

Singularities in Gaussian Random Fields.

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# Singularities in Gaussian random fields

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## Abstract

In this paper we discuss a Gaussian random field that arises in pattern analysis. This random field exhibits phase transitive behavior for a particular value of the temperature parameter. We analyze this kind of non singular behavior and the effect that it has on the field random variables. The limiting specific heat also exhibits a phase transition with a power law behavior.

### Section 1. Introduction

One of the principal aims of statistical mechanics is to derive the thermodynamic behavior of macroscopic bodies beginning from a description of their microscopic components. A good deal of work has been done on modelling ferromagnetic and antiferromagnetic behavior. A magnet can be considered to have a large number of magnetic domains, to each of which a magnetic spin is associated that represents the direction of magnetization at that domain. We usually assume that the spins take two values, 0 and 1. The physical models usually postulate that these domains are sites (or vertices) in a graph.

An undirected graph  $\mathcal{G} = (\Lambda, e)$  consists of a set of vertices,  $\Lambda$  and an edge set,  $e$ . The elements of  $e$  are unordered pairs  $(x, y)$ ,  $x, y \in \Lambda$ ; when  $(x, y) \in e$  we say that there is an edge of the graph between  $x$  and  $y$ , or that  $x$  and  $y$  are neighbors. We shall assume that  $(x, x) \notin e$ , i.e. the graph has no loops. As an example, consider a 4 neighbor  $n \times n$  lattice graph in the plane. A vertex of this graph is an ordered pair  $(i, j)$ ,  $0 \leq i, j \leq n - 1$ , the edge set is defined as follows; each point has four neighbors, where  $(i - 1, j)$ ,  $(i + 1, j)$ ,  $(i, j - 1)$ ,  $(i, j + 1)$ ,  $i - 1$ ,  $i + 1$  etc. are calculated modulo  $n$ . Thus  $(n, n)$  is identified with  $0$ , and this graph is actually a torus. We shall be seeing this graph again in Section 3. of this essay. In general, the way we define the graph neighborhood structure is dictated by our knowledge of the influence of different sites on each other.

We have spins at every site in the graph, and a probability model is fully specified as soon as we put a joint distribution on these spins. These models are supposed to tell us which configurations of spins are more likely than the others. Physical models usually assume that the joint distribution of the spins is a Markov random field. A Markov random field is a probability distribution on the set of spins in the graph for which the conditional distribution of spins on a set  $A$ , given all other spins in the graph equals the conditional distribution of the spins on  $A$ , given the spins immediately bordering  $A$ . Markov random fields are identical to the so called Gibbs distributions with nearest neighbor potentials, provided that every  $A$  is given positive probability. Presently, we shall give a precise definition of Gibbs distributions with nearest neighbor potentials. Preston [1] has a more complete discussion of Markov random fields and Gibbs states.

A nearest neighbor Gibbs distribution is defined as follows:

Let  $\mathcal{G} = (\Lambda, e)$  be the finite graph with vertex set  $\Lambda$  and edge set  $e$ . A set  $B \subseteq \Lambda$  is called a simplex of  $\mathcal{G}$  if for all  $x \in B, y \in B, x \neq y$ , there is an edge between  $x$  and  $y$  in the graph  $\mathcal{G}$ . Simplices are also sometimes referred to as cliques.

Let  $J$  be a real valued function defined on subsets of  $\Lambda$  such that  $J(\emptyset) = 0$ . The function  $J(\cdot)$  is called a potential. Let  $A$  be a non-empty subset of  $\Lambda$ . Define the probability of  $A$ ,

$$\pi(A) = Z^{-1} \exp \left( \sum_{\substack{B \subseteq A \\ B \text{ a simplex of } \mathcal{G}}} J(B) \right) \quad (1.1)$$

This is the probability that all sites in  $A$  have spin 1 and the rest have spin 0. We could generalize this to let the spins assume arbitrary real (or complex) values. We shall consider these kinds of distributions below. (see (1.2)).

### Definition 1.1

Let

$$L_n^d = \{(i_1, \dots, i_d) : 0 \leq i_k \leq n-1, k = 1, \dots, d\}.$$

For

$$i = (i_1, \dots, i_d) \in L_n^d \text{ and } j = (j_1, \dots, j_d) \in L_n^d$$

let,  $i + j = (i_1 \oplus j_1, \dots, i_d \oplus j_d)$ , where  $\oplus$  is addition modulo  $n$ .

Note For the rest of this essay,  $i, j, k, l$  will denote  $d$  dimensional vectors in  $L_n^d$ .

We shall consider graphs with vertex set  $L_n^d$ . The edge set will be defined as follows. First specify a neighborhood  $N_0$  of 0, and then define a neighborhood of  $i \in L_n^d$  by  $i + j, j \in N_0$ . These graphs are said to be isotropic, i.e. the neighborhood structure is the same for all vertices. In fact, the example that we had introduced earlier was just a special case with  $d = 2$ , and  $N_0 = (0, 1), (1, 0), (0, -1), (-1, 0)$ .  $N_0$  is usually taken to be a symmetric

neighborhood of 0. Let  $\{X_i^n, i \in L_n^d\}$  be a collection of random variables on the lattice  $L_n^d$  with joint distribution defined by

$$Z_n(T)^{-1} \exp(-H(x)T^{-1}) \times \prod_{i \in L_n^d} Q(x_i), \quad (1.2)$$

where  $Q(\cdot)$  is a density on  $\mathfrak{R}$ .  $Z_n(T)$  is called the partition function. The function  $H : \mathfrak{R}^{L_n^d} \rightarrow \mathfrak{R}$  is called the Hamiltonian. We can look at two different types of Hamiltonians, ferromagnetic and antiferromagnetic. Ferromagnetic Hamiltonians increase as neighboring spins become more alike, and antiferromagnetic Hamiltonians increase as neighboring spins become less alike. For instance, if the Hamiltonian in (1.2) is given by

$$H(x) = \sum_{i \in L_n^d} \sum_{j \in N_0} c_j (x_i - x_{i+j})^2, \quad (1.3)$$

then it is ferromagnetic if the  $c_j$ 's are positive, and is antiferromagnetic if the  $c_j$ 's are negative.

## Section 2. Summary of results.

Our study is based on an unpublished manuscript of Grenander and Sethuraman [2]. They defined a class of Gaussian Markov random fields on the lattice  $L_n^d$ . These fields were trigonometrically interpolated to  $[0, 1]^d$  and the convergence of these processes were studied in some simple cases. Grenander and Sethuraman [2] studied probability distributions like those specified in (1.2) with Hamiltonians given by (1.3). The  $c_j$  were allowed to depend on  $n$ . However, we shall assume that the  $c_j$  are fixed. Our fields are also Gaussian Markov random fields. A parameter  $T$  is present which will play the role of temperature in one of our models. We obtained the following results:

1. The variance of the field variables grows faster for  $T > 16$ , than for  $T = 16$ .
2. The limiting specific heat diverges as  $T \rightarrow 16$ .
3. This divergence takes place at a rate proportional to  $(T - 16)^{-1}$ . This is called a power law behavior at  $T = 16$ .
4. The sum of squares of the field random variables satisfy a different central limit law for  $T = 16$  as compared to  $T > 16$ .

## Section 3. A Gaussian model.

We shall define a Gaussian model as shown below. Let  $Y^{(n)}$  be a collection of random variables indexed by the lattice  $L_n^2$  with a joint p.d.f given by

$$\exp \left( \frac{\sum_{i \in L_n^2} \sum_{j \in N_0} c_j (y_i - y_{i+j})^2}{2T} - \frac{1}{2} \sum_{i \in L_n^2} y_i^2 \right) / Z_n(T). \quad (3.1)$$

$N_0 = M_0 \cup -M_0$  is a symmetric neighbourhood of 0. Here  $c_j$  represent the interaction between  $y_i$  and  $y_{i+j}$ , are independent of  $n$  and reflect the isotropic structure of the graph.  $T$  is the usual temperature parameter that we saw in (1.2).

This joint density can be put in the form

$$\exp\left(-\frac{y^{(n)T} A y^{(n)}}{2}\right) / Z_n(T)$$

where  $A$  is of the form

$$\begin{aligned} A_{i,i} &= A_{0,0} = 1 - 2T^{-1} \sum_{j \in N_0} c_j \quad \forall i \in L_n^2 \\ A_{i,i+j} &= 0 \text{ if } j \notin N_0 \\ A_{i,i+j} &= A_{0,j} = \frac{c_j + c_{-j}}{T} \text{ if } j \in N_0 \quad \forall i \in L_n^2. \end{aligned} \tag{3.2}$$

$A$  is a circulant matrix, and has eigen-vectors

$$\zeta_k = (\exp(i2\pi \langle k, j \rangle / n), j \in L_n^2), k \in L_n^2.$$

The eigen values of  $A$ ,  $\lambda_k$  are given by

$$\begin{aligned} \lambda_k &= \sum_{j \in L_n^2} A_{0,j} \exp(i2\pi \langle k, j \rangle / n) \\ &= \sum_{j \in N_0} A_{0,j} \exp(i2\pi \langle k, j \rangle / n) \\ &= A_{0,0} + 2 \sum_{j \in M_0} (c_j + c_{-j}) \cos(2\pi \langle k, j \rangle / n) \\ &= 1 - 2T^{-1} \sum_{j \in M_0} (c_j + c_{-j}) [1 - \cos(2\pi \langle k, j/n \rangle)] \text{ (by (3.2)).} \end{aligned} \tag{3.3}$$

The eigenvalues and eigenvectors depend on  $n$ , which for reasons of clarity has not been included in the notation.

We will now specialize this model to the four neighbor lattice graph that we had defined in Section 1., that is, we will assume that  $M_0 = \{(1, 0), (0, 1)\}$ , we will also assume that all the  $c_j=1$ , and that  $n$  is odd. Since the  $c_j$  are positive, the model is antiferromagnetic.

Now by (3.3), the eigen values of  $A$  are given by

$$\lambda_k = 1 - \frac{4}{T} [1 - \cos(\frac{2\pi k_1}{n}) + 1 - \cos(\frac{2\pi k_2}{n})]. \tag{3.4}$$

Since  $A$  is the inverse of the variance-covariance matrix of the  $Y$ 's, we have that

$$\text{Var}(Y_i^{(n)}) = \text{Var}(Y_0^{(n)}) = n^{-2} \sum_{k \in L_n^2} \frac{1}{\lambda_k}. \quad (3.5)$$

Notice that the  $Y$ 's have a legitimate p.d.f if  $T \geq 16$  since all the eigen values of  $A$  are positive, by (3.4). The two theorems below will study the 'rate of growth of the variance' of the  $Y$ 's.

### Theorem 3.1

Suppose that  $T > 16$ , then, for each  $i \in L_n^2$ ,

$$\text{Var}(Y_i^{(n)}) \rightarrow \int_{[0,1]^2} (1 - 4T^{-1}(1 - \cos(2\pi x) + 1 - \cos(2\pi y)))^{-1} dx dy \text{ as } n \rightarrow \infty. \quad (3.6)$$

Proof. By (3.5),

$$\text{Var}(Y_i^{(n)}) = n^{-2} \sum_{k \in L_n^2} \frac{1}{[1 - \frac{4}{T}[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]]}. \quad (3.7)$$

The rest is trivial.  $\square$

Remark When  $T = 16$  the integral in (3.6) diverges. However, with a different normalisation, the variance of the normalised  $Y$ 's converges to a finite constant, as studied in Theorem 3.2 below.

### Theorem 3.2

Let  $T = 16$ , then

$$\text{Var}((\sqrt{\log n})^{-1} Y_i^{(n)}) \rightarrow 4/\pi \text{ as } n \rightarrow \infty.$$

Proof. By (3.5), we can write

$$\begin{aligned} V_n &= \text{Var}((\sqrt{\log n})^{-1} Y_i^{(n)}) \\ &= (n\sqrt{\log n})^{-2} 4 \sum_{k \in L_n^2} \frac{1}{1 + \cos(2\pi k_1/n) + 1 + \cos(2\pi k_2/n)}. \end{aligned} \quad (3.8)$$

Let  $0 < \epsilon < 1$  and let  $0 < \delta < 1/2$  be such that

$$2\pi^2 x^2(1 - \epsilon) < 1 - \cos(2\pi x) < 2\pi^2 x^2(1 + \epsilon) \text{ if } 0 < |x| < \delta. \quad (3.9)$$

Let  $l_1 = k_1 - (n-1)/2$ ,  $l_2 = k_2 - (n-1)/2$ , and  $a(l_1, l_2) = [1 - \cos(2\pi l_1/n) + 1 - \cos(2\pi l_2/n)]$ . Then  $V_n$  can be written as

$$\begin{aligned}
V_n &= 4(n^2 \log n)^{-1} \left[ \sum_{\substack{-(n-1)/2 \leq l_1 \leq (n-1)/2-1 \\ -(n-1)/2 \leq l_2 \leq (n-1)/2-1}} \frac{1}{[1 - \cos(2\pi l_1/n) + 1 - \cos(2\pi l_2/n)]} \right] \\
&= 4(n^2 \log n)^{-1} \sum_{\substack{|l_1| \geq n\delta \text{ OR} \\ |l_2| \geq n\delta}} \frac{1}{a(l_1, l_2)} \\
&\quad + 4(n^2 \log n)^{-1} \sum_{\substack{0 < |l_1| < n\delta \\ 0 < |l_2| < n\delta}} \frac{1}{a(l_1, l_2)} \\
&= V_{n,1} + V_{n,2}, \text{ respectively.}
\end{aligned} \tag{3.10}$$

Now observe that  $a(l_1, l_2) \geq (1 - \cos(2\pi\delta))$  in the region  $\{|l_1| \geq n\delta \text{ or } |l_2| \geq n\delta\}$ . Hence,  $V_{n,1} \leq 4(\log n)^{-1}(1 - \cos(2\pi\delta))^{-1} = o(1)$  as  $n \rightarrow \infty$ .

Then by (3.9) we have that

$$2\pi^2(1 - \epsilon)(l_1^2 + l_2^2) \leq n^2 a(l_1, l_2) \leq 2\pi^2(1 + \epsilon)(l_1^2 + l_2^2), \tag{3.11}$$

if  $|l_1| < n\delta, |l_2| < n\delta$ .

By (3.11),

$$\begin{aligned}
&4(\pi^2(1 + \epsilon) \log n)^{-1} \sum_{\substack{0 < |l_1| < n\delta \\ 0 < |l_2| < n\delta}} \frac{1}{(l_1^2 + l_2^2)} \\
&\leq V_{n,2} \\
&\leq 4(\pi^2(1 - \epsilon) \log n)^{-1} \sum_{\substack{0 < |l_1| < n\delta \\ 0 < |l_2| < n\delta}} \frac{1}{(l_1^2 + l_2^2)}.
\end{aligned} \tag{3.12}$$

From Lemma 3.1 (below) and (3.12), it follows that  $V_{n,2} \rightarrow 4/\pi$  as  $n \rightarrow \infty$ . Hence, by (3.10) we know that  $V_n$  converges to  $4/\pi$  as  $n \rightarrow \infty$ . This completes the proof of the theorem.  $\square$

### Lemma 3.1

Let  $K_n = (\log n)^{-1} \sum_{\substack{0 < |l_1| < n \\ 0 < |l_2| < n}} \frac{1}{(l_1^2 + l_2^2)}$ . Then,  $K_n \rightarrow 2\pi$  as  $n \rightarrow \infty$ .

Proof. This lemma is proved by finding an upper and lower bound to  $K_n$ . Using the inequality,

$$\frac{1}{(l_1 - 1)^2 + (l_2 - 1)^2} < \frac{1}{x^2 + y^2} \leq \frac{1}{(l_1^2 + l_2^2)} \quad (3.13)$$

for  $(l_1 - 1, l_2 - 1) < (x, y) \leq (l_1, l_2)$ , it follows that  $K_n$  is bounded above by

$$\begin{aligned} & (\log n)^{-1} \int_{1 \leq |x|, |y| < n} \frac{dx dy}{x^2 + y^2} \\ & \leq (\log n)^{-1} \int_{1 \leq x^2 + y^2 \leq 2n^2} \frac{dx dy}{x^2 + y^2} \quad (3.14) \\ & = (\log n)^{-1} \int_0^{2\pi} \int_1^{n\sqrt{2}} \frac{dr d\theta}{r} \quad (\text{by a polar transformation}) \\ & = 2\pi(1 - (\log n)^{-1}(1 - \log \sqrt{2})) = 2\pi + o(1). \end{aligned}$$

To obtain a lower bound, define  $D_n = \{1 \leq x < n \text{ or } 1 \leq y < n \text{ and } (x, y) \neq (1, 1)\}$ . Then by using (3.13) a lower bound on  $K_n$  is obtained as follows

$$\begin{aligned} K_n &= 4(\log n)^{-1} \sum_{\substack{0 < l_1 < n \\ 0 < l_2 < n}} \frac{1}{(l_1^2 + l_2^2)} \\ &= 4(\log n)^{-1} \sum_{\substack{2 \leq l_1 < n+1 \\ 2 \leq l_2 < n+1}} \frac{1}{(l_1 - 1)^2 + (l_2 - 1)^2} \\ &= 4(\log n)^{-1} \left( \sum_{\substack{2 \leq l_1 < n+1 \\ \text{or } 2 \leq l_2 < n+1 \\ (l_1, l_2) \neq (2, 2)}} \frac{1}{(l_1 - 1)^2 + (l_2 - 1)^2} + O(1) \right) \quad (3.15) \end{aligned}$$

$$\begin{aligned} & \geq 4(\log n)^{-1} \left( \int_{D_n} \frac{dx dy}{x^2 + y^2} + O(1) \right) \\ & \geq 4(\log n)^{-1} \left( \int_{1 < x^2 + y^2 \leq n^2} \frac{dx dy}{x^2 + y^2} + O(1) \right) \\ & \asymp 2\pi. \end{aligned}$$



This completes the proof of Lemma 3.1.  $\square$

We shall now study the behavior of the specific heat, which is defined by  $G_n(T) = T \frac{\partial^2}{\partial T^2} \left( \frac{T \log Z_n}{n^2} \right)$ . We shall show that  $G_n(T) \rightarrow G(T)$  where  $G(T)$  is called the limiting specific heat. The limiting specific heat  $G(T)$  is proportional to  $(T - 16)^{-1}$  near  $T = 16$ . This is called a power law behavior at  $T = 16$  in the statistical mechanics literature.

Now note that  $A$  is the inverse of the variance covariance matrix of  $Y$ 's, and the determinant of  $A^{-1}$  is the product of the eigenvalues of  $A^{-1}$ . These eigen values are the reciprocals of the eigen values of  $A$ , since  $A$  is square symmetric. Hence the partition function  $Z_n$  is given by

$$Z_n = (2\pi)^{\frac{n^2}{2}} \times \prod_{\mathbf{k} \in L_n^2} (\lambda_{\mathbf{k}})^{-1/2}.$$

For  $T > 16$ ,

$$\begin{aligned} \log Z_n &= \frac{n^2 \log 2\pi}{2} - 2^{-1} \sum_{\mathbf{k} \in L_n^2} \log(\lambda_{\mathbf{k}}) \\ &= \frac{n^2 \log 2\pi}{2} - 2^{-1} n^2 \left( \frac{1}{n^2} \sum_{\mathbf{k} \in L_n^2} \log(\lambda_{\mathbf{k}}) \right). \end{aligned} \tag{3.16}$$

We shall now calculate the specific heat,

$$G_n(T) = T \frac{\partial^2}{\partial T^2} \left( \frac{T \log Z_n}{n^2} \right).$$

By (3.16) we have

$$\begin{aligned} \frac{\partial}{\partial T} \left( \frac{T \log Z_n}{n^2} \right) &= \frac{\log 2\pi}{2} - 2^{-1} n^{-2} \sum_{\mathbf{k} \in L_n^2} \log(\lambda_{\mathbf{k}}) \\ &\quad - 2(n^2 T)^{-1} \sum_{\mathbf{k} \in L_n^2} \frac{([1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)])}{([1 - \frac{4}{T}[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)])}. \end{aligned}$$

Differentiating again, we have

$$\begin{aligned}
G_n(T) &= \frac{T \partial^2 (T \log Z_n)}{\partial T^2} \\
&= T \left[ \frac{-2}{T^2 n^2} \sum_{k \in L_n^2} \frac{[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]}{[1 - \frac{4}{T}[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]]} \right. \\
&\quad + \frac{2}{T^2 n^2} \sum_{k \in L_n^2} \frac{[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]}{[1 - \frac{4}{T}[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]]} \\
&\quad \left. + \frac{8}{T^3 n^2} \sum_{k \in L_n^2} \frac{[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]^2}{[1 - \frac{4}{T}[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)]]^2} \right].
\end{aligned}$$

The first two terms in the above expression will cancel out and leave us with

$$G_n(T) = \frac{8}{T^2 n^2} \sum_{k \in L_n^2} \frac{([1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)])^2}{([1 - \frac{4}{T}[1 - \cos(2\pi k_1/n) + 1 - \cos(2\pi k_2/n)])^2}.$$

This will converge to

$$G(T) = \frac{8}{T^2} \int_0^1 \int_0^1 \frac{[1 - \cos(2\pi x) + 1 - \cos(2\pi y)]^2}{[1 - 4T^{-1}[1 - \cos(2\pi x) + 1 - \cos(2\pi y)]]^2} dx dy \text{ as } n \rightarrow \infty. \quad (3.17)$$

This integral is divergent at  $T = 16$  and in fact,  $G(T) \propto (T - 16)^{-1}$  for  $T > 16$  as  $T \rightarrow 16$ .

Thus the specific heat for this model diverges.

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#### Section 4.

In this section, we shall study the behavior of the sum of squares of the random variables  $Y_i^{(n)}$ . We shall show that this sum of squares obeys a different central limit theorem when  $T = 16$ , as compared to  $T > 16$ . The reason for this result, is due to the asymptotic behavior of the eigen values  $\lambda_k$  being different at  $T = 16$ .

Define

$$Q_n = \sum_{k \in L_n^2} (Y_k^{(n)})^2.$$

Then,  $Q_n \stackrel{\mathcal{D}}{=} \sum_{k \in L_n^2} V_k / \lambda_k$ , where  $V_k$  are i. i. d. random variables with a  $\chi_1^2$  distribution.

#### Theorem 4.1

If  $T > 16$ , then

$$\frac{Q_n - \sum_{k \in L_n^2} \lambda_k^{-1}}{\sqrt{3 \sum_{k \in L_n^2} \lambda_k^{-2}}} \xrightarrow{\mathcal{D}} N(0, 1) \text{ as } n \rightarrow \infty.$$

Proof. We shall check that Liapounov's conditions [4] for asymptotic normality hold. Let  $K_n = \text{Var}(Q_n)$ , then  $(Q_n - E(Q_n)) / \sqrt{K_n} \xrightarrow{\mathcal{D}} N(0, 1)$  as  $n \rightarrow \infty$  if  $K_n^{-2} \sum_{k \in L_n^2} E(V_k - 1)^4 / \lambda_k^4 \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $V_k$  are i. d.,  $E(V_k - 1)^4$  is a constant independent of  $k$ . The terms  $K_n$  and  $\sum_{k \in L_n^2} E(V_k - 1)^4 / \lambda_k^4$  are both  $O(n^2)$ . Hence Liapounov's condition holds.

This proves the theorem  $\square$

We shall now study the behavior of  $Q_n$  when  $T = 16$ . Theorem 4.2 is a central limit theorem for  $Q_n$  when  $T = 16$ .

#### Theorem 4.2

If  $T = 16$ , then

$$(Q_n - E(Q_n)) / n^2 \xrightarrow{\mathcal{D}} B \text{ as } n \rightarrow \infty \text{ where } B \text{ has an m.g.f. given by}$$

$$\psi(t) = \exp\left(-2^{-1} \sum_{j=2}^{\infty} \frac{(8t)^j}{j(j-1)\pi^{2j-1}}\right).$$

Proof. We shall calculate the m.g.f of  $(Q_n - E(Q_n)) / n^2$  and show that this m.g.f. converges to the m.g.f. of  $B$ . This is sufficient to prove convergence in distribution of  $(Q_n - E(Q_n)) / n^2$  to  $B$ .

Let  $\psi_n(t)$  be the m.g.f. of  $Q_n - E(Q_n) / n^2$ . Then,

$$\psi_n(t) = \frac{\exp(-c_n^{-1} \sum_{k \in L_n^2} \lambda_k^{-1})}{\prod_k (1 - 2t\lambda_k/n^2)^{1/2}}.$$

Taking logarithms in the above,

$$\begin{aligned}
\log(\psi_n(t)) &= -n^{-2} \sum_{k \in L_n^2} \lambda_k^{-1} - 2^{-1} \sum_{k \in L_n^2} \log(1 - 2t/\lambda_k n^2) \\
&= 2^{-1} \sum_{j=2}^{\infty} \frac{(2t)^j}{j} \left( n^{-2j} \sum_{k \in L_n^2} \lambda_k^{-j} \right).
\end{aligned} \tag{4.1}$$

By Lemma 4.1 the term

$$n^{-2j} \sum_{k \in L_n^2} \lambda_k^{-j} \rightarrow (4^j \pi^{2j-1} (j-1)) \text{ as } n \rightarrow \infty.$$

Hence, by equation (4.1) and the above,

$$\psi_n(t) \rightarrow \exp\left(-4^{-1} \sum_{j=2}^{\infty} \frac{(16t/\pi^2)^j}{j(j-1)}\right)$$

This completes the proof of the theorem.  $\square$

Lemma 3.2 Let  $T = 16$  and let  $j \geq 2$  be an integer. Then

$$n^{-2j} \sum_{k \in L_n^2} \lambda_k^{-j} \rightarrow (4^j \pi^{2j-1} (j-1)) \text{ as } n \rightarrow \infty.$$

Proof. The proof of this lemma is substantially the same as the proof of Theorem 3.1 and so we will only sketch the proof. Since  $T = 16$ ,

$$n^{-2j} \sum_{k \in L_n^2} \lambda_k^{-j} = 8^j n^{-2j} \sum_{\substack{0 < |l_1| < n/2 \\ 0 < |l_2| < n/2}} \frac{1}{(1 - \cos(2\pi l_1/n) + 1 - \cos(2\pi l_2/n))^j}. \tag{4.2}$$

Let  $\delta$  be as in the proof of Theorem 3.1. Then the right hand side of (4.2) is approximately equal to

$$\begin{aligned}
&8^j (2\pi^2)^{-j} \sum_{\substack{0 < |l_1| < n \\ 0 < |l_2| < n}} \frac{1}{(l_1^2 + l_2^2)^j} \\
&\asymp 8^j (2\pi^2)^{-j} \int_{1 \leq x^2 + y^2 \leq 2n^2 \delta^2} \frac{1}{(x^2 + y^2)^j} \\
&\asymp 8^j (2\pi^2)^{-j} \pi / (j-1).
\end{aligned}$$

This completes the proof of the lemma.  $\square$

Remark. The limit in distribution of  $Q_n$  is different for  $T = 16$  as compared to  $T > 16$ , and the normalization constant for  $Q_n$  at  $T = 16$  is  $n^2$  instead of  $n$  when  $T > 16$ . This suggests that the field random variables  $Y_k^{(n)}$  vary much more for  $T = 16$  than for  $T > 16$ .

However the critical behavior is at the endpoint of definition of the model and so it is our opinion that this does not seriously restrict the application of the Gaussian model in pattern analysis.

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<p>In this paper we discuss a Gaussian random field that arises in pattern analysis. This random field exhibits phase transitive behavior for a particular value of the temperature parameter. We analyze this kind of non singular behavior and the effect that it has on the field random variables. The limiting specific heat also exhibits a phase transition with a power law behavior.</p>			
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