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NOTES ON SEARCH, DETECTION
AND LOCALIZATION MODELING

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Preface

This report is a collection of material that has been used in courses on search, detection and localization modeling. Its organization follows to some extent material by S. M. Pollock in Selected Methods and Models in Military Operations Research which is listed in the report bibliography. The report is not intended to be a text on these subjects. In particular, in some areas it does not provide the depth of coverage that is found in the book Search and Detection by Alan R. Washburn which is cited as Reference 27 in the report.

In the fourth revision, typographical and other errors have been corrected and, in addition, changes and additions have been made to a number of the sections in the report.

Table of Contents

I. Detection Models and Signal Detection Theory

Signal detection theory is the basis for analyzing the detection models that are described in this report. In signal detection theory, the decision making portion of a detection system is called the **receiver** and a detection experiment is the analysis by a receiver of input data observed during some time interval. The data that is related to a target is called **signal.** The data that is not related to the target is called **noise.** In general, the target data is associated with a localization region that in some cases is called a **resolution cell.** In a detection experiment, either the event H_0 = **{the receiver input is noise}** or its compliment H_1 = **{the receiver input is signal and noise}** occur. In the first detection models that are described in this report, after analysis of the input data by the receiver, either the event D_0 =**{the receiver decides the input is noise}** or its complement D_1 = **{the receiver decides the input is signal and noise}** also occurs. Detection models for which D_1 is the complement of D_0 are called binary detection models or forced choice detection models. Eight events which are important in binary detection models are indicated in the Venn diagram of Figure 1.

The Venn diagram emphasizes a decision problem that is associated with a receiver that can be modeled using a binary detection model in a forced choice situation. The problem is this: Under what conditions should the event D_1 occur? That is, under what conditions should a receiver decide that the input data

H_0	H_1	
$D_0 \cap H_0$ missed false alarm	$D_0 \cap H_1$ missed detection	D_0
false alarm $D_1 \cap H_0$	detection $D_1 \cap H_1$	D_1

Figure 1. Eight events that are important in binary detection models.

accumulated during the observation time interval is signal and noise? Four criteria that provide a basis for answering this question are discussed in Section II. In the discussion and in the development of the detection models that are based these criteria, the following notation and terminology is used: $p_f = P(D_1|H_0)$, the conditional probability of D_1 given H_0, is called the **false alarm probability**; $p_d = P(D_1|H_1)$, the conditional probability of D_1 given H_1, is called the **detection probability** and $P = P(H_1)$, the probability of H_1, is called the **prior probability.**

In the detection models, the input to a receiver is determined by a stochastic process that has the following characteristics: It is a random noise process when there is no target data and it is a random noise process plus a signal process when there is target data. Although the receiver input in some cases may appear to be determined by a continuous parameter stochastic process, because of the finite amount of data contained in a bounded sequence of finite length, a discrete parameter stochastic process is sufficient to determine the receiver input in these cases. This is established formally by the stochastic sampling theorem. Consequently, in these models, the input to a receiver is determined by a sequence of random variables $Y_1, \cdots, Y_m$ and an observation yields a sequence of values $y_1, \ldots, y_m$.

Three detection models are described in Section III. In the first model, the signal process is a deterministic process. That is, the signal data can be determined prior to its generation. In the second and third models, the signal process is a random process. Therefore, the noise data can not be determined prior to its generation except in terms of its statistical characteristics. To define a random noise process or a random signal process, only the joint distribution of the finite sequence of random variables that determine the process needs to be specified. If the signal process is a deterministic process, the signal values can be determined before an observation is performed. To define the process in this case, only these values need to be specified.

II. Decision Criteria

To simplify the discussion of decision criteria and decision rules, a receiver's input will be assumed to be determined by a single decision random variable Y. In this case, the input process in determined by the conditional distribution function $F_Y(y|H_0)$ when the input is noise alone and by the conditional distribution function $F_Y(y|H_1)$ when the input is signal plus noise.

The condition that a receiver's input is required to satisfy in order that the event D_1 will occur can be specified in terms of a decision rule. For the assumed case, a decision rule is a rule which determines for every observable value of Y the decision that the receiver is to make. The decision rule can be considered to be a function $\phi(y)$ which relates each observable value of y to one or the other of the following two decisions:

d_0 = "the receiver input was noise"

d_1 = "the receiver input was signal and noise".

Choosing a decision rule $\phi(y)$ defines a set Ω of observable values of Y such that the event $D_1 = \{ Y \in \Omega \}$.

The problem which was considered in Section I can now be restated in the following way: What criterion should be adopted in order to determine a decision rule or, what is equivalent, its corresponding set Ω ? A desirable characteristic for a criterion is suggested by the following argument: Consider the odds in favor of H_1 given y is observed. That is, consider

$P(H_1|Y = y)/P(H_0|Y = y)$. One might expect that y would be a member of the set Ω if and only if y made this ratio equal to or greater than some value k. But this is equivalent to defining Ω as follows: $\Omega = \{ y : L(y) \geq K \}$ where $L(y)$ is the likelihood ratio associated with an observed value y and K is a constant related to the constant k. This suggests that choosing an optimum criterion is equivalent to choosing an optimum value for K.

Four specific decision criteria are defined next in terms of K. For each criterion, Ω has the above form. But for each criterion the choice of K is different. The decision criteria are:

1. **The Neyman-Pearson Criterion:** Choose Ω so that p_d is a maximum subject to the constraint that $p_f \leq \alpha$ where α is a specified value. For a continuous decision random variable, the constant K is chosen so that $p_f = \alpha$.

2. **The Bayes Criterion:** Choose Ω so that the expected cost of a receiver's decision is a minimum. For a continuous decision random variable, if $c_{10} > c_{00}$ and $c_{01} > c_{11}$ where c_{ij} is the cost of $D_i \cap H_j$, then $K = [(c_{10}-c_{00})/(c_{01}-c_{11})](1-P)/P$.

3. **The Ideal Observer Criterion:** Choose Ω so that the probability that the receiver makes an incorrect decision is a minimum. For a continuous decision random variable, $K = (1-P)/P$.

4. **The Minimax Criterion:** Choose Ω when P is unknown so that the maximum expected cost of a receiver's decision is a minimum. For a continuous decision random variable, if $c_{10} > c_{00}$ and $c_{01} > c_{11}$, then $K = [(c_{10}-c_{00})/(c_{01}-c_{11})](1-P^*)/P^*$. Here, P^* is

the value of the prior probability P that would make the expected cost of a receiver's decision a maximum if P were known and the Bayes Criterion were used.

If a model which specifies the conditional distributions $F_Y(y|H_0)$ and $F_Y(y|H_1)$ and a decision rule are adopted, then the value of p_f and the value of p_d are determined. This pair of values (p_f, p_d) is called a receiver operating point. If the decision rule results from using a likelihood ratio criterion such as one of the four listed above, then it will involve the parameter K since $\Omega = \{y: L(y) \geq K\}$. And, for a given value of K, since Ω uniquely determines the pair (p_f, p_d), a single operating point results. By varying K, a set of operating points can be generated which determines a **receiver operating characteristic curve** or **ROC curve.** Different ROC curves can be produced by changing either one or both of the conditional distributions which implies either a change in the signal process or a change in the noise process.

A decision rule which results from using a likelihood ratio criterion in a model in which the input process is determined by a set of m random variables can be expressed in terms of a set Ω as follows: $\Omega = \{ (y_1, \cdots, y_m) : L(y_1, \cdots, y_m) \geq K \}$ where K is specified in the same way that it is when $m = 1$.

III. Three Binary Detection Models

Three detection models are examined in this section. For the first two detection models, the input stochastic process for an observation is defined by a time sequence of continuous random variables. The random variables represent a sample from a continuous parameter stochastic process which is sampled at times such that the random variables are independent. For the third detection model, the input stochastic process is a counting process and it is defined by a single discrete random variable that is equal to the number of events that are counted during the observation.

Model I: In the first detection model, a sampled noise value is a value of a normally distributed random variable with mean zero and with known variance σ^2. And a sampled signal value is a known value of a deterministic variable. Thus, the input process corresponding to an observation consists of some number m of independent normal random variables $Y_1, \cdots, Y_m$ each with variance σ^2. And, for $i = 1, 2, \cdots, m$, when a signal is not present the mean of Y_i is zero and when a signal is present the mean is s_i. The result of using a likelihood ratio decision rule in the model can be expressed in terms of a random variable Z. This random variable is called a crosscorrelation statistic and it is defined by $Z = \Sigma\, s_i \cdot Y_i$ where the sum index $i = 1, 2, \cdots, m$ here and in the remainder of this section. However, it is more convenient to express the result in terms of a statistic V which is defined by $V = Z/\sigma_z$. In terms of V, the two conditional probabilities p_f and p_d are determined by: $\mathbf{p_f = 1 - \Phi(v*)}$ and $\mathbf{p_d = 1 - \Phi(v^* - d^{1/2})}$ where Φ is the standard normal cumulative distribution function, $v_* = (1/\sigma_z)(\sigma^2 \ln K + (1/2) \Sigma s_i^2)$, is determined by the decision rule and $d = \Sigma s_i^2/\sigma^2$ is called the **detection index.**

Often, the input stochastic process represents a quantity whose square is proportional to power. In such a case, the average receiver input power is the random variable $\Sigma Y_i^2/m$. The expected average receiver input noise power is $N = \Sigma \sigma^2/m = \sigma^2$ where **N** is called the **noise.** The average receiver input signal power is $S = \Sigma s_i^2/m$ where **S** is called the **signal.** In terms of these two quantities, $d = m\cdot(S/N)$ where **S/N** is called the **signal-to-noise ratio.**

If a receiver's input data can be considered to be a time sequence of current or voltage values, in some cases a frequency representation can be used that involves the concept of receiver bandwidth. In these cases, the noise process is assumed to be such that $m = t/\delta t$ where **t** is the **integration time** (the duration of an observation) and δt is the **time between samples** with $\delta t = 1/[2(BW)]$ where **BW** is the **bandwidth** and δt is determined by the sampling theorem. This implies that the detection index $\mathbf{d = 2t\cdot(BW)(S/N)}$. By defining N_0 as the **power spectral density** where $N_0 = N/BW$, this becomes $\mathbf{d = 2t\cdot(S/N_0)}$.

In Reference 2, the conditions required for this form of the first model are called Case I. In the following sections, this form of the first model: $\mathbf{p_f = 1 - \Phi(v^*)}$ and $\mathbf{p_d = 1 - \Phi(v^* - d^{1/2})}$

where $d = t \cdot (BW)(S/N)$ is called the **Case I model.** A receiver that processes data such that it would implement a likelihood ratio decision rule under the conditions of the first model is called a matched filter or crosscorrelation detector. If the description of the input noise is adequate, a Case I model can be used to obtain an estimate of an upper bound on a detection system's performance, since all the information necessary to define the signal is assumed to be known.

Model II: In the second detection model, a sampled noise value is an independent normal random variable with mean zero and known variance σ^2. And, a sampled signal value is an independent random variable with mean zero and known variance σ_s^2. Thus, the input process corresponding to an observation consists of some number m of independent normal random variables $Y_1, \cdots, Y_m$ each with mean zero and each with variance σ^2 when a signal is not present and each with variance $\sigma^2 + \sigma_s^2$ when a signal is present. The result of applying a likelihood ratio decision rule in this model can be expressed in terms of a statistic X which is defined by $X = \Sigma Y_i^2$.

When a signal is not present, the statistic X/N has a chi-square distribution with m degrees of freedom. When a signal is present, the statistic $X/(N+S)$ has a chi-square distribution with m degrees of freedom. So, in terms of these statistics, the two conditional probabilities p_f and p_d are: $p_f = P(X^2{}_m \geq x^*/N)$ and $p_d = P\{X^2{}_m \geq (x^*/N)[1/(1+S/N)]\}$ where $X^2{}_m$ is a chi-square random variable with m degrees of freedom, x^* is a number which is determined by the decision rule and S/N is the signal-to-noise ratio. A receiver that would implement a likelihood ratio decision rule under the conditions of the second model is called an energy detector or square law detector.

The mean of a chi-square random variable with m degrees of freedom is m and the variance is 2m. By the central limit theorem, as the number of degrees of freedom of a chi-square random variable becomes large, it can be approximated by a normal random variable that has the same mean and variance. For m sufficiently large, after using this approximation, $p_f = 1 - \Phi[(x^*/N - m)/(2m)^{1/2}]$ and $p_d = 1 - \Phi\{[1/(1+S/N)][x^*/N - m - m\cdot(S/N)]/(2m)^{1/2}\}$. And, with $v^* = [x^*/N - m)/(2m)^{1/2}]$ and $d = (m/2)(S/N)^2$, the approximations are: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi\{[1/(1+S/N)](v^* - d^{1/2})\}$. If the noise N is significantly larger than the signal S, then p_d can be further approximated by: $p_d = 1 - \Phi(v^* - d^{1/2})\}$. The concept of bandwidth is applicable so that $m = 2t\cdot(BW)$, then the detection index $d = t\cdot(BW)(S/N)^2$. In Reference 2, the conditions required for these approximations are called Case II. In the following sections, the last limiting form of the second model: $\mathbf{p_f = 1 - \Phi(v^*)}$ and $\mathbf{p_d = 1 - \Phi(v^* - d^{1/2})}$ where $\mathbf{d = t\cdot(BW)(S/N)^2}$, $\mathbf{m >> 1}$ and $\mathbf{S/N << 1}$ is called the **Case II model.**

Model III: In the third detection model, a sampled noise value and a sampled signal value are values of independent random variables that are determined by independent Poisson processes that are observed for a time interval t. The noise process is characterized by a counting rate α, the signal process is characterized by a counting rate α_s and the noise and signal processes are additive. This implies that when the input is noise alone, the input is a Poisson random variable with parameter αt, the expected number of noise counts, and when the input is signal and noise, the input is a Poisson random variable with parameter $(\alpha + \alpha_s)t$, the expected number of noise and signal counts.

For a likelihood ratio decision rule, $p_f = 1 - P(y^*;\alpha t)$ and $p_d = 1 - P[y^*;(\alpha + \alpha_s)t]$ where y^* is a threshold value that is determined by the decision rule and $P(y;\theta)$ represents the Poisson cumulative distribution function with parameter θ. When θ is large, the cumulative distribution function can be approximated by the cumulative distribution function of a normal random variable that has the same mean and variance. Using this approximation for cases where αt is sufficiently large, since both the mean and variance of a Poisson random variable are equal, $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi\{[1/(1 + \alpha_s/\alpha)^{1/2}](v^* - d^{1/2})\}$ where $v^* = (y^* - \alpha t)/(\alpha t)^{1/2}$ and $d = \alpha t \cdot (\alpha_s/\alpha)^2$. If, in addition, α is significantly larger than α_s then p_f and p_d can be approximated by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{1/2})$ which is identical to the form of the expression for p_f and p_d for the Case I and Case II models. And, the approximations: $\mathbf{p_f = 1 - \Phi(v^*)}$ and $\mathbf{p_d = 1 - \Phi(v^* - d^{1/2})}$ where $\mathbf{\alpha_s/\alpha << 1}$ and $\mathbf{\alpha t >> 1}$ could be called the Case III model.

This Case III model could be used to describe a receiver whose input for an observation is the number of photons counted by a radiation detector in situations where αt, the expected number of counts when no signal is present, is of the order of thirty or more.

When a likelihood ratio decision rule is used in the three models discussed above, for the first model and under limiting conditions for the second and third models, the following result is obtained: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{½})$ where the value of v^* depends on the noise power N for the first and second models. For a sonar receiver described by the first model, that is, by the Case I model: $d = 2t\cdot(BW)(S/N)$. For a sonar receiver described under the limiting conditions for the second model, that is, by the Case II model, $d = t\cdot(BW)(S/N)^2$. This implies that in either a Case I model or a Case II model of a sonar receiver, the detection index d is a function of the time bandwidth product $t\cdot(BW)$ and the signal-to-noise ratio S/N. Since sonar equations relate S/N to system, target and environmental parameters, a sonar equation can be used to relate S/N to these parameters in a model of a sonar receiver.

IV. General Detection Models

The detection models that have been considered to this point are based on binary detection theory. After each observation, a receiver decides either that the input data corresponding to the observation was noise or else it decides it was signal plus noise. However, in some detection systems this decision is not prior to the next observation. In a computational sense, a model of such a detection system is generally more complex than a binary detection model. To illustrate this, consider an active sonar system whose receiver includes an operator. Suppose the probability that the operator will detect a target echo has been determined in a laboratory experiment in which the operator was required to decide after each observation that either the input was signal and noise or the input was noise alone. In addition, suppose that under operational conditions the operator normally delays this decision. Then, in general, the probability that the operator will decide that the input corresponding to a resolution cell that contains a target is a target echo and noise will not be equal to the probability of the event in the forced choice experiment. And, in addition, the probability that the operator will decide the input corresponding to a resolution cell that does not contain a target is a target echo and noise will not be equal to the probability of this event in the forced choice experiment. Consequently, in general, the value of both p_d and p_f for an operational environment will be different than that for the laboratory environment.

One model that has been proposed to deal with this kind of situation defines the event that a receiver decides that the input corresponding to a resolution cell is signal and noise to be equivalent to the event that out of n consecutive observations at least k of them would result in the decision that the input was signal and noise in a forced choice experiment. The model is said to be based on an k-out-of-n detection criterion. With this criterion, the probability that a target will be first detected on the j^{th} observation can be found as follows: Determine the 2^j sequences of forced choice responses that could result for a sequence of j consecutive observations. Next, determine the probability of occurrence for each sequence that first satisfies the k-out-of-n detection criterion on the j^{th} observation. The probability of first detection on the j^{th} observation is equal to the sum of these probabilities. The cumulative probability of detection at the j^{th} observation is the sum of the probabilities of first detection on the i^{th} observation for $i = 1, 2, \cdots, j$.

V. Signal-to-Noise Ratio Detection Models

In some radar and sonar detection models, for a specified value of p_f, a minimum acceptable value of p_d is defined. This minimum acceptable value of p_d and the specified value of p_f define what can be called a minimum acceptable signal-to-noise ratio $(S/N)_m$ if p_d is a nondecreasing function of the signal-to-noise ratio. In some sonar detection models, $(S/N)_m$ in decibels is called the **detection threshold DT**. In symbols, the detection threshold $\mathbf{DT = 10 \log(S/N)_m}$. If the minimum acceptable value of p_d is .5, then DT is usually called the **recognition differential RD**. The difference between the signal-to-noise ratio in decibels and RD (or DT) is called the **signal excess SE**. In symbols, the signal excess $\mathbf{SE = 10 \log(S/N) - RD}$.

One interpretation of signal excess is that for a localization region containing a target detection occurs with probability one if $SE \geq 0$ and with probability zero if $SE < 0$. This interpretation provides the basis for defining detection in the three encounter detection models that are discussed in Section VII. A more consistent interpretation is: If $SE \geq 0$, then the probability of detection p_d is greater than or equal to the minimum acceptable value. (The minimum acceptable value is .5 if recognition differential RD is used to define signal excess.) For cases where p_d increases rapidly with signal excess in the neighborhood of zero signal excess, the two interpretations may be operationally equivalent. For a discussion of this point as well as a discussion of an operational case in which receiver decisions are delayed, see Reference 3.

The signal excess (signal-to-noise ratio) detection model provides a basis for detection models describing nonstationary noise and signal processes and randomly changing decision rules. This is illustrated by the models discussed in Section VII. In addition, the signal excess model provides a basis for detection models describing delayed receiver decision models. This is illustrated by the active sonar detection models in Reference 4 and Reference 5 that are based on a k-out-of-n detection criterion. In both of these models, the signal-to-noise ratio and the recognition differential are random variables.

Using $X(t)$ to represent a random variable corresponding to an index time t and a subscript to identify the random variable in such models, for a passive sonar receiver, the signal-to-noise ratio in decibels associated with a decision at the index time is: $X_{SL}(t) - X_{TL}(t) - [X_{NL}(t) - X_{DI}(t)]$. In this expression, SL represents source level, TL represents transmission loss, NL represents noise level and DI represents directivity index. Since **signal excess SE** is defined to be the difference in decibels between the signal-to-noise ratio and the recognition differential (or detection threshold), it too is a random variable and, for any decision time t, one can write:

(1) $X_{SE}(t) = X_{SL}(t) - X_{TL}(t) - [X_{NL}(t) - X_{DI}(t)] - X_{RD}(t).$

The distributions of the random variables on the right side of Equation 1 determine the distribution of the signal excess. In the passive sonar detection model described in Reference 6, $X_{SL}(t)$, $X_{RD}(t)$ and, in effect, $X_{NL}(t)$ are normally distributed random variables while $X_{TL}(t)$ is a uniformly distributed random variable. In the three signal excess models that are described in Section VII, all of the random variables in Equation 1 are normally distributed.

It is sometimes convenient to write Equation 1 as:

(2) $X_{SE}(t) = SE(t) + X(t)$.

In Equation 2, $SE(t)$ is the expected value of the signal excess determined by the following expected value equation:

(3) $SE(t) = SL(t) - TL(t) - [NL(t) - DI(t)] - RD(t)$

where each term on the right represents the expected value of the indicated random variable and $X(t)$ is a random variable that determines the stochastic character of the signal excess. Since $SE(t)$ is the mean of $X_{SE}(t)$; by Equation 2, the mean of $X(t)$ is equal to zero and the standard deviation of $X(t)$ is equal to the standard deviation of $X_{SE}(t)$. If the quantities on the right side of Equation 1 are independent random variables, it implies that $\sigma^2 = \sigma_{SL}^2 + \sigma_{TL}^2 + \sigma_{NL}^2 + \sigma_{DI}^2 + \sigma_{RD}^2$ where σ represents the standard deviation of $X_{SE}(t)$. This relation has been used to determine a standard deviation for the signal excess in operational models.

VI. General Encounter Models

A basic problem associated with search modeling is that of determining the probability that a target will be detected by a detection system during an encounter with one or more detection systems. In the encounter models that are considered in this report, during a search, observations are made of a series of localization regions. The probability of detection resulting from an observation is $P(D_1 \cap H_1)$. And, the probability of a false alarm is $P(D_1 \cap H_0)$. In these models, the time to resolve a false alarm is ignored. However, p_d and p_f are assumed to be determined by some criterion such that p_f is an operationally reasonable value.

Using the order number of a decision rather than its time as an index and a random variable N to represent the order number at which detection first occurs, the probability of detection during an encounter can be written as:

$$P(N \le n) = P(N \le m) + P(N = m+1) + \cdots + P(N = n)$$

or as:

$$P(N \le n) = 1 - [1 - P(N \le m)](1 - g_{m+1}) \cdots (1 - g_n)$$

where $g_i = P(N = i | \overline{N \le i-1})$ is the probability of the event detection at the i^{th} decision conditioned on the event no detection at an earlier decision and $1 \le m \le n$. The second expression is generally of greater interest than the first expression, since g_i can usually be more directly related to operational parameters such as range and environmental conditions that determine a target's detectability than can $P(N = i)$.

With a time rather than the order number to index a decision and a random variable T to represent the time index at which detection first occurs, $P(N \le n)$ becomes $P(T \le t_n)$ with

$$P(T \le t_n) = 1 - [1 - P(T \le t_m)][1 - g(t_{m+1})] \cdots [1 - g(t_n)]$$

where $g(t_i) = P(T = t_i | \overline{T \le t_{i-1}})$.

If $g(t_i) << 1$ for $i = 1,2, \cdots ,n,$ then, to a first approximation, $\ln[1 - g(t_i)] = -g(t_i)$ for $1 = 1,2, \cdots ,n$ and $P(T \le t_n) = 1 - [1 - P(T \le t_m)]\cdot\exp[-\Sigma\, g(t_i)]$. This follows since $P(T \le t_n) = 1 - [1 - P(T \le t_m)]\cdot\exp[\Sigma \ln[1 - g(t_i)]$ where the sum index $i = m+1, \cdots , n$. A continuous analog to this approximation can be used to describe an encounter for cases where $g(t_i) << 1$ for $i = m, m+1, \cdots , n$ and decisions during the encounter can be considered to occur continuously. That is, the time of an observation corresponding to a decision and the time between decisions are both negligible relative to the time of the encounter.

The analog can be developed as follows: First, let δt be the time between decisions, then $t_i = i\cdot\delta t$ and the probability of detection $P(T \le t_n) = 1 - [1 - P(T \le t_m)]\cdot\exp[-\Sigma\, \tau(t_i)\cdot\delta t]$ where $\tau(t_i) = (1/\delta t)g(t_i)$ is a detection rate function (a probability of detection per unit time) and, in terms of δt, the probability $g(t_i) = P[T = i\cdot\delta t | \overline{T \le (i-1)\cdot\delta t}]$.

If T is considered to be a continuous random variable, the expression for $P(T \le t_i)$ above indicates that the sum in the exponent should be replaced by an integral whose integrand is a continuous function $\tau(t)$. If $\tau(t)$ can be determined, then, with

$g(t_i)$ as a guide, $P(T \leq t)$, the cumulative probability of detection, can be defined by:

(4) $\quad \tau(t) = \lim \{(1/\delta t)\ P(t < T \leq t+\delta t | \overline{T \leq t})\}$

where the limit is for δt approaching zero. Equation 4 implies the following differential equation: $dp(t)/dt = [1 - p(t)]\cdot\tau(t)$ where $p(t) = P(T \leq t)$. A solution to this equation is:

(5) $$P(T \leq t_n) = 1 - [1 - P(T \leq t_m)]\exp[-\int_{t_m}^{t_n} \tau(t)\,dt]$$

where t is the time index for a decision during an encounter, tm is some time during the encounter and tn > tm. A $\tau(t)$ that is based on a visual detection model is described in Reference 7. If the detection capability of a detection system is assumed to depend on a target's position relative to the detection system during an encounter but not to depend on the clock time, then the time index of a decision can be a relative index that determines the target position that is associated with a decision rather than the clock time associated with the decision.

The above results apply to the case of an encounter between a target and a collection of detection systems. However, if the detection systems are not collocated, it is generally convenient to describe encounters of this kind in terms of encounters between the target and the individual detection systems. In either case, if the event target detection for a detection system is not independent of the event for other detection systems, then in order to describe this in an encounter model the correlation between the input to the detection system and the inputs to the other detection systems must be specified. This has been done in some models as follows: First determine the probability of detection for each system acting alone. Let P_i be the probability that the i^{th} system detects the target during the encounter under this condition. Next, consider two cases: In the first case, the random factors that determine detection for a system are independent of those that determine detection for the remaining systems. In the second case, the random factors that determine detection for the systems are completely dependent. In the first case, the probability that at least one system detects the target is given by: $P_I = 1 - (1 - P_1)(1 - P_2) \cdots (1 - P_n)$ where n is the number of detection systems involved. In the second case, the probability that none of the systems detect the target is given by: $1 - P_D = 1 - P_m$ where $P_m \geq P_i$ for $i = 1, 2, \cdots, n$ since if the m^{th} system does not detect the target, none of the remaining systems will detect it. The probability that at least one system detects the target is given by: $P = \alpha \cdot P_D + (1 - \alpha) \cdot P_I$ where α determines the degree of correlation and $0 \leq \alpha \leq 1$. A way to determine a value for α is described in Reference 8.

VII. Three Signal Excess Encounter Models

In the three models described in this section, detection is defined in terms of signal excess as it is in Section V. Each model determines a cumulative probability of detection for a target in an encounter with a passive sonar system. An observation in the models is indexed by time and the index can usually be considered to be the time at the end of the observation. During an encounter, observations are made of one or a series of localization regions. By implication, a false alarm can occur for a localization region that does not contain a target during an observation since the value of RD (or DT) is determined by some specified false alarm probability. However, as they are generally used, signal excess models do not account for false alarms. This can be viewed as equivalent to modeling the time to resolve a false alarm to be effectively zero.

To determine signal excess in the models, it is convenient to use Equation 2. For each decision in an encounter, there is a random variable $X(t)$ defined by Equation 2 that determines the random character of the signal excess. For a sequence of decisions, the set of these random variables ordered by their time index constitutes a stochastic process. And the joint distributions of these random variables determines the nature of the stochastic process. In the three encounter models described in this section, the stochastic process is called a lambda-sigma jump process. The time series that are generated by lambda-sigma jump processes are represented by the plot in Figure 2 below. The jumps in the time series occur at times determined by a Poisson process with a mean jump rate λ. This implies that the time between jumps is a random variable with an exponential distribution and that the expected times between jumps τ is equal to the reciprocal of λ.

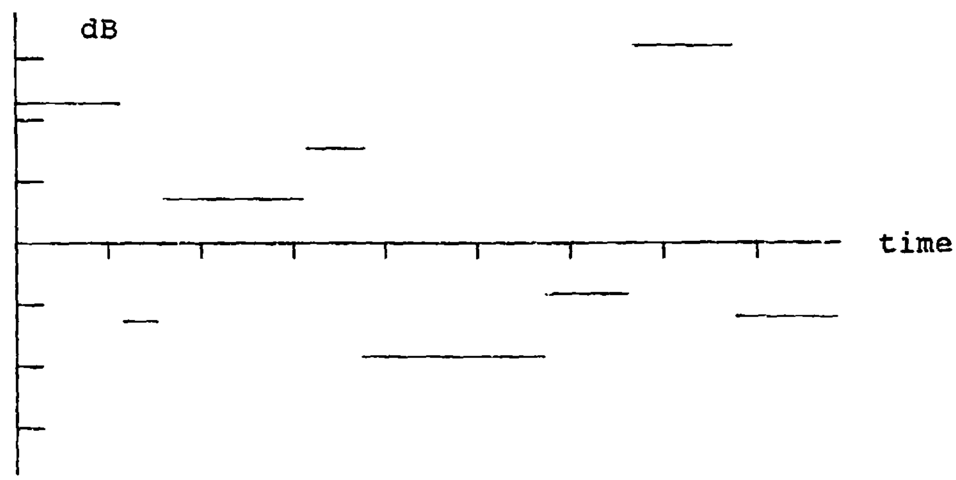

Figure 2. A time series representing a realization of a lambda-sigma jump process. On the plot, σ in dB equals one unit on the vertical axis and τ equals one time unit on the horizontal axis.

From Figure 2, note that the observed values of neighboring random variables are equal unless a jump has occurred between the observations. When a jump occurs, the first random variable after the jump is normally distributed with mean zero and variance σ^2 and it is independent of all the random variables before the jump. Conditioned on a jump pattern, this random variable and all the random variables between it and the next jump are dependent and the correlation coefficient between any pair is one. However, since the jumps occur randomly, knowing the value of the signal excess with certainty at some time does not determine the values of the signal excess with certainty at neighboring times. In the unconditioned case, the correlation coefficient between the random variables $X(t)$ and $X(t+\tau)$ is equal to $1/e$. For this reason, τ is referred to as a relaxation time.

It appears that the use of the lambda-sigma jump process to describe the stochastic character of signal excess is based more on past practice than on experimental justification. In this regard, see Reference 9. By referring to Equation 1, it can be seen that the signal excess stochastic process is determined by the sum of the stochastic processes that determine the random variables on the right side of this equation. Although the sum of a collection of normal random variables is a normal random variable, in general, the sum of a collection of lambda-sigma jump processes is not a lambda-sigma jump process. This suggests that if the lambda-sigma jump process does adequately describe the variability of the signal excess, then the majority of the variability of the signal excess is due to a single one of its components. If that component is transmission loss, then there is an additional complication: The internal wave field appears to be a major factor in determining the temporal variability of the transmission loss. For example, see Reference 10. If this is true, the spatial variability of the internal wave field should also be a factor. In this case, movement relative to the internal wave field by a source or a receiver should generate variability in the signal excess which would depend on the spatial variability of the field and the relative motion.

In the three encounter models described below, detection is defined in terms of signal excess and decisions are indexed by a time that can usually be considered to be the time of the decision. During an encounter, observations are made sequentially of one or a series of localization regions (resolution cells). For a localization region that does not contain a target, the signal observed during the observation of the region is zero. For these observations, the time to resolve a false alarm is zero. However, since the value of RD (or DT) is finite and consequently the false alarm probability is not zero, by implication, the cost associated with a false alarm is not zero.

The First Passive Sonar Encounter Detection Model: This model describes an encounter in terms of a series of decisions with each decision based on the signal excess $X_{SE}(t)$ at a time corresponding to the end of an observation. The observations are of equal duration and the integration time that determines the recognition differential is equal to the duration of the observations. In the model, $X_{SE}(t)$ is determined by a lambda-sigma jump process. For an encounter of m observations in which SE(t) is unimodal and in which the time of the single maximum is prior to or at the end of the encounter, it is shown in Reference 11 that the probability p that detection will occur during the encounter is given by the following equation:

(6) $p = 1 - [(1 - p_c)/(1 - \beta \cdot p_c)](1 - \beta \cdot p_1) \cdots (1 - \beta \cdot p_m)$

where $\beta = 1 - \exp(-\delta t/\tau)$ and where $p_i = \Phi[SE(t_i)/\sigma]$ for the index $i = 1, 2, \cdots, m$. Here, δt indicates the duration of an observation and Φ indicates the standard normal cumulative distribution function as before. The integer c is the index of a decision time t_c for which $SE(t_c)$ is greater than or equal to $SE(t_i)$ for any time t_i and $t_1 \leq t_c \leq t_m$.

As τ approaches zero, β approaches one and Equation 6 approaches:

(7) $p = 1 - (1 - p_i) \cdots (1 - p_m)$.

In this limit, the signal excess random variables are all independent. Note that Equation 7 applies without the condition that $SE(t)$ be unimodal.

As τ approaches infinity, β approaches zero and, in this case, Equation 6 approaches:

(8) $p = p_c$.

In this limit, the correlation coefficient between any pair of signal excess random variables is equal to one. Note that Equation 8 also applies without the condition that $SE(t)$ be unimodal. Equation 8 defines a complete dependence encounter model.

The Second Passive Sonar Encounter Detection Model: This model is in a sense a third limiting form of the first passive sonar encounter detection model. In this limit, the time δt between decisions approaches zero. However, in this limit the integration time that determines the recognition differential is not equal to δt and it does not approach zero. It is, in effect, chosen by the user of the model through the user's choice of the value for the recognition differential. For an encounter that begins at t_1 and ends at t_2 and for which $X_{SE}(t)$ is determined by a lambda-sigma jump process and SE(t) is unimodal, it is shown in Reference 11 that for this limit, Equation 6 has the following form:

$$(9) \qquad p=1-[1-p(t_c)]\exp[-\lambda\int_{t_1}^{t_2}p(t)\,dt]$$

where $p(t) = \Phi[SE(t)/\sigma]$ and where now t_c is the encounter time such that $SE(t_c)$ is greater than equal to SE(t) for any other encounter time t and $t_1 \leq t_c \leq t_2$.

The Third Passive Sonar Encounter Detection Model: This model describes an encounter between a target and a passive sonar detection system in which detection occurs during an encounter if the average value of the square of the continuously observed signal-to-noise ratio over a time interval of length u is greater than or equal to the square of the signal-to-noise ratio that determines the recognition differential for an integration time equal to u. With R(s) the random signal-to-noise ratio at a time s and $R_m(u)$ the random signal-to-noise ratio that determines R(s) for an integration time u, detection occurs at the first time t that the following inequality is satisfied:

(10) $$(1/u)\int_{t-u}^{t}[R(s)/R_m(u)]^2 ds \geq 1$$

Here, the time origin is chosen so that $t \geq 0$ and the integration time $u = t$ for $t < t_0$ and $u = t_0$ for $t \geq t_0$ where t_0 is a maximum integration time. The random integrand in the inequality is related to the random signal excess at the time s for an integration time u. The relation is:

(11) $10 \log [R(s)/R_m(u)]^2 = 2[SE(s;u) + X(s)]$

where $SE(s;u)$ is the expected value of the signal excess at a time s for an integration time u and $X(s)$ is the random component of the signal excess at the time s. In the model, $X(s)$ is determined by a lambda-sigma jump process and $SE(s;u)$ is determined by an expected value sonar equation with a recognition differential $RD(u) = 10 \log r_m(u)$. Here, $r_m(t)$ is the value of the signal-to-noise ratio that gives a probability of detection equal to .5 for an integration time t and a specified probability of false alarm p_f. With the signal detection process described by a Case II signal detection model, the detection index necessary to give the required operating point $(p_f, .5)$ is related to the integration time t and the signal-to-noise ratio $r_m(t)$ by:

(12) $d = u \cdot (BW)[r_m(t)]^2$

where BW is the bandwidth of the receiver. For a spectrum analyzer, BW would be the bandwidth corresponding to a given frequency resolution and d would be the detection index required in order to be at the operating point $(p_f, .5)$ for a signal that was contained within a bandwidth BW. Since d in Equation 12 must be the same for $t = u$ and $t = t_0$,

(13) $RD(u) = 5 \log(t_0/u) + RD(t_0)$

where t_0 is the maximum integration time. Then, since $SE(s;u) - SE(s;t_0) = RD(t_0) - RD(u)$, by using Equation 13 and Equation 11, Relation 10 becomes:

$$(14) \qquad \int_{t-u}^{t} 10^{(1/5)[X(s)+SE(s;t_0)-5\cdot\log(t_0)]} ds \geq 1$$

where as above the time origin is chosen so that $t \geq 0$, the integration time $u = t$ for $t < t_0$ and $u = t_0$ for $t \geq t_0$ and where $SE(s;t_0)$ is the expected value of the signal excess at the time s for a recognition differential determined by an integration time t_0. In an encounter, detection occurs the first time that Relation 14 is satisfied.

As noted in Reference 12, the appeal of the Third Passive Sonar Encounter Detection Model relative to the Second and First Passive Sonar Encounter Detection Models is that it appears to more closely describe the detection process in passive sonar detection systems that display their processed data to an operator in a continuous manner over a time window of duration t_0. However, results reported in Reference 13 indicate that the difference between the three models may not be significant in some types of encounters.

VIII. Straight Line Encounters

In general, a range r_m can be defined beyond which the probability of detecting a target based on an observation is effectively zero. For example, the range to a radar horizon. In this report, an encounter between a target and a detection system exists when the range between the target and the detection system is less than or equal to r_m. Suppose r_m is small enough so that when the target and the detection system are having an encounter they can be considered to be moving on planes parallel to a tangent plane to the earth's surface at a point in their vicinity. In this case, if the target and detection system maintain a constant course and speed during the encounter, it is called a **straight line encounter**.

A straight line encounter can be described in terms of a two dimensional rectangular coordinate system whose plane is parallel to the tangent plane to the earth. If the coordinate system is stationary relative to the detection system with the detection system located at the origin and is oriented so that the target's motion is parallel to the y-axis and is in the positive y-direction, then the target's x-coordinate during a straight line encounter will be constant. The constant is equal to the target's horizontal range at the **closest point of approach (CPA)** on the straight line track on which the target is moving relative to the detection system during the encounter. This range is called the target's **lateral range**.

A **complete straight line encounter** is a straight line encounter that begins at a range from a detection system that is greater than or equal to r_m and continues past CPA to a range from the detection system that is again equal to or greater than r_m. Let p(x) be the cumulative probability that a target is detected by a detection system in a complete straight line encounter in which the target's lateral range is x. Then the function **p(x)** defines what is called a **lateral range curve** or **lateral range function.**

Let p be the probability that a target is detected during a complete straight line encounter. If the lateral range of a target in a straight line encounter is assumed to be a continuous random variable X with a uniform distribution with $f_X(x) = 1/a$ for $|x| \leq a/2$ and $p(x) = 0$ for $|x| > a/2$, then the probability that a target will be detected during a complete straight line encounter is given by:

(15) $$p=(1/a)\int_{-\infty}^{\infty} p(x)\,dx$$

where the limits of integration can be used since the value of p(x) is zero for $|x| > a/2$. Equation 15 suggests a measure of a detection system's capability to detect a target in a straight line encounter. The measure **W** is called **sweep width** and it is defined in Reference 6 as:

(16) $$W=\int_{-\infty}^{\infty} p(x)\,dx.$$

In an application of the definition, the infinite limits are replaced by a number that corresponds to a maximum detection range for the circumstances involved. To do this may require some analysis. For example, consider a detection system described by the Case II model and a signal such that the maximum encounter signal-to-noise ratio approached zero as x increased. In this case, $p(x)$ would approach p_f and the integrand in defining Equation 16 would not approach zero and W would increase without limit as x increased.

IX. Two Intermittent Signal Encounter Models

In the intermittent signal encounter models that are described here, a straight line encounter takes place between a detection system and a target that at various times either emits a signal (an acoustic transient) or is the cause of a signal (a visible wake) during the encounter. Two cases are considered: In the first case, the signals occur periodically, the signals are of length δt and the time between the occurrence of signals is τ where $\tau > \delta t$. And, for the detection system, prior to the detection of a signal, the time at which a signal will be emitted in a time interval of length τ is uniformly distributed over that interval. In the second case, $\delta t = 0$ (the signals are instantaneous) and the signals occur at times determined by a Poisson process with τ the expected time between signals.

In the models, the detectability of a target signal depends on the target's horizontal range r from the detection system where r is determined by the characteristics of the detection system and the target signal. If a signal is present while the target is within horizontal range r, it will be detected. Otherwise, it will not be detected. The geometry for an encounter is shown in Figure 3.

For an intermittent signal, the exposure time of a target relative to a detection system is $(2/w)(r^2 - x^2)^{1/2} + \delta t$. The models are based on the assumption that the encounters are such that a target is exposed for this time between two consecutive signals.

For a periodic intermittent signal, if $r \geq w \cdot (\tau - \delta t)/2$, the lateral range function for an encounter is:

$$
\begin{aligned}
p(x) &= 0 \quad \text{for} \quad |x| > r \\
p(x) &= 1 \quad \text{for} \quad |x| \leq \{r^2 - [w \cdot (\tau - \delta t)/2]^2\}^{1/2} \qquad (17) \\
p(x) &= [2/(w\tau)](r^2 - x^2)^{1/2} + \delta t/\tau \quad \text{otherwise}
\end{aligned}
$$

If $r < w \cdot (\tau - \delta t)/2$, the middle equality in Equation 17 does not apply.

For intermittent signals whose occurrence is determined by a Poisson process and for which $\delta t = 0$, the lateral range function for an encounter is:

$$
\begin{aligned}
p(x) &= 1 - \exp\{-[2/(w\tau)](r^2 - x^2)^{1/2}\} \quad \text{for} \quad |x| \leq r \qquad (18) \\
p(x) &= 0 \quad \text{for} \quad |x| > r.
\end{aligned}
$$

For signals whose occurrence is determined by a Poisson process and for which $\delta t > 0$, signals can overlap. If this is allowed,

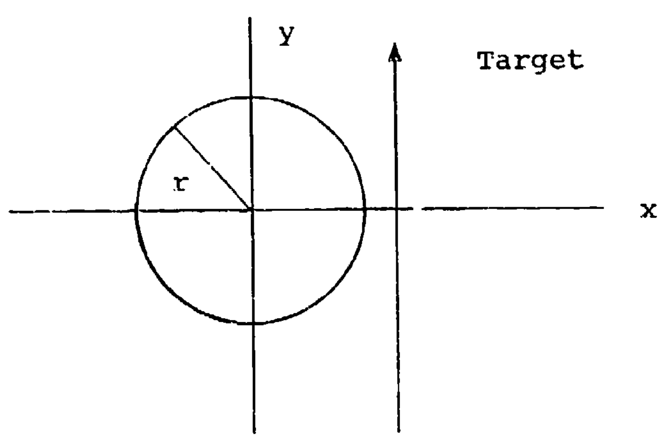

Figure 3. The encounter geometry for the two intermittent signal models described here.

then Equation 18 can be modified to describe this case by adding $\delta t/\tau$ to the term in the exponent of Equation 18 that is within the square brackets. In particular, note that a modified Equation 18 can be approximated by the bottom equality in Equation 17 when $(2/w\tau)(r^2 - x^2)^{1/2} + \delta t/\tau << 1$. This implies that when the expected time τ between signals is large relative to the exposure time $(2/w)(r^2 - x^2)^{1/2} + \delta t$, the periodic signal model and the Poisson random signal model are essentially equivalent.

If $r < w\cdot(\tau - \delta t)/2$, for a periodic intermittent signal and an encounter such that between two consecutive signals the exposure time is $(2/w)(r^2 - x^2)^{1/2} + \delta t$ for $|x| \le r$, the sweep width for the encounter $W = \pi r^2/(w\tau) + 2r\delta t/\tau$.

X. A Random Search Model

As the term is used here, a random search of a region is one in which a detection system's track relative to a target consists of a series of straight line segments which, in a limiting sense, are placed randomly within a search region. Figure 4 represents the track of a detection system performing a random search for a fixed target in a search region bounded by a circle.

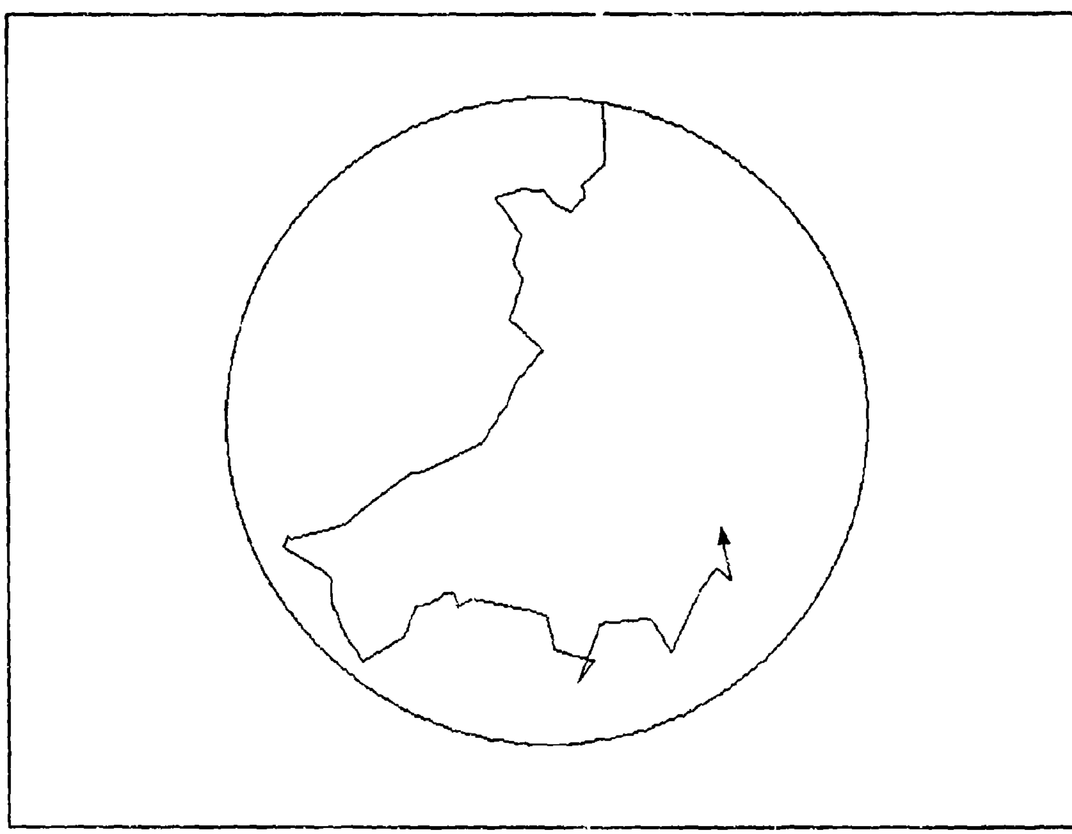

Figure 4. A search region and a track that could be described as a random search track.

Three models that represent this kind of search are described in this section. The first model is based on the following conditions: 1. A search consists of the search of a series of rectangular subregions that are completely contained within the search region, whose width is determined by the maximum detection range of the searcher's detection system on a track segment and whose length is equal to the length of the detection system's corresponding track segment. 2. Given a target is within the search region, the probability that the target will be within a rectangle during its search is equal to the ratio of the area of the rectangle to the area of the search region. 3. The track segments are located in such a way that the event the target is in a track segment's corresponding rectangle is independent of the event that is in any other rectangle. 4. If the target is within a rectangle being searched, a complete straight line encounter occurs in which the relative track of the detection system (the corresponding track segment) is parallel to the long axis of the rectangle and the lateral range of the target is uniformly distributed across the width of the rectangle. 5. The probability that the searcher's detection system will detect the target in the encounter is $p(x)$ where x is the lateral range and $p(x)$ is the lateral range function for the encounter. 6. The probability that the searcher's detection system will detect a target that is not in the rectangle being searched is zero.

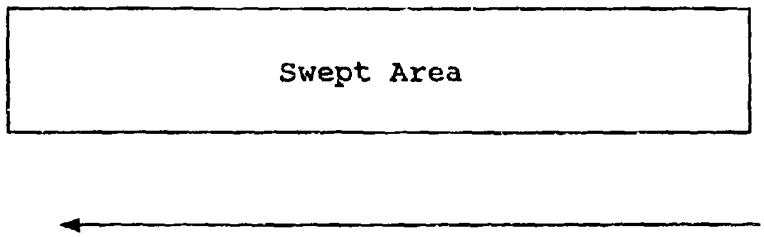

Figure 5. A search rectangle and the track of a searching aircraft with a side looking detection system.

Based on the above conditions, the probability that a target will be detected given it is in a search rectangle that is being searched is given by:

$$\int_{-\infty}^{\infty} p(x)\, f_X(x)\, dx = W/b \tag{19}$$

where $f_X(x) = 1/b$ for $-b/2 \le x \le b/2$ and where $f_X(x) = 0$ and $p(x) = 0$ otherwise. Note that the left side of Equation 19 applies to any complete straight line encounter in which the target's lateral range for the encounter is a random variable with a distribution corresponding to the probability density function $f_X(x)$. The unconditional probability that the target will be detected on the track segment is: $(W/b)(\delta A/A)$ where δA is the area of the search rectangle associated with the track segment and A is the area of the search region. With ℓ the length of the rectangle, $\delta A = b \cdot \ell$ and the probability becomes: $(W \cdot \ell)/A$. Then, since the event that the target will be in the search rectangle of a track segment is independent of the event that it is rectangle of a track segment is independent of the event that it is in the search rectangle of any other track segment, the probability p that a random search consisting of m track segments will detect the target is given by:

$1 - [1 - (W \cdot \ell_1)/A][1 - (W \cdot \ell_2)/A] \cdots [1 - (W \cdot \ell_n)/A]$ where ℓ_i is the length of the i^{th} track segment. The probability is also given by: $p = 1 - \exp(\Sigma \ln[1 - (W \cdot \ell_i)/A])$ where $i = 1, 2, \cdots,$ n. If $(W \cdot \ell_i)/A << 1$ for $i = 1, 2, \cdots, n$, this expression can be approximated by:

(20) $$p = 1 - \exp[-(W \cdot \ell)/A]$$

where $\ell = \Sigma \ell_i$ is the track length of the search. Equation 20 is known as the **random search formula.**

The second model of a random search is based on Equation 5 and a random search detection rate function: $\tau(t) = W \cdot v(t)/A$ where $v(t)$ is the detection system's or the target's speed. With this detection rate function and Equation 5, the random search formula is:

(20a) $$P(T \leq t) = 1 - \exp\{-[W \cdot \ell(t)]/A\}$$

where $\ell(t)$ is the track length for a random search that starts at time 0 and ends at time t and

(20b) $$\ell(t) = \int_0^t v(s)\,ds.$$

Replacing $P(T \leq t)$ by p and $\ell(t)$ by ℓ gives Equation 20. In the form of Equation 20a, the random search formula indicates explicitly the relation between the probability of detection and the duration of a random search. Note that Equation 20a implies that the sweep width is independent of speed over the range of speeds in the encounter. Reference 15 contains an example of an extension of this model to determine the probability of detecting a target in a random search with $\tau(t) = W \cdot v/A(t)$ where $A(t)$ is the area of a disk whose radius increases with time.

A random search model can be used to determine the probability of detecting an intermittent target using a sweep width determined with one of the intermittent target models described in Section IX. For the periodic intermittent target model, $W = [\pi r^2/(v\tau) + 2r\delta t/\tau]$ and Wvt is the area that is approximately equal to the area searched for a track $\ell = vt$ if $t >> \tau$.

The two models each imply that the time to resolve a false alarm is zero in a random search. However, for each model, p_d and p_f can be assumed to be determined by a criterion such that p_f is less than one. Consequently, although the time to resolve false alarms is ignored in each model, the cost associated with a false alarm is not zero. (A model that accounts for the time to resolve false alarms is described in Reference 14.)

XI. Ladder and Barrier Search Models

In some barrier searches, the barrier search track is a ladder search track relative to a reference system that moves with the target. This fact is used in the barrier search model development that follows the two ladder search model developments below. The first ladder search model is referred to as an ideal ladder search model. It can be considered to describe a ladder search with precise navigation. The second ladder search model is referred to as a degraded ladder search. It can be considered to describe a ladder search track in which navigational errors result in omissions and overlaps in coverage.

An Ideal Ladder Search Model: The model is based on the following conditions: 1. A ladder search region is a rectangle that contains a fixed target. 2. During a search of the region, a searcher's detection system searches a set of m adjacent parallel rectangular strips of width s and length b that just cover the ladder search region. 3. There is a complete straight line encounter between the target and the detection system during the search of a strip. 4. The target's position in a strip is uniformly distributed across the width of the strip. 5. If the target is not in a rectangular strip, then the probability that the target will be detected during the search of the strip is zero. Because of Condition 5, targets outside of the rectangular strip that corresponds to a track segment cannot be detected while a detection system is on the track segment. This implies that the lateral range function for an encounter must satisfy the relation

$p(x) = 0$ for values of the lateral range x for which the target is outside of the strip and, consequently, $W \leq s$. If $W = s$, then the detection system detect the target with

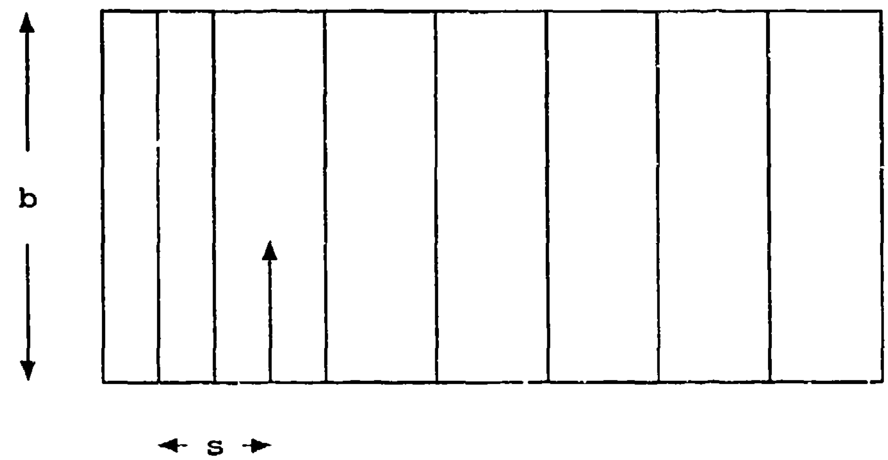

Figure 6. A schematic representation of a ladder search geometry for a case in which the searcher's track bisects the rectangular strips.

probability one if the ladder search is completed.

The ideal ladder search model implies that if the conditions of the model are satisfied, the probability p that a target will be detected by a an ideal ladder search is given by:

(21) $\quad p = W/s$

where $W/s \leq 1$. The quantity **W/s** is called the **coverage factor** in this case.

A Degraded Ladder Search Model: The ideal ladder search model implies precise navigation. A model of a ladder search is given in Reference 6 that can be used for cases in which this is a poor assumption. The model which is referred to here as a degraded ladder search model can be considered to describe navigational inaccuracies in terms of omissions and overlaps of coverage of the rectangular strips. It is develcped as follows: Consider a random search in the ladder search region whose track length is equal to the search track length required to complete an ideal ladder search, that is, a track length $\ell = mb$. The degraded ladder search model describes the result of omissions and overlaps in a ladder search to be such that the probability of detection for the ladder search is equal to the probability of detection for this random search. Consequently, since the area of the ladder search region is msb, for the degraded ladder search model:

(22) $$p = 1 - \exp(-W/s).$$

Although the requirement that the coverage factor $W/s \leq 1$ can be relaxed for Equation 22, it is still an approximate condition.

The condition that the target be fixed within the rectangular search region is critical to the models that determine both Equation 21 and Equation 22. However, these equations are also applicable to a search for a moving target under the conditions that are described next.

A Barrier Search Model: A target moves with a constant course and a constant speed u. Both the target's course and the target's speed are known by a searcher. The searcher establishes a barrier of width b that is perpendicular to the target's track and moves on the barrier with a speed $v > u$. The barrier is designed so that in a reference system relative to the target the barrier search is a ladder search that satisfies the conditions for a ladder search that are given above. There are two cases to consider: 1. The barrier is established in front of the target. 2. The barrier is established behind the target.

From the search geometry for a barrier established in front of the target, it can be seen from Figure 7 below that $\theta = \sin^{-1}(u/v)$ and $d = v\tau$ where $\tau = s/(v + u)$ is the time to move from one search leg to the next. The angle θ and the perpendicular distance d which depend on u, v and s, and the width of the barrier b are the quantities that determine the implementation of the barrier.

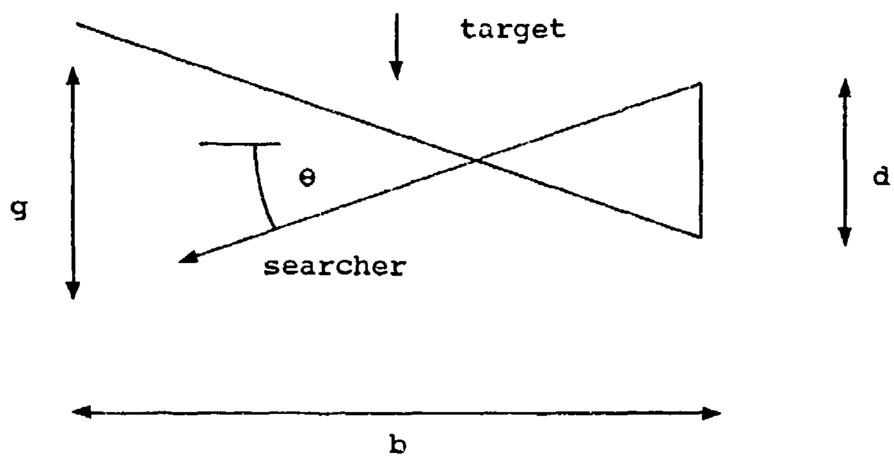

Figure 7. A barrier search track for a barrier established in front of the target. The track is shown in a reference system fixed relative to the earth.

For a barrier that is established in front of a target, one of three barrier types will result. A barrier's type is determined by the relation of the distance d to the distance $g = ut$ where the time $t = b/(v^2 - u^2)^{1/2}$ is the time to complete a search leg (cross the barrier). The barrier type is determined as follows: 1. For $g < d$, the barrier is an advancing barrier. 2. For $g = d$, the barrier is a stationary barrier. 3. For $g > d$, the barrier is a retreating barrier.

For a barrier established behind the target, there is only one barrier type and it is called an overtaking barrier. For an overtaking barrier, $\theta = \sin^{-1}(u/v)$ as for a barrier established in front of the target. But, for an overtaking barrier, $\tau = s/(v - u)$ and $d = v \cdot s/(v - u)$.

Given that a target crosses a barrier, the probability of detection for an ideal barrier search is given by Equation 21 and the probability for a degraded barrier search is given by Equation 22 where the terms ideal and degraded refer to the nature of the ladder search in the reference system moving with the target. A discussion of an application of these two equations to a search for a magnetic dipole target is given in Reference 16.

XII. A Target State Estimation Procedure

A target state estimation procedure based on bearing observations is developed in this section that generates point estimates of a target's position and velocity vector coordinates in a rectangular coordinate system. The procedure is based on a model in which bearing errors are unknown and are not determined by random variables with known distributions. Because of this, confidence regions for the estimates are not generated by the procedure. However, for a moving target, it illustrates general characteristics of bearings only **target motion analysis (TMA)**. The model is defined as follows: 1. The target moves in a plane with constant but unknown course and speed. 2. Observations of the target are made from known positions at known times. 3. The observations provide only target bearings with unknown errors.

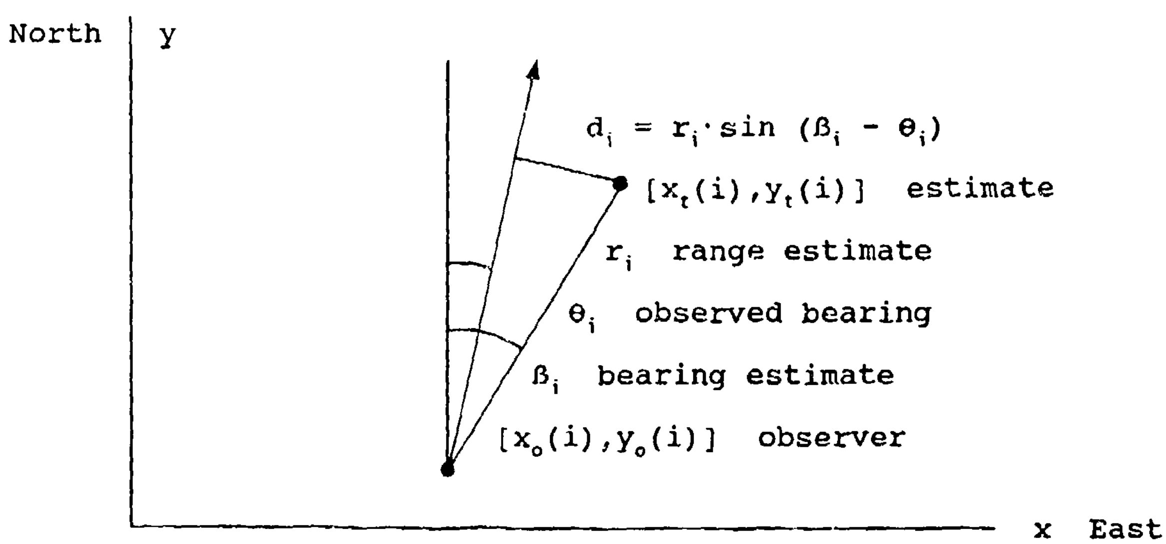

Figure 8. The geometry of the target motion analysis model.

The procedure criterion is: For observations from n positions, choose target position estimates $x_t(i)$ and $y_t(i)$ and target velocity component estimates u_x and u_y that make the sum of the squares of the algebraic distance d_i between the estimated positions and their corresponding observed bearing lines a minimum. From Figure 8, the algebraic distance can be written as $d_i = [x_t(i) - x_o(i)]\cdot\cos\theta_i - [y_t(i) - y_o(i)]\cdot\sin\theta_i$. Because of the requirement that the target move with constant course and speed during the encounter, the number of independent estimates is reduced from $2n$ to 4: u_x, u_y and any two position estimates $x_t(j)$, $y_t(j)$. In the following, $j = 1$ and, for $i = 2, 3, \cdots, n$, the estimates are given by:

$x_t(i) = x_t(1) + u_x\cdot(t_i - t_1)$ and $y_t(i) = y_t(1) + u_y\cdot(t_i - t_1)$. To determine "best" estimates of the target state parameters, take the partial derivative of the sum $S = \Sigma d_i^2$ with respect to each of them. Then set the four partial derivatives equal to zero. This creates four linear equations in $x_t(1)$, $y_t(1)$, u_x and u_y whose solution are the desired estimates $x_t(1)$, $y_t(1)$, u_x and u_y. In matrix notation, the equations can be represented by $AX = B$ where the elements of X are: $x_{11} = x_t(1)$, $x_{21} = y_t(1)$, $x_{31} = u_x$ and $x_{41} = u_y$. The determinant of A will be equal to zero if $n < 4$. Therefore, a necessary condition for a unique solution for X is that $n \geq 4$.

The procedure described above can also be used to estimate a target's position at various times when both the target's course and speed are constant and known. In particular, it can be used if the target is stationary so that u_x and u_y are both equal to zero. In this case, since the number of unknowns is two, the number of linear equations is also two and a necessary condition for a unique solution is $n \geq 2$.

Now, suppose the observations are at positions and times that correspond to the positions and times of an observer moving on some constant course at some constant speed (including zero speed). In this case, the observation position coordinates are related by: $x_o(i) = x_o(1) + v_x \cdot (t_i - t_1)$ and $y_o(i) = y_o(1) + v_y \cdot (t_i - t_1)$ where v_x and v_y are the required velocity components of the observer. Using these equations of motion, the matrix equation $AX = B$ can be transformed to the matrix equation $AX' = 0$ where the elements of X are related to the elements of the matrix X' by the equqtions: $x'_{11} = x_t(1) - x_o(1)$, $x'_{21} = y_t(1) - y_o(1)$, $x'_{31} = u_x - v_x$ and $x'_{41} = u_y - v_y$.

Since the linear equations represented by $AX' = 0$ are homogenous, they do not have a unique solution. Therefore, neither do the equations represented by $AX = B$. Consequently, in this case, the condition $n \geq 4$ is clearly not a sufficient condition for a unique solution However, if there is at least one observation whose time and position is not determined by the above equations of motion, the transformation from X to X' cannot be made. If the observations are made from a platform that is moving with a constant course and speed, the requirement can be achieved by changing the course, the speed or both prior to completing the observations. (That the condition $n \geq 4$ is not a sufficient condition in this case has been established by a counter example for $n = 4$. See Reference 17.)

Estimation models that describe bearing error as a random variable provide a basis for determining confidence regions for point estimates such as those discussed above. For example, see Reference 18 for a fixed target or simultaneous observation model and Reference 19 for more general cases.

XIII. Position Distributions That Change with Motion

Target motion models provide a basis for determining position distributions that change with target motion. In this section, two classes of target motion models are considered. In the first class, a target moves in a plane with a constant course and speed and the course and speed are independent of the target's position. In the second class, a target moves in a plane but its course or speed changes during the motion. Three models of the first class are developed first. This is followed by a discussion of some models of the second class.

Motion Models of the First Class: For the first class of motion models, the joint density function of the distribution that determines a target's coordinates $X(t)$ and $Y(t)$ at some time $t \geq 0$ can be determined by:

$$(23) \qquad f_{X(t),Y(t)}(x,y;t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X(0),Y(0)}(x-u_x t, y-u_y t; 0)\, f_{U_x,U_y}(u_x, u_y)\, du_x du_y$$

Equation 23 can be developed as follows: To first order, $f_{X(t),Y(t)}(x,y;t)\ \delta x \delta y$ is the probability that a target's coordinates x and y are in an element of area $\delta x \delta y$. To first order, the integrand of Equation 23 multiplied by $\delta q \delta s$ and $\delta u_x \delta u_y$ is the probability that the target's position at time 0 has coordinates $q = x - u_x t$ and $s = y - u_y t$ in an element of area $\delta q \delta s$. To first order, $\delta q \delta s$ is identical in size and shape to $\delta x \delta y$ because of the transformation from q and s to x and y. And, to first order, the sum of such probabilities for all pairs of values of u_x and u_y is also the probability that the target's coordinates at time t are in the element of area $\delta x \delta y$. In the limit, after equating the two expressions for this probability and canceling the common factor $\delta x \delta y$, Equation 23 results.

The First Motion Model: In the first model, $X(0)$ and $Y(0)$ are both independent normal random variables with means μ_x and μ_y and equal standard deviations σ. However, U_x and U_y are not normal and they are not independent random variables. In this model, $U_x = u \cdot \sin \Phi$ and $U_y = u \cdot \cos \Phi$ where Φ is the random variable that determines the target's course and u is the targets speed which is known. So, only a value for the random variable Φ is required to determine the target's velocity. In the model, Φ has a uniform distribution over the interval 0 to 2π and it is convenient to choose the rectangular coordinate system so that the means μ_x and μ_y are each equal to 0. Then, with the circular normal distribution determining the random position coordinates and with the distribution that is described above determining the random velocity components , in the coordinates u and ϕ, the integral of Equation 23 is a single integral over ϕ and the integrand of the integral is $(1/2\pi\sigma^2)\exp[-(q^2 + s^2)/2\sigma^2]\,(1/2\pi)$ where now $q = x - ut \cdot \sin \phi$ and $s = y - ut \cdot \cos \phi$. Integration gives:

$$(24) \qquad f_{X(t),Y(t)}(x,y;t) = \frac{1}{2\pi\sigma^2}\exp\{-[x^2+y^2+(ut)^2]/2\sigma^2\}I_0\{(x^2+y^2)^{1/2}ut/\sigma^2\}$$

where $t \geq 0$ and I_0 indicates the hyperbolic Bessel function of zeroth order. In Reference 6, $f_{X(t),Y(t)}(x,y;t)$ is plotted for several values of t in terms of $r = (x^2 + y^2)^{1/2}$, the target's range from the origin. The plots show a characteristic of the distribution that can be indicated as follows: First, replace $(x^2 + y^2)^{1/2}$ by r in $f_{X(t),Y(t)}(x,y;t)$. Next, multiply and then divide $f_{X(t),Y(t)}(r;t)$ by $\exp(-rut/\sigma^2)$. This gives:

$$\frac{1}{2\pi\sigma^2}\exp\left\{\left(\frac{1}{2\sigma^2}\right)(r-ut)^2\right\}I_0\left(\frac{rut}{\sigma^2}\right) \tag{25}$$

where $t \geq 0$. As noted in Reference 20, $I_0(z)\cdot\exp(-z)$ is a slowly decreasing function that asymptotically approaches $1/(2\pi z)^{1/2}$ as z increases. Because of this, a plot of $f_{X(t),Y(t)}(r;t)$ against r for values of t greater than $4\sigma/u$ has the appearance of a normal density function.

A target's random rectangular coordinates $X(t)$ and $Y(t)$ and its random bearing $\Theta(t)$ and range $R(t)$ from the origin are related by: $X(t) = R(t)\cdot\sin\Theta(t)$ and $Y(t) = R(t)\cdot\cos\Theta(t)$. Using these relations, $f_{X(t),Y(t)}(x,y;t)$ can be transformed to the joint density function $f_{R(t),\Theta(t)}(r,\alpha;t)$ of the random variables $R(t)$ and $\Theta(t)$. To do this, replace $x^2 + y^2$ by r^2 in Expression 24. Then multiply by r, the Jacobian of the transformation. Next, integrate the resulting joint density function $f_{R(t),\Theta(t)}(r,\alpha;t)$ over the interval 0 to 2π. This gives the marginal density function for the target's range $R(t)$:

$$f_{R(t)}(r;t) = \frac{r}{\sigma^2}\exp\left\{-\left(\frac{1}{\sigma^2}\right)[r^2+(ut)^2]\right\}I_0\left(\frac{rut}{\sigma^2}\right) \tag{26}$$

where $0 \leq r$.

The Second Motion Model: In the second model, X(0) and Y(0) are independent normal random variables with means μ_x and μ_y and standard deviations σ_x and σ_y that determine a target's random position coordinates at time 0. And U_x and U_y are independent normal random variables with means $\hat{u}_x$ and $\hat{u}_y$ and standard deviation σ_u that determine a target's random velocity components. Because of these conditions, the target coordinates are $X(t) = X(0) + t \cdot U_x$ and $Y(t) = Y(0) + t \cdot U_y$ at time t. This implies that X(t) and Y(t) are independent normal random variables with means $\mu_x + \hat{u}_x t$ and $\mu_y + \hat{u}_y t$ and with standard deviations $\sigma^2{}_x + \sigma_u{}^2 t^2$ and $\sigma^2{}_y + \sigma_u{}^2 t^2$. The model describes a bivariate normal position distribution whose center moves with a constant velocity determined by $\hat{u}_x$ and $\hat{u}_y$ and which becomes more and more circular as its standard deviations increase with the passage of time. Although the targets joint density can be found by using Equation 23, this procedure is more direct. For another discussion of the first and second models, see Reference 7.

The Third Motion Model: In the third model, the target is at the origin of a rectangular coordinate system at time zero. After that, its position is uniformly distributed on a circular disk of radius $u_m t$ centered at the origin. This implies that

(27) $$f_{X(t),Y(t)}(x, y : t) = \frac{1}{\pi u_m^2 t^2}$$

for $t > 0$ where $x^2 + y^2 \le u_m^2 t^2$ and that the joint density function of the distribution of the random variables $\Theta(t)$ and $R(t)$ that determine a target's bearing and range is:

$$f_{R(t),\Theta(t)}(r,\alpha;t) = \frac{r}{\pi u_m^2 t^2} \tag{28}$$

for $t > 0$ where $0 \le \alpha < 2\pi$ and $0 < r \le u_m t$. Since the ranges of α and r are independent and their joint density function is equal to the product of $1/(2\pi)$ and $2r/(u_m^2 t^2)$, the random variables $\Theta(t)$ and $R(t)$ are independent and $f_{\Theta(t)}(\alpha) = 1/(2\pi)$ and $f_{R(t)} = 2r/(u_m^2 t^2)$ where $0 \le \alpha < 2\pi$ and $0 < r \le u_m t$. These two marginal distributions define the motion model: At time 0, choose a course ϕ from a uniform distribution defined by the density function $f_\Phi(\phi) = 1/(2\pi)$ where ϕ is in radians and where $0 \le \phi < 2\pi$ and a speed u from a triangular distribution defined by the density function $f_U(u) = 2u/u_m^2$ where $0 \le u \le u_m$.

Motion Models of the Second Class: For the second class of motion models, a target's course or speed or both can change. In a limited number of cases, a description in terms of a closed form position distribution is possible. An example which is due to Washburn is described in Reference 21. In this model, a target's speed is known and its position is known at time 0 but is unknown at any future time. In addition, although the target's course ϕ at any time is unknown, it is known that the course is chosen from a uniform distribution and $f_\Phi(\phi) = 1/(2\pi)$ for ϕ in radians where $0 \le \phi < 2\pi$. A new course is chosen at times determined by a Poisson process with rate parameter λ. This idealized motion is referred to as a random tour. With $\rho^2 = (x^2 + y^2)/(ut)^2$, a target position distribution at a time $t > 0$ is defined as follows: For $\rho < 1$,

$$(29) \qquad f_{X(t),Y(t)}(x,y;t) = \frac{1}{2\pi u^2 t^2}\left\{\frac{\lambda t}{(1-\rho^2)^{1/2}}\right\}\exp\{-\lambda t[1-(1-\rho^2)^{1/2}]\}.$$

For $\rho > 1$, $f_{X(t),Y(t)}(x,y;t) = 0$. For $\rho = 1$, $f_{X(t),Y(t)}(x,y;t)$ is not defined, since $P\{X^2(t) + Y^2(t) = u^2t^2\} = \exp(-\lambda t)$ for $\rho = 1$. By transforming to bearing and range coordinates, a target range distribution at a time $t > 0$ can be determined by inspection and is defined as follows: For $\rho < 1$,

$$(30) \qquad f_{R(t)}(r;t) = \frac{r}{u^2 t^2}\left\{\frac{\lambda t}{(1-\rho^2)^{1/2}}\right\}\exp\{-\lambda t[1-(1-\rho^2)^{1/2}]\}.$$

For $\rho > 1$, $f_{R(t)}(r;t) = 0$. For $\rho = 1$, $P[R(t) = u^2t^2] = \exp(-\lambda t)$. An example that can not be defined in this manner which is a modification of the third model of the first class is described in Reference 22.

Two additional ways of defining target position distributions are by means of state transitions and by means of monte carlo simulations. Reference 23 gives an example of the use of state transitions and Reference 24 gives an example of the use of monte carlo simulations.

XIV. Position Distributions That Change with Search

For the models that are considered here, a target is within a region that has been divided into n subregions or cells and the event S_i = {the target is in the i^{th} cell}. For each cell, a number is assigned that is interpreted as the probability that the target is in the cell at the time of the search. This set of numbers define a target position distribution at that time. In a search of the region, suppose that a search planner is told that the target has been detected. Or suppose the search planner is told the target has not been detected. In the first case, positive information is available that can be used by the search planner to modify the position distribution. In the second case, negative information is available that can be used to do this. Models are developed below that provide ways of utilizing this kind of positive and negative information to modify a target position distribution. In the development, three random variables are determined by a search: N_c, the number of unresolved contacts; N_t, the number of unresolved true contacts and N_f the number of unresolved false contacts. Since $N_c = N_t + N_f$, this implies that

(31) $P(N_c=\ell) = \sum_{j+k=\ell} P[(N_t=j)\cap(N_f=k)]$ where $j \geq 0$ and $k \geq 0$.

A Positive Information Model: The event $\{N_c = 1\}$ is the union of two mutually exclusive events: $\{N_t = 1\} \cap \{N_f = 0\}$ and $\{N_t = 0\} \cap \{N_f = 1\}$. Let the event $C = \{N_c = 1\}$ and the events $TC = \{N_t = 1\} \cap \{N_f = 0\}$ and $FC = \{N_t = 0\} \cap \{N_f = 1\}$ where TC represents a true contact and FC a false contact.

Since $P(S_i|C) = [P(S_i \cap TC) + P(S_i \cap FC)]/P(C)$ and $TC = TC \cap C$ and $FC = FC \cap C$, $P(S_i|C) = P(S_i|TC)P(TC|C) + P(S_i|FC)P(FC|C)$ where $i = 1, 2, \cdots, n$. The probability $p = P(TC|C)$ has been called the credibility of the contact. In terms of p,

(32) $\quad P(S_i|C) = P(S_i|TC)\cdot p + P(S_i|FC)\cdot(1 - p)$

for $i = 1, 2, \cdots, n$. If $\{N_t = 1\} \cap S_i$ and $\{N_f = 0\}$ are independent for $i = 1, 2, \cdots, n$, then $P(S_i|TC)$ is given by

(32a) $$P(S_i|TC) = \frac{P(N_t{=}1|S_i)}{P(N_t{=}1)}P(S_i), \quad \text{since}$$

$$P[(N_t{=}1)\cap(N_f{=}0)] = P(N_f{=}0)\sum_1^n P[(N_t{=}1)\cap S_j] \quad \text{and} \quad P(N_t{=}1) = \sum_1^n P[(N_t{=}1)\cap S_j]$$

imply that $\{N_t = 1\}$ and $\{N_f = 0\}$ are independent events. If, in addition, the events $\{N_t = 0\} \cap S_i$ and $\{N_f = 1\}$ are independent for $i = 1, 2, \cdots, n$ then $P(S_i|FC)$ is given by

(32b) $$P(S_i|FC) = \frac{P(N_t{=}0|S_i)}{P(N_t{=}0)}P(S_i), \quad \text{since}$$

$$P[(N_t{=}0)\cap(N_f{=}1)] = P(N_f{=}1)\sum_1^n P[(N_t{=}0)\cap S_j] \quad \text{and} \quad P(N_t{=}0) = \sum_1^n P[(N_t{=}0)\cap S_j]$$

imply that $\{N_t = 0\}$ and $\{N_f = 1\}$ are independent events. To illustrate how Equation 32 might be used, suppose that the conditions for Equation 32a and Equation 32b are satisfied, that a contact is a line of bearing contact or an omnidirectional sensor contact and that the cells are range cells. With r_i a range identifying the i^{th} cell and the random variable R the target's range, suppose that $P(N_t = 1|R = r_i) = \Phi[SE(r_i)/\sigma]$. In this case,

$$P(N_t=1)=\sum_1^n \Phi[SE(r_j)/\sigma]P(R=r_j) \quad \text{and} \quad P(N_t=0)=\sum_1^n\{1-\Phi[SE(r_j)/\sigma]\}P(R=r_j).$$

A classical analogue to the above is obtained if R is taken to be continuous; r, $f_R(r|C)$ and $f_R(r)$ replace r_i, $P(R = r_i|C)$ and $P(R = r_i)$ and, in addition,

$$P(N_t=1)=\int_{r_1}^{r_2}\Phi[SE(r)/\sigma]f_R(r)\,dr \quad \text{and} \quad P(N_t=0)=\int_{r_1}^{r_2}\{1-\Phi[SE(r)/\sigma]\}f_R(r)\,dr.$$

A Negative Information Model: The event $\{N_c = 0\}$ is the intersection of the events $\{N_t = 0\}$ and $\{N_f = 0\}$. Let the event $NC = \{N_c = 0\}$ and the events $NTC = \{N_t = 0\}$ and $NFC = \{N_f = 0\}$. Then, $NC = NTC \cap NFC$ and

(34) $P(S_i|NC) = P(NFC|NTC \cap S_i)P(NTC|S_i)P(S_i)/P(NC)$.

If $NTC \cap S_i$ and NFC are independent for $i = 1, 2, \cdots, n$, then $P(S_i|NC) = P(NTC|S_i)P(S_i)/P(NTC)$ or, equivalently,

(34a) $$P(S_i|NC)=\frac{P(N_t=0|S_i)}{P(N_t=0)}P(S_i)$$ since

$$P[(N_t=0)\cap(N_f=0)]=P(N_f=0)\sum_1^n P[(N_t=0)\cap S_j] \quad \text{and} \quad P(N_t=0)=\sum_1^n P[(N_t=0)\cap S_j]$$

imply that NTC and NFC also are independent. To illustrate how it might be used, suppose the conditions for Equation 34a are satisfied and a search in a cell is a random search. With A_i the area, W_i the sweep width and ℓ_i the track length of the i^{th} cell:

$$P(S_i|N_c=0)=\frac{\exp[-(W_i\ell_i)/A_i]\,P(S_i)}{\sum_1^n \exp[-(W_j\ell_j)/A_j]\,P(S_j)}.$$

XV. Search Models and Search Theory

Search theory provides a basis for determining optimal search plans for a target whose state is determined within some bounds. Here, an optimal search plan is one for which the probability of finding a target within a given length of time is a maximum, the expected time to find a target is a minimum given the target is found or a search plan for which some other optimal search criterion is satisfied.

Search theory results are based on models of the search process. To the degree that a search model describes a search process, an optimal search plan for a target that is based on the search model should provide guidance for the development of an operationally feasible search plan. However, because of the limitations of analytical search models, an optimal search plan that is based on an analytical search model may give only initial guidance in this regard. The optimal search plans that are described below illustrate this. The search plans are based on the random search model. Because of this, the requirement on the location of search track segments is not realizable and the time to resolve false alarms is ignored.

Optimal search plans based on search models implemented through a monte carlo simulation are not considered here. However, with sufficient information, such a plan could be superior to an optimal search plan based on an analytical search model in some cases.

Three Optimal Search Plans: The three optimal search plans differ in their criterion for an optimal search plan. However, each one is based on the following search model: A target is fixed at some point in a region that consists of n subregions. A search in a subregion is a random search in the sense of the definition in Section X and a searchers sweep width there is a constant. In addition, a search of a subregion will not detect a target which is in another subregion. To determine a plan, let S_i = {the target is in subregion i} for $i = 1, 2, \cdots, n$ and let $p_i = P(S_i)$ be the prior probability that the target is in the i^{th} subregion. Let W_i be the sweep width in the i^{th} subregion. Let $\delta_i = A_i/W_i$ where A_i is the area of the i^{th} subregion and δ_i is the expected track length to find the target by a search of the i^{th} subregion given the target is in the i^{th} subregion, a characteristic length. The probability P that the target will be detected by a random search is given by:

(35) $$P = \Sigma\, [1 - \exp(-\ell_i/\delta_i)]\cdot p_i$$

where the sum index $i = 1, 2, \cdots, n$ and ℓ_i is the track length of the search in the i^{th} subregion.

The first criterion: Choose ℓ_i so that P is a maximum subject to the two constraints: 1. $\ell = \Sigma\, \ell_i$ and 2. $\ell_i \geq 0$ where the index $i = 1, 2, \cdots, n$. Determining this choice is a nonlinear optimization problem whose solution is given in Reference 25. It is:

(36) $$\begin{aligned} \ell_i^*/\delta_i &= \ln(p_i/\delta_i) - \lambda(k) \qquad & i &= 1, 2, \cdots, k \\ \ell_i^*/\delta_i &= 0 & i &= k+1, k+2, \cdots, n \end{aligned}$$

where $\lambda(k) = (\ell/\Sigma\delta_j)\cdot\Sigma[\delta_j\cdot\ln(p_j/\delta_j)] - \ell/\Sigma\delta_j$ and the sum index $j = 1, 2, \cdots, k$ where the subregions are relabeled so that the following order relation holds: $p_1/\delta_1 > p_2/\delta_2 > \cdots > p_n/\delta_n$ and where k is chosen so that for $k+1$ the solution for ℓ_{k+1} using $\lambda(k+1)$ is either negative or zero.

The second criterion: Choose ℓ_i so that P is a maximum subject to the two constraints: 1. $c = \Sigma\, c_i$ and 2. $c_i \geq 0$ where the index $i = 1, 2, \cdots, n$, $c_i = k_i\cdot\ell_i$ is the cost of the search in the i^{th} subregion and k_i is the cost per unit track length in that subregion. For this criterion, the solution to the corresponding nonlinear optimization problem can be obtained from Equation 36 by replacing δ_i by $\epsilon_i = k_i\cdot\delta_i$ and labeling the subregions so that $p_1/\epsilon_i > p_2/\epsilon_i > \cdots > p_n/\epsilon_i$. The basis for this can be seen by replacing ℓ_i/δ_i by its equivalent c_i/ϵ_i in the exponential term in Equation 35.

The third criterion: Choose ℓ_i so that the expected utility of the search is a maximum subject to the following two constraints: 1. $\ell = \Sigma\, \ell_i$ and 2. $\ell_i \geq 0$ where the index $i = 1, 2, \cdots, n$. For this criterion, the solution to the corresponding nonlinear optimization problem can be obtained from Equation 36, first, by replacing p_i by q_i where $q_i = u_i\cdot p_i$ and u_i is the utility of detecting the target given it is in the i^{th} subregion. And, second, by labeling the subregions so that the following inequalities hold: $q_1/\delta_1 > q_2/\delta_2 > \cdots > q_n/\delta_n$. The basis for this can be seen by multiplying the i^{th} summation term in Equation 35 by u_i for $i = 1, 2, \cdots, n$ so that the resulting equation gives the expected utility of the search given the utility of not detecting the target is zero.

Equation 36 can be used to determine an order of search for the subregions which will effectively minimize the expected track length required to detect a target given it 's detected. To do this, divide the available track length ℓ into units small enough so that with a single unit only the 1^{st} subregion would be searched. Then allocate one unit to the search of the 1^{st} subregion. If the search is unsuccessful, determine the optimum allocation for two units. Then search with a second unit so that the first search with the first unit plus the second search with the second unit satisfy the optimum allocation for two units. If the search is unsuccessful, continue in this fashion until either the target is found or all the track length is expended. That this allocation order will effectively minimize the expected track length required to detect a target given it is detected can be argued as follows: Let L be the track length at detection, let $\Delta\ell$ be a unit of track length and let n be the number of units. Then the value of the probability $P(L \leq i\cdot\Delta\ell)$ that the target will be detected on or before the i^{th} step of the search for the given allocation order will be greater than or equal to its value for any other allocation order with the same allocation step size. Since the value of the probability $P(L \leq \ell)$ will be equal to its value for any other allocation order of the optimum allocation. And, in addition, since $P(L \leq i\cdot\Delta\ell|L \leq 1) = P(L \leq i\cdot\Delta\ell)/P(L \leq \ell)$, the value of the distribution function $F_L(i\cdot\Delta\ell|L \leq \ell) = P(L \leq i\cdot\Delta\ell|L \leq \ell)$ will be greater than or equal to its value for any other allocation order. This implies that the expected track length given detection occurs is given by: $E(L|L \leq \ell) = \Sigma\ [1 - F_L(i \cdot \Delta\ell | L \leq \ell)]$, where the sum index $i = 1, 2, \cdots, n$, is effectively a minimum for the given allocation order. A search based on the optimum allocation given by Equation 36 and the given allocation order is equivalent to the following search: After an allocation of track length $\Delta\ell$ and an unsuccessful search, new values for $P(S_i)$ are calculated using Equation 34 and then Equation 36 is used with these new values to determine the next optimum allocation. A discussion of this procedure is given in Reference 6. And an example of its application is given in Reference 26.

Equation 36 also defines an optimal search plan for a detection system that searches beams and can be described by Equation 35 by replacing ℓ_i by t_i where t_i is the time the i^{th} beam is searched and by replacing δ_i by τ_i where τ_i, a characteristic time, is the expected time to detect the target by a search of the i^{th} beam given the target is in the i^{th} beam.

For a more extensive discussion of search theory and its application to military operations research, see Reference 27.

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