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TIME RATES OF GENERALIZED STRAIN TENSORS PART II: APPROXIMATE BASIS-FREE FORMULAS

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Utilizing the component formulas derived in Part I, we obtain approximate basis-free formulas for the material time derivative of a generalized Lagrangian strain tensor and for the Jaumann rate of a generalized Eulerian strain tensor. These formulas provide good estimates when the shear strains are small, regardless of the volumetric strain and the strain rate. We also derive explicit bounds for the error in the approximate formulas. By applying our general results to the logarithmic strain tensors, we obtain improved versions of approximate basis-free formulas due to Hill and Gurtin and Spear.

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Contents

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1 Introduction

This is the second in a series of three papers dealing with the time rates of generalized strain tensors. The main results of Part I (Scheidler [6]) may be summarized as follows. Let $\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$ and $\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$ denote the right and left stretch tensors, respectively, corresponding to a C^2 motion. Here $\{\lambda_i\}$ are the principal stretches, $\{\mathbf{u}_i\}$ is any principal basis of $\mathbf{U}$, and $\{\mathbf{v}_i\}$ is the corresponding principal basis of $\mathbf{V}$. Let $\mathbf{R}^+$ denote the positive reals. Then for any C^1 function $f : \mathbf{R}^+ \to \mathbf{R}$, the material time derivative of $\mathbf{f}(\mathbf{U}) \equiv \sum_{i=1}^{3} f(\lambda_i)\mathbf{u}_i \otimes \mathbf{u}_i$ and the Jaumann rate of $\mathbf{f}(\mathbf{V}) \equiv \sum_{i=1}^{3} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i$ are given by the component formulas

$$\mathbf{f}(\mathbf{U})^{\cdot} = \sum_{i,j=1}^{3} \hat{F}(\lambda_i, \lambda_j) D_{ij} \mathbf{u}_i \otimes \mathbf{u}_j \,, \tag{1.1}$$

$$\mathbf{f}(\mathbf{V})^{\circ} = \sum_{i,j=1}^{3} F(\lambda_i, \lambda_j) D_{ij} \mathbf{v}_i \otimes \mathbf{v}_j \,, \tag{1.2}$$

where

$$\hat{F}(\lambda_i, \lambda_j) = F(\lambda_i, \lambda_j) = \lambda_i f'(\lambda_i) \quad \text{if } \lambda_i = \lambda_j \,, \tag{1.3}$$

$$\hat{F}(\lambda_i, \lambda_j) = \frac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \quad \text{if } \lambda_i \neq \lambda_j \,, \tag{1.4}$$

$$F(\lambda_i, \lambda_j) = \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i + \lambda_j} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \quad \text{if } \lambda_i \neq \lambda_j \,, \tag{1.5}$$

and $\{D_{ij}\}$ are the components of the stretching tensor $\mathbf{D}$ relative to $\{\mathbf{v}_i\}$. Recall that the Jaumann rate $\mathbf{f}(\mathbf{V})^{\circ}$ is related to the material time derivative $\mathbf{f}(\mathbf{V})^{\cdot}$ by

$$\mathbf{f}(\mathbf{V})^{\cdot} = \mathbf{f}(\mathbf{V})^{\circ} + \mathbf{W}\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{W} \,, \tag{1.6}$$

where $\mathbf{W}$ denotes the spin tensor. When $f(1) = 0$, $f'(1) = 1$ and $f' > 0$, f is called a *strain measure*, and $\mathbf{f}(\mathbf{U})$ and $\mathbf{f}(\mathbf{V})$ are called the *generalized Lagrangian* and *Eulerian strain tensors*, respectively, corresponding to the strain measure f.

Because of their simplicity these formulas are useful in theoretical analyses, as demonstrated for the formula (1.1) by Hill [3,4]. On the other hand, to actually compute $\mathbf{f}(\mathbf{V})^{\circ}$, say, for a given motion from the formulas above requires the calculation of the eigenvalues and eigenvectors of $\mathbf{V}$ at each place and time. Hence it would also be useful to have a simple expression for the tensor $\mathbf{f}(\mathbf{V})^{\circ}$ directly in terms of the tensors $\mathbf{V}$ and $\mathbf{D}$, or in terms of the left Cauchy-Green tensor $\mathbf{B}$ and $\mathbf{D}$. Unfortunately, as we will show in Part III, the coefficients in such basis-free formulas are extremely complicated when the principal stretches are distinct. Thus we are led to seek simple approximations to the basis-free formulas for $\mathbf{f}(\mathbf{U})^{\cdot}$ and $\mathbf{f}(\mathbf{V})^{\circ}$. In principle, these approximate formulas could be derived from the exact basis-free formulas obtained in Part III. However, the complexity of the exact basis-free formulas makes this approach impractical. Instead, we will derive the approximate basis-free formulas from the exact component formulas above. Our approximate formulas involve an arbitrary parameter λ which can always be chosen in such a way that the formulas provide good estimates when the shear strains are small, regardless of the volumetric strain and the strain rates.

In Section 2 we introduce some notation and state the assumptions used in deriving the approximate formulas. In Section 3 we present the main results of this paper, namely the approximate basis-free formulas for $\mathbf{f}(\mathbf{U})^{\cdot}$ and $\mathbf{f}(\mathbf{V})^{\circ}$, together with explicit bounds for the error in these formulas. In Section 4 we list some approximate formulas involving the spin $\boldsymbol{\Omega}$ of the rotation tensor, as well as some approximate formulas for the tensors $\mathbf{f}(\mathbf{U})$ and $\mathbf{f}(\mathbf{V})$. The derivation of the results in Sections 3 and 4 is sketched in Section 5. In Section 6 we apply the general results of the preceding sections to the logarithmic strain tensors. We give a rigorous proof of an approximate formula for $(\ln \mathbf{U})^{\cdot}$ due to Hill [4], and we obtain an improved version of an approximate formula for $(\ln \mathbf{V})^{\circ}$ due to Gurtin and Spear [2].

2 Notation and basic assumptions

The derivation of the approximate formulas involves the use of Taylor's Theorem to express $f(\lambda_i)$ and $f'(\lambda_i)$ $(i = 1, 2, 3)$ in terms of f and its derivatives evaluated at some number λ. Our results take the general form

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_n(\lambda) + \varepsilon^n \boldsymbol{\Delta}_n \,, \tag{2.1}$$

where n is a positive integer (usually 1 or 2), $\boldsymbol{\Gamma}_n(\lambda)$ is the basis-free approximation to some tensor $\boldsymbol{\Gamma}$ such as $\mathbf{f}(\mathbf{U})^{\cdot}$ or $\mathbf{f}(\mathbf{V})^{\circ}$, and

$$\varepsilon \equiv \max_{i=1,2,3} |\lambda_i - \lambda| \,. \tag{2.2}$$

We determine explicit bounds on the norm of the term $\boldsymbol{\Delta}_n$. This allows us to bound the error in approximating $\boldsymbol{\Gamma}$ by $\boldsymbol{\Gamma}_n(\lambda)$. Roughly speaking, $\boldsymbol{\Gamma}_n(\lambda)$ is a good approximation to $\boldsymbol{\Gamma}$ when ε is sufficiently small, i.e., when all principal stretches are sufficiently close to λ. Thus for a given set of principal stretches a reasonable choice for λ is the one which minimizes ε. However, in view of the dependence of $\boldsymbol{\Gamma}_n$ on λ, the most useful choice for λ will generally depend on the intended applications of the approximate formula; some examples are given at the end of this section. Unless stated otherwise, we regard λ as arbitrary. Note that since the principal stretches are time-dependent scalar fields, in general λ, ε and the other scalars introduced below are also time-dependent scalar fields.

Let

$$\lambda_{min} \equiv \min\{\lambda_1, \lambda_2, \lambda_3\}, \quad \lambda_{max} \equiv \max\{\lambda_1, \lambda_2, \lambda_3\}. \tag{2.3}$$

Then

$$\varepsilon = \max\{\lambda_{max} - \lambda, \lambda - \lambda_{min}\}. \tag{2.4}$$

For any symmetric tensor $\mathbf{A}$ let $s(\mathbf{A})$ denote the *maximum orthogonal shear component* of $\mathbf{A}$:

$$s(\mathbf{A}) \equiv \max_{\mathbf{a},\mathbf{b}}(\mathbf{a} \cdot \mathbf{A}\mathbf{b}), \tag{2.5}$$

where $\mathbf{a}$ and $\mathbf{b}$ range over all pairs of orthogonal unit vectors. Then (Ericksen [1, §46])

$$s(\mathbf{U}) = s(\mathbf{V}) = (\lambda_{max} - \lambda_{min})/2 \equiv s. \tag{2.6}$$

Thus s is the largest shear strain corresponding to the stretch tensors $\mathbf{U}$ and $\mathbf{V}$. From (2.4) it follows that

$$s \le \varepsilon \tag{2.7}$$

for all choices of λ, and

$$\varepsilon \le 2s, \quad \text{if } \lambda_{min} \le \lambda \le \lambda_{max}. \tag{2.8}$$

Hence, if ε is small then the shear strains are small. Conversely, if λ satisfies $\lambda_{min} \le \lambda \le \lambda_{max}$ and if the shear strains are small, then ε is small. Also note that

$$\varepsilon \le ||\mathbf{U} - \lambda\mathbf{I}|| = ||\mathbf{V} - \lambda\mathbf{I}|| \le \sqrt{3}\,\varepsilon. \tag{2.9}$$

Here $||\cdot||$ denotes the norm of a tensor; i.e., for any tensor $\mathbf{H}$,

$$||\mathbf{H}|| \equiv [\mathrm{tr}(\mathbf{H}^T\mathbf{H})]^{1/2} = \left[\sum_{i,j=1}^{3}(H_{ij})^2\right]^{1/2}, \tag{2.10}$$

where $\{H_{ij}\}$ are the components of $\mathbf{H}$ relative to any orthonormal basis. The inequality (2.9) implies that ε is small iff $\mathbf{U}$ and $\mathbf{V}$ are close to the dilatation $\lambda\mathbf{I}$. All the results in this paper are valid under the following conditions:

1. $\varepsilon < \lambda$; in view of (2.4) this is equivalent to $\lambda > \lambda_{max}/2$.

2. The motion is C^2.

3. The function $f : \mathbb{R}^+ \to \mathbb{R}$ is C^3.[1]

[1]The one exception is equation (4.6), which involves the first k derivatives of f for arbitrary k. However, only the cases $k \le 3$ will be utilized in the derivation of other formulas.

In particular, we do not require that f be a strain measure, although this is the case of most interest. The condition $\varepsilon < \lambda$ is imposed solely to obtain simple error bounds. For some choices of λ this condition is automatically satisfied, while for other choices of λ it imposes a constraint on the principal stretches (see the examples below).

The condition $\varepsilon < \lambda$ is equivalent to the existence of a scalar field r such that

$$\varepsilon \leq r\,\lambda\,, \qquad 0 \leq r < 1\,. \tag{2.11}$$

This, together with the definition of ε, implies that

$$\lambda_i \in I_r(\lambda) \equiv [(1-r)\lambda, (1+r)\lambda] \subset \mathbf{R}^+\,, \quad i = 1,2,3. \tag{2.12}$$

The factor $1/(1-r)$ will appear in some of the error bounds; hence these bounds are useful only for those motions for which r is bounded away from 1. Some of our results take a simpler form when expressed in terms of the scalar

$$\tilde{\varepsilon} \equiv \max_{i=1,2,3} \left| \frac{\lambda_i}{\lambda} - 1 \right| . \tag{2.13}$$

By definition (2.2) we have

$$\varepsilon = \lambda\tilde{\varepsilon}\,, \tag{2.14}$$

so that (2.11) is equivalent to the condition

$$\tilde{\varepsilon} \leq r < 1\,. \tag{2.15}$$

In the remainder of this section we discuss some appropriate choices for λ.

Example I. By (2.4) we see that the value of λ which minimizes ε and $\tilde{\varepsilon}$ is

$$\lambda = (\lambda_{min} + \lambda_{max})/2\,, \tag{2.16}$$

in which case $\varepsilon = (\lambda_{max} - \lambda_{min})/2 = s$. Thus ε is small iff the shear strains are small. Note that the condition $\varepsilon < \lambda$ is satisfied for all motions. Equivalently, (2.11) holds for

$$r \equiv \tilde{\varepsilon} = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}}\,, \tag{2.17}$$

in which case

$$\frac{1}{1-r} = \frac{1}{2}\left(\frac{\lambda_{max}}{\lambda_{min}} + 1\right) . \tag{2.18}$$

Thus $1/(1-r)$ is bounded iff the motion is such that $\lambda_{max}/\lambda_{min}$ is bounded.

Example II. At the other extreme, we may simply choose $\lambda = 1$. When f is a strain measure this choice yields the simplest expressions for the coefficients in the approximate formulas. As we will see in §6, this choice is sometimes useful for comparison with other approximate formulas in the literature. Note that the bounds $\lambda_{min} \leq \lambda \leq \lambda_{max}$ need not hold in this case, and that the condition $\varepsilon < \lambda$ reduces to $\varepsilon < 1$, which would usually be imposed anyway. Also note that ε is small iff all principal stretches are sufficiently close to 1 iff all principal strains are sufficiently close to zero. Hence for this choice of λ not only the shear strains but also the volumetric strain must small be to obtain good approximations.

Example III. A natural choice for λ is the geometric mean of the principal stretches:

$$\lambda = (\lambda_1 \lambda_2 \lambda_3)^{1/3}. \tag{2.19}$$

Then

$$\lambda = (\det \mathbf{U})^{1/3} = (\det \mathbf{V})^{1/3} = (\det \mathbf{F})^{1/3} = \left(\frac{\rho_0}{\rho}\right)^{1/3} = \left(\frac{v}{v_0}\right)^{1/3}, \tag{2.20}$$

where $\mathbf{F}$ denotes the deformation gradient. ρ and v denote the mass density and the specific volume, respectively, in the deformed configuration, and ρ_0 and v_0 denote the corresponding quantities in the reference configuration. Thus λ may also be interpreted as the *volumetric stretch*. Unlike (2.16), this choice of λ does not require knowledge of the principal stretches since λ can be determined directly from (2.20). In particular, $\lambda = 1$ for any isochoric motion. Since (2.19) implies $\lambda_{min} \leq \lambda \leq \lambda_{max}$, it follows from (2.7) and (2.8) that $s \leq \varepsilon \leq 2s$. Thus ε is small iff the shear strains are small. Note that the condition $\varepsilon < \lambda$ does not hold in general; e.g., if $\lambda_1 = \lambda_2/2$ and $\lambda_3 = 2\lambda_2$, then $\epsilon = \lambda = \lambda_2$. However, since a small ε is required for good approximations, for many applications the constraint $\varepsilon < \lambda$, or equivalently, the condition (2.11), is fairly mild. For example, in many solids elastic shear strains cannot exceed 1%, and density increases by a factor greater than 2 are extremely difficult to achieve. Hence, for elastic deformations of these solids we may take $s \leq 0.01$ and $\rho/\rho_0 \leq 2$, in which case $\varepsilon \leq 2s \leq 0.02$ and, by (2.20), $\lambda \geq 0.79$, so that (2.11) is satisfied for $r = 0.03$.

For λ given by (2.19) it is useful to introduce the *right* and *left distortional stretch tensors* $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$, respectively:

$$\tilde{\mathbf{U}} \equiv \frac{1}{\lambda}\mathbf{U}, \quad \tilde{\mathbf{V}} \equiv \frac{1}{\lambda}\mathbf{V}. \tag{2.21}$$

By (2.20) it follows that $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ are unaffected by the dilatational part of the deformation in the sense that the deformation gradients $\mathbf{F}$ and $a\mathbf{F}$ have the same distortional stretch tensors for any $a > 0$; in particular, $\det \tilde{\mathbf{U}} = \det \tilde{\mathbf{V}} = 1$. The eigenvalues $\{\lambda_i/\lambda\}$ of $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$, and the eigenvalues $\{\lambda_i/\lambda - 1\}$ of $\tilde{\mathbf{U}} - \mathbf{I}$ and $\tilde{\mathbf{V}} - \mathbf{I}$, are the *principal distortional stretches* and *strains*, respectively. These scalars, and hence $\tilde{\varepsilon}$, are also unaffected by the dilatational part of the deformation. The condition $\varepsilon < \lambda$, or equivalently, the condition (2.15), is simply the requirement that the principal distortional strains have magnitude less than 1.

Example IV. There are other choices for λ which retain the interpretation of a mean stretch. Examples are

$$\lambda = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) = \frac{1}{3}\mathrm{tr}\,\mathbf{U} = \frac{1}{3}\mathrm{tr}\,\mathbf{V} \tag{2.22}$$

and

$$\lambda = \frac{1}{\sqrt{3}}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right]^{1/2} = \frac{1}{\sqrt{3}}(\mathrm{tr}\,\mathbf{C})^{1/2} = \frac{1}{\sqrt{3}}(\mathrm{tr}\,\mathbf{B})^{1/2}, \tag{2.23}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ are the right and left Cauchy-Green tensors.

3 Main results

Let $\mathbf{R}$ denote the rotation tensor and let $\hat{\mathbf{D}}$ denote the rotated stretching tensor: $\hat{\mathbf{D}} \equiv \mathbf{R}^T\mathbf{D}\mathbf{R}$. Let

$$\alpha(\lambda) \equiv \frac{1}{2}[f'(\lambda) + \lambda f''(\lambda)] = \frac{1}{2}\frac{d}{d\lambda}[\lambda f'(\lambda)]. \tag{3.1}$$

In particular, $\alpha(\lambda) = (1 + f''(1))/2$ when $\lambda = 1$ and f is a strain measure. We will show that

$$\begin{aligned} \mathbf{f}(\mathbf{U})^{\cdot} &= \lambda f'(\lambda)\,\hat{\mathbf{D}} + \varepsilon\,\mathbf{\Phi}_1 \\ &= -\lambda^2 f''(\lambda)\,\hat{\mathbf{D}} + \alpha(\lambda)(\hat{\mathbf{D}}\mathbf{U} + \mathbf{U}\hat{\mathbf{D}}) + \varepsilon^2\mathbf{\Phi}_2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \mathbf{f}(\mathbf{V})^{\circ} &= \lambda f'(\lambda)\,\mathbf{D} + \varepsilon\,\mathbf{\Phi}_3 \\ &= -\lambda^2 f''(\lambda)\,\mathbf{D} + \alpha(\lambda)(\mathbf{D}\mathbf{V} + \mathbf{V}\mathbf{D}) + \varepsilon^2\mathbf{\Phi}_4. \end{aligned} \tag{3.3}$$

Here and below, $\mathbf{\Phi}_k$ $(k = 1, 2, \ldots, 8)$ is a symmetric tensor satisfying

$$\|\mathbf{\Phi}_k\| \le \varphi_k(\lambda)\,\|\mathbf{D}\| \tag{3.4}$$

for some continuous function φ_k. When the terms of order ε or ε^2 are neglected, equations (3.1)-(3.3) yield approximate basis-free formulas for $\mathbf{f}(\mathbf{U})^{\cdot}$ and $\mathbf{f}(\mathbf{V})^{\circ}$.

For any tensor field $\mathbf{H}$, we have

$$\begin{aligned} \mathbf{H}\,\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{H} &= \varepsilon\,\mathbf{\Psi}_1 \\ &= f'(\lambda)(\mathbf{H}\mathbf{V} - \mathbf{V}\mathbf{H}) + \varepsilon^2\mathbf{\Psi}_2. \end{aligned} \tag{3.5}$$

Here and below, $\mathbf{\Psi}_k$ ($k = 1, 2, 3$) is a tensor satisfying

$$||\mathbf{\Psi}_k|| \leq \psi_k(\lambda)\, ||\mathbf{H}|| \tag{3.6}$$

for some continuous function ψ_k. Then (3.5) with $\mathbf{H} = \mathbf{W}$, together with (3.3) and (1.6), yield approximate basis-free formulas for $\mathbf{f}(\mathbf{V})^{\cdot}$.

Let $\mathcal{O}$ denote a tensor-valued function of a nonnegative real variable such that $||\mathcal{O}(h)|| \leq Mh$ for some constant M and all sufficiently small $h \geq 0$. Then $(3.3)_1$ and (3.4) imply that $\mathbf{f}(\mathbf{V})^{\circ} = \lambda f'(\lambda)\,\mathbf{D} + \mathcal{O}(\varepsilon)$. Similarly, we could replace the term $\varepsilon^2\mathbf{\Phi}_4$ in $(3.3)_2$ by $\mathcal{O}(\varepsilon^2)$. This is typically the way that approximate formulas are stated in the literature; cf. the examples in §6. While such results are of theoretical interest, they are of little practical use. If we know only that $\mathbf{f}(\mathbf{V})^{\circ} = \mathbf{\Gamma}_n + \mathcal{O}(\varepsilon^n)$, then we may conclude that $||\mathbf{f}(\mathbf{V})^{\circ} - \mathbf{\Gamma}_n||/\varepsilon^{n-1} \to 0$ as $\varepsilon \to 0$, but for a given value of ε we cannot determine the error in approximating $\mathbf{f}(\mathbf{V})^{\circ}$ by $\mathbf{\Gamma}_n$. To do this we need explicit bounds for the remainder terms. These bounds can be obtained from (3.4) and (3.6) provided that we have explicit for formulas for the functions φ_k and ψ_k. Such formulas are given below.

The stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are irrational functions of the deformation gradient $\mathbf{F}$. For computational purposes it would be useful to have formulas analogous to $(3.2)_2$, $(3.3)_2$ and $(3.5)_2$, but involving the Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ instead. Let

$$\beta(\lambda) \equiv \frac{\lambda}{2}[f'(\lambda) - \lambda f''(\lambda)]\,, \tag{3.7}$$

$$\gamma(\lambda) \equiv \frac{1}{4}\left[\frac{f'(\lambda)}{\lambda} + f''(\lambda)\right] = \frac{\alpha(\lambda)}{2\lambda}\,. \tag{3.8}$$

In particular, $\beta(\lambda) = (1 - f''(1))/2$ and $\gamma(\lambda) = (1 + f''(1))/4$ when $\lambda = 1$ and f is a strain measure. The analogs of $(3.2)_2$, $(3.3)_2$ and $(3.5)_2$ are

$$\begin{aligned} \mathbf{f}(\mathbf{U})^{\cdot} &= \beta(\lambda)\,\hat{\mathbf{D}} + \gamma(\lambda)(\hat{\mathbf{D}}\mathbf{C} + \mathbf{C}\hat{\mathbf{D}}) + \varepsilon^2\mathbf{\Phi}_5\,, && (3.9)\\ \mathbf{f}(\mathbf{V})^{\circ} &= \beta(\lambda)\,\mathbf{D} + \gamma(\lambda)(\mathbf{D}\mathbf{B} + \mathbf{B}\mathbf{D}) + \varepsilon^2\mathbf{\Phi}_6\,, && (3.10) \end{aligned}$$

and

$$\mathbf{H}\,\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{H} = \frac{f'(\lambda)}{2\lambda}(\mathbf{H}\mathbf{B} - \mathbf{B}\mathbf{H}) + \varepsilon^2\mathbf{\Psi}_3 \tag{3.11}$$

for any tensor field $\mathbf{H}$.

Now suppose that $f'(\lambda) \neq 0$, as is the case when f is a strain measure. Let

$$\mu(\lambda) \equiv \frac{1}{2}\left[1 + \frac{\lambda f''(\lambda)}{f'(\lambda)}\right]\,, \quad \nu(\lambda) \equiv \lambda f'(\lambda) - 2\mu(\lambda)f(\lambda)\,. \tag{3.12}$$

In particular, $\mu(\lambda) = (1 + f''(1))/2$ and $\nu(\lambda) = 1$ when $\lambda = 1$ and f is a strain measure. We will show that

$$\mathbf{f}(\mathbf{U})^{\cdot} = \nu(\lambda)\,\hat{\mathbf{D}} + \mu(\lambda)[\hat{\mathbf{D}}\,\mathbf{f}(\mathbf{U}) + \mathbf{f}(\mathbf{U})\hat{\mathbf{D}}] + \varepsilon^2\mathbf{\Phi}_7 \tag{3.13}$$

and

$$\begin{aligned}\mathbf{f}(\mathbf{V})^{\cdot} \;=\;& \nu(\lambda)\,\mathbf{D} + [\mu(\lambda)\,\mathbf{D} + \mathbf{W}]\,\mathbf{f}(\mathbf{V}) + \mathbf{f}(\mathbf{V})\,[\mu(\lambda)\,\mathbf{D} - \mathbf{W}] \\ & + \varepsilon^2\mathbf{\Phi}_8\,.\end{aligned} \tag{3.14}$$

When the terms of order ε^2 are neglected, (3.13) and (3.14) yield approximate linear ordinary differential equations for $\mathbf{f}(\mathbf{U})$ and $\mathbf{f}(\mathbf{V})$. For a given material point $\mathbf{X}$, integration of (3.14) (with $\varepsilon^2\mathbf{\Phi}_8$ neglected) requires only the time histories of λ and the stretching and spin tensors corresponding to $\mathbf{X}$, together with the value of $\mathbf{f}(\mathbf{V}(\mathbf{X}, t_0))$ at some initial time t_0. Note that if λ is taken to be the volumetric stretch (cf. 2.20), then λ satisfies the ordinary differential equation

$$\dot{\lambda} = \frac{1}{3}(\mathrm{tr}\,\mathbf{D})\lambda\,. \tag{3.15}$$

To obtain bounds for the error when the remainder terms are neglected in the formulas above, we need explicit expressions for the functions φ_k and ψ_k in (3.4) and (3.6). Recall the definition (2.12) of the interval $I_r(\lambda)$. Let $f^{(k)}$ denote the kth derivative of f, and let

$$f_k(\lambda) \equiv \max_{x\in I_r(\lambda)} |f^{(k)}(x)|\,. \tag{3.16}$$

Then we may take

$$\begin{aligned}
\psi_1(\lambda) &= 2f_1(\lambda)\,, \quad \psi_2(\lambda) = f_2(\lambda)\,, && (3.17)\\
\psi_3(\lambda) &= \frac{|f'(\lambda)|}{2\lambda} + f_2(\lambda)\,, && (3.18)\\
\varphi_1(\lambda) &= \varphi_3(\lambda) = \frac{|f'(\lambda)| + 2\lambda f_2(\lambda)}{1-r}\,, && (3.19)\\
\varphi_2(\lambda) &= \varphi_4(\lambda) = \frac{1}{1-r}\left[\frac{|f'(\lambda)|}{\lambda} + |f''(\lambda)| + \lambda f_3(\lambda)\right], && (3.20)\\
\varphi_5(\lambda) &= \varphi_6(\lambda) = 3\varphi_2(\lambda)/2\,, && (3.21)\\
\varphi_7(\lambda) &= \varphi_8(\lambda) = \varphi_2(\lambda) + 2|\mu(\lambda)|f_2(\lambda)\,. && (3.22)
\end{aligned}$$

If f'' has constant sign, then $f_2(\lambda)$ may be replaced by $f_2(\lambda)/2$ in $(3.17)_2$, (3.18) and (3.19). If f''' has constant sign, then $f_3(\lambda)$ may be replaced by $f_3(\lambda)/2$ in (3.20). In view of (2.14), we may replace ε by $\hat{\varepsilon}$ in the remainder terms, provided that the right-hand side of $(3.17)_1$ and (3.19) is multiplied by

λ, and the right-hand side of $(3.17)_2$, (3.18) and (3.20)–(3.22) is multiplied by λ^2.

For the special case $f(x) = x$ we have $\mathbf{f}(\mathbf{V}) = \mathbf{V}$, and the results in this section reduce to

$$\begin{aligned} \mathbf{V}^\circ &= \lambda \mathbf{D} + \varepsilon\, \mathbf{\Phi}_3 \\ &= \frac{1}{2}(\mathbf{DV} + \mathbf{VD}) + \varepsilon^2 \mathbf{\Phi}_4 \\ &= \frac{\lambda}{2}\mathbf{D} + \frac{1}{4\lambda}(\mathbf{DB} + \mathbf{BD}) + \varepsilon^2 \mathbf{\Phi}_6 , \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \dot{\mathbf{V}} &= \mathbf{V}^\circ + \mathbf{WV} - \mathbf{VW} \\ &= \mathbf{V}^\circ + \frac{1}{2\lambda}(\mathbf{WB} - \mathbf{BW}) + \varepsilon^2 \mathbf{\Psi}_3 , \end{aligned} \tag{3.24}$$

where

$$||\mathbf{\Psi}_3|| \le \frac{1}{2\lambda}||\mathbf{W}|| , \tag{3.25}$$

$$||\mathbf{\Phi}_3||,\ \ \lambda||\mathbf{\Phi}_4||,\ \ \frac{2\lambda}{3}||\mathbf{\Phi}_6|| \ \le \frac{||\mathbf{D}||}{1-r} . \tag{3.26}$$

4 Some additional formulas

Let $\mathbf{\Omega}$ denote the spin of the rotation tensor: $\mathbf{\Omega} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{\Omega}^{\mathrm{T}}$. In addition to the Jaumann rate of $\mathbf{f}(\mathbf{V})$, the corotational rate $\mathbf{f}(\mathbf{V})^{*}$ is also of interest. Recall (Part I, §5) that $\mathbf{f}(\mathbf{V})^{*}$ is defined in terms of $\mathbf{\Omega}$ and the material time derivative of $\mathbf{f}(\mathbf{V})$ by

$$\mathbf{f}(\mathbf{V})^{\cdot} = \mathbf{f}(\mathbf{V})^{*} + \mathbf{\Omega}\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{\Omega} . \tag{4.1}$$

Approximate formulas for the term $\mathbf{\Omega}\,\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{\Omega}$, as well as bounds for the error in these formulas, are obtained by setting $\mathbf{H} = \mathbf{\Omega}$ in (3.5), (3.6) and (3.11). Moreover, we will show that $\mathbf{f}(\mathbf{V})^\circ$ *may be replaced by* $\mathbf{f}(\mathbf{V})^{*}$ *in* (3.3) *and* (3.10). Of course, the tensors $\mathbf{\Phi}_4$ and $\mathbf{\Phi}_6$ in the remainder terms will generally be different, but they satisfy the same bounds as before. It follows that $\mathbf{f}(\mathbf{V})^{*}$ and $\mathbf{f}(\mathbf{V})^\circ$ agree to within terms of order ε^2. In fact, we have

$$\mathbf{f}(\mathbf{V})^{*} = \mathbf{f}(\mathbf{V})^\circ + \tilde{\varepsilon}^{\,2}\tilde{\mathbf{\Phi}}_1 . \tag{4.2}$$

Here and below, $\tilde{\mathbf{\Phi}}_k$ $(k = 1, 2, 3, 4)$ is a symmetric tensor satisfying

$$||\tilde{\mathbf{\Phi}}_k|| \le \tilde{\varphi}_k(\lambda)||\mathbf{D}|| \tag{4.3}$$

for some continuous function $\tilde{\varphi}_k$.

Since equations (3.2), (3.9) and (3.13) for $\mathbf{f}(\mathbf{U})^{\cdot}$ involve the rotation tensor $\mathbf{R}$ through the rotated stretching tensor $\hat{\mathbf{D}}$, and since $\dot{\mathbf{R}} = \mathbf{\Omega R}$, approximate formulas for $\mathbf{\Omega}$ in terms of $\mathbf{W}$ are also of interest. We have

$$\begin{aligned} \mathbf{\Omega} &= \mathbf{W} + \tilde{\varepsilon}\,\tilde{\mathbf{\Phi}}_2 \\ &= \mathbf{W} + \frac{1}{2\lambda}(\mathbf{DV} - \mathbf{VD}) + \tilde{\varepsilon}^{\,2}\tilde{\mathbf{\Phi}}_3 \\ &= \mathbf{W} + \frac{1}{4\lambda^2}(\mathbf{DB} - \mathbf{BD}) + \tilde{\varepsilon}^{\,2}\tilde{\mathbf{\Phi}}_4 \,. \end{aligned} \tag{4.4}$$

For the functions $\tilde{\varphi}_k$ in the above formulas we may take

$$\tilde{\varphi}_1 = \frac{2\lambda f_1(\lambda)}{(1-r)^3}, \quad \tilde{\varphi}_2 = \tilde{\varphi}_3 = \frac{4}{5}\tilde{\varphi}_4 = \frac{1}{1-r}. \tag{4.5}$$

Finally, we consider some approximate formulas for $\mathbf{f}(\mathbf{V})$ and $\mathbf{f}(\mathbf{U})$. If f is C^k ($k \geq 1$) then

$$\mathbf{f}(\mathbf{V}) = \sum_{n=0}^{k-1} \frac{f^{(n)}(\lambda)}{n!}(\mathbf{V} - \lambda\mathbf{I})^n + \varepsilon^k \mathbf{\Lambda}_k \,. \tag{4.6}$$

where $\mathbf{\Lambda}_k$ is a symmetric tensor satisfying

$$\|\mathbf{\Lambda}_k\| \leq \frac{2}{k!} f_k(\lambda) \,. \tag{4.7}$$

Recall that f_k is defined by (3.16). Let

$$\eta(\lambda) \equiv f''(\lambda) - \frac{f'(\lambda)}{\lambda} \,. \tag{4.8}$$

Then we also have

$$\begin{aligned} \mathbf{f}(\mathbf{V}) &= f(\lambda)\mathbf{I} + \frac{f'(\lambda)}{2\lambda}(\mathbf{B} - \lambda^2\mathbf{I}) + \varepsilon^2\bar{\mathbf{\Lambda}}_2 \,, \\ &= f(\lambda)\mathbf{I} + \frac{f'(\lambda)}{2\lambda}(\mathbf{B} - \lambda^2\mathbf{I}) + \frac{\eta(\lambda)}{8\lambda^2}(\mathbf{B} - \lambda^2\mathbf{I})^2 + \varepsilon^3\bar{\mathbf{\Lambda}}_3 \,, \end{aligned} \tag{4.9}$$

where $\bar{\mathbf{\Lambda}}_2$ and $\bar{\mathbf{\Lambda}}_3$ are symmetric tensors satisfying

$$\|\bar{\mathbf{\Lambda}}_2\| \leq \frac{3}{2}\frac{|f'(\lambda)|}{\lambda} + f_2(\lambda), \quad \|\bar{\mathbf{\Lambda}}_3\| \leq \frac{1}{3}\left[\frac{4|\eta(\lambda)|}{\lambda} + f_3(\lambda)\right]. \tag{4.10}$$

To obtain formulas for $\mathbf{f}(\mathbf{U})$, simply replace $\mathbf{V}$ by $\mathbf{U}$ and $\mathbf{B}$ by $\mathbf{C}$ in (4.6) and (4.9). Note that $\eta(\lambda) = f''(1) - 1$ when $\lambda = 1$ and f is a strain measure.

5 Derivation of the results

Equations (3.1)–(3.3), (3.4) for $k \leq 4$, (3.19) and (3.20) follow from (1.1)–(1.5) and Taylor's Theorem for f and f' with Lagrange's formula for the remainder. In deriving (3.2) we have used the fact that $\{D_{ij}\}$ are also the components of $\hat{\mathbf{D}}$ relative to $\{\mathbf{u}_i\}$. In deriving (3.20) we have also used the following result due to Hummel and Seebeck [5]:

$$\frac{f(y)-f(x)}{y-x} = \frac{f'(y)+f'(x)}{2} - \frac{f'''(\theta)}{12}(y-x)^2 \tag{5.1}$$

for some θ between x and y.

We illustrate the method by deriving $(3.3)_2$. We must show that

$$F(\lambda_i, \lambda_j) = -\lambda^2 f''(\lambda) + \frac{1}{2}[f'(\lambda) + \lambda f''(\lambda)](\lambda_i + \lambda_j) + \delta_{ij}\varepsilon^2 , \tag{5.2}$$

where

$$|\delta_{ij}| \leq \varphi_4(\lambda) \tag{5.3}$$

for $\varphi_4(\lambda)$ satisfying $(3.20)_2$. Then substitution of (5.2) into (1.2) yields $(3.3)_2$ with $\alpha(\lambda)$ given by (3.1) and

$$\mathbf{\Phi}_4 = \sum_{i,j=1}^{3} \delta_{ij} D_{ij} \mathbf{v}_i \otimes \mathbf{v}_j . \tag{5.4}$$

Then (3.4) with $k = 4$ follows from (5.4) and (5.3). Thus it remains to establish (5.2) and (5.3). We will use the following inequalities, which are consequences of (2.2), (2.11) and (2.12):

$$|\lambda_i - \lambda_j| \leq 2\varepsilon , \tag{5.5}$$

$$2(1-r)\lambda \leq \lambda_i + \lambda_j \leq 2\lambda + 2\varepsilon \leq 2(1+r)\lambda , \tag{5.6}$$

$$\varepsilon^2 \leq r\lambda\varepsilon \leq r^2\lambda^2 . \tag{5.7}$$

Suppose that $\lambda_i \neq \lambda_j$. Then $F(\lambda_i, \lambda_j)$ is given by (1.5). By (5.5) and $(5.6)_1$,

$$\begin{aligned} \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i + \lambda_j} &= \frac{\lambda_i + \lambda_j}{2} + p_{ij} \\ &= \lambda + \frac{1}{2}(\lambda_i + \lambda_j - 2\lambda) + p_{ij} , \end{aligned} \tag{5.8}$$

$$p_{ij} = \frac{(\lambda_i - \lambda_j)^2}{2(\lambda_i + \lambda_j)} \leq \frac{\varepsilon^2}{(1-r)\lambda} . \tag{5.9}$$

By applying the Taylor-Lagrange formula to f' we obtain

$$f'(\lambda_i) = f'(\lambda) + f''(\lambda)(\lambda_i - \lambda) + \frac{1}{2}f'''(z_i)(\lambda_i - \lambda)^2 \tag{5.10}$$

for some number z_i between λ_i and λ. By setting $y = \lambda_i$ and $x = \lambda_j$ in (5.1), and using (5.10) and an analogous formula for $f'(\lambda_j)$, we obtain

$$\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} = f'(\lambda) + \frac{1}{2}f''(\lambda)(\lambda_i + \lambda_j - 2\lambda) + q_{ij}\,, \tag{5.11}$$

$$q_{ij} = \frac{1}{4}f'''(z_i)(\lambda_i - \lambda)^2 + \frac{1}{4}f'''(z_j)(\lambda_j - \lambda)^2 - \frac{1}{12}f'''(\theta)(\lambda_i - \lambda_j)^2\,. \tag{5.12}$$

Since z_i, z_j and θ all lie in the interval $I(\lambda)$, by (5.12), (5.5), (3.16) and (2.2) we have

$$|q_{ij}| \le f_3(\lambda)\varepsilon^2\,. \tag{5.13}$$

From (1.5), (5.8) and (5.11) we obtain (5.2) with

$$\begin{aligned} \delta_{ij}\varepsilon^2 \;\; &= \;\; p_{ij}f'(\lambda) + \left[\frac{1}{2}(\lambda_i + \lambda_j - 2\lambda) + p_{ij}\right]\cdot\frac{1}{2}f''(\lambda)(\lambda_i + \lambda_j - 2\lambda) \\ &\quad + \left[\frac{1}{2}(\lambda_i + \lambda_j) + p_{ij}\right]q_{ij}\,. \end{aligned} \tag{5.14}$$

Then (5.3), with $\varphi_4(\lambda)$ given by $(3.20)_2$, follows from (5.14), (5.13), (5.9), (5.7) and (5.6). A similar analysis shows that (5.2) and (5 3) also hold when $\lambda_i = \lambda_j$. This completes the proof of $(3.3)_2$.

Equations (3.5), (3.6) for $k \le 2$, and (3.17) follow from

$$\mathbf{H}\,\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{H} = \sum_{i,j=1}^{3}[f(\lambda_j) - f(\lambda_i)]H_{ij}\mathbf{v}_i \otimes \mathbf{v}_j\,, \tag{5.15}$$

and the Taylor-Lagrange formula for f.

Equations (3.7)-(3.11), (3.4) for $k = 5$ and 6, (3.6) for $k = 3$, (3.18) and (3.21) follow from the corresponding results in terms of $\mathbf{U}$ and $\mathbf{V}$, the identity

$$\mathbf{V} = \frac{\lambda}{2}\mathbf{I} + \frac{1}{2\lambda}\mathbf{B} - \frac{1}{2\lambda}(\mathbf{V} - \lambda\mathbf{I})^2\,, \tag{5.16}$$

and an analogous identity involving $\mathbf{U}$ and $\mathbf{C}$. Equations (3.9) and (3.10) can also be derived without recourse to (3.2) and (3.3). For example, (3.10) can be obtained directly from the component formula I-$(5.20)^2$ with $h(x^2) = f(x)$.

[2] An equation number prefixed by "I" denotes an equation in Part I.

Equations (4.6) and (4.7) follow from $\mathbf{f}(\mathbf{V}) = \sum_{i=1}^{3} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i$ and the Taylor-Lagrange formula for f. Equations (4.8)—(4.10) follow from (4.6) and (4.7) (with $k = 2, 3$), the identity (5.16), and (2.9).

By solving (4.6) with $k = 2$ for $\mathbf{V}$, substituting the result into $(3.3)_2$, and using (3.4), (4.7) and (1.6), we obtain (3.14), (3.12), (3.4) for $k = 8$, and $(3.22)_2$. Equations (3.13) and $(3.22)_1$ are derived similarly.

To see that $\mathbf{f}(\mathbf{V})^{\circ}$ may be replaced by $\mathbf{f}(\mathbf{V})^{*}$ in (3.3) and (3.10), use $\mathbf{f}(\mathbf{V})^{*} = \mathbf{R}\mathbf{f}(\mathbf{U})^{\cdot}\mathbf{R}^{T}$ (cf. I-(5.9)), (3.2), (3.9) and $\|\mathbf{R}\mathbf{\Phi}_k\mathbf{R}^T\| = \|\mathbf{\Phi}_k\|$. Equations (4.2), (4.3) for $k = 1$, and $(4.5)_1$ follow from the component formula I-(5.16). Equations (4.4), (4.5) and (4.3) for $k \geq 2$ follow from the component formula I-(5.14) and the identity (5.16).

6 The logarithmic strain tensors

Approximate basis-free formulas for $(\ln \mathbf{U})^{\cdot}$, $(\ln \mathbf{V})^{\circ}$ and $(\ln \mathbf{V})^{*}$ are obtained from (3.1), $(3.2)_2$, $(3.3)_2$, (3.4), (3.20), the comments following (3.22), and the comments preceding (4.2). The results are

$$(\ln \mathbf{U})^{\cdot} = \hat{\mathbf{D}} + \tilde{\varepsilon}^{2}\mathbf{\Theta}_1 \tag{6.1}$$

and

$$(\ln \mathbf{V})^{\circ} = \mathbf{D} + \tilde{\varepsilon}^{2}\mathbf{\Theta}_2, \quad (\ln \mathbf{V})^{*} = \mathbf{D} + \tilde{\varepsilon}^{2}\mathbf{\Theta}_3, \tag{6.2}$$

where the symmetric tensors $\mathbf{\Theta}_k$ satisfy

$$\|\mathbf{\Theta}_k\| \leq \frac{3}{(1-r)^4}\|\mathbf{D}\|, \quad k = 1, 2, 3. \tag{6.3}$$

From (3.5), (3.6), (3.11), (3.17), (3.18) and the comments following (3.22), we obtain the following results. For any tensor field $\mathbf{H}$,

$$\begin{aligned} \mathbf{H}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{H} &= \tilde{\varepsilon}\,\mathbf{\Psi}_1 \\ &= \frac{1}{\lambda}(\mathbf{HV} - \mathbf{VH}) + \tilde{\varepsilon}^{2}\mathbf{\Psi}_2 \\ &= \frac{1}{2\lambda^2}(\mathbf{HB} - \mathbf{BH}) + \tilde{\varepsilon}^{2}\mathbf{\Psi}_3, \end{aligned} \tag{6.4}$$

where the tensors $\mathbf{\Psi}_k$ satisfy

$$\|\mathbf{\Psi}_k\| \leq \psi_k \|\mathbf{H}\|, \quad k = 1, 2, 3, \tag{6.5}$$

for

$$\psi_1 = \frac{2}{1-r}, \quad 2\psi_2 = \psi_3 = \frac{1}{(1-r)^2}. \tag{6.6}$$

Since

$$\begin{aligned}(\ln \mathbf{V})^{\cdot} &= (\ln \mathbf{V})^{\circ} + \mathbf{W}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{W} \\ &= (\ln \mathbf{V})^{*} + \mathbf{\Omega}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{\Omega}, \end{aligned} \tag{6.7}$$

approximate basis-free formulas for $(\ln \mathbf{V})^{\cdot}$ follow from (6.2) and (6.4) with $\mathbf{H} = \mathbf{W}$ or $\mathbf{H} = \mathbf{\Omega}$.

The approximate formula (6.1) is essentially due to Hill [3,4], although his derivation is generally valid only for the case of distinct principal stretches. Hill noted that for $f = \ln$, the coefficient $\hat{F}(\lambda_i, \lambda_j)$ in (1.1) satisfies $\hat{F}(\lambda_i, \lambda_j) = 1$ if $i = j$ and $\hat{F}(\lambda_i, \lambda_j) = 1 - (\lambda_i/\lambda_j - 1)^2/6 + \cdots$ if $i \neq j$; cf. I-(6.6) through I-(6.9). He concluded that $(\ln \mathbf{U})^{\cdot} = \hat{\mathbf{D}} + \mathcal{O}(\|\mathbf{f}(\mathbf{U})\|^2)$ for any strain measure f, and that the remainder term is unaffected by the dilatational part of the deformation. Hill did not derive a bound for the remainder term.

Gurtin and Spear [2] proved that

$$\mathbf{F} - \mathbf{I} = \mathcal{O}(h) \ \&\ \dot{\mathbf{F}} = \mathcal{O}(h) \Rightarrow (\ln \mathbf{V})^{\circ} = \mathbf{D} + \mathcal{O}(h^3); \tag{6.8}$$

they did not derive a bound for the remainder term. Their proof is valid for any C^2 motion satisfying the conditions on the left-hand side of (6.8). We can easily recover their result from ours. Indeed, by setting $\lambda \equiv 1$ and using (6.2), (6.3), (2.9) and (2.14), we find that

$$\mathbf{V} - \mathbf{I} = \mathcal{O}(h) \ \&\ \mathbf{D} = \mathcal{O}(h) \Rightarrow \begin{cases} (\ln \mathbf{V})^{\circ} = \mathbf{D} + \mathcal{O}(h^3) \\ (\ln \mathbf{V})^{*} = \mathbf{D} + \mathcal{O}(h^3). \end{cases} \tag{6.9}$$

Then Gurtin and Spear's result follows since, as shown in the course of their proof, the conditions on the left-hand side of (6.8) imply that $\mathbf{V} - \mathbf{I} = \mathcal{O}(h)$, $\mathbf{D} = \mathcal{O}(h)$ and $\mathbf{W} = \mathcal{O}(h)$. Note that our result (6.9) places no restrictions on the rotation tensor $\mathbf{R}$ or the spin tensor $\mathbf{W}$.

Now assume that λ is defined by (2.16), (2.19), (2.22) or (2.23); then given any $x > 0$ there is some motion for which $\lambda = x$. Equations $(6.2)_1$ and (6.3) imply that $(\ln \mathbf{V})^{\circ} = \mathbf{D} + \mathcal{O}(\bar{\varepsilon}^2)$. We claim that this condition characterizes the logarithmic strain measure in the sense that if f is a strain measure and $\mathbf{f}(\mathbf{V})^{\circ} = \mathbf{D} + \mathcal{O}(\bar{\varepsilon}^2)$ for any motion, then $f = \ln$. For by (2.14), (3.1), $(3.3)_2$ and (3.4), we see that this condition can be satisfied iff $-\lambda^2 f''(\lambda) = 1$ and $f'(\lambda) + \lambda f''(\lambda) = 0$ for all $\lambda > 0$. Since f is a strain measure it follows that $f = \ln$. There are a number of weaker conditions which also suffice to characterize the logarithmic strain measure. By using (2.9) and arguing as above, we see that if a strain measure f satisfies any one of the following conditions then $f = \ln$:

$$\begin{aligned}\mathbf{f}(\mathbf{V})^{\circ} &= \mathbf{D} + \mathcal{O}(\|\mathbf{V} - \lambda\mathbf{I}\|), \\ \mathbf{f}(\mathbf{V})^{\circ} &= a(\lambda)\,\mathbf{D} + \mathcal{O}(\|\mathbf{V} - \lambda\mathbf{I}\|^2) \quad \text{for some function } a, \\ \mathbf{f}(\mathbf{V})^{\circ} &= \mathbf{D} + b(\lambda)(\mathbf{D}\mathbf{V} + \mathbf{V}\mathbf{D}) + \mathcal{O}(\|\mathbf{V} - \lambda\mathbf{I}\|^2) \quad \text{for some function } b. \end{aligned}$$

In view of (4.2), we arrive at the same conclusion if $\mathbf{f}(\mathbf{V})^{\circ}$ is replaced by $\mathbf{f}(\mathbf{V})^{*}$ in the above.

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