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VARIATIONAL PRINCIPLE FOR SOME NONSTANDARD ELASTIC PROBLEMS*

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Variational Principles for Some Nonstandard Elastic Problems

Variational principles are derived for some nonstandard problems involving elastic bodies in smooth contact. For these problems, the portions of the surfaces where one boundary condition holds rather than another must be determined as part of the solution to the problem. Cases considered include a body containing a crack or delamination, indentation by a rigid punch, and contact with an elastic foundation.

Introduction

The principles of minimum potential energy and minimum complementary energy for infinitesimal elastic deformations are well known for problems in which the tractions or the displacements are specified at each point of the surface of an elastic body (see Sokolnikoff, 1956, for example). The principles can lead to bounds on quantities of physical interest and can be used to obtain approximate analytical and numerical solutions. For some problems, the portions of the surface where one boundary condition holds rather than another must be determined as part of the solution to the problem. For example, in problems involving contact between elastic bodies, the shape of the contact area can vary with the loading (for references see Gladwell, 1980). Uniqueness of solution for typical problems of this type was considered by Shield (1982) for problems involving smooth contact between surfaces of elastic bodies. Here we again consider elastic problems involving smooth contact and develop variational principles for some typical situations: loading of a body containing a crack across which there is no cohesion, bodies in smooth contact, indentation by a smooth rigid punch, and contact with a smooth elastic foundation. The examples can be combined to treat more complex problems, such as the indentation of an elastic body containing a crack.

The principles rest on the positive-definiteness of the strain energy and they show that the potential and complementary energies attain absolute minimum values only when the trial functions generate the strains or stresses of the actual solution. Weaker stationary principles apply without the assumption of positive-definiteness of the strain energy. It is assumed that the integrals involved are convergent if they are improper and this requires the states considered to have finite total strain energy or total complementary energy. The elastic material can be inhomogeneous. At an interface between two materials in a composite, the displacements and traction are assumed to be continuous across the interface but delaminations can be included if they are modeled as cracks across which there is no cohesion.

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Elastic Body With a Crack

We assume that the strain-energy density W of the body is positive-definite and we write

$$2W(e) = c_{ijkl}e_{ij}e_{kl} \quad (c_{ijkl} = c_{klij} = c_{jikl}),$$

where e_{ij} are the infinitesimal strains referred to rectangular Cartesian axes x_i. The stresses t_{ij} are related to the strains through

$$t_{ij} = c_{ijkl}e_{kl}, \quad e_{ij} = C_{ijkl}t_{kl},$$

where C_{ijkl} have the same symmetries as c_{ijkl}. The strain-energy is also a positive-definite function W_C of the stresses,

$$2W_C(t) = C_{ijkl}t_{ij}t_{kl}.$$

In the unstressed reference state, the body occupies a region V with surface S and we suppose that it contains a crack across which the material has no cohesion. The crack is defined by a surface C in V, and we use $\mathbf{n}$ to denote the unit normal to one side of the surface C. We use $\pm$ signs to indicate values of quantities on the two sides of C, with the $+$ sign referring to the side of C with exterior normal $\mathbf{n}$. We shall also use square brackets to indicate the difference in the values of a quantity across C, so that for the displacement field $\mathbf{u}$, for example,

$$[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^- .$$

Under loading of the body, we assume that at points of the crack surface C either (i) the crack opens with no traction transmitted across C or (ii) the two sides remain in smooth contact. Then at points of C we have

$$(\mathbf{u}^+ - \mathbf{u}^-)\cdot\mathbf{n} = [\mathbf{u}]\cdot\mathbf{n} \le 0, \tag{1}$$

and we require on C

$$\left.\begin{array}{ll} \text{either } (i)\ \mathbf{T}^+ = -\mathbf{T}^- = 0 & \text{when } [\mathbf{u}]\cdot\mathbf{n} < 0 \\ \text{or } (ii)\ \mathbf{T}^+ = -\mathbf{T}^+ = -p\mathbf{n} & \text{when } [\mathbf{u}]\cdot n = 0, \end{array}\right\} \tag{2}$$

where $\mathbf{T}$ is the surface traction and $p(x)$ is the (nonnegative) pressure transmitted across C. We set $p=0$ at points of C where (i) holds.

The loading of the body is assumed to be caused by a body force $\mathbf{F}$, prescribed surface tractions $\mathbf{T}^G$ on a portion S_T of S and prescribed displacements $\mathbf{u}^G$ on the remainder S_U of S. The displacement field $\mathbf{u}$ is unique except possibly for a rigid displacement, depending on the conditions on S_U (Shield, 1982).

We define the potential energy P as the functional

$$P\{u'\} = \int_V W(e')dV - \int_{S_T} \mathbf{T}^G \cdot \mathbf{u}' dS - \int_V \mathbf{F} \cdot \mathbf{u}' dV \qquad (3)$$

for fields $\mathbf{u}'$ such that

$$\mathbf{u}' = \mathbf{u}^G \text{ on } S_U, \quad [\mathbf{u}'] \cdot \mathbf{n} \le 0 \text{ on } C. \qquad (4)$$

We set

$$\mathbf{u}' = \mathbf{u} + \Delta\mathbf{u}.$$

Then $\Delta\mathbf{u}$ is zero on S_U and we have

$$W(e') = W(e) + t_{ij}\Delta e_{ij} + W(\Delta e). \qquad (5)$$

The stresses t_{ij} satisfy equilibrium with body force $\mathbf{F}$ and with the divergence theorem and boundary conditions we get

$$P\{u'\} - P\{u\} = \int_V W(\Delta e)dV + \int_C \mathbf{T} \cdot [\Delta\mathbf{u}]dS,$$

where contributions from both sides of C have been included in the integral over C. For the actual solution $\mathbf{T} \cdot [\mathbf{u}]$ is zero on C in view of equations (2) and the integral over C has the value

$$\int_C \mathbf{T} \cdot [\mathbf{u}']dS = -\int_C p\mathbf{n} \cdot [\mathbf{u}']dS.$$

From equation (4) this is seen to be greater than or equal to zero and with W positive-definite, we have

$$P\{u'\} \ge P\{u\}$$

with equality if and only if $\mathbf{u}$ and $\mathbf{u}'$ have the same strains and $\mathbf{n} \cdot [\mathbf{u}']$ is zero where p is nonzero.

Thus we have: *For displacements which satisfy the displacement boundary conditions and have no interpenetration of material across the crack, the potential energy* P *is least for the displacements of the actual solution.*

The complementary energy Q is defined as

$$Q\{t'\} = \int_V W_C(t')dV - \int_{S_U} \mathbf{T}' \cdot \mathbf{u}^G dS \qquad (6)$$

for stresses t'_{ij} in equilibrium with $\mathbf{F}$ and such that

$$t_{ij}n_j = T_i^G \text{ on } S_T, \ \mathbf{T}'^{+} = -\mathbf{T}'^{-} = -p'\mathbf{n} \text{ on } C,$$

where $p' \ge 0$. We set

$$t'_{ij} = t_{ij} + \Delta t_{ij},$$

so that the stresses Δt_{ij} satisfy equilibrium with no body force and have zero traction on S_T. We have

$$W_C(t') = W_C(t) + \Delta t_{ij}e_{ij} + W(\Delta t), \qquad (7)$$

and with the divergence theorem we obtain

$$Q\{t'\} - Q\{t\} = \int_V W_C(\Delta t)dV + \int_C \Delta T \cdot [\mathbf{u}]dS.$$

As before $\mathbf{T} \cdot [\mathbf{u}]$ is zero on C and the integral over C becomes

$$\int_C \mathbf{T}' \cdot [\mathbf{u}]dS = -\int_C p'\mathbf{n} \cdot [\mathbf{u}]dS \ge 0.$$

It follows that

$$Q\{t'\} \ge Q\{t\}$$

with equality if and only if $t'_{ij} = t_{ij}$ in V, and we have: *For stress fields with the given surface tractions and in equilibrium with the given body force and which transmit at most pressure across the crack, the complementary energy Q is least for the stresses of the actual solution.*

For the actual solution we have

$$P\{u\} + Q\{t\} = 2\int_V WdV - \int_S \mathbf{T} \cdot \mathbf{u}dS - \int_V \mathbf{F} \cdot \mathbf{u}dV = 0, \qquad (8)$$

using the divergence theorem and $\mathbf{T} \cdot [\mathbf{u}] = 0$ on the crack (Clapeyron's theorem). We then have

$$P\{u'\} \ge P\{u\} \ge -Q\{t'\}, \qquad (9)$$

and the principles provide upper and lower bounds for the potential energy.

Smooth Contact Between Elastic Bodies

For simplicity we consider contact between two elastic bodies occurring over parts of the surfaces of the bodies which are nearly plane, but the approach is easily generalized. In the reference configuration the bodies touch at the origin 0 of coordinates and the plane $x_3 = 0$ is tangent to both bodies at 0. The bodies occupy regions V_1 and V_2 with the x_3 axis pointing into V_2. Under loading, contact may occur over surfaces C_1 and C_2 of the bodies, defined as the nearly plane surfaces

$$C_1: x_3 = f(x_1, x_2), \quad C_2: x_3 = g(x_1, x_2),$$

where x_1, x_2 lie in a region C of the $x_1 - x_2$ plane enclosing the origin and

$$f(x_1, x_2) \le g(x_1, x_2).$$

The contact is smooth and we denote the pressure between the two bodies by the nonnegative function $p(x_1, x_2)$, defined over C. Then

$$T_3^2 = -T_3^1 = p(x_1, x_2) \text{ on } C, \qquad (10)$$

where the superscripts indicate values for the two bodies. The bodies do not penetrate each so that the displacement compo- other
nent u_3 satisfies

$$u_3^2 - u_3^1 \ge f - g \text{ on } C. \qquad (11)$$

At each point of C we require

$$p = 0 \text{ when } u_3^2 - u_3^1 > f - g, \ u_3^2 - u_3^1 = f - g \text{ when } p > 0. \qquad (12)$$

On the remaining portions S_1 and S_2 of the surfaces of the two bodies, we suppose that tractions $\mathbf{T}^G$ are specified on parts S_{1T}, S_{2T} and displacements $\mathbf{u}^G$ are specified on parts S_{1U}, S_{2U} of S_1, S_2, respectively. We use S_T and S_U to denote $S_{1T} + S_{2T}$ and $S_{1U} + S_{2U}$. The body force $\mathbf{F}$ is assumed known in $V = V_1 + V_2$. The solution is unique except possibly for a rigid body displacement (Shield, 1982).

The potential energy P is again defined by equation (3) for displacements which satisfy the boundary conditions on S_U and which satisfy (11) on C. As in the previous section, we look at the difference between $P\{u'\}$ and $P\{u\}$, use equation (5) and the divergence theorem, and find that the difference involves the integral

$$\int_{C_1 + C_2} \mathbf{T} \cdot (\mathbf{u}' - \mathbf{u})dS.$$

In view of equations (10) and (12), the integral becomes

$$\int_C p\{u_3'^2 - u_3'^1 - (f - g)\}dS,$$

and this is nonnegative because u'_3 satisfies (11). Thus we can show that: *For displacements which satisfy the displacement boundary conditions and which satisfy (11) on C, the potential energy is least for the displacements of the actual solution.*

The complementary energy Q is defined by

$$Q\{t'\} = \int_V W_C(t')dV - \int_{S_U} \mathbf{T}' \cdot \mathbf{u}^G dS - \int_C p'(f - g)dS \qquad (13)$$

for stresses t'_{ij} in equilibrium with the given body force and given tractions and which involve at most a pressure p' between the bodies across C. The difference between $Q\{t'\}$ and $Q\{t\}$ is transformed as in the previous section and we are led to consider the sign of

$$\int_{C_1+C_2} (\mathbf{T}'-\mathbf{T})\cdot\mathbf{u}dS - \int_C (p'-p)(f-g)dS.$$

This can be written as

$$\int_C (p'-p)\{u_3^2 - u_3^1 - (f-g)\}dS.$$

and with equations (11) and (12) we see that the integral is nonnegative. Thus we can show that: *For stress fields in equilibrium with the given body force and given tractions and which involve at most a pressure between the bodies across C, the complementary energy is least for the stresses of the actual solution.*

For the actual solution we again have $P\{u\} = -Q\{t\}$ and the principles lead to upper and lower bounds on $P\{u\}$.

Indentation by a Smooth Rigid Punch

In order to illustrate problems in which an elastic body can come into contact with a rigid body of known shape, we consider indentation by a smooth rigid punch when the possible area of contact is a region C of the x_1-x_2 plane enclosing the origin 0. The exterior normal to the body at 0 is along the x_3 axis, and the remainder of the surface of the body is denoted by S.

When the movement of the punch is known, we will have

$$u_3 \le g(x_1, x_2) \text{ on } C, \tag{14}$$

where g is a known function, and we require at each point of C

$$p=0 \text{ when } u_3 < g,\ u_3 = g \text{ when } p > 0, \tag{15}$$

where p is the contact pressure. Tractions are prescribed on a part S_T of S and displacements on the remainder S_U of S, with a known body force in the region V occupied by the body.

The problem can be considered as a limiting case of contact between two elastic bodies. The potential energy is defined to be the functional (3) for displacements which satisfy the conditions on S_U and (14) on C. We can then proceed as in the previous section to show that: *The potential energy is least for the displacements of the actual solution.*

The complementary energy Q is defined to be

$$Q\{t'\} = \int_V W_C(t')dV - \int_{S_U}\mathbf{T}'\cdot\mathbf{u}^G dS + \int_C p'g dS$$

for stresses in equilibrium with the given body force and surface tractions and which involve a pressure p' in the contact area C. Then: Q *will be an absolute minimum for the stresses of the actual solution.*

Instead of prescribing the movement of the punch, we may prescribe the downward force L on the punch and the moments M_1, M_2 of the force about the x_1, x_2 axes, with prescribed loading on S as before. The contact pressure p must then satisfy

$$L = \int_C p dS,\quad M_1 = -\int_C p x_2 dS,\quad M_2 = \int_C p x_1 dS. \tag{16}$$

For a known punch shape $g(x_1, x_2)$ the solution will satisfy

$$u_3 \le g(x_1, x_2) - d + ax_2 - bx_1 \text{ on } C, \tag{17}$$

with equality where p is nonzero. The constants d, a, b are determined as part of the solution, which will be unique except possibly for a rigid body displacement (Shield, 1982). (Other problems may be treated; for example, we may require the punch to indent without tilting and then a, b are zero and M_1, M_2 are not prescribed.)

The potential energy is defined as

$$P\{u'\} = \int_V W(e')dV - \int_{S_T}\mathbf{T}^G\cdot\mathbf{u}'dS - \int_V \mathbf{F}\cdot\mathbf{u}'dV - Ld' - M_1a' - M_2b' \tag{18}$$

for fields $\mathbf{u}'$ which satisfy the displacement conditions on S_U and which are such that

$$\mathbf{u}'_3 \le g(x_1, x_2) - d' + a'x_2 - b'x_1 \text{ on } C, \tag{19}$$

where d', a', b' are constants. After transformation, we find that

$$P\{\mathbf{u}'\} - P\{u\} = \int_V W(\Delta e)dV - \int_C p(\mathbf{u}_3' - u_3)dS - L(d'-d) - M_1(a'-a) - M_2(b'-b).$$

If we set

$$v = u_3 + d - ax_2 + bx_1,\ v' = u_3' + d' - a'x_2 + b'x_1$$

and use equations (16), then apart from the strain-energy term the right-hand side becomes

$$-\int_C p(v'-v)dS.$$

Now where $p > 0$, equality holds in (17) and $v = g$. Because $v' \le g$ from (19), we then see that the integrand is nonpositive. Thus: *The potential energy is least for the displacements of the actual solution.*

The complementary energy is defined to be

$$Q\{t'\} = \int_V W_C(t')dV - \int_{S_U}\mathbf{T}'\cdot\mathbf{u}^G dS + \int_C p'g dS$$

for stresses in equilibrium with the given body force and surface tractions and which involve a pressure p' in the contact area C satisfying the loading conditions (16). We can then show that

$$Q\{t'\} - Q\{t\} = \int_V W_C(\Delta t)dV - \int_C (p'-p)(u_3 - g)dS.$$

Because p' and p apply the same resultant force and moment over C, the integral over C can be written as

$$\int_C (p'-p)(u_3 - g + d - ax_2 + bx_1)dS.$$

Equality holds in (17) where $p > 0$ so that there is no contribution to the integral from p, and the contribution from p' is nonpositive in view of (17). It follows that: *The complementary energy is least for the stresses of the actual solution.*

For the actual solution we have $P\{u\} = -Q\{t\}$ for both punch problems of this section.

Smooth Contact With an Elastic Foundation

Variational principles also hold for elastic bodies which can receive support from a foundation of the Winkler type. For simplicity we assume here that the foundation has a plane surface and lies below the plane $x_3 = 0$. The portion of the surface of an elastic body that can come into contact with the foundation is the nearly plane surface

$$x_3 = f(x_1, x_2)$$

touching the plane $x_3 = 0$ but lying entirely above it, where x_1, x_2 lie in a region C of the x_1-x_2 plane. The reactive pressure p of the foundation is proportional to the downward displacement of the surface, so that on C we require

$$p=0 \text{ when } u_3 \geq -f, p=-K(u_3+f) \text{ when } u_3 < -f, \quad (20)$$

where $K \geq 0$ is the stiffness of the foundation (K may vary with x_1, x_2). If we define $q(u_3)$ by

$$q(u_3)=0 \text{ when } u_3 \geq -f, q(u_3)=1 \text{ when } u_3 < -f,$$

then we can write

$$T_3=p=-K(u_3+f)\ q(u_3) \text{ on } C, \quad (21)$$

while the tangential tractions are zero on C. Boundary conditions on the remaining surface S of the body and a body force field are prescribed as before.

The potential energy is defined to be

$$P\{u'\}=\int_V W(e')dV-\int_{S_T}\mathbf{T}^G\cdot\mathbf{u}'dS$$

$$-\int_V \mathbf{F}\cdot\mathbf{u}'dV+\frac{1}{2}\int_C K(u_3'+f)^2q'dS,$$

where $q' = q(u_3')$, for displacement fields u_3' which satisfy the displacement boundary conditions on S_U. Using equation (5) and the divergence theorem, we find that

$$P\{u'\}-P\{u\}=\int_V W(\Delta e)dV$$

$$+\frac{1}{2}\int_C K(v'^2q'+v^2q-2vqv')dS \quad (22)$$

in which

$$v=u_3+f, \quad v'=u_3'+f.$$

By considering the various possibilities for the signs of v and v' together with the corresponding values for the step functions q and q', we find that the integrand of the integral over C in equation (22) is nonnegative. Thus we again have: *The potential energy is an absolute minimum for the displacements of the actual solution.*

For the complementary energy we take

$$Q\{t'\}=\int_V W_C(t')dV-\int_{S_U}\mathbf{T}'\cdot\mathbf{u}^G dS$$

$$+\int_C (p'^2/2K+p'f)dS$$

for stresses in equilibrium with the given body force and surface tractions and which involve a pressure p' on the foundation interface C. Using equation (7) and the divergence theorem, we can show that

$$Q\{t'\}-Q\{t\}=\int_V W_C(\Delta t)dV$$

$$+\int_C \{(p'^2-p^2)/2K+(p'-p)(u_3+f)\}dS.$$

From equations (20), the integrand of the integral over C is found to be nonnegative and we see that: *The complementary energy is an absolute minimum for the stresses of the actual solution.*

We can also show that $P\{u\} = -Q\{t\}$, so that the principles can be used to bound $P\{u\}$.

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