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AN IMPROVED FIRST BOREL-CANTELLI LEMMA

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DONALD R. HOOVER

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FOR THE OFFICE OF NAVAL RESEARCH

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INTRODUCTION

Let $E_1, E_2, \ldots, E_i, \ldots$ be an infinite sequence of events. Define $\{E_n \text{ i.o}\}$ to be the event that an infinite number of the E_i occur. The well known First Borel–Cantelli Lemma states that:

$$\sum_{i=1}^{\infty} P\{E_i\} < \infty \Rightarrow P\{E_n \text{ i.o.}\} = 0 \text{ or equivalently} \tag{1}$$

$$\lim_{m \to \infty} \sum_{i=m}^{\infty} P\{E_i\} = 0 \Rightarrow P\{E_n \text{ i.o.}\} = 0. \tag{2}$$

In words this lemma states that if the total sum of probability for all events is finite, then the probability of events occurring infinitely often is zero.

The First Borel–Cantelli Lemma gives only a sufficient condition for $P\{E_n \text{ i.o.}\}$ to be zero. A trivial example where the First Borel–Cantelli Lemma conditions are not met yet $P\{E_n \text{ i.o.}\} = 0$ is now given: Let $P\{E_i\} = \frac{1}{i}$ and let $E_i \supset E_{i+1}$ for all i. Since events after E_m occur only if E_m occurs, $P\{E_n \text{ i.o.}\}$ is less than the limit as m goes to infinity of $P\{E_m\}$, which is zero. Yet the summation of probabilities for the individual events is infinite, hence the First Borel–Cantelli condition is not met.

If, however, in this above example, the events E_i were independent then $P\{E_n \text{ i.o.}\}$ would be one. $P\{E_n \text{ i.o.}\}$ changes dramatically even though the probabilities for the individual events are the same. This underscores the importance of event overlap in infinite occurrence. In one case, the overlap was so large that future events did not occur unless previous events had occurred. Since the probability of an event occurring went to zero, so did $P\{E_n \text{ i.o.}\}$. In the other case, the independence of events acted to ensure that future events would always occur.

In the next section, known overlap between occurrence of adjacent events is incorporated into the First Borel–Cantelli Lemma expanding its application.

IMPROVING THE FIRST BOREL–CANTELLI LEMMA

By definition of i.o., $P\left\{E_n \text{ i.o.}\right\} = \lim_{m\to\infty} P\left\{\bigcup_{i=m}^{\infty} E_i\right\}$. (3)

Or equivalently for any positive integer p,

$$P\left\{E_n \text{ i.o.}\right\} = \lim_{m\to\infty}\left[P\left\{\bigcup_{i=m}^{m+p-1} E_i\right\} + \sum_{i=m+p}^{\infty} P\left\{E_i \cap \left[\bigcup_{j=m}^{i-1} E_j\right]^c\right\}\right]. \tag{4}$$

Applying (4) with $p = 1$ to the previous example with $E_i \supset E_{i+1}$ gives

$$P\left\{E_n \text{ i.o.}\right\} = \lim_{m\to\infty}\left[P\left\{E_m\right\} + \sum_{i=m+p}^{\infty} 0\right] = \lim_{m\to\infty} \frac{1}{m} = 0.$$

Similarly applying (3) or (4) with $p = 1$ to the previous example with the E_i being independent gives $P\left\{E_n \text{ i.o.}\right\} = \lim_{m\to\infty} P\left\{\bigcup_{i=m}^{\infty} E_i\right\} = 1$.

The usefulness of (3) and (4) in practice are limited since the right–hand side of (3) and the second term in the right–hand side of (4) involve probabilities of infinite unions. These probabilities may not be calculable.

It often will be possible to calculate probabilities involving up to p elements. When this is the case, the second term in the right–hand side of (4) can be bounded by a summation of terms involving p events. It sometimes can be shown that this term vanishes as $m \to \infty$ resulting in the Improved First Borel–Cantelli Lemma (IFBCL).

THE IFBCL:

If for any positive integer p, $\sum_{i=p+1}^{\infty} P\left\{E_i \bigcap \left[\bigcup_{j=1}^{p-1} E_{i-j}\right]^c\right\}$ is finite, (5)

it follows that $\lim_{m \to \infty} P\left\{\bigcup_{i=m}^{m+p-1} E_i\right\}$ exists and equals $P\{E_n \text{ i.o.}\}$. (5.1)

Proof:

From (4), $P\left\{E_n \text{ i.o.}\right\} = \lim_{m \to \infty} \left[P\left\{\bigcup_{i=m}^{m+p-1} E_i\right\} + \sum_{i=m+p}^{\infty} P\left\{E_i \bigcap \left[\bigcup_{j=m}^{i-1} E_j\right]^c\right\}\right]$.

But $0 \leq \lim_{m \to \infty} \left[\sum_{i=m+p}^{\infty} P\left\{E_i \bigcap \left[\bigcup_{j=m}^{i-1} E_j\right]^c\right\}\right] \leq \lim_{m \to \infty} \left[\sum_{i=m+p}^{\infty} P\left\{E_i \bigcap \left[\bigcup_{j=1}^{p-1} E_{i-j}\right]^c\right\}\right]$. (6)

The assumptions stated in (5) imply that the limit of the right–most term of (6) is zero, thus the limit of the middle term in (6) is also zero. Hence, this same term vanishes from the right–hand side of (4) as $m \to \infty$ giving (5.1).

Notes:

(a) For $p = 1$, if one uses standard convention and considers $\left[\bigcup_{j=1}^{p-1} E_{i-j}\right]$ to be ϕ, then the IFBCL reduces to the Standard First Borel–Cantelli Lemma.

(b) For $p = 1$, under the assumptions in (5), the limit of (5.1) must always be zero. For $p > 1$, the limit of (5.1) can be any number c in $[0,1]$, as is shown by the trivial example where E_i occurs if and only if E_1 occurs for all $i \geq 1$ and the probability E_1 occurs is c.

(c) For $p_2 > p_1$, $\sum_{i=p_1+1}^{\infty} P\left\{E_i \cap \left[\bigcup_{j=1}^{p_1-1} E_{i-j}\right]^c\right\} < \infty$ implies that $\sum_{i=p_2+1}^{\infty} P\left\{E_i \cap \left[\bigcup_{j=1}^{p_2-1} E_{i-j}\right]^c\right\} < \infty$. Thus, increasing the value of p results in a loosening of the restrictions.

(d) The IFBCL could be generalized by making the condition in (5) that $\sum_{i=2}^{\infty} P\left\{E_i \cap \left[\bigcup_{j=1}^{q_i} B_{i,j}\right]^c\right\}$ is finite, where q_i is any positive integer and $B_{i,1}, B_{i,2}, \ldots, B_{i,q_i}$ are any events taken from $E_1, \ldots E_{i-1}$. The result would then be that:

$$P\left\{E_n \text{ i.o.}\right\} = \lim_{m\to\infty} P\left\{\bigcup_{i=1}^{q_m} B_{m,i}\right\}.$$

<u>An Example with $p = 2$</u>

Consider the probability space (F, Ω, P) where Ω is Angular measure in radians on a circle $[0, 2\pi)$, F is Borel Sets and P is Lebesgue Measure$/2\pi$. Let c be a constant with $0 \leq c \leq 1$. Let $r_i = \left[\left[2\pi \cdot \sum_{j=1}^{i-1} \left[\frac{1}{j} - \frac{1}{j+1} + \left(\frac{1}{j+1}\right)^{\alpha}\right]\right] \quad \text{MOD} \quad 2\pi\right]$ for $\alpha > 1$, $f_i(w) = I_{\{w \in [r_i, (r_i + 2\pi/i) \text{ MOD } 2\pi] \cup [0, 2\pi c]\}}$ and E_i occur if and only if $f_i(w)$ is 1. Then E_i

is a sequence with $P\{E_i\} \geq \min(1/i, c)$ and $P\{E_i \cap E^c_{i-1}\} \leq (1/i)^\alpha$ for $i \geq 2$. Since $\sum_{i=1}^{\infty} P\{E_i\}$ diverges, the Standard First Borel–Cantelli Lemma may not be used. But $\sum_{i=2}^{\infty} P\{E_i \cap E^c_{i-1}\} \leq \sum_{i=2}^{\infty} (\frac{1}{i})^\alpha$ which, for $\alpha > 1$, is finite. So by the IFBCL with $p = 2$, $P\Big\{E_n \text{ i.o.}\Big\} = \lim_{m\to\infty} P\Big\{E_m \cup E_{m+1}\Big\} = \lim_{m\to\infty} (c + \epsilon)$ where ϵ is $0(\frac{1}{m})$ and thus the limit is c.

In this example, the Improved First Borel–Cantelli Lemma compensates for overlap of successive events to give an exact probability for infinite occurrence in a sequence of events with individual probabilities which are too large to enable application of the standard First Borel–Cantelli Lemma.

RELATION TO IMPROVED BONFERRONI INEQUALITIES

The standard First Borel–Cantelli Lemma and improved versions presented here are asymptotic analogs of the first order Bonferroni upper bound and improved forms of it that have recently appeared in the literature. The standard Bonferroni upper bound written for an infinite number of ordered events starting at event m is:

$$\sum_{i=m}^{\infty} P\Big\{E_i\Big\} \geq P\Big\{\bigcup_{i=m}^{\infty} E_i\Big\}. \tag{8}$$

If the summation on the left–hand side of (8) is finite, then taking the limit as $m \to \infty$ on both sides of (8) results in (2). Thus the standard First Borel–Cantelli Lemma arises from taking a limit on the standard Bonferroni inequality applied to an infinite number of events with a finite sum of individual probabilities.

The standard Bonferroni upper bound has been improved to incorporate dependency structure for up to p events (Hunter (1976) for $p = 2$, Hoover (1990) for $p > 2$)). This bound applied to an infinite number of ordered events starting at event m gives:

$$P\left\{\bigcup_{i=m}^{m+p-1} E_i\right\} + \sum_{i=m+p}^{\infty} P\left\{E_i \cap \left[\bigcup_{j=1}^{p-1} E_{i-j}\right]^c\right\} \geq P\left\{\bigcup_{i=m}^{\infty} E_i\right\}. \tag{9}$$

Taking the limit of the right–hand side of (9) as $m \to \infty$ gives $P\{E_n \text{ i.o.}\}$. If the second term on the left–hand side of (9) is finite for any m, then it vanishes as $m \to \infty$ and thus $\liminf_{m \to \infty} P\left\{\bigcup_{i=m}^{m+p-1} E_i\right\}$ is an upper bound for $P\{E_n \text{ i.o.}\}$. Clearly $\limsup_{m \to \infty} P\left\{\bigcup_{i=m}^{m+p-1} E_i\right\}$ is a lower bound for $P\{E_n \text{ i.o.}\}$. Thus both limits must equal each other and $P\{E_n \text{ i.o.}\}$. But this is restating the IFBCL using the improved Bonferroni upper bound applied to an infinite number of events.

While Bonferroni's (1936) other inequalities can be extended to produce Borel–Cantelli type lemmas, doing so will usually produce trivial results, such as $P\{E_n \text{ i.o.}\} \geq 0$ or $P\{E_n \text{ i.o.} \leq d\}$ for d some positive integer larger than 1.

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An improved First Borel-Cantelli Lemma incorporating dependency structure between subsets of events is presented and applied to an example where the First Borel-Cantelli Lemma can't be used. Analogies to improved Bonferroni upper bounds are given.