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NUMERICAL METHODS FOR SCATTERING FROM ELECTRICALLY LARGE OBJECTS (UNCLASSIFIED)

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Prepared for:

Air Force Office of Scientific Research
Bolling AFB
Washington, DC 20332-6448

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A new and computationally very efficient integral equation numerical method for computing electromagnetic scattering and Radar Cross Section (RCS) was developed. The operation count for this method was $O(n^{4/3})$, where n is the number of sample points on the boundary of the scatterer. A theory of higher order impedance boundary conditions was derived to handle single and multiple dielectric coatings around conductors. The method was tested in two dimensions using a 14,000-line FORTRAN program and was found to be very promising for electrically large objects. Initial ideas for extensions to three dimensions were explored. Treatments of trailing edge and corner singularities were developed.

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A. Research Objectives

The research objectives in this contract are to develop and test new integral-equation methods for solving electromagnetic scattering problems. One important application of the methods is in calculating Radar Cross Section (RCS). Because the methods are designed to be extremely efficient computationally, they will ultimately be able to be applied to calculations of the RCS of realistically large vehicles and platforms. The specific research objectives for the program were:

(1) Develop, test and evaluate codes to calculate electromagnetic scattering from conducting bodies in two dimensions, employing the Fast Multipole Method (FMM) originally developed by Rokhlin (Ro85, Ro90) for acoustic scattering. The FMM reduces computation time for electromagnetic scattering problems from $O(n^3)$ to $O(n^{4/3})$, and potentially $O(n \log n)$ per iteration, where n is the number of sample points on the boundary of the scatterer. Using complexification and extrapolation, the FMM can be made *globally* $O(n^{4/3})$, and potentially $O(n \log n)$.

(2) Develop the theory of impedance boundary conditions so that scattering may be computed from conducting bodies coated with thin dielectrics.

(3) Develop methods to improve the accuracy of integral-equation computations of electromagnetic scattering. Extend and improve a scattering code using fourth-order-convergent quadrature formulas. Development of this code was begun under a related prior IR&D project in FY1988.

(4) Explore approaches to computing scattering in three dimensions.

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B. Research Accomplishments

(1) A FORTRAN code (FMM1) implementing the Fast Multipole Method for the two-dimensional calculation of electromagnetic scattering from conductors was developed, tested, and evaluated. The code represents a significant accomplishment because of its extraordinary computational efficiency. The operation count has been reduced to $O(n^{4/3})$ (*globally*) by using the code in conjunction with the method of complexification and extrapolation. Conventional iterative methods have operation counts of $O(n^2)$ per iteration, and Gaussian elimination is globally $O(n^3)$. The FMM for electromagnetic scattering is reported in our publication En91 (see Section C and Appendix A).

(2) The theory of impedance boundary conditions has been developed for multiple thin material layers. This theory allows for the calculation of electromagnetic scattering from coated conductors.

(3) A code calculating electromagnetic scattering from conductors has been developed, using an integral-equation formulation employing highly accurate, fourth-order-convergent quadrature formulas. This is perhaps the most accurate integral-equation code in existence for electromagnetic scattering. Development of this code (SKIE1) was begun in a related prior IR&D program in FY1988 and reported in our publication Mu89 (see section C and Appendix A). However, the code was extended and perfected under this contract, because many of its features and subroutines were necessary for the proper design of the code implementing the Fast Multipole Method (Accomplishment (1)).

(4) The code (SKIE1) as perfected and extended under this contract was used in an IR&D investigation of the resonance problem in electromagnetic scattering computation, as reported in our publication Mu90 (see Section C and Appendix A).

(5) Preliminary exploratory work was performed on the extremely difficult problem of three-dimensional scattering computation.

C. Publications

[1] En91: Engheta, N., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1990) "The Fast Multipole Method for Electromagnetic Scattering Problems", submitted to *J. Appl. Phys..* **Included in Appendix A**

[2] En91a: Engquist, B., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), "Higher Order Impedance Boundary Conditions for Electromagnetic Scattering," to be submitted to *Journal of Applied Physics.* **Included in Appendix A**

[3] En91b: Engquist, B. (1991), "Effective Boundary Conditions for Helmholtz's Equation with Thin Layers", to be submitted to *SIAM J. Numer. Anal.*

[4] Mu90: Murphy, W. D., V. Rokhlin, and M. S. Vassiliou (1990), "Solving the Resonance Problem in Electromagnetic Scattering Computation", *J. Appl. Phys.* **67** (10), 15 May 1990, 6061-6065. **Included in Appendix A**

[5] Mu89: Murphy, W. D., V. Rokhlin, and M. S. Vassiliou (1989), "Numerical Second-Kind-Integral-Equation Solutions of Electromagnetic Scattering Problems", *Electronics Lett.* **25**, 643. (This is a relevant publication produced under a related FY1988 IR&D project). **Included in Appendix A**

D. Personnel

• WILLIAM D. MURPHY, PhD. Member of the Technical Staff, Computational Sciences Function, Rockwell International Science Center. Program Manager and Co–Principal

Investigator. Dr. Murphy has built on the foundations created by Dr. Rokhlin for acoustic scattering to develop effective new numerical techniques for electromagnetic scattering.

- MARIUS S. VASSILIOU, Ph.D. Member of the Technical Staff, Computational Sciences Function, Rockwell International Science Center. Co-Principal Investigator. Dr. Vassiliou's role has been primarily in the computer implementation of the new numerical methods.

- BJORN ENGQUIST, PhD. Professor of Mathematics, University of California, Los Angeles. Co-Principal Investigator. Professor Engquist has developed the theory of homogenization and concentration allowing the calculation of scattering from conductors with thin coatings.

- VLADIMIR ROKHLIN, Ph.D. Professor, Dept. of Computer Science, Yale University. Co-Principal Investigator. Dr. Rokhlin's role has been in mathematical analysis and program design.

E. Interactions

(1). Papers Presented at Conferences:

[1] Engheta, N., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), "The Fast Multipole Method for Scattering from Electrically Large Objects," accepted for presentation at the 1991 National Radio Science Meeting/ IEEE/AP-S Symposium, University of Western Ontario, London, Ontario, Canada, June 24-28.

[2] Engquist, B., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), "New Methods for Developing Higher-Order Impedance Boundary Conditions on Curved Surfaces," accepted

for presentation at the 1991 National Radio Science Meeting/ IEEE/AP-S Symposium, University of Western Ontario, London, Ontario, Canada, June 24-28.

[3] Engheta, N., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), " What is The Fast Multipole Method (FMM) and How can it Help You Solve Electromagnetic Scattering Problems More Effectively?", accepted for presentation at the Progress in Electromagnetics Research Symposium (PIERS), Cambridge, Massachusetts, July 1-5.

[4] Murphy, W.D., V. Rokhlin, and M. S. Vassiliou (1990), "The Fast Multipole Method (FMM) for Electromagnetic Scattering Problems", presented at the URSI/National Radio Science Meeting, Jan. 3-5, Boulder, Colorado.

[5] Murphy, W.D., V. Rokhlin, and M. S. Vassiliou (1990), "Solving the Resonance Problem in electromagnetic Scattering Computation" , presented at the URSI/National Radio Science Meeting, Jan. 3-5, Boulder, Colorado.

(2). Other Laboratories and Agencies

Harold Brooks of the Naval Weapons Laboratory, China Lake, CA, has been made aware of our work and is interested in receiving copies of the code FMM1.

F. New Discoveries

We have developed the first Fast Multipole Code that solves electromagnetic scattering problems. Previous implementations of the method solved acoustic or potential problems. We have developed a new method of calculating electromagnetic scattering and RCS that is both highly efficient and highly accurate. The Fast Multiple Method, used in conjunction with complexification and extrapolation, reduces computation time for electromagnetic scattering problems from $O(n^3)$ to $O(n^{4/3})$, and potentially $O(n \log n)$, where n is the number of sample points on the boundary of the scatterer.

We have also developed new theory on impedance boundary conditions for multilayered dielectrics, and begun work on the necessary theory to better handle corner singularities.

G. Background Information, Theory, and Computational Results

G1. The Fast Multipole Method (FMM) Applied to Electromagnetic Scattering

Our paper En91 (reproduced in Appendix A) should be consulted for full details and results. Below, we present some background information, partially excerpted from that paper. Essentially, the Fast Multipole Method that we have developed and applied is one of the most efficient methods available for computing electromagnetic scattering. The method reduces the operation count for solving the Magnetic-Field Integral Equation (MFIE) from $O(n^3)$ for Gaussian elimination to $O(n^{4/3})$ per conjugate-gradient iteration. If our method of "complexification" and extrapolation of the wavenumber (mentioned below and described in detail in the paper in Appendix A) is applied, then the condition number of the discrete system is bounded; consequently, the operation count of the *entire* FMM (all iterations,) becomes *globally* $O(n^{4/3})$.

The Fast Multipole Method (FMM) was developed by Rokhlin (Ro85, Ro90) to solve acoustic scattering problems very efficiently. We have modified and adapted it to the second-kind-integral-equation formulation of electromagnetic scattering problems in two dimensions. The present implementation treats the Dirichlet (TM) problem for two-dimensional closed conducting objects of arbitrary geometry.

Consider a scatterer with n nodes on its boundary. Divide the boundary into p segments, each containing n/p nodes. Instead of calculating n^2 interactions among n current sources on the boundary, consider each segment to be a cluster of n/p sources. For segments that are close together, the exact interactions must be calculated. For segments sufficiently far apart, however, we may combine the sources in each segment, approximating their radiation fields by the first N multipoles. We describe each segment via an equivalent

multipole located at the segment's center. We can calculate the contribution of each such equivalent expansion to the field at the center of any sufficiently separated receiver segment, and in that receiver segment we can use a partial wave expansion to obtain the field at all the individual n/p nodes. The radiation field at any particular node on the scatterer boundary is the sum of contributions of N multipole expansions of each of the far-away segments, and the direct contribution of very close segments. The method reduces the operation count for solving the Magnetic-Field Integral Equation (MFIE) from $O(n^3)$ for Gaussian elimination to $O(n^{4/3})$ per conjugate-gradient iteration.

We also present a simple technique for accelerating convergence of the iterative method: "complexifying" k, the wave number. This has the effect of bounding the condition number of the discrete system; consequently, the operation count of the *entire* FMM (all iterations) becomes $O(n^{4/3})$. We present computational results for moderate values of ka, where a is the characteristic size of the scatterer. Methods and results are described in detail in En91, included in Appendix A.

Consider a two-dimensional conducting body whose axis is aligned with the z coordinate axis. A monochromatic electromagnetic wave incident on this structure with an electric field vector parallel to the axis of the body is referred to as the transverse-magnetic (TM) case. The incident and scattered fields both satisfy the following Helmholtz equation

$$\Delta E_z + k^2 E_z = 0. \tag{1}$$

The boundary condition for equation (1) is that the total **E** field vanish on the surface of this conductor, i.e.,

$$E_z^{tot} = 0 \qquad \text{on} \quad C \tag{2}$$

or more explicitly

$$E_z^{inc} + E_z^{scat} = 0 \quad \text{on} \quad C. \tag{3}$$

The above Helmholtz equation can be reduced to the following second-kind integral equation

$$\Psi(\mathbf{r}) + 2 \int_C \frac{\partial G(k|\mathbf{r} - \mathbf{r}'|)}{\partial n(\mathbf{r}')} \Psi(\mathbf{r}') \, dl' = -2E_z^{inc}(\mathbf{r}) \tag{4a}$$

$$E_z^{scat}(\mathbf{r}) = \int_C \frac{\partial G(k|\mathbf{r} - \mathbf{r}'|)}{\partial n(\mathbf{r}')} \Psi(\mathbf{r}') \, dl' \tag{4b}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are both on the boundary C, and G is the free-space Green's function in two dimensions, i.e.,

$$G(k|\mathbf{r} - \mathbf{r}'|) = iH_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)/4 \tag{5}$$

with $H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)$ being the Hankel fuction of the first kind of order zero. See Co83 for a derivation of equation (4). Equation (4), the TM case, is often referred to as the Dirichlet problem in the mathematical literature (Kr89, Co83). If we discretize the boundary into n points, then the above integral equation (4a) is converted to the following linear system (via Nyström's method (Kr89)):

$$\Psi(\mathbf{r}_i) + 2 \sum_{j=1}^{n} A_{ij} \Psi(\mathbf{r}_j) = -2E_z^{inc}(\mathbf{r}_i) \tag{6}$$

where the matrix $A = (A_{ij})$ is $n \times n$ and the vectors $(\Psi(\mathbf{r}_i))$ and $(E_z^{inc}(\mathbf{r}_i))$ are column vectors having n rows. Applying normal matrix multiplication, $A\vec{\Psi}$ requires $O(n^2)$ operations. The FMM algorithm reduces this to $O(n^{4/3})$ or ultimately to $O(n \log n)$.

Consider n nodes on the boundary of the scatterer. Divide the boundary into p equal segments, where $2 \leq p < n$. In each segment, there are n/p nodes. If the length of the boundary is L, each segment has length L/p. The center of each segment is located at

$z_i(i = 1, 2, \ldots, p)$. In scattering problems, each node can be treated conceptually as if it were a source of radiation.

If we have sources within a finite region of space, the radiation emitted from these sources in the far zone can be approximated using a collection of multipoles located at the center of the region (Ro85, Ro90, Gr87). The multipole approximation converges rapidly outside any circle D containing all sources and separated from D by at least one wavelength. In fact, once a sufficiently large number of multipoles is included, the accuracy of the approximation increases superalgebraically (faster than any negative power of N (Ro90)).

Consider each segment on the boundary as a cluster of n/p sources. The sources in each segment are treated as a single aggregate source, and the radiation field of that equivalent source is approximated using the first N multipoles located at the center of the segment. For each pair of sufficiently-separated segments, the radiation of the N multipoles of one segment can be represented as an analytical partial field expansion around the center of the other segment. Then from this information, the field at the other nodes of that segment can be evaluated using equation (16). For nearby segments, the direct contribution must be calculated to evaluate the radiation field. The radiation field at any particular node on the boundary is the sum of the contributions of N multipoles of each of the far-away segments and the direct contribution of the nearby segments. Ro90 considers the precise mathematical description of the process.

To illustrate the above verbal description mathematically, let us consider the field of a two-dimensional magnetic surface current distributed over a two-dimensional body. Since we have an exterior Dirichlet problem, a double-layer potential is needed. For the two-dimensional TM problem, a magnetic current density at any point of the boundary is a

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vector in the x-y plane tangent to the body. The electric field of such a current distribution is given by

$$E^{scat}(\vec{\rho}) = \nabla \times \int_C G(k|\vec{\rho} - \vec{\rho}'|)K(\vec{\rho}')\,dl' \qquad (7)$$

where $K(\vec{\rho}')$ is the magnetic surface current density, $\vec{\rho} = (\rho, \theta)$, and C is the contour of integration. In two dimensions equation(7) can be written as

$$E_z^{scat}(\vec{\rho}) = \int_C \frac{\partial G(k|\vec{\rho} - \vec{\rho}'|)}{\partial n(\vec{\rho}')}|K(\vec{\rho}')|\,dl' \qquad (8)$$

which is identical with equation(4b) if $|K(\vec{\rho}')|$ is taken to be $\Psi(\vec{\rho}')$. Substituting equation(5) into (8), we get

$$E_z^{scat}(\vec{\rho}) = (i/4)\int_C \frac{\partial H_0^{(1)}(k|\vec{\rho} - \vec{\rho}'|)}{\partial n(\vec{\rho}')}|K(\vec{\rho}')|\,dl' \qquad (9)$$

The Hankel fuction $H_0^{(1)}(k|\vec{\rho} - \vec{\rho}'|)$ can be expanded in terms of higher order Hankel functions:

$$H_0^{(1)}(k|\vec{\rho} - \vec{\rho}'|) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(k\rho)J_m(k\rho')\exp(im(\theta - \theta')) \qquad (10)$$

Substituting equation (10) into equation (9) yields

$$E_z^{scat}(\rho, \theta) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(k\rho)\exp(im\theta)\int_C (i/4)\frac{\partial J_m(k\rho')\exp(-im\theta')}{\partial n(\rho', \theta')}|K(\rho', \theta')|\,dl' \qquad (11)$$

This can be regarded as the multipole expansion of the source $K(\rho', \theta')$. For a discretized source at n points located at $x_j' = (\rho_j', \theta_j')\,(j = 1, 2, \ldots, n)$ over a segment of the boundary, equation(11) reduces to

$$E_z^{scat}(\rho, \theta) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{n} (i/4)\frac{\partial J_m(k\rho_j')\exp(-im\theta_j')}{\partial n(x_j')}H_m^{(1)}(k\rho)\exp(im\theta)|K(x_j')|\Delta l_j' \qquad (12)$$

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where $\Delta l'_j$ is the discretized element of arc length containing the source $|K(x'_j)|$. For a given accuracy, we can truncate the infinite sum in equation(12) at N, and thus calculate the first N multipoles of the source.

Historically, the FMM algorithm was first applied to Poisson's equation for n point charge lines at locations $x_i(i = 1, 2, \ldots, n)$ with strengths κ_i. This is mathematically equivalent to solving the equation

$$\Delta\phi = \sum_{i=1}^{n} \delta(x - x_i)\kappa_i \tag{13}$$

where $\delta(x)$ is the Dirac delta function and x and x_i are two-dimensional points. The solution to equation (13) is

$$\phi(x) = \sum_{i=1}^{n} \kappa_i \log(|x - x_i|)/(2\pi) \tag{14}$$

If we evaluate equation (14) at each point $x_i(i = 1, 2, \ldots, n)$, then this computation requires $O(n^2)$ operations. However, if large numbers of particles are combined into single computational elements, then this operation count can be reduced if an approximate answer (to a specified accuracy) is desired. When a cluster of particles is "far away" from a particular point, then the potential of the cluster is approximated by the potential induced by a single computational element located inside the cluster (Gr87). In the FMM algorithm the computational element is a Laurent expansion centered at a circle containing the cluster of particles. Given a cluster of charges located at points $z_i(i = 1, 2, \ldots, n_c)$, the expansion is given by

$$\phi(z) = Re(\sum_{i=1}^{n_c} \log(z - z_i)) \approx Re(a_0 \log(z - z_0) + \sum_{k=1}^{p} a_k/(z - z_0)^k) \tag{15}$$

Here p is the order of the multipole and the $a_k s$ are coefficients chosen so that the truncated series is an accurate approximation of the potential. For equation (15), the computational

effort is only $O(p)$ operations and is much less than $O(n_c)$ for the direct approach. The region must now be organized into well-separated points and very-near points. For near points, the direct evaluation of (14) is used. See Ro85, Ro90, and Gr87 for details of decomposing the regions into boxes of different sizes. When applying the FMM to Helmholtz's equation instead of Poisson's equation the expansion (15) is replaced by the standard Hankel function expansion, which is then truncated to obtain a given accuracy requirement. That is,

$$E_z^{scat}(\rho, \theta) = \begin{cases} \sum_{m=-\infty}^{+\infty} \alpha_m H_m^{(1)}(k\rho)\exp(im\theta), & \text{if } \rho > a \quad (16a) \\ \sum_{m=-\infty}^{+\infty} \beta_m J_m(k\rho)\exp(im\theta), & \text{if } \rho < a \quad (16b) \end{cases}$$

depending on whether the calculations are to be done outside or inside the circle of radius a (Ro90). Obviously, any prescribed accuracy in the series can be guaranteed by taking more terms in the expansion at the expense of more CPU time. We refer the reader to En91 (included in Appendix A) for more details.

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G2. Impedance Boundary Conditions for Thin Dielectrics

Higher order impedance boundary conditions for thin coatings about closed conductors in two dimensions are derived using Fourier integral techniques. Using a single-layer potential and these impedance boundary conditions, second-kind weakly singular integral equations are derived that model TE electromagnetic scattering problems. These integral equations are solved using Nyström's method and approximately fourth-order convergent quadrature formulas.

Consider a two-dimensional closed perfect electrical conductor coated with a thin layer of dielectric and/or magnetic material. The classical way of solving the electromagnetic scattering problem from such an object is to develop an integral equation in which the contour of integration contains both the conductor and the outer surface of the dielectric. The difficulty with this approach is that as the thickness of the dielectric layer approaches zero, an ill-conditioned equation may result. In addition, the size of the discrete linear system is twice as large as in the method described below. Our procedure will translate the boundary condition on the surface of the conductor to the dielectric-air interface by developing an impedance boundary condition on the interface. The resulting integral equation will only have to be integrated along the interface, thereby reducing the number of unknowns for the discrete problem by a factor of two and possibly removing the ill-conditioning caused by grid points on the conductor and dielectric being too close together. Other work in this area can be found in Ro89, Se89, Vo89, Ha75, Ka65, Se81, Se62, Se91, and Ba90.

G.2.1 Derivation of Higher Order Impedance Boundary Conditions

The following treatment is taken from our paper En91a, in preparation for submission to the *Journal Applied Physics*. A draft of the paper is included in Appendix A.

Consider Helmholtz's equation written in co-ordinates normal (n) and tangential (t) to an infinite flat scatterer along the t axis, i.e.,

$$\frac{\partial^2 u^{tot}}{\partial n^2} + \frac{\partial^2 u^{tot}}{\partial t^2} + k_1^2 u^{tot} = 0 \tag{1}$$

where k_1 is the wave number in the dielectric ($k_1 = k_{air}\sqrt{\epsilon_1 \mu_1}$). Here the superscript *tot* denotes the total field. Thus

$$u^{tot} = u + u^{inc} \tag{2}$$

where *inc* denotes the incident field. No superscript is used for the scattered field. The boundary condition on the conductor for the TE polarization case is

$$\frac{\partial u^{tot}}{\partial n} = 0 \tag{3}$$

and on the dielectric-air interface we have

$$\frac{\partial u_2^{tot}}{\partial n} = \frac{\epsilon_2}{\epsilon_1} \frac{\partial u_1^{tot}}{\partial n} \tag{4}$$

where ϵ_1 is the dielectric constant in the thin layer and ϵ_2 is that in air. We assume the layer thickness is δ.

Taking Fourier integral transforms along the scatterer and assuming functions decay at $\pm\infty$, we can derive the following equation, where $\hat{\ }$ denotes Fourier transformation:

$$\frac{\partial^2 \hat{u}^{tot}}{\partial n^2} = (\omega^2 - k_1^2)\hat{u}^{tot} \equiv \alpha \hat{u}^{tot}, \qquad 0 < n < \delta \tag{5}$$

Let $\hat{u}_0^{tot}$ denote the unknown value of $\hat{u}^{tot}$ at $n = 0$ (on the metal-dielectric interface). Then we obtain

$$\hat{u}^{tot}(0) = \hat{u}_0^{tot} \tag{6a}$$

$$\frac{\partial^{2r}\hat{u}^{tot}(0)}{\partial n^{2r}} = \alpha^r \hat{u}_0^{tot} \tag{6b}$$

$$\frac{\partial \hat{u}^{tot}(0)}{\partial n} = \frac{\partial^{2r+1}\hat{u}^{tot}(0)}{\partial n^{2r+1}} = 0, \qquad r = 1, 2, \cdots \tag{6c}$$

where we have used equations (3) and (5). Expanding in a Taylor series in δ, we have

$$\hat{u}^{tot}(\delta - 0) = \hat{u}_0^{tot} + \frac{\delta^2 \alpha \hat{u}_0^{tot}}{2} + \cdots \tag{7}$$

$$\frac{\partial \hat{u}^{tot}(\delta - 0)}{\partial n} = \delta \alpha \hat{u}_0^{tot} + \frac{\delta^3 \alpha^2 \hat{u}_0^{tot}}{6} + \cdots \tag{8}$$

Substituting (8) into (4) gives (at $n = \delta + 0$)

$$\partial \hat{u}_2^{tot}/\partial n = \epsilon_2 \delta(\alpha \hat{u}_0^{tot} + (\alpha\delta)^2 \hat{u}_0^{tot}/6 + \cdots)/\epsilon_1$$

$$= \epsilon_2 \delta \alpha \hat{u}_2^{tot}/\epsilon_1 + O(\delta^2) \tag{9}$$

where we have used equation (7) and the continuity of $\hat{u}^{tot}$. Note that the subscript 2 represents the point $n = \delta + 0$. Taking inverse transforms of equation (9) yields the desired impedance boundary condition

$$\frac{\partial u_2^{tot}}{\partial n} = \frac{-\delta\epsilon_2\left(\partial^2 u_2^{tot}/\partial t^2 + k_1^2 u_2^{tot}\right)}{\epsilon_1} + O(\delta^2) \tag{10}$$

In terms of scattered and incident fields equation (10) may be re-written as

$$\frac{\partial u_2}{\partial n} + \delta(\epsilon_2/\epsilon_1)\left[\frac{\partial^2 u_2}{\partial t^2} + k_1^2 u_2\right] = -\frac{\partial u_2^{inc}}{\partial n} - \delta(\epsilon_2/\epsilon_1)\left[\frac{\partial^2 u_2^{inc}}{\partial t^2} + k_1^2 u_2^{inc}\right]$$

$$\equiv g \tag{11}$$

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where we have dropped the $O(\delta^2)$ term in equation (11).

Although equation(11) has been derived for a flat surface, it remains true to $O(\delta^2)$ for a curved surface as well. This can be seen by making a rotation of angle α between the flat boundary and the curved boundary, i.e.,

$$t = y\cos\alpha - x\sin\alpha$$

$$n = x\cos\alpha + y\sin\alpha.$$

Then to leading order $\alpha = y/R$ where R is the radius of curvature. Specializing to the point $(x,y) = (0,0)$, the origin, we have

$$\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2}{\partial y^2}((1 - \frac{y^2}{2R^2})y - \frac{y}{R}x) + O(\delta^2)$$

or

$$\left.\frac{\partial^2 t}{\partial y^2}\right|_{(x,y)=(0,0)} = 0$$

and

$$\frac{\partial^2 n}{\partial y^2} = \frac{\partial^2}{\partial y^2}((1 - \frac{y^2}{2R^2})x + \frac{y}{R}y) + O(\delta^2)$$

or

$$\left.\frac{\partial^2 n}{\partial y^2}\right|_{(x,y)=(0,0)} = \frac{2}{R}.$$

Using the above and the chain rule in equation(10) leads to

$$\frac{\partial u_2^{tot}}{\partial n} = -\delta(\epsilon_2/\epsilon_1)[\frac{\partial^2 u_2^{tot}}{\partial t^2} + \frac{2}{R}\frac{\partial u_2^{tot}}{\partial n} + k_1^2 u_2^{tot}] + O(\delta^2)$$

or

$$(1 + \delta\frac{\epsilon_2}{\epsilon_1}\frac{2}{R})\frac{\partial u_2^{tot}}{\partial n} = -\delta\frac{\epsilon_2}{\epsilon_1}[\frac{\partial^2 u_2^{tot}}{\partial t^2} + k_1^2 u_2^{tot}] + O(\delta^2).$$

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Finally,

$$\frac{\partial u_2^{tot}}{\partial n} = -\delta\frac{\epsilon_2}{\epsilon_1}[\frac{\partial^2 u_2^{tot}}{\partial t^2} + k_1^2 u_2^{tot}] + O(\delta^2)$$

for a curved surface.

To develop a third order impedance boundary condition on a curved surface, start with Helmholtz's equation in normal and tangential coordinates as

$$\frac{\partial^2 u^{tot}}{\partial n^2} + 2H\frac{\partial u^{tot}}{\partial n} + \frac{\partial^2 u^{tot}}{\partial t^2} + k_1^2 u^{tot} = 0 \qquad (12)$$

where H is the curvature at the boundary point. Let x and y be points separated by a distance δ on the dielectric and conductor, respectively. Then expanding in Taylor series along the normal yields

$$u^{tot}(y) = u^{tot}(x) - \delta\frac{\partial u^{tot}(x)}{\partial n} + \frac{\delta^2}{2}\frac{\partial^2 u^{tot}(x)}{\partial n^2} + O(\delta^3). \qquad (13)$$

Integrate equation (12) along the normal and apply equation (3):

$$\frac{\partial u^{tot}}{\partial n} + 2H\int_0^\delta \frac{\partial u^{tot}}{\partial n}\,ds + \frac{\partial^2}{\partial t^2}\int_0^\delta u^{tot}\,ds + k_1^2\int_0^\delta u^{tot}\,ds = 0 \qquad (14)$$

where H is treated as a constant to $O(\delta^3)$.

From equation (13), we have

$$\int_0^\delta u^{tot}\,ds = \delta u^{tot}(x) - \frac{\delta^2}{2}\frac{\partial u^{tot}(x)}{\partial n} + O(\delta^3) \qquad (15)$$

$$\int_0^\delta \frac{\partial u^{tot}}{\partial n}\,ds = \delta\frac{\partial u^{tot}(x)}{\partial n} - \frac{\delta^2}{2}\frac{\partial^2 u^{tot}(x)}{\partial n^2} + O(\delta^3). \qquad (16)$$

Inserting these expansions into (14) gives

$$\frac{\partial u^{tot}(x)}{\partial n} + 2H(\delta\frac{\partial u^{tot}(x)}{\partial n} - \frac{\delta^2}{2}\frac{\partial^2 u^{tot}(x)}{\partial n^2}) + \delta\frac{\partial^2 u^{tot}(x)}{\partial t^2} + \delta k_1^2 u^{tot}(x) + O(\delta^3) = 0. \qquad (17)$$

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Note that since $\partial u^{tot}/\partial n = O(\delta)$, we have that

$$\int_0^\delta u^{tot}\,ds = \delta u^{tot}(x) + O(\delta^3) \tag{18}$$

and from equation (12)

$$\frac{\partial^2 u^{tot}(x)}{\partial n^2} + O(\delta) + \frac{\partial^2 u^{tot}(x)}{\partial t^2} + k_1^2 u^{tot}(x) = 0. \tag{19}$$

Using equation (19) in equation (17) leads to

$$\frac{\partial u^{tot}(x)}{\partial n} + 2H\delta\frac{\partial u^{tot}(x)}{\partial n} + (\delta + \delta^2 H)\frac{\partial^2 u^{tot}(x)}{\partial t^2} + k_1^2(\delta + \delta^2 H)u^{tot}(x) + O(\delta^3) = 0. \tag{20}$$

The last expression may be re-written as

$$\frac{\partial u_1^{tot}}{\partial n} = -\frac{\delta}{1 + \delta H}\left(\frac{\partial^2 u_1^{tot}}{\partial t^2} + k_1^2 u_1^{tot}\right) + O(\delta^3). \tag{21}$$

The interface condition (4) may be used as before to obtain the appropriate condition at the dielectric-air interface.

Now introduce the single-layer potential

$$u(x) = \int_C \Phi(x,y)\phi(y)\,ds(y) \tag{22}$$

where x and y are two-dimensional points, C represents the outer contour (around the dielectric), and Φ is the two-dimensional free stream Green's function given by

$$\Phi(x,y) = iH_0^{(1)}(k|x - y|)/4 \tag{23}$$

Here $H_0^{(1)}$ denotes the first kind Hankel function of order zero, and $k = k_{air}$. If equation (22) is substituted into (11) and the appropriate jump conditions (Co83) are enforced at

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the dielectric surface, we can derive the following weakly singular second-kind integral equation:

$$\phi(x) - 2\int_C \left[\partial\Phi(x,y)/\partial n(x) + \delta(\epsilon_2/\epsilon_1)(\partial^2\Phi(x,y)/\partial t^2(x) + k_1^2\Phi(x,y))\right]\phi(y)\,ds(y)$$

$$= -2g(x) \tag{24}$$

See Co83 for more details on derivations of integral equations in the form of equation (24). If δ is set to zero, equation (24) reduces to the TE polarization case for scattering from a perfect electrical conductor. If the term $\partial^2\Phi/\partial t^2$ is removed from equation (24), we have the standard first order impedance boundary condition integral equation (Co83). Finally, as written, we call equation (24) the second order ($O(\delta^2)$) impedance boundary condition integral equation for modeling a thin coating about a metal conductor. The second derivative term obviously models the curvature of the scatterer.

The above derivation can easily be extended to multiple dielectric surfaces. Suppose we have m overlapping coatings with physical parameters $(\delta_i, \epsilon_i, k_i)$ $(i = 1, 2, \ldots, m)$. Let $\delta = \delta_1 + \delta_2 + \cdots + \delta_m$ and set $\beta_i = \epsilon_{i+1}/\epsilon_i$ where $\epsilon_{m+1} = \epsilon_{air}$. Then following the same steps as above, we can derive the following impedance boundary condition at the m-th dielectric-air interface:

$$\partial u/\partial n = -\sum_{j=1}^{m} \beta_{j+1}\delta_j(\partial^2 u/\partial t^2 + k_j^2 u) + g + O(\delta^2) \tag{25}$$

where we have dropped the subscript 2, and g is now defined to be equal to the sum in equation (25) for the incident field. An integral equation analogous to equation (24) can be easily written.

In addition, fourth and higher order impedance boundary conditions may be developed by allowing more terms in the Taylor series expansions (7) and (8). For example, the fourth

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order impedance boundary condition is given by

$$\partial u/\partial n + (\epsilon_2/\epsilon_1)(\delta^3/6)[\partial^2/\partial t^2 + k_1^2][\partial^2 u/\partial t^2 + k_1^2 u] = g \qquad (26)$$

where $-g$ now takes the form of the second term in equation (26) for the incident field and the subscript 2 has been dropped.

G2.2 Examples

Assume the incident field is in the form of a plane wave with the incident angle β, i.e.,

$$u^{inc}(x) = \exp[ik(x_1 \cos\beta + x_2 \sin\beta)] \qquad (27)$$

where $x = (x_1, x_2)$.

Fig. 1 shows the results (expressed as Radar Cross Section per incident wavelength, in dB) for first and second-order impedance boundary conditions, for a perfectly conducting elliptical cylinder (axes $a = 2, b = 1$; $kb = 5$) having a thin coating of thickness $\delta = 0.005$. The coating has material parameters $\epsilon = \mu = 14 + 1.4i$. The figure also shows a perfectly conducting elliptical cylinder without a coating, for comparison.

G3. Achieving High Accuracy in Second-Kind Integral Equation Solutions

In this section we describe the application of highly accurate quadrature formulas to the computation of electromagnetic scattering via second-kind integral equations. Our code using this method is perhaps the most accurate integral-equation method in existence for computing electromagnetic scattering. The method handles the singularity in the kernel (Green's function) by employing quadrature formulas with endpoint corrections (Ro85a, Mu89) that are $O(1/n^k)$ where n is the number of grid points and k is a positive integer. Thus, these quadrature formulas are convergent even for singularities of the form $\log x$ and $x^a(-1 < a < 0)$. In our computer program SKIE1, k is approximately 4 (but can be made higher).

In two dimensions, electromagnetic scattering from a closed conducting object defined by the curve γ can be described by solving Helmholtz's equation

$$\nabla^2 \phi + k^2 \phi = 0 \tag{1}$$

where k is the wave number of the incident radiation. In the case of TM polarization, ϕ represents E_z, the z-component of the electric field, and the boundary condition on γ is given by

$$\phi = f = -E_z^{\text{inc}} \quad \text{on } \gamma \tag{2}$$

where E_z^{inc} is the z-component of the incident electric field. We refer to problem (1-2) as an exterior Dirichlet problem. For TE polarization, ϕ represents H_z, the z-component of the magnetic field, and the boundary condition on γ is given by

$$\frac{\partial \phi}{\partial \nu} = g = -\frac{\partial H_z^{\text{inc}}}{\partial \nu} \quad \text{on } \gamma \tag{3}$$

where ν is in the outward normal direction from γ and H_z^{inc} is the z-component of the incident magnetic field. We refer to the problem defined by equations (1) and (3) as the exterior Neumann problem. For both the exterior Dirichlet and Neumann problems a radiation boundary condition at ∞ is required to guarantee the uniqueness of the solution. This condition is given by

$$\lim_{r\to\infty} r^{1/2} \left(\frac{\partial \phi}{\partial r} - ik\phi \right) = 0 \quad . \tag{4}$$

By developing local or global boundary conditions that are applicable at a finite distance R from the scatterer, it is possible to replace (4) with such conditions and solve the finite domain problem. Although the discrete problem has matrices which are sparse, the condition number increases with the inverse of the square of the mesh size as it approaches zero and eventually produces ill-conditioned systems especially as k, the wave number, increases. Instead we prefer an integral equation formulation which yields smaller matrices (unknowns are only on γ and not exterior to γ), although unfortunately these matrices are dense.

We define the field $\phi_{x_0}^k : R^2 \setminus \{x_o\} \to C^1$ of a unit charge located at the point $x_0 \in R^2$ by the formula

$$\phi_{x_0}^k(x) = H_0(k\|x - x_0\|), \tag{5}$$

where H_0 denotes the Hankel function of order zero. We define the field $\phi_{x_0,h}^k$ of a unity dipole located at x_0 and oriented in the direction $h \in R^2$ by the formula

$$\phi_{x_0,h}^k(x) = -H_1(k\|x - x_0\|) \cdot \frac{k(x - x_0, h)}{\|x - x_0\|}. \tag{6}$$

For a continous function $\sigma : [0, L] \to C^1$, the potential of a single layer of density σ on a

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curve γ is a mapping $P_{k\sigma}^0 : R^2 \to C^1$ defined by the formula

$$P_{k\sigma}^0(x) = \int_0^L \phi_{\gamma(t)}^k(x)\sigma(t)dt \tag{7}$$

and the potential of a double layer of density σ on a curve γ is a mapping $P_{k\sigma}^1 : R^2 \to C^1$ defined by the formula

$$P_{k\sigma}^1(x) = \int_0^L \phi_{\gamma(t),N(t)}^k(x)\sigma(t)dt. \tag{8}$$

Co83 show that by deriving appropriate jump conditions as $x \to \gamma$, the function σ satisfies the integral equation

$$2i\sigma(x) - P_{k\sigma}^1(x) = -f(x) \tag{9}$$

Thus, the TM problem is solved by solving the integral equation (9), and the resulting field is generated by evaluating the integral (8).

Similarly, for the TE problem (Neumann case), using the single-layer potential (7) we must solve

$$2i\sigma(x) + \frac{\partial}{\partial N}P_{k\sigma}^0(x) = g(x) \tag{10}$$

Consider a function Ψ satisfying Helmholtz's equation (1) and the radiation condition (4). In polar coordinates it may be expanded as

$$\Psi(x) = \sum_{m=-\infty}^{\infty} \beta_m H_m(k\rho)e^{im\theta} \tag{11}$$

where H_m denotes the Hankel function of order m. Asymptotic forms of the radiation field can be developed. First define

$$F_{\Psi,x_0}(\theta) = \lim_{t\to\infty} \Psi(tx + x_0)\sqrt{t}e^{-ikt}\frac{\sqrt{k\pi}}{\sqrt{2}}e^{(\pi/4)i} \tag{12}$$

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where $x = (\cos\theta, \sin\theta)$. Substituting (11) into (12) and using the asymptotic decay of the Hankel functions, we may write

$$F_{\Psi,x_0}(\theta) = \sum_{m=-\infty}^{\infty} \beta_m e^{-(m\pi/2)} e^{im\theta} \tag{13}$$

which provides an explicit expression for F_{Ψ,x_0} via the coefficients $\{\beta_m\}$, and we will refer to F_{Ψ,x_0} as the asymptotic representation of the field Ψ with the origin at x_0.

In order to solve eq. (9) or (10) numerically using Nyström's method, it is necessary to develop quadrature formulas that can handle the logarithmic singularity of the Hankel function. Roughly fourth-order-convergent quadrature formulas are derived (Ro85a, Mu89) by starting with the Euler-Maclaurin formula with the singular point ($x = 0$) removed. That is, let

$$\int_0^1 f(x)dx = h[\sum_{i=1}^n f(x_i) - \frac{f(x_n)}{2}] - \frac{1}{12}h^2 f'(x_n) \tag{14}$$

where $h = 1/n$ and $x_i = i/n$. To correct for the singular point, a concentration of points of the form $\sum_{j=1}^6 \lambda_j f(\chi_j)$ is introduced in the first interval, where $\chi_j = jh/6$ for $j = 1, 2, ..., 6$. The derivative term is approximated by the one-sided difference formula

$$f'(x_n) = \frac{1}{2h}(3f(x_n) - 4f(x_{n-1}) + f(x_{n-2})) \tag{15}$$

Combining these terms yields

$$\int_0^1 f(x)dx = h(\sum_{i=1}^n f(x_i) + \sum_{j=1}^6 \lambda_j f(\chi_j) - f(x_n)/2$$
$$- \frac{1}{24}(f(x_{n-2}) - 4f(x_{n-1}) + 3f(x_n))) \tag{16}$$

The unknown weights λ_j ($j = 1, 2, ..., 6$) are determined by solving the 6×6 linear system that results when Eq. (16) is assumed exact for the following candidate functions $f(x)$:

$1, x, x^2, \log x, x \log x$, and $x^2 \log x$, using analytic integration rules. Once computed, the quadrature weights can be stored and looked up numerically when needed. The proof of approximately fourth–order convergence can be found in Ro85a.

Fig. 2 shows the approximately fourth-order convergence in several electromagnetic scattering solutions. The relative errors are calculated by a method which is described in Mu89. The method essentially amounts to making the code numerically verify Green's second identity, and is a very good way to check the accuracy of the computations. As can be seen on the figure, accuracy at 10 points per wavelength is at least one percent, and can be much better. Dramatically increased accuracy can be obtained by using more points. This is important, because integral-equation methods commonly used today (most of which are method-of-moments implementations) do not have nearly as favorable convergence properties, and some of them even diverge.

G4. Approaches to the Three-Dimensional Problem

The extension of the FMM to three dimensions is non-trivial because of the difficulty of working with spherical Hankel functions. The essential idea is that one replaces the computation of the field induced by a large number of monopoles or dipoles by the computation of the field induced by a new computational element which is a spherical Hankel function expansion. The cost of evaluating this expansion is much cheaper than the direct evaluation of the field induced by all the mono-poles or dipoles. The cost of evaluating the matrix-vector products are reduced when the accuracy requirements are small because the number of terms in the spherical Hankel function expansion can be minimized.

In three dimensions, the formulation in equations (1-5) of Section G1 is modified to use the three dimensional free-space Green's function given by

$$G(k|\mathbf{r} - \mathbf{r}'|) = \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{5'}$$

In addition, the curve C in equation(4) now becomes a surface in three dimensional space. The source distribution $\psi(\mathbf{r})$ on C can be approximated by linear functions over each triangular element which subdivides the surface C, i.e.,

$$\psi(\mathbf{r}) = \sum_n \alpha_n L_n(\mathbf{r})$$

where L_n is the hat function at the point $\mathbf{r}_n$, the center of the n-th triangular element. The singularities in the integral in equation(4) at the point $\mathbf{r} = \mathbf{r}'$ may be handled by the quadrature formulas in Mu89.

Once equation(4) is discretized using the above expansion and quadrature formulas, a linear system results. The l-th component of this system is identical to the field evaluated

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at the point $\mathbf{r}_l$ which is induced by the monopoles or dipoles around the surface C. The fast multipole method replaces this matrix-vector operation by a computational element induced by these monopoles or dipoles. Such a formula may be derived by using a Poisson formula for Helmholtz's equation on a sphere S given by

$$I(x) = \int_S I(s) H(x,s)\, ds$$

with

$$H(x,s) = \frac{1}{4\pi a^2} \sum_{n=0}^{\infty} (2n+1) \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} P_n(\cos\theta)$$

Here $h_n^{(1)}$ is the first spherical Hankel function, P_n is the Legendre polynomial of order n, θ is the angle between Ox and Os, $r = |x|$, $a = |s|$ and x is a three dimensional point. Using an appropriate integration formula (e.g., Ro85a) and truncating the above series to quarantee a prescribed accuracy requirement leads to the desired computational element which may be used in the FMM to reduce the number of calculations.

G5 Numerical treatment of corner singularities.

The numerical methods that we have employed to solve the integal equations of scattering theory assume the boundaries of the scatterer are smooth. For trailing edges of airfoils or corner singularities, we must apply different quadrature formulas. One approach is given below. Assume the parameterization variable is s with $s = 0$ at the corner and choose a uniform mesh, h. Then we write

$$\int_0^1 g(x)\, dx = \sum_{j=0}^l \alpha'_j g(x_j) + h \sum_{j=l+1}^N g(x_j). \qquad (1)$$

Suppose the corner singularity can be approximated by a function like

$$g(x) = a_0 + a_1 x^\sigma + a_2 x + \cdots \qquad (2)$$

where $0 < \sigma < 1$ and σ is a function of the corner angle or the trailing edge angle. The unknown values α'_j in equation (1) can be determined by assuming the formula is exact for the trial functions $g(x) = 1, x^\sigma, x, \ldots, x^{l-1}$. If, in addition, the function $g(x)$ has a logarithmic singularity due to the Hankel function kernel in the integral equation, we must include in the trial functions $g(x)$ those of the form $\ln x, x \ln x, x^2 \ln x, \ldots$ as in Ro85a, Mu89, and Mu90. For example, with $\sigma = 1/2$ and $l = 2$, the weights in equation (1) are $\alpha'_0 = .195h$, $\alpha'_1 = 1.61h$, and $\alpha'_2 = .695h$. This can be seen by letting $\alpha'_0 = h\alpha_0$, $\alpha'_1 = h\alpha_1$, and $\alpha'_2 = h\alpha_2 + h/2$ and performing the integration over the interval [0,2h]. For $g = 1$, $g = x^\sigma$, and $g = x$, we obtain

$$h\alpha_0 + h\!\ _\lrcorner + h\alpha_2 = 2h \qquad (3a)$$

$$h\alpha_1 h^\sigma + h\alpha_2 (2h)^\sigma = (2h)^{\sigma+1}/(\sigma+1) \qquad (3b)$$

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$$h\alpha_1 h + h\alpha_2(2h) = (2h)^2/2. \tag{3b}$$

Simplifying equation (3) gives

$$\alpha_0 + \alpha_1 + \alpha_2 = 2 \tag{4a}$$

$$\alpha_1 + 2^\sigma \alpha_2 = 2^{\sigma+1}/(\sigma + 1) \tag{4b}$$

$$\alpha_1 + 2\alpha_2 = 2. \tag{4c}$$

Solving equation (4) yields

$$\alpha_0 = r \tag{5a}$$

$$\alpha_1 = 2 - 2r \tag{5b}$$

$$\alpha_2 = r \tag{5c}$$

where

$$r = [2 - 2^{\sigma+1}/(\sigma + 1)]/(2 - 2^\sigma). \tag{5d}$$

Substituting $\sigma = 1/2$ into equation (5d) gives $r = 0.195$, and the result follows.

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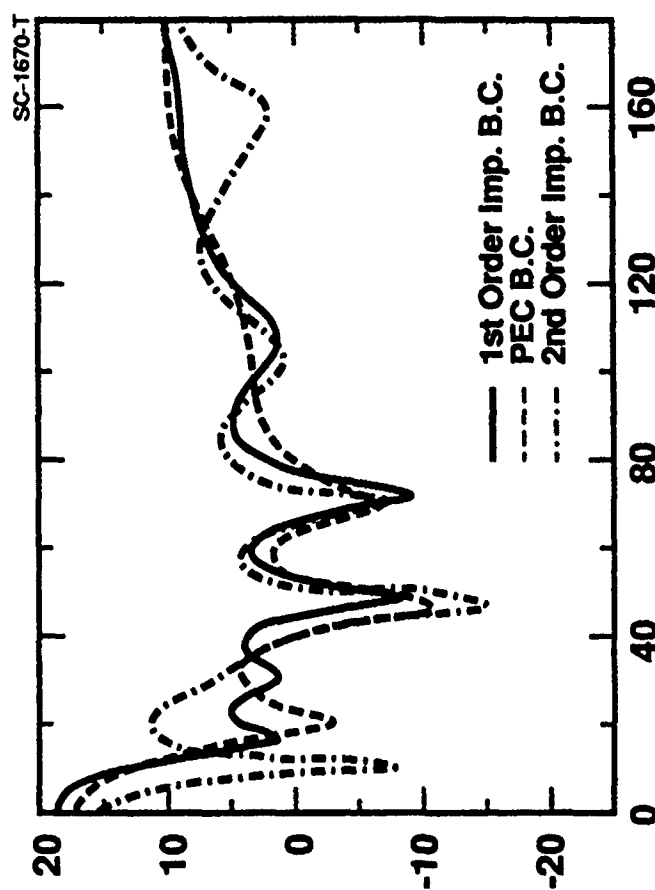Vo89: Volakis, J. L. and T.B.A. Senior, Proc. IEEE **77**, 796 (1989).

Fig. 1 Radar Cross Section per incident wavelength (in dB) for first and second-order impedance boundary conditions, for a perfectly conducting elliptical cylinder (axes $a = 2, b = 1$; $kb = 5$) having a thin coating of thickness $\delta = 0.005$. The coating has material parameters $\epsilon = \mu = 14 + 1.4i$. The figure also shows a perfectly conducting elliptical cylinder without a coating, for comparison.
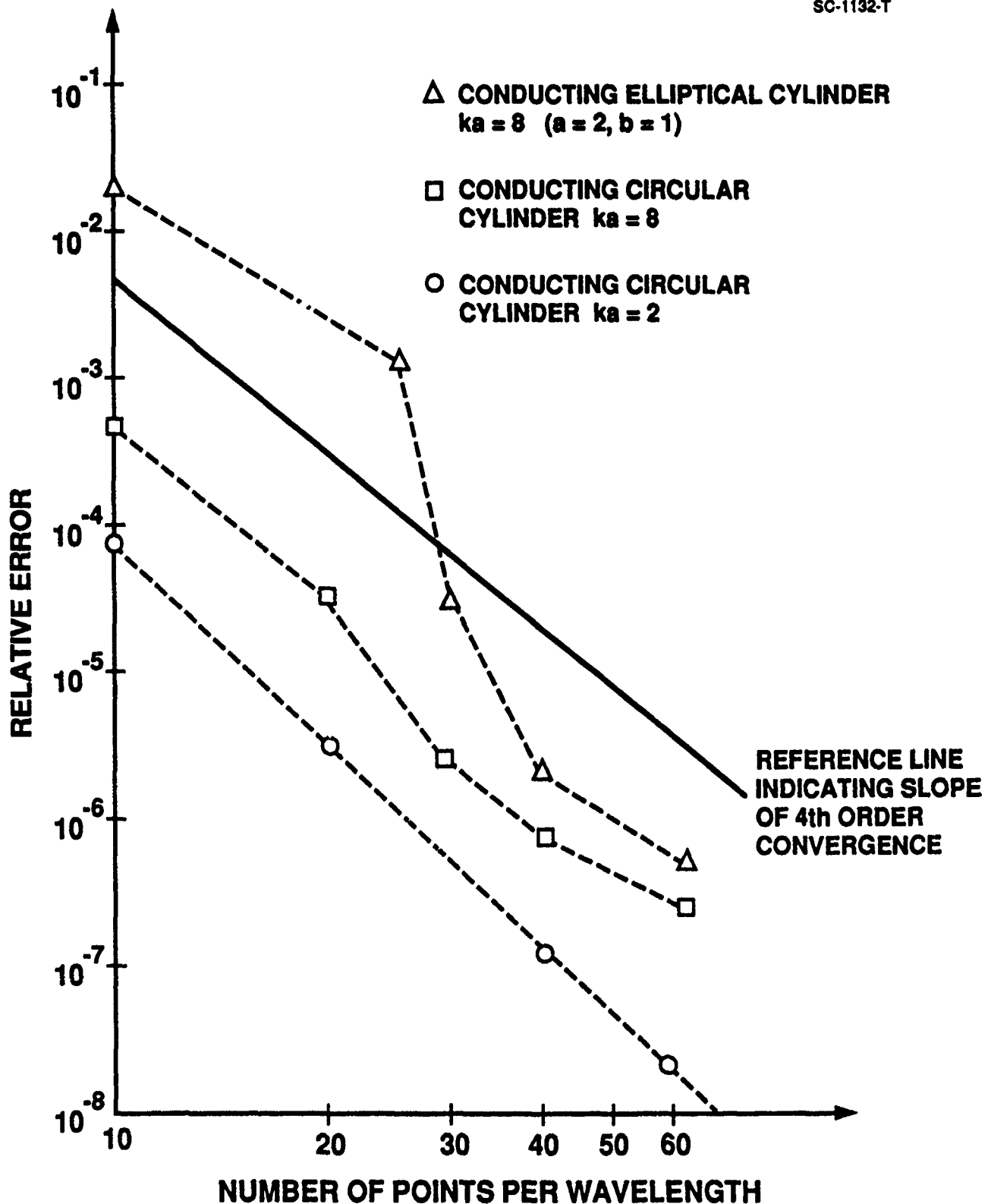
Fig. 2 Relative error vs. number of points per wavelength on scatterer boundary, for various two-dimensional conducting scatterers. The reference line shows the slope of fourth-order convergence. The quadrature formulas used are approximately fourth-order convergent.

APPENDIX A

1. Abstracts of Papers Presented at Conferences

[1] Engheta, N., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), "The Fast Multipole Method for Scattering from Electrically Large Objects," accepted for presentation at the 1991 National Radio Science Meeting/ IEEE/AP-S Symposium, University of Western Ontario, London, Ontario, Canada, June 24-28.

[2] Engquist, B., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), "New Methods for Developing Higher-Order Impedance Boundary Conditions on Curved Surfaces," accepted for presentation at the 1991 National Radio Science Meeting/ IEEE/AP-S Symposium, University of Western Ontario, London, Ontario, Canada, June 24-28.

[3] Engheta, N., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), " What is The Fast Multipole Method (FMM) and How can it Help You Solve Electromagnetic Scattering Problems More Effectively?", accepted for presentation at the Progress in Electromagnetics Research Symposium (PIERS), Cambridge, Massachusetts, July 1-5.

[4] Murphy, W.D., V. Rokhlin, and M. S. Vassiliou (1990), "The Fast Multipole Method (FMM) for Electromagnetic Scattering Problems", presented at the URSI/National Radio Science Meeting, Jan. 3-5, Boulder, Colorado.

[5] Murphy, W.D., V. Rokhlin, and M. S. Vassiliou (1990), "Solving the Resonance Problem in electromagnetic Scattering Computation" , presented at the URSI/National Radio Science Meeting, Jan. 3-5, Boulder, Colorado.

2. Publications

[1] En91: Engheta, N., W. D.Murphy, V. Rokhlin, and M. S. Vassiliou (1990) "The Fast Multipole Method for Electromagnetic Scattering Problems", submitted to *J. Appl. Phys.*.

[2] En91a: Engquist, B., W. D. Murphy, V. Rokhlin, and M. S. Vassiliou (1991), "Higher Order Impedance Boundary Conditions for Electromagnetic Scattering," to be submitted to *Journal of Applied Physics*.

[3] Mu89: Murphy, W. D., V. Rokhlin, and M. S. Vassiliou (1990), "Solving the Resonance Problem in Electromagnetic Scattering Computation", *J. Appl. Phys.* **67** (10), 15 May 1990, 6061-6065.

[4] Mu90: Murphy, W. D., V. Rokhlin, and M. S. Vassiliou (1989), "Numerical Second-Kind-Equation Solutions of Electromagnetic Scattering Problems", *Electronics Lett.* **25**, 643. (This is a relevant publication produced under a related FY1988 IR&D project).

THE FAST MULTIPOLE METHOD FOR SCATTERING FROM ELECTRICALLY LARGE OBJECTS

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The Fast Multipole Method (FMM) was developed by Rokhlin (*J. Comp. Phys.* 60, 187-207, 1985; *Yale Univ. Research Repor:* YALEU/DCS/RR-440, 1985) to solve acoustic scattering problems very efficiently. We have modified and adapted it to electromagnetic scattering problems in two dimensions.

Consider a 2-D closed conducting scatterer with n nodes on its boundary. Divide the boundary into p segments, each containing n/p nodes. Instead of calculating n^2 interactions among n current sources on the boundary, consider each segment to be a cluster of n/p sources. For segments that are close together, the exact interactions must be calculated. For segments sufficiently far apart, however, we may do the following: we combine the sources in each segment, approximating their radiation fields by the first N multipoles. We describe each segment via an equivalent source located at the segment's center. We can calculate the contribution of each such equivalent source to the field at the center of any sufficiently separated receiver segment, and in that receiver segment we can use a Taylor expansion to obtain the field at all the individual n/p nodes. The radiation field at any particular node on the scatterer boundary is the sum of contributions of N multipoles of each of the far-away segments, and the direct contribution of very close segments.

The method reduces the operation count for solving the Magnetic-Field Integral Equation (MFIE) from $O(n^3)$ for Gaussian elimination to $O(n^{4/3})$ per conjugate-gradient iteration. It has proven useful in calculating the scattering from electrically large objects difficult to compute by many other methods.

NEW METHODS FOR DEVELOPING HIGHER ORDER IMPEDANCE BOUNDARY CONDITIONS ON CURVED SURFACES[1]

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We have developed techniques for computing electromagnetic scattering from closed 2-D conductors coated with multiple thin layers of possibly lossy dielectric and/or magnetic material. Using Fourier integral techniques, we derive higher-order impedance boundary conditions of $O(\delta^2)$ and $O(\delta^4)$ in the thickness parameter δ. We develop second-kind, weakly singular integral equations. These integral equations are discretized and solved using Nyström's method and approximately fourth-order-convergent quadrature formulas. Solutions compare favor'ably with analytical results. We have used our techniques to study the effects of layer thickness, body geometry, and incidence angle on the scattered fields.

WHAT IS THE FAST MULTIPOLE METHOD (FMM), AND HOW CAN IT HELP YOU SOLVE ELECTROMAGNETIC SCATTERING PROBLEMS MORE EFFICIENTLY?[1]

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We describe the Fast Multipole Method (FMM) developed by Rokhlin (*J. Comp. Phys.* 60, 187-207, 1985; *Yale Univ. Research Report* YALEU/DCS/RR-440, 1985) to reduce the computation time required for solving scattering problems modeled in the form of integral equations. The method was developed first for acoustics. We have modified and adapted it to electromagnetic scattering problems in two dimensions. Briefly, the method relies on dividing the boundary of the scatterer into segments, each containing several nodes, and performing exact computations only for the interactions of neighboring segments. When segments are sufficiently far from each other, approximations are used, expanding "source segments" into multipoles.

Our present computer code reduces the operation count for solving the Magnetic-Field Integral Equation (MFIE) from $O(n^3)$ to $O(n^{4/3})$ per conjugate-gradient iteration. It has achieved dramatic results in solving scattering from electrically large objects. The implementation is presently limited to computing scattering from closed conducting objects in two dimensions. Impedance boundary conditions which have been developed and tested in a simpler computer code will soon be merged into the FMM implementation to allow computation of conductors coated with dielectrics. In addition, work is proceeding on a three-dimensional implementation.

THE FAST MULTIPOLE METHOD (FMM) FOR ELECTROMAGNETIC SCATTERING PROBLEMS[1]

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The Fast Multipole Method (FMM) was developed by Rokhlin (*J. Comp. Phys.* 60, 187-207, 1985; *Yale Univ. Research Report* YALEU/DCS/RR-440, 1985) to reduce the computation time required for solving scattering problems modeled in the form of integral equations. When such integral equations are discretized, a full dense linear system of size $n \times n$ results. If direct methods such as Gaussian Elimination are used to solve the system, $O(n^3)$ arithmetic operations are needed. If indirect methods such as Generalized Conjugate Residual (GCR) are employed, $O(n^2)$ arithmetic operations are needed for each iteration. Rokhlin has shown that the operation count may be reduced to $O(n \log n)$ per iteration, by using the structure of the Green's function and the fact that only an approximation to the matrix product is needed. The algorithm is analogous to the evaluation of the field created by a charge and dipole distribution (hence the name "Fast Multipole Method"). The actual implementation thus far uses a simpler algorithm which achieves $O(n^{4/3})$, although $O(n \log n)$ is achievable theoretically. For moderately large problems ($n \approx 20000$), the two algorithms yield similar CPU times, although for very large problems, implementing the full $O(n \log n)$ algorithm is desirable.

Originally, Rokhlin's work was for acoustic scattering problems where a fluid scatterer is embedded in a two-dimensional fluid space. This mathematical problem is formulated as a coupled system of integral equations between the interior and exterior problems. Rokhlin extended these ideas also to two and three-dimensional potential theory (Laplace's equation) where even faster computations are possible ($O(n)$ operations).

We review some of these results, and extend their application to the area of electromagnetic scattering. Initial results with the $O(n^{4/3})$ algorithm have been obtained for TM scattering from arbitrarily shaped two-dimensional closed conductors. The FMM, combined with fourth-order convergent quadrature formulas in Nystrom's method (Murphy, Rokhlin and Vassiliou, *Electronics Letters* 25, 543-644, 1989) represents a very fast and accurate method to solve electromagnetic scattering problems from electrically large objects.

SOLVING THE RESONANCE PROBLEM IN ELECTROMAGNETIC SCATTERING COMPUTATION[1]

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The "resonance problem" considered here is that at certain values of the wavenumber k, the second-kind integral equation for solving scattering problems can become extremely ill-conditioned. This affects both the accuracy and speed of numerical solutions. The convergence of iterative methods in particular is adversely affected by large condition numbers. We consider the second-kind integral equation, derived using double-layer potentials, for TM scattering from a conductor. We solve an exterior Dirichlet problem. It has nonunique solutions for values of k at which the corresponding homogeneous interior Neumann problem has a nontrivial solution. These values of k are the resonant wavenumbers.

Numerically, we discretize the integral directly using quadrature formulas (Nystrom's method). We employ fourth-order convergent quadrature formulas which handle the logarithmic singularities in the problem (Murphy, Rokhlin, and Vassiliou, *Electron Lett.* 25, 643, 1989). We tested the solution procedure at resonant k's for circular and elliptical scatterers (roots of derivatives, respectively, of Bessel functions and modified Mathieu functions). We found very large condition numbers for the discrete matrices (up to $O(10^7)$), generally leading to poor solutions.

We apply two approaches to alleviate the resonance problem. The first is to use a different integral equation, based on both single and double-layer potentials, analogous to the combined-field equation (CFIE). This leads to low condition numbers, and good solutions, at resonant k.. The second method is to use the original second-kind integral equation, introduce a small imaginary part in k, and extrapolate back to the real axis. Solutions obtained by the two methods are in excellent agreement.

By solving the resonance problem, we ensure that fast and accurate solutions are obtainable at any arbitrary wavenumber.

THE FAST MULTIPOLE METHOD (FMM) FOR ELECTROMAGNETIC SCATTERING PROBLEMS

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ABSTRACT

The Fast Multipole Method (FMM) was developed by Rokhlin to solve acoustic scattering problems very efficiently. We have modified and adapted it to the second-kind-integral-equation formulation of electromagnetic scattering problems in two dimensions. The present implementation treats the Dirichlet (TM) problem for two-dimensional closed conducting objects of arbitrary geometry. Consider a scatterer with n nodes on its boundary. Divide the boundary into p segments, each containing n/p nodes. Instead of calculating n^2 interactions among n current sources on the boundary, consider each segment to be a cluster of n/p sources. For segments that are close together, the exact interactions must be calculated. For segments sufficiently far apart, however, we may combine the sources in each segment, approximating their radiation fields by the first N multipoles. We describe each segment via an equivalent multipole located at the segment's center. We can calculate the contribution of each such equivalent expansion to the field at the center of any sufficiently separated receiver segment, and in that receiver segment we can use a partial wave expansion to obtain the field at all the individual n/p nodes. The radiation field at any particular node on the scatterer boundary is the sum of contributions of N multipole expansions of each of the far-away segments, and the direct contribution of very close segments. The method reduces the operation count for solving the Magnetic-Field Integral Equation (MFIE) from $O(n^3)$ for Gaussian elimination to $O(n^{4/3})$ per conjugate-gradient iteration. We also present a simple technique for accelerating convergence of the iterative method: "complexifying" k, the wave number. This has the effect of bounding the condition number of the discrete system; consequently, the operation count of the *entire* FMM (all iterations) becomes $O(n^{4/3})$. We present computational results for moderate values of ka, where a is the characteristic size of the scatterer.

I. INTRODUCTION

Electromagnetic scattering comprises an important class of problems in physics and engineering. It is desirable to have efficient techniques to compute scattering accurately. In this paper we present the first application of Rokhlin's Fast Multipole Method (FMM) to the problem of electromagnetic scattering from two-dimensional closed conducting objects of arbitrary geometry.

In many two-dimensional scattering problems, it is customary to reduce the scalar Helmholtz equation to a second-kind integral equation. The resulting integral equation can generally be treated using various numerical techniques[1,2]. One of the standard methods for this numerical treatment of scattering problems is to discretize the second kind integral equation using an appropriate quadrature formula (Nyström's method)[1,3]. Such discretization leads to systems of linear algebraic equations which may be solved by Gaussian elimination or iterative methods such as conjugate gradient or generalized conjugate residual (GCR)[4]. These iterative methods require the full dense matrix to operate on a sequence of recursively generated vectors. Consequently, the operation count is $O(n^2)$ where n is the dimension of the matrix. There have been many successful efforts to reduce the operation count and storage requirements and to introduce "fast" algorithms[5,6,7,8]. The FMM is a particularly promising method among these. In the FMM, the operation count for each iteration is reduced to $O(n^{4/3})$, which is significantly smaller than $O(n^2)$, especially for large $n(> 10,000)$. This algorithm can be further improved to one that has an operation count of $O(n \log n)$ per iteration. However, we have not yet implemented the fastest algorithm. When these algorithms are combined with a GCR or conjugate gradient algorithm, the resulting procedure only requires a small number of iterations to converge to a solution to the scattering problem. This is the case even at resonance frequencies, if the method of "complexification" is applied (see discussion below).

The purpose of the present paper is to explain this algorithm in simple terms and to explore its application to electromagnetic scattering. Rokhlin's FMM, which was first

employed for Poisson's equation[5] and acoustic scattering[6] in two dimensions has been extended to three dimension for Poisson's equation[9] and is currently being extended for Helmholtz's equation.

II. PROBLEM STATEMENT

Consider a two-dimensional conducting body whose axis is aligned with the z coordinate axis. A monochromatic electromagnetic wave incident on this structure with an electric field vector parallel to the axis of the body is referred to as the transverse-magnetic (TM) case. The incident and scattered fields both satisfy the following Helmholtz equation

$$\Delta E_z + k^2 E_z = 0. \tag{1}$$

The boundary condition for equation (1) is that the total **E** field vanish on the surface of this conductor, i.e.,

$$E_z^{tot} = 0 \quad \text{on} \quad C \tag{2}$$

or more explicitly

$$E_z^{inc} + E_z^{scat} = 0 \quad \text{on} \quad C. \tag{3}$$

The above Helmholtz equation can be reduced to the following second-kind integral equation

$$\Psi(\mathbf{r}) + 2 \int_C \frac{\partial G(k|\mathbf{r} - \mathbf{r}'|)}{\partial \mathrm{n}(\mathbf{r}')} \Psi(\mathbf{r}') \, \mathrm{dl}' = -2\mathrm{E}_z^{inc}(\mathbf{r}) \tag{4a}$$

$$E_z^{scat}(\mathbf{r}) = \int_C \frac{\partial G(k|\mathbf{r} - \mathbf{r}'|)}{\partial \mathrm{n}(\mathbf{r}')} \Psi(\mathbf{r}') \, \mathrm{dl}' \tag{4b}$$

where **r** and **r**' are both on the boundary C, and G is the free-space Green's function in two dimensions, i.e.,

$$G(k|\mathbf{r} - \mathbf{r}'|) = \mathrm{i}H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)/4 \tag{5}$$

with $H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)$ being the Hankel fuction of the first kind of order zero. See Ref.[10] for a derivation of equation (4). Equation (4), the TM case, is often referred to as the Dirichlet problem in the mathematical literature[1,10]. If we discretize the boundary into n

points, then the above integral equation (4a) is converted to the following linear system (via Nyström's method[1]):

$$\Psi(\mathbf{r_i}) + 2\sum_{j=1}^{n} A_{ij}\Psi(\mathbf{r_j}) = -2E_z^{inc}(\mathbf{r_i}) \tag{6}$$

where the matrix $A = (A_{ij})$ is $n \times n$ and the vectors $(\Psi(\mathbf{r_i}))$ and $(E_z^{inc}(\mathbf{r_i}))$ are column vectors having n rows. Applying normal matrix multiplication, $A\vec{\Psi}$ requires $O(n^2)$ operations. The FMM algorithm reduces this to $O(n^{4/3})$ or ultimately to $O(n\log n)$.

III. RAPID SOLUTION OF INTEGRAL EQUATIONS

The detailed mathematics behind the FMM is presented in [5] and [6]. The development is quite complex. Below, we offer a simplified version, with more physical intuition relevant to electromagnetic scattering.

Consider n nodes on the boundary of the scatterer. Divide the boundary into p equal segments, where $2 \leq p < n$. In each segment, there are n/p nodes. If the length of the boundary is L, each segment has length L/p. The center of each segment is located at $z_i (i = 1, 2, \ldots, p)$. In scattering problems, each node can be treated conceptually as if it were a source of radiation.

If we have sources within a finite region of space, the radiation emitted from these sources in the far zone can be approximated using a collection of multipoles located at the center of the region[5,6,11]. The multipole approximation converges rapidly outside any circle D containing all sources and separated from D by at least one wavelength. In fact, once a sufficiently large number of multipoles is included, the accuracy of the approximation increases superalgebraically (faster than any negative power of N)[6].

Consider each segment on the boundary as a cluster of n/p sources. The sources in each segment are treated as a single aggregate source, and the radiation field of that equivalent source is approximated using the first N multipoles located at the center of the segment. For each pair of sufficiently-separated segments, the radiation of the N multipoles of one segment can be represented as an analytical partial field expansion around the center of the other segment. Then from this information, the field at the other nodes of that segment can be evaluated using equation (16). For nearby segments, the direct contribution must be calculated to evaluate the radiation field. The radiation field at any particular node on the boundary is the sum of the contributions of N multipoles of each of the far-away segments and the direct contribution of the nearby segments. Reference [6] considers the precise mathematical description of the process.

To illustrate the above verbal description mathematically, let us consider the field of a two-dimensional magnetic surface current distributed over a two-dimensional body. Since we have an exterior Dirichlet problem, a double-layer potential is needed. For the two-dimensional TM problem, a magnetic current density at any point of the boundary is a vector in the x-y plane tangent to the body. The electric field of such a current distribution is given by

$$\mathbf{E}^{scat}(\vec{\rho}) = \nabla \times \int_C G(k|\vec{\rho} - \vec{\rho}'|) \mathbf{K}(\vec{\rho}') \, dl' \tag{7}$$

where $\mathbf{K}(\vec{\rho}')$ is the magnetic surface current density, $\vec{\rho} = (\rho, \theta)$, and C is the contour of integration. In two dimensions equation(7) can be written as

$$E_z^{scat}(\vec{\rho}) = \int_C \frac{\partial G(k|\vec{\rho} - \vec{\rho}'|)}{\partial n(\vec{\rho}')} |\mathbf{K}(\vec{\rho}')| \, dl' \tag{8}$$

which is identical with equation(4b) if $|\mathbf{K}(\vec{\rho}')|$ is taken to be $\Psi(\vec{\rho}')$. Substituting equation(5) into (8), we get

$$E_z^{scat}(\vec{\rho}) = (i/4) \int_C \frac{\partial H_0^{(1)}(k|\vec{\rho} - \vec{\rho}'|)}{\partial n(\vec{\rho}')} |\mathbf{K}(\vec{\rho}')| \, dl' \tag{9}$$

The Hankel fuction $H_0^{(1)}(k|\vec{\rho} - \vec{\rho}'|)$ can be expanded in terms of higher order Hankel functions:

$$H_0^{(1)}(k|\vec{\rho} - \vec{\rho}'|) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(k\rho) J_m(k\rho') \exp(im(\theta - \theta')) \tag{10}$$

Substituting equation (10) into equation (9) yields

$$E_z^{scat}(\rho, \theta) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(k\rho) \exp(im\theta) \int_C (i/4) \frac{\partial J_m(k\rho') \exp(-im\theta')}{\partial n(\rho', \theta')} |\mathbf{K}(\rho', \theta')| \, dl' \tag{11}$$

This can be regarded as the multipole expansion of the source $\mathbf{K}(\rho', \theta')$. For a discretized source at n points located at $x_j' = (\rho_j', \theta_j')$ $(j = 1, 2, \ldots, n)$ over a segment of the boundary, equation(11) reduces to

$$E_z^{scat}(\rho, \theta) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{n} (i/4) \frac{\partial J_m(k\rho_j') \exp(-im\theta_j')}{\partial n(x_j')} H_m^{(1)}(k\rho) \exp(im\theta) |\mathbf{K}(x_j')| \Delta l_j' \tag{12}$$

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where $\Delta l'_j$ is the discretized element of arc length containing the source $|K(x'_j)|$. For a given accuracy, we can truncate the infinite sum in equation(12) at N, and thus calculate the first N multipoles of the source.

Historically, the FMM algorithm was first applied to Poisson's equation for n point charge lines at locations $x_i (i = 1, 2, \ldots, n)$ with strengths κ_i. This is mathematically equivalent to solving the equation

$$\Delta \phi = \sum_{i=1}^{n} \delta(x - x_i)\kappa_i \tag{13}$$

where $\delta(x)$ is the Dirac delta function and x and x_i are two-dimensional points. The solution to equation (13) is

$$\phi(x) = \sum_{i=1}^{n} \kappa_i \log(|x - x_i|)/(2\pi) \tag{14}$$

If we evaluate equation (14) at each point $x_i (i = 1, 2, \ldots, n)$, then this computation requires $O(n^2)$ operations. However, if large numbers of particles are combined into single computational elements, then this operation count can be reduced if an approximate answer (to a specified accuracy) is desired. When a cluster of particles is "far away" from a particular point, then the potential of the cluster is approximated by the potential induced by a single computational element located inside the cluster[12]. In the FMM algorithm the computational element is a Laurent expansion centered at a circle containing the cluster of particles. Given a cluster of charges located at points $z_i (i = 1, 2, \ldots, n_c)$, the expansion is given by

$$\phi(z) = Re(\sum_{i=1}^{n_c} \log(z - z_i)) \approx Re(a_0 \log(z - z_0) + \sum_{k=1}^{p} a_k/(z - z_0)^k) \tag{15}$$

Here p is the order of the multipole and the $a_k s$ are coefficients chosen so that the truncated series is an accurate approximation of the potential. For equation (15), the computational

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effort is only $O(p)$ operations and is much less than $O(n_c)$ for the direct approach. The region must now be organized into well separated points and very-near points. For near points, the direct evaluation of (14) is used. See Refs.[5,6,12] for details of decomposing the regions into boxes of different sizes. When applying the FMM to Helmholtz's equation instead of Poisson's equation the expansion (15) is replaced by the standard Hankel function expansion, which is then truncated to obtain a given accuracy requirement. That is,

$$
E_z^{scat}(\rho, \theta) = \begin{cases} \sum_{m=-\infty}^{+\infty} \alpha_m H_m^{(1)}(k\rho) \exp(im\theta), & \text{if } \rho > a \quad (16\text{a}) \\ \sum_{m=-\infty}^{+\infty} \beta_m J_m(k\rho) \exp(im\theta), & \text{if } \rho < a \quad (16\text{b}) \end{cases}
$$

depending on whether the calculations are to be done outside or inside the circle of radius a[6]. Obviously, any prescribed accuracy in the series can be guaranteed by taking more terms in the expansion at the expense of more CPU time.

IV. OPERATION COUNT

We illustrate the computational work required for the FMM algorithm using a simple example. Consider p segments around the boundary of the scatterer. Assume that for each segment, the two adjacent segments are "nearby," and thus require direct calculation of the radiation field. All other segments are considered "far away" for this example and the multipole expansion will be used. The following steps are taken:

Step 1: Find the first N multipoles of sources in each segment. Since to evaluate each multipole, all the sources are involved, the first N multipoles of sources in each segment require Nn/p operations. For p segments, we have

$$N(n/p)p = Nn \tag{17}$$

As we mentioned earlier, the number of multipoles used is a function of the accuracy needed in the calculation. This number is typically proportional to kd where $k = 2\pi/\lambda$ and d is the length of the source region. Thus, $N \approx kL/p$. Substituting this value of N in o equation (17) gives

$$Nn \approx k(L/p)n \tag{18}$$

as the operation count for this step.

Step 2: For each pair of "far away" segments, evaluate the radiation fields of N multipoles of one segment at the center of the other. This is an $O(N)$ operation for each pair. The number of pairs is almost p^2. In actuality, for each segment, the number of far away segments is $p - 3$. Therefore, the number of far away pairs is $p(p - 3)$, which for large p is almost p^2. Therefore, the operation count for this step is

$$Np^2 \approx k(L/p)p^2 \approx kLp \tag{19}$$

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Step 3: In this step, for each segment, add the contribution of N multipoles of any one of the far away segments evaluated at the center of the chosen segment. For any chosen segment, the number of far away segments is $p - 3$, or approximately p, for large p. Thus, this step requires

$$Np \approx k(L/p)p \approx kL \tag{20}$$

operations.

Step 4: Here, the radiation field is known at the center of each segment. The field at the other nodes in the segment can be evaluated using a partial field expansion. For each neighboring node, this is an $O(N)$ operation. Thus, for $n/p - 1$ nodes in each segment, the number of operations is

$$N(n/p - 1) \approx Nn/p \tag{21}$$

For $p - 3$ segments, we have

$$(Nn/p)(p - 3) \approx Nn \approx k(L/p)n \tag{22}$$

Step 5: Finally for the nearby (neighboring) segments, the direct contributions must be evaluated. For q sources, the number of operations is q^2. Here, in each segment there are n/p sources. For a particular segment in question and its two near-neighbors, the work is

$$(3n/p)^2 \approx n^2/p^2 \tag{23}$$

For p segments, the count is

$$(n^2/p^2)p \approx n^2/p \tag{24}$$

Adding the above five steps and optimizing with respect to p the resulting expression for total operation count (as in Ref.[6]), the optimal count T is

$$T = O(n^{3/2}) \tag{25}$$

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The operation count can be further reduced by applying the above procedure recursively, with each of the nearby segments subdivided with appropriately chosen p'. The new estimate so obtained shows that the FMM algorithm is $O(n^{4/3})$. Our current FMM code implements the $O(n^{4/3})$ algorithm. In Ref.[5] the above subdivision is used recursively to obtain an $O(n)$ algorithm when applied to Poisson's equation (13). By reproducing the construction of Section VII in Ref.[5] for Helmholtz's equation an $O(n\log(n))$ algorithm is theoretically achievable.

The essential feature of the FMM algorithm is that it performs the matrix-vector operation

$$A\vec{\Psi}^{(m)} \quad (m = 0, 1, 2, \ldots) \tag{26}$$

in $O(n^{4/3})$ operations. Here the superscript m is the iteration counter. *Note that A is never computed, so that the algorithm only requires vector storage* ! The storage requirement is $O(n^{4/3})$, like the operation count[6]. The actual solution to the system (6) is obtained using a conjugate gradient or GCR[4] algorithm in which the most computationally extensive step is that of forming the vector expression (26) by the FMM technique. GCR must be used on the matrix $I + 2A$ or conjugate gradient on the normal equation because $I + 2A$ is nonsymmetric.

The FMM algorithm is in no way restricted to the TM case (4). In fact, an almost identical algorithm can be applied to the TE case (Neumann problem in the mathematical literature[1,10]) or to the combined field integral equation (CFIE), which does not have a resonance problem (see, e.g., [3]). Rather than solve the CFIE, we use a "lazy man's" technique that employs the method of "complexification" on equation (4) directly. This "trick", the reader will soon see, works remarkably well and allows us to avoid dealing with the more complicated CFIE. Although "complexification" has been used in electromagnetics to satisfy the radiation condition, here it is used to reduce the condition number of the matrix A and thereby reduce the number of iterations for convergence of the conjugate gradient algorithm.

V. COMPUTATIONAL RESULTS

In Fig. 1a, we show the comparison of the FMM algorithm with the analytic solution for the scattering of a plane wave incident on a circular cylinder for the case $ka = 80$. We present our computational results in a form often used in electrical engineering: that is, as plots of differential scattering cross section or Radar Cross Section (RCS) versus observation angle, for a given angle of incidence. The RCS is related to the magnitude of the electric field in the far zone. In two dimensions, "RCS" is something of a misnomer: the more proper term is "scattering width," but the label RCS is commonly applied anyway. The two-dimensional definition is[13]

$$\text{RCS} = \sigma = 2\pi \lim_{r \to \infty} r \frac{|E_z^{scat}|^2}{|E_z^{inc}|^2} \tag{27}$$

where E_z^{scat} is the scattered field and E_z^{inc} is the incident field. The quantity we plot is the ratio of σ to the wavelength of the incident wave, expressed in decibels (dB).

Fig. 1a shows the agreement of the RCS between the two solutions is exact for the observation angle ϕ between $0°$ and $40°$, where most of the rapid changes occur. In Fig. 1b, we plot the entire FMM solution from $0° \leq \phi \leq 180°$, which also agrees exactly with the series solution, but we leave out the latter to avoid too much congestion in the figure. In this example, we have used 10 points per wavelength on the scatterer boundary, so $n = 800$. In Table 1, we list some results from the FMM code for various values of k and n for scattering from two-dimensional conducting circular and elliptical cylinders (a=semi-major axis, b=semi-minor axis). The expansions (16) were truncated to give an accuracy of 10^{-4} and the convergence tolerance for the conjugate gradient algorithm was set to 10^{-3}. In the table, N_{iter} denotes the actual number of iterations in the conjugate gradient algorithm to achieve a relative error tolerance of 10^{-3}, ERR is the actual relative error tolerance between the last two iterations, and CPU is the average CPU time in seconds on a VAX 6410 computer for one iteration of the conjugate gradient FMM algorithm using

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double precision arithmetic. In most cases, for "complexified" k-values an eror tolerance of 10^{-4} instead of 10^{-3} would only add one or two more iterations to N_{iter}. However, for "non-complexified" k-values, an error tolerance of 10^{-4} would require many more iterations because of the large condition number involved in those cases. Since the matrix A is never explicitly calculated in the FMM algorithm, the total CPU time for one incident angle is roughly equal to the product of ITER and CPU. To calculate the number of points N_λ per incident wavelength in Table 1, we have used an approximate formula for the perimeter of an ellipse, obtaining

$$N_\lambda \approx \frac{n}{k\sqrt{(a^2 + b^2)/2}} \tag{28}$$

Note that for some of the cases in Table 1, we have used values of k, the wave number, that have imaginary parts. This was done to test the method of "complexification". Equation (4) has resonances[10] only at real distinct values of k and as k gets larger, root clusters become more dense; at these resonance frequencies, the condition number of the matrix A becomes large and iterative methods require many more iterations to converge to a given tolerance. We have shown in Ref.[3] that by moving k slightly into the complex plane("complexifying" k) the condition number can be reduced by four or five orders of magnitude and consequently, the convergence rate of most iterative methods can be greatly improved. This is clearly demonstrated in Table 1. Furthermore, by using two complex values of k (say $k = 100 + .1i$ and $k = 100 + .2i$) we have shown in Ref.[3] that the extrapolated values of the RCS to the real axis are accurate to a value in dB roughly equal to the amount of movement into the complex plane (in this case .1dB). Of course, parabolic extrapolation would be even more accurate, but would require computations for three complex values of k. However, this is not a severe problem when using iterative methods because the iteration scheme for the second value of k can be started with the final solution from the first value of k and so on. Thus two solutions for two close values of k require only about 1.1 times as much CPU time as does one case. Complexification is hence a viable and realtively cheap method for accelerating convergence in such problems.

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Parabolic extrapolation is done in the following way. Suppose the imaginary parts of three "complexified" k-values are k_0, k_1, and k_2 and the corresponding RCS values for one observation angle are σ_0, σ_1, and σ_2, respectively. We compute the Lagrange interpolation polynomial through the points (k_0, σ_0), (k_1, σ_1), and (k_2, σ_2) and set k equal to zero in this quadratic polynomial in k(i.e., we extrapolate to the real axis) obtaining

$$\sigma(0) = p_0 \sigma_0 + p_1 \sigma_1 + p_2 \sigma_2 \tag{29}$$

where $p_0 = k_1 k_2 / [(k_0 - k_1)(k_0 - k_2)]$, $p_1 = k_0 k_2 / [(k_1 - k_0)(k_1 - k_2)]$, and $p_2 = k_0 k_1 / [(k_2 - k_0)(k_2 - k_1)]$. Parabolic extrapolation is second-order accurate provided the function $\sigma(k)$ is "sufficiently smooth" (has three continuous derivatives). This means that for uniform spacing ($\Delta k = k_2 - k_1 = k_1 - k_0$) the error in σ due to the extrapolation is $O((\Delta k)^2)$. Linear extrapolation is only $O(\Delta k)$ if σ has two continuous derivatives in k. In practice, we have found that the accuracy of the extrapolation procedure is even better than these estimates would suggest.

In Fig. 2, we show some RCS plots for scattering from a two-dimensional conducting circular cylinder for linear extrapolation from "complexified" k values. To remove the effect of the "complexification" in computing the RCS in the far field, we must rescale $E_z^{scat}(r_{FF})$ where r_{FF} is the value of r in the far field, typically about 10,000. If $k = k_{re} + ik_{im}$ where k_{im} is the imaginary term added to k to accelerate convergence of the iterative procedure, then we rescale $E_z^{scat}(r_{FF})$ by multiplying it by $\exp(k_{im} r_{FF})$ from the asymptotic expansion of $H_\nu^{(1)}(k_{im} r_{FF})$. The RCS is then computed using equation (27) and the rescaled values of $E_z^{scat}(r_{FF})$. Extrapolation of the RCS to the real axis is also done using the rescaled values. In Fig. 2, the linearly extrapolated values (from $k = 80 + 0.5i$ and $k = 80 + 1.0i$) of the RCS give exact comparisons with the unextrapolated ones (for $k = 80$). The parabolically extrapolated values give the same excellent agreement. The number of iterations required to converge to an error tolerance of 10^{-3} for these three cases

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are 79, 26, and 16, respectively, and again for faster convergence it pays to "complexify". Note also that large condition numbers not only imply slow convergence but also poor accuracy, giving another reason to "complexify". Of course, one could always solve the CFIE rather than equation (4) and avoid the problem of large condition numbers due to resonance. Unfortunately, solving the CFIE for TE polarization requires dealing with the second derivative of the free space Green's fuction, which has a nasty singularity. "Complexification" is easier!

In Fig. 3a and Fig. 3b, we plot the RCS for scattering from a conducting elliptical cylinder using linear extrapolation ($a = 2, b = 1, k = 100 + 0.5i, k = 100 + 1.0i$) with an incident plane wave at 90°. As can be seen in Table 1, $n = 1600$ for these cases. The agreement between extrapolated values and unextrapolated values of the RCS is exact. Again we merely show the extrapolated values in the figures to avoid too much congestion. Parabolic extrapolation using $k = 100 + 0.3i, 100 + 0.6i, 100 + 0.9i$ gives the same excellent results.

As we have stated, the FMM algorithm of this paper is $O(n^{4/3})$ per conjugate gradient iteration. This means that the CPU time is proportional to $n^{4/3}$, i.e.,

$$T = \alpha_{VAX}\, n^{4/3} \tag{30}$$

where we have emphasized that the proportionality constant α is a fuction of the computer being used (in this case a VAX 6410 in double precision). Using equation (30) and some of the data in Table 1, we can obtain a rough estimate for the value of α_{VAX}. In this case

$$\alpha_{VAX} \approx .003 \tag{31}$$

Suppose now we would like to estimate the CPU time required on our VAX to compute the electromagnetic scattering problem for an example having 10.000 unknowns. We

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assume that without too much "complexification" convergence occurs in 10 iterations or less. Thus, for this example

$$T^{tot} \approx \alpha_{VAX}(10)n^{4/3} = .03(10,000)^{4/3} = 1.8 \text{ hours} \tag{32}$$

Although this last CPU time may seem large, we are only using a VAX in double precision, and a vectorized version of this code on a CRAY computer would do the same calculation in minutes. Furthermore, ordinary Gaussian elimination for a matrix of order 10,000 would require about $(10,000)^3/3$ operations and would use considerably more CPU time than that in equation (32).

Finally, the most important statement about the FMM and our approach to "complexification" and extrapolation of the RCS values for complex ks to the real axis is that for complex k ($Re(k) > 0$ and $Im(k) > 0$), the solution to equation (4), a second kind integral equation, is unique for sufficiently smooth scatterers and consequently, the condition number of A is bounded[14] and independent of n. This means, when we use our extrapolation procedure (either linear or parabolic), N_{iter} is small (say, $N_{iter} \leq 10$) and *independent of n* for error tolerances of 10^{-3}. Of course, if the accuracy requirements are increased, so will N_{iter}, but in either case the FMM is *globally $O(n^{4/3})$* and we can write

$$T^{tot} = N_{iter} \, \alpha_{computer} \, n^{4/3} \tag{33}$$

where N_{iter} is only a function of ϵ, the error tolerance, and *not a function of n* , and $\alpha_{computer}$ is a function of the computer speed for arithmetic operations and, also *independent of n* . For further proof, we present more evidence in Table 2, where we have set the "complexification" to $5i$ in all cases and maintained approximately 10 points per wavelength for various geometries. Note the small number of iterations to convergence, indicating a bounded condition number independent of n. Finally, in Table 3, we list some

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preliminary results of the FMM code on the CRAY 2. This code is not yet fully vectorized, so we don't give CPU times.

Although we have only considered circular and elliptical cylinders for the test cases in this paper, the code can handle any closed two-dimensional metal scatterer having a unique outward normal at every point on C. Therefore, at present, we cannot deal with the trailing edge of an airfoil that comes to a point (non-existence of $\partial G/\partial n$ at the trailing edge); currently, we must round this region off. However, we are working on replacing the current at this singular point with its asymptotic expansion, allowing us then to deal with such points. We are also considering thin coatings about metal scatterers using higher order impedance boundary conditions, similar to our treatment in Ref.[15].

Our closing observation is that in computational electromagnetic scattering, almost anyone can obtain reasonable results using a bad algorithm if ka is small. It is much more difficult to get good results for large values of ka, and it is doubly difficult to do so efficiently. We hope that by choosing moderate values of ka, we have demonstrated the robustness of the FMM algorithm for effectively solving electromagnetic scattering problems.

ACKNOWLEDGEMENTS

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Table 1
Fast Multipole Results for Scattering from Two-Dimensional Conducting
Circular and Elliptical Cylinders
(DEC VAX 6410 Computer, Double Precision)

kb	a	b	n	n_λ	N_{iter}	CPU	Err
50	1	1	500	10	66	11.99	.441 (-3)
80	1	1	800	10	79	22.35	.856 (-3)
80 + 1.6i	1	1	800	10	13	22.43	.200 (-3)
80 + 2.4i	1	1	800	10	9	21.73	.655 (-3)
80 + 3.2i	1	1	800	10	8	21.89	.198 (-3)
80 + 4.8i	1	1	800	10	6	21.27	.750 (-3)
80 + 6.4i	1	1	800	10	6	21.22	.223 (-3)
150 + 6.4i	1	1	1500	10	6	51.94	.633 (-3)
100	2	1	1600	10.12	184	34.89	.992 (-3)
100 + 2i	2	1	1600	10.12	11	35.57	.293 (-3)
100 + 3i	2	1	1600	10.12	9	36.53	.478 (-3)
100 + 4i	2	1	1600	10.12	8	35.84	.494 (-3)
100 + 6i	2	1	1600	10.12	6	35.35	.998 (-3)
50	3	1	1200	10.73	95	32.61	.950 (-3)
50 + 2i	3	1	1200	10.73	10	33.98	.863 (-3)
100 + 2i	3	1	2237	10	12	86.93	.639 (-3)

k = Wavenumber
a = Semimajor axis
b = Semiminor axis
n = Total number of unknowns (sample points on scatterer boundary)
n_λ = Number of unknowns per wavelength of incident radiation
N_{iter} = Number of iterations to convergence
CPU = CPU time per iteration, in seconds
Err = Relative Error
NOTATION: .441 (-3) means 0.441×10^{-3}

Table 2
Fast Multipole Method: CPU Time for Fixed "Complexification"
(Scattering from Two-Dimensional Conducting Circular and Elliptical Cylinders)
(DEC VAX 6410 Computer, Double Precision)

kb	a	b	n	N_{iter}	CPU	Err
50 + 5i	1	1	500	6	11.97	.251 (-3)
100 + 5i	1	1	1000	6	22.46	.973 (-3)
150 + 5i	1	1	1500	7	42.28	.313 (-3)
200 + 5i	1	1	2000	7	51.90	.686 (-3)
50 + 5i	2	1	791	7	22.22	.998 (-3)
150 + 5i	2	1	2372	8	78.59	.408 (-3)
100 + 5i	3	1	2237	9	86.28	.999 (-3)

k = Wavenumber
a = Semimajor axis
b = Semiminor axis
n = Total number of unknowns (sample points on scatterer boundary)
N_{iter} = Number of iterations to convergence
CPU = CPU time per iteration, in seconds
Err = Relative Error

NOTATION: .251 (-3) means 0.251×10^{-3}

Table 3

Fast Multipole Method: CRAY Results with Fixed "Complexification"

(Scattering from Two-Dimensional Conducting Circular and Elliptical Cylinders)

(CRAY-2 Computer, Single Precision)

kb	a	b	n	N_{iter}	Err
200 + 5i	1	1	2000	7	.686 (-3)
300 + 5i	1	1	3000	8	.331 (-3)
400 + 5i	1	1	4000	8	.803 (-3)
500 + 5i	1	1	5000	9	.327 (-3)
100 + 5i	2	1	1582	7	.997 (-3)
200 + 5i	2	1	3163	6	.995 (-3)
300 + 5i	2	1	4744	6	.991 (-3)
100 + 5i	3	1	2237	8	.995 (-3)
200 + 5i	3	1	4473	7	.997 (-3)
100 + 5i	4	1	2916	9	.998 (-3)
50 + 5i	10	1	3554	8	.996 (-3)

k = Wavenumber
a = Semimajor axis
b = Semiminor axis
n = Total number of unknowns (sample points on scatterer boundary)
N_{iter} = Number of iterations to convergence
Err = Relative Error

NOTATION: .686 (-3) means 0.686×10^{-3}

FIGURE CAPTIONS

Fig. 1a. Radar cross section (RCS) for a circular cylinder of unit radius. Incident field is a plane wave with wave number $k = 80$. Solid line shows solution from the FMM code, while dotted line is that from the series solution.

Fig. 1b. RCS of previous case from the FMM code for more observation angles.

Fig. 2. RCS for a circular cylinder of unit radius. Incident fields are plane waves with with wave numbers $k = 80$, $k = 80 + .5i$, and $k = 80 + 1.0i$. Solid line shows solution from the FMM code with no extrapolation ($k = 80$). Dotted line shows solution of the linearly extrapolated RCS from $k = 80 + .5i$ and $k = 80 + 1.0i$.

Fig. 3a. RCS results from the FMM code for an elliptical cylinder with semimajor axis $a = 2$ and semiminor axis $b = 1$ linearly extrapolated from incident field plane waves at 90° with wave numbers $k = 100 + .5i$ and $k = 100 + 1.0i$.

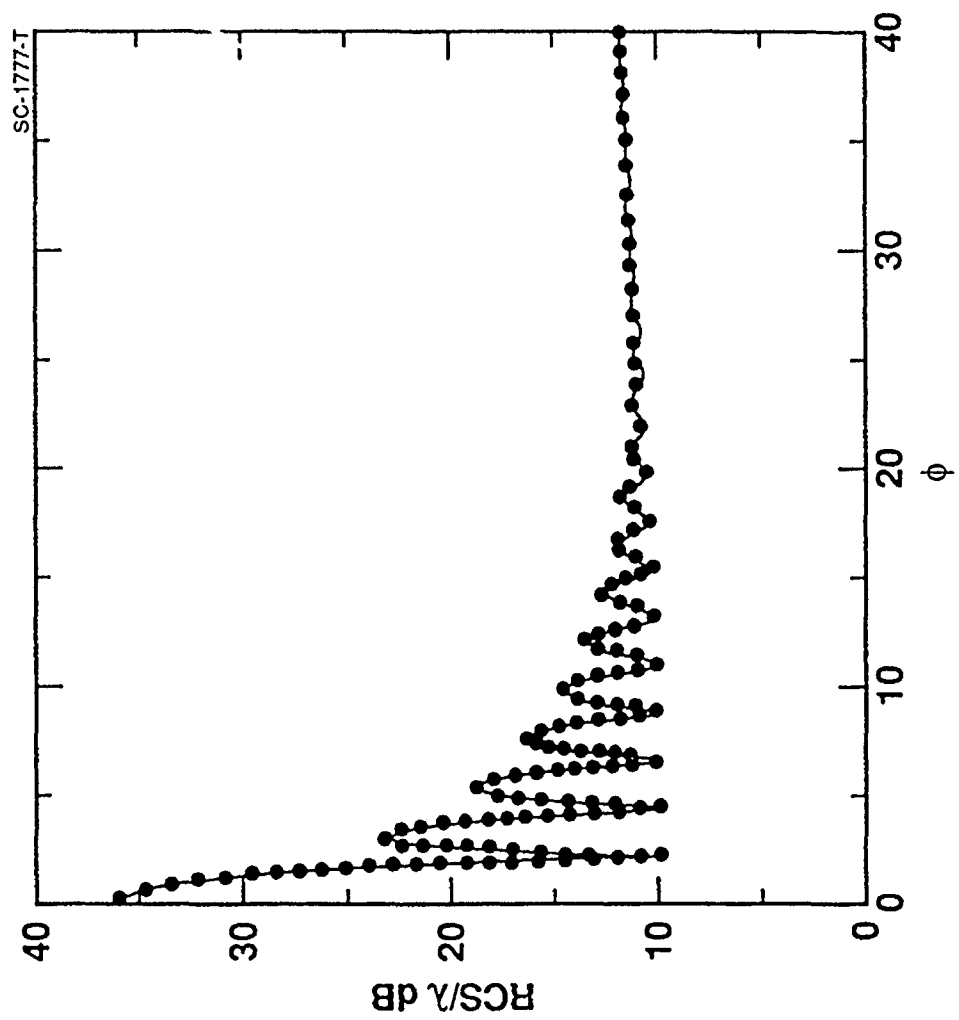Fig. 3b. RCS of previous elliptical case for more observation angles.

SC-1778-T

RCS/λ dB

φ

Fig. 4.

E

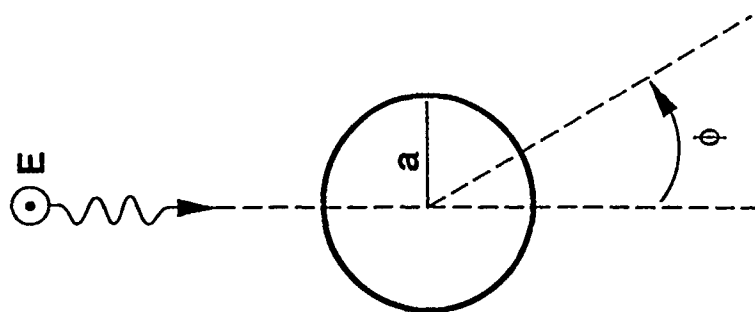a

φ

Fig. 16

Fig. 2

SC-1780-T
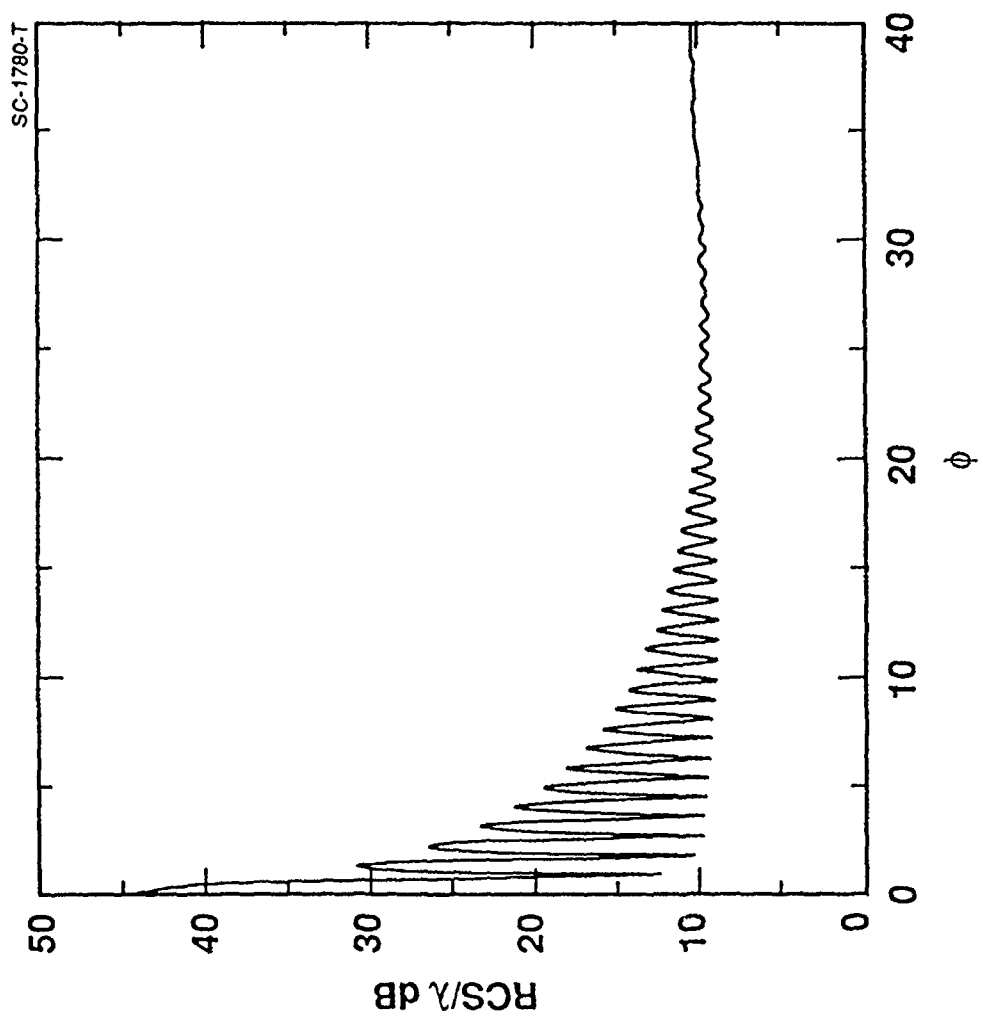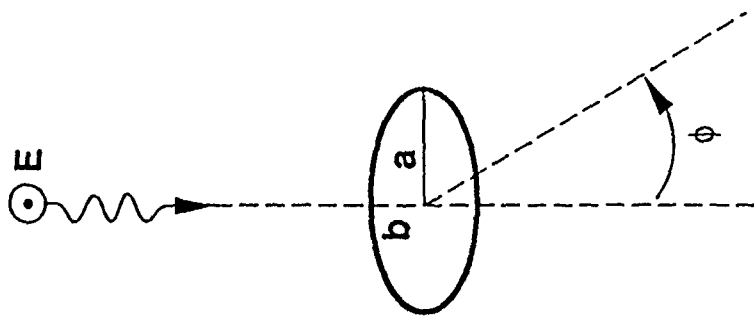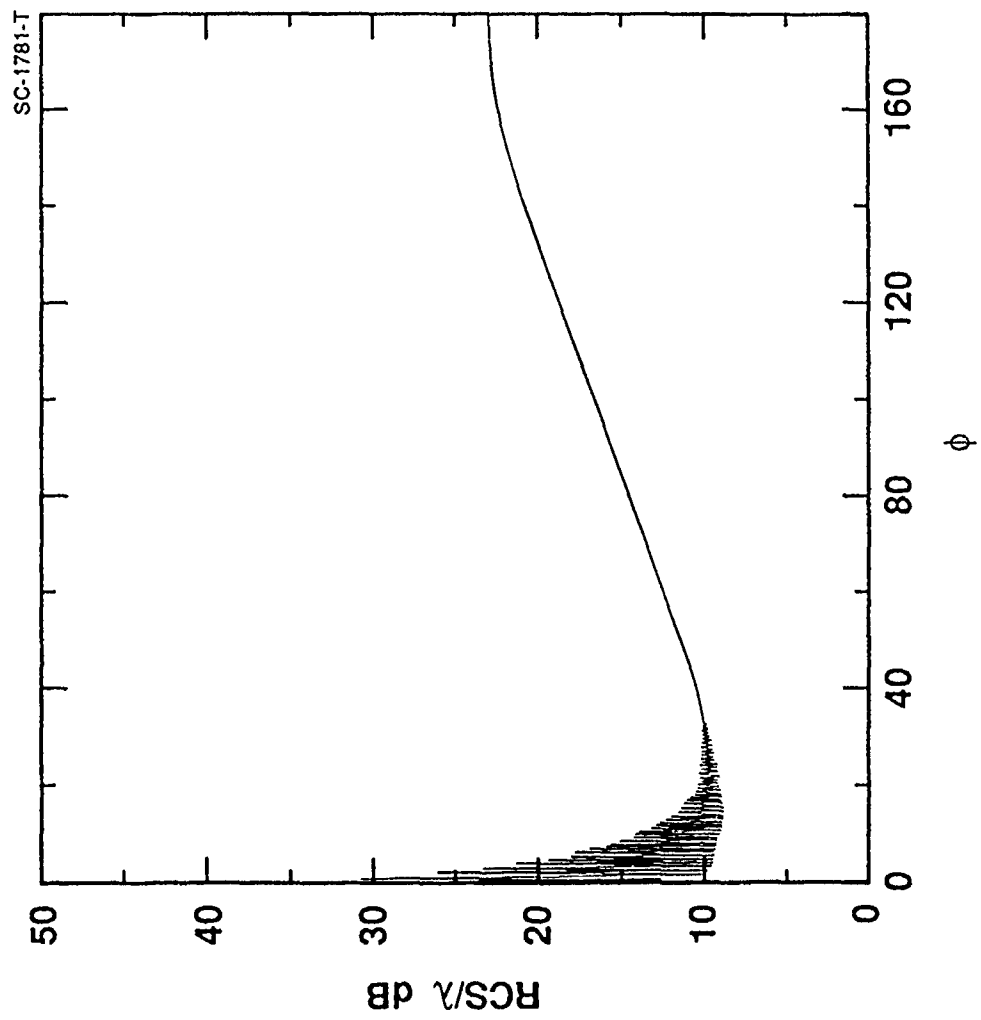
RCS/λ dB

φ

E

a

b

φ
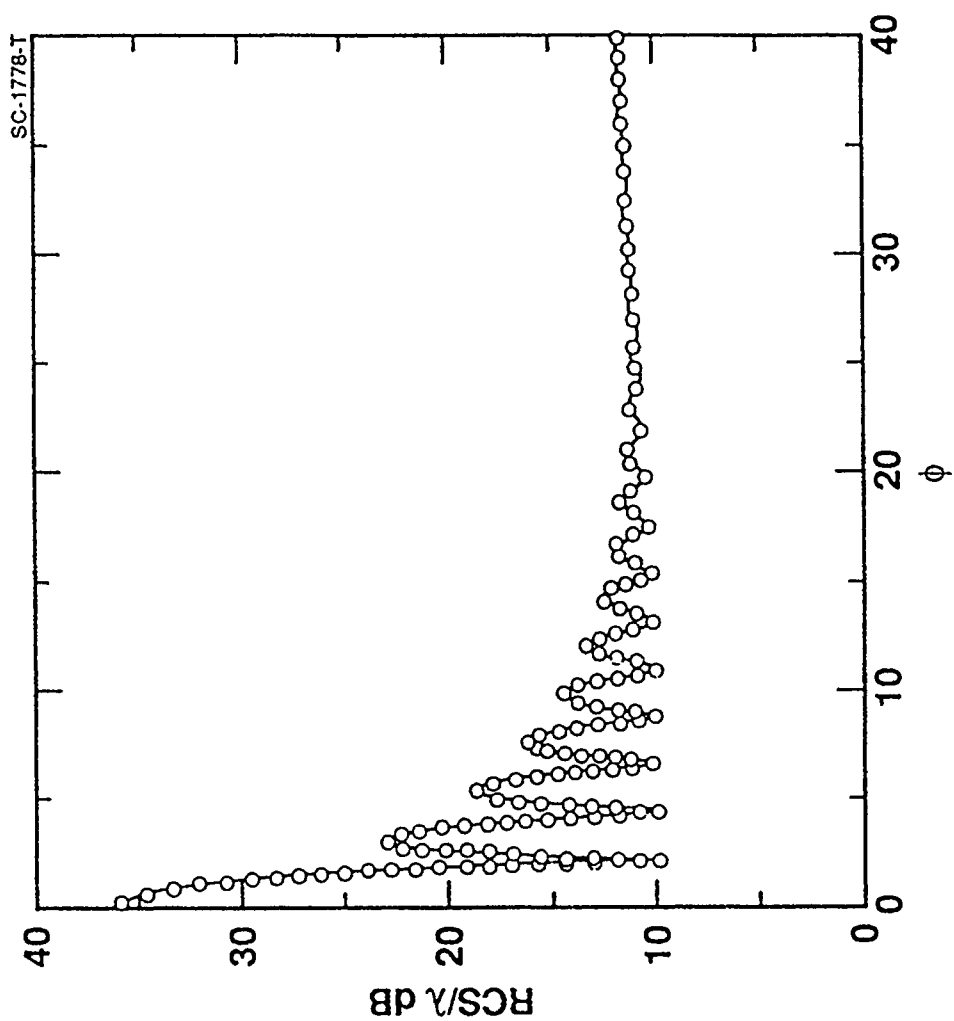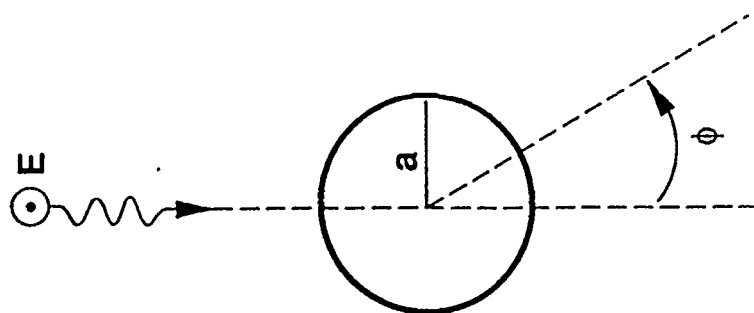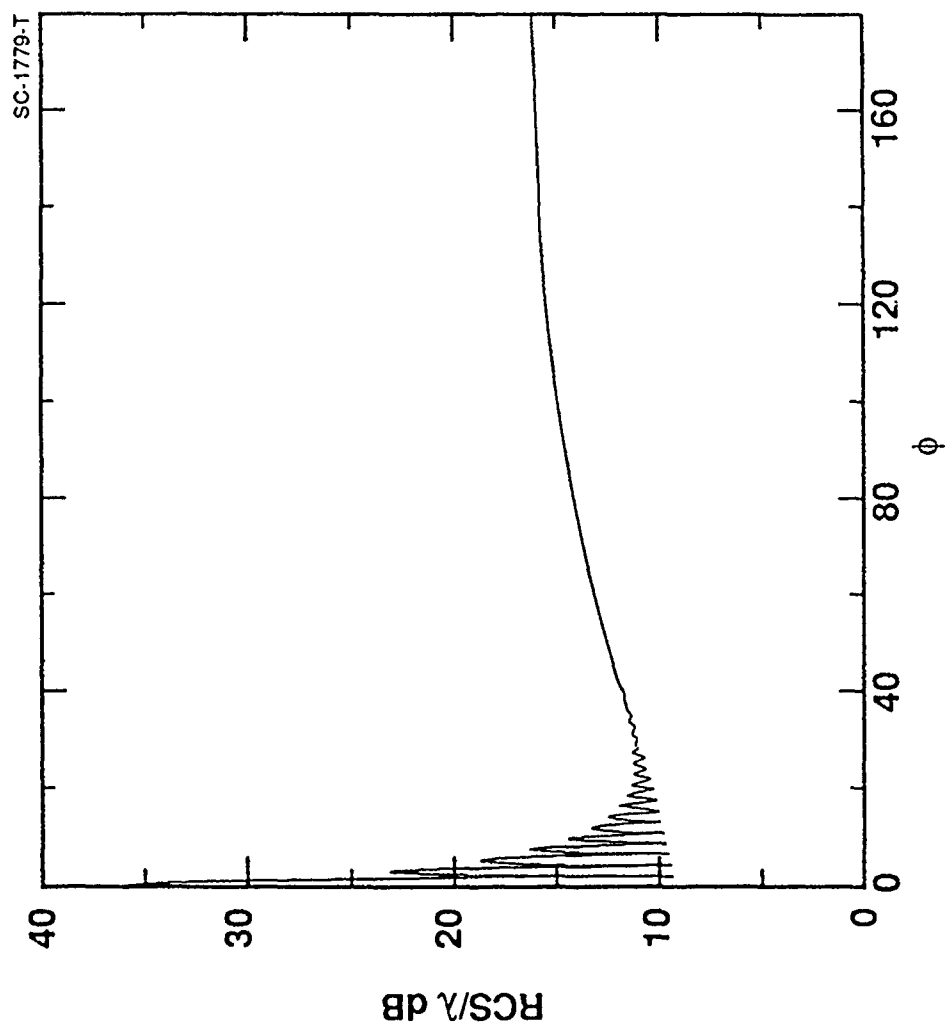
Fig. 3a

Fig. 3b

HIGHER ORDER IMPEDANCE BOUNDARY CONDITIONS
FOR ELECTROMAGNETIC SCATTERING

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ABSTRACT

Higher order impedance boundary conditions for thin coatings about closed conductors in two dimensions are derived using Fourier integral techniques. Using a single-layer potential and these impedance boundary conditions, second-kind weakly singular integral equations are derived that model TE electromagnetic scattering problems. These integral equations are solved using Nyström's method and approximately fourth-order convergent quadrature formulas.

I. INTRODUCTION

Consider a two-dimensional closed perfect electrical conductor coated with a thin layer of dielectric and/or magnetic material. The classical way of solving the electromagnetic scattering problem from such an object is to develop an integral equation in which the contour of integration contains both the conductor and the outer surface of the dielectric. The difficulty with this approach is that as the thickness of the dielectric layer approaches zero, an ill-conditioned equation may result. In addition, the size of the discrete linear system is twice as large as in the method described below. Our procedure will translate the boundary condition on the surface of the conductor to the dielectric-air interface by developing an impedance boundary condition on the interface. The resulting integral equation will only have to be integrated along the interface, thereby reducing the number of unknowns for the discrete problem by a factor of two and possibly removing the ill-conditioning caused by grid points on the conductor and dielectric being too close together. Other work in this area can be found in Rojas and Al-hekail[1], Senior and Volakis[2,3], Harrington and Mautz[4], Karp and Karal[5], Senior[6,7,8], and Barkeshli and Volakis[9].

II. DERIVATION OF HIGHER ORDER IMPEDANCE BOUNDARY CONDITIONS

Consider Helmholtz's equation written in co-ordinates normal (n) and tangential (t) to the scatterer, i.e.,

$$\frac{\partial^2 u^{tot}}{\partial n^2} + \frac{\partial^2 u^{tot}}{\partial t^2} + k_1^2 u^{tot} = 0 \tag{1}$$

where k_1 is the wave number in the dielectric ($k_1 = k_{air}\sqrt{\epsilon_1 \mu_1}$). Here the superscript tot denotes the total field. Thus

$$u^{tot} = u + u^{inc} \tag{2}$$

where inc denotes the incident field. No superscript is used for the scattered field. The boundary condition on the conductor for the TE polarization case is

$$\frac{\partial u^{tot}}{\partial n} = 0 \tag{3}$$

and on the dielectric-air interface we have

$$\frac{\partial u_2^{tot}}{\partial n} = \frac{\epsilon_2}{\epsilon_1} \frac{\partial u_1^{tot}}{\partial n} \tag{4}$$

where ϵ_1 is the dielectric constant in the thin layer and ϵ_2 is that in air. We assume the layer thickness is δ.

Taking Fourier integral transforms around the scatterer and assuming periodic conditions, we can derive the following equation, where ^ denotes Fourier transformation:

$$\frac{\partial^2 \hat{u}^{tot}}{\partial n^2} = (\omega^2 - k_1^2)\hat{u}^{tot} \equiv \alpha \hat{u}^{tot}, \qquad 0 < n < \delta \tag{5}$$

Let $\hat{u}_0^{tot}$ denote the unknown value of $\hat{u}^{tot}$ at $n = 0$ (on the metal-dielectric interface). Then we obtain

$$\hat{u}^{tot}(0) = \hat{u}_0^{tot} \tag{6a}$$

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$$\frac{\partial^{2r} \hat{u}^{tot}(0)}{\partial n^{2r}} = \alpha^r \hat{u}_0^{tot} \qquad (6b)$$

$$\frac{\partial \hat{u}^{tot}(0)}{\partial n} = \frac{\partial^{2r+1} \hat{u}^{tot}(0)}{\partial n^{2r+1}} = 0, \qquad r = 1, 2, \cdots \qquad (6c)$$

where we have used equations (3) and (5). Expanding in a Taylor series in δ, we have

$$\hat{u}^{tot}(\delta - 0) = \hat{u}_0^{tot} + \frac{\delta^2 \alpha \hat{u}_0^{tot}}{2} + \cdots \qquad (7)$$

$$\frac{\partial \hat{u}^{tot}(\delta - 0)}{\partial n} = \delta \alpha \hat{u}_0^{tot} + \frac{\delta^3 \alpha^2 \hat{u}_0^{tot}}{6} + \cdots \qquad (8)$$

Substituting (8) into (4) gives (at $n = \delta + 0$)

$$\partial \hat{u}_2^{tot} / \partial n = \epsilon_2 \delta (\alpha \hat{u}_0^{tot} + (\alpha \delta)^2 \hat{u}_0^{tot} / 6 + \cdots) / \epsilon_1$$

$$= \epsilon_2 \delta \alpha \hat{u}_2^{tot} / \epsilon_1 + O(\delta^2) \qquad (9)$$

where we have used equation (7) and the continuity of $\hat{u}^{tot}$. Note that the subscript 2 represents the point $n = \delta + 0$. Taking inverse transforms of equation (9) yields the desired impedance boundary condition

$$\frac{\partial u_2^{tot}}{\partial n} = \frac{-\delta \epsilon_2 \left(\partial^2 u_2^{tot} / \partial t^2 + k_1^2 u_2^{tot} \right)}{\epsilon_1} + O(\delta^2) \qquad (10)$$

In terms of scattered and incident fields equation (10) may be re-written as

$$\frac{\partial u_2}{\partial n} + \delta (\epsilon_2 / \epsilon_1) \left[\frac{\partial^2 u_2}{\partial t^2} + k_1^2 u_2 \right] = -\frac{\partial u_2^{inc}}{\partial n} - \delta (\epsilon_2 / \epsilon_1) \left[\frac{\partial^2 u_2^{inc}}{\partial t^2} + k_1^2 u_2^{inc} \right]$$

$$\equiv g \qquad (11)$$

where we have dropped the $O(\delta^2)$ term in equation (11).

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Now introduce the single-layer potential

$$u(x) = \int_C \Phi(x,y)\phi(y)\,ds(y) \tag{12}$$

where x and y are two-dimensional points, C represents the outer contour (around the dielectric), and Φ is the two-dimensional free stream Green's function given by

$$\Phi(x,y) = iH_0^{(1)}(k|x-y|)/4 \tag{13}$$

Here $H_0^{(1)}$ denotes the first kind Hankel function of order zero, and $k = k_{air}$. If equation (12) is substituted into (11) and the appropriate jump conditions[10] are enforced at the dielectric surface, we can derive the following weakly singular second-kind integral equation:

$$\phi(x) - 2\int_C \left[\partial\Phi(x,y)/\partial n(x) + \delta(\epsilon_2/\epsilon_1)(\partial^2\Phi(x,y)/\partial t^2(x) + k_1^2\Phi(x,y))\right]\phi(y)\,ds(y)$$

$$= -2g(x) \tag{14}$$

See Colton and Kress[10] for more details on derivations of integral equations in the form of equation (14). If δ is set to zero, equation (14) reduces to the TE polarization case for scattering from a perfect electrical conductor. If the term $\partial^2\Phi/\partial t^2$ is removed from equation (14), we have the standard first order impedance boundary condition integral equation[10]. Finally, as written, we call equation (14) the second order ($O(\delta^2)$) impedance boundary condition integral equation for modeling a thin coating about a metal conductor. The second derivative term obviously models the curvature of the scatterer.

The above derivation can easily be extended to multiple dielectric surfaces. Suppose we have m overlapping coatings with physical parameters $(\delta_i, \epsilon_i, k_i)$ $(i = 1, 2, \ldots, m)$. Let $\delta = \delta_1 + \delta_2 + \cdots + \delta_m$ and set $\beta_i = \epsilon_{i+1}/\epsilon_i$ where $\epsilon_{m+1} = \epsilon_{air}$. Then following the same

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steps as above, we can derive the following impedance boundary condition at the m-th dielectric-air interface:

$$\partial u/\partial n = -\sum_{j=1}^{m} \beta_{j+1}\delta_j(\partial^2 u/\partial t^2 + k_j^2 u) + g + O(\delta^2) \tag{15}$$

where we have dropped the subscript 2, and g is now defined to be equal to the sum in equation (15) for the incident field. An integral equation analogous to equation (14) can be easily written.

In addition, fourth and higher order impedance boundary conditions may be developed by allowing more terms in the Taylor series expansions (7) and (8). For example, the fourth order impedance boundary condition is given by

$$\partial u/\partial n + (\epsilon_2/\epsilon_1)(\delta^3/6)[\partial^2/\partial t^2 + k_1^2][\partial^2 u/\partial t^2 + k_1^2 u] = g \tag{16}$$

where $-g$ now takes the form of the second term in equation (16) for the incident field and the subscript 2 has been dropped.

III. NUMERICAL METHOD

We discretize equation (14) using Nystrom's method[11] and the approximately fourth-order convergent quadrature formulas that handle logarithmic singularities derived in Refs.[12,13,14]. The main advantages of Nyström's method over the method of moments[15] or finite elements are that matrix fill is less expensive and higher order convergent quadrature formulas are easier to employ than corresponding high order basis functions such as piecewise cubic Hermite polynomials or cubic B-splines[16].

The resulting linear system is then solved by Gaussian elimination with partial pivoting or generalized conjugate residual(GCR)[17] iterative methods. The latter algorithm performs well for moderate condition numbers ($\kappa < 1000$). For larger condition numbers, we use the theory of "complexification". See below and Ref.[14] where we have seen that this theory is applicable even at resonance frequencies ($\kappa \to \infty$).

IV. COMPUTATIONAL RESULTS

Assume the incident field is in the form of a plane wave with the incident angle β, i.e.,

$$u^{inc}(x) = \exp[ik(x_1 \cos\beta + x_2 \sin\beta)] \tag{17}$$

where $x = (x_1, x_2)$. For our first example we consider a thin dielectric around a metal circular cylinder, since the analytic solution[18] is available for this case.

ACKNOWLEDGEMENTS

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Solving electromagnetic scattering problems at resonance frequencies

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The "resonance problem" is that at certain values of the wave number k (the resonant k's), the second-kind integral equation for solving scattering problems can become extremely ill-conditioned. This adversely affects both the accuracy and speed of numerical solutions. We consider transverse-magnetic scattering from a conductor (Dirichlet problem). The integral equation (derived using double-layer potentials) is discretized using approximately fourth-order convergent quadrature formulas. At resonant k's for circular and elliptical scatterers, we find very large condition numbers for the discrete matrices [up to $O(10^7)$], generally leading to poor solutions. We apply two approaches to alleviate the resonance problem. The first is to use a different integral equation, based on both single- and double-layer potentials. This leads to low condition numbers and good solutions at resonant k. The second method is to use the original second-kind integral equation, introduce a small imaginary part in k, and extrapolate back to the real axis. Solutions obtained by the two methods are in excellent agreement. The extrapolation technique will be particularly useful in the case of the exterior Neumann problem, when the application of the first technique will be numerically more difficult. By solving the resonance problem, we ensure that fast and accurate solutions are obtainable at any arbitrary wave number.

I. INTRODUCTION

A. What is the resonance problem?

Briefly, the resonance problem is that at certain values of wave number, the second-kind integral equation for solving scattering problems can become extremely ill-conditioned. The presence of resonances adversely affects both the accuracy and the speed of the solution method: The accuracy is affected because of the ill-conditioning of the problem. The speed may be affected because large condition numbers tend to make it difficult for iterative solution techniques to converge.[1] The resonance problem *must be solved* if reliable and efficient solutions are to be obtainable at any arbitrary frequency.

The resonance problem arises in many numerical integral-equation methods for solving scattering problems. Previous investigators have discussed the problem for moment-method solutions of the magnetic-field integral equation (MFIE),[2] electric-field integral equation (EFIE),[3] and combined-field integral equation (CFIE).[2,4]

In this paper we concentrate on alleviating the resonance problem in the Nyström-method solution of the second-kind integral equation for electromagnetic scattering derived using double-layer potentials. The basic numerical method is described by Murphy, Rokhlin, and Vassiliou[5] and is summarized in Secs. I B and II A below.

B. A more precise definition

Consider a two-dimensional closed conducting object in the (x,y) plane. We treat transverse-magnetic (TM) scattering from such an object. By TM we mean that the electric field of the incident radiation is in the z direction: i.e., parallel to the axis of the scatterer (in transverse electric, or TE, scattering the magnetic field is parallel to the axis).

One way of modeling TM scattering from a closed conducting object defined by the curve C is to introduce the double-layer potential

$$u(x) = \int_C \frac{\partial \Phi(x,y)}{\partial \nu(y)} \Psi(y)ds(y),\qquad (1)$$

where x and y are points in the plane, x is outside C and y is on C. Φ is the two-dimensional Green's function

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)\qquad (2)$$

where k is the wave number of the incident radiation and $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Physically, u represents the scattered electric field E_z perpendicular to the plane containing the scatterer. u satisfies Helmholtz's equation, with a radiation condition at infinity and with the boundary condition determined by the incident electric field $E_z^{(inc)}$:

$$u \equiv f(x) = -E_z^{(inc)}\quad \text{on } C.\qquad (3)$$

This is a Dirichlet problem. Colton and Kress[6] show that the function Ψ satisfies the integral equation

$$\Psi(x) + 2\int_C \frac{\partial \Phi(x,y)}{\partial \nu(y)} \Psi(y)ds(y) = 2f(x),\quad x \in C.\qquad (4)$$

Unfortunately, Eq. (4) does not have a unique solution for all values of k. In fact, nonunique solutions exist for those values of k where the interior Neumann problem

$$\nabla^2 v + k^2 v = 0 \quad \text{in } C, \tag{5a}$$

$$\frac{\partial v}{\partial v} = 0 \quad \text{on } C \tag{5b}$$

has a nontrivial solution. These values of k are referred to as interior resonances of the Dirichlet (TM) problem, or interior Dirichlet eigenvalues.

Numerically, when one tries to solve Eq. (4) at or near a resonance, the condition number κ of the resulting discretized linear system becomes large. The solution becomes much more sensitive to computer round-off errors, and iterative solution procedures have more difficulty converging. In short, efficient solution techniques and accurate results become much more difficult to achieve.

C. What can be done?

Below, we study the condition number of the discrete system for the case when C is a circle and the case when C is an ellipse. We then outline two approaches to alleviate the resonance problem: (1) using a combined-potential equation in place of Eq. (4) and (2) introducing a small imaginary part in k and extrapolating back to the real axis.

II. THE ILL-CONDITIONING OF SCATTERING PROBLEMS NEAR A RESONANCE

A. Numerical method: Discretization and quadrature formulas

We discretize Eq. (4) using Nyström's method; that is, we approximate the integral directly with a quadrature formula. We use roughly fourth-order-convergent quadrature formulas that handle logarithmic singularities.[5,7] The high-order accuracy is important because in the vicinity of a resonance, solutions are extremely sensitive to round-off and truncation errors.

Roughly fourth-order-convergent quadrature formulas that handle logarithmic singularities at $x = 0$ are developed[7] by starting with the Euler–Maclaurin formula with the singular point removed. That is, let

$$\int_0^1 f(x)dx = h\left(\sum_{i=1}^{n} f(x_i) - \frac{f(x_n)}{2}\right) - \frac{1}{12}h^2 f'(x_n), \tag{6}$$

where $h = 1/n$ and $x_i = i/n$. To correct for the singular point, a concentration of points of the form $\sum_{j=1}^{6} \lambda_j f(\chi_j)$ is introduced in the first interval, where $\chi_j = jh/6$ for $j = 1,2,...,6$. The derivative term is approximated by the one-sided difference formula

$$f'(x_n) = \frac{1}{2h}\left[3f(x_n) - 4f(x_{n-1}) + f(x_{n-2})\right]. \tag{7}$$

Combining these terms yields

$$\int_0^1 f(x)dx = h\left(\sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{6} \lambda_j f(\chi_j) - \frac{1}{2}f(x_n)\right.$$
$$\left. - \frac{1}{24}[f(x_{n-2}) - 4f(x_{n-1}) + 3f(x_n)]\right). \tag{8}$$

The unknown weights λ_j ($j = 1,2,...,6$) are determined by

solving the 6×6 linear system that results when Eq. (8) is assumed exact for the following candidate functions $f(x)$: $1, x, x^2, \log x, x \log x$, and $x^2 \log x$, using analytic integration rules. Once computed, the quadrature weights can be stored and looked up numerically when needed. The proof of approximately fourth-order convergence can be found in Rokhlin.[7]

B. The condition number

The discrete system is of the form

$$Ax = b, \tag{9}$$

where A is a nonsymmetric complex matrix of order n and x and b are vectors of length n. The condition number κ is defined as

$$\kappa = \|A\| \, \|A^{-1}\|, \tag{10}$$

where $\| \ \|$ can be any finite-dimensional norm. We choose the 2-norm, in which case we have[8]

$$\kappa = \frac{\sigma_{max}}{\sigma_{min}}, \tag{11}$$

where σ_{max} and σ_{min} are, respectively, the largest and smallest singular values of the matrix A. We can thus compute the condition number of the matrix by performing a singular value decomposition. This is a somewhat expensive operation to perform routinely, but it is indispensible for the present study. As a resonance frequency is approached, σ_{min} approaches zero, and the condition number approaches infinity.

C. Resonances of a circular conductor of unit radius

For a circle of unit radius, the resonant wave numbers are determined from the roots of the derivatives of Bessel functions; that is,

$$J'_n(k_m) = 0, \quad n = 0,1,2,..., \quad m = 0,1,2,... . \tag{12}$$

In Table I, we show some computational results for κ with k at or near a resonance, with the scatterer being a circle of unit radius. The value of n denotes the number of sample points along the boundary, separated by equal arc lengths. The number of points per wavelength is n/k. The three resonant wave numbers k_1, k_2, and k_3 in Table I have values 3.831 705 970 2, 7.015 586 669 8, and 19.615 858 510 5, respectively. More roots of J'_n can be found in Abramowitz and Stegun.[9]

Notice in Table I that a slight movement of k into the complex plane reduces the condition number κ by orders of magnitude, even for the highest wave number k_3. We will use this fact in our extrapolation technique.

D. Resonances of an elliptical conductor

For an ellipse, the resonant wave numbers are determined from the roots of the derivative of a modified Mathieu function. We generated these roots ourselves: to our knowledge they are not satisfactorily tabulated. Mathieu functions arise often in problems with elliptical geometry and have an extensively developed theory,[10] which we cannot present here. Since, however, they are not as generally known or as

TABLE I. Condition numbers κ for a circle of radius $a = 1$, using Eq. (4). Notes: 0.10 (3) means 0.1×10^3. n is the number of sample points on the scatterer boundary, k is the wave number, and a is the radius; $a = 1$.

n	ka	κ	Number of points Wavelength on scatterer boundary
40	k_1	0.34790 (5)	10.4
80	k_1	0.92251 (6)	20.8
100	k_1	0.29271 (7)	26.1
200	k_1	0.50210 (7)	52.2
100	$k_1 + 0.05i$	0.20355 (2)	26.1
100	$k_1 + 0.025i$	0.19506 (2)	26.1
100	$k_1 + 0.01i$	0.10163 (3)	26.1
100	$k_1 + 0.002i$	0.50803 (3)	26.1
100	$k_1 + 0.001i$	0.10160 (4)	26.1
100	$k_1 + 0.0004i$	0.25401 (4)	26.1
100	$k_1 + 0.0002i$	0.50804 (4)	26.1
40	k_2	0.38677 (4)	5.7
80	k_2	0.96858 (5)	11.4
100	k_2	0.27361 (6)	13.7
200	k_2	0.83871 (7)	27.4
100	$k_2 + 0.05i$	0.20126 (2)	13 7
100	$k_2 + 0.001i$	0.10062 (4)	13.7
100	k_3	0.63729 (4)	5.1
200	k_3	0.16200 (6)	10.2
200	$k_3 + 0.05i$	0.20194 (2)	10.2

widely tabulated as Bessel functions, we discuss them briefly. When we apply the method of separation of variables to Helmholtz's equation in elliptical coordinates, we obtain the following equations:

$$\frac{d^2y}{du^2} + (\lambda - 2q \cos 2u)y = 0 \qquad (13a)$$

and

$$\frac{d^2y}{du^2} - (\lambda - 2q \cosh 2u)y = 0, \qquad (13b)$$

which are, respectively, the Mathieu and modified Mathieu equations. In the present problem, the equation of interest is the second one, the modified Mathieu equation. Solutions to this equation are the modified Mathieu functions, which have the form[11]

$$Ce_{2n}(u,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2ru, \qquad (14a)$$

$$Ce_{2n+1}(u,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)u, \qquad (14b)$$

$$Se_{2n+1}(u,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh(2r+1)u, \qquad (14c)$$

$$Se_{2n+2}(u,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh(2r+2)u. \qquad (14d)$$

The notation Ce and Se (on which there are variations) arose from the notion of "elliptic cosine" and "elliptic sine."

The coefficients A and B must be found through recurrence relationships. First, however, one must determine the "characteristic number" λ for which a periodic solution exists. One must do this for each value of the parameter q (given u, which is fixed by the eccentricity of the ellipse in ques-

tion). To find the characteristic number one must find the root of a continued-fraction equation. Our approach is to obtain tabulated values of the characteristic numbers,[10] interpolate them using a natural cubic spline, and use the interpolated value as an initial guess in the method of bisection to find the root of the continued-fraction equation. This gives us the characteristic number for any value of q to high precision. We can then determine the coefficients and calculate the values of the derivative of the modified Mathieu function for various values of q, and use the secant method to obtain the derivative's root. The root q_0 is related to the resonant wave number k via $q_0 = (c^2/4)k^2$, where $2c$ is the interfocal distance for the ellipse in question. We have generated roots for the function Ce_0' for use in this study.

Our present computer implementation requires that the sample points on the boundary of the scatterer be equispaced in arc length. In the case of a circular scatterer, this is simple: $\Delta s = a\Delta\theta$, where Δs is the arc length interval and a is the radius of the circle. However, for other closed contours a numerical algorithm must be invoked to produce equally spaced points. This resampling procedure, which uses fast Fourier transforms, perturbs the discrete problem very slightly (typically 0.01% or less); however, these small perturbations are often great enough so that the resulting linear system has a different resonance from the original problem.

To obtain the resonant wave numbers of the discrete problem resulting from a slightly perturbed scatterer shape, we use the secant method to drive the minimum singular value σ_{min} to 0. Suppose the resonant wave number is defined as the root of a nonlinear function $\sigma_{min} = f(k)$. Then, using the secant method, an improved root is given by

$$k^{(n+1)} = k^{(n)} - \sigma_{min}^{(n)} \frac{k^{(n)} - k^{(n-1)}}{\sigma_{min}^{(n)} - \sigma_{min}^{(n-1)}}, \qquad (15)$$

where the superscript (n) denotes the iteration counter. In practice, we have found that only one or two iterations are required to obtain an accurate resonant wave number for an ellipse, when we use the root of the derivative of the appropriate modified Mathieu function to obtain a starting value for the algorithm. Table II lists the continuous and discrete resonant wave numbers for a number of ellipses. The values k_4, k_5, k_6, and k_7 refer to the discrete k's in this table in order from top to bottom, and are used in the tables following.

Table III lists some condition numbers for various values of k near a discrete resonance for an ellipse with semimajor axis a and semiminor axis b. We compute the number of points per wavelength using an approximate formula for the perimeter of an ellipse, which yields

TABLE II. Resonant wave numbers for various ellipses. Notes: a is the semimajor axis, b the semiminor axis, and k the wave number.

Continuous kb	Discrete kb	a	b
3.380 123 647 98	3.377 841 278 90 ($= k_4$)	2	1
3.467 850 747 85	3.467 387 362 51 ($= k_5$)	1.5	1
3.562 999 874 37	3.558 639 339 65 ($= k_6$)	1.25	1
3.301 504 871 94	3.300 332 241 01 ($= k_7$)	3	1

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n	kb	a	b	κ	Number of points Wavelength on scatterer boundary
150	k_4	2	1	0.15107 (8)	28.07
150	$k_4 + 0.002i$	2	1	0.50537 (3)	28.07
150	$k_4 + 0.001i$	2	1	0.10107 (4)	28.07
150	$k_4 + 0.02i$	2	1	0.50542 (2)	28.07
150	$k_4 + 0.01i$	2	1	0.10108 (3)	28.07
150	$k_4 + 0.2i$	2	1	0.51228 (1)	28.07
150	$k_4 + 0.1i$	2	1	0.10141 (2)	28.07
150	k_5	1.5	1	0.12572 (8)	33.93
150	$k_5 + 0.002i$	1.5	1	0.50698 (3)	33.93
150	$k_5 + 0.001i$	1.5	1	0.10139 (4)	33.93
150	k_6	1.25	1	0.36088 (7)	37.24
150	$k_6 + 0.002i$	1.25	1	0.50202 (3)	37.24
150	$k_6 + 0.001i$	1.25	1	0.10040 (4)	37.24
150	k_7	3	1	0.19492 (8)	20.32
150	$k_7 + 0.002i$	3	1	0.52592 (3)	20.32
150	$k_7 + 0.001i$	3	1	0.10522 (4)	20.32

$$\frac{\text{points}}{\lambda} = \frac{n}{k\sqrt{\tfrac{1}{2}(a^2 + b^2)}}. \tag{16}$$

Table III shows, as does Table I, the bad conditioning that occurs near resonance. Again, notice how a slight movement of k into the complex plane reduces κ by *orders of magnitude*.

III. SOLVING THE RESONANCE PROBLEM USING THE COMBINED-POTENTIAL EQUATION

A different integral equation from Eq. (4) can be formed by combining single- and double-layer potentials. Let

$$u(x) = \int_C \left(\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta\Phi(x,y)\right)\Psi(y)ds(y), \tag{17}$$

where $\eta \neq 0$ is an arbitrary real number such that .

$$\eta \, \text{Re } k \geqslant 0. \tag{18}$$

We use $\eta = 1$ in our computations. Using (17), Colton and Kress[6] derive the integral equation

$$\Psi + K\Psi - i\eta S\Psi = 2f, \tag{19}$$

where K and S are operators given by

$$(K\Psi)(x) = 2\int_C \frac{\partial \Phi(x,y)}{\partial \nu(y)} \Psi(y)ds(y), \tag{20}$$

$$(S\Psi)(x) = 2\int_C \Phi(x,y)\Psi(y)ds(y). \tag{21}$$

Equation (19) is the modified Dirichlet problem (TM) that has solutions for all wave numbers k satisfying Im $k \geqslant 0$.[4] Equation (19) is analogous to the combined-field integral equation (CFIE) of electrical engineering practice, but it is not the same. We refer to it in this paper as the combined-potential equation.

n	ka	κ	Number of points Wavelength on scatterer boundary
40	k_1	3.8933	10.4
80	k_1	3.8930	20.8
100	k_1	3.8930	26.1
40	k_2	7.0648	5.7
80	k_2	7.0507	11.4
100	k_2	7.0505	13.7
200	k_2	7.0505	27.4

If we discretize Eq. (19) using the same highly accurate quadrature formulas we used for Eq. (4), a linear system similar to (9) results, but with a much better-conditioned coefficient matrix A. We illustrate this point in Tables IV and V. Equation (19), the combined-potential equation, is somewhat more complicated than Eq. (4), the integral equation obtained by using only double-layer potentials. However, the linear system is so much better conditioned that the additional work required to discretize the integral equation is justified. Since most iterative methods for solving the linear system (9) have a convergence time that is an increasing function of κ, the condition number, κ should be minimized.

Unfortunately, in the transverse-electric (TE) case, which is a Neumann problem, the combined-potential equation analogous to Eq. (19) contains the second derivative of the Green's function Φ. This makes the equation much more difficult to solve numerically. Thus, we investigate an alternative approach to solving the resonance problem.

IV. SOLVING THE RESONANCE PROBLEM BY EXTRAPOLATION FROM THE COMPLEX PLANE

In this simple but powerful approach, we take the resonant wave number, add to it a small imaginary part, and solve the scattering problem. We repeat this for a different small imaginary part. We then have two different solutions that we can use to extrapolate to a solution corresponding to the resonant wave number with zero imaginary part. We show an example in Fig. 1. Here, we use solutions for $k = k_1 + i0.002$ and $k = k_1 + i0.001$ to obtain the radar

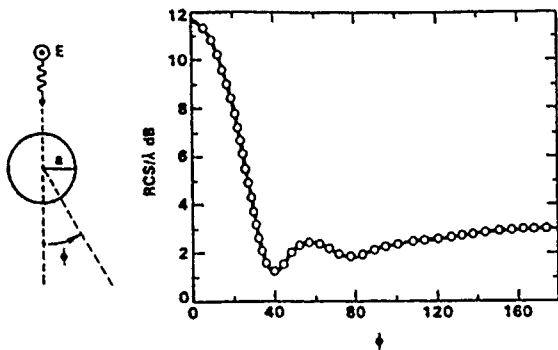| n	kb	a	b	κ	Number of points Wavelength on scatterer boundary
150	k_4	2	1	3.2821	28.07
150	k_5	1.5	1	3.3914	33.93
150	k_6	1.25	1	3.4160	37.24
150	k_7	3	1	3.5451	20.32

FIG. 1. Radar cross section (RCS) for a circle of unit radius at resonant wave number k_1. Solid line shows solution obtained using the combined-potential equation (19). Dotted line shows solution obtained using extrapolation from the complex plane. The error between the two methods is less than 0.002 dB and cannot be seen on the plots. The series solution for a circle is also indistinguishable on the plot.

cross section (RCS) for a circle of unit radius at resonant wave number k_1. As can be seen in Fig. 1, the agreement between this method and that using Eq. (19) described above is excellent. The error between the two methods is less than 0.002 dB and cannot be seen on the plots.

As another example, we consider an elliptical scatterer with parameters $a = 2$, $b = 1$ (where a is the semimajor axis and b is the semiminor axis), $n = 150$, and $k = k_4$, illuminated by a plane wave with incidence angle 90° (broadside incidence). Three separate linear extrapolations are computed: (1) $k_4 + i0.002$ and $k_4 + i0.001$, (2) $k_4 + i0.02$ and $k_4 + i0.01$, and (3) $k_4 + i0.2$ and $k_4 + i0.1$. The maximum RCS errors (for 100 observation angles) between the solutions obtained using these three extrapolations and those obtained using the combined-potential equation (19) are, respectively, 0.003, 0.002, and 0.146 dB. Even the last extrapolation, a crude one that probably would not be used in practice, yields a good solution at resonance. Note also from Table III that the condition numbers κ for the cases $k_4 + i0.2$ and $k_4 + i0.1$ are almost as small as that for k_4 in Table V. Similarly good results are obtained for the other ellipses (with different values of a and b) in Table III. Iterative methods such as the generalized conjugate residual (GCR) method for solving linear systems can be expected to converge rapidly for such small values of κ.[12,13]

The RCS results for the ellipse $a = 2$, $b = 1$ at resonance k_4 are plotted in Fig. 2 [this curve is the same on a plot of this size whether computed via Eq. (19) or via the extrapolation technique]. These results are compared in the figure with the RCS results computed at resonance using Eq. (4) with no extrapolation. The enormous difference between the two curves highlights the possibly disastrous results that can be obtained when condition numbers reach values of $O(10^7)$. Blindly solving Eq. (4) without considering resonance is not wise.

V. SUMMARY

We have solved the problem of resonance in integral-equation scattering methods using two different approaches: (1) By solving a combined-potential equation [Eq. (19)] and (2) by extrapolation from the complex plane. In this paper we have solved a Dirichlet (TM) problem for which the combined-potential equation is relatively easy to treat numerically, so that the extrapolation technique may not appear to offer any overwhelming advantages. However, when we solve the Neumann (TE) problem, the combined-potential equation analogous to Eq. (19) contains the second derivative of the Green's function Φ. This makes the equation much more difficult to solve numerically. In the TE case, the extrapolation technique will offer a significant advantage.

By solving this problem, we have assured the possibility of obtaining accurate and efficient solutions at all frequencies. We have thus significantly strengthened a highly accurate implementation of the second-kind integral equation for scattering.
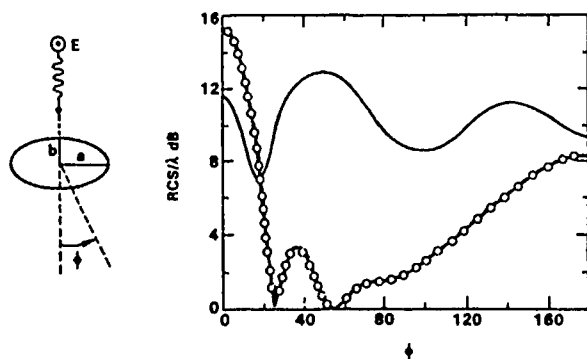
FIG. 2. RCS results for an ellipse with semimajor axis $a = 2$ and semiminor axis $b = 1$ at resonance k_4. The extrapolation technique (dotted line) and combined-potential equation technique agree very well with each other and would be indistinguishable if different symbols were not used. The other solid line represents the incorrect results computed at resonance using Eq (4) with no extrapolation.

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NUMERICAL SECOND-KIND-INTEGRAL-EQUATION SOLUTIONS OF ELECTROMAGNETIC SCATTERING PROBLEMS

Indexing terms: Electromagnetic waves, Scattering, Numerical methods and number theory, Radar cross-sections

A new, highly accurate numerical method based on second-kind integral equations has been developed to solve electromagnetic scattering problems for closed conducting bodies in two dimensions. The method is approximately fourth-order convergent, owing to the use of accurate new quadrature formulas

Introduction: We have developed a second-kind integral equation solver (the SKIE method) for transverse-magnetic (TM) electromagnetic scattering from perfect electrical conductors of arbitrary closed geometry in two dimensions. The method uses accurate, roughly fourth-order-convergent quadrature formulas. The resulting discrete matrix has a condition number bounded by a constant as the sampling is refined (for a given scatterer size and nonresonant frequency).

Method: We begin with a second-kind integral equation derived from Helmholtz's equation

$$\int_B \frac{\partial G(p, q)}{\partial n} \mu(q)\, dq + 2\pi\mu(p) = \phi(p) \tag{1}$$

We solve a Dirichlet problem, with ϕ the prescribed value of the potential on the closed boundary B of the scatterer. In the TM case we are solving, ϕ is the electric field normal to the plane of the scatterer, q and p are, respectively, source and field evaluation points on B, $\partial G/\partial n$ is the normal derivative of the Green's function for Helmholtz's equation and μ is the unknown double-layer distribution on the boundary of the scatterer. Once μ is known, the field at a point exterior to the scatterer can be obtained by evaluating

$$\phi(p) = \int_B \frac{\partial G(p, q)}{\partial n} \mu(q)\, dq \tag{2}$$

Eqn. 1 is discretised by replacing the integral with quadrature formulas, leading to a matrix equation that can be solved numerically. We have solved the linear system using both Gaussian elimination and an iterative generalised conjugate residual technique.[1] Roughly fourth-order-convergent quadrature formulas that handle logarithmic singularities at $x = 0$ are developed[2] by starting with the Euler–Maclaurin formula with the singular point removed. That is, let

$$\int_0^1 f(x)\, dx = h\left[\sum_{i=1}^{n} f(x_i) - \frac{f(x_n)}{2}\right] - \frac{1}{12} h^2 f'(x_n) \tag{3}$$

where $h = 1/n$ and $x_i = i/n$. To correct for the singular point, a concentration of points of the form $\sum_{j=1}^{6} \lambda_j f(\chi_j)$ is introduced in the first interval, where $\chi_j = jh/6$ for $j = 1, 2, \ldots, 6$. The derivative term is approximated by the one-sided difference formula

$$f'(x_n) = \frac{1}{2h}\left[3f(x_n) - 4f(x_{n-1}) + f(x_{n-2})\right] \tag{4}$$

Combining these terms yields

$$\int_0^1 f(x)\, dx = h\left\{\sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{6} \lambda_j f(\chi_j) - f(x_n)/2\right.$$
$$\left. - \frac{1}{24}\left[f(x_{n-2}) - 4f(x_{n-1}) + 3f(x_n)\right]\right\} \tag{5}$$

The unknown weights $\lambda_j (j = 1, 2, \ldots, 6)$ are determined by solving the 6×6 linear system that results when eqn. 5 is presumed exact for the following candidate functions $f(x)$: 1, x, x^2, $\log x$, $x \log x$, and $x^2 \log x$, using analytic integration rules. Once computed, the quadrature weights can be stored and looked up numerically when needed. The proof of approximately fourth-order convergence can be found in Reference 2.

Our method involves a direct discretisation of the integral using a quadrature formula, and does not explicitly involve an expansion in basis functions as in the method of moments.[3] Common implementations of the method of moments typically use pulse, triangle or low-order trigonometric basis functions. Such choices of basis functions lead at best to first or second-order accuracy, provided the singularity in the Green's function is handled adequately. (In theory, it is possible to obtain higher accuracy by using higher-order basis functions.)

Results: The method as implemented at present can calculate TM electromagnetic scattering from closed, perfectly conducting bodies in two dimension. Some bistatic RCS calculations are shown in Figs. 1 and 2. Fig. 1 shows the results for a cylinder of size $ka = 20$ where a is the radius, and Fig. 2 shows the results for an ellipse of size $ka = 27.7$ where a is the semimajor axis.

Accuracy: One way to verify the accuracy of the numerical method is to compare numerical results with known analytical solutions for canonical shapes, as done for the cylinder of Fig. 1. We also adopt an approach where the accuracy can be verified for any shape of object. The approach is essentially one of testing how closely the method verifies Green's second
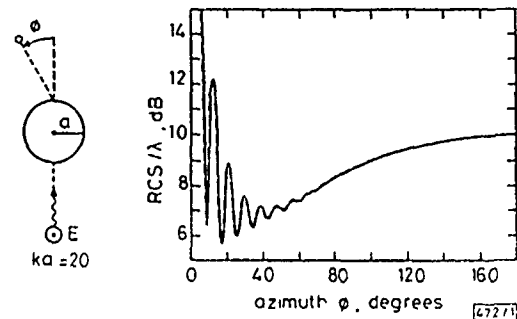

Fig. 1 *Bistatic RCS (TM) for conducting cylinder of size $ka = 20$, where a is radius*

To show detail, lobe at 0° (forward scattering) is cut off in plot, and plot shows only first half of azimuth range
——— Numerical
– – – Analytic

identity. Suppose that we are solving an exterior Dirichlet problem on some two-dimensional boundary, and the right-hand side of the problem is equal to the field generated by a dipole source located inside the boundary. The solution of this Dirichlet problem at receivers exterior to the boundary must be equal to the field of the source at those receivers, as if the boundary were not present. A relative error can be computed between the fields calculated by solving the Dirichlet problem and the fields of the source.
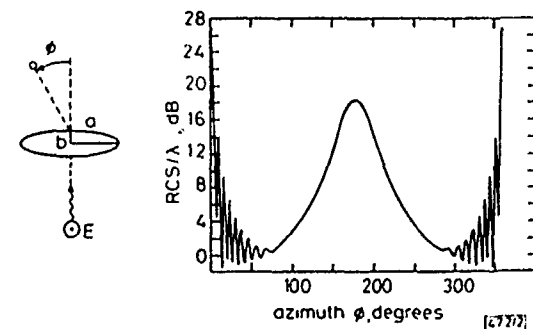

Fig. 2 *Bistatic RCS (TM) for broadside incidence on conducting ellipse of size 27.7 where a is semimajor axis*
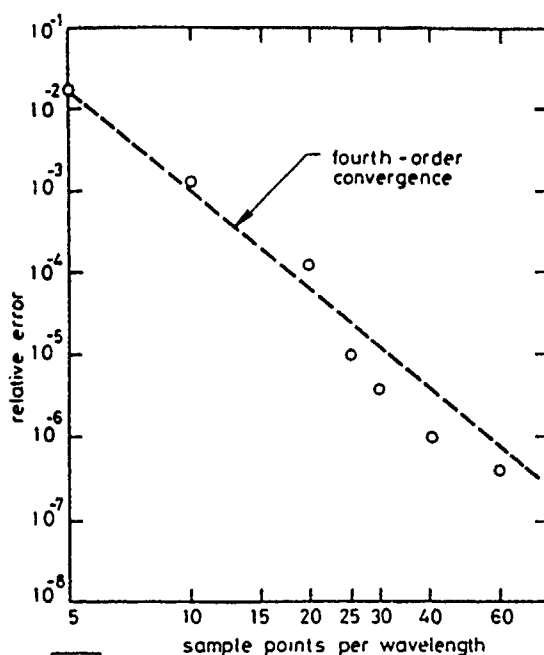
Fig. 3 *Relative accuracy against fineness of sampling for ellipse (not same ellipse as in Fig. 2)*

The accuracy of the method has been verified for two-dimensional scatterers of many different shapes. An example of the results of one accuracy test, for an ellipse of eccentricity 0·87 and $ka = 5$ (a is the semimajor axis) is shown in Fig. 3. The average value, over 360°, of the absolute value of the relative error as defined above is plotted against the number of points per wavelength on the scatterer. As the number of
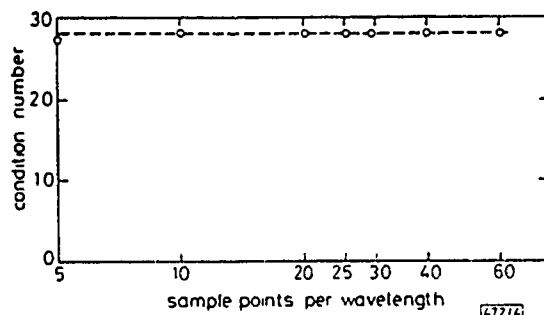

Fig. 4 *Condition number of discrete matrix against number of sample points per wavelength ellipse of Fig 3*

points per wavelength is increased from 5 to 60, the relative error decreases from order 10^{-2} to order 10^{-6}. The dotted line shows fourth-order convergence, and is drawn for reference. At 10 points per wavelength, the accuracy is of order 10^{-3}.

Condition number: Fig. 4 illustrates another important property of the SKIE method. For the ellipse mentioned above in connection with Fig. 3, Fig. 4 plots the condition number of the matrix against the number of sample points per wavelength. As Fig. 4 shows, the condition number is low, and it does not change as the number of sample points is increased. The SKIE method produces problems that are well-conditioned away from resonances. A low condition number is advantageous both because it indicates a stable method[4] and because it tends to lead to faster convergence for iterative methods for linear systems.[5] Note that the condition number is bounded by a constant for a given scatterer size and non-resonant frequency.

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