b^2\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]}\left[\nu\frac{M^2\pi^2}{4}+\left(\frac{n\pi b}{c}\right)^2\frac{\left(\frac{l\pi}{a}\right)^2}{\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2}\right]\tanh\alpha_{Mn}a \\
&+\frac{K_{nl}ab\delta_2}{\beta_{nl}^2 c}\left[2\nu n^2\pi^2+\left(\frac{l\pi c}{a}\right)^2\Big\{(1-\beta_{nl}b\tanh\beta_{nl}b)+\beta_{nl}b\coth\beta_{nl}b\Big\}\right] \\
&+\frac{K_{nl}bc\delta_2}{\beta_{nl}^2 a}\left[2\nu l^2\pi^2+\left(\frac{n\pi a}{c}\right)^2\Big\{(1-\beta_{nl}b\tanh\beta_{nl}b)+\beta_{nl}b\coth\beta_{nl}b\Big\}\right] \\
&+\sum_{M}\frac{4\delta_3 L_{lM}ac(-1)^{n+(M-1)/2}}{\gamma_{lM}b^2\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]}\left[\nu\frac{M^2\pi^2}{4}+\left(\frac{l\pi b}{a}\right)^2\frac{\left(\frac{n\pi}{c}\right)^2}{\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2}\right]\tanh\gamma_{lM} \\
&+\sum_{M}\frac{4\delta_3 L_{lM}\gamma_{lM}^3 ac(-1)^{n+(M-1)/2}}{\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]^2}\tanh\gamma_{lM}c
\end{aligned}
$$

Simplifying this equation gives the following result, as presented in equation (22).

$$
\begin{aligned}
&\frac{1}{abc}\int_{-c}^{+c}\int_{-a}^{+a}(\sigma_x+\sigma_z)\Big|_{y=b}\cos\frac{l\pi x}{a}\cos\frac{n\pi z}{c}\,dx\,dz \\
&\quad=\sum_{M}\frac{4H_{Mn}\delta_1(-1)^{l+(m-1)/2}}{b\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\left[\alpha_{Mn}^3+\frac{1}{\alpha_{Mn}b^2}\left\{\nu\frac{M^2\pi^2}{4}\left[\alpha_{Mn}^2+\left(\frac{l\pi}{a}\right)^2\right]\right.\right. \\
&\qquad\qquad\left.\left.+\left(\frac{n\pi b}{c}\right)^2\left(\frac{l\pi}{a}\right)^2\right\}\right]\tanh\alpha_{Mn}a \\
&\quad+\sum_{M}\frac{4\delta_3 L_{lM}(-1)^{n+(M-1)/2}}{b\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]^2}\left[\gamma_{lM}^3+\frac{1}{\gamma_{lM}b^2}\left\{\nu\frac{M^2\pi^2}{4}\left[\gamma_{lM}^2+\left(\frac{n\pi}{c}\right)^2\right]\right.\right. \\
&\qquad\qquad\left.\left.+\left(\frac{l\pi b}{a}\right)^2\left(\frac{n\pi}{c}\right)^2\right\}\right]\tanh\gamma_{lM}c \\
&\quad+\frac{K_{nl}\delta_2\pi^2}{\beta_{nl}^2 a^2c^2}\left(a^2n^2+c^2l^2\right)\left(2\nu+1-\beta_{nl}b\tanh\beta_{nl}b+\beta_{nl}b\coth\beta_{nl}b\right)
\end{aligned}
$$

DERIVATION OF EQUATION (24)

From Equation (23)

$$
\begin{aligned}
\sigma_x + \sigma_z = &\sum_m \sum_N \frac{H_{mN} a}{\cosh \alpha_{mN} a} \left\{ (1 + \alpha_{mN} a \coth \alpha_{mN} a) \cosh \alpha_{mN} x - \alpha_{mN} x \sinh \alpha_{mN} x \right\} \\
&\times \cos \frac{m\pi y}{b} \sin \frac{N\pi z}{2c} + \sum_N \sum_l \frac{K_{Nl} b}{\beta_{Nl}^2 c^2 \cosh \beta_{Nl} b} \left[2\nu \frac{N^2 \pi^2}{4} \cosh \beta_{Nl} y + \left(\frac{l\pi c}{a} \right)^2 \right. \\
&\times \left. \left\{ (1 - \beta_{Nl} b \coth \beta_{Nl} b) \cosh \beta_{Nl} y + \beta_{Nl} y \sinh \beta_{Nl} y \right\} \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} \\
&+ \sum_l \sum_m \frac{L_{lm} c}{\gamma_{lm}^2 b^2 \sinh \gamma_{lm} c} \left[2\nu m^2 \pi^2 \sinh \gamma_{lm} z + \left(\frac{l\pi b}{a} \right)^2 \left\{ (1 - \gamma_{lm} c \tanh \gamma_{lm} c) \right. \right. \\
&\times \left. \left. \sinh \gamma_{lm} z + \gamma_{lm} z \cosh \gamma_{lm} z \right\} \right] \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} \\
&+ \sum_l \sum_m \frac{L_{lm} c}{\sinh \gamma_{lm} c} \left\{ (1 + \gamma_{lm} c \tanh \gamma_{lm} c) \sinh \gamma_{lm} z - \gamma_{lm} z \cosh \gamma_{lm} z \right\} \\
&\times \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} + \sum_m \sum_N \frac{H_{mN} a}{\alpha_{mN}^2 b^2 \cosh \alpha_{mN} a} \left[2\nu m^2 \pi^2 \cosh \alpha_{mN} x + \left(\frac{N\pi b}{2c} \right)^2 \right. \\
&\times \left. \left\{ (1 - \alpha_{mN} a \coth \alpha_{mN} a) \cosh \alpha_{mN} x + \alpha_{mN} x \sinh \alpha_{mN} x \right\} \right] \cos \frac{m\pi y}{b} \sin \frac{N\pi z}{2c} \\
&+ \sum_N \sum_l \frac{K_{Nl} b}{\beta_{Nl}^2 a^2 \cosh \beta_{Nl} b} \left[2\nu l^2 \pi^2 \cosh \beta_{Nl} y + \left(\frac{N\pi a}{2c} \right)^2 \left\{ (1 - \beta_{Nl} b \coth \beta_{Nl} b) \right. \right. \\
&\times \left. \left. \cosh \beta_{Nl} y + \beta_{Nl} y \sinh \beta_{Nl} y \right\} \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a}
\end{aligned}
$$

On a lateral surface of $y = b$, $\sigma_y = 0$ and so the bulk stress on this surface becomes $\sigma_b = \sigma_x + \sigma_z$. Putting $y = b$ into the above equation, and noting that $\cos m\pi = (-1)^m$ for $m = 0, 1, 2...$, yields the following result.

$$\sigma_b = \sum_m \sum_N \frac{H_{mN} a(-1)^m}{\cosh \alpha_{mN} a} \left\{ (1 + \alpha_{mN} a \coth \alpha_{mN} a) \cosh \alpha_{mN} x - \alpha_{mN} x \sinh \alpha_{mN} x \right\} \sin \frac{N\pi z}{2c} \qquad \text{(I)}$$

$$+ \sum_N \sum_l \frac{K_{Nl} b}{\beta_{Nl}^2 c^2 \cosh \beta_{Nl} b} \left[2\nu \frac{N^2 \pi^2}{4} \cosh \beta_{Nl} b + \left(\frac{l\pi c}{a} \right)^2 \left\{ (1 - \beta_{Nl} b \coth \beta_{Nl} b) \right. \right.$$

$$\left. \left. \times \cosh \beta_{Nl} b + \beta_{Nl} b \sinh \beta_{Nl} b \right\} \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} \qquad \text{(II)}$$

$$+ \sum_l \sum_m \frac{L_{lm} c(-1)^m}{\gamma_{lm}^2 b^2 \sinh \gamma_{lm} c} \left[2\nu m^2 \pi^2 \sinh \gamma_{lm} z + \left(\frac{l\pi b}{a} \right)^2 \left\{ (1 - \gamma_{lm} c \tanh \gamma_{lm} c) \right. \right.$$

$$\left. \left. \times \sinh \gamma_{lm} z + \gamma_{lm} z \cosh \gamma_{lm} z \right\} \right] \cos \frac{l\pi x}{a} \qquad \text{(III)}$$

$$+ \sum_l \sum_m \frac{L_{lm} c(-1)^m}{\sinh \gamma_{lm} c} \left\{ (1 + \gamma_{lm} c \tanh \gamma_{lm} c) \sinh \gamma_{lm} z - \gamma_{lm} z \cosh \gamma_{lm} z \right\} \cos \frac{l\pi x}{a} \qquad \text{(IV)}$$

$$+ \sum_m \sum_N \frac{H_{mN} a(-1)^m}{\alpha_{mN}^2 b^2 \cosh \alpha_{mN} a} \left[2\nu m^2 \pi^2 \cosh \alpha_{mN} x + \left(\frac{N\pi b}{2c} \right)^2 \left\{ (1 - \alpha_{mN} a \coth \alpha_{mN} a) \right. \right.$$

$$\left. \left. \times \cosh \alpha_{mN} x + \alpha_{mN} x \sinh \alpha_{mN} x \right\} \right] \sin \frac{N\pi z}{2c} \qquad \text{(V)}$$

$$+ \sum_N \sum_l \frac{K_{Nl} b}{\beta_{Nl}^2 a^2 \cosh \beta_{Nl} b} \left[2\nu l^2 \pi^2 \cosh \beta_{Nl} b + \left(\frac{N\pi a}{2c} \right)^2 \left\{ (1 - \beta_{Nl} b \coth \beta_{Nl} b) \right. \right.$$

$$\left. \left. \times \cosh \beta_{Nl} b + \beta_{Nl} b \sinh \beta_{Nl} b \right\} \right] \sin \frac{N\pi z}{2c} \cos \frac{l\pi x}{a} \qquad \text{(VI)}$$

To determine the Fourier coefficients H_{mN}, K_{Nl} and L_{lm} it is necessary to take the double Fourier transform of the above equation.

$$\int_{-c}^{+c} \int_{-a}^{+a} (\sigma_x + \sigma_z) \cos \frac{p\pi x}{a} \sin \frac{Q\pi z}{2c} \, dx \, dz$$
$$= \int_{-c}^{+c} \int_{-a}^{+a} \Big[(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}) \Big] \cos \frac{p\pi x}{a} \sin \frac{Q\pi z}{2c} \, dx \, dz$$

Note that p and Q are now fixed positive integers.

Integrating the RHS, taking term (I)

$$\int_{-c}^{+c}\int_{-a}^{+a}(\mathrm{I})\cos\frac{p\pi x}{a}\sin\frac{Q\pi z}{2c}\,dx\,dz$$

$$=\sum_m\sum_N\frac{H_{mN}a(-1)^m}{\cosh\alpha_{mN}a}\left[(1+\alpha_{mN}a\coth\alpha_{mN}a)\int_{-a}^{+a}\cosh\alpha_{mN}x\cos\frac{p\pi x}{a}\,dx\right.$$
$$\left.-\alpha_{mN}\int_{-a}^{+a}x\sinh\alpha_{mN}x\cos\frac{p\pi x}{a}\,dx\right]\int_{-c}^{+c}\sin\frac{N\pi z}{2c}\sin\frac{Q\pi z}{2c}\,dz$$

$$=\sum_m\frac{H_{mN}ac(-1)^m}{\cosh\alpha_{mN}a}\left[(1+\alpha_{mN}a\coth\alpha_{mN}a)\frac{2\alpha_{mN}(-1)^p}{\alpha_{mN}^2+\left(\frac{p\pi}{a}\right)^2}\sinh\alpha_{mN}a\right.$$
$$\left.-\frac{2\alpha_{mN}^2a(-1)^p}{\alpha_{mN}^2+\left(\frac{p\pi}{a}\right)^2}\cosh\alpha_{mN}a+\frac{2\alpha_{mN}(-1)^p\left[\alpha_{mN}^2-\left(\frac{p\pi}{a}\right)^2\right]}{\left[\alpha_{mN}^2+\left(\frac{p\pi}{a}\right)^2\right]^2}\sinh\alpha_{mN}a\right]$$

$$=\sum_m\frac{2H_{mN}\alpha_{mN}ac(-1)^{m+p}}{\alpha_{mN}^2+\left(\frac{p\pi}{a}\right)^2}\left[(1+\alpha_{mN}a\coth\alpha_{mN}a)\tanh\alpha_{mN}a\right.$$
$$\left.-\alpha_{mN}a+\frac{\alpha_{mN}^2-\left(\frac{p\pi}{a}\right)^2}{\alpha_{mN}^2+\left(\frac{p\pi}{a}\right)^2}\tanh\alpha_{mN}a\right]$$

Put $l=p$ and simplify to give :

$$=\sum_m\frac{4H_{mN}\alpha_{mN}^3ac(-1)^{m+l}}{\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\tanh\alpha_{mN}a$$

Taking term (II)

$$\int_{-c}^{+c}\int_{-a}^{+a}(\mathrm{II})\cos\frac{p\pi x}{a}\sin\frac{Q\pi z}{2c}\,dx\,dz$$

$$=\sum_N\sum_l\frac{K_{Nl}b}{\beta_{Nl}^2c^2\cosh\beta_{Nl}b}\left[2\nu\frac{N^2\pi^2}{4}\cosh\beta_{Nl}b+\left(\frac{l\pi c}{a}\right)^2\left\{(1-\beta_{Nl}b\coth\beta_{Nl}b)\right.\right.$$
$$\left.\left.\times\cosh\beta_{Nl}b+\beta_{Nl}b\sinh\beta_{Nl}b\right\}\right]\int_{-a}^{+a}\cos\frac{l\pi x}{a}\cos\frac{p\pi x}{a}\,dx\int_{-c}^{+c}\sin\frac{N\pi z}{2c}\sin\frac{Q\pi z}{2c}\,dz$$

$$=\frac{K_{Nl}ab\delta_1}{\beta_{Nl}^2c\cosh\beta_{Nl}b}\left[2\nu\frac{N^2\pi^2}{4}\cosh\beta_{Nl}b+\left(\frac{l\pi c}{a}\right)^2\left\{(1-\beta_{Nl}b\coth\beta_{Nl}b)\cosh\beta_{Nl}b+\beta_{Nl}b\sinh\beta_{Nl}b\right\}\right]$$

$$=\frac{K_{Nl}ab\delta_1}{\beta_{Nl}^2c}\left[2\nu\frac{N^2\pi^2}{4}+\left(\frac{l\pi c}{a}\right)^2\left\{(1-\beta_{Nl}b\coth\beta_{Nl}b)+\beta_{Nl}b\tanh\beta_{Nl}b\right\}\right]$$

where

$$\delta_1 = 2 \quad \text{when} \quad l = 0, \quad \delta_1 = 1 \quad \text{when} \quad l \neq 0$$

Taking term (III)

$$\int_{-c}^{+c}\int_{-a}^{+a} (\text{III}) \cos\frac{p\pi x}{a} \sin\frac{Q\pi z}{2c}\, dx\, dz$$

$$= \sum_l \sum_m \frac{L_{lm} c(-1)^m}{\gamma_{lm}^2 b^2 \sinh\gamma_{lm} c}\left[\left\{2\nu m^2\pi^2 + \left(\frac{l\pi b}{a}\right)^2 (1 - \gamma_{lm} c \tanh\gamma_{lm} c)\right\}\right.$$
$$\times \int_{-c}^{+c} \sinh\gamma_{lm} z \sin\frac{Q\pi z}{2c}\, dz + \left(\frac{l\pi b}{a}\right)^2 \gamma_{lm} \left.\int_{-c}^{+c} z\cosh\gamma_{lm} z \sin\frac{Q\pi z}{2c}\, dz\right]$$
$$\times \int_{-a}^{+a} \cos\frac{l\pi x}{a} \cos\frac{p\pi x}{a}\, dx$$

$$= \sum_m \frac{L_{lm}\delta_1 ac(-1)^m}{\gamma_{lm}^2 b^2 \sinh\gamma_{lm} c}\left[\left\{2\nu m^2\pi^2 + \left(\frac{l\pi b}{a}\right)^2 (1 - \gamma_{lm} c \tanh\gamma_{lm} c)\right\}\right.$$
$$\times \frac{2\gamma_{lm}(-1)^{(Q-1)/2}}{\gamma_{lm}^2 + \left(\frac{Q\pi}{2c}\right)^2} \cosh\gamma_{lm} c + \left(\frac{l\pi b}{a}\right)^2 \gamma_{lm} \frac{2\gamma_{lm} c(-1)^{(Q-1)/2}}{\gamma_{lm}^2 + \left(\frac{Q\pi}{2c}\right)^2} \sinh\gamma_{lm} c$$
$$\left. - \left(\frac{l\pi b}{a}\right)^2 \gamma_{lm} \frac{2(-1)^{(Q-1)/2}\left[\gamma_{lm}^2 - \left(\frac{Q\pi}{2c}\right)^2\right]}{\left[\gamma_{lm}^2 + \left(\frac{Q\pi}{2c}\right)^2\right]^2} \cosh\gamma_{lm} c\right]$$

$$= \sum_m \frac{2L_{lm}\delta_1 ac(-1)^{m+(Q-1)/2}}{\gamma_{lm} b^2\left[\gamma_{lm}^2 + \left(\frac{Q\pi}{2c}\right)^2\right]}\left[\left\{2\nu m^2\pi^2 + \left(\frac{l\pi b}{a}\right)^2 (1 - \gamma_{lm} c \tanh\gamma_{lm} c)\right\}\right.$$
$$\left.\times \coth\gamma_{lm} c + \left(\frac{l\pi b}{a}\right)^2 \gamma_{lm} c - \left(\frac{l\pi b}{a}\right)^2 \frac{\gamma_{lm}^2 - \left(\frac{Q\pi}{2c}\right)^2}{\gamma_{lm}^2 + \left(\frac{Q\pi}{2c}\right)^2} \coth\gamma_{lm} c\right]$$

Put $N = Q$, re-arrange and simplify to give:

$$= \sum_m \frac{4\delta_1 L_{lm} ac(-1)^{m+(N-1)/2}}{\gamma_{lm} b^2\left[\gamma_{lm}^2 + \left(\frac{N\pi}{2c}\right)^2\right]} \cdot \left[\nu m^2\pi^2 + \left(\frac{l\pi b}{a}\right)^2 \frac{\left(\frac{N\pi}{2c}\right)^2}{\gamma_{lm}^2 + \left(\frac{N\pi}{2c}\right)^2}\right] \coth\gamma_{lm} c$$

where

$$\delta_1 = 2 \quad \text{when} \quad l = 0, \quad \delta_1 = 1 \quad \text{when} \quad l \neq 0$$

APPENDIX B

The integrals of parts (IV) (V) and (VI) were derived using the same approach as shown in (I) (II) and (III).

$$\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{IV}) \cos\frac{p\pi x}{a} \sin\frac{Q\pi z}{2c}\, dx\, dz$$
$$= \sum_m \frac{4\delta_1 L_{lm} \gamma_{lm}^3 ac(-1)^{m+(N-1)/2}}{\left[\gamma_{lm}^2 + \left(\frac{N\pi}{2c}\right)^2\right]^2} \coth \gamma_{lm} c$$

$$\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{V}) \cos\frac{p\pi x}{a} \sin\frac{Q\pi z}{2c}\, dx\, dz$$
$$= \sum_m \frac{4H_{mN}ac(-1)^{m+l}}{\alpha_{mN} b^2\left[\alpha_{mN}^2 + \left(\frac{l\pi}{a}\right)^2\right]} \left[\nu m^2\pi^2 + \left(\frac{N\pi b}{2c}\right)^2 \frac{\left(\frac{l\pi}{a}\right)^2}{\alpha_{mN}^2 + \left(\frac{l\pi}{a}\right)^2}\right] \tanh \alpha_{mN} a$$

$$\int_{-c}^{+c}\int_{-a}^{+a} (\mathrm{VI}) \cos\frac{p\pi x}{a} \sin\frac{Q\pi z}{2c}\, dx\, dz$$
$$= \frac{K_{Nl} bc\delta_1}{\beta_{Nl}^2 a} \left[2\nu l^2\pi^2 + \left(\frac{N\pi a}{2c}\right)^2 \left\{(1 - \beta_{Nl} b \coth \beta_{Nl} b) + \beta_{Nl} b \tanh \beta_{Nl} b\right\}\right]$$

where

$$\delta_1 = 2 \quad \text{when} \quad l = 0, \quad \delta_1 = 1 \quad \text{when} \quad l \neq 0$$

The individual integral components may now be summed to give the following.

$$\int_{-c}^{+c}\int_{-a}^{+a}\Big[(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})+(\mathrm{VI})\Big]\cos\frac{p\pi x}{a}\sin\frac{Q\pi z}{2c}\,dx\,dz$$

$$=\sum_m \frac{4H_{mN}\alpha_{mN}^3 ac(-1)^{m+l}}{\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\tanh\alpha_{mN}a$$

$$+\sum_m \frac{4H_{mN}ac(-1)^{m+l}}{\alpha_{mN}b^2\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]}\left[\nu m^2\pi^2+\left(\frac{N\pi b}{2c}\right)^2\frac{\left(\frac{l\pi}{a}\right)^2}{\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2}\right]\tanh\alpha_{mN}a$$

$$+\frac{K_{Nl}ab\delta_1}{\beta_{Nl}^2 c}\left[2\nu\frac{N^2\pi^2}{4}+\left(\frac{l\pi c}{a}\right)^2\Big\{(1-\beta_{Nl}b\coth\beta_{Nl}b)+\beta_{Nl}b\tanh\beta_{Nl}b\Big\}\right]$$

$$+\frac{K_{Nl}bc\delta_1}{\beta_{Nl}^2 a}\left[2\nu l^2\pi^2+\left(\frac{N\pi a}{2c}\right)^2\Big\{(1-\beta_{Nl}b\coth\beta_{Nl}b)+\beta_{Nl}b\tanh\beta_{Nl}b\Big\}\right]$$

$$+\sum_m \frac{4\delta_1 L_{lm}ac(-1)^{m+(N-1)/2}}{\gamma_{lm}b^2\left[\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2\right]}\left[\nu m^2\pi^2+\left(\frac{l\pi b}{a}\right)^2\frac{\left(\frac{N\pi}{2c}\right)^2}{\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2}\right]\coth\gamma_{lm}$$

$$+\sum_m \frac{4\delta_1 L_{lm}\gamma_{lm}^3 ac(-1)^{m+(N-1)/2}}{\left[\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2\right]^2}\coth\gamma_{lm}c$$

Simplifying this equation gives the following result, as presented in equation (24).

$$\frac{1}{abc}\int_{-c}^{+c}\int_{-a}^{+a}(\sigma_x+\sigma_z)\big|_{y=b}\cos\frac{l\pi x}{a}\sin\frac{N\pi z}{2c}\,dx\,dz$$

$$=\sum_m \frac{4H_{mN}(-1)^{m+l}}{b\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]^2}\left[\alpha_{mN}^3+\frac{1}{\alpha_{mN}b^2}\left\{\nu m^2\pi^2\left[\alpha_{mN}^2+\left(\frac{l\pi}{a}\right)^2\right]+\left(\frac{N\pi b}{2c}\right)^2\left(\frac{l\pi}{a}\right)^2\right\}\right]\tanh\alpha_{mN}a$$

$$+\sum_m \frac{4\delta_1 L_{lm}(-1)^{m+(N-1)/2}}{b\left[\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2\right]^2}\left[\gamma_{lm}^3+\frac{1}{\gamma_{lm}b^2}\left\{\nu m^2\pi^2\left[\gamma_{lm}^2+\left(\frac{N\pi}{2c}\right)^2\right]+\left(\frac{l\pi b}{a}\right)^2\left(\frac{N\pi}{2c}\right)^2\right\}\right]\coth\gamma_{lm}c$$

$$+\frac{K_{Nl}\delta_1\pi^2}{\beta_{Nl}^2a^2c^2}\left[\frac{N^2a^2}{4}+c^2l^2\right]\Big(2\nu+1-\beta_{Nl}b\coth\beta_{Nl}b+\beta_{Nl}b\tanh\beta_{Nl}b\Big)$$

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A mathematical method is presented for the determination of the normal antisymmetric end loads from known values of the bulk stress on a lateral surface of a three dimensional rectangular prism. This extends previous work which considered only the symmetric loading case. A computer program has been developed to implement this method, and results for an arbitrarily chosen case are presented.

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