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TECHNICAL REPORT ARCCB-TR-91018

# GOOD, BETTER, AND BEST MESHES IN PIECEWISE LINEAR INTERPOLATION

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20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

We first describe the "good" meshes obtained by C. deBoor which roughly equidistribute the classical error bound in piecewise linear interpolation. We then proceed to obtain "better" meshes which more precisely equidistribute this bound. Finally, we obtain meshes which equidistribute the maximum absolute error (not bound) and refer to these meshes as "best." Some graphical results are then presented comparing these three types of meshes.

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## INTRODUCTION

The classical bound on the error in linear interpolation of function f on interval (a,b) is given by

$$\frac{1}{8}(b-a)^2\|f''\|_{(a,b)}$$

where

$$\|f''\|_{(a,b)} = \sup_{a\leqslant x\leqslant b} |f''(x)|$$

We wish to obtain meshes ($x_i$, $1 \leqslant i \leqslant n$) whicn will somehow equilibrate the "error" over each subinterval.

## GOOD MESHES

C. deBoor (ref 1) has supplied us with a computationally simple method for generating what he calls "good" meshes. His idea is to make the classical bound roughly constant

$$\frac{1}{8}(x_{i+1}-x_i)^2\|f''\|_{(x_i,x_{i+1})} = \text{constant} \qquad 1 \leqslant i < n$$

This is equivalent to

$$(x_{i+1}-x_i)\|f''\|^{\frac{1}{2}}_{(x_i,x_{i+1})} = \text{constant}$$

or

$$\int_{x_i}^{x_{i+1}} \|f''\|^{\frac{1}{2}}_{(x_i,x_{i+1})} dx = c$$

As n becomes large and $x_{i+1}-x_i \rightarrow 0$ for all i, we approximate the integrand by $|f''(x)|^{\frac{1}{2}} = g(x)$.

Thus, we have a relatively simple problem to solve

$$\int_{x_i}^{x_{i+1}} g(x)dx = c \qquad 1 \leqslant i < n$$

If we define $G(x) = \int_{x_1}^{x} g(t)dt$, we have $G(x_i) = (i-1)c$, therefore

$$\frac{G(x_i)}{G(x_n)} = \frac{i-1}{n-1} \text{ and } x_i = G^{-1}\left[\frac{i-1}{n-1}\, G(x_n)\right] \quad 1 < i < n$$

In practice, we may only have a positive, continuous, piecewise linear estimate of g over some mesh u. We denote this estimate of g by v. G as defined by

$$G(x) = \int_{u_1}^{x} v(t)dt \qquad u_1 < x < u_m$$

would then be piecewise quadratic and invertible in the following manner:

$$G^{-1}(G^*) = x^* = u_i + 2\,\frac{(G^*-G_i)}{v_i+\sqrt{D}}$$

where

$$G_1 = 0,\ G_{j+1} = G_j + (u_{j+1}-u_j)(v_j+v_{j+1})/2 \qquad 1 \leqslant j < m$$

$$G_i \leqslant G^* \leqslant G_{i+1}$$

$$\rho = (G^*-G_i)/(G_{i+1}-G_i)$$

and

$$D = (1-\rho)v_i^2 + \rho v_{i+1}^2$$

All this is common knowledge. Unfortunately, good meshes do not always seem quite as good as we might like them to be. The shorter subintervals have a fairly uniform error bound pattern, but the lengths of the longer subintervals are overestimated, yielding larger error bounds. This is due to the fact that the integral of the norm of g is underestimated by the integral of g. In fact, it is easy to prove that for $f(x) = x^p (p > 2,\ 0 \leqslant x \leqslant 1)$, the largest error bound on a good mesh is exactly equal to the largest error bound on the corresponding uniform mesh ($x_{i+1}-x_i$ = const).

In order to get what we might call "better" meshes, we go back to the original problem:

Find n-2 x's ($x_1$ and $x_n$ fixed) such that

$$(x_{i+1}-x_i)\|g\|_{(x_i,x_{i+1})} = \int_{x_i}^{x_{i+1}} \|g\|_{(x_i,x_{i+1})}dx = c \qquad 1 \leqslant i < n$$

This problem is described in Reference 1 as being rather difficult to solve in general. Even if we knew what c was, solving

$$(x_{i+1}-x_i)\|g\|_{(x_i,x_{i+1})} = c$$

for $x_{i+1}$ given $x_i$ would not be easy. The problem obtained upon substituting v for g, however, is quite easy to solve (ref 2). In addition, if v is a very good approximation to g (with m >> n), we get a virtually constant error bound for the entire mesh!

We refer to the following equation as the "stepping" equation:

$$(\beta-\alpha)\|v\|_{(\alpha,\beta)} = c$$

The solution of the stepping equation for $\beta$ given $\alpha$ and c represents the central part of our algorithm for obtaining better meshes. Although the stepping equation is nonlinear in $\beta$, the piecewise linearity of v enables us to solve it noniteratively. For given $\alpha$ and c, we solve the stepping equation in the following manner. Suppose $\alpha \in (u_i,u_{i+1})$ and we have located j such that

$$(u_j-\alpha)\|v\|_{(\alpha,u_j)} < c < (u_{j+1}-\alpha)\|v\|_{(\alpha,u_{j+1})}$$

Hence,

$$\beta \in (u_j,u_{j+1})$$

To locate j, we simply search from left to right, computing the norms as we go and checking the previous inequality. We use

$$\|v\|_{(\alpha,u_{i+1})} = \mathrm{Max}(v(\alpha),v_{i+1})$$

and

$$\|v\|_{(\alpha,u_{j+1})} = \mathrm{Max}(\|v\|_{(\alpha,u_j)},v_{j+1})$$

Now if

$$\|v\|(\alpha,u_j) = \|v\|(\alpha,u_{j+1})$$

we have

$$(\beta-\alpha)\|v\|(\alpha,u_j) = c$$

hence

$$\beta = \alpha + c/\|v\|(\alpha,u_j)$$

But if

$$\|v\|(\alpha,u_j) < \|v\|(\alpha,u_{j+1})$$

there is a

$$t \in (u_j,u_{j+1})$$

such that

$$\|v\|(\alpha,u_j) = \|v\|(\alpha,t) = v(t)$$

and this t is given by

$$t = u_j + (\|v\|(\alpha,u_j)-v_j)/s$$

where

$$s = (v_{j+1}-v_j)/(u_{j+1}-u_j)$$

Now if

$$(t-\alpha)\|v\|(\alpha,u_j) > c$$

$\beta$ must lie to the left of t and

$$\beta = \alpha + c/\|v\|(\alpha,u_j)$$

as before, but if

$$(t-\alpha)\|v\|(\alpha,u_j) < c$$

$\beta$ lies to the right of t and

$$(\beta-\alpha)\|v\|(\alpha,\beta) = c$$

However, in this case,

$$\|v\|(\alpha,\beta) = v(\beta) = v_j + s(\beta-u_j)$$

Therefore,

$$(\beta-\alpha)(v_j+s(\beta-u_j)) = c$$

or

$$(\beta-\alpha)(v_j+s(\beta-\alpha+\alpha-u_j)) = c$$

or

$$s(\beta-\alpha)^2 + k(\beta-\alpha) - c = 0$$

where

$$k = v_j + s(\alpha-u_j)$$

Solving this simple quadratic equation for $\beta - \alpha$ yields

$$\beta = \alpha + (\sqrt{k^2+4sc}-k)/(2s) \text{ for } k < 0$$

and

$$\beta = \alpha + 2c/(k+\sqrt{k^2+4sc}) \text{ for } k > 0$$

Having elaborated the solution to the stepping equation for arbitrary c, we now consider obtaining the correct value of c by defining the function $\mu$

$$\mu(c) = \nu-n$$

where $\nu$ is the number of x's we get by solving the stepping equation $\nu-2$ times ($x_1$ and $x_\nu = x_n$ being fixed). It is intuitively clear that for small c, $\mu$ will be positive and for large c, $\mu$ will be negative. Since $\mu$ is a step function, we are interested only in its leftmost zero, the correct value of c. When the correct value of c has been obtained, we shall have concurrently obtained the better or uniform error bound mesh. Therefore, we see that to find the better mesh, we need only solve a single nonlinear equation in a single unknown (by modified bisection), where each evaluation of $\mu$ involves an O(m) search through the (u,v) data and O(n) solutions of simple linear or quadratic equations.

Suppose we are not satisfied with better meshes and decide to take the additional step of finding the best possible mesh. We define this best mesh as one in which the maximum absolute error is constant, independent of subinterval. Now we must work in terms of exact error instead of error bounds. The exact error in linear interpolation of f on interval (a,b) is given by

$$e(x) = \frac{b-x}{b-a} \int_a^x (t-a)f''(t)dt$$

$$+ \frac{x-a}{b-a} \int_x^b (b-t)f''(t)dt$$

If $x_m$ is the maximizing or minimizing point of e,

$$e'(x_m) = 0$$

implies

$$\int_a^{x_m} (t-a)f''(t)dt = \int_{x_m}^b (b-t)f''(t)dt$$

which, in turn, implies that

$$e(x_m) = \int_a^{x_m} (t-a)f''(t)dt = \int_{x_m}^b (b-t)f''(t)dt$$

These last two equations tell us that for given a, E, and f", we can, in principle, solve

$$E = \int_a^{x_m} (t-a)f''(t)dt$$

for $x_m$ and then solve

$$E = \int_{x_m}^b (b-t)f''(t)dt$$

for b. We therefore see that finding the best mesh is not a very different process from finding the better mesh. The major difference is that two equations must be solved in the stepping process instead of one. This stepping process is further simplified to solving simple quadratic equations if we use a piecewise constant approximation to f".

The first half of the stepping process amounts to solving

$$R(\beta) = \pm E \text{ for } \beta$$

given $\alpha$ where

$$R(\beta) = \int_\alpha^\beta (t-\alpha)f''(t)dt$$

if $f''(x) = c_i$ on $(u_i,u_{i+1})$, we have

$$R(\beta) = \int_\alpha^{u_i} (t-\alpha)f''(t)dt + \int_{u_i}^\beta (t-\alpha)c_i dt \quad (\beta\epsilon(u_i,u_{i+1}))$$

$$R(\beta) = R(u_i) + \frac{1}{2} c_i(t-\alpha)^2 \Big|_{u_i}^{\beta}$$

$$= R(u_i) + \frac{1}{2} c_i((\beta-\alpha)^2-(u_i-\alpha)^2)$$

$$= R(u_i) + \frac{1}{2} c_i(\beta-u_i)(\beta+u_i-2\alpha)$$

The recursion for R is

$$R_{i+1} = R_i + \frac{1}{2} c_i(u_{i+1}-u_i)(u_i+u_{i+1}-2\alpha)$$

The second half of the stepping process amounts to solving

$$S(\gamma) = \pm E \text{ for } \gamma$$

given $\beta$ where

$$S(\gamma) = \int_\beta^\gamma (\gamma-t)f''(t)dt$$

For $\gamma \in (u_i,u_{i+1})$, we have

$$S(\gamma) = \int_\beta^{u_i} (\gamma-t)f''(t)dt + \int_{u_i}^\gamma (\gamma-t)c_i dt$$

$$= \int_\beta^{u_i} (u_i-t+\gamma-u_i)f''(t)dt - \frac{1}{2} c_i(\gamma-t)^2\Big|_{u_i}^{\gamma}$$

$$= S(u_i) + (\gamma-u_i) \int_\beta^{u_i} f''(t)dt + \frac{1}{2} c_i(\gamma-u_i)^2$$

Letting

$$T_i = \int_\beta^{u_i} f''(t)dt$$

we have

$$T_{i+1} = T_i + c_i(u_{i+1}-u_i)$$

and the recursion for S

$$S_{i+1} = S_i + (u_{i+1}-u_i)T_i + \frac{1}{2} c_i(u_{i+1}-u_i)^2$$

We mention in passing that after we have computed $\beta$, and as we search for $\gamma$, we will need to compute additional $\beta$'s if there are inflection points present in f.

Having given this rather brief sketch of the best mesh process, we proceed to some interesting graphical results shown in Figures 1 through 4. We have obtained good, better, and best meshes for the simple test functions $x^3(1-x)^6$ and $x^{10}$, and we may describe our little odyssey as follows. Starting with deBoor's good mesh with its predictably large bounds on the long subintervals, we then proceed to flatten these bounds out almost perfectly through the better mesh. Proceeding one additional step to obtain the best mesh (whose maximum error can indeed be seen to be constant), we then note the striking similarity between the good mesh and the best mesh. One might therefore say that in going from good to better to best meshes, we have come nearly full circle and can see the wisdom of the old French proverb: "The more things change, the more they remain the same." It would seem that the only thing deBoor can be criticized for is excessive modesty in referring to his meshes as merely good when, in fact, they are nearly best.

## REFERENCES

1. C. deBoor, A Practical Guide to Splines, Springer-Verlag, New York, 1978.

2. R.W. Soanes, "Uniform Error Bound Meshes in Piecewise Linear Interpolation," Transactions of the Sixth Army Conference on Applied Mathematics and Computing, ARO-89-1, U.S. Army Research Office, Research Triangle Park, NC, 1989.

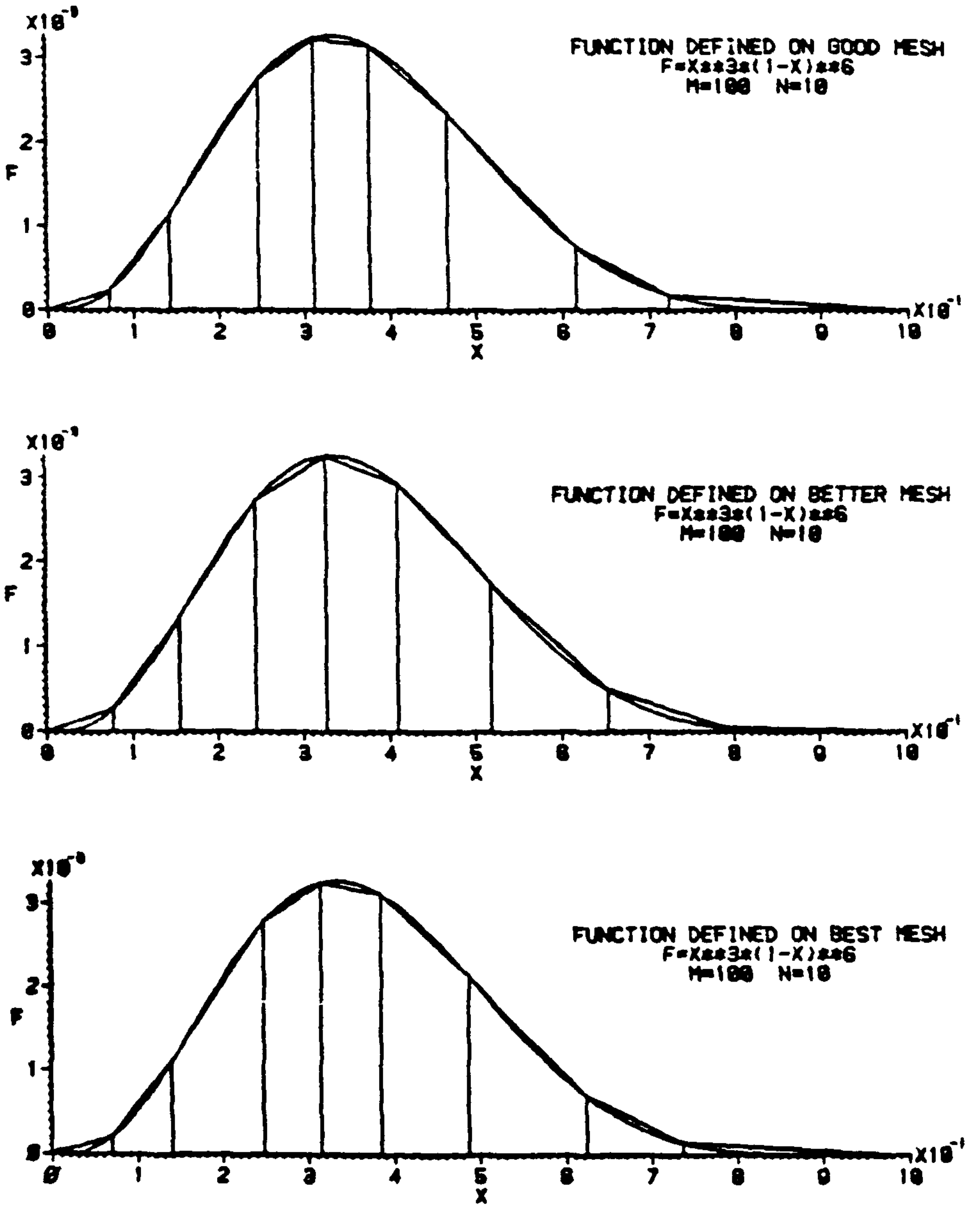


Figure 1. First test function defined on good, better, and best meshes.

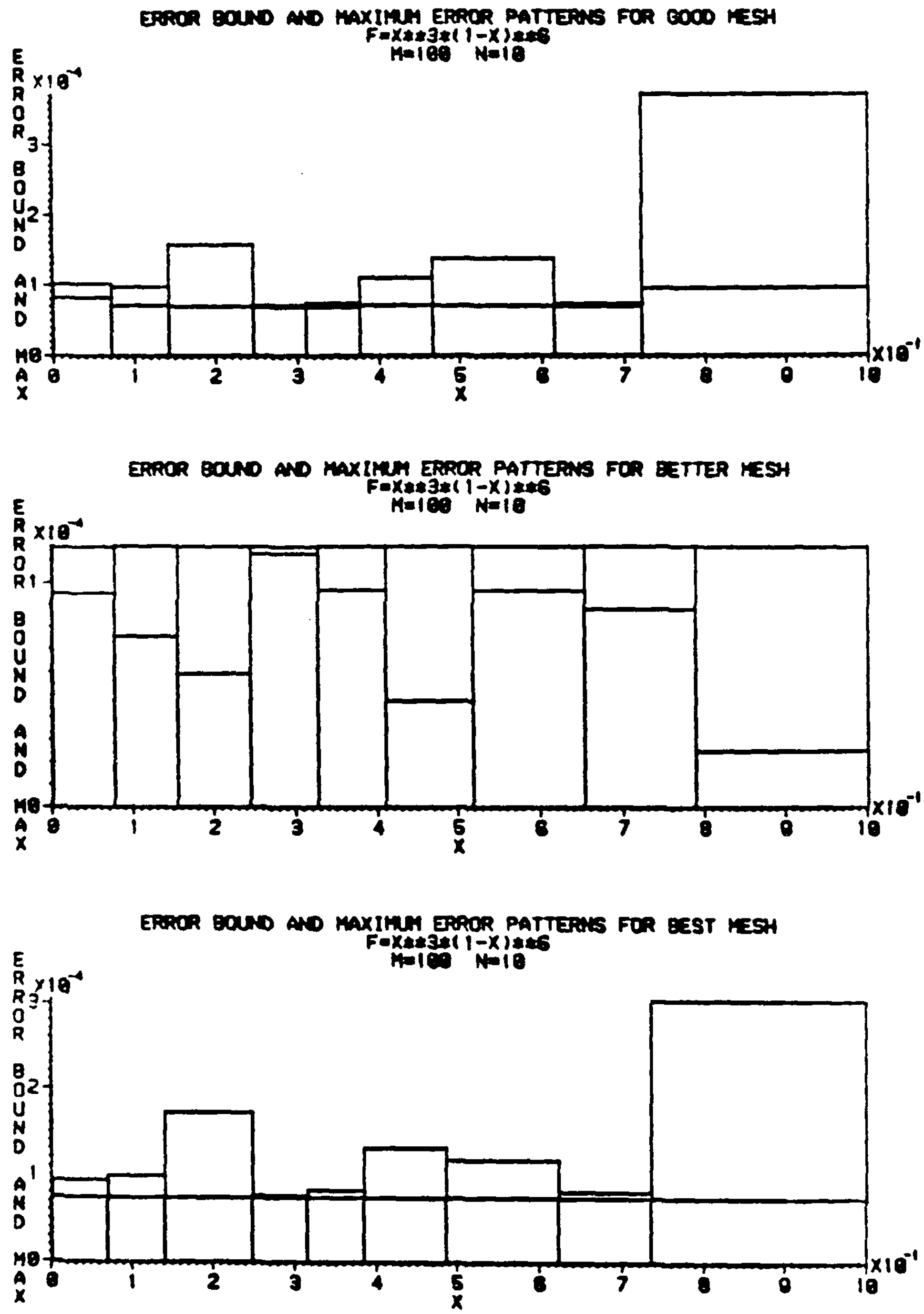


Figure 2. First test function error bound and maximum error patterns for good, better, and best meshes.

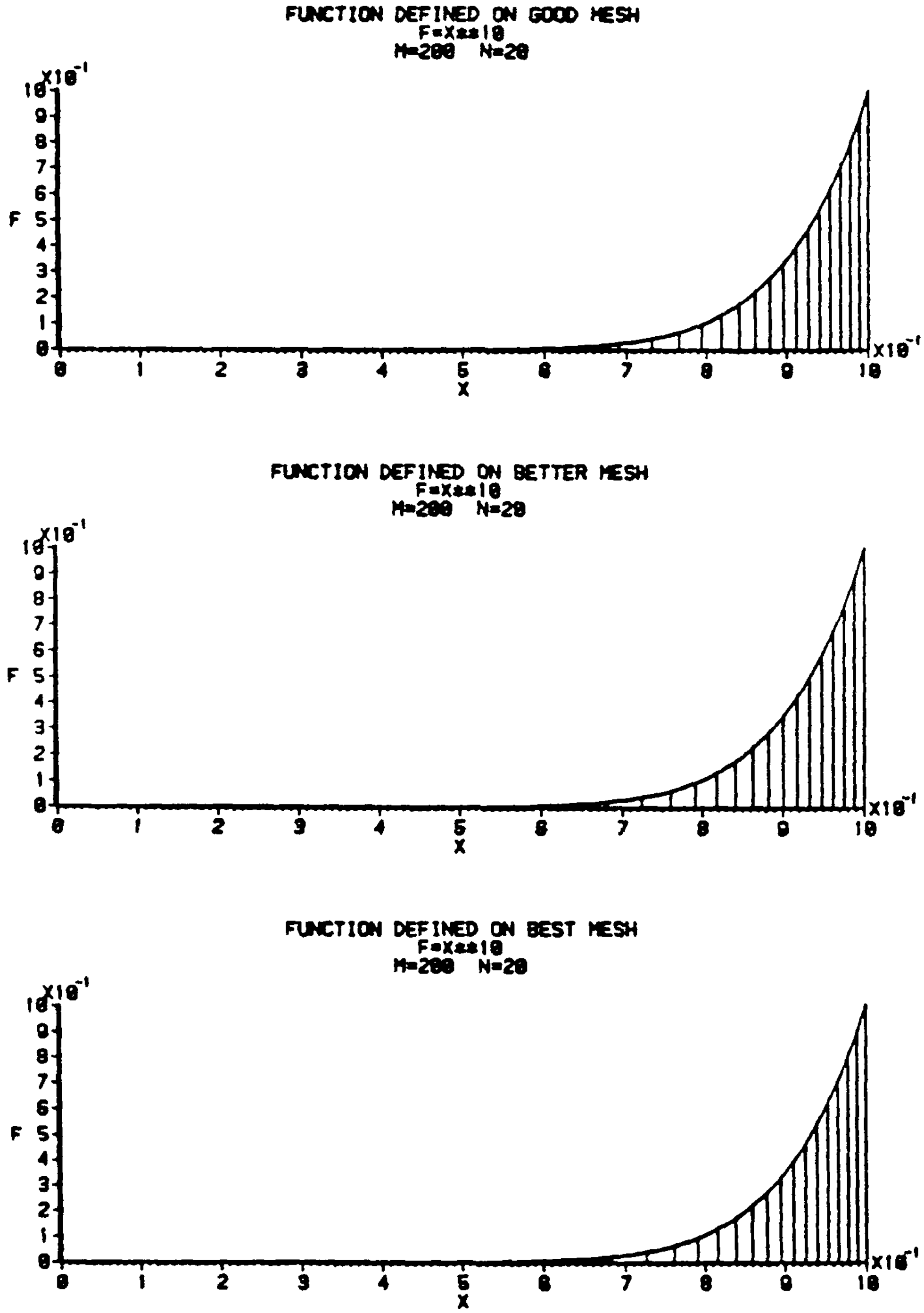


Figure 3. Second test function defined on good, better, and best meshes.

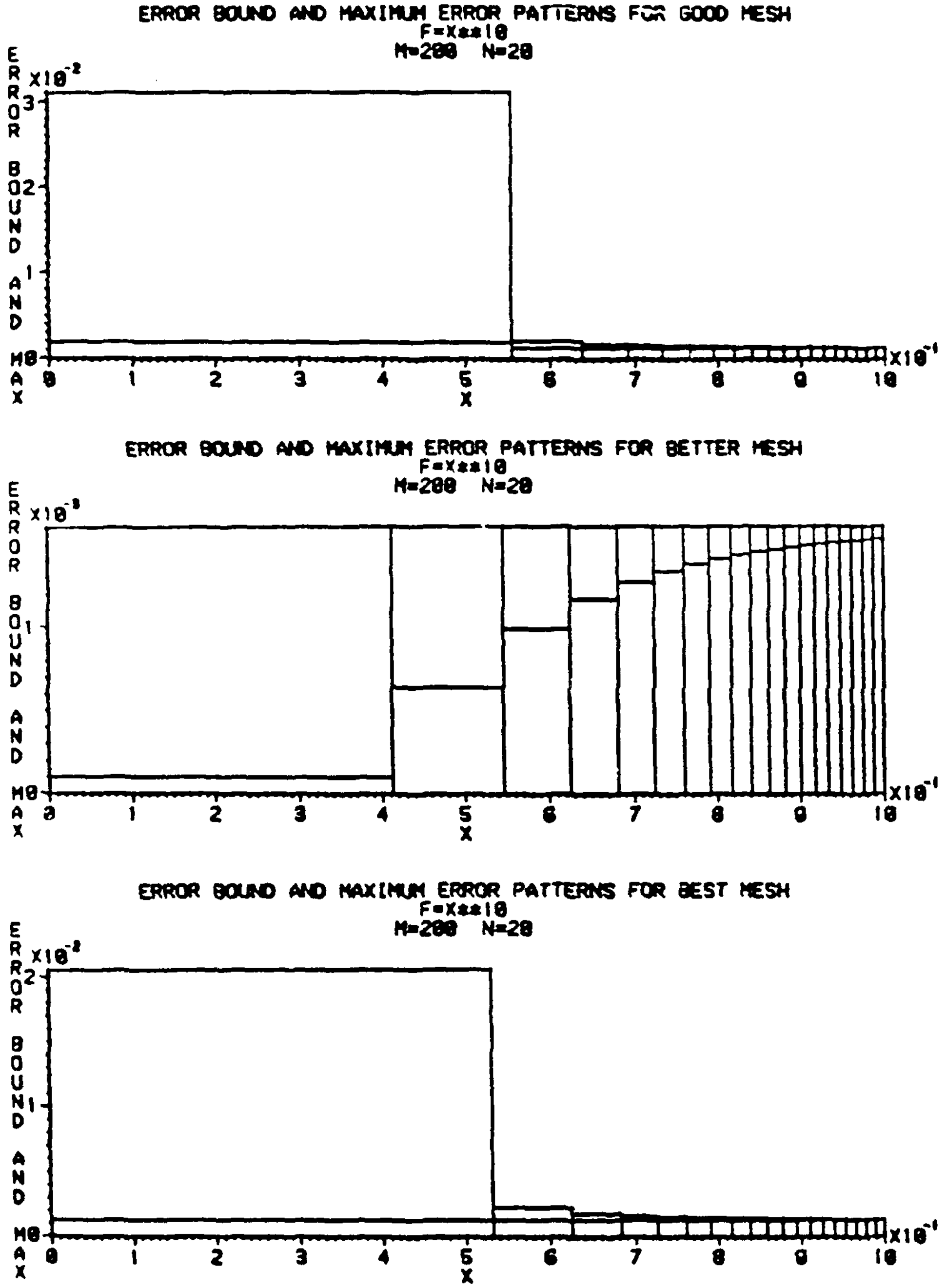


Figure 4. Second test function error bound and maximum error patterns for good, better, and best meshes.

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