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Form Approved
OMB No. 0704-0188

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1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE December 1990		3. REPORT TYPE AND DATES COVERED memo	
4. TITLE AND SUBTITLE Continuous Stochastic Cellular Automata that have a Stationary Distribution and No Detatiled Baiance				5. FUNDING NUMBERS DACA76-85-C-0010 N00014-85-K-0124	
6. AUTHOR(S) Tomaso Poggio Federico Girosi					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Artificial Intelligence Laboratory 545 Technology Square Cambridge, Massachusetts 02139				8. PERFORMING ORGANIZATION REPORT NUMBER AIM 1168	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Office of Naval Research Information Systems Arlington, Virginia 22217				10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES None					
12a. DISTRIBUTION / AVAILABILITY STATEMENT Distribution of this document is unlimited				12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) Marroquin and Ramirez (1990) have recently discovered a class of discrete stochastic cellular automata with Gibbsian invariant measure that have a non-reversible dynamic behavior. Practical applications include more powerful algorithms than the Metropolis algorithm to compute MRF models. In this paper we describe a large class of stochastic dynamical systems that has a Gibbs asymptotic distribution but does not satisfy reversibility. We characterize sufficient properties of a sub-class of stochastic differential equations in terms of the associated Fokker-Planck equation for the existence of an asymptotic probability distribution in the system of coordinates which is given. Practical implications include VLSI analog circuits to compute coupled MRF models.					
14. SUBJECT TERMS (key words) MRFs cellular automata Fokker-Planck VLSI analog circuits				15. NUMBER OF PAGES 6	
				16. PRICE CODE \$3.00	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED		18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED		19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	
20. LIMITATION OF ABSTRACT UNCLASSIFIED					

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
ARTIFICIAL INTELLIGENCE LABORATORY
and
CENTER FOR BIOLOGICAL INFORMATION PROCESSING
WHITAKER COLLEGE

A.I. Memo No.1168
C.B.I.P. Paper No.

December 1990

**Continuous Stochastic Cellular Automata That Have
a Stationary Distribution and No Detailed Balance**

Tomaso Poggio and Federico Girosi

Abstract

Marroquin and Ramirez (1990) have recently discovered a class of discrete stochastic cellular automata with Gibbsian invariant measure that have a non-reversible dynamic behavior. Practical applications include more powerful algorithms than the Metropolis algorithm to compute MRF models. In this paper we describe a large class of stochastic dynamical systems that has a Gibbs asymptotic distribution but does not satisfy reversibility. We characterize sufficient properties of a sub-class of stochastic differential equations in terms of the associated Fokker-Planck equation for the existence of an asymptotic probability distribution in the system of coordinates which is given. Practical implications include VLSI analog circuits to compute coupled MRF models.

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This paper describes research done within the Center for Biological Information Processing, in the Department of Brain and Cognitive Sciences, and at the Artificial Intelligence Laboratory. This research is sponsored by a grant from the Office of Naval Research (ONR), Cognitive and Neural Sciences Division; by the Artificial Intelligence Center of Hughes Aircraft Corporation (S1-801534-2). Support for the A. I. Laboratory's artificial intelligence research is provided by the Advanced Research Projects Agency of the Department of Defense under Army contract DACA76-85-C-0010, and in part by ONR contract N00014-85-K-0124.

It is well known (see Stratonovitch, 1963, for instance) that one can associate, under some conditions, to a stochastic continuous automata (i.e., a stochastic differential equation) a so-called Fokker-Planck (F-P) equation in the probability distribution of the state variables. In this note, we wish to characterize conditions under which the F-P equation admits a stationary solution of the Gibbs type.

Let \mathbf{x} a n -dimensional vector of state variables, and $W(\mathbf{x}, t)$ the probability distribution of the state variables described by \mathbf{x} at time t . The F-P equation is:

$$\frac{\partial}{\partial t} W(\mathbf{x}, t) = \left(- \sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} d_{\alpha}(\mathbf{x}) + \frac{1}{2} \sum_{\alpha, \beta=1}^n \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} K_{\alpha\beta}(\mathbf{x}) \right) W(\mathbf{x}, t)$$

where $d_{\alpha}(\mathbf{x})$ is the *drift vector* and $K_{\alpha\beta}(\mathbf{x})$ is the *diffusion matrix* (see Stratonovitch, 1963, p. 76).

The stationary solution $w(\mathbf{x})$ of the F-P satisfies the equation:

$$\sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} G_{\alpha}(\mathbf{x}) = 0, \quad (1)$$

where we have defined the *probability current* $G_{\alpha}(\mathbf{x})$:

$$G_{\alpha}(\mathbf{x}) = d_{\alpha}(\mathbf{x})w(\mathbf{x}) - \frac{1}{2} \frac{\partial}{\partial x_{\beta}} K_{\alpha\beta}(\mathbf{x})w(\mathbf{x}). \quad (2)$$

In order to find the stationary solution, we do *not* assume, as Stratonovitch and everybody else does, that $G_{\alpha}(\mathbf{x}) = 0$ and set $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ in equation (1), obtaining:

$$\begin{aligned} & \sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} \left(d_{\alpha}(\mathbf{x})e^{-U(\mathbf{x})} - \frac{1}{2} \frac{\partial}{\partial x_{\beta}} e^{-U(\mathbf{x})} K_{\alpha\beta}(\mathbf{x}) \right) = \\ & = \sum_{\alpha=1}^n \frac{\partial}{\partial x_{\alpha}} e^{-U(\mathbf{x})} \left(d_{\alpha}(\mathbf{x}) + \frac{1}{2} K_{\alpha\beta}(\mathbf{x}) \frac{\partial}{\partial x_{\beta}} U(\mathbf{x}) - \frac{1}{2} \frac{\partial}{\partial x_{\beta}} K_{\alpha\beta}(\mathbf{x}) \right) = 0 \end{aligned}$$

Assuming that the diffusion matrix is constant, that is $K_{\alpha\beta}(\mathbf{x}) = K_{\alpha\beta}$, we obtain:

$$e^{-U(\mathbf{x})} \sum_{\alpha=1}^n \left[-\frac{\partial}{\partial x_{\alpha}} U(\mathbf{x}) \left(d_{\alpha}(\mathbf{x}) + \frac{1}{2} K_{\alpha\beta} \frac{\partial}{\partial x_{\beta}} U(\mathbf{x}) \right) + \frac{\partial}{\partial x_{\alpha}} d_{\alpha}(\mathbf{x}) + \frac{1}{2} K_{\alpha\beta} \frac{\partial^2}{\partial x_{\beta} \partial x_{\alpha}} U(\mathbf{x}) \right] = 0$$

Provided that the diffusion matrix K is invertible we can effectuate the coordinate transformation

$$x_{\alpha} \rightarrow \frac{1}{2} K_{\beta\alpha} x_{\beta}$$

and defining the vector $(\mathbf{d})_{\alpha} = d_{\alpha}(\mathbf{x})$ we rewrite the previous equation as

$$\begin{aligned} -\nabla U \cdot \mathbf{d} - \nabla U \cdot \nabla U + \nabla \cdot \mathbf{d} + \nabla^2 U &= \\ = -\nabla U(\mathbf{d} + \nabla U) + \nabla(\mathbf{d} + \nabla U) &= 0. \end{aligned}$$

We finally obtain

$$(\nabla - \nabla U) \cdot (\nabla U + \mathbf{d}) = 0, \quad (3)$$

which is the condition for stationary distribution.

Thus, one solution is:

$$\nabla U + \mathbf{d} = 0 \Leftrightarrow \mathbf{d} = -\nabla U, \quad (4)$$

which is equivalent to the so called *potential conditions*, that amount to say that \mathbf{d} is the gradient of a potential. If the potential conditions are satisfied the probability current $G_{\alpha}(\mathbf{x})$ is identically zero, and thus *detailed balance* holds. Therefore we recover the well known result that detailed balance implies the existence of a stationary Gibbs distribution $w(\mathbf{x}) = e^{-U(\mathbf{x})}$. However condition (3) shows that the converse is not true. In fact equation (3) has also the solution

$$(\nabla - \nabla U) \cdot \mathbf{f} = 0, \quad (5)$$

with $\mathbf{f} = \nabla U + \mathbf{d}$ and this solution is *not trivial only if* $\mathbf{f} \neq 0$, that is if \mathbf{d} is not the gradient of a function. Equation (3) has therefore a "larger" space of solutions than the one represented by the potential conditions. Of course in

both cases the solution U must be such that $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ is a probability distribution, and therefore the following additional condition must hold:

$$\int d\mathbf{x} e^{-U(\mathbf{x})} < \infty$$

A simple and interesting example that proves the existence of non-trivial solutions U such that $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ is the following.

Example of existence

Consider the stochastic differential equation in \mathcal{R}^2

$$\begin{cases} \dot{x} = -2x + y + \xi_x(t) \\ \dot{y} = -x - 2y + \xi_y(t) \end{cases} \quad (6)$$

where $\xi_x(t)$ and $\xi_y(t)$ are Gaussian noise terms, that is

$$\langle \xi_x(t) \xi_x(t') \rangle = \langle \xi_y(t) \xi_y(t') \rangle = 2\delta(t - t') .$$

The F-P equation associated to (6) is

$$\frac{\partial}{\partial t} W(\mathbf{x}, t) = -\nabla \cdot (W(\mathbf{x}, t) \mathbf{d}(\mathbf{x})) + 2\nabla^2 W(\mathbf{x}, t)$$

where the drift vector is $\mathbf{d}(\mathbf{x}) = (-2x + y, -x - 2y)$. It is easy to verify that $\mathbf{d}(\mathbf{x})$ is not a conservative field, so that detailed balance does not hold. However a stationary solution of the F-P exists, with $w(\mathbf{x}) = e^{-U(\mathbf{x})}$ and

$$U(\mathbf{x}) = x^2 + y^2 .$$

In fact, defining $\mathbf{f} = \nabla U + \mathbf{d}$ we have

$$\mathbf{f} = (2x - (2x - y), 2y - (x + 2y)) = (y, -x)$$

and therefore equation (5) is satisfied, since

$$(\nabla - \nabla U) \cdot \mathbf{f} = \nabla \cdot \mathbf{f} - \nabla U \cdot \mathbf{f} = 2(x, y) \cdot (y, -x) = 0 .$$

Notice that in absence of noise the differential equation (6) is linear, with characteristic eigenvalues $\lambda = -2 \pm i$, and the associated trajectories are inward spirals. This makes perfectly plausible the fact that, in presence of

noise, the probability distribution of the variables is a Gaussian centered in the origin.

Remarks:

- In the linear case, that is when $d(x) = Ax$ and A is a symmetric matrix, detailed balance always holds, because $d = -\nabla U$ with $U(x) = -\frac{1}{2}xAx$. However the stationary solution $w(x) = e^{-U(x)}$ exists only if the matrix A is negative definite, that is if $w(x)$ is integrable.
- Stratonovitch (1963, p. 79) says that even when potential conditions are not met but d is a linear function of x and $K_{\alpha\beta}(x)$ are independent of x , the F-P equations can be solved. In fact it is easy to see that if $d(x) = Ax$ and A is *not* symmetric the potential conditions do not hold but the function $U(x) = -\frac{1}{2}xAx$ is a solution of equation (5).
- If the forces d_α in the Langevin equation are conservative, i.e., $d = -\nabla U$, then, if the fluctuations are thermic-like, detailed balance is satisfied and a Gibbs stationary distribution exists (Equation 4 is satisfied).
- It appears that our results may be derivable from the formulation of Graham (1980) and the more general case considered by Jauslin (1984) and Zeeman (1988). An in-depth analysis of many properties of the Fokker-Planck equation relevant for this note can be found in Tan and Wyatt (1985).

Acknowledgements We are grateful to J. Wyatt who read this note and provided several remarks that go well beyond the scope of it.

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