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James F. Geer
Carl M. Andersen

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Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665-5225

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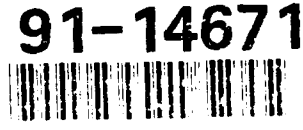

RESONANT FREQUENCY CALCULATIONS USING A HYBRID PERTURBATION-GALERKIN TECHNIQUE

James F. Geer[1]
Department of Systems Science
Watson School of Engineering, Applied Science and Technology
State University of New York
Binghamton, NY 13902

Carl M. Andersen
Department of Mathematics
College of William and Mary in Virginia
Williamsburg, VA 23185

ABSTRACT

A two step hybrid perturbation-Galerkin technique is applied to the problem of determining the resonant frequencies of one- or several-degree(s)-of-freedom nonlinear systems involving a parameter. In step one, the Lindstedt-Poincaré method is used to determine perturbation solutions which are formally valid about one or more special values of the parameter (e.g. for small or large values of the parameter). In step two, a subset of the perturbation coordinate functions determined in step one is used in a Galerkin type approximation. The technique is illustrated for several one-degree-of-freedom systems, including the Duffing and van der Pol oscillators, as well as for the compound pendulum. For all of the examples considered, it is shown that the frequencies obtained by the hybrid technique using only a few terms from the perturbation solutions are significantly more accurate than the perturbation results on which they are based, and they compare very well with frequencies obtained by purely numerical methods.

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1. INTRODUCTION

In this paper we present and discuss a two-step hybrid perturbation-Galerkin technique for the computation of the resonant frequencies of nonlinear oscillating systems with a finite number of degrees of freedom involving a scalar parameter ϵ. In previous papers, we have developed and applied different versions of the hybrid technique to several classes of two-point boundary-value problems for ordinary differential equations [2,6,7], to some boundary value problems for elliptic partial differential equations [8], and to some integral equations of the first kind which arise in slender body theory [5]. For each of these classes of problems, the method has yielded results which are typically far more accurate than the perturbation solutions on which they are based, and often provide at least reasonable solutions even for values of the expansion parameter for which the perturbation solution is meaningless.

The idea of exploiting perturbation expansions in conjunction with Galerkin or variational techniques was introduced by Noor and Peters in 1979 [12] and developed by Noor and his collaborators in a number of papers (see e.g. [11], as well as many other references cited in [5]). Noor's "reduced basis method" is a combination of finite element or other discretization techniques, perturbation expansions, and Galerkin (or variational) techniques.

In general terms, our two-step hybrid technique consists of computing a few terms in the perturbation expansion of the solution about one or more values of the parameter ϵ and then using a subset of these functions, with new amplitudes, in a Galerkin type approximation. This manner of combining the perturbation and Galerkin approaches (which we will describe in more detail below) seems to overcome some of the drawbacks associated with each of the methods when they are applied by themselves, while preserving some of the good features of each one. In particular, the perturbation method has at least two significant drawbacks. The first is that, for most practical problems, only a few terms in a perturbation expansion can be computed because of the rapidly increasing amount of "labor" that is required to compute each additional term. Secondly, the expansion parameter must usually be restricted to values which lie close to the point about which the expansion was constructed, in order to obtain approximations of acceptable accuracy. A drawback of the Galerkin method is the problem, from a practical point of view, of selecting a small number of "good" basis functions. As we shall demonstrate below, the functions determined by the perturbation method appear to be very effective basis functions and hence our method overcomes the main drawback associated with the Galerkin method. Also, the new amplitudes determined by the method produce approximations which are typically much more accurate than the perturbation solutions on which they are based.

In the following sections we describe our method in the context of the problem of determining the resonant frequencies of nonlinear oscillating systems. For simplicity and in

order to illustrate explicitly some of the salient features of the method, we consider first systems with only one degree of freedom. The method is described in detail for such systems and then applied to several model one-degree-of-freedom problems. This allows us to obtain several explicit results and expressions, which provide insight into the application of the method to more complicated systems. We then describe the method in the context of a general (conservative) system with a finite number of degrees of freedom which oscillates about a stable equilibrium state. The method is then applied to the classical problem of a compound pendulum. Observations about the method are presented in the final section.

2. ONE–DEGREE–OF–FREEDOM SYSTEMS

We consider first the problem determining the resonant frequency ν of a one-degree-of-freedom nonlinear oscillator which we write in nondimensional form as

$$\nu^2 u'' + u + (\epsilon/a) f(a u, a \nu u', \epsilon) = 0, \tag{1a}$$

$$u(0) = 1, \qquad u'(0) = 0, \tag{1b}$$

$$u(x + 2\pi) = u(x) \quad \text{for all} \quad x > 0, \tag{1c}$$

$$\int_0^{2\pi} f(a u, a \nu u', \epsilon)\, u'\, dx = 0. \tag{2}$$

In Eq (1a), a is the maximum amplitude of the response, ϵ is a "small" parameter, and f is a specified nonlinear function of its arguments. (Here the dimensional response y of the oscillator has been nondimensionalized by defining $u = y/a$. Also, the primes in Eqs (1)–(2) denote differentiation with respect to the nondimensional time $x = \omega t$, where t is time, $\omega = 2\pi/T$ is the (unknown) resonant frequency of the oscillator, and T is the (unknown) period of the oscillation. Then $\nu = \omega/\omega_0$, where ω_0 is the linear natural frequency of the oscillator.) Conditions (1b) and (1c) insure that u has its maximum amplitude at $x = 0$ and is 2π periodic. Condition (2) is a necessary and sufficient condition for Eq (1) to have a 2π periodic solution. We note that in the special case when f does not depend explicitly on u', i.e. $f = f(a u, \epsilon)$, then Eq (1a) represents a conservative system. Then condition (2) is satisfied automatically and Eqs (1) can be used to express ν as

$$\begin{aligned} \nu &= \pi \left[\int_{-c}^{1} [F(u, \epsilon)]^{-1/2}\, du \right]^{-1}, \\ F(u, \epsilon) &= 1 - u^2 + \frac{2\epsilon}{a} \int_u^1 f(a x, \epsilon)\, dx, \end{aligned} \tag{3}$$

where c is defined by the requirements that $c > 0$ and $F(-c, \epsilon) = 0$. Thus, for this special case, the computation of ν reduces to a quadrature, which will be useful for comparison purposes with our perturbation and hybrid results (see [9]).

3. THE HYBRID PERTURBATION–GALERKIN TECHNIQUE

We now apply a slightly modified version of a two step hybrid perturbation-Galerkin technique which has been introduced and discussed in a series of papers [2,5–8]. In step one of the method, we use the Lindstedt-Poincaré method [10] to find approximate solutions for u, ν, and a in the form

$$\begin{aligned} u &= \sum_{j=0}^{N-1} u_j(x)\,\epsilon^j + O(\epsilon^N), \\ \nu &= 1 + \sum_{j=1}^{N-1} \nu_j\,\epsilon^j + O(\epsilon^N), \\ a &= \sum_{j=0}^{N-1} a_j\,\epsilon^j + O(\epsilon^N), \end{aligned} \tag{4}$$

which are formally valid as $\epsilon \to 0$. In (4), the $\{u_j(x)\}$ are the unknown perturbation coordinate functions, while the constants $\{\nu_j\}$ and $\{a_j\}$, which determine the perturbation approximations to the frequency and initial amplitude, respectively, are also unknown. To determine these quantities, we substitute the expansions (4) into Eqs (1), expand the resulting equations in power series in ϵ about $\epsilon = 0$, and then equate the coefficients of like powers of ϵ on each side of these expressions. In this manner, we obtain a sequence of problems to solve for each of the unknowns. In particular, we find easily that

$$u_0(x) = \cos(x), \tag{5}$$

while each of the functions $u_j(x)$ with $j > 1$ satisfies a problem of the form

$$u_j'' + u_j = 2\,\nu_j \cos(x) + g_j, \tag{6}$$

with $u_j(0) = u_j'(0) = 0$ and $u_j(x + 2\pi) = u_j(x)$. Here g_j depends on u_k, ν_k, and a_k with $k < j$. Thus, in order for u_j to be 2π periodic, the right side of (6) must be orthogonal to both $\cos(x)$ and $\sin(x)$, since otherwise "secular" terms proportional to $x\,\sin(x)$ and $x\,\cos(x)$ will appear in the solution for $u_j(x)$. These conditions yield the relations

$$\nu_j = -\frac{1}{2\pi}\int_0^{2\pi} g_j\,\cos(x)\,dx, \quad \int_0^{2\pi} g_j\,\sin(x)\,dx = 0, \tag{7}$$

which are two equations for the two unknowns ν_j and a_{j-1}. In particular, for $j = 1$ Eqs (7) become

$$\begin{aligned} \nu_1 &= -\frac{1}{2\pi\,a_0}\int_0^{2\pi} f(a_0\,\cos(x), -a_0\,\sin(x), 0)\,\cos(x)\,dx, \\ 0 &= \int_0^{2\pi} f(a_0\,\cos(x), -a_0\,\sin(x), 0)\,\sin(x)\,dx. \end{aligned} \tag{8}$$

The second equation in (8) is an equation for a_0, while the first equation expresses ν_1 as a function of a_0.

In the examples which follow, we shall exhibit several explicit expressions for the various terms in the perturbation expansions (4), as well as comment on the accuracy of the approximations computed using them.

In step two of the hybrid method, we use a subset of the perturbation coordinate functions $\{u_j\}$ determined in step one as both trial and test functions in a Galerkin type approximation. In particular, we seek new approximate solutions $\tilde{u}$, $\tilde{\nu}$, and $\tilde{a}$ for u, ν, and a, respectively, with

$$\tilde{u} = u_0(x) + \sum_{j=1}^{N-1} \delta_j \, u_j(x). \tag{9}$$

In (9), the functions $\{u_j\}$ are the perturbation coordinate functions determined by the Lindstedt-Poincaré method in step one, while the $\{\delta_j\}$ represent new "amplitudes" of these coordinate functions. (We note that (9) satisfies conditions (1b) and (1c) for any choice of the $\{\delta_j\}$.) To determine the amplitudes $\{\delta_j\}$, as well as the quantities $\tilde{\nu}$ and $\tilde{a}$, we substitute (9) into (1a) and require that the residual is orthogonal to each u_k, $0 \le k \le N-1$. Also, we require that (2) is satisfied when u is replaced by $\tilde{u}$. Thus we obtain the conditions

$$\begin{aligned} \int_0^{2\pi} [\tilde{\nu}^2 \, \tilde{u}'' + \tilde{u} + (\epsilon/\tilde{a}) \, f(\tilde{a}\,\tilde{u}, \, \tilde{a}\,\tilde{\nu}\,\tilde{u}', \, \epsilon)] \, u_k \, dx &= 0, \qquad 0 \le k \le N-1, \\ \int_0^{2\pi} f(\tilde{a}\,\tilde{u}, \, \tilde{a}\,\tilde{\nu}\,\tilde{u}', \, \epsilon) \, \tilde{u}' \, dt &= 0. \end{aligned} \tag{10}$$

Equations (10) are a system of $N+1$ equations to determine the $N+1$ unknowns $\delta_1, \ldots, \delta_{N-1}$, $\tilde{\nu}$, and $\tilde{a}$. Although these equations must, in general, be solved numerically, we note that the solution to this system is a point in $(N+1)$-dimensional space, where N is reasonably small, while the solution to Eq (1) is a continuous function. Also, for small values of ϵ, it is reasonable to assume that the unknown quantities in (10) are "close to" the corresponding values in the perturbation solution (4) (e.g., $\delta_j \approx \epsilon^j$). Hence, beginning with small values of ϵ and then proceeding to larger values of ϵ, good starting values are available for the unknown quantities in (10), which can be used with an iterative method of solution, such as Newton's method.

If we set $N = 1$ in (9), we find that $\tilde{u} = u_0(x) = \cos(x)$, while Eqs (10) reduce to

$$\begin{aligned} \tilde{\nu}^2 &= 1 + \frac{\epsilon}{\pi \, \tilde{a}} \int_0^{2\pi} f(\tilde{a} \, \cos(x), \, -\tilde{a}\,\tilde{\nu} \, \sin(x), \, \epsilon) \, \cos(x) \, dx, \\ &\int_0^{2\pi} f(\tilde{a} \, \cos(x), -\tilde{a}\,\tilde{\nu} \, \sin(x), \, \epsilon) \, \sin(x) \, dx = 0. \end{aligned} \tag{11}$$

Equations (11) are a system of two nonlinear equations for the two unknowns $\tilde{\nu}$ and $\tilde{a}$. In particular, if we examine the solution to these equations for small values of ϵ and let $\tilde{\nu} = 1 + \tilde{\nu}_1\,\epsilon + O(\epsilon^2)$ and $\tilde{a} = \tilde{a}_0 + O(\epsilon)$, we find from (11) that the equations for $\tilde{\nu}_1$ and $\tilde{a}_0$ are identical to Eqs (8) for ν_1 and a_0. Thus, for small values of ϵ, our hybrid solution (with $N = 1$) reproduces the perturbation approximations for ν and a, at least to within terms which are $O(\epsilon^2)$ and $O(\epsilon)$, respectively.

In the special case when f is independent of u', we see that the second equation in (11) is satisfied for any choice of $\tilde{a}$ and hence $\tilde{a}$ is an arbitrary parameter. Then the first equation in (11) yields the explicit expression

$$\tilde{\nu} = \left[1 + \frac{\epsilon}{\tilde{a}} \int_0^{2\pi} f(\tilde{a}\,\cos(x),\,\epsilon)\,\cos(x)\,dx\right]^{1/2}. \tag{12}$$

In the next section we shall comment on the precise form of Eqs (10) for several examples, as well as comment on the accuracy of the hybrid approximations generated by their solution.

4. EXAMPLES – ONE–DEGREE–OF–FREEDOM SYSTEMS

We now illustrate some of the basic features of the hybrid solutions outlined above with examples of several one-degree-of-freedom systems. The small parameter ϵ which appears in the nondimensional version of the oscillator Eq (1a) can have several different physical interpretations. The examples which follow illustrate some of these different interpretations and hence indicate some of the classes of problems for which we feel the hybrid method will be especially useful. We will discuss these classes of problems, as well as some possible variations of the basic method, more fully in the discussion section at the end of the paper.

Duffing Oscillator

As our first example, we consider the Duffing oscillator which, in dimensional form, can be written as

$$\ddot{y} + \omega_0^2\,y + \alpha\,y^3 = 0, \qquad y(0) = a, \quad \dot{y}(0) = 0, \tag{13}$$

where ω_0 is the (linearized) natural frequency of the oscillator, α is a specified parameter, and the dots denote differentiation with respect to time t. We let T be the (unknown) period of the oscillation and then define $\omega = 2\,\pi/T$, $\nu = \omega/\omega_0$, $x = \omega\,t$, and $u = y/a$. With these definitions, Eq (13) can be written in the form of Eq (1a) with $f = u^3$, i.e.

$$\nu^2\,u'' + u + \epsilon\,u^3 = 0, \qquad u(0) = 1, \quad u'(0) = 0, \qquad \epsilon = \alpha\,a^2/\omega_0^2. \tag{14}$$

Thus, in this example, ϵ can be interpreted as a measure of the amount of nonlinearity in the restoring force in the system.

For this case condition (2) is satisfied automatically. Thus, the initial amplitude a is arbitrary and hence can be incorporated into our expansion parameter ϵ. The Lindstedt-Poincaré method yields

$$\begin{aligned} u_0 &= \cos(x), \\ u_1 &= -\left[\cos(x) - \cos(3x)\right]/32, \\ u_2 &= \left[23\,\cos(x) - 24\,\cos(3x) + \cos(5x)\right]/1024, \\ &\cdots \\ \nu &= 1 + \frac{3}{8}\epsilon - \frac{21}{256}\epsilon^2 + \frac{81}{2048}\epsilon^3 - \frac{6549}{262144}\epsilon^4 \\ &\quad + \frac{37737}{2097152}\epsilon^5 - \frac{936183}{67108864}\epsilon^6 + O(\epsilon^7). \end{aligned} \tag{15}$$

The hybrid approximation $\tilde{\nu}$ is obtained from Eqs (10) which, in general, will be cubically nonlinear in the amplitudes $\{\delta_j\}$. In particular, for $N = 1$, Eq (11) yields the solution

$$\tilde{\nu} = \left[1 + \frac{3\,\epsilon}{4}\right]^{1/2}, \tag{16}$$

while for $N = 2$, Eqs (10) yield the relations

$$\begin{aligned} \tilde{\nu} &= \left[1 + 3\,\epsilon/4 - 3\,\epsilon\,\delta_1/128 + 3\,\epsilon\,\delta_1^2/2048\right]^{1/2}, \\ \delta_1 &- \epsilon + 3\,\epsilon\,\delta_1/4 - 9\,\epsilon\,\delta_1^2/512 + 23\,\epsilon\,\delta_1^3/16384 = 0. \end{aligned} \tag{17}$$

The first equation in (17) expresses $\tilde{\nu}$ as a function of δ_1 and ϵ, while the second equation is a (cubic) equation for δ_1 as a function of ϵ. For small values of ϵ, we see that (16) agrees with (15) to within terms which are $O(\epsilon^2)$, while the solution for $\tilde{\nu}$ from (17) agrees with (15) to within terms which are $O(\epsilon^3)$.

In Figs 1 and 2, we have plotted several approximations to ν as a function of the parameter ϵ. In particular, we have plotted perturbation approximations (denoted by $P[N]$), where N, the number of terms in expansion (15), ranges from 1 to 6; hybrid approximations $\tilde{\nu}$ (denoted by $H[N]$) determined from (16) and (17) with $N = 1$ and $N = 2$, respectively; and numerical solutions obtained by evaluating the integral representation (3) for ν. As the figures illustrate, the perturbation results (15) are useless for values of ϵ above 1 (the radius of convergence of the perturbation series), while the one and two term hybrid results are in excellent agreement with the numerical results, even for values of ϵ as large as 500.

Simple Pendulum

The equation of motion of a simple pendulum can be expressed as

$$\ddot{\theta} + \omega_0^2 \sin(\theta) = 0, \qquad \theta(0) = a, \quad \dot{\theta}(0) = 0, \tag{18}$$

with $\omega_0^2 = g/L$, where g is the acceleration due to gravity and L is the length of the pendulum. Here $\theta(t)$ is the angle the pendulum makes with the vertical. Using the same definitions of T, ω, ν, and x as in the first example and letting $u = \theta/a$, we find that (18) can be written in the form of (1a) as

$$\nu^2 u'' + u + (1/\epsilon)\,[\sin(\epsilon\, u) - \epsilon\, u] = 0, \quad \epsilon \equiv a. \tag{19}$$

Thus, in this case, ϵ can be interpreted simply as the maximum amplitude of the oscillation. (Condition (2) is again satisfied automatically and hence the initial amplitude is arbitrary.) Also, since the nonlinear term in (19) can be expanded for small values of ϵ as a power series in ϵ^2, we see that the series in (4) involve only even powers of ϵ. In particular, using the symbolic manipulation system Mathematica [15], the Lindstedt-Poincaré method yields the results

$$\begin{aligned}
u_0 &= \cos(x), \\
u_2 &= [\cos(x) - \cos(3x)]\,/192, \\
u_4 &= [17\,\cos(x) - 20\,\cos(3x) + 3\,\cos(5x)]\,/61440, \\
&\cdots \\
\nu &= 1 - \frac{1}{16}\,\epsilon^2 + \frac{1}{3072}\,\epsilon^4 - \frac{23}{737280}\,\epsilon^6 - \frac{2519}{1321205760}\,\epsilon^8 + O(\epsilon^{10}).
\end{aligned} \tag{20}$$

This expansion converges for $|\epsilon| < \pi$.

The hybrid approximations $\tilde{u}$ and $\tilde{\nu}$ are obtained from Eqs (10) with $(\epsilon/a)\,f$ replaced by $[\sin(\epsilon\, u) - \epsilon\, u]/\epsilon$. In particular, for $N = 1$, Eq (11) yields

$$\begin{aligned}
\tilde{\nu} &= \Big[\frac{1}{\epsilon\,\pi}\int_0^{2\pi} \sin(\epsilon\,\cos(x))\,\cos(x)\,dx\Big]^{1/2} \\
&= 1 - \epsilon^2/16 + \epsilon^4/1536 + O(\epsilon^6), \quad \text{as} \quad \epsilon \to 0,
\end{aligned} \tag{21}$$

which agrees with (20) to within terms which are $O(\epsilon^4)$. In a similar manner, when the solutions to Eqs (10) with $N = 2$ are expanded for small values of ϵ, we find that our expression for $\tilde{\nu}$ agrees with (20) to within terms which are $O(\epsilon^6)$, and the solution for δ_2 is $\delta_2 = \epsilon^2 + \epsilon^4/16 + O(\epsilon^6)$. Thus, our hybrid solution $\tilde{u}$ (with $N = 2$) agrees with the perturbation solution for u to within terms which are $O(\epsilon^4)$.

In Fig 3 we have plotted various approximations to ν as a function of ϵ. Included are perturbation solutions (20) (denoted by $P[N]$) of order $O(\epsilon^{2N})$ where $N = 1, 2, 6$; hybrid approximations (10) (denoted by $H[N]$) where $N = 1, 2$ based on the first N nonzero functions $\{u_j\}$; and numerical solutions obtained by numerically evaluating Eq (3) for the case of the pendulum equation. The figure illustrates that the perturbation method gives good results for $\epsilon = \theta_{max}$ up to approximately $\epsilon = 3\pi/4$. For ϵ near π, the perturbation results converge very slowly and give little hint that the frequency must go to zero as $\epsilon \to \pi$. In the region near $\epsilon = 7\pi/8$ the $H[2]$ solution seems definitely superior to the $P[2]$ solution, but for larger amplitudes the accuracy of the $H[2]$ solution is not good, though it does hint that the frequency is tending to zero at $\epsilon = \pi$. The $H[3]$ solution (not shown) is little better than the $H[2]$ solution.

Large amplitude oscillations about an unstable equilibrium

We now consider the problem of determining approximations to the frequency of "large" amplitude oscillations about an unstable equilibrium of a nonlinear system. As a model problem in this area, we consider the (non-dimensional) problem

$$\nu^2 u'' - u + u^3 = 0, \quad u(0) = a, \quad u'(0) = 0, \tag{22}$$

where u is 2π periodic and a is now interpreted as the nondimensional initial amplitude. The equilibrium state $u \equiv 0$ is unstable for the system described by (22), while the states $u \equiv \pm 1$ are stable. However, stable oscillations about $u = 0$ exist for $a^2 > 2$.

To study the stable oscillations about $u = 0$, we consider the modified problem

$$\begin{gathered} \nu^2 u'' + u + \epsilon(u^3 - 2u) = 0, \qquad u(0) = a, \quad u'(0) = 0, \\ u(x + 2\pi) = u(x), \end{gathered} \tag{23}$$

where ϵ is an unspecified parameter. We now make two observations. First, we note that Eq (23) is of the form of Eqs (1) with $f/a = u^3 - 2u$ and hence we can construct approximate solutions which will be formally valid as $\epsilon \to 0$. Secondly, we note that, by setting $\epsilon = 1$ in (23), we recover our original Eq (22). Thus, our (formal) procedure will be to apply our two step hybrid method, as outlined above, and then set $\epsilon = 1$ to obtain an approximation to the solution to (22). In this sense, we can interpret ϵ in this example as a type of "homotopy" parameter. In particular, as ϵ varies from 0 to 1, we can think of our problem (23) as varying from a trivial problem, corresponding to $\epsilon = 0$, which we can easily solve, to the problem we really want to solve, i.e. problem (22), corresponding to $\epsilon = 1$.

Following the general method outlined above, we first use the Lindstedt-Poincaré method

on problem (23) to obtain

$$\begin{aligned} u_0 &= a\cos(x), \\ u_1 &= -(a^3/32)\,[\cos(x) - \cos(3x)]\,, \\ u_2 &= (a^3/1024)\,[(23\,a^2 - 64)\,\cos(x) \\ &\quad -(24\,a^2 - 64)\,\cos(3x) + a^2\cos(5x)], \\ &\quad \cdots \\ \nu &= 1 + (3\,a^2 - 1)\,\epsilon - \left(\frac{21\,a^4}{256} - \frac{3\,a^2}{8} + \frac{1}{2}\right)\epsilon^2 \\ &\quad + \left(\frac{81\,a^6}{2048} - \frac{63\,a^4}{256} + \frac{9\,a^2}{16} - \frac{1}{2}\right)\epsilon^3 \\ &\quad - \left(\frac{6549\,a^8}{262144} - \frac{405\,a^6}{2048} + \frac{315\,a^4}{512} - \frac{15\,a^2}{16} + \frac{5}{8}\right)\epsilon^4 + O(\epsilon^5). \end{aligned} \tag{24}$$

The hybrid approximation $\tilde{u}$ is given by (9), where the amplitudes $\{\delta_j\}$ and $\tilde{\nu}^2$ are determined by Eqs (10) with the term $(\epsilon/\tilde{a})\,f$ replaced by $\epsilon\,(\tilde{u}^3 - 2\,\tilde{u})$. In particular, setting $N = 1$ and $\epsilon = 1$ in (11) we find

$$\begin{aligned} \tilde{\nu} &= [3\,a^2/4 - 1]^{1/2} \\ &= (\sqrt{3}/2)\,|a| + O(1/|a|), \quad \text{as } |a| \to \infty. \end{aligned} \tag{25}$$

From the integral representation (3) of ν, we find for this case that

$$\begin{aligned} \nu &= \frac{\pi}{2^{3/2}}\Big[\int_0^{\pi/2}[a^2\,(1 + \sin^2(x)) - 2]^{-1/2}\,dx\Big]^{-1} \\ &= \frac{\pi\,|a|}{2\sqrt{2}}\Big[\int_0^{\pi/2}(1 + \sin^2(x))^{-1/2}\,dx\Big]^{-1} + O(1/|a|) \\ &= 0.8472\,|a| + O(1/|a|), \quad \text{as } a \to \infty. \end{aligned} \tag{26}$$

Thus, from (25) we see that $\tilde{\nu} = 0.866\,|a| + O(1/|a|)$ as $|a| \to \infty$, which compares well with the exact asymptotic result (26).

In Figs 4 and 5 we have plotted various approximations to ν as functions of the amplitude a. In Fig 4 it is seen from the numerical solution that the frequency goes rapidly to zero as $a \to \sqrt{2}$ from above. Also in Fig 4 the perturbation expansions (24) ($P[N]$ for $N = 1, 2, 4, 8, 16$) indicate that the convergence of the perturbation solutions takes place only for

a very limited range of a values, perhaps $\sqrt{2} < a < 1.75$. The hybrid solutions ($H[N]$ for $N = 1, 2, 3, 4$) show rapid convergence for $a > 1.7$ and slower convergence nearer $a = \sqrt{2}$. Even so in the region just above $a = \sqrt{2}$, $H[4]$ seems much more accurate that $P[16]$. Figure 5 shows that the hybrid solution, $H[2]$ continues to have good accuracy up to $a = 50$ and that even $H[1]$ is a fairly good approximation for $a = 50$.

Self excited oscillations – the van der Pol oscillator

As an example of a nonconservative system, we now consider the limit cycle of the van der Pol oscillator, for which $f(a\,u, \nu\,a\,u', \epsilon) = (a^2\,u^2 - 1)\,a\,\nu\,u'$ in Eq (1a), i.e.

$$\nu^2\,u'' + u + \epsilon\,(a^2\,u^2 - 1)\,\nu\,u' = 0, \qquad u(0) = 1, \quad u'(0) = 0, \tag{27}$$

$$u(x + 2\,\pi) = u(x) \quad \text{for all} \quad x > 0.$$

For this case, ϵ is a "tuning" parameter and can be interpreted as a measure of the amount of a special kind of nonlinear damping in the system.

Several terms in the perturbation expansions of u, ν, and a have been reported [1,4]. In particular, we have

$$\begin{aligned}
u_0 &= 2\cos(x), \\
u_1 &= [3\sin(x) - \sin(3x)]/4, \\
u_2 &= [12\cos(x) - 18\cos(3x) + 5\cos(5x)]/96, \\
&\cdots \\
\nu &= 1 - \frac{1}{16}\epsilon^2 + \frac{17}{3072}\epsilon^4 + \frac{35}{884736}\epsilon^6 + O(\epsilon^8) \\
a &= 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{55738368000}\epsilon^6 + O(\epsilon^8).
\end{aligned} \tag{28}$$

The radius of convergence for this expansion is known to be approximately $\epsilon = 1.85$ (see Andersen and Geer [1]).

In Fig 6 we have plotted several approximations to ν, including the approximations $P[N]$, obtained by including all of the terms in the perturbation expansion (28b) up to those which are $O(\epsilon^{N+1})$, the approximations $H[N]$, which are the hybrid solutions based on the first N perturbation coordinate functions $\{u_j\}$, and the approximations of Zonnefeld [16], obtained using purely numerical methods. The figure illustrates that the hybrid results are superior to the perturbation results on which they are based, and appear to be converging to the numerical approximations as N increases. However, the overall quality of the hybrid

approximation is not as good as in the previous three examples. In the next section we shall discuss a general technique to improve the overall quality of the hybrid approximations and then apply it specifically to the van der Pol oscillator.

5. COMBINING TWO OR MORE PERTURBATION EXPANSIONS

In many practical applications, it may be possible to construct perturbation expansions of the solution u and the frequency ν about more than one value of the perturbation parameter ϵ. That is, it may be possible to expand both u and ν as formal asymptotic series of the form

$$\begin{aligned} u &= \sum_{j=0}^{N_p} u_j^p \, \alpha_j^p(\epsilon) + O(\alpha_{N_p+1}^p(\epsilon)), \\ \nu &= \sum_{j=0}^{N_p} \nu_j^p \, \alpha_j^p(\epsilon) + O(\alpha_{N_p+1}^p(\epsilon)), \end{aligned} \tag{29}$$

which will be formally valid as $\epsilon \to \epsilon_p$, for $p = 1, 2, \ldots, P$. Here $\{\alpha_j^p(\epsilon)\}$ is an appropriate asymptotic sequence of gauge functions and each u_j^p can be determined completely by a standard perturbation method (e.g. a composite expansion of inner and outer expansions). For example, in the Lindstedt-Poincaré method, we have constructed expansions of the form of (29) with $\epsilon_p = \epsilon_1 = 0$ and $\alpha_j^1(\epsilon) = \epsilon^j$. In addition, it may also be possible to construct expansions of the form of (29) which will be valid as $\epsilon \to \epsilon_2 = \infty$, where the gauge functions $\{\alpha_j^2(\epsilon)\}$ might now include terms such as ϵ^{-q}, where q may be a positive integer or a positive rational number, and might also involve appropriate transcendental functions, such as $\log(\epsilon)$ (see e.g., [13]).

A subset of all of the perturbation functions u_j^p are now chosen as coordinate functions for the hybrid technique and an approximation $\tilde{u}$ for u is sought in the form of (9), where each u_j is now one of the perturbation coordinate functions u_l^p. To determine the unknown amplitudes δ_j, we again apply the conditions (10).

To illustrate the application of our method when more than one perturbation expansion of the solution is available to us, we consider first the problem of determining the resonant frequency of a simple mechanical system consisting of a mass m restrained by two identical springs, each having natural length L and spring constant k, midway between two parallel walls a distance $2d$ apart, as discussed by Arnold and Case [3]. Thus, if we let h be the maximum displacement of the mass, its displacement along the centerline between the two

planes is described by $h\,u(x)$, where

$$\nu^2\,u'' + u + \mu\,u\left[1 - \tfrac{1}{(1+\epsilon^2\,u^2)^{1/2}}\right] = 0, \qquad u(0) = 1, \qquad u'(0) = 0, \tag{30}$$
$$u(x + 2\,\pi) = u(x).$$

Here $\epsilon = h/d$, $\mu = \lambda/(1-\lambda)$, $\lambda = L/d < 1$, $\nu = 2\,\pi/T\,\omega_0$, $\omega_0 = [2\,k\,(1-\lambda)/m]^{1/2}$, and $x = \omega_0\,\nu\,t$, where t is time. Thus ϵ can again be interpreted as a measure of the maximum amplitude of the oscillation of the system.

For small values of ϵ, u and ν have expansions of the form (29), with $\alpha_j^1 = \epsilon^{2j}$. Using the Lindstedt-Poincaré method, we find

$$\begin{aligned} u &= \cos(x) + (\mu/64)[\cos(3x) - \cos(x)]\,\epsilon^2 + O(\epsilon^4), \\ \nu &= 1 + (3\mu/8)\,\epsilon^2 + O(\epsilon^4), \quad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{31}$$

For large values of ϵ, u has a (composite) expansion in the general form of (29) with $\alpha_j^2 = \epsilon^{-j}$, in which the formal outer expansion, valid for $|\epsilon\,u| > 1$, must be supplemented by inner expansions around $x = \pi/2$ and $x = 3\,\pi/2$, where u vanishes. In this case, it is easy to show that u is still well approximated by the first term on the right side of (31a), while

$$\nu = 1 + \mu + O(\epsilon^{-1}), \text{ as } \epsilon \to \infty. \tag{32}$$

To apply our hybrid method to this problem, we look for an approximate solution $\tilde{u}$ for u in the form

$$\tilde{u} = \sum_{j=0}^{N} u_j(x)\,\delta_j, \qquad \delta_0 = 1, \tag{33}$$

where $u_0(x) = \cos(x)$ and the remaining $\{u_j(x)\}$ can be selected from either the small or large-ϵ expansions of u. Substituting (33) into (10), we see that the orthogonality conditions become

$$\begin{aligned} &F_j(\nu^2, \delta_1, \delta_2, \ldots, \delta_N) = 0, \qquad j = 0, 1, \ldots, N, \\ &F_j = \int_0^{2\,\pi} \left[\tilde{\nu}^2\,\tilde{u}'' + \tilde{u} + \mu\,\tilde{u}\,(1 - \frac{1}{(1+\epsilon^2\,\tilde{u}^2)^{1/2}})\right] u_j\,dx. \end{aligned} \tag{34}$$

Equations (34) are a system of $N+1$ equations for the $N+1$ unknowns $\tilde{\nu}^2$ and δ_j, $(j = 1, 2, \ldots, N)$. For the special case when $N = 0$, we can solve (34) explicitly and find

$$\nu^2 = 1 + \frac{\mu}{\pi}\int_0^{2\,\pi}\left[1 - \frac{1}{(1+\epsilon^2\,\cos^2(x))^{1/2}}\right]\cos^2(x)\,dx. \tag{35}$$

If we can expand (35) for small values of ϵ, we recover the expansion for ν in (31b), while if we expand it for large values of ϵ we recover the expansion (32). In Fig 7 we have plotted

ν determined by (35) as a function of ϵ, and have also plotted some corresponding values determined by the numerical evaluation of Eq (3). As the figure indicates, the agreement of the hybrid results with the frequencies determined from Eq (3) is excellent.

As a second example, we again consider the van der Pol oscillator. For large values of ϵ, both u and ν have expansions of the form of (29) (see e.g. [13]), where the gauge functions now include ϵ^{-1}, $\epsilon^{-7/3}$, $\epsilon^{-3}\log(\epsilon)$, ... In particular,

$$\nu = \frac{3 - \log(4)}{2\,\pi}\,\epsilon^{-1} + O(\epsilon^{-7/3}), \qquad \text{as } \epsilon \to \infty. \tag{36}$$

The leading term in the corresponding perturbation solution for u involves several inner and outer expansions, which can be combined in a straightforward manner to form the leading term in a composite expansion of the solution. We shall denote the leading term in this expansion by $u_\infty(x, \epsilon)$.

We now use $u_\infty(x, \epsilon)$ as one of our coordinate functions in our hybrid solution (9), which we write in the form

$$\tilde{u} = \delta_0\, u_0 + \sum_{j=1}^{N-1} \delta_j\, u_j + (1 - \delta_0)\, u_\infty, \tag{37}$$

where $\{u_j,\ j = 0, 1, 2, \ldots\}$ represent the perturbation coordinate functions determined from the (regular) perturbation expansion about $\epsilon = 0$. Thus, our solution (37) combines N terms from the small-ϵ perturbation expansion of u with the leading term in the large-ϵ perturbation expansion of u. The $N+2$ unknowns $\delta_0, \ldots, \delta_{N-1}$, a and ν are then determined by substituting (37) into the $N+2$ conditions (10), with N replaced by $N+1$ and with u_N formally replaced by u_∞. In Fig 8 we have plotted two perturbation approximations to ν: the three-term small-ϵ approximation denoted by $P_0[3]$ and the one-term large-ϵ approximation (Eq (36)) denoted by $P_\infty[1]$. We also show three hybrid approximations: $H[3,0]$, which is based on the three small-ϵ perturbation functions; $H[0,1]$, which is based on the one large-ϵ perturbation function; and $H[3,1]$, which combines the information from the three small-ϵ perturbation functions with the one large-ϵ perturbation function. As the figure illustrates, the hybrid approximation $H[0,1]$ is a considerable improvement over P_∞, but it gives poor results for $\epsilon < 2$. The hybrid approximation $H[3,0]$ is a better approximation than $P[3]$, but it gives poor results for $\epsilon > 3$. Finally, $H[3,1]$ gives good results for very small or very large values of ϵ and reasonable (but not accurate) results for "intermediate" (i.e. neither "small" nor "large") values of ϵ where neither perturbation expansion is accurate.

6. SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

We now wish to study periodic solutions $\vec{Q}(t, \epsilon)$ to systems of nonlinear equations describing oscillating systems with several degrees of freedom, which can be expressed in the

form

$$A\,\ddot{\vec{Q}} + B\,\vec{Q} + \epsilon\,\vec{f}(\vec{Q}, \dot{\vec{Q}}, \ddot{\vec{Q}}, \epsilon) = 0. \tag{38}$$

Here ϵ is a small, dimensionless parameter, the dots denote differentiation with respect to time t, $\vec{Q}$ is an M-component vector of dependent variables (where $M > 1$ is the number of degrees of freedom of the system), A and B are real, symmetric, positive definite, M by M, constant matrices, and $\vec{f}$ is an M-component vector. (Here it is convenient to require the components of A to be dimensionless, while the components of B and $\vec{f}$ are required to have the units of t^{-2}.) We assume that, for small values of ϵ, the term $\epsilon\,\vec{f}$ can be expanded in a series of the form

$$\epsilon\,\vec{f} = \epsilon\,\vec{f}_1 + \epsilon^2\,\vec{f}_2 + \ldots \tag{39}$$

where each $\vec{f}_i$ is independent of ϵ. We also assume that the eigenvalues λ_j of the generalized eigenvalue problem associated with the linearized version of Eqs (38), i.e.

$$[B - \lambda_j\,A]\,\vec{\alpha}_j = 0, \qquad j = 1, 2, \ldots, M, \tag{40}$$

are distinct and simple, and that the corresponding eigenvectors $\vec{\alpha}_j$ have been normalized so that $(A\,\vec{\alpha}_j,\, \vec{\alpha}_j) = 1$ for each j. (Here $(\ ,\ )$ is the usual Euclidean vector inner product.) Consequently, the set of eigenvectors $\{\vec{\alpha}_j, 1 \le j \le M\}$ is complete and A-orthonormal. Then, for $\epsilon = 0$, Eqs (38) have periodic solutions of the form

$$\vec{Q}(t) = a\,\cos(\omega_p\,t - \beta)\,\vec{\alpha}_p, \qquad \omega_p = \lambda_p^{1/2}, \tag{41}$$

where a and β are arbitrary constants and $1 \le p \le M$.

Let T be the (unknown) period of a periodic solution of (38) which reduces to the solution (41) as $\epsilon \to 0$. Then we define

$$\omega = \frac{2\,\pi}{T}, \quad \nu = \frac{\omega}{\omega_p}, \quad x = \omega\,t = \nu\,\omega_p\,t, \quad \vec{q}(x) = \vec{Q}(t). \tag{42}$$

In terms of these variables, Eqs (38) become

$$\nu^2\,\omega_p^2\,A\,\vec{q}'' + B\,\vec{q} + \epsilon\,\vec{f}(\vec{q},\, \nu\,\omega_p\,\vec{q}',\, \nu^2\,\omega_p^2\,\vec{q}'',\ ) = 0,$$

$$\vec{q}(x + 2\,\pi) = \vec{q}(x), \tag{43}$$

where the primes denote differentiation with respect to x.

In the first step of our hybrid technique, we again use the Lindstedt-Poincaré method to construct perturbation solutions which are formally valid for small values of ϵ. Thus, we seek solutions to Eqs (43) in the form

$$\vec{q}(x, \epsilon) = \vec{q}_0(x) + \epsilon\,\vec{q}_1(x) + \epsilon^2\,\vec{q}_2(x) + \ldots,$$

$$\vec{q}_0(x) = a\,\cos(x-\beta)\,\vec{\alpha}_p, \tag{44}$$

$$\nu = 1 + \epsilon\,\nu_1 + \epsilon^2\,\nu_2 + \ldots.$$

In Eqs (44), each $\vec{q}_j$ is 2π periodic in x, and each ν_j is a constant.

Substituting (44) into (43) we find that the terms which are $O(1)$ on the left side of (43) sum to zero (from the definition of $\vec{q}_0$), while the terms which are $O(\epsilon^k)$ with $k \geq 1$ yield the relations

$$\omega_p^2\,A\,\vec{q}_k'' + B\,\vec{q}_k = 2\,\nu_k\,\omega_p^2\,a\,\cos(x-\beta)\,A\,\vec{\alpha}_p + \vec{g}_k, k = 1, 2, \ldots, \tag{45}$$

where $\vec{g}_k$ involves $\vec{q}_j$ and ν_j with $j < k$. In particular, $\vec{g}_1 = -\vec{f}_1(\vec{q}_0, \omega_p\,\vec{q}_0', \omega_p^2\,\vec{q}_0'')$. The general solution for $\vec{q}_k$ will be the sum of a particular solution to Eqs (45) and an arbitrary multiple of $\vec{q}_0$, which is the solution to the homogeneous version of Eqs (45). To make our solution unique, we impose the orthogonality condition

$$\int_0^{2\pi} (A\,\vec{q}_k,\,\vec{q}_0)\,dx = 0, \quad \text{for } k \geq 1. \tag{46}$$

We suppose at this point that k is fixed and that $\vec{q}_j$ and ν_j are known for $j < k$. If we now take the inner product of Eqs (45) with $\cos(x-\beta)\,\vec{\alpha}_p$, integrate the resulting expression with respect to x between 0 and 2π, and use the periodicity condition for $\vec{q}_k$, we find that the left side vanishes and hence we can express ν_k as

$$\nu_k = -\frac{1}{2\,\pi\,\omega_p^2\,a}\int_0^{2\pi} (\vec{g}_k, \vec{\alpha}_p)\,\cos(x-\beta)\,dx. \tag{47}$$

In particular, for $k = 1$ in (47) we obtain

$$\nu_1 = \frac{1}{2\,\pi\,\omega_p^2\,a}\int_0^{2\pi} \left(\vec{f}_1(\vec{q}_0, \omega_p\,\vec{q}_0', \omega_p^2\,\vec{q}_0''),\,\vec{\alpha}_p\right)\cos(x-\beta)\,dx. \tag{48}$$

With ν_k determined by (47), we look for a solution for $\vec{q}_k$ in the form

$$\vec{q}_k = \sum_{j=0}^{J_k} \cos(j(x-\beta))\left[\sum_{\rho=1}^{M} \gamma_{k,j,\rho}\,\vec{\alpha}_\rho\right], \tag{49}$$

where the positive integer J_k and the constants $\gamma_{k,j,\rho}$ are to be determined. To determine these quantities, we first express the right side of Eq (45) as a trigonometric polynomial in the form

$$2\,\nu_k\,\omega_p^2\,a\,\cos(x-\beta)\,A\,\vec{\alpha}_p + \vec{g}_k = \sum_{j=0}^{J_k} \vec{g}_{k,j}\,\cos(j(x-\beta)), \tag{50}$$

where the vectors $\{\vec{g}_{k,j}\}$ are independent of x and the inner product $(\vec{g}_{k,1}, \vec{\alpha}_p) = 0$. This follows from the definition of ν_k in (47). Substituting (49) and (50) into (45), using the

relations (40), and taking the inner product of the resulting expression (40) with $\vec{\alpha}_m$, we find that

$$\gamma_{k,j,m} = \frac{(\vec{g}_{k,j},\, \vec{\alpha}_m)}{\omega_m^2 - j^2\,\omega_p^2}, \qquad 0 \le j \le J_k,$$
$$1 \le m \le M, \quad (j-1)^2 + (m-p)^2 > 0, \tag{51}$$

while the constant $\gamma_{k,1,p}$ is zero from condition (46). In deriving Eq (51), we have assumed that $\omega_m^2 \neq j^2\,\omega_p^2$, unless $j = 1$ and $m = p$.

In the second step of our hybrid method, we seek new approximations $\tilde{\vec{q}}$ and $\tilde{\nu}$ to $\vec{q}$ and ν, respectively, where

$$\tilde{\vec{q}} = \vec{q}_0(x) + \sum_{j=1}^{N-1} \delta_j(\epsilon)\,\vec{q}_j(x). \tag{52}$$

In (52), the $\vec{q}_j(x)$ are the perturbation coordinate functions which appear in the expansion (44), while the constants $\{\delta_j\}$ represent new "amplitudes", which must be determined. We note that $\tilde{\vec{q}}$ is 2π periodic for any choice of the $\{\delta_j\}$, since each $\vec{q}_j$ is 2π periodic. To determine the amplitudes $\{\delta_j\}$, as well as $\tilde{\nu}$, we substitute (52) into (43) and demand that the residual is orthogonal to each of the perturbation coordinate functions $\vec{q}_k$, i.e.

$$\int_0^{2\pi} \Big(\Big[\tilde{\nu}^2\,\omega_p^2\,A\,\tilde{\vec{q}}'' + B\,\tilde{\vec{q}} + \epsilon\,\vec{f}(\tilde{\vec{q}},\, \tilde{\nu}\,\omega_p\,\tilde{\vec{q}}',\, \tilde{\nu}^2\,\omega_p^2\,\tilde{\vec{q}}'',\, \epsilon) \Big], \vec{q}_k \Big)\, dx = 0, \qquad k = 0, 1, \ldots, N-1. \tag{53}$$

Equations (53) are a system of N equations for the N unknowns $\delta_1, \ldots, \delta_{N-1}$, and $\tilde{\nu}$. In particular, setting $N = 1$ in (52) we find that $\tilde{\vec{q}} = \vec{q}_0$ while (53) yields the expression

$$\tilde{\nu} = \left[1 + \frac{\epsilon}{\pi\,\omega_p^2\,a} \int_0^{2\pi} \Big(\vec{f}(\vec{q}_0,\, \tilde{\nu}\,\omega_p\,\vec{q}_0',\, \tilde{\nu}^2\,\omega_p^2\,\vec{q}_0''),\, \vec{\alpha}_p \Big)\, \cos(x - \beta)\, dx \right]^{1/2}. \tag{54}$$

We note that, for small values of ϵ, from (54) we can write $\tilde{\nu} = 1 + \epsilon\,\nu_1 + O(\epsilon^2)$, where ν_1 is given by (48). Thus, for small values of ϵ, our hybrid approximation (with $N = 1$) agrees with our perturbation result up to terms which are $O(\epsilon^2)$.

7. EXAMPLE – COMPOUND PENDULUM

As an application of the results of the previous section, we consider two pendula, each of length L and each with attached mass m. They are attached in such a way as to form a classical compound pendulum. Letting $\theta_1(t)$ and $\theta_2(t)$ denote the angles the two pendula make with the vertical, we find that θ_1 and θ_2 satisfy the relations

$$2\,\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + 2\,\omega_\theta^2 \sin(\theta_1) = 0$$
$$\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \ddot{\theta}_2 - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \omega_\theta^2 \sin(\theta_2) = 0. \tag{55}$$

In (55), $\omega_\theta^2 = g/L$, where g is the acceleration due to gravity. We now let $\theta_1 = \epsilon Q_1$ and $\theta_2 = \epsilon Q_2$ to write (55) in the form of Eq (38) with ϵ replaced by ϵ^2, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \omega_\theta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \tag{56}$$

and

$$\epsilon^2 \vec{f} = \begin{bmatrix} \ddot{Q}_2 \left[\cos(\epsilon(Q_1 - Q_2)) - 1\right] + \epsilon \dot{Q}_2^2 \sin(\epsilon(Q_1 - Q_2)) \\ +2\epsilon^{-1}\omega_\theta^2 \left[\sin(\epsilon Q_1) - \epsilon Q_1\right] \\ \ddot{Q}_1 \left[\cos(\epsilon(Q_1 - Q_2)) - 1\right] - \epsilon \dot{Q}_1^2 \sin(\epsilon(Q_1 - Q_2)) \\ +\epsilon^{-1}\omega_\theta^2 \left[\sin(\epsilon Q_2) - \epsilon Q_2\right] \end{bmatrix}$$

$$= \epsilon^2 \begin{bmatrix} -(1/2)\ddot{Q}_2(Q_1 - Q_2)^2 + \dot{Q}_2^2(Q_1 - Q_2) - (1/3)\omega_\theta^2 Q_1^3 \\ -(1/2)\ddot{Q}_1(Q_1 - Q_2)^2 - \dot{Q}_1^2(Q_1 - Q_2) - (1/6)\omega_\theta^2 Q_2^3 \end{bmatrix} + O(\epsilon^4). \tag{57}$$

For this problem we find

$$\begin{gathered} \omega_1 = \sqrt{2 - \sqrt{2}}\,\omega_\theta, \qquad \omega_2 = \sqrt{2 + \sqrt{2}}\,\omega_\theta, \\ \vec{\alpha}_1 = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \qquad \vec{\alpha}_2 = c_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}, \\ c_1 = (1/2)(2 - \sqrt{2}), \qquad c_2 = (1/2)(2 + \sqrt{2}). \end{gathered} \tag{58}$$

Thus, for small amplitude oscillations, the periodic mode corresponding to $p = 1$ (i.e. the mode with frequency ω_1) represents an oscillation for which, at any instant of time, the pendula are always on the same side of the vertical. We shall refer to this mode as the "symmetric" mode of oscillation. In a similar manner, the periodic mode corresponding to $p = 2$ (with frequency ω_2) represents an oscillation for which the pendula are on opposite sides of the vertical, which we shall refer to as the "asymmetric" mode of oscillation. Then, setting $p = 1$ in our formulae of the previous section (with $a = 1$ and $\beta = 0$ in the definition of $\vec{q}_0$), we find that we can express the perturbation solutions for the frequency ω of oscillation of the symmetric mode, as well as the angular displacements θ_1 and θ_2, as

$$\begin{aligned} \omega^2/\omega_\theta^2 \;&=\; (2 - \sqrt{2})\,\nu^2 = 2 - \sqrt{2} + \\ &\epsilon^2 \frac{(-102 + 71\sqrt{2})}{16} + \epsilon^4 \frac{(470602 - 332533\sqrt{2})}{14336} + \\ &\epsilon^6 \frac{(-2676206382 + 1892439443\sqrt{2})}{12845056} + O(\epsilon^8), \end{aligned} \tag{59}$$

$$
\begin{aligned}
\theta_1(x) &= \epsilon\, q_1(x) = \epsilon\, \cos(x) + \\
&\epsilon^3\Big[\frac{(1-\sqrt{2})\,\cos(x)}{32} + \frac{23(-19+24\sqrt{2})\,\cos(3x)}{2688}\Big] + \\
&\epsilon^5\Big[\frac{(233+1375\sqrt{2})\,\cos(x)}{28672} + \frac{(812575-535236\sqrt{2})\,\cos(3x)}{802816} + \\
&\frac{3(970497-751760\sqrt{2})cos(5x)}{9748480}\Big] + O(\epsilon^7), \\
\theta_2(x) &= \epsilon\, q_2(x) = \epsilon\,\sqrt{2}\,\cos(x) + \\
&\epsilon^3\Big[\frac{(2-\sqrt{2})\,\cos(x)}{32} + \frac{(576-781\sqrt{2})\,\cos(3x)}{2688}\Big] + \\
&\epsilon^5\Big[\frac{(-2750-233\sqrt{2})\,\cos(x)}{28672} + \frac{(-1148584+756783\sqrt{2})\,\cos(3x)}{802816} + \\
&\frac{(-4066960+3111099\sqrt{2})\,\cos(5x)}{9748480}\Big] + O(\epsilon^7).
\end{aligned}
\tag{60, 61}
$$

It follows that the perturbation expansions for ν and for the (total) energy $m\,\omega_\theta^2\,L^2\,E$ for the symmetric mode are given by

$$
\begin{aligned}
\nu &= 1 + \epsilon^2\,\frac{(-31+20\sqrt{2})}{32} + \epsilon^4\,\frac{(-61839+46888\sqrt{2})}{28672} \\
&\quad + \epsilon^6\,\frac{(-284017367+197610484\sqrt{2})}{25690112} + O(\epsilon^8), \\
E &= 2\,\epsilon^2 + \epsilon^4\,\frac{3\,(29+8\sqrt{2})}{32} \\
&\quad + \epsilon^6\,\frac{(2738185-550288\sqrt{2})}{401408} + O(\epsilon^8).
\end{aligned}
\tag{62}
$$

If $\theta_1(x)$ and $\theta_2(x)$ are computed to order $O(\epsilon^{2N-1})$, then the energy E is conserved (in time) to $O(\epsilon^{2N})$.

The corresponding expressions for the asymmetric mode of oscillation are the $\sqrt{2}$ conjugates of these expressions, i.e. everywhere $\sqrt{2}$ appears in Eqs (59)-(62) it is replaced by $-\sqrt{2}$.

For both the symmetric and asymmetric modes ν is a monotonic decreasing function of E in the interval $0 < E < 6$, the energy interval in which the pendulum exhibits a back-and-forth motion. For any given energy in this interval the frequency of the asymmetric

mode is higher than the frequency of the symmetric mode. Parametric plots of $\nu(\epsilon)$ vs. $E(\epsilon)$ as computed to orders $O(\epsilon^2)$, $O(\epsilon^4)$, $O(\epsilon^6)$, $O(\epsilon^8)$, $O(\epsilon^{10})$, and $O(\epsilon^{12})$, are shown in Fig 9 for the symmetric case and in Fig 10 for the asymmetric case. These curves are labelled $P[1]$ through $P[6]$. The hybrid results are useful well past the radii of convergence for the perturbation results. They appear to be converging to the correct values for the energy interval shown.

8. DISCUSSION AND CONCLUSIONS

Each of the examples we have considered provides some insights into the hybrid method which we now discuss briefly.

The Duffing oscillator illustrates the fact that the usefulness of a perturbation expansion can often be limited because the expansion has a finite radius of convergence. In this case, the perturbation series converges only for $|\epsilon| < 1$, since the solution u, when regarded as a function of both x and ϵ, has a singularity at $\epsilon = -1$. (This follows from the fact that the solution to (14) is non-oscillatory for $\epsilon < -1$.) Thus, the perturbation expansion fails to converge over the range $1 < \epsilon < \infty$, which is most of the region of physical interest. The hybrid method overcomes this limitation and provides meaningful and, in this case, at least, very accurate approximations to the resonant frequency, even for ϵ well beyond the radius of convergence of the perturbation solution (see Figs 1 and 2).

The perturbation expansion of the frequency (and solution u) for the simple pendulum converges over the entire interval of interest, i.e. $-\pi < \epsilon < \pi$, and hence, in principle, could be used to compute approximations to the resonant frequency for any value of ϵ in this range. However, the rate of convergence is very slow for values of ϵ close to $\pm\pi$ and a great many terms in the perturbation series would be needed to provide a solution of acceptable accuracy (see Fig 3). By contrast, the hybrid approximation, based on only two terms of the perturbation expansion, provides an approximation to the resonant frequency which is essentially as accurate as the perturbation solution based on six terms for ϵ not close to $\pm\pi$, and is reasonably close to the numerically computed frequencies, even when ϵ is close to $\pm\pi$. Thus, we see that the hybrid method has the effect of accelerating the convergence of an otherwise slowly converging series.

In the example of oscillations about an unstable equilibrium, we saw that the perturbation expansions depend on the parameter a (the maximum initial amplitude), as well as upon ϵ. Thus, the convergence of the perturbation solutions will also depend upon the parameter a. In particular, for $\epsilon = 1$, which corresponds to the case of interest in this problem, we conclude from Fig 4 that the series solution for the resonant frequency converges only for a very limited range of values of a, this range being from $a = \sqrt{2}$ to approximately $a \approx 1.75$.

This is in contrast to the hybrid solutions which appear to converge to the exact frequencies for all values of $a > \sqrt{2}$. However, the rate of convergence, even for the hybrid solutions, is much slower for values of a close to $\sqrt{2}$ (see Fig 4) than for larger values of a (see Fig 5).

The van der Pol oscillator is an example of a nonconservative system, for which both the frequency ν and the initial amplitude a of the motion appear as unknowns in the problem formulation. For this case, we again note (see Fig 6) that the hybrid solutions appear to be converging to the "exact" (numerically determined) frequencies as the number of terms in the approximation is increased. This convergence appears to hold even for values of ϵ greater than the radius of convergence of the perturbation expansion, which is approximately 1.85. However, the rate of convergence of the hybrid solutions for $\epsilon > 1.85$ is not as dramatic as in the previous examples. Although we do not as yet have a good explanation for this behavior, we feel (intuitively) that it is due to the rather complicated form of the response u as ϵ becomes large (see e.g. [13]). It simply appears that this behavior cannot be well approximated by the perturbation coefficient functions which have been included so far in our hybrid approximation.

When perturbation expansions of the solution about more than one value of the parameter ϵ are available, it has been our experience that it is usually advantagous to use at least one term from each of these expansions. This is illustrated in the springs-and-mass example as well as for the van der Pol oscillator.

In the springs-and-mass example, the exact form of the first term in the small-ϵ perturbation expansion, i.e. $u_0 = \cos(x)$, was used in the hybrid solution, along with an approximation to the exact form of the first term in the large-ϵ perturbation expansion, which is again $\cos(x)$. In particular, we did not include in our hybrid approximation the various inner expansions which should be combined with the outer expansion $\cos(x)$ to form a uniformly valid first approximation to the solution when ϵ is large. Nonetheless, the hybrid results are very accurate for all values of the expansion parameter ϵ, as well as for an entire range of values of the parameter μ (see Fig 7). This illustrates the fact that apparently one does not need to determine the "complete" form of the perturbation coefficient functions and that merely a good approximation to these functions may suffice for use in the hybrid approximation.

For the van der Pol oscillator, the exact form of the large-ϵ perturbation solution is very complicated, although several terms in the various inner and outer expansions necessary to construct this expansion have been computed [13]. However, in our approximation, we used only the leading term from this expansion (which is "difficult" to compute), along with several of the small-ϵ perturbation coordinate functions (which are "easy" to compute) to form our hybrid approximation. As illustrated in Fig 8, there is a definite improvement

in the quality of the hybrid approximation, although the agreement with the numerically determined frequencies is by no means entirely satisfactory. Presumably, the accuracy of our approximations could be improved by including more terms from either the small- or large-ϵ perturbation expansions. However, we have not carried out these calculations.

The compound pendulum is an example of the application of our method to systems with more than one degree of freedom. We note from Figs 9 and 10 that the perturbation expansions of the different resonant frequencies appear to have significantly different radii of convergence. In particular, the perturbation expansion of the higher resonant frequency, corresponding to the asymmetric mode of oscillation, has a much smaller radius of convergence (approximately $\epsilon^2 = 0.092$) than the expansion of the lower resonant frequency, corresponding to the symmetric mode of oscillation (approximately $\epsilon^2 = 1.2$). Hence, the two perturbation expansions will be useful for calculating approximations to the corresponding resonant frequency over different ranges of values of energy E (approximately $0 \leq E < 0.17$ for the asymmetric case vs. $0 \leq E < 2.3$ for the symmetric case). In both cases, however, the hybrid approximations based on only a few perturbation functions appear to converge to the numerically determined solutions, even for values of ϵ well beyond the respective radius of convergence. We are currently developing the hybrid method in the context of general Hamiltonian or Lagrangian systems (and even more general systems whose solution can be characterized in terms of a variational principle) and are applying the methods to several other examples. The results of these investigations will be reported elsewhere.

We note that the perturbation parameters ϵ employed in the above-mentioned problems have a variety of interpretations, but in each case $\epsilon = 0$ corresponds to simple harmonic motion.

While we have for reasons of space limitation only reported on frequency functions, the hybrid technique appears to converge to correct mode shapes as well as to correct frequencies.

The hybrid method is more successful in some of the examples reported here than in others. We think it is particularly effective for large values of ϵ in Fig 2, for large values of a in Fig 5, and for all values of μ in Fig 7. It seems to be less effective, as in the simple pendulum problem, when "directly" approaching a singularity rather than merely approaching a radius of convergence.

We also note that the equations, such as (10) and (53), which determine the hybrid amplitudes $\{\delta_j\}$ and $\tilde{\nu}$ for nonlinear systems will, in general, be nonlinear and have multiple solutions. Some of these solutions may be ruled out on the basis that they involve complex numbers. Of the remaining solutions, it appears that the only ones of interest are those for which the $\delta_j(\epsilon)$ coincide with the gauge functions of the perturbation expansion in the limit $\epsilon \to 0$. Therefore following a solution path starting at $\epsilon = 0$ seems to be an essential part of

the method. In our experience this leads in each case to a unique solution.

It is interesting to note that the hybrid method provides a nice supplement to some of the methods and ideas presented by Van Dyke [14] for improving the usefulness of a perturbation expansion. In particular, Van Dyke's ideas are applicable when a fairly large number of terms in a perturbation expansion can be computed (usually with the aid of numerical or symbolic computation). The coefficients in the series are then analyzed to help uncover some of the analytical structure of the solution and then this information is used to recast the series into a different form which, in general, will be valid for a wider range of parameter values. In contrast, the hybrid method requires only a few terms (often only one or two terms) in the perturbation expansion to be computed and then uses these terms to construct an "improved" approximation to the solution.

In conclusion, it appears that the hybrid perturbation-Galerkin technique is a useful way to enhance the usefulness of perturbation solutions to resonant frequency calculation problems. In particular, the hybrid solutions $\tilde{\nu}$ provide useful approximations to the resonant frequencies for the examples illustrated here, even for parameter values well beyond the radius of convergence of the perturbation solutions on which they are based. We are currently investigating application of the method to more general Hamiltonian and Lagrangian systems with a finite number of degrees of freedom, as well as to systems with an infinite number of degrees of freedom, i.e. to partial differential equations. The initial results of these investigations have been very promising.

While there are still many theoretical questions to be answered about the hybrid method, it is based on an intuitively plausible idea, it is relatively simple to implement, and it appears to provide reasonable and often very accurate approximate solutions.

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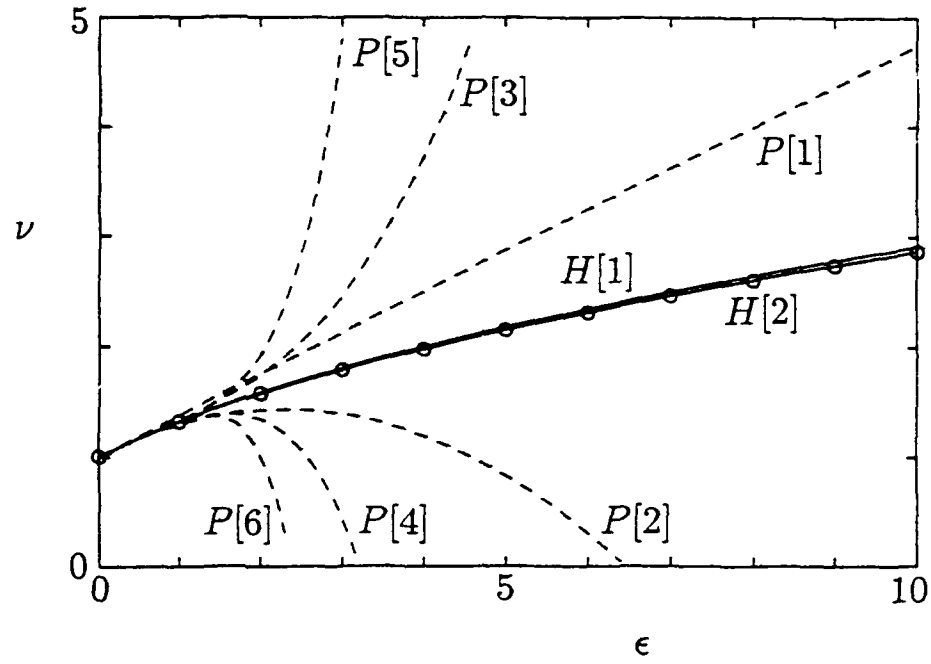

FIG 1. Comparison of hybrid (solid lines) frequency computations for the Duffing oscillator with perturbation (dashed lines) and numerical (circles) computations for a small range of ϵ values.

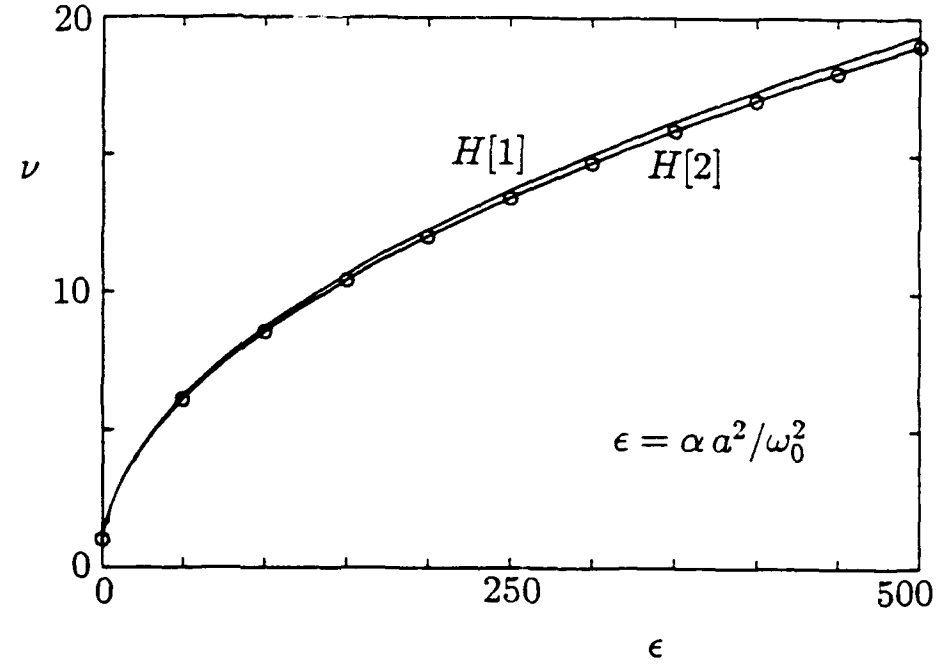

FIG 2. Comparison of hybrid (solid lines) frequency computations for the Duffing oscillator with numerical (circles) computations for a large range of ϵ values.

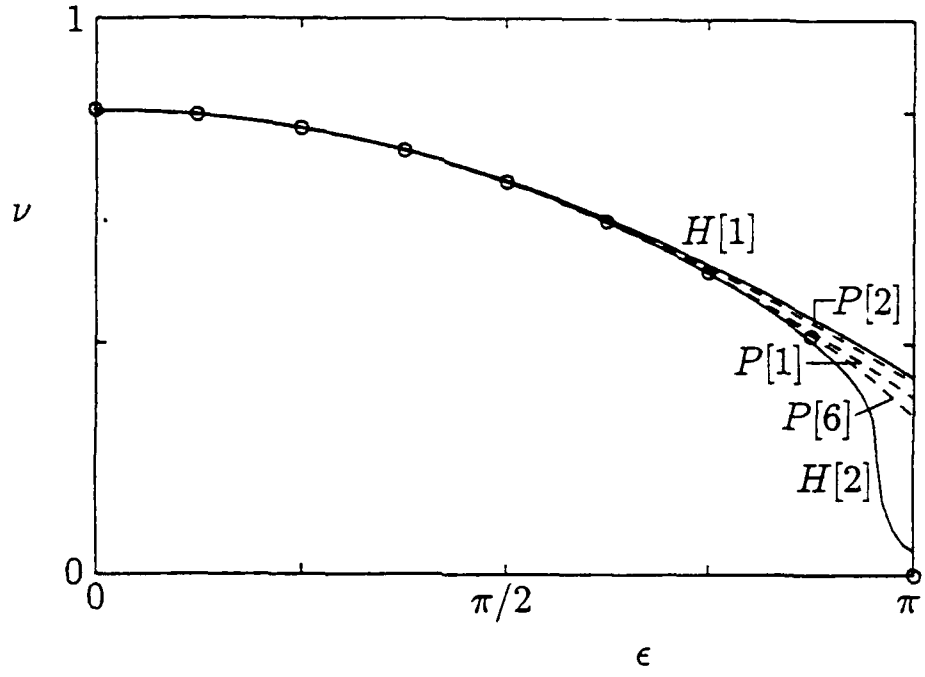

FIG 3. Comparison of hybrid (solid lines) frequency computations for the simple pendulum with perturbation (dashed lines) and numerical (circles) computations. The amplitude ranges from 0 to π.

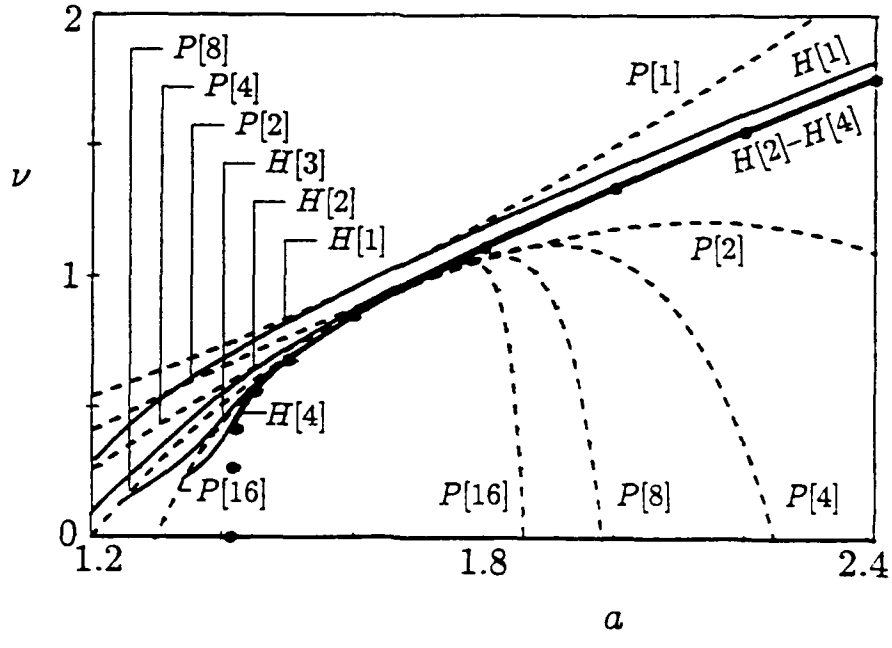

FIG 4. Comparison of hybrid (solid lines) frequency computations with perturbation (dashed lines) and numerical (dots) computations for a small range of ϵ values.

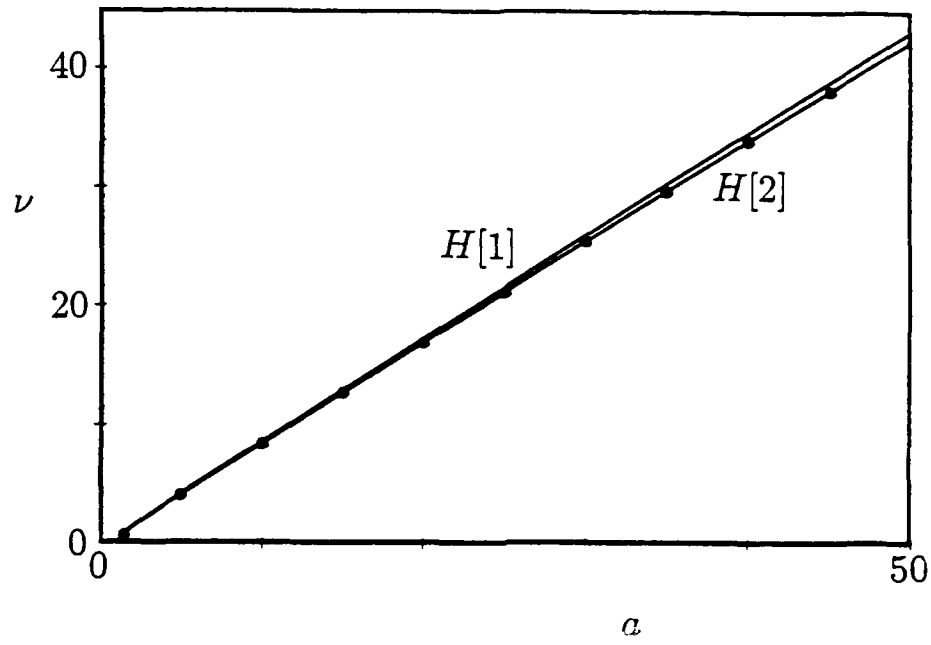

FIG 5. Comparison of hybrid (solid lines) frequency computations with numerical (dots) computations for a large range of ϵ values.

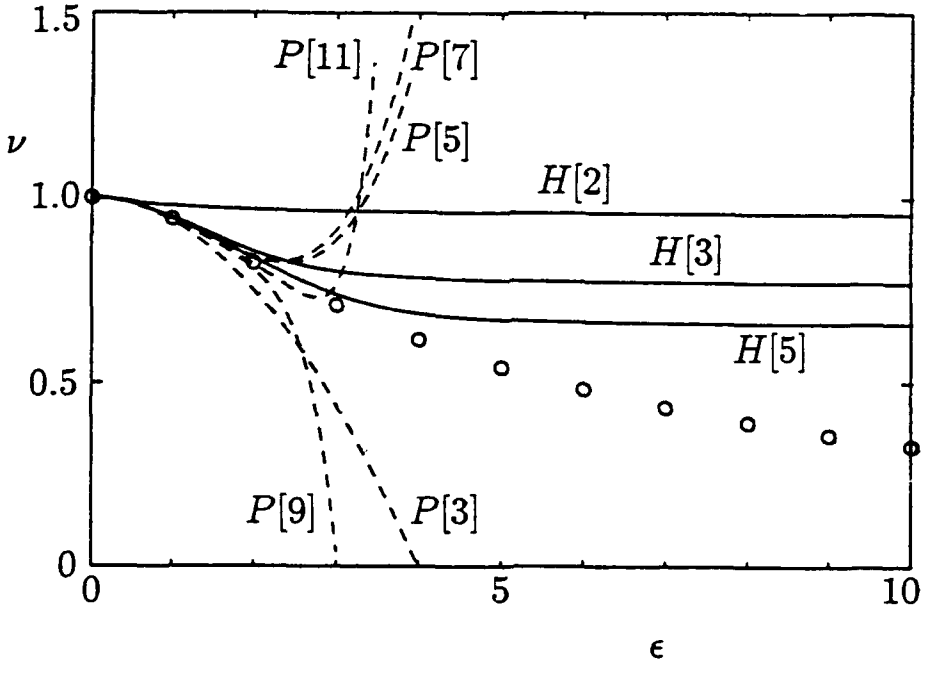

FIG 6. Comparison of hybrid (solid lines) frequency computations for the limit cycle of the van der Pol oscillator with perturbation (dashed lines) and numerical (circles) computations for a range of ϵ values.

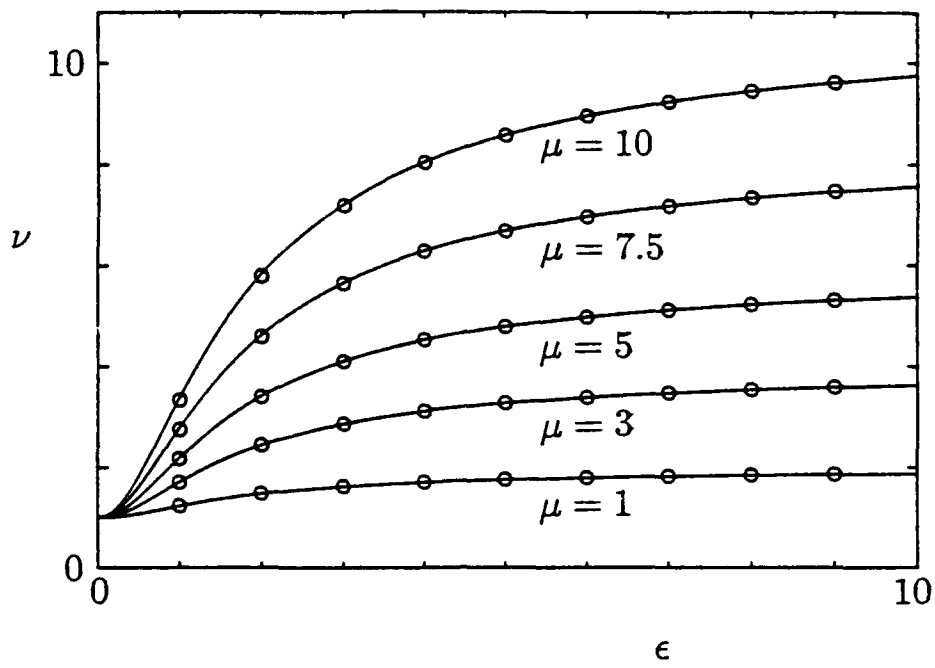

FIG 7. Comparison of hybrid (solid lines) frequency computations as a function of ϵ for the springs-and-mass system with numerical (circles) computations for a variety of μ values.

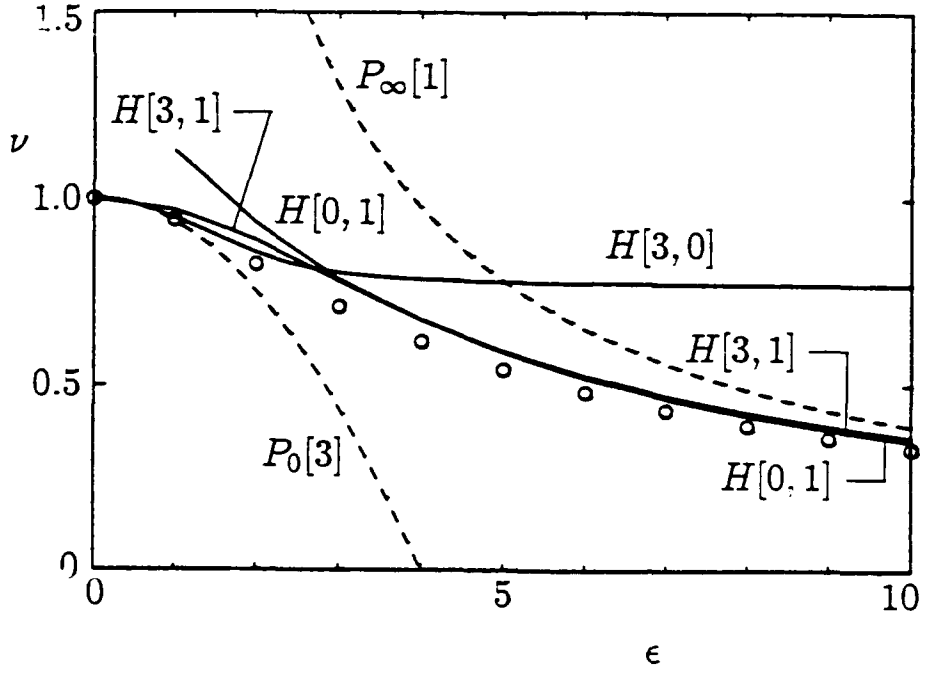

FIG 8. Comparison of hybrid (solid lines) frequency computations as a function of ϵ for the limit cycle of the van der Pol oscillator with perturbation (dashed lines) and numerical (circles) computations. The $H[3, 1]$ solution is based on two different perturbation expansions.

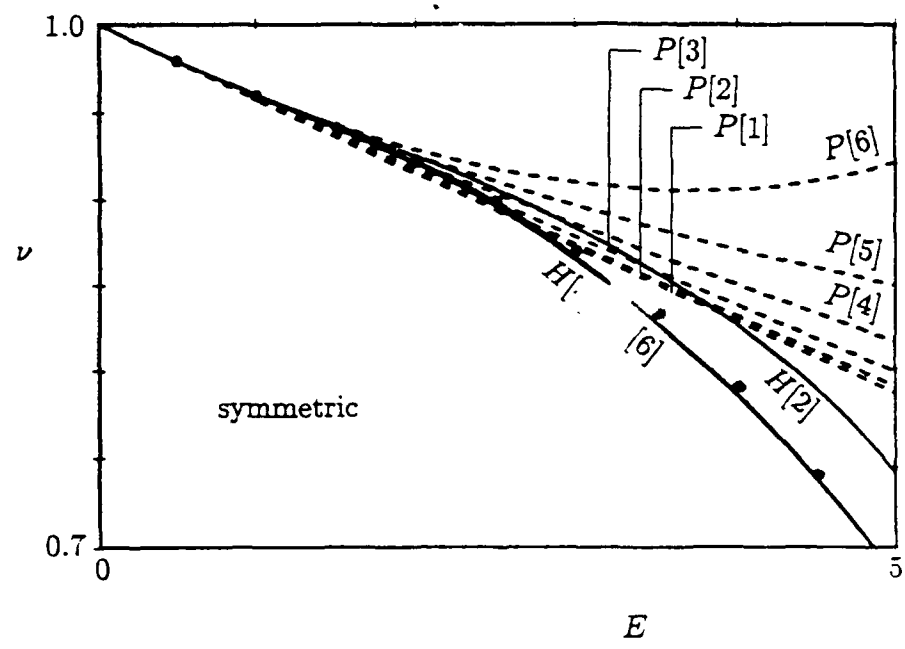

FIG 9. Comparison of hybrid (solid lines) frequency computations as a function of E for the "symmetric" mode of the double pendulum with perturbation (dashed lines) and numerical (dots) computations.

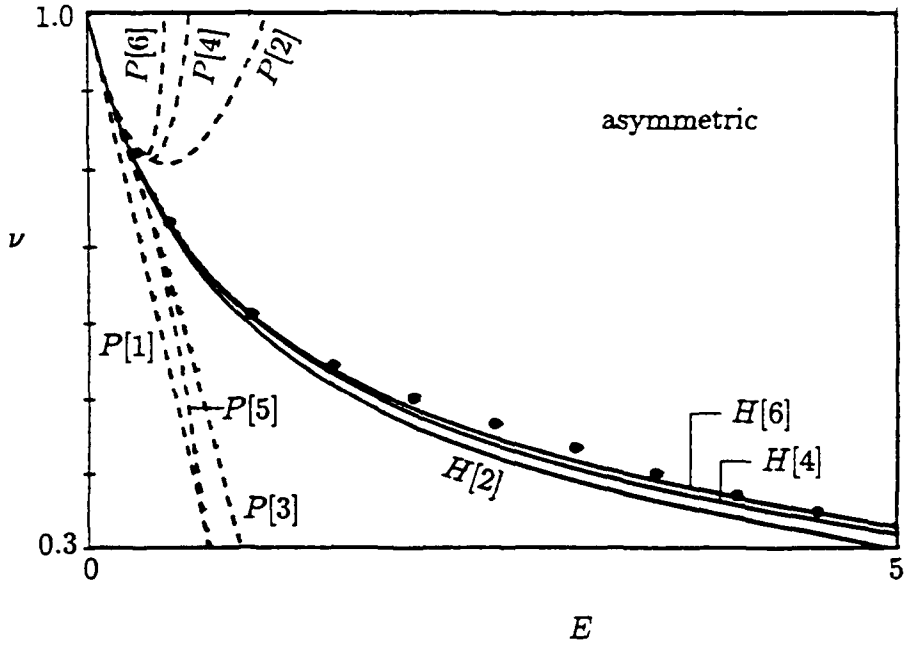

FIG 10. Comparison of hybrid (solid lines) frequency computations as a function of E for the "asymmetric" mode of the double pendulum with perturbation (dashed lines) and numerical (dots) computations.

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Final Report

16. Abstract

A two step hybrid perturbation-Galerkin technique is applied to the problem of determining the resonant frequencies of one- or several-degree(s)-of-freedom nonlinear systems involving a parameter. In step one, the Lindstedt-Poincaré method is used to determine perturbation solutins which are formally valid about one or more special values of the parameter (e.g. for small or large values of the parameter). In step two, a subset of the perturbation functions determined in step one is used in a Gelerkin type approximation. The technique is illustrated for several one-degree-of-freedom systems, including the Duffing and van der Pol oscillators, as well as for the compound pendulum. For all of the examples considered, it is shown that the frequencies obtained by the hybrid technique using only a few terms from the perturbation solutions are significantly more accurate than the perturbation results on which they are based, and they compare very well with frequencies obtained by purely numerical methods.

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