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On Singular Semilinear Elliptic Equations

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Abstract: - For the semilinear elliptic equation $\Delta u + p(x)u^{-\gamma} = 0$, $x \in R^n$, $n \geq 3$, $\gamma > 0$, we show via the barrier method the existence of a positive entire solution behaving like $|x|^{2-n}$ near ∞.

1 Introduction

We study the singular semilinear elliptic equation

$$(1) \qquad \Delta u + p(x)u^{-\gamma} = 0$$

in R^n. This type of equation arises in the boundary layer theory of viscous fluids [3,4]. From the results of Fulks and Maybee [8], Crandall, Rabinowith, and Tartar [5], Gomes [9], and recently Lazer and McKenna [14], it follows that (1) has a unique classical solution within a bounded domain Ω, where $p(x)$ is a sufficiently regular function which is positive on $\bar{\Omega}$. Kusano and Swanson [12] gave the existence proof on exterior domains. As for the existence of entire solutions, not much is known. Edelson [7], Kusano and Swanson [13] have been able to show the existence of entire solutions of (1) with $\gamma \in (0,1)$, and $p(x)$ sufficiently regular. In this paper we show via the upper and lower solution method, which is also referred to as the barrier method, that (1) has a bounded positive entire solution vanishing at ∞ in R^n for $n \geq 3$ and all $\gamma > 0$.

The author learned after this paper was finished that a similar result was given earlier by R. Dalmasso [6], but by a different approach.

2 Preliminaries

We first state the theorem by Kusano and Swanson [13] for the case $0 < \gamma < 1$.

Lemma 1. *Equation (1) has an entire bounded positive solution $u(x)$ in R^n for $n \geq 3$,and $|x|^{n-2}u(x)$ is bounded and bounded away from zero near ∞ if $p(x)$ satisfies the following conditions:*
(H1) $p(x) \in C^{\alpha}_{loc}(R^n)$, $n \geq 3$, $p(x) > 0$, $x \in R^n \setminus \{0\}$,
(H2) $\exists C > 0$, *such that* $C\phi(|x|) \leq p(x) \leq \phi(|x|)$, $\phi(x) = max_{|x|=t} p(x)$, $0 \leq t \leq \infty$,
(H3) $\int_1^{\infty} t^{n-1+\gamma(n-2)}\phi(t)dt < \infty$.

The term "entire" has often been used for solutions of equation (1) in R^n. To avoid confusion with the traditional definition for entire functions, we use the term "$C^{2+\alpha}$-entire". A $C^{2+\alpha}$-entire solution of (1) is defined to be a function $u(x) \in C^{2+\alpha}_{loc}(R^n)$ that satisfies (1) pointwise in R^n.

The method that we shall be using heavily in our proof is the so-called **barrier method**, or **upper-lower solution** method.

We consider the elliptic boundary value problem

$$(2) \qquad \begin{cases} Lu + f(x,u) = 0 & \text{in } D \\ Bu = a\partial u/\partial\nu + bu = g & \text{on } \partial D \end{cases}$$

where D is a smoothly bounded domain in R^N and $\nu = (\nu_1, \cdots, \nu_n)$ is a smoothly varying outward normal vector field on ∂D which is of class $C^{2+\alpha}$, while a and b are positive constants. We also assume that $f \in C^{\alpha}$ and that g has an extension $\hat{g}$ to the interior of D such that $\hat{g} \in C^{2+\alpha}$.

An **upper solution** to the above problem is a function ϕ satisfying

$$\begin{cases} L\phi + f(x,\phi) \leq 0 & \text{in } D \\ B\phi \geq g & \text{on } \partial D. \end{cases}$$

A **lower solution** to the above problem is a function ψ satisfying

$$\begin{cases} L\psi + f(x,\psi) \geq 0 & \text{in } D \\ B\psi \leq g & \text{on } \partial D. \end{cases}$$

We assume that $\partial D, f, g$, and the coefficients of L are smooth in what follows

Lemma 2. *(Theorem 2.3.1 of [16]) Let ϕ be an upper solution and ψ a lower solution with $\psi \leq \phi$ on D. Then there exists a solution u to the above boundary value problem with $\psi \leq u \leq \phi$.*

We consider the following example:

$$\begin{cases} u'' + \lambda u - u^3 = 0 & x \in (0,\pi) \\ u = 0 & x = 0,\pi \end{cases}$$

By the above theorem, if $\lambda > 1$, then the problem has at least three solutions.

Actually, $\underline{u} = \epsilon \sin x$ with ϵ small is a lower solution, and $\bar{u} = Rx^{1/2}$ with R large is an upper solution. Therefore there exists a solution u such that $\underline{u} \leq u \leq \bar{u}$ in $(0,\pi)$. Clearly $-u$ and 0 are also solutions to this problem.

The following lemma on the barrier method for $D = R^n$ is due to Ni [15] in 1982. A special case was proved earlier by Ako and Kusano [1] in 1964. The proof is standard. Using the well known result on the upper-lower solution approach in bounded regions (see Sattinger [16]), we first solve the equation

$$Lu + F(x,u) = 0$$

on B_R. Then by letting, $R \to \infty$, we obtain a solution on R^n by a diagonal process.

Lemma 3. *Let $u_1 \geq u_2$ in R^n be such that*

$$\begin{cases} Lu_1 + f(x,u_1) \leq 0 \\ Lu_2 + f(x,u_2) \geq 0 \end{cases} \tag{3}$$

where f is locally Hölder continuous in (x,u) and locally Lipschitz in u, and L is an elliptic operator of second order. Then there exists a solution u of $Lu + f(x,u) = 0$ with $u_1 \geq u \geq u_2$.

3 Main Result

Theorem 1. *Under the same conditions as given in Lemma 1, the equation (1) has a $C^{2+\alpha}$-entire positive solution in $R^N, N \geq 3$, vanishing at ∞ at the rate of at least $|x|^{q(N-2)}$ with some $q \in (0,1)$ for any $\gamma > 0$.*

The difficulty in constructing the proof is to find an appropriate upper solution of equation (1). In order to use the barrier method we first study the nonsingular equation

$$\Delta u + p(x)[\delta + u]^{-\gamma} = 0.$$

For each fixed γ there corresponds a solution $u_\gamma(x)$. Letting $\gamma \to \infty$, we show that the limiting function is the desired solution.

Proof: By Lemma 1, for $\gamma = \gamma_1 \in (0,1)$, equation (1) has a $C^{2+\alpha}$-entire positive solution $u_1(x)$ in R^n, $n \geq 3$, vanishing at ∞ at the rate r^{2-n}. We claim that $\bar{u} = cu_1^q$ is an upper solution of the equation (1) for $\gamma \geq 1$, where

$$q < \frac{1+\gamma_1}{1+\gamma} < 1,$$

$$c > (\frac{M^{1+\gamma_1-q(1+\gamma)}}{q})^{\frac{1}{1+\gamma}}, \quad M = max_{x\in R^n}|u(x)|.$$

In fact:

$$\begin{aligned} &\Delta\bar{u} + \frac{p(x)}{\bar{u}^\gamma} \\ &= cq(q-1)u^{q-2}|\nabla u|^2 - cqu^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma q} \\ &\leq -cqu^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma q} \\ &= \frac{p(x)}{c^\gamma u^{\gamma q}}(1 - \frac{c^{1+\gamma}q}{u^{1+\gamma_1-q(1+\gamma)}}) \\ &\leq \frac{p(x)}{c^\gamma u^\gamma}(1 - (\frac{M^{1+\gamma_1-q(1+\gamma)}}{q})\frac{q}{u^{1+\gamma_1-q(1+\gamma)}}) \\ &\leq \frac{p(x)}{c^\gamma u^\gamma}(1-1) = 0. \end{aligned}$$

Let δ be a fixed positive number. We then observe that $\bar{u}$ is an upper solution of the equation

$$\Delta u(x) + p(x)[u(x)+\delta]^{-\gamma} = 0, \; x \in R^n. \tag{4}$$

$\underline{u} = 0$ is a lower solution of (4). Since $\bar{u} = cu_1^q > 0$, $\bar{u} \geq \underline{u}$ in R^n. By Lemma 2, (4) has a solution u such that $\underline{u} \leq u \leq \bar{u}$.

For $\hat{\delta} < \delta$, u is a lower solution of (4) with $\delta = \hat{\delta}$. Lemma 2 then implies that (4) has a solution $\hat{u}$ for $\delta = \hat{\delta}$ such that $\underline{u} \leq \hat{u} \leq \bar{u}$.

Let $\{\delta_n\}_1^\infty$ be a sequence of strictly decreasing positive numbers, and let $u_n(x)$ be a smooth positive solution of (4) when $\delta = \delta_n$. From the construction of our lower solutions, it is clear that $u_n(x) \geq u_{n-1}(x)$ for all n. So $\lim_{n\to\infty} u_n(x) = u(x)$ exists for all $x \in R^n$ and

$$\underline{u} \leq u \leq \bar{u} \tag{5}$$

for $x \in R^n$.

We can now assert that $u \in C^{2+\alpha}(R^n)$ and that

$$\Delta u + p(x)u^{-\gamma} = 0 \tag{6}$$

for $x \in R^n$. This follows from more or less standard arguments.

Let $x_o \in R^n$ and $r > 0$. We consider the ball of radius r centered at x_o, $B(x_o, r)$ in R^n. Let Ψ be a C^∞ function which is equal to 1 on $\overline{B(x_o, r/2)}$ and equal to 0 off $B(x_o, r)$. We have

$$\Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n$$

for $n \geq 1$, where p_n is a term whose L^∞ norm is bounded independently of n. Therefore for $n \geq 1$ we have

$$\Psi u_n \Delta(\Psi u_n) = \sum_{j=1}^{N} b_{nj} \frac{\partial(\Psi u_n)}{\partial x_j} + q_n,$$

where $b_{nj}, j = 1, \cdots, n$ and q_n are terms bounded independently of n for $n \geq 1$. Integrating the above equation, we have that there exist constants $c_1 > 0$ and $c_2 > 0$ independent of n such that

$$\int_{B(x_o,r)} |\nabla u_n|^2 dx \leq c_1 (\int_{B(x_o,r)} |\nabla u_n|^2 dx)^{1/2} + c_2.$$

From this, it follows that the $L^2(B(x_o, r))$-norm of $|\nabla \Psi u_n|$ is bounded independently of n. Hence, the $L^2(B(x_o, r/2))$-norm of $|\nabla u_n|$ is bounded independently of n. Let Ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_o, r/4)}$ and equal to 0 iff $B(x_o, r/2)$. We have for $n \geq 1$,

$$\Delta(\Psi_1 u_n) = 2\nabla\Psi_1 \cdot \nabla u_n + p_{1n}.$$

where p_{1n} is a term whose $L^\infty(B(x_o, r/2))$- norm is bounded independently of n. From standard elliptic theory, the $W^{2,2}(B(x_o, r/2))$-norm of $\Psi_1 u_n$ is also bounded independently of n and hence, the $W^{2,2}(B(x_o, r/4))$-norm of u_n is bounded independently of n. Since the $W^{1,2}(B(x_o, r/4))$-norm of the components of ∇u_n are bounded independently of n, it follows from the Sobolev embedding theorem that if $q = 2n/(n-2) > 2$ for $n > 2$ and in addition if $q > 2$ is arbitrary for $n \leq 2$, then the $L^q(B(x_o, r/4))$-norm of $|u_n|$ is bounded independently of n. Let Ψ_2 be a C^∞ function which is equal to 1 on $\overline{B(x_o, r/8)}$ and equal to 0 iff $B(x_o, r/4)$. We have for $n \geq 1$,

$$\Delta(\Psi_2 u_n) = 2\nabla\Psi_2 \cdot \nabla u_n + p_{2n},$$

where p_{2n} is a term whose $L^\infty(B(x_o, r/4))$- norm is bounded independently of n. Since the right hand side of the above equation is bounded in $L^q(B(x_o, r/4))$ independently of n, the $W^{2,q}(B(x_o, r/4))$-norm of $\Psi_2 u_n$ is also bounded independently of n. Hence, the $W^{2,q}(B(x_o, r/8))$-norm of u_n is bounded independently of n. Continuing this line of reasoning, after a finite number of steps, we find a number $r_1 > 0$ and $q_1 > n/(1-\alpha)$ such that the $W^{2,q_1}(B(x_o, r_1))$-norm of u_n is bounded independently of n. Hence, there is a subsequence of $\{u_n\}_1^\infty$, which we may assume is the sequence itself, which converges in $C^{1+\alpha}(\overline{B(x_o, r_1)})$. If θ is a C^∞ function which is equal to 1 on $\overline{(B(x_o, r_1/2))}$ and 0 off $B(x_o, r_1)$, then

$$\Delta(\theta u_n) = 2\nabla\theta \cdot \nabla u_n + \hat{p_n}, \text{ where } \hat{p_n} = \theta\Delta u_n + u_n\Delta\theta.$$

The right-hand side of the above equation converges in $C^\alpha(\overline{B(x_o, r_1)})$. Hence by Schauder theory, $\{\theta u_n\}_1^\infty$ converges in $C^{2+\alpha}(B(x_o, r_1))$ and thus $\{u_n\}_1^\infty$ converges in $C^{2+\alpha}(\overline{B(x_o, r_1/2)})$. Since x_o was arbitrary, this shows that $u \in C^{2+\alpha}(R^n)$. Clearly (6) holds.

4 Some remarks

Remark 1:

For $n = 1$, the properties of positive solutions of equation (1) have been studied by Taliaferro [17], and Gatica [10]. For $n = 2$, no entire positive solution of equation (1) exists regardless of its asymptotic behavior at ∞ (see [13]).

Remark 2:

It is observed by Callegari, Friedman and Nachman [2], [3,4] that if the partial differential equations describing the boundary layer behind a rarefaction or shock wave (with viscosity proportional to the temperature) traveling down, and perpendicular to, a flat plate are written in terms of a stream function and a similarity variable the following Blasius-type equation emerges [18].

$$f'''(\eta) + f(\eta)f''(\eta) = 0,$$

where

$$f(0) = 0, \;\; f'(0) = K, \;\; f'(\infty) = 1.$$

Here, $0 < K < 1$, for rarefaction waves and, $1 < K < 6$, for shock waves. ($K = 0$ corresponds to the classical Blasius problem.) Adopting the Crocco variables

$$x = f'(\eta), \;\; g = f''(\eta)$$

results in the system

$$gg'' + x = 0,$$

$$g'(K) = 0, \;\; g(1) = 0,$$

which falls into the class of equation discussed in this paper.

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