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A GEOMETRIC TREATMENT OF IMPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS[1]

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ABSTRACT. A differential-geometric approach for proving the existence and uniqueness of solutions of implicit differential-algebraic equations is presented. It provides for a significant improvement of an earlier theory developed by the authors as well as for a completely intrinsic definition of the index of such problems. The differential-algebraic equation is transformed into an explicit ordinary differential equation by a reduction process that can be abstractly defined for specific submanifolds of tangent bundles here called reducible π-submanifolds. Local existence and uniqueness results for differential-algebraic equations then follow directly from the final stage of this reduction by means of an application of the standard theory of ordinary differential equations.

1. Introduction.

An implicit differential equation

$$F(t, x, \dot{x}) = 0, \quad F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \tag{1.1}$$

with a sufficiently smooth mapping F is usually referred to as a differential-algebraic equation (DAE) when the partial derivative $D_pF(t, x, p)$ has constant rank $\rho < n$ on the (open) domain of F, or, more generally, when the constant rank condition holds in some open neighborhood of $F^{-1}(0)$ in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$.

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In [11] we presented a general existence and uniqueness theory for such DAEs which confirms the commonly accepted idea that, in some way, any reasonable DAE can be reduced – at least locally – to an explicit ODE. Broadly speaking, a sequence of equations

$$(1.2) \qquad F_j(t,x,\dot{x}) = 0, \; F_0 \equiv F, \; F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \; j = 0,1,\dots,$$

is constructed provided each equation is a DAE: that is, D_pF_j has constant rank. The sequence terminates with $j = \nu$ when D_pF_ν has full rank. Then the equation (1.2) with $j = \nu$ is reducible to an explicit ODE and ν is called the index of the problem. The solutions of (1.1) automatically satisfy all equations (1.2) of the sequence and, conversely, every solution of a particular equation (1.2) with consistent initial data solves (1.1). Here the consistency condition reflects the well-known fact that, in contrast with ODEs, a DAE (1.1) does not have locally defined solutions for arbitrary initial data in $F^{-1}(0)$. The reduction process fails when one of the equations (1.2) is a singular differential equation, for which the, as yet quite incomplete, existence theory is substantially different from that for ODEs or DAEs (see [10]).

The concept of an index plays an important role in the study of DAEs. For linear DAEs

$$(1.3) \qquad F(t,x,p) = Ax + Bp - f(t), \; f : \mathbb{R} \to \mathbb{R}^n$$

with constant $n \times n$ matrices A and B such that rank $B = \rho < n$, the index has been defined (see [3], [7]) as the index of the matrix pencil (A, B) in the sense of [5]. For linear systems (1.3) the theory in [11] requires the same hypotheses and leads to the same value of the index as the classical theory. For the general case (1.1) various authors (see, e.g., the monographs [2], [9], and [6], [8] for references) have introduced definitions which generalize the linear index. These definitions serve well for classifying DAEs and for characterizing properties of numerical procedures, but, by themselves, they do not incorporate any existence results for solutions of the equations.

Although the reduction process in [11] provides a general setting for a solvability theory of DAEs, it turns out not to be intrinsic since the sequence $\{F_j\}$ is not uniquely determined by the mapping F. In fact, for the construction of the F_j, various equally reasonable approaches are feasible and it is by not clear whether the resulting solvability criteria are equivalent and, in particular, whether the index of the problem is independent of $\{F_j\}$. A principal aim of this paper is the development of a completely intrinsic geometric approach for the existence and uniqueness of DAEs that extends the analytic treatment of [11].

By adding, as usual, the equation $\dot{t} = 1$ we can transform (1.1) into an autonomous problem. Thus without loss of generality we consider here equations of the form

$$(1.4) \qquad F(x, \dot{x}) = 0, \quad F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.$$

Then, formally, and without any of the technicalities involved, our approach is based on the following general idea: As in [11] suppose that the zero set $M = F^{-1}(0)$ is a smooth submanifold of $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. In practice, this is guaranteed by assuming that F is a submersion on its zero set. Now (1.4) may be written in the form

$$(1.5) \qquad (x, \dot{x}) \in M$$

which in turn implies that any solution $x = x(t)$ of (1.4) has to satisfy $x(t) \in W = \pi(M)$ where $\pi : T\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection onto the first factor. If W is a submanifold of $\mathbb{R}^n$ then $(x(t), \dot{x}(t))$ belongs to the tangent bundle TW of W here identified with a submanifold of $T\mathbb{R}^n$. In other words, any solution $x = x(t)$ of (1.4) has to satisfy not only (1.5) but also the more restricted inclusion $(x(t), \dot{x}(t)) \in M_1 = TW \cap M$. The step from (1.5) to this new restricted problem represents a reduction of (1.4) similar to that involved in the construction of the equation (1.2) with $j = 1$. Hence it is appropriate to call M_1 the first reduction of M. If M_1 and $W_1 = \pi(M_1)$ are submanifolds of $T\mathbb{R}^n$ and $\mathbb{R}^n$, respectively, then the reduction can be continued a step further and the same argument yields $(x(t), \dot{x}(t)) \in M_2 \equiv TW_1 \cap M_1$. Hence, under suitable conditions, we obtain a decreasing sequence $M_0 = M, M_1, M_2, \ldots$ of manifolds and find that for every solution $x = x(t)$ of (1.4), $(x(t), \dot{x}(t))$ has to belong to the 'core' $\bigcap_{j \geq 0} M_j$ of M.

It is reasonable to expect the decreasing sequence of manifolds M_j to become stationary. The first ν such that $M_\nu = M_{\nu+1}$ is then called the index of (1.4). This index definition was proposed by S. Reich (see [12], [13]) without further results about the existence of the manifolds or their properties. Clearly the practical value of the approach depends largely upon satisfactory answers to the following two questions:

1) Are there realistic conditions for ensuring that M_{j+1} and W_{j+1} are submanifolds of M_j and W_j or, equivalently of $T\mathbb{R}^n$ and $\mathbb{R}^n$, respectively?

2) Does the reduction lead to a conclusion about the existence of solutions of (1.4)?

There are significant technical and conceptual difficulties in providing positive answers to both these questions. As a typical example consider the problem of deciding whether $M_{j+1} = TW_j \cap M_j$ is a submanifold which may appear to be resolvable by a standard transversality argument. But one soon finds that dimensional considerations dictated by the second question rule out at once transversality of M_j and TW_j in the natural ambient manifold TW_{j-1}. In fact, prior to anything else, it turns out that the global approach in our expository sketch must be replaced by a local one. This also makes the 'subimmersion theorem' available which turns out to play here a critical role.

After some preliminaries in Section 2, a framework for the local approach is established in Section 3. Then, in Section 4 a reducibility concept is introduced and geometric conditions for reducibility are given which then provide a positive answer to the first question (or, rather, its analog in the local setting). Next, in Section 5, we settle (the analog of) the second question by means of a concept of complete reducibility. The entire development makes no reference to DAEs although occasionally we use a simple DAE to motivate some of the general definitions. However, Section 6 does present the application of the general theory to DAEs. In particular, the connection to Reich's index and an existence theory for (1.4) is obtained as an immediate corollary to the result that, locally, the analog of the above indicated 'core' of $M = F^{-1}(0)$ is the image of a section of a suitable tangent bundle, and therefore that, on this core the DAE (1.4) reduces locally to an explicit ODE. For practical applications and, in particular, for numerical purposes it is desirable to express the various geometric conditions in analytic terms. This is the topic of Section 7 and the results given there allow, in Section 8, for a comparison between the geometric theory and the results of [11].

Altogether, it appears that the new geometric treatment offers three important advantages. First of all, unlike in [11], constant rank conditions are required only where they are unquestionably needed; that is, on appropriate submanifolds and not on open subsets of the ambient space. For instance, it becomes legitimate to call (1.4) a differential-algebraic equation if the constant rank condition for D_pF is satisfied only at points of $M = F^{-1}(0)$. This was suspected to be true but could not be proved by the methods of [11]. Next, as discussed in Section 8, a rather complicated condition in [11] involving second derivatives can now be replaced by a much simpler one involving only first derivatives. In fact, it was not noticed in [11], and could hardly have been expected there, that the second derivatives play an entirely passive role. Finally, the geometric treatment achieves its original goal of supplying an intrinsic definition for the index of DAEs which directly incorporates a local existence theory for the equations. It appears to be fair to say that as long as singularities are ruled out, the theory provides a nearly optimal answer to the existence and uniqueness question for differential-algebraic equations.

2. Preliminaries.

Throughout this presentation we consider only finite dimensional, separable, Hausdorff manifolds which, for simplicity of exposition, are assumed to be of class C^∞. But it should be evident that finite regularity will suffice in general. As usual, it is explicitly allowed that different connected components of a manifold M may have different dimensions and we denote the maximal dimension of the connected components of M by $\dim M$. When all connected components of M have the same dimension we follow [4] and say that M has pure dimension $\dim M$.

For any n-dimensional manifold X we denote the tangent bundle by TX and the canonical projection by $\pi : TX \to X$. Points of TX will be written in the form (x, p) with $x \in X$ and $p \in T_xX$. In particular, for simplicity, we always write $T\mathbb{R}^n$ for $\mathbb{R}^n \times \mathbb{R}^n$. For any submanifold Y of X and any point $x \in Y$ the tangent space T_xY is canonically identified with a subspace of T_xX and hence the tangent bundle TY is identified with a submanifold of TX. Note that this is not a sub-bundle since the base manifolds X and Y of TX and TY, respectively, are different.

As in the case of manifolds we assume for simplicity that all mappings under consideration will be of class C^∞ although, once again, this condition can be reduced easily. For convenience, the notation $f : \mathbb{R}^k \to \mathbb{R}^m$ will be used for any mapping f defined on some open subset U of $\mathbb{R}^k$ with values in $\mathbb{R}^m$, even if $U \neq \mathbb{R}^k$. This slight abuse of notation should not lead to any confusion.

Recall that a mapping $f : X \to Y$ between the manifolds X and Y is a subimmersion if the rank of the linear map $T_x f : T_x X \to T_{f(x)} Y$ is constant in some open neighborhood of every point $x \in X$. Then the rank r of $T_x f$ has a constant value on each connected component Ξ of X called the rank of f on Ξ. For simplicity we call $f : X \to Y$ a local subimmersion at $x \in X$ if there exists an open neighborhood U of x in X such that the restriction of f to the submanifold U of X is a subimmersion on U.

The following subimmersion theorem plays a key role in several of our arguments; for a proof see, for instance, [1], [4]:

Theorem 2.1. (subimmersion theorem): *Let X be a connected manifold of dimension m and suppose that the mapping $f : X \to Y$ from X into another manifold Y is a subimmersion of rank r. Then, for any $x \in X$ and $y = f(x) \in Y$, the following statements hold:*

(i) The subset $f^{-1}(y)$ is a closed, $(m-r)$-dimensional submanifold of X and the tangent space $T_x(f^{-1}(y))$ coincides with $ker T_x f$.

(ii) There exists an open neighborhood V of x in X such that $f(V)$ is an r-dimensional submanifold of Y. Moreover, if N is any r-dimensional submanifold of X such that $x \in N$ and $T_x N \cap \ker T_x f = \{0\}$ then the restriction $f_{|N}$ is a local diffeomorphism of some neighborhood of x in N onto $f(V)$.

For the application to DAEs the following characterization theorem for certain subimmersions will be needed:

Theorem 2.2. *For the mapping $G : T\mathbb{R}^m \to \mathbb{R}^q$, $m \leq q \leq 2m$, suppose that $DG(x,p)$ has full rank q at a point $(x_0, p_0) \in G^{-1}(0)$ and hence that for some open neighborhood U of this point in $T\mathbb{R}^m$ the set $M = U \cap G^{-1}(0)$ is a $2m - q$ dimensional submanifold of $T\mathbb{R}^m$.*

Then the restriction $\pi_{|M} : M \to \mathbb{R}^m$ of the canonical projection is a local subimmersion at (x_0, p_0) if and only if rank $D_pG(x,p) = \rho \leq m$ *is constant in a neighborhood of (x_0, p_0) in M. In that case, locally near (x_0, p_0), the rank of $\pi_{|M}$ equals $m + \rho - q$.*

Proof. For $(x,p) \in M$ locally near (x_0, p_0) we have

$$T_{(x,p)}M = \ker DG(x,p) = \{(h,k) \in T\mathbb{R}^m : D_xG(x,p)h + D_pG(x,p)k = 0\}. \tag{2.1}$$

Since $\pi : T\mathbb{R}^m \to \mathbb{R}^m$ is linear the differential of $\pi_{|M}$ at (x,p) is simply the restriction of π to $T_{(x,p)}M$; that is, the mapping

$$(h,k) \in T_{(x,p)}M \;\mapsto\; h \in \mathbb{R}^m. \tag{2.2}$$

Because $\dim T_{(x,p)}M = \dim \ker DG(x,p) = 2m - q$ is constant in a neighborhood of (x_0, p_0), the mappings (2.2) will have constant rank exactly if their null-spaces are of constant dimension locally near (x_0, p_0). From (2.1) we see that $(0,k) \in T_{(x,p)}M$ if and only if $D_pG(x,p)k = 0$; that is, exactly if $k \in \ker D_pG(x,p)$, and $\ker D_pG(x,p)$ has constant dimension for $(x,p) \in M$ near (x_0, p_0) if and only if $D_pG(x,p)$ has constant rank on M locally near (x_0, p_0). Moreover, if this rank is equal to ρ then $\ker D_pG(x,p)$ has dimension $m - \rho$ and hence the mapping (2.2), and equivalently $\pi_{|M}$, has locally near (x_0, p_0) the rank $2m - q - (m - \rho) = m + \rho - q$. $\square$

As noted already, the development in the next four sections will be independent of DAEs. But the following simple DAE shall occasionally be used to motivate some of the concepts:

$$F(x, \dot{x}) \equiv \begin{pmatrix} x_1 - \cos x_2 \\ \dot{x}_1 - x_3 \\ \dot{x}_2 - 1 \end{pmatrix} = 0, \quad F : T\mathbb{R}^3 \to \mathbb{R}^3. \tag{2.3}$$

Obviously, $DF(x,p)$ has rank 3 everywhere and hence $M = F^{-1}(0)$ is a 3-dimensional submanifold of $T\mathbb{R}^3$. If $x = x(t)$ is any C^1-solution of (2.3) then by differentiating the algebraic equation $x_1(t) - \cos x_2(t) = 0$ and using the differential equations we see that this solution must also satisfy the equation $x_3(t) + \sin x_2(t) = 0$. In other words, any such solution must be contained in the set

$$(2.4) \qquad Y = g^{-1}(0) \subset \mathbb{R}^3, \quad g(x) = (x_1 - \cos x_2,\ x_3 + \sin x_2)^T,\ x \in \mathbb{R}^3,$$

which, because of rank $Dg(x) = 2,\ x \in \mathbb{R}^3$, is a one-dimensional submanifold of $\mathbb{R}^3$. Thus, if $x = x(t)$ is a solution of (2.3), then $(x(t), \dot{x}(t)) \in TY \cap F^{-1}(0)$ and it is obvious that the converse is true.

3. π-submanifolds.

For the DAE (1.4) suppose, as in the Introduction, that $M = F^{-1}(0)$ is an n-dimensional submanifold of $T\mathbb{R}^n$ and hence that (1.4) may be written in the form (1.5); that is,

$$(3.1) \qquad (x, \dot{x}) \in M.$$

In fact, a central feature of our approach is the use of (3.1) and the study of its geometric implications. The reduction process sketched in the Introduction replaces (3.1) by a sequence of problems of the same form but with different submanifolds M. Thus, for the development of this process we need to characterize the submanifolds M that will be allowed to arise in (3.1).

Clearly, for a given problem (3.1) with an arbitrarily given submanifold M of $T\mathbb{R}^n$ there may be no solution at all. For instance, in the case $n = 2$ this is certainly true for the one-dimensional submanifold $M = \{(x,p) \in T\mathbb{R}^2;\ x_2 = x_1,\ p_1 = 1,\ p_2 = x_1\}$. Moreover, when M has dimension larger than n then there are usually many solutions through a given point.

In connection with (3.1) our guiding situation will be the choice $M = \varphi(Y)$ where $\varphi : Y \to TY$ is a section of the tangent bundle TY of some connected submanifold Y of $\mathbb{R}^n$. Then the resulting problem (3.1) is locally equivalent to an explicit ODE and hence is locally solvable. This occurred for the example (2.3) where on the tangent bundle of the submanifold Y of $\mathbb{R}^3$ given by (2.4) the canonical projection $\pi : TY \cap F^{-1}(0) \to Y$ has the global inverse

$$\varphi : Y \to TY, \quad \varphi(x) = (x, p), \quad p = (x_3, 1, -\cos x_2)^T,$$

whence – in this case globally – $M = TY \cap F^{-1}(0)$ coincides with $\varphi(Y)$ and the DAE (2.3) is equivalent to the ODE $\dot{x} = (x_3, 1, -\cos x_2)^T$ with initial data on Y.

This guiding case suggests that, in general, we should require that M is embedded in the tangent bundle TY of some submanifold Y of $\mathbb{R}^n$ of the same dimension as M. Of course, as noted before, this dimensional restriction will have to be formulated locally.

As indicated in the Introduction, the class of equations of the general form (1.4) also includes the singular differential equations for which the existence theory is substantially different from that of ODEs and DAEs. This has to be reflected in the allowable choice of the manifolds in the problem (3.1). In the existence theory of [11] these singular equations where essentially excluded by the assumption that rank $D_pF(x,p) = \rho < n$ in some open set in the domain of F. This condition certainly holds for our example (2.3). As mentioned before it will suffice to introduce such a constant rank condition for D_pF only on neighborhoods of points of M. As Theorem 2.2 shows this is essentially equivalent with the assumption that locally the restriction of the canonical projection π to the manifold is a subimmersion.

In line with these introductory remarks we now introduce the following class of submanifolds M that shall be allowed in problems of the form (3.1):

Definition 3.1. *Let X be an n-dimensional manifold. A submanifold M of TX is a π-submanifold (of TX) if for each connected component Ξ of M the following conditions hold:*

(i) For any $(x,p) \in \Xi$ there exists an open neighborhood U in Ξ of (x,p) and a submanifold Y of X such that $\dim Y = \dim \Xi$ and $U \subset TY$.

(ii) The mapping $\pi_{|\Xi} : \Xi \to X$ is a subimmersion in some neighborhood of any point $(x,p) \in \Xi$.[2]

[2] Since Ξ is connected this is equivalent to assuming that $\pi_{|\Xi}$ is a subimmersion on Ξ.

If Y is a connected submanifold of X and $\varphi : Y \to TY$ a section of TY then $M = \varphi(Y)$ turns out to be a π-submanifold of TX. If X has pure dimension n, every pure n-dimensional submanifold of TX satisfies the condition (i) of the definition with $Y = X$ but not necessarily the condition (ii). However, in our simple example (2.3) the 3-dimensional submanifold $M = F^{-1}(0)$ of $\mathbb{R}^3$ is a π-submanifold of $T\mathbb{R}^3$. In fact, we have everywhere rank $D_pF(x,p) = 2$ which by Theorem 2.2 implies that $\pi_{|M}$ is a subimmersion on all of M. This example shows that there exist π-submanifolds that are not the image of a section.

One of our principal aims will be to prove that under certain assumptions a general π-submanifold contains another 'maximal' π-submanifold which, at least locally, is the image of a section $\varphi : Y \to TY$ for some submanifold Y of X. This maximal π-submanifold will be obtained by the recursive reduction procedure to be described below. In preparation we introduce some needed terminology derived from Definition 3.1.

Let Ξ be any connected component of the π-submanifold M of TX and $(x,p) \in \Xi$ any given point. Then, by Definition 3.1 (ii), the rank of $\pi_{|\Xi}$ is constant on some neighborhood of (x,p) and therefore on all of Ξ due to the connectedness of Ξ. Moreover, by the subimmersion theorem (Theorem 2.1) there exists an open neighborhood V of (x,p) in Ξ (and therefore also in M) such that $W = \pi(V)$ is a submanifold of X of dimension equal to the rank r of $\pi_{|\Xi}$ on Ξ. Obviously, the rank r cannot exceed the dimension of Ξ.

Definition 3.2. *Let Ξ be any connected component of the π-submanifold M of TX. For any $(x,p) \in \Xi$ the rank of $\pi_{|\Xi}$ at (x,p) is the order of that point in M and is denoted by $\mathrm{ord}_M(x,p)$. The order of all points of Ξ is the same and we write $\mathrm{ord}_M\Xi$. If V is an open neighborhood in Ξ of $(x,p) \in \Xi$ such that $W = \pi(V)$ is a submanifold of X with $\dim W = \dim \Xi$ then W is called a local projection of Ξ (or M) at (x,p).*

The observations preceeding the definition show that local projections exist at each point of M. The following remark provides a useful technical tool for later use:

Remark 3.1. Let Ξ be a connected component of a π-submanifold M of TX and at a point $(x_0,p_0) \in \Xi$ choose – in accordance with Definition 3.1 (i) – an open neighborhood U in Ξ and a submanifold Y of X such that $\dim Y = \dim \Xi$ and $U \subset TY$. Then TY is a submanifold of TX and therefore U is a submanifold of TY. Now Definition 3.1 (ii) requires that $\pi_{|\Xi} : \Xi \to X$, and therefore also $\pi_{|U} : U \to X$, is a subimmersion in some neighborhood of (x_0, p_0). This holds if and only if $\pi_{|U} : U \to Y$ is a subimmersion in a neighborhood of (x_0, p_0). In other words, it does not matter whether U is viewed as a submanifold of TX or of TY since for $(x, p) \in U$ near (x_0, p_0) we have $T_{(x,p)}(\pi_{|U}) = (T_{(x,p)}\pi)_{|T_{(x,p)}U}$ and $T_{(x,p)}U$ is contained in both $T_{(x,p)}(TX)$ and $T_{(x,p)}(TY)$. This argument also proves that the order of (x_0, p_0) equals the rank of $\pi_{|U} : U \to Y$.

The next theorem provides a rather simple condition for a π-submanifold to be – locally – the image of a section.

Theorem 3.1. *Let M be a π-submanifold such that*

$$\dim \Xi = \mathrm{ord}_M \Xi, \tag{3.2}$$

for each connected component Ξ of M. Then for any $(x_0, p_0) \in M$ there exists a local projection $W = \pi(V)$ of M at (x_0, p_0) and a section $\varphi : W \to TW$ such that $V = \varphi(W)$.

Proof. Let Ξ be a connected component of M and $(x_0, p_0) \in \Xi$ a given point. By Remark 3.1 and the hypothesis (3.2) it follows that $\pi_{|U} : U \to Y$ has full rank $m = \dim \Xi$ at (x_0, p_0) and hence that $\pi_{|U}$ is a diffeomorphism of some open neighborhood $V \subset U$ of (x_0, p_0) onto the open subset $W = \pi(V)$ of Y. Let $\varphi : W \to V$ be the inverse diffeomorphism, then it follows that $(x, p) \in V$ if and only if $x = \pi(x, p) \in W$ and $(x, p) = \varphi(x)$.

Now note that V is a submanifold of TW. Indeed, W is open in Y whence $TW = (\pi_{|Y})^{-1}(W) = \pi^{-1}(W) \cap TY$. Moreover $V \subset TY$ and $V \subset \pi^{-1}(W) = \pi^{-1}(\pi(V))$ together imply that $V \subset TW$. Hence, since both V and TW are submanifolds of TY, we see that V is a submanifold of TW. Thus, instead of viewing φ as a mapping with values in V, we may consider it to be a mapping with values in TW. Finally, φ is a section of TW since $\pi \circ \varphi(x) = x$ holds for all $x \in W$ by definition of φ. $\square$

Clearly, the assumption that M is a π-submanifold is crucial to the proof. Naturally, for a general π-submanifold M the condition (3.2) may not hold for all connected components

Ξ. Our reduction process will ensure that under appropriate assumptions this condition will hold for some submanifold of M.

4. Reducible π-submanifolds.

As before we formulate the implicit DAE (1.4) as a problem of the form (3.1) where now M is a given π-submanifold of TX. Let $x = x(t)$ be a local solution of (3.1) through a point $(x(0), p(0)) \equiv (x_0, p_0) \in M$. Because M is a π-submanifold there is a neighborhood V of (x_0, p_0) in M such that $W = \pi(V)$ is a local projection at that point. Hence, for all sufficiently small t we have $(x(t), \dot{x}(t)) \in V$ which implies that $x(t) \in W$ and, since W is a submanifold of X, that $(x(t), \dot{x}(t)) \in TW$. In particular, we must have $(x_0, p_0) \in TW \cap M$ which, evidently, represents a necessary condition for the existence of a local solution through that point. This situation is reflected in the following terminology:

Definition 4.1. *For the manifold X let M be any π-submanifold of TX. A point $(x, p) \in M$ is a point of reducibility of M if for some local projection $W = \pi(V)$ of M at (x, p) we have $p \in T_x W$ and hence $(x, p) \in TW \cap M$. The subset of all points of reducibility of M is the reduction of M and is denoted by M'.*

Obviously, if the condition $p \in T_{(x,p)} W$ holds for some local projection W of M at (x, p) then it has do so for all others as well. Hence the concept of a point of reducibility is independent of the particular choice of the local projection $W = \pi(V)$ of M at (x, p).

In the following we shall be concerned only with the reduction M' of M and the structure of M away from M' will be irrelevant. In particular, the hypothesis that the rank of $\pi_{|M}$ is locally constant will never be called upon at any point of $M \backslash M'$. Nevertheless, local constancy of the rank of $\pi_{|M}$ near *every* point of M is a requirement that cannot be weakened, for it is needed in the first place to check whether any given point of M is or is not a point of M'!

As an example of a reduction consider the 3-dimensional π-submanifold $M = F^{-1}(0)$ of $T\mathbb{R}^3$ for the simple DAE (2.3). Obviously, its projection

$$W = \pi(M) = \{x \in \mathbb{R}^3 : x_1 - \cos x_2 = 0\}$$

is a 2-dimensional submanifold of $\mathbb{R}^3$ with tangent bundle

$$TW = \{(x,p) \in T\mathbb{R}^3 : \; x_1 - \cos x_2 = 0, \; p_1 + p_2 \sin x_2 = 0\}.$$

Thus the global projection of M is here a manifold and the reduction M' of M coincides with $TW \cap M$ and is globally characterized by the system of equations

$$x_1 - \cos x_2 = 0, \; p_1 + p_2 \sin x_2 = 0, \; p_1 = x_3, \; p_2 = 1.$$

By substituting the third and fourth equation into the second equation it follows that $M' = F_1^{-1}(0)$ where

$$F_1 : T\mathbb{R}^3 \to \mathbb{R}^4, \quad F_1(x,p) = \begin{pmatrix} x_1 - \cos x_2 \\ x_3 + \sin x_2 \\ p_1 - x_3 \\ p_2 - 1 \end{pmatrix}.$$

Obviously we have rank $DF_1(x,p) = 4$ and rank $D_pF_1(x,p) = 2$ everywhere. Thus M' is a 2-dimensional submanifold of $T\mathbb{R}^3$ and, since Definition 3.1 (i) holds with $Y = W$ and Theorem 2.2 applies, we see that M' is again a π-submanifold.

In general, the situation is not so simple and some additional conditions are needed to ensure that the reduction is again a π-submanifold. We begin with a basic topological property of M'.

Theorem 4.1. *The reduction M' of a π-submanifold M is a closed subset of M.*

Proof. We prove that $M \backslash M'$ is open in M which is obvious if $M' = M$. Let $(x_0, p_0) \in M \backslash M'$ and Ξ the connected component of M containing that point. By assumption there exists an open neighborhood V of (x_0, p_0) in Ξ such that $W = \pi(V)$ is a local projection and hence W is a submanifold of X of dimension $\rho = \mathrm{ord}_M \Xi$. Upon shrinking, if needed, V and hence also W we may assume that there is a chart $[\Omega, \psi]$ of X with domain $\Omega \supset W$ for which $\psi(\Omega)$ and $\psi(W)$ are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^\rho$, respectively. Hence, by means of the corresponding bundle chart, $T\Omega$ can be viewed as $\Omega \times \mathbb{R}^n$ and TW as $W \times \mathbb{R}^\rho$. Under this transformation V becomes a submanifold of $\Omega \times \mathbb{R}^n$ which is projected onto W and the condition $p_0 \notin T_x W$ corresponding to $(x_0, p_0) \notin M'$ is now $p_0 \notin \mathbb{R}^\rho$. As a result we have $p \notin \mathbb{R}^\rho$ and hence $(x,p) \notin TW$ for all $(x,p) \in V$ sufficiently close to (x_0,p_0). This is equivalent with $(x,p) \notin M'$ and the result follows. $\square$

The following result motivates a condition under which the reduction M' of a π-submanifold does retain a differentiable structure:

Proposition 4.1. *Let M be a π-submanifold and (x,p) a point in the reduction M' of M. Then for any local projection W of M at (x,p) we have*

$$\dim[T_{(x,p)}TW \cap T_{(x,p)}M] \geq \mathrm{ord}_M(x,p). \tag{4.1}$$

Proof. Let Ξ be the connected component of M containing a given point $(x_0,p_0) \in M'$ and V an open neighborhood of the point in Ξ such that $W = \pi(V)$ is a local projection of Ξ at that point. By Definition 3.1 (i) there exists a neighborhood U of (x_0,p_0) in Ξ and a submanifold Y of X with $\dim Y = \dim \Xi$ such that $U \subset TY$. Without loss of generality, we may assume that $V \subset U$ whence $W \subset \pi(U) \subset Y$. Since W and Y are both submanifolds of X we see that W is a submanifold of Y and hence that $T_x W$ is a subspace of $T_x Y$ for any $x \in W$. This shows that TW is a subset of the bundle

$$\Sigma = \bigcup_{x \in W} [\{x\} \times T_x Y]. \tag{4.2}$$

In fact, Σ is just the pull-back bundle of TY under the canonical embedding $W \subset Y$. Moreover Σ is a submanifold of TX. Indeed, let $(y,q) \in \Sigma$; that is, $y \in W$ and $q \in T_y Y$. Since W and Y are submanifolds of X it follows that, locally near y, we have $W = g^{-1}(0)$ and $Y = h^{-1}(0)$ for some submersions $g : X \to \mathbb{R}^{n-\rho}$ and $h : X \to \mathbb{R}^{n-m}$ where $\rho = \mathrm{ord}_M \Xi$ and $m = \dim_Y = \dim_\Xi$. Using a chart $[\Omega, \Psi]$ of X near y and the corresponding bundle chart $[T\Omega, T\Psi]$ of TX near (y,q) we may assume $X = \mathbb{R}^n$. Thus in some neighborhood of

(y,q), Σ coincides with the zero set of the mapping

$$(x,p) \in T\mathbb{R}^n \;\mapsto\; (g(x), Dh(x)p) \in \mathbb{R}^{2n-(\rho+m)}.$$

It is easily checked that this mapping is a submersion at (y,q) whence Σ is a $(\rho+m)$-dimensional submanifold of $T\mathbb{R}^n$. Hence, TW is not only a subset but a submanifold of Σ. Moreover, from $V \subset U \subset TY$ and $(x,p) \in V$ it follows that $x \in W$ and $p \in T_xY$. In other words, we have $V \subset \Sigma$ and thus V is also a submanifold of Σ. Therefore, (4.1) is a direct consequence of $\dim TW = 2\rho$, $\dim V = \dim \Xi$, and $\dim \Sigma = \rho + \dim \Xi$. $\square$

Since generically two subspaces of dimensions 2ρ and m, respectively, of a $(\rho+m)$-dimensional space intersect along a ρ-dimensional subspace, Proposition 4.1 indicates that equality in (4.1) should be expected in most cases. This partly justifies the following concept:

Definition 4.2. *Let X be a given manifold. A π-submanifold M of TX is reducible if for every point (x,p) of the reduction M' of M there exists a local projection W of M at (x,p) such that*

$$\dim T'_{(x,p)} = \mathrm{ord}_M(x,p), \quad T'_{(x,p)} = T_{(x,p)}TW \cap T_{(x,p)}M \tag{4.3}$$

and if on some neighborhood U of (x,p) in $TW \cap M$

$$\text{rank } \pi_{|T'_{(y,q)}} = \text{constant}, \quad \forall (y,q) \in U. \tag{4.4}$$

Obviously (4.3) is independent of the local projection W and this is essentially also true of (4.4) if we confine attention to sufficiently small neighborhoods of the point.

The term reducible in the above definition is justified by the following result:

Theorem 4.2. *Let M be a reducible π-submanifold. Then, the reduction M' of M is either empty or a closed submanifold of M and a π-submanifold of TX. Moreover for any*

$(x,p) \in M'$ the dimension of the connected component of M' containing (x,p) equals the order of (x,p) in M and for every local projection $W = \pi(V)$ of M at (x,p) we have

$$T_{(x,p)}M' = T_{(x,p)}(TW \cap V) = T_{(x,p)}TW \cap T_{(x,p)}M. \tag{4.5}$$

Proof. The closedness of M' in M was shown in Theorem 4.1. If $W = \pi(V)$ is a local projection of M at the given point $(x,p) \in M'$ then the condition (4.3) expresses the fact that the submanifolds TW and V of the bundle Σ in (4.2) intersect transversally at (x,p) since $T_{(x,p)}M = T_{(x,p)}V$. This implies that $TW \cap V$ is a ρ-dimensional submanifold of TW and M and also that the second equality in (4.5) holds. On the other hand, since $W = \pi(V)$ is a local projection of M not only at (x,p) but also at any other point of V and since V can be chosen so small that the rank of $\pi_{|V}$ is constant on all of V, we see that M' and $TW \cap V$ must coincide . This implies that M' is a submanifold of M and that the first equality in (4.5) holds. Moreover, the dimension of the connected component of M' containing (x,p) has to equal $\dim T_{(x,p)}M'$ which, by (4.3) and (4.5), is $\rho = \mathrm{ord}_M(x,p)$.

Since M' and $TW \cap V$ coincide there exists an open connected neighborhood U' of (x,p) in M' contained in TW. Hence we have $U' \subset \Xi'$ where Ξ' is the connected component of M' containing (x,p). Now for $Y' = W$ it follows from the relation $\dim \Xi' = \rho = \dim Y'$ that the pair (U', Y') satisfies the conditions required of the pair (U, Y) in Definition 3.1 (i). Finally, because of (4.5), the relation (4.4) implies that $\pi_{|M'}$ has constant rank near (x,p) and hence that M' is a π-submanifold. □

From Theorem 4.2 and its proof, condition (4.3) in Definition 4.2 represents a transversality condition for the intersection $TW \cap M$ in an appropriate local setting and the condition (4.4) ensures that $\pi_{|M'}$ has locally constant rank. As observed earlier, transversality is not true, in general, in the 'natural' ambient manifold TX.

With this we can reformulate Theorem 3.1 as follows:

Theorem 4.3. *Let M be a reducible π-submanifold of TX for which the reduction M' is identical with M. Then for every point $(x,p) \in M$ there exists a local projection $W = \pi(V)$ of M at this point and a section $\varphi : W \to TW$ such that $V = \varphi(W)$.*

Proof. The connected component Ξ of M containing (x, p) is also the connected component of M' containing (x, p) whence by Theorem 4.2 the dimension of Ξ equals the order of (x, p) in M and hence

$$\dim \Xi = \operatorname{ord}_M(x, p) = \operatorname{ord}_M \Xi.$$

Thus the dimension of every connected component Ξ of M is equal to the order of Ξ in M and the result follows from Theorem 3.1. □

We end this section with a result characterizing reducible π-submanifolds that coincide with their reduction.

Proposition 4.2. *Let M be a submanifold of TX and suppose that for every $(x, p) \in M$ there exists an open neighborhood U in the connected component Ξ of M containing (x, p) and a submanifold Y of X with $\dim TY = \dim \Xi$ such that $U \subset TY$. Then, M is a reducible π-submanifold with $M' = M$ if and only if $T_{(x,p)}\pi$ is a linear isomorphism from $T_{(x,p)}M$ to T_xY irrespective of the choice of $(x, p) \in M$.*

Proof. The necessity is obvious and we prove only the sufficiency. If the isomorphism condition holds then $\pi_{|\Xi} : \Xi \to Y$ has maximum rank at (x, p) and hence is a local diffeomorphism on some neighborhood of that point. In particular, $\pi_{|\Xi}$ has maximum rank at all points of M near (x, p) so that condition (ii) of Definition 3.1 holds. Moreover, a local projection $W = \pi(V)$ of M at (x, p) is an open neighborhood of x in Y whence $TW = \pi^{-1}(W)$ certainly contains (x, p). In other words, every point of M is a point of reducibility of M; that is, $M' = M$. The conditions (4.3) and (4.4) require here that $\dim T_{(x,p)}M = \dim Y$ and that the rank of $\pi_{|T_{(y,q)}M}$ is constant for all (y, q) in some neighborhood of (x, p). But these relations have already been proved above and hence the result follows. □

5. Completely reducible π-submanifolds.

As before, let X be an n-dimensional manifold and M a reducible π-submanifold of TX. Then by Theorem 4.2 the reduction M' of M is either empty or a π-submanifold of

TX. Of course, if $M \neq \emptyset$ we do not know whether M' is again reducible, but if that is the case, then the reduction $M_2 \equiv M_1'$ of $M_1 \equiv M'$ is again either empty or a π-submanifold of TX. This suggests the following recursive definition:

Definition 5.1. *The reducible π-submanifold $M = M_0$ of TX is completely reducible if for every index $j \geq 0$ such that $M_j \neq \emptyset$ the reduction M_{j+1} of M_j is either empty or a reducible π-submanifold. For $M_j = \emptyset$ we set $M_{j+1} = \emptyset$. Then the sequence $\{M_j\}_{j\geq 0}$ is well-defined and called the reduction chain of M.*

Let M be a completely reducible π-submanifold of TX with reduction chain $\{M_j\}_{j\geq 0}$. If $M_{k+1} = M_k$ for some index $k \geq 0$ then it is obvious that $M_j = M_k$ for $j \geq k$; that is, the reduction chain becomes stationary. Since reduction never increases the local dimension near any point (x,p) it is intuitively evident that every reduction chain should become stationary. The following result proves this fact and introduces rigorously the index concept given by S. Reich in [12], [13].

Theorem 5.1. *Let M be a completely reducible π-submanifold of TX with $\dim M = m$ and reduction chain $\{M_j\}_{j\geq 0}$. For any non-empty, connected component Ξ_{m+1} of M_{m+1} define Ξ_j, $j = m, m-1, \ldots, 0$, recursively as the connected component of M_j containing Ξ_{j+1}. Then there exists a smallest integer ν, $0 \leq \nu \leq \dim \Xi_0 \leq m$, the index of Ξ_{m+1} in M, such that*

$$\Xi_j' = \Xi_j, \; j = \nu, \nu+1, \ldots, m+1. \tag{5.1}$$

In particular, the reduction chain of M always satisfies

$$M_{m+2} = M_{m+1}. \tag{5.2}$$

Proof. For $M_{m+1} = \emptyset$ the first part of the theorem is vacuous and (5.2) is obvious. Suppose therefore that there exists a non-empty connected component Ξ_{m+1} of M_{m+1}. Then the connected components $\Xi_j \in M_j$ constructed in the theorem satisfy $\emptyset \neq \Xi_{m+1} \subset \Xi_{j+1} \subset \Xi_j$

for $j = 0, \dots, m$. Evidently, the reduction Ξ'_j of Ξ_j equals $\Xi'_j = M_{j+1} \cap \Xi_j$ and, since Ξ_{j+1} is a connected component of M_{j+1}, it is also a connected component of $\Xi'_j = M_{j+1} \cap \Xi_j$. Thus we have

$$\Xi_{j+1} \subset \Xi'_j = M_{j+1} \cap \Xi_j \subset \Xi_j, \quad j = 0, \dots, m. \tag{5.3}$$

and, because connected components of closed sets are closed, it follows by Theorem 4.2 that Ξ_j is closed in M for all j. In particular, Ξ_{j+1} is closed in Ξ_j. Now $\Xi_j \neq \emptyset$ ensures that $\mu_j = \dim \Xi_j$ is defined for $j = 0, \dots, m+1$ and (5.3) implies that

$$0 \leq \mu_{m+1} \leq \mu_m \leq \dots \leq \mu_0 \leq m. \tag{5.4}$$

The last inequality follows from the fact that Ξ_0 is a connected component of M and hence has dimension at most m. Hence, by necessity, two of the integers in (5.4) must be equal and there exists a smallest index ν among $0 \leq \nu \leq m$ for which $\mu_{\nu+1} = \mu_\nu$. This implies that $\Xi_{\nu+1}$ is an open submanifold of Ξ_ν and, since it is also a nonempty and closed subset of Ξ_ν and Ξ_ν is connected, that $\Xi_\nu = \Xi_{\nu+1}$. But then (5.3) shows that $\Xi_{\nu+1} = \Xi'_\nu = \Xi_\nu$; that is, reduction does not affect Ξ_ν and, of course, neither will repeated reduction. In other words, we must have $\Xi_j = \Xi'_j$ for $j = \nu, \nu+1, \dots, m+1$ which is (5.1). For $j = m+1$ it follows that for any nonempty connected component Ξ_{m+1} of M_{m+1} the reduction Ξ'_{m+1} is equal to Ξ_{m+1} which proves (5.2). □

Definition 5.2. *For the completely reducible π-submanifold M with reduction chain $\{M_j\}_{j\geq 0}$ the core $C(M)$ of M is the intersection $\bigcap_{j\geq 0} M_j$.*

Hence, for any completely reducible π-submanifold M of TX Theorem 5.1 implies that $C(M) = M_{m+1}$ with $m = \dim M$ and that $C(M)$ is either empty or a (completely) reducible π-submanifold with $C(M)' = C(M)$. Thus Theorems 4.2 and 4.3 at once provide the following result:

Theorem 5.2. *Let M be a completely reducible π-submanifold. Then, the core $C(M)$ of M is either empty or a closed submanifold of M and, moreover, $C(M)$ is a reducible*

π-submanifold equal to its reduction. In particular, for every $(x,p) \in C(M)$ there exists a local projection $W = \pi(V)$ of $C(M)$ at (x,p) and a section $\varphi : W \to TW$ such that $V = \varphi(W)$.

Theorem 5.1 defines the index of any nonempty connected component Ξ of $C(M)$ and shows that this index is at most equal to the dimension $\dim \Xi_0$ of the connected component of M containing Ξ. Hence, for any point $(x,p) \in C(M)$ the index may be defined as the index of the connected component of $C(M)$ containing (x,p). Then, again, the index of (x,p) never exceeds the dimension of the connected component of M containing the point.

At a first sight, Theorem 5.1 may appear to imply that the relation (5.2) could be improved to $M_{m+1} = M_m$. This is not the case, for it may happen that the reduction Ξ'_m of some connected component Ξ_m of M_m is empty. All we can say is that when $\Xi'_m \neq \emptyset$ then $\Xi'_m = \Xi_m$ which is weaker.

More generally, note that any open subset of a π-submanifold M obviously is a π-submanifold with the same dimension as M.This fact was used implicitly in the proof of Theorem 5.1. Thus, if a point $(x,p) \in M$ has an open neighborhood U that is a completely reducible π-submanifold and for which (x,p) belongs to the core $C(U)$ of U, then we may define a local index of (x,p) as the index of (x,p) in $C(U)$. It is easily checked that shrinking U has no effect on this definition and that when (x,p) has a local index and M is completely reducible, then $(x,p) \in C(M)$ and the index and local index of the point coincide. Thus, the concept of a local index is useful only when not all of M is completely reducible. It is equally straightforward to verify that if each point of M is contained in a completely reducible neighborhood, then M is completely reducible.

6. Application to Differential-Algebraic Equations.

As an application of our theory we consider now an implicit differential equation

$$F(x(t), \dot{x}(t)) = 0, \tag{6.1}$$

where $\dot{x}$ again stands for dx/dt. Here $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be a C^∞-mapping and we recall our agreement that F may only be defined on an open subset of $\mathbb{R}^n \times \mathbb{R}^n$.

A C^1-solution of (6.1) is a C^1 function

$$x : J \to \mathbb{R}^n, \quad J \subset \mathbb{R} \text{ open interval,} \tag{6.2}$$

such that (6.1) holds for all $t \in J$.

As indicated at the end of the Introduction the geometrical theory allows us to weaken the definition of an implicit differential-algebraic equation:

Definition 6.1. *The implicit differential equation (6.1) is a nonsingular differential - algebraic equation (DAE) if $M \equiv F^{-1}(0)$ is a completely reducible π-submanifold of $\mathbb{R}^n \times \mathbb{R}^n \equiv T\mathbb{R}^n$.*

Then we obtain the following basic existence and uniqueness result:

Theorem 6.1. *Let (6.1) be a nonsingular DAE and denote the core of $M = F^{-1}(0)$ by $C(M)$.*

(i) Any C^1-solution (6.2) of (6.1) satisfies $(x(t), \dot{x}(t)) \in C(M)$ for all $t \in J$.

(ii) Conversely, for any $(x_0, p_0) \in C(M)$ there exists a local projection $W = \varphi(V)$ of $C(M)$ at (x_0, p_0) and a section $\varphi : W \to TW$ such that for any C^1 function (6.2) we have

$$(x(t), \dot{x}(t)) \in V \;\Leftrightarrow\; \{x(t) \in W \text{and } (x(t), \dot{x}(t)) = \varphi(x(t))\}. \tag{6.3}$$

Hence, locally near any $(x_0, p_0) \in C(M)$, (6.1) is equivalent to an explicit ODE on W. In particular, for any $t_0 \in \mathbb{R}$ there exists an open interval J containing t_0 and a unique C^1-solution (6.2) of (6.1) such that $(x(t_0), \dot{x}(t_0)) = (x_0, p_0)$. Moreover, x is of class C^∞.

Proof. For the proof of (i) let x be any C^1-solution (6.2) of (6.1) and for any fixed $t_0 \in J$ set $(x(t_0), \dot{x}(t_0)) = (x_0, p_0)$. Then there exists a local projection $W = \pi(V)$ of M at (x_0, p_0) and obviously, for all $t \in J$ near t_0 we have $x(t) \in W$ and therefore $(x(t), \dot{x}(t)) \in TW$; that is, $(x(t), \dot{x}(t)) \in TW \cap M$. But $TW \cap M$ coincides with the reduction M_1 of M near (x_0, p_0) so that $(x(t), \dot{x}(t)) \in M_1$ for t near t_0 and hence for all $t \in J$ since t_0 was arbitrary. Let M_j, $j = 0, 1, \ldots$ denote the reduction chain of $M = M_0$. By proceeding inductively, we then find that $(x(t), \dot{x}(t)) \in M_j$ for $j = 0, 1, \ldots$ and all $t \in J$. Thus, in particular, $(x(t), \dot{x}(t)) \in C(M)$ since, by definition $C(M) = \bigcap_{j \geq 0} M_j$. This proves (i).

For the proof of the first part of (ii), recall that by Theorem 5.2 there exists a local projection $W = \pi(V)$ of $C(M)$ at (x_0, p_0) and a section $\varphi : W \to TW$ such that

$$(x, p) \in V \Leftrightarrow \{x \in W \text{ and } (x, p) = \varphi(x)\}.$$

This proves (6.3). Moreover, since φ is a section of TW there exists an open interval J and a unique C^1 function $x : J \to W$, of class C^∞, for which $x(t_0) = x_0$ and $(x(t), \dot{x}(t)) = \varphi(x(t))$ for all $t \in J$. Here $\dot{x}(t)$ denotes $T_t x \cdot 1$, but, because W is a submanifold of $\mathbb{R}^n$, this is just the usual derivative $(dx/dt)(t)$. Hence we have $(x(t), \dot{x}(t)) \in V \subset C(M) \subset M$, so that $F(x(t), \dot{x}(t)) = 0$ for all $t \in J$. In other words, x is a C^1-solution of (6.1) of class C^∞. The uniqueness follows from part (i) and (6.3), and the uniqueness of the constructed solution x as an integral curve of φ since, necessarily, (6.3) must hold for t in some open subinterval of J around t_0. □

By using arguments closely related to those in the proof of Theorem 3.1 of [11], we can formulate also global results for the solutions of (6.1):

Theorem 6.2. *Let (6.1) be a nonsingular DAE. Then any C^1-solution[3](6.2) of (6.1) can be extended to a C^1-solution of (6.1) on an interval (a, b) that is maximal under set inclusion. Moreover at the endpoints this solution has the properties:*
(i) If $b < \infty$ (or $a > -\infty$) and $\dot{x}(t)$ is bounded for $t \in (a, b)$ near b (or a), then $\lim_{t \to b_-} x(t) = x_0$ (or $\lim_{t \to a_+} x(t) = x_0$) exists.
(ii) If $b < \infty$ (or $a > -\infty$) and $\lim_{t \to b_-} x(t) = x_0$ (or $\lim_{t \to a_+} x(t) = x_0$) exists, then $\lim_{t \to b_-} |\dot{x}(t)| = \infty$ (or $\lim_{t \to a_+} |\dot{x}(t)| = \infty$).
As a result, if $b < \infty$ (or $a > -\infty$), then $\dot{x}(t)$ is unbounded for $t \in (a, b)$ near b (or a).

Since the proof follows very closely that of Theorem 3.1 in [11] we give here only a brief sketch. By Theorem 6.1 (ii) any two solutions of (6.1) defined in open intervals J_1 and

[3]Note that by Theorem 6.1 any C^1-solution (6.2) of (6.1) is necessarily of class C^∞.

J_2, respectively, with $(x_1(t_0), \dot{x}_1(t_0)) = (x_2(t_0), \dot{x}_2(t_0))$ for some $t_0 \in J_1 \cap J_2$ must coincide in $J_1 \cap J_2$. Hence they continue each other in $J_1 \cup J_2$ as a solution of (6.1). In turn, this argument yields, for instance by Zorn's lemma, the existence of a maximal interval of definition (a, b) for any C^1-solution x of (6.1).

Now part (i) of the statement follows readily with the help of the integral mean value theorem $x(t) - x(s) = \int_s^t \dot{x}(\tau)d\tau$. For part (ii) we argue by contradiction. Suppose that $b < \infty$ and that there is a sequence $t_k \in (a, b)$, $\lim_{k\to\infty} t_k = b$ such that $\dot{x}(t_k)$ is bounded. By extracting a subsequence we find a $p_0 \in \mathbb{R}^n$ such that $\lim_{k\to\infty} \dot{x}(t_k) = p_0$ whence $\lim_{k\to\infty} (x(t_k), \dot{x}(t_k)) = (x_0, p_0)$. Note that $(x_0, p_0) \in C(M)$ since, by Theorem 5.2, $C(M)$ is closed in $M = F^{-1}(0)$ and M is closed in $\mathbb{R}^n$. Moreover, the same theorem ensures that $C(M)$ coincides with the graph of a section $\varphi : W \to TW$ in a neighborhood of (x_0, p_0) where $W = \pi(V)$ is a local projection of $C(M)$ at (x_0, p_0). Thus it follows that for $p \in \mathbb{R}^n$, sufficiently close to p_0, we have

$$(6.4) \qquad (x_0, p) \in C(M) \Rightarrow p = p_0.$$

Consider now any other sequence $s_k \in (a, b)$ such that $\lim_{k\to\infty} s_k = b$. By extracting a subsequence we may assume either that $\lim_{k\to\infty} |\dot{x}(s_k)| = \infty$ or that $\lim_{k\to\infty} \dot{x}(s_k) = \hat{p}_0 \in \mathbb{R}^n$. In the first case, as well as in the second case for $\hat{p}_0 \neq p_0$, the same arguments as in the proof of Theorem 3.1 of [11] can be used, together with the closedness of $C(M)$ to show that for every sufficiently small $\delta > 0$ there exists a $p_\delta \in \mathbb{R}^n$ such that $(x_0, p_\delta) \in C(M)$ and $|p_0 - p_\delta| = \delta$. But this contradicts (6.4) and thus we necessarily have $\lim_{t\to b_-} \dot{x}(t) = p_0$.

Altogether we find that $\lim_{t\to b_-} (x(t), \dot{x}(t)) = (x_0, p_0)$. Moreover, using once again the fact that $C(M)$ is the graph of the section $\varphi : W \to TW$ near (x_0, p_0), we see that $(x(t), \dot{x}(t))$ belongs to V for all $t \in (a, b)$ sufficiently close to b. Hence x coincides with the unique solution $\tilde{x}$ of the initial value problem

$$(\tilde{x}(t), \dot{\tilde{x}}(t)) = \varphi(\tilde{x}(t)), \quad (\tilde{x}(b), \dot{\tilde{x}}(b)) = (x_0, p_0).$$

But then $\tilde{x}$ extends x beyond b as a solution of (6.1), in contradiction with the maximality of (a, b). The case when $t \longrightarrow a_+$ in (ii) of the theorem can be handled in the exact same way. $\square$

Let x denote a C^1-solution (6.2) of a nonsingular DAE (6.1). By Theorem 6.1 (i) we then have $(x(t), \dot{x}(t)) \in C(M)$ for all $t \in J$ where $M = F^{-1}(0)$ and all points $(x(t), \dot{x}(t))$ for $t \in J$ must belong to the same connected component Ξ of $C(M)$. Theorem 5.1 defines the index of any such component Ξ and it is natural to call it the *index of the solution* x. Thus, for the DAE (6.1) the index of a solution x is simply the number of successive reductions of $M = F^{-1}(0)$ needed to obtain x (locally) as a solution of an explicit ODE. Note that this index is of a solution is not a local concept at a point of x but, in fact, is independent of the point on the trajectory.

Nonautonomous problems of the form $F(t, x(t), \dot{x}(t)) = 0$ can easily be included in this theory. In fact, after using again the trick of adding the scalar equation $\dot{t} = 1$, we may apply the results to the autonomous equation $\mathcal{F}(X(\tau), \dot{X}(\tau)) = 0$ with

$$\mathcal{F} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad \mathcal{F}(\hat{x}, \hat{p}) = (s - 1, F(t, x, p)), \; \hat{x} = (t, x), \; \hat{p} = (s, p).$$

We conclude this section with the simple example of a planar pendulum. Of course, our interest does not center on the particular application but, instead, on illuminating the details of the reduction process. This example was chosen for its simple form which permits an explicit manipulation of all equations arising in the reduction process and allows for a direct identification of the entire chain. Moreover, the reduction chain turns out to end a forteriori with a system equivalent to the classical 'pendulum equation' which therefore is recovered here without any guess of the proper variables.

Suppose a mass m is attached at the end of a rigid massless wire of length ℓ hanging from the origin.If λ is the tension of the wire and g the gravity constant then the problem may be modelled by the second order DAE

$$\begin{aligned} & x_1^2 + x_2^2 = \ell^2, \\ (6.5) \qquad & \ddot{x}_1 = -(\lambda/m)x_1, \\ & \ddot{x}_2 = -(\lambda/m)x_2 - g, \end{aligned}$$

which with $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$, and $x_5 = \lambda/m$ transforms into the form (6.1) where

$$(6.6) \qquad F : T\mathbb{R}^5 \to \mathbb{R}^5, \quad F(x,\dot{x}) = \begin{pmatrix} x_1^2 + x_2^2 - \ell^2 \\ \dot{x}_1 - x_3 \\ \dot{x}_2 - x_4 \\ \dot{x}_3 + x_1 x_5 \\ \dot{x}_4 + x_2 x_5 + g \end{pmatrix}.$$

Clearly rank $DF(x,p) = 5$ and rank $D_pF(x,p) = 4$ on $M = F^{-1}(0)$ and hence M is a pure 5-dimensional submanifold and, by Theorem 2.2, also a π-submanifold of $T\mathbb{R}^5$.

The projection of M is $W = \pi(M) = \{x \in \mathbb{R}^5 : x_1^2 + x_2^2 - \ell^2 = 0\}$ and hence is a 4-dimensional submanifold of $\mathbb{R}^5$ with tangent bundle

$$TW = \{(x,p) \in T\mathbb{R}^5 : \ x_1^2 + x_2^2 - \ell^2 = 0, \ x_1 p_1 + x_2 p_2 = 0\}.$$

Since the global projection of M is a manifold all local arguments are here of a global nature and the reduction M_1 of M coincides with $TW \cap M$ and is characterized by the equations $x_1^2 + x_2^2 - \ell^2 = 0$, $x_1 p_1 + x_2 p_2 = 0$ coupled with the system

$$(6.7) \qquad \begin{aligned} p_1 - x_3 &= 0, \\ p_2 - x_4 &= 0, \\ p_3 + x_1 x_5 &= 0, \\ p_4 + x_2 x_5 + g &= 0. \end{aligned}$$

With the first and second equations of (6.7), $x_1 p_1 + x_2 p_2 = 0$ becomes $x_1 x_3 + x_2 x_4 = 0$ and hence we find that $M_1 = F_1^{-1}(0)$ where

$$F_1(x,p) = \begin{pmatrix} x_1^2 + x_2^2 - \ell^2 \\ x_1 x_3 + x_2 x_4 \\ p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1 x_5 \\ p_4 + x_2 x_5 + g \end{pmatrix}, \quad F_1 : T\mathbb{R}^5 \to \mathbb{R}^6.$$

Here $DF_1(x,p)$ has full rank 6 at every point of $F_1^{-1}(0)$ and hence M_1 is a pure 4-dimensional submanifold of $T\mathbb{R}^5$. Moreover, Definition 3.1 (i) holds with $Y = W$ and by Theorem 2.2 it follows that M_1 is a π-submanifold of $T\mathbb{R}^5$. The projection of M_1 is the pure 3-dimensional submanifold

$$W_1 = \pi(M_1) = \{x \in \mathbb{R}^5 : \ x_1^2 + x_2^2 - \ell^2 = 0, \ x_1x_3 + x_2x_4 = 0\}$$

of $\mathbb{R}^5$ which has the tangent bundle

$$\begin{aligned} TW_1 = \{(x,p) \in T\mathbb{R}^5 : \ & x_1^2 + x_2^2 - \ell^2 = 0, \ x_1x_3 + x_2x_4 = 0, \\ & x_1p_1 + x_2p_2 = 0, \ x_3p_1 + x_4p_2 + x_1p_3 + x_2p_4 = 0\}. \end{aligned}$$

Hence we see that $(x,p) \in M_2 = TW_1 \cap M_1$ exactly if (x,p) satisfies the system of differential equations (6.7) now coupled with the equations

$$\begin{aligned} & x_1^2 + x_2^2 - \ell^2 = 0, \\ & x_1x_3 + x_2x_4 = 0, \\ & x_1p_1 + x_2p_2 = 0, \\ & x_3p_1 + x_4p_2 + x_1p_3 + x_2p_4 = 0, \end{aligned} \tag{6.8}$$

and again M_2 turns out to be the reduction of M_1.

By using (6.7) we can express p_1, p_2, p_3, p_4 in all four equations (6.8) in terms of x. Then the third equation is the same as the second one and can be dropped. Thus we see that $M_2 = F_2^{-1}(0)$ with

$$F_2(x,p) = \begin{pmatrix} x_1^2 + x_2^2 - \ell^2 \\ x_1x_3 + x_2x_4 \\ gx_2 - x_3^2 - x_4^2 + \ell^2 x_5 \\ p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1x_5 \\ p_4 + x_2x_5 + g \end{pmatrix}, \quad F_2 : T\mathbb{R}^5 \to \mathbb{R}^7,$$

and a trivial verification shows that $DF_2(x,p)$ has full rank 7 on $F_2^{-1}(0)$ while $D_pF_2(x,p)$ has constant rank 4 everywhere. Therefore, in the same manner as before we conclude that M_2 is a pure 3-dimensional π-submanifold of $T\mathbb{R}^5$.

A further step uses the projection $W_2 = \pi(M_2)$ of M_2; that is,

$$W_2 = \{x \in \mathbb{R}^5 : \; x_1^2 + x_2^2 - \ell^2 = 0, \; x_1x_3 + x_2x_4 = 0, \; gx_2 - x_3^2 - x_4^2 + \ell^2 x_5 = 0\},$$

which is a pure 2-dimensional submanifold of $\mathbb{R}^5$ with the tangent bundle

$$\begin{aligned} TW_2 = &\{(x,p) \in T\mathbb{R}^5 : x_1^2 + x_2^2 - \ell^2 = 0, \; x_1x_3 + x_2x_4 = 0, \\ & gx_2 - x_3^2 - x_4^2 + \ell^2 x_5 = 0, \; x_1p_1 + x_2p_2 = 0, \\ & x_3p_1 + x_4p_2 + x_1p_3 + x_2p_4 = 0, \; gh_2 - 2x_3p_3 - 2x_4p_4 + \ell^2 p_5 = 0\}. \end{aligned}$$

Hence it follows that $(x,p) \in M_3 = TW_3 = TW_2 \cap M_2$ if and only if (x,p) satisfies the system (6.7) coupled with the equations

$$\begin{aligned} & x_1^2 + x_2^2 - \ell^2 = 0, \\ & x_1x_3 + x_2x_4 = 0, \\ & gx_2 - x_3^2 - x_4^2 + \ell^2 x_5 = 0, \\ & x_1p_1 + x_2p_2 = 0, \\ & x_3p_1 + x_4p_2 + x_1p_3 + x_2p_4 = 0, \\ & gp_2 - 2x_3p_3 - 2x_4p_4 + \ell^2 p_5 = 0, \end{aligned} \tag{6.9}$$

and that M_3 is the reduction of M_2.

Once again, we use (6.7) to express p_1, p_2, p_3, p_4 in all equations (6.9) in terms of x. Then the fourth equation is the same as the second one and can be eliminated. Moreover, with the help of the first equation the fifth equation turns out to be identical with the third one and can also be dropped. Finally, by means of the second equation the sixth equation reduces to $3gx_4 + \ell^2 p_5 = 0$. Hence, altogether, it follows that $M_3 = F_3^{-1}(0)$ with

$$
(6.10) \qquad F_3(x,p) = \begin{pmatrix} x_1^2 + x_2^2 - \ell^2 \\ x_1x_3 + x_2x_4 \\ gx_2 - x_3^2 - x_4^2 + \ell^2 x_5 \\ p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1x_5 \\ p_4 + x_2x_5 + g \\ p_5 + \frac{3g}{\ell^2}x_4 \end{pmatrix}, \quad F_3 : T\mathbb{R}^5 \to \mathbb{R}^8.
$$

The derivative $DF_3(x,p)$ has rank 8 on $M_3 = F^{-1}(0)$ and $D_pF_3(x,p)$ has rank 5 everywhere. Thus by the same reasoning as before M_3 turns out to be a pure 2-dimensional π-submanifold of $T\mathbb{R}^5$.

Finally we note that the projection of M_3 is

$$
\begin{aligned}
W_3 = \pi(M_3) = \{x \in \mathbb{R}^5 : x_1^2 + x_2^2 - \ell^2 = 0, \\
x_1x_3 + x_2x_4 = 0, gx_2 - x_3^2 - x_4^2 + \ell^2 x_5 = 0\}.
\end{aligned}
$$

and hence that $W_3 = W_2$. As a result we have

$$
M_3 \cap TW_3 = M_3 \cap TW_2 = (M_2 \cap TW_2) \cap TW_2 = M_2 \cap TW_2 = M_3;
$$

that is, M_3 equals its reduction M_4 and, from this point on, the process becomes stationary. Altogether we see here that $C(M) = M_3$.

It follows that $C(M) = M_3$ and that each point of $C(M)$ has index 3. By Theorem 6.1, the DAE reduces locally to a first order ODE on $\pi(C(M)) = \pi(M_3) = W_3 = W_2$. In this simple case, this ODE can be found explicitly. In fact, for instance, in the neighborhood of the equilibrium position $x_1 = 0$, $x_2 = -\ell$, we can solve $F_3(x,p) = 0$ by expressing $p_1, \ldots, p_5$ and x_2, x_4, x_5 in terms of x_1 and x_3. This shows that $(x,p) \in C(M) = M_3$ if and only if

$$
x_2 = -(\ell^2 - x_1^2)^{1/2},
$$

$$
(6.11) \qquad x_4 = x_1 x_3 (\ell^2 - x_1^2)^{-1/2}
$$

$$
x_5 = (g/\ell^2)(\ell^2 - x_1^2)^{1/2} + x_3^2(\ell^2 - x_1^2)^{-1},
$$

and

$$
\begin{aligned}
p_1 &= x_3, \\
p_2 &= x_1 x_3 (\ell^2 - x_1^2)^{-1/2}, \\
p_3 &= -(g/\ell^2) x_1 (\ell^2 - x_1^2)^{1/2} - x_1 x_3^2 (\ell^2 - x_1^2)^{-1}, \\
p_4 &= -(g/\ell^2)(\ell^2 - x_1^2) - x_3^2 (\ell^2 - x_1^2)^{-1/2}, \\
p_5 &= -(3g/\ell^2) x_1 x_3 (\ell^2 - x_1^2)^{-1/2},
\end{aligned} \tag{6.12}
$$

Now note that for $x \in W_2$ the right-hand side of (6.12) automatically belongs to $T_x W_2$. Hence by replacing p by $\dot{x}$ we obtain explicitly the following ODE on W_2:

$$
\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_1 x_3 (\ell^2 - x_1^2)^{-1/2}, \\
\dot{x}_3 &= -(g/\ell^2) x_1 (\ell^2 - x_1^2)^{1/2} - x_1 x_3^2 (\ell^2 - x_1^2)^{-1}, \\
\dot{x}_4 &= -(g/\ell^2)(\ell^2 - x_1^2) - x_3^2 (\ell^2 - x_1^2)^{-1/2}, \\
\dot{x}_5 &= -(3g/\ell^2) x_1 x_3 (\ell^2 - x_1^2)^{-1/2}.
\end{aligned} \tag{6.13}
$$

Interestingly, (6.13) reduces to a system consisting only of the first and third equations. In fact, for given initial data x_1^0 and x_3^0, x_1 and x_3 are uniquely determined by these two equations and then x_2, x_4, x_5 are explicitly derived from (6.11). In turn, this 2×2 system is equivalent to the single scalar second order equation

$$
\ddot{x}_1 = -(g/\ell^2) x_1 (\ell^2 - x_1^2)^{1/2} - x_1 \dot{x}_1^2 (\ell^2 - x_1^2)^{-1},
$$

which, with $x_1 = \ell \sin\theta$, becomes

$$
\ddot{\theta} = -(g/\ell)\sin\theta;
$$

that is, the classical 'pendulum equation'.

7. Analytic Characterizations.

For the pendulum example all considerations were of a global nature and all equations characterizing the manifolds of the reduction chain could be manipulated explicitly. But this is hardly a typical situation. For most practical applications it will be important to obtain analytic criteria corresponding to the various geometric conditions contained in the Definitions 3.1, 4.1, and 4.2.

The first condition of Definition 3.1 requires that locally near the point (x_0, p_0) under consideration the manifold M is embedded in TY where Y is a submanifold of X with the same dimension m as M. Clearly, this condition can hardly be translated into analytic terms and must be assumed. This is not a major inconvenience because the reduction procedure of Sections 4 and 5 automatically preserves this property and provides explicit information about the choice of Y. Moreover, as we saw in the previous section, at the beginning of the process Y is typically given by $Y = X = \mathbb{R}^n$.

All the geometric conditions are local in nature at a point $(x_0, p_0) \in Y$ and are unaltered by natural tangent bundle isomorphisms; that is, those occuring as tangent maps of local diffeomorphisms defined in the neighborhood of x_0 in X. Hence, there will be no loss of generality in assuming that the problem has been represented in the following form:

Local Assumption. *The space Y is given by $Y = \mathbb{R}^m$ (or some open subset of $\mathbb{R}^m$) and there exists an open neighborhood $\mathcal{U}$ in TY of $(x_0, p_0) \in TY$ such that $M = \mathcal{U} \cap G^{-1}(0)$ where $G : TY \to Y$ is a (smooth) submersion on $\mathcal{U}$; that is, DG has full rank m on $\mathcal{U}$.*

This assumption simply means that, locally at the given point (x_0, p_0), we have replaced the manifold M by its connected component (of dimension m) containing the point and that we used a natural tangent bundle chart for TY at (x_0, p_0) (in the terminology of [1]). Note that with $Y = \mathbb{R}^m$ we have $TY = T\mathbb{R}^m \equiv \mathbb{R}^m \times \mathbb{R}^m$ and that the canonical projection $\pi : TY \to Y$ is the projection onto the first factor.

From Theorem 2.2 and Remark 3.1 we obtain immediately the following result:

Proposition 7.1. *Under the Local Assumption. condition (ii) of Definition 3.1 holds if and only if $D_pG(x, p)$ has constant rank $\rho \leq m$ for all $(x, p) \in M$ in some neighborhood* *of* (x_0,p_0)*and in that case we have* $\mathrm{ord}_M(x_0,p_0)=\rho$.

Before turning to a characterization of reducibility in Definition 4.2, we need analytic representations of local projections in the setting of our Local Assumption. For this suppose that M is a π-submanifold and hence, in accordance with Proposition 7.1, that $D_pG(x,p)$ has constant rank ρ locally near (x_0,p_0).

By Definition 3.2 a local projection W of M near (x_0,p_0) is the projection $\pi(V)$ of a sufficiently small open neighborhood V of (x_0,p_0) in M such that W is a submanifold of $Y=\mathbb{R}^m$ of the same dimension ρ as the order of (x_0,p_0) in M. Thus, locally near x_0, the manifold W can be represented in the form $M=g^{-1}(0)$ where $g:\mathbb{R}^m\to\mathbb{R}^{m-\rho}$ is a submersion on some neighborhood of x_0; that is Dg has full rank $m-\rho$. The tangent bundle $TW\subset T\mathbb{R}^m$ is then characterized by

$$TW=\{(x,u)\in T\mathbb{R}^m : g(x)=0,\ Dg(x)u=0\}. \tag{7.1}$$

Hence any point $(x,p)\in TW\cap V$ satisfies $g(x)=0$, $Dg(x)p=0$, and $G(x,p)=0$. But the latter equation states that $(x,p)\in V$ and hence implies that $x\in W$ which is $g(x)=0$. Therefore we have

$$TW\cap V=\{(x,p)\in T\mathbb{R}^m :\ Dg(x)p=0,\ G(x,p)=0\}. \tag{7.2}$$

Now the question arises how g can be obtained from G. Clearly $\dim\ker D_pG(x_0,p_0)=m-\rho$ and hence there exists a linear map

$$A\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^{m-\rho}),\quad \ker A\cap\ker D_pG(x_0,p_0)=\{0\} \tag{7.3}$$

with full rank $m-\rho$. Using A we construct the mapping

$$\tilde G:T\mathbb{R}^m\to\mathbb{R}^m\times\mathbb{R}^{m-\rho},\quad \tilde G(x,p)=(G(x,p),A(p-p_0)), \tag{7.4}$$

for which evidently $\tilde G(x_0,p_0)=0$ and

$$D\tilde G(x_0,p_0)=\Lambda_0=\begin{pmatrix} D_xG(x_0,p_0) & D_pG(x_0,p_0)\\ 0 & A\end{pmatrix}\in\mathcal{L}(T\mathbb{R}^m,\mathbb{R}^m\times\mathbb{R}^{m-\rho}), \tag{7.5}$$

has full rank $2m - \rho$. Hence, $\tilde{G}^{-1}(0)$ (restricted to some neighborhood of (x_0, p_0)) is a ρ-dimensional submanifold N of M which at that point has the tangent space $T_{(x_0,p_0)}N = \ker \Lambda_0$. As in the proof of Theorem 2.2 note that the differential of $\pi_{|M}$ at (x_0, p_0) is simply the restriction of π to $T_{(x_0,p_0)}M$. Hence we have

$$T_{(x_0,p_0)}N \cap \ker \pi_{|T_{(x_0,p_0)}M} = \ker \Lambda_0 \cap [\{0\} \times \ker D_pG(x_0,p_0)].$$

In other words, any element (h, k) of the space on the left has to satisfy the relations $\Lambda_0(h, k) = 0$, $h = 0$, and $D_pG(x_0, p_0)k = 0$ which by (7.3) imply that $k = 0$. Thus by the second part of Theorem 2.1 it follows that $\pi_{|N}$ is a local diffeomorphism of some open neighborhood of (x_0, p_0) in M to W.

For the construction of a local coordinate system on N let $Q_0 \in \mathcal{L}(\mathbb{R}^m)$ be a projection onto some complement Z_0 of the range of $D_pG(x_0, p_0)$ so that $\dim Z_0 = m - \rho$ and

$$Z' = \ker Q_0 D_xG(x_0, p_0). \tag{7.6}$$

has at least dimension ρ. From $Q_0DG(x_0,p_0) = Q_0D_xG(x_0,p_0)$ it follows that $(h, k) \in \ker \Lambda_0$ implies $h \in Z'$. Conversely, for $h \in Z'$ we have $D_xG(x_0,p_0)h \in \operatorname{rge} D_pG(x_0,p_0)$ so that $DG(x_0,p_0)(h,k) = 0$ for some $k \in \mathbb{R}^m$ and, since A is an isomorphism of $\ker D_pG(x_0,p_0)$ onto $\mathbb{R}^{m-\rho}$ there exists a $k' \in \ker D_pG(x_0,p_0)$ for which $Ak' = -Ak$. But then we have $\Lambda_0(h, k + k') = 0$ which shows that the projection π maps $\ker \Lambda_0$ onto Z' and hence that $\dim Z' \le \rho$. It follows that $\dim Z' = \rho$.

For any complement Z'' of Z' in $\mathbb{R}^m$ we now have $\dim Z'' = m - \rho$ and

$$Q_0D_xG(x_0,p_0)_{|Z''} \in \text{Isom}\,(Z'', Z_0). \tag{7.7}$$

In accordance with the decomposition $\mathbb{R}^m = Z' \oplus Z''$ we shall henceforth write $x = z' + z''$ for any $x \in \mathbb{R}^m$ and, in particular, set $x_0 = z_0' + z_0''$.

It is easily checked that Λ_0 is an isomorphism of $Z'' \times \mathbb{R}^m$ to $\mathbb{R}^m \times \mathbb{R}^{m-\rho}$. Hence the implicit function theorem applies to the equation $\tilde{G}(z' + z'', p) = 0$, and we obtain for N

the local coordinate representation

$$(x,p) \in N \;\Leftrightarrow\; z'' = \varphi(z'),\; p = \psi(z'), \tag{7.8}$$

where φ and ψ are smooth functions in some neighborhood of z_0' such that $z_0'' = \varphi(z_0')$, $p_0 = \psi(z_0')$. Evidently

$$g : \mathbb{R}^m = Z' \oplus Z'' \to Z'' \simeq \mathbb{R}^{m-\rho},\; g(x) = z'' - \varphi(z'). \tag{7.9}$$

satisfies $D_{z''}g(x_0) = I_{Z''}$ and therefore is a submersion at x_0. From (7.8), and after shrinking W if necessary, it follows that

$$x \in W \;\Leftrightarrow\; z'' = \varphi(z')$$

which implies that $W = g^{-1}(0)$.

For later purposes, we require the derivatives $Dg(x_0)$ and $D^2g(x_0)(p_0,\cdot)$ when $(x_0,p_0) \in TW \cap M$. Because of (7.9) it suffices to calculate $D\varphi(z_0')$ and $D^2\varphi(z_0')$ which can be done by implicit differentiation of $\tilde{G}(z' + \varphi(z'), \psi(z')) = 0$. By a straightforward calculation using (7.6) we obtain

$$D\varphi(z_0') = 0, \quad D\psi(z_0') = 0, \tag{7.10}$$

and therefore

$$Dg(x_0)h = h'',\; h = h' + h'' \in Z' \oplus Z'' = \mathbb{R}^m. \tag{7.11}$$

From this, in turn, an equally simple calculation provides

$$D^2g(x_0)(h,k) = [Q_0 D_x G(x_0,p_0)_{|Z''}]^{-1} Q_0 D_x^2 G(x_0,p_0)(h',k'),$$
$$h = h' + h'',\; k = k' + k'' \in Z' \oplus Z'' = \mathbb{R}^m.$$

Now for $(x_0, p_0) \in TW \cap M$ it follows from (7.1) that $Dg(x_0)p_0 = 0$ which by (7.11) implies that $p_0 = p_0'$ and thus with $k = p_0$ above

$$
\begin{aligned}
D^2g(x_0)(p_0, h) = [Q_0 D_x G(x_0, p_0)_{|Z''}]^{-1} Q_0 D_x^2 G(x_0, p_0)(p_0, h'), \\
h = h' + h'' \in Z' \oplus Z'' = \mathbb{R}^m.
\end{aligned}
\tag{7.12}
$$

With these results we obtain now the following characterization of the conditions in Definition 4.2.

Proposition 7.2. *Suppose that at $(x_0, p_0) \in TW \cap M$ with $W = \pi(V)$, the Local Assumption holds. For any (x, p) in V let $Q(x, p)$ be the projection of $\mathbb{R}^m$ onto an arbitrary complement $Z_{(x,p)}$ of the range of $D_pG(x, p)$ and write $Z_0 = Z_{(x_0,p_0)}$ and $Q_0 = Q(x_0, p_0)$. Then the conditions (4.3) and (4.4) of Definition 4.2 hold if and only if the mapping*

$$
\begin{pmatrix} Q_0 D_x^2 G(x_0, p_0)(p_0, \cdot) & Q_0 D_x G(x_0) \\ D_x G(x_0, p_0) & D_p G(x_0, p_0) \end{pmatrix} \in \mathcal{L}(T\mathbb{R}^m, Z_0 \times \mathbb{R}^m)
\tag{7.13}
$$

has full rank $2m - \rho$ and, after shrinking V if necessary, for all $(x, p) \in TW \cap M$ the relation

$$
\begin{aligned}
\dim\,[\ker Q(x,p) D_x G(x,p) \cap \ker D_p G(x,p)] \; = \\
\dim\,[\ker Q_0 D_x G(x_0, p_0) \cap \ker D_p G(x_0, p_0)],
\end{aligned}
\tag{7.14}
$$

holds.[4]

Proof. Since g is a submersion at x_0 it follows that the mapping

$$
(x, u) \in T\mathbb{R}^m \mapsto (g(x), Dg(x)u) \in \mathbb{R}^{m-\rho} \times \mathbb{R}^{m-\rho}
$$

is a submersion at (x_0, u) irrespective of u. Hence, because of (7.1) and $(x_0, p_0) \in TW$, we find that

$$
T_{(x_0,p_0)}TW = \{(h, k) \in T\mathbb{R}^m : Dg(x_0)h = 0,\; D^2g(x_0)(p_0, h) + Dg(x_0)k = 0\},
$$

[4] Obviously (7.14) is independent of the specific choice of the projections $Q(x, p)$.

and therefore $(h,k) \in T_{(x_0,p_0)}TW \cap T_{(x_0,p_0)}M$ exactly if

$$\begin{aligned}
&Dg(x_0)h = 0,\\
&D^2g(x_0)(p_0,h) + Dg(x_0)k = 0,\\
&D_xG(x_0,p_0)h + D_pG(x_0,p_0)k = 0.
\end{aligned}$$

By (7.11) the first relation states that $h \in Z' = \ker Q_0 D_x G(x_0,p_0)$ which is also a consequence of the third relation. In other words, the first relation is redundant and the above set of conditions reduces to $(h,k) \in \ker L(x_0,p_0)$ where

$$L(x,p) = \begin{pmatrix} D^2g(x)(p,\cdot) & Dg(x) \\ D_xG(x,p) & D_pG(x,p) \end{pmatrix}. \tag{7.15}$$

Thus the condition (4.3) is equivalent with $\dim\ker L(x_0,p_0) = \rho$. From (7.11) and (7.12) together with (7.7) we find that $(h,k) \in \ker L(x_0,p_0)$ exactly if

$$\begin{aligned}
&Q_0 D_x^2 G(x_0,p_0)(p_0,h') + Q_0 D_x G(x_0,p_0)k'' = 0,\\
&D_xG(x_0,p_0)h + D_pG(x_0,p_0)k = 0.
\end{aligned}$$

By multiplying the second equation with Q_0 we see that $Q_0 D_x G(x_0,p_0)h = 0$ and hence that $h = h' \in Z'$. Thus, in the first equation h' may be replaced by h. Since k'' may, of course, be replaced by k it follows that $\ker L(x_0,p_0)$ is the null-space of the mapping (7.13) and that $\dim\ker L(x_0,p_0) = \rho$ if and only if the rank of (7.13) (and also of $L(x_0,p_0)$) equals $2m-\rho$ and hence is full.

Suppose now that the condition (4.3) holds and therefore, as we just saw, that $L(x,p)$ has rank $2m-\rho$ for all $(x,p) \in T\mathbb{R}^m$ near (x_0,p_0). In order to show that (7.14) is equivalent to (4.4), note that $L(x,p)$ is the derivative of the mapping $(x,p) \mapsto (Dg(x)p, G(x,p))$ whose local zero set near (x_0,p_0) is precisely $TW \cap V$. This means that, locally near (x_0,p_0), $TW \cap V$ is a ρ-dimensional submanifold of $T\mathbb{R}^m$ whose tangent space at (x,p) is $\ker L(x,p)$. The condition (4.4) now states that the rank of the linear mapping $\pi_{|T_{(x,p)}(TW\cap V)}$ is constant for all $(x,p) \in TW \cap V$ near (x_0,p_0). Equivalently, this means that the mapping

$$(h,k) \in \ker L(x,p) \mapsto h \tag{7.16}$$

has constant rank for $(x,p) \in TW \cap V$ near (x_0, p_0). Since $\ker L(x,p)$ has dimension ρ it follows that (4.4) holds exactly if the dimension of the null-space of (7.16) is constant for all $(x,p) \in TW \cap V$ near (x_0,p_0). This null-space consists of those pairs $(0,k) \in T\mathbb{R}^m$ for which

$$Dg(x)k = 0, \quad D_pG(x,p)k = 0.$$

But $Dg(x)k = 0$ means that $k \in T_xW = \pi(T_{(x,p)}M)$. Then there exists some $\ell \in \mathbb{R}^m$ such that $D_xG(x,p)k + D_pG(x,p)\ell = 0$ and by multiplying with $Q(x,p)$ we see that $Q(x,p)D_xG(x,p)k = 0$. Thus, $\ker Dg(x) = \ker Q(x,p)D_xG(x,p)$ and, since the null-space of (7.16) is $\{0\} \times [\ker Dg(x) \cap \ker D_pG(x,p)]$, its dimension will be constant exactly if (7.14) holds. $\square$

Propositions 7.1 and 7.2 show how reducibility of a π-submanifold M can be checked in the neighborhood of a given point (x_0,p_0) after a natural tangent bundle chart has been chosen to reduce the problem to that characterized by the Local Assumption. When (x_0,p_0) belongs to the reduction M' of M and has order ρ in M, then an analytic characterization of M' near (x_0,p_0) is given by

$$\begin{aligned} Dg(x)p &= 0 \quad (\in \mathbb{R}^{m-\rho}), \\ G(x,p) &= 0 \quad (\in \mathbb{R}^m), \end{aligned} \tag{7.17}$$

where we constructed $g : \mathbb{R}^m \to \mathbb{R}^{m-\rho}$ as a submersion at x_0 for which the zero set $g^{-1}(0)$ coincides locally near x_0 with a local projection W of M at (x_0,p_0). Then $\dim W = \rho$ and M' is localy near (x_0,p_0) a ρ-dimensional submanifold of TW which therefore can be represented as $H^{-1}(0)$ with some submersion $H : TW \to \mathbb{R}^\rho$ at (x_0,p_0).

Once H is known we can use a natural tangent bundle chart on TW to introduce the representations $W = \mathbb{R}^\rho$, $TW = T\mathbb{R}^\rho$ which means that, once again the Local Assumption holds but with M replaced by $M_1 = M'$ and G replaced by the representation G_1 of H in the bundle chart. Thus Propositions 7.1 and 7.2 can be applied again. A natural tangent bundle chart for TW is easily constructed once g has been determined from G as discussed above. This leaves us with the question of finding H in terms of the data available from the Local Assumption; that is, in terms of the mapping G characterizing M near (x_0, p_0). For this it may be noted that although the system (7.17) characterizes M_1 near (x_0, p_0) and $Dg(x)p = 0$ holds whenever $(x,p) \in TW$, we cannot choose $H = G$ for then H maps into $\mathbb{R}^m$ and not into $\mathbb{R}^\rho$.

We begin by representing the manifold M locally near (x_0, p_0) as the zero set of another submersion $\hat{G}$ from which H can then be obtained. Let $B \in \mathcal{L}(\mathbb{R}^{m-\rho}, \mathbb{R}^m)$ be chosen such that

$$\text{rge } B \cap \text{rge } D_pG(x_0, p_0) = \{0\}. \tag{7.18}$$

Such a mapping B is necessarily one-to-one. Moreover, set

$$V' = \ker D_pG(x_0, p_0), \quad \dim V' = m - \rho, \tag{7.19}$$

and let V'' be any complement of V' in $\mathbb{R}^m$. We shall henceforth write $p \in \mathbb{R}^m$ in the form $p = p' + p''$ in accordance with the decomposition $\mathbb{R}^m = V' \oplus V''$. There should be no confusion with the corresponding notation used earlier relative to the decomposition $\mathbb{R}^m = Z' \oplus Z''$ since the two decompositions will not appear at the same time.

By the choice of V'' the mapping

$$(k'', \ell) \in V'' \times \mathbb{R}^{m-\rho} \mapsto D_pG(x_0, p_0)k'' + B\ell \in \mathbb{R}^m,$$

is one-to-one and hence a linear isomorphism. It thus follows from the implicit function theorem that near $(x_0, p_0, 0)$ the zero set of the mapping

$$(x, p, q) \in T\mathbb{R}^m \times \mathbb{R}^{m-\rho} \mapsto G(x,p) + Bq \in \mathbb{R}^m,$$

is a $(2m - \rho)$-dimensional submanifold $\hat{M}$ of $T\mathbb{R}^m \times \mathbb{R}^{m-\rho}$ characterized by

$$(x,p,q) \in \hat{M} \Leftrightarrow \{p'' = \lambda(x, p'),\ q = \mu(x, p')\}, \tag{7.20}$$

where λ and μ are smooth functions near (x_0, p'_0) such that $\lambda(x_0, p'_0) = p''_0$, $\mu(x_0, p'_0) = 0$. For $(x, p) \in T\mathbb{R}^m$ near (x_0, p_0) let

$$\hat{H}(x,p) = p'' - \lambda(x,p') \in V'' \simeq \mathbb{R}^\rho, \tag{7.21}$$

and

$$\hat{G}(x,p) = (g(x), \hat{H}(x,p)) \in \mathbb{R}^{m-\rho} \times V'' \simeq \mathbb{R}^m. \tag{7.22}$$

Since $D_{p''}\hat{H}(x_0,p_0) = I_{V''}$ we see that $D_p\hat{H}(x_0,p_0)$ is surjective and hence, because g is a submersion at x_0, that $\hat{G}$ is a submersion at (x_0, p_0). Now, observe that for $(x,p) \in M$ near (x_0, p_0) we have $G(x,p) = 0$ whence $(x,p,0) \in \hat{M}$. In particular, (7.20) implies $p'' = \lambda(x,p)$; that is, $\hat{H}(x,p) = 0$. On the other hand, from $G(x,p) = 0$ we also obtain $g(x) = 0$ and therefore $\hat{G}(x,p) = 0$. Thus, locally near (x_0, p_0), we have

$$(x,p) \in M \Rightarrow (x,p) \in \hat{G}^{-1}(0).$$

But since $\hat{G}$ is a submersion at (x_0, p_0) its zero set near (x_0, p_0) is a submanifold of $T\mathbb{R}^m$ with the same dimension m as M so that M and $\hat{G}^{-1}(0)$ coincide in the vicinity of (x_0, p_0). This means that, in all previous considerations we may use $\hat{G}$ instead of G when desirable. More specifically, the conditions expressed in Propositions 7.1 and 7.2 in terms of G are equivalent to the corresponding conditions expressed in terms of $\hat{G}$ (because with both choices these conditions translate geometric, hence intrinsic, properties). Therefore if $(x_0, p_0) \in TW \cap M$ and the mapping (7.13) has full rank $2m - \rho$ (as it must if M is reducible), then the mapping obtained by replacing G by $\hat{G}$ in (7.13) has full rank $2m - \rho$ as well. As was seen in the proof of Proposition 7.2, the full rank condition for the mapping (7.13) is equivalent to

$$\operatorname{rank} \begin{pmatrix} D^2g(x_0)(p_0, \cdot) & Dg(x_0) \\ D_xG(x_0,p_0) & D_pG(x_0,p_0) \end{pmatrix} = 2m - \rho,$$

and hence to

$$\text{(7.23)} \qquad \operatorname{rank} \begin{pmatrix} D^2g(x_0)(p_0,\cdot) & Dg(x) \\ D_x\hat{G}(x_0,p_0) & D_p\hat{G}(x_0,p_0) \end{pmatrix} = 2m-\rho.$$

Hence for $v'' \in V''$ and with (7.22) and (7.23) we see that there exists some $(h,k) \in \mathbb{R}^m \times \mathbb{R}^m$ such that

$$D^2g(x_0)(p_0,h) + Dg(x_0)k = 0,$$
$$Dg(x_0)h = 0,$$
$$D_x\hat{H}(x_0,p_0)h + D_p\hat{H}(x_0,p_0)k = v''.$$

Here the first two conditions mean that $(h,k) \in T_{(x_0,p_0)}(TW)$, and, since v'' is arbitrary, it follows that $D\hat{H}(x_0,p_0)_{|T_{(x_0,p_0)}TW}$ maps onto V''; that is, that $\hat{H}_{|TW}$ is a local submersion at (x_0,p_0) in $TW \cap M$. This shows that $H = \hat{H}_{|TW}$ can be chosen to characterize $TW \cap M$ and hence M' as $H^{-1}(0)$ in a neighborhood of (x_0,p_0) and, as desired, by (7.21) H maps into $\mathbb{R}^\rho$.

8. Relationship with the Earlier Theory.

We use again the setting of the previous section and discuss now the special case when the point (x_0,p_0) has local index 1 and relate the results to our earlier theory developed in [11]. This will show that in this case the two approaches essentially coincide and that some apparent discrepancies can be fully explained.

As defined in Section 5, the point $(x_0,p_0) \in M$ has local index 1 if there exists an open neighborhood U of (x_0,p_0) in M such that (i) U is a completely reducible π-submanifold, (ii) (x_0,p_0) belongs to the core $C(U)$ of U, and (iii) the connected component of $C(U)$ containing (x_0,p_0) is a connected component of the reduction $U_1 = U'$ of U but not of U itself (for otherwise the index would be 0). In particular, this means that U_1 is reducible and the reduction U_1' of U_1 equals U_1.

Under the conditions of Propositions 7.1 and 7.2 and, in particular, if $(x_0,p_0) \in TW \cap M$, then there exists an open neighborhood U of (x_0,p_0) in M which is a reducible π-submanifold and $(x_0,p_0) \in U_1$. By shrinking U and therefore also U_1 we may assume that $U_1 \subset TW$ where W is the local projection of M near (x_0,p_0) used before. Hence, a necessary and sufficient condition for U_1 to be reducible with $U_1' = U_1$ is given by Proposition 4.2 with M replaced by U_1 and Y by W. More specifically, since π is here linear, Proposition 4.2 requires that π be a linear isomorphism from $T_{(x_0,p_0)}U_1$ to $T_{x_0}W$. Note that it is actually required that the condition holds for every $(x,p) \in U_1$. But because U_1 can be shrunk to arbitrarily small size, it suffices indeed to require the condition only at (x_0,p_0) since it then holds automatically at all nearby points. In summary, under the conditions of Propositions 7.1 and 7.2 we have

$$(8.1) \qquad (x_0,p_0) \in TW \cap M \quad \text{has local index 1} \quad \Leftrightarrow \quad \ker \pi_{|T_{(x_0,p_0)}(TW \cap M)} = \{0\}.$$

This provides us with the following characterization:

Theorem 8.1. *Under the Local Assumption the point $(x_0,p_0) \in M$ has local index 1 if and only if*

(i) rank $D_pG(x,p) \equiv \rho$ is constant for all (x,p) near (x_0,p_0) in M.

(ii) $D_xG(x_0,p_0)p_0 \in$ rge $D_pG(x_0,p_0)$.

(iii) For any $k \in \mathbb{R}^m$ the implication

$$(8.2) \qquad \left.\begin{array}{l} D_xG(x_0,p_0)k \in \text{ rge } D_pF(x_0,p_0) \\ D_pG(x_0,p_0)k = 0 \end{array}\right\} \quad \Rightarrow \quad k = 0.$$

holds.

Proof. The necessity of (i) and (ii) is obvious since (i) is required by Proposition 7.1 and (ii) merely expresses the assumption that $(x_0,p_0) \in TW \cap M$. Now note that (8.2) is an equivalent formulation of the condition on the right of (8.1) since in the proof of Proposition 7.2 the null-space of $\pi_{|T_{(x_0,p_0)}(TW \cap M)}$ was seen to be equal to

$$\{0\} \times [\ker Q_0 D_xG(x_0,p_0) \cap \ker D_pG(x_0,p_0)]$$

where Q_0 is a projection onto a complement of rge $D_pG(x_0,p_0)$. Hence the necessity of (iii) follows from the equivalence (8.1).

For the proof of the sufficiency we have to show that all the conditions of Propositions 7.1 and 7.2 are satisfied. Then (8.2) will ensure that the condition on the right side of (8.1) holds.

Clearly the conditions of Proposition 7.1 hold and so does the condition $(x_0, p_0) \in TW \cap M$ of Proposition 7.2 which is our condition (ii). We now show that the mapping (7.13) has the full rank $2m - \rho$. For this let Z_0 and Q_0 be as given in Proposition 7.2 and $(u, v) \in Z_0 \times \mathbb{R}^m$. Since G is a submersion at (x_0, p_0) there exists some $(h, k) \in T\mathbb{R}^m$ such that $D_xG(x_0, p_0)h + D_pG(x_0, p_0)k = v$ and k may be replaced by $k + \ell$ with any $\ell \in \ker D_pF(x_0, p_0)$ without affecting this relation. By (8.2) we have $\ker Q_0D_xG(x_0, p_0) \cap \ker D_pG(x_0, p_0) = \{0\}$; that is, $Q_0D_xG(x_0, p_0)$ is one-to-one on $\ker D_pG(x_0, p_0)$ and hence a linear isomorphism of $\ker D_pG(x_0, p_0)$ onto Z_0. Thus, $\ell \in \ker D_pG(x_0, p_0)$ can be found so that

$$Q_0D_xG(x_0, p_0)\ell = u - Q_0D_x^2G(x_0, p_0)(p_0, h) - Q_0D_xG(x_0, p_0)k,$$

which proves the claim.

Finally, we show that (7.14) holds. As noted above the right-hand side of (7.14) is zero and it suffices to prove that for $(x, p) \in TW \cap M$ near (x_0, p_0) we have, with the same notation as in Proposition 7.2,

$$\ker Q(x, p)D_xG(x, p) \cap \ker D_pG(x, p) = \{0\}. \tag{8.3}$$

We observed that (7.14) is independent of the specific choice of the projections $Q(x, p)$. In particular, we may choose $Q(x, p)$ to depend continuously on $(x, p) \in M$. In fact, let $\{e_1^0, \ldots, e_\rho^0\}$ be an orthonormal basis of rge $D_pG(x_0, p_0)$, relative, say, to the canonical inner product of $\mathbb{R}^m$, and choose $w_1, \ldots, w_\rho \in \mathbb{R}^m$ such that $e_i^0 = D_pG(x_0, p_0)w_i$ for $i = 1, \ldots, \rho$. By continuity, the vectors $e_i(x, p) = D_pG(x, p)w_i$, $i = 1, \ldots, \rho$ remain linearly independent for $(x, p) \in M$ near (x_0, p_0) and hence by (i) they span a space of dimension $\rho = \dim$ rge $D_pG(x, p)$. Therefore, we have rge $D_pG(x, p) =$ span $\{e_1(x, p), \ldots, e_\rho(x, p)\}$ and by applying the Gram-Schmidt process to the vectors $e_i(x, p)$ we obtain an orthonormal basis of rge $D_pG(x, p)$.[5] As a sum of dyadic products of the vectors of this basis, the

[5] Recall that the Gram-Schmidt process involves only continuous operations.

orthogonal projection $P(x,p)$ of rge $D_pG(x,p)$ depends continuously on $(x,p) \in M$ near (x_0, p_0), and hence the same is true of $Q(x,p) = I - P(x,p)$.

It only remains to show that (8.3) holds for all $(x,p) \in TW \cap W$ near (x_0, p_0). Suppose in the contrary that there exist sequences $(x_j, p_j) \in TW \cap M$ and $k_j \in \mathbb{R}^m$ such that, for all $j \geq 0$, $(x_j, p_j) \to (x_0, p_0)$ and k_j is a unit vector under some norm, for which $Q(x_j, p_j)D_xG(x_j, p_j)k_j = D_pG(x_j, p_j)k_j = 0$. By extracting a subsequence we may assume that $k_j \to k$ where $k \in \mathbb{R}^m$ again is a unit vector. By continuity it then follows that $k \in \ker Q(x_0, p_0)D_xG(x_0, p_0) \cap \ker D_pG(x_0, p_0)$ which is a contradiction. $\square$

Theorem 8.1 can be compared with a related result in [11] where a different approach is taken and a different definition for the index is given. It turns out that the result in [11] corresponding to Theorem 8.1 – when phrased in our present notation – requires the conditions:

Condition (i)'. *rank* $D_pG(x,p) \equiv \rho$ *is constant for all* (x,p) *near* (x_0, p_0) *in* $T\mathbb{R}^m$.

Condition (ii)'. $D_xG(x_0, p_0)p_0 \in$ rge $D_pG(x_0, p_0)$.

Condition (iii)'. *If for any vectors* $k \in \mathbb{R}^m$ *and* $q_0 \in \mathbb{R}^m$ *such that* $D_xF(x_0, p_0)p_0 + D_pF(x_0, p_0)q_0 = 0$ *the relations*

$$D^2_{x,p}G(x_0, p_0)(p_0, k) + D^2_pG(x_0, p_0)(q_0, k) + D_xG(x_0, p_0)k \in \text{rge } D_pG(x_0, p_0)$$
$$D_pG(x_0, p_0)k = 0$$

hold, then $k = 0$.

In condition (iii)' it may be noted here that the requirement for the vector q_0 is consistent with (ii)' and that $D^2_{x,p}G(x_0, p_0)(p_0, k)$ stands for $D^2G(x_0, p_0)((p_0, 0), (0, k))$.

It immediately appears that condition (i) of Theorem 8.1 is weaker than (i)' since constancy of the rank of $D_pG(x,p)$ is required only in a neighborhood of (x_0, p_0) in M. The approach taken in [11] does not go through under this weaker assumption. Conditions (ii) and (ii)' are the same for both results but an obvious discrepancy exists among the conditions (iii) and (iii)' since the second derivative of G plays no role in Theorem 8.1. However, the discrepancy is removable. In fact, suppose that for $(x_0, p_0) \in M$ the conditions (i)', (ii)', and (iii)' hold. Then for any $k \in \ker D_pF(x_0, p_0)$ we have

$$D^2_{x,p}G(x_0, p_0)(p_0, k) + D^2_pG(x_0, p_0)(q_0, k) \ \in \ \text{rge } D_pG(x_0, p_0),$$

and hence the condition (iii)' coincides with condition (iii) of Theorem 8.1.

In order to see this observe that

$$\begin{aligned} & D^2_{x,p}G(x_0, p_0)(p_0, k) + D^2_pG(x_0, p_0)(q_0, k) \\ (8.4) \qquad & = D^2G(x_0, p_0)((p_0, 0), (0, k)) + D^2G(x_0, p_0)((0, q_0), (0, k)) \\ & = D^2G(x_0, p_0)((p_0, q_0), (0, k)) = D[D_pG(x, p)k](p_0, q_0)_{|(x,p)=(x_0,p_0)} \\ & = \frac{d}{dt}[D_pG(x_0 + tp_0, p_0 + tq_0)k]_{|t=0}. \end{aligned}$$

By a method identical to the one used in the proof of Theorem 8.1 it was shown in [11] that the condition (i)' implies that for all $(x, p) \in T\mathbb{R}^m$ sufficiently close to (x_0, p_0) a projection $Q(x, p)$ onto a complement of rge $D_pG(x, p)$ can be chosen which depends not only continuously but smoothly upon (x, p). Thus, with

$$(8.5) \qquad q(t) = Q(x_0 + tp_0, p_0 + tq_0), \quad \varphi(t) = D_pG(x_0 + tp_0, p_0 + tq_0)k,$$

it follows that $q(t)\varphi(t) \equiv 0$ for all sufficiently small $|t|$ and by differentiating this identity at $t = 0$ we obtain

$$\frac{dq}{dt}(0)\varphi(0) + q(0)\frac{d\varphi}{dt}(0) = 0.$$

Since for $k \in \ker D_pG(x_0, p_0)$, $\varphi(0) = 0$, this relation reduces to $q(0)(d\varphi/dt)(0) = 0$ and because $q(0) = Q(x_0, p_0)$ projects onto a complement of rge $D_pG(x_0, p_0)$ this reads

$$\frac{d\varphi}{dt}(0) \ \in \ \text{rge } D_pG(x_0, p_0),$$

and the conclusion follows from (8.4) and (8.5).

For problems with arbitrary index, there is also some similarity between the methods of this paper and those of [11]. Indeed, for (x_0, p_0) in the core $C(M)$ of M it follows from the analytic results of Section 6 that the structure of $C(M)$ near (x_0, p_0) can be determined from a sequence of mappings $G_0, \ldots, G_\nu$ where ν is the index of (x_0, p_0) and $G_\nu^{-1}(0)$ represents $C(M)$ near (x_0, p_0) in a local chart. This is reminiscent of what was done in [11] where also a sequence of mappings $F_0, \ldots, F_{\nu'}$ is defined inductively. But an important difference is that the G_j are defined over, and map into, spaces with smaller and smaller dimension while no reduction of dimension occurs with the F_j in [11].

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A differential-geometric approach for proving the existence and uniqueness of solutions of implicit differential-algebraic equations is presented. It provides for a significant improvement of an earlier theory developed by the authors as well as for a completely intrinsic definition of the index of such problems. The differential-algebraic equation is transformed into an explicit ordinary differential equation by a reduction process that can be abstractly defined for specific submanifolds of tangent bundles here called reducible $\ddot{\pi}$-submanifolds. Local existence and uniqueness results for differential-algebraic equations then follow directly from the final stage of this reduction by means of an application of the standard theory of ordinary differential equations.

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