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Analysis of Truss by Method of the Stiffness Matrix

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ANALYSIS OF TRUSS FRAMES BY METHOD OF THE STIFFNESS MATRIX

by

Ronald Laverne Kruse

A Thesis Presented in Partial Fulfillment
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ANALYSIS OF TRUSS FRAMES BY METHOD

OF THE STIFFNESS MATRIX

by

Ronald Laverne Kruse

has been approved

December 1990

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ABSTRACT

The development of the general stiffness coefficients and load constants are presented for the flat Pratt and gabled Pratt truss frames. The stiffness coefficients and load constants are derived through the application of the theorem of least work.

The elemental stiffness matrices for the flat and gabled Pratt truss frames are assembled using the respective stiffness coefficients for each type of truss.

Two examples illustrate the procedures for computing numerical solutions for each type of truss frame.

In memory of my brother

Kenneth

ACKNOWLEDGEMENTS

The writer wishes to express his gratitude to the following individuals for their guidance and support during his studies toward a Master of Science in Engineering degree:

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NOMENCLATURE

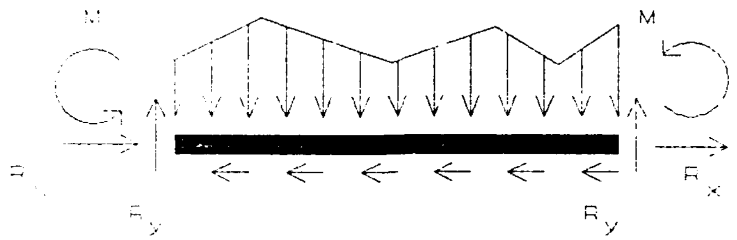
a	Half-length of a truss
A	Cross-sectional area of a given member
c	Moment arm of the horizontal redundant, H_o
CM	Cantilever moment due to applied loads
C_x, C_y, C_θ	Truss framing constants
d	Truss height
D_x, D_y, D_θ	Truss load constants
E	Modulus of elasticity
FH	Fixed end horizontal force (thrust)
FM	Fixed end moment
FV	Fixed end shear
h	Truss depth
H, U	Horizontal force
H_o	Horizontal redundant force = 1
i	Specified truss element or member property
I	Moment of inertia
Kc	Column stiffness coefficient
Kt	Truss stiffness coefficient
L_i	Length of a specified member
M, Z	Moment
N_i	Total normal force in a specified truss member
P_x	Applied external load--horizontal
P_y	Applied external load--vertical
R	Reaction

SN_i	Normal force in truss member due to loading
U_i	Internal energy
U_e	External energy
U_r	Energy of reactions
U_1	Energy of applied loads
V	Shear
W	Resultant of external loads
Y_o	Location of the elastic center
α_i	Influence in a specific member due to $H_o = 1$
β_i	Influence in a specific member due to $V_o = 1$
γ_i	Influence in a specific member due to $M_c = 1$
Θ	Angular rotation of a joint
Δ_x	Linear displacement--horizontal direction
Δ_y	Linear displacement--vertical direction
λ_i	Unit deformation of a truss member

SIGN CONVENTION

Stiffness--Method Sign Convention

Positive Forces and Moments



Positive Displacements

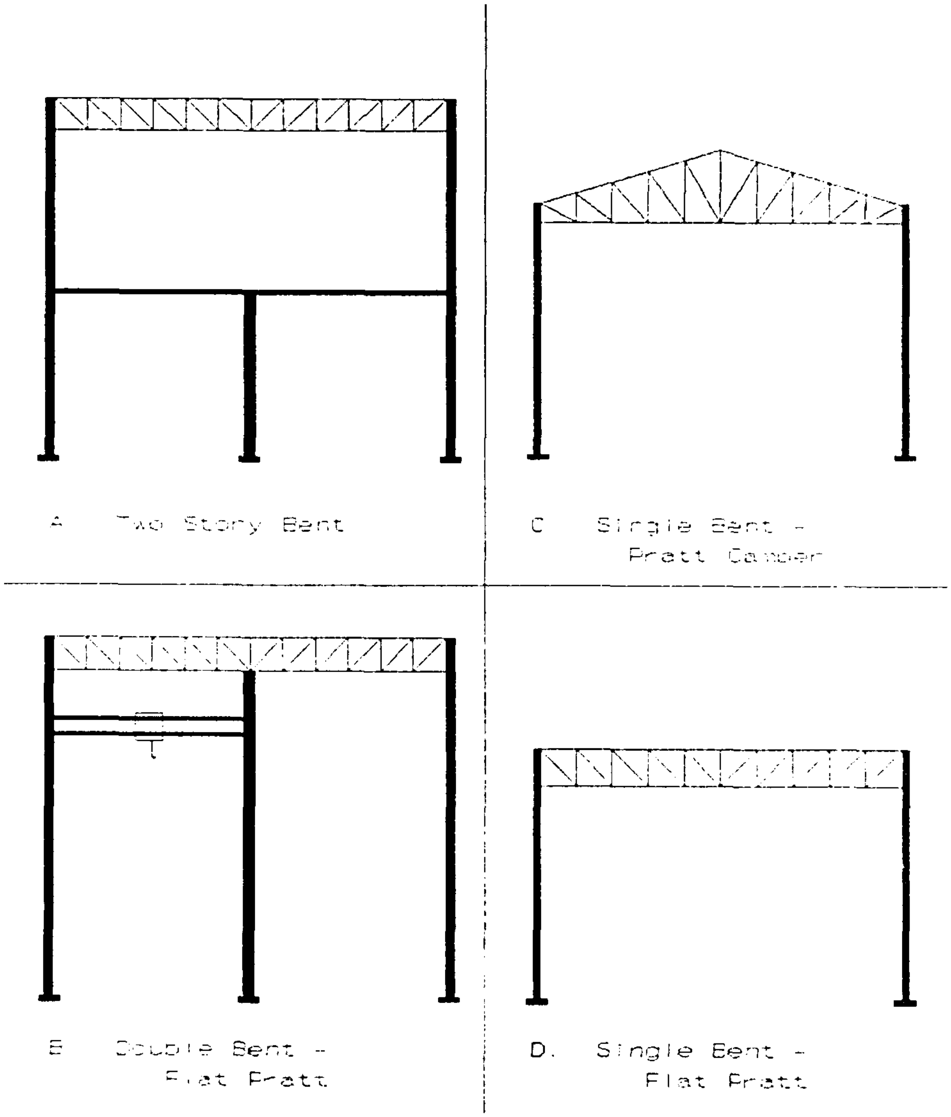
CHAPTER 1

Introduction

1.1 Concept of the Truss Frame

The truss frame is described as a structural steel framing system utilizing a truss as the load carrying horizontal member supported at its ends by columns. The truss frame has historically been utilized in what some authors and designers have referred to as "industrial buildings." Though these buildings are typically single story structures in which the roof is supported by the upper chords of the truss, the truss frame concept can also be incorporated into multi-story buildings as well. Figure 1.1 shows several examples.

The truss frames are generally lighter in weight than the typical beam-column framing system and thus provide a cost savings in material. In addition, due to the high stiffness of the truss, truss frame structures typically can span longer distances and therefore provide for larger open floor areas free of interior support columns found in most standard beam-column structural framing systems. The truss frame structure is generally best utilized when the



<u>Figure 1.1</u> Examples--truss frame configurations

clear span column spacing is greater than 40 feet but not larger than 120-140 feet (White & Solman, 1987). Normally for spans of less than 40 feet, many designers recommend the use of standard wide flange sections in a standard beam-column frame since these sections are readily available from standard steel pre-engineered building supply companies. For shorter spans, the use of the truss frame in steel weight is normally offset by the increased cost of labor to manufacture the truss, especially if done so as a special order. The designer should check with a standard products supplier to see if the shorter truss is readily available, if they in fact feel justified in its use for spans shorter than 40 feet.

1.2 Classification of Truss Frames

There are as many variations of the truss framed structure as given by the style of truss used. Examples of the variations include the Warren, Fink (or W), Pratt, and Howe. Each of these in general refer to the geometric configuration of the web members of the truss.

There also are variations in the truss frame given by the geometric shape of the frame, also referred to in some texts as a bent. These variations include the single story and multi-story previously mentioned, the multiple bent given by the use of interior supporting columns, and geometric shape variation given in the use of the gabled, flat, flat-cambered, and arched truss. (See Figure 1.2 for examples of the various shapes and styles of trusses.)

The choice as to which truss frame shape is used is normally one of architectural aesthetics or simply one of owner's preference. The selection of which truss type used is normally a preference of the designer. However, the Warren and Pratt are used most for flat roofs while the Fink and Pratt styles are used most for the large rise gabled roofs (White & Solman, 1987).

1.3 Historical Background

Early examples of truss-frame structures include steel mills, train-locomotive repair and maintenance shops, automotive assembly plants, and aircraft maintenance facilities and factories. Each of these examples typifies the requirement for a large open floor space with adequate overhead space to allow the use of an overhead traveling crane (Grinter, 1955).

Other applications and benefits of the truss frame can also be seen for smaller scale manufacturing plants, commercial retail sales buildings, and warehouses. Through the use of the truss frame, the interior columns are eliminated or at least significantly reduced. In the case of its use in a warehouse, this reduces the number of obstacles which forklift operators and delivery trucks

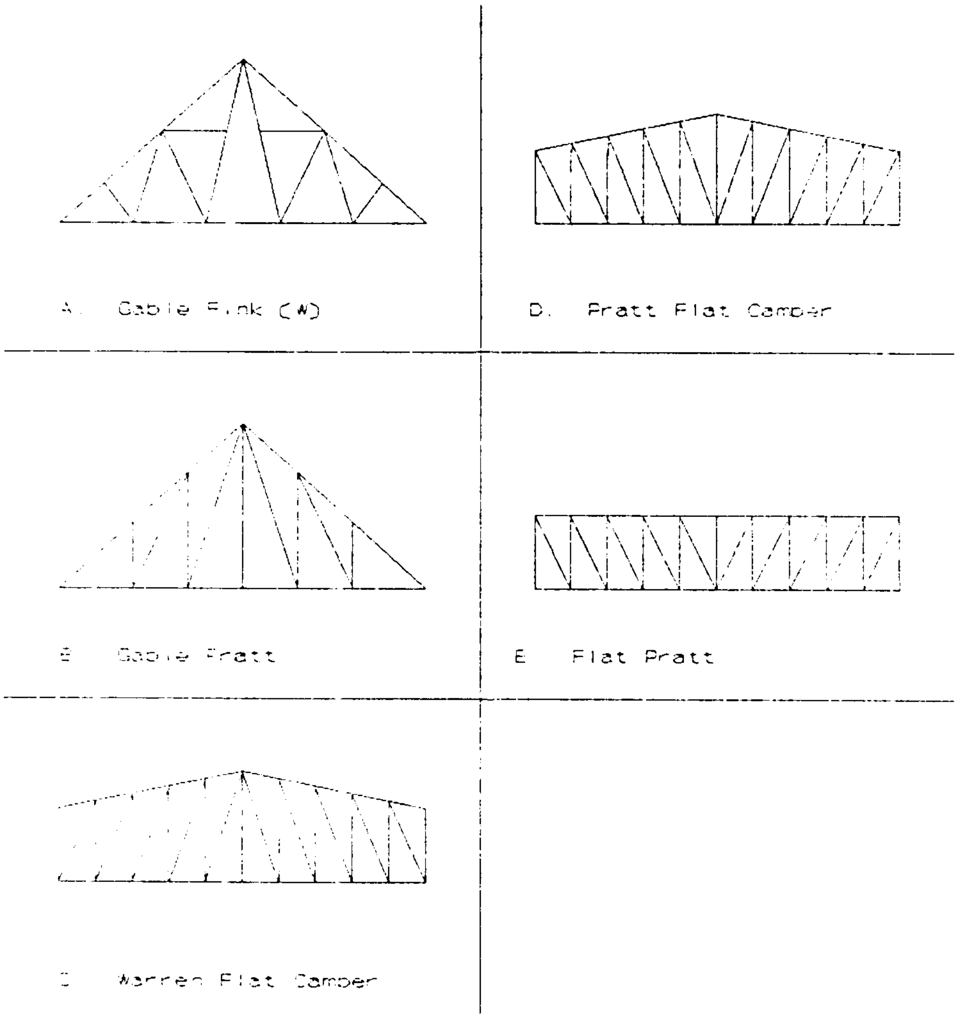


Figure 1.2 Examples--various truss shapes and styles

navigate around and thus reduces the opportunity for structural damage to occur to the structure caused by impact on the columns from forklifts and trucks operating in the warehouse. In the case of the commercial retail sales facilities, the use of the truss frame allows the building owner to erect a clear span facility without the worry of interior columns dictating the final layout of equipment or manufacturing processes. In the case of the commercial property development companies who lease their buildings to other users, the benefit of the clear space offered by the truss frame allows the developer to construct the facility without limiting the use or layout of the facility because of interior column constraints. It should be pointed out, however, that if the end user intends to suspend mechanical equipment such as an overhead hoist from the truss or other special mechanical systems not initially included in the designer's calculations, a design investigation will be required to determine what, if any, structural changes and modifications will be required to accommodate the special equipment and associated loadings.

CHAPTER 2

Stiffness Coefficients and Load Functions

2.1 Introduction to the Load and Stiffness Coefficients

The load and stiffness coefficients for a parallel (flat) Pratt truss and a gabled truss are derived in Sections 2.2 and 2.3, respectively.

The development of the load and stiffness coefficients follows the derivation of the slope-deflection equations contained in the unpublished Masters of Science theses of Morrisett (1957) and Smith (1957), prepared at the School of Civil Engineering, Oklahoma State University, in 1957 under the direction of Dr. J. Tuma.

The derivations are based on the energy methods of Castigliano's theorem of least work. In each of the derivations, the properties of the elastic center are employed in order to simplify the number of terms contained in the deformation equations.

The necessary changes have been made to the slope--deflection derivation to reflect the stiffness sign convention defined by Dr. J. Tuma (1987). Also, several symbols in the nomenclature in the original theses are changed to simplify or clarify the equations and to once again conform to the sign convention used by Dr. Tuma (1987).

The final load and stiffness coefficient equations are located at the end of each section and are presented in a tabular form.

The load and stiffness coefficients for the truss frame columns are not derived in this paper. Their derivation is contained in several structural analysis text books offered at the undergraduate level, following the standard beam analogy. The load and stiffness coefficients for the columns are presented in tabular form in Section 2.4.

2.2 Derivation of the Load and Stiffness Coefficients: Flat Pratt Truss

2.2.1 Statics

A typical flat Pratt truss beam removed from a continuous elastic system, loaded by a general system of forces, is shown in Figure 2.2.1. The truss has constant depth and is fixed at both ends.

In the analysis of this truss, the following assumptions have been made:

1. All members are connected by frictionless hinges.

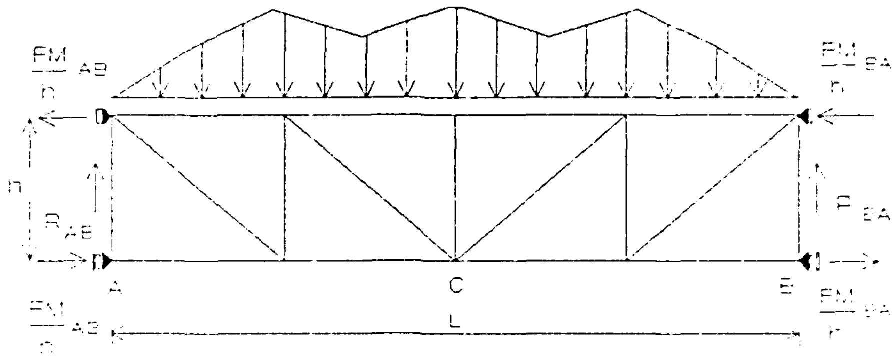


Figure 2.2.1 Truss beam with general system of loading

2. All members are subjected to axial forces only, and the influence of shear and bending moment is neglected.

3. The truss and the loads are forming a coplanar system.

4. All loads are applied at joints.

5. The deformations of the truss are elastic and small.

The structure has four reactions: two reactive forces, R_{AY} and R_{BY}, and two reactive moments, FM_{AB} and FM_{BA}. The problem is statically indeterminate to the second degree and requires two equations of deformation to form a solution.

The general displacements of the supports are given by Δ_{AY}, Δ_{BY}, Θ_A and Θ_B. Figure 2.2.2 shows the free body sketch of segments AC and CB. The resultant of the loads corresponding to part AC and CB are denoted by W_1 and W_2, respectively. The redundant forces at the center of the cross-section are V_o and M_o/h. Assuming all displacements and reactions are positive and using conditions of static equilibrium, the end reactions of parts AC and CB are:

$$R_{AY} = W_1 + V_o, \qquad M_{AB} = -M_o + aV_o + CM_{AC}$$
$$R_{BY} = W_2 - V_o, \qquad M_{BA} = M_o + aV_o - CM_{BC} \qquad (1)$$

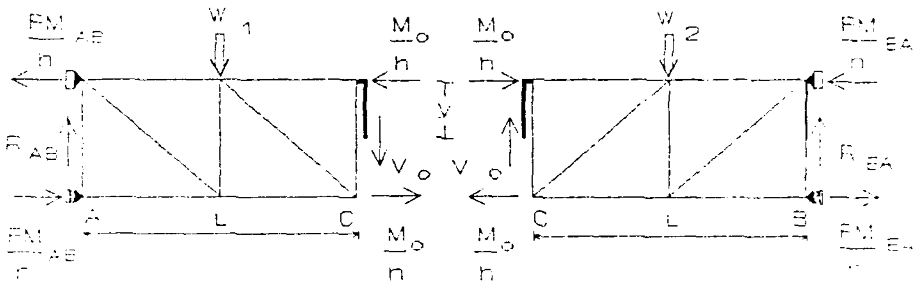


Figure 2.2.2 Free-body trusses AC and CB with displaced internal forces

CM_{AC} and CM_{BC} are the cantilever moments at c due to W_1 and W_2, respectively. Since there are no horizontal loads applied to the truss, R_{AX} and $R_{AY} = 0$.

The normal force for any member in the truss due to the applied loads and the redundants is:

$$N_i = SN_i + \alpha_i H_o + \beta_i V_o + \gamma_i M_o \tag{2}$$

where SN_i = normal force in a given member due to loading on the truss

α_i = normal force in a given member due to $H_o = 1$

β_i = normal force in a given member due to $V_o = 1$

and γ_i = normal force in a given member due to $M_o = 1$

Due to the geometry and loadings of the truss, the axial deformation and normal force due to H_o will be neglected. As such, $\alpha_i H_o = 0$ will be eliminated at this point in the derivation.

2.2.1 <u>Least Work</u>

The Principle of Minimum Potential Energy is given as:

$$U_i = U_e \tag{3}$$

where U_e = the external work

and U_i = the internal work

The internal work, U_i, is formed by:

$$U_i = U_s + U_v \tag{3a}$$

where U_s = the strain energy of the structure

U_v = the strain energy due to volume change

The energy due to volume change will be neglected. Only energy of normal forces will be considered. Hence, Equation (3a) becomes:

$$U_i = U_s = \sum_A^B \frac{N_i^2 L_i}{2A_i E} \tag{3b}$$

where L_i = length of any member

A_i = cross-sectional area of any member

and E = modulus of elasticity

If we let $\lambda_i = \frac{L_i}{A_i E}$, then Equation (3b) may be rewritten as:

$$U_i = U_s = \sum_A^B \frac{N_i^2 \lambda_i}{2} \tag{3c}$$

The external work is given by:

$$U_e = U_l + U_r \tag{3d}$$

where $U_l = \sum_A^B W\Delta + \sum_A^B W\Theta$ = work due to applied loads

and $U_r = \sum_A^B R\Delta + \sum_A^B M\Theta$ = work due to reactions

The work of the supports in terms of reactions and displacements given by Equation (1) is:

$$U_r = R_{AY}\Delta_{AY} + M_{AB}\Theta_A + R_{BY}\Delta_{BY} + M_{BA}\Theta_B$$

Substituting the equivalents for each of the reactions from Equation (1):

$$U_r = (W_1 + V_o)\Delta_{AY} + (-M_o + aV_o + CM_{AC})\Theta_A + (W_2 - V_o)\Delta_{BY} + (M_o + aV_o - CM_{CB})\Theta_B \quad (3e)$$

From Castigliano's theorems, the first partial derivative of the strain energy of a truss with unyielding supports, with respect to a redundant, is equal to zero. Allowing displacement of supports, we have:

$$\frac{\partial U_r}{\partial M_o} = \frac{\partial U_s}{\partial M_o} \qquad \frac{\partial U_r}{\partial V_o} = \frac{\partial U_s}{\partial V_o} \quad (4)$$

The partial derivatives of Equation (3c) with respect to each redundant are:

$$\frac{\partial U_s}{\partial M_o} = \sum_A^B N_i \frac{\partial N_i}{\partial M_o} = \sum_A^B N_i \gamma_i \lambda_i \quad (4a)$$

$$\frac{\partial U_s}{\partial V_o} = \sum_A^B N_i \frac{\partial N_i}{\partial V_o} = \sum_A^B N_i \beta_i \lambda_i \quad (4b)$$

The partial derivatives of Equation (3e) with respect to each redundant are:

$$\frac{\partial U_r}{\partial M_o} = -\Theta_A + \Theta_B \tag{4c}$$

$$\frac{\partial U_r}{\partial V_o} = \Delta_{AY} + a\Theta_A + a\Theta_B - \Delta_{BY}$$

If we let $\Delta_Y = \Delta_{AY} - \Delta_{BY}$

then,

$$\frac{\partial U_r}{\partial M_o} = -\Theta_A + \Theta_B$$

$$\frac{\partial U_r}{\partial V_o} = \Delta_Y + a\,(\Theta_A + \Theta_B) \tag{4d}$$

2.2.3 Deformation Equations

Equations (4) in terms of Equations (4a, 4b, 4c, 4d) may now be written as:

$$-\Theta_A + \Theta_B = \sum_A^B SN_i\gamma_i\lambda_i + M_o \sum_A^B \gamma_i^2\lambda_i + V_o \sum_A^B \beta_i\gamma_i\lambda_i \tag{4e}$$

$$\Delta_Y + a(\Theta_A + \Theta_B) = \sum_A^B SN_i\beta_i\lambda_i + M_o \sum_A^B \gamma_i\beta_i\lambda_i + V_o \sum_A^B \beta_i^2\lambda_i$$

From symmetry of the truss and making use of the properties of the elastic center:

$$\sum_A^B \gamma_i\beta_i\lambda_i = \sum_A^B \beta_i\gamma_i\lambda_i = 0$$

The equations in (4e) can be reduced to:

$$- \Theta_A + \Theta_B = \sum_A^B SN_i \gamma_i \lambda_i + M_o \sum_A^B \gamma_i^2 \lambda_i$$

(4f)

$$\Delta_Y + a\ (\Theta_A + \Theta_B) = \sum_A^B SN_i \beta_i \lambda_i + V_o \sum_A^B \beta_i^2 \lambda_i$$

Denoting:

$$D_\Theta = \sum_A^B SN_i \gamma_i \lambda_i, \qquad D_Y = \sum_A^B SN_i \beta_i \lambda_i$$

$$C_\Theta = \sum_A^B \gamma_i^2 \lambda_i, \qquad C_Y = \sum_A^B \beta_i^2 \lambda_i$$

where

D_Θ = truss load factor--rotations

D_Y = truss load factor--shear

C_Θ = truss framing constant--rotation

C_Y = truss framing constant--shear

Substituting the terms into Equation (4f), the deformation equations become:

$$- \Theta_A + \Theta_B = D_\Theta + M_o C_\Theta$$

(4g)

$$\Delta_Y + a\ (\Theta_A + \Theta_B) = D_Y + V_o C_Y$$

Solving these two equations, the redundants are:

$$M_o = - \frac{D_\Theta}{C_\Theta} - \frac{\Theta_A}{C_\Theta} + \frac{\Theta_B}{C_\Theta}$$

(5)

$$V_o = \frac{\Delta_Y}{C_Y} + \frac{a\ (\Theta_A + \Theta_B)}{C_Y} - \frac{D_Y}{C_Y}$$

Substituting the results of Equation (5) into Equation (1):

$$M_{AB} = \left(\frac{1}{C_\Theta} + \frac{a^2}{C_Y}\right)\Theta_A + \left(\frac{a^2}{C_Y} - \frac{1}{C_\Theta}\right)\Theta_B + \frac{a}{C_Y}(\Delta_{AY} - \Delta_{BY}) + \frac{D_\Theta}{C_\Theta} - a\frac{D_Y}{C_Y} + CM_{AC} \tag{6}$$

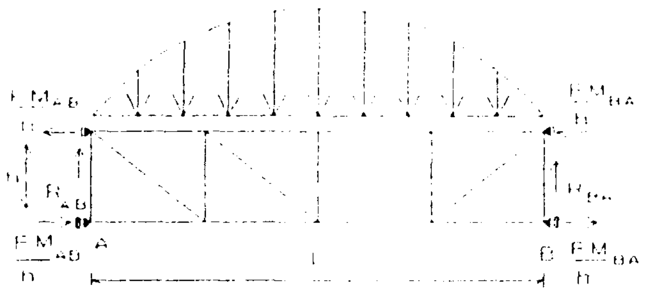
$$M_{BA} = \left(\frac{a^2}{C_Y} - \frac{1}{C_\Theta}\right)\Theta_A + \left(\frac{1}{C_\Theta} + \frac{a^2}{C_Y}\right)\Theta_B + \frac{a}{C_Y}(\Delta_{AY} - \Delta_{BY}) - \frac{D_\Theta}{C_\Theta} - a\frac{D_Y}{C_Y} - CM_{CB} \tag{7}$$

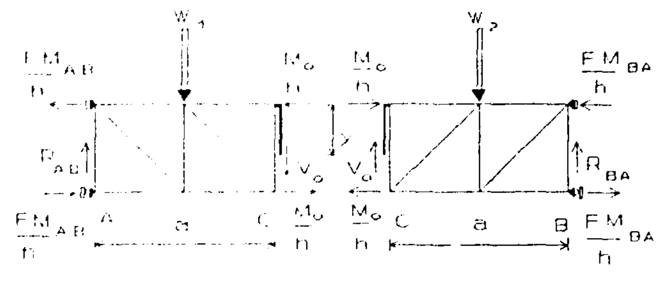
$$R_{AY} = W_1 + \frac{\Delta_{AY} - \Delta_{BY}}{C_Y} + \frac{a}{C_Y}(\Theta_A + \Theta_B) - \frac{D_Y}{C_Y} \tag{8}$$

$$R_{BY} = W_2 - \frac{\Delta_{AY} - \Delta_{BY}}{C_Y} - \frac{a}{C_Y}(\Theta_A + \Theta_B) + \frac{D_Y}{C_Y} \tag{9}$$

Note:

If the truss is loaded symmetrical with respect to the axis of symmetry of the truss, the load constant D_Θ will equal zero.





a. Flat truss structural system b. Flat truss free body diagram

Geometry: Symmetrical-flat truss of constant h. Elastically restrained at A and B. Loaded by a coplanar system of forces.

Load Constants:	Frame Constants:	Elastic Center:
$D_X = 0$	$C_X = 0$	$\overline{Y}_o$ = Located at the center of the girder, $\frac{h}{2}$.
$D_Y = \sum_{A}^{B} SN_i\beta_i\lambda_i$	$C_Y = \sum_{A}^{B} \beta_i^2\lambda_i$	
$D_\Theta = \sum_{A}^{B} SN_i\gamma_i\lambda_i$	$C_\Theta = \sum_{A}^{B} \gamma_i^2\lambda_i$	$\overline{X}_o = a$ (mid span-symmetric truss)

Figure 2.2.3 Structural constants--flat Pratt truss

Stiffness Matrix Coefficients:

$$Kt_{0,AB} = Kt_{0,BA} = 0$$

$$Kt_{1,AB} = Kt_{1,BA} = 0$$

$$Kt_{2,AB} = Kt_{2,BA} = \frac{1}{C_Y}$$

$$Kt_{3,AB} = Kt_{3,BA} = \frac{a}{C_Y}$$

$$Kt_{4,AB} = Kt_{4,BA} = \frac{1}{C_\Theta} + \frac{a^2}{C_Y}$$

$$Kt_{5,AB} = Kt_{5,BA} = - \frac{1}{C_\Theta} + \frac{a^2}{C_Y}$$

Load Functions:

$$FH_{AB} = W_{1X} \qquad FH_{BA} = W_{2X}$$

$$FV_{AB} = W_{1Y} - \frac{D_Y}{C_Y} \qquad FH_{BA} = W_{2Y} + \frac{D_Y}{C_Y}$$

$$FM_{AB} = \frac{D_\Theta}{C_\Theta} - a\frac{D_Y}{C_Y} + CM_{AC} \qquad FM_{BA} = - \frac{D_\Theta}{C_\Theta} - a\frac{D_Y}{C_Y} - CM_{CB}$$

2.3 Derivation of the Load and Stiffness Coefficients: Gabled Pratt Truss

2.3.1 Statics

A fixed end gabled truss removed from a truss-frame system, loaded by a general system of loads, is shown in Figure 2.3.1. Once again, the basic assumptions made in the analysis of the gabled truss are:

1. All joints are pin connected.
2. Truss members are subjected to axial loads only.
3. Truss members and loads are in one plane only.
4. All deformations of the truss are small in comparison with the dimensions of the truss.

Under the action of loading, the vertical reactions R_{AY} and R_{BY} are induced. The end thrust of the truss produces the horizontal reactions R_{AX} and R_{BX}. Since the ends of the truss are restrained, the end moments FM_{AB} and FM_{BA} are also developed.

The truss is then divided at its centerline as shown in Figure 2.3.2, and the internal forces at the central section are then replaced by an infinitely rigid arm to some arbitrary point 0, the elastic center of the truss. The displaced forces are denoted as H_o, V_o, and M_o (Figure 2.3.2). This displacement of the internal forces will allow the simplifications of the analysis. The general

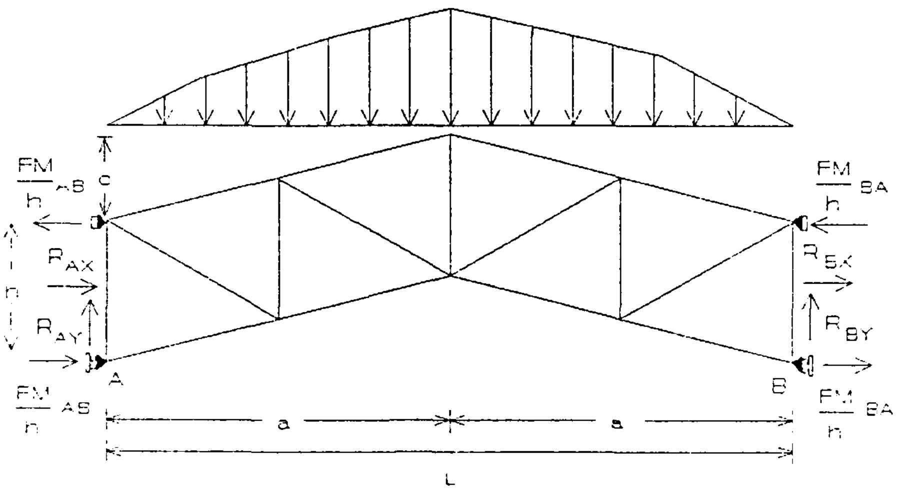


Figure 2.3.1 Gabled truss with general system of loading

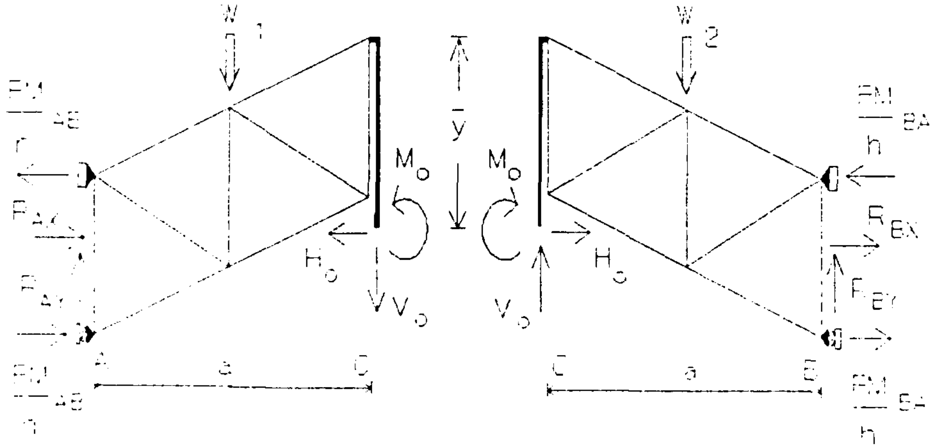


Figure 2.3.2 Free-body gabled trusses AC and CB with displaced internal forces

system of loads is resolved into two parts, $\overline{AC}$ and $\overline{CB}$, each corresponding to one-half of the symmetrical truss. The resultants of the system of loads acting on parts $\overline{AC}$ and $\overline{CB}$ are denoted as W_1 and W_2, respectively. The general displacement of the supports are given by Δ_{AX}, Δ_{AY}, Θ_A and Δ_{BX}, Δ_{BY}, Θ_B is introduced. If we again assume all displacements and reactions to be positive, the new reactions, due to applied loads and internal forces at the elastic center, O (Figure 2.3.2), are:

$$
\begin{aligned}
R_{AX} &= W_{1X} + H_o, \\
R_{BX} &= W_{2X} - H_o, \\
R_{AY} &= W_{1Y} + V_o, \qquad (1) \\
R_{BY} &= W_{2Y} - V_o, \\
M_{AB} &= - M_o + cH_o + aV_o + CM_{AC}, \\
M_{BA} &= M_o - cH_o + aV_o - CM_{BC}
\end{aligned}
$$

where CM_{AC} and CM_{BC} represent the cantilever moments of the applied loads at the

There are six unknown reactions in Equation (1). With only three equations of static equilibrium, the truss is statically indeterminate to the third degree. The three additional equations required to solve the problem must be derived from the deformation relationships corresponding to H_o, V_o, and M_o.

The normal force for any given member in the truss in terms of the applied loads and the redundants is:

$$N_i = SN_i + \alpha_i H_o + \beta_i V_o + \gamma_i M_o \quad (2)$$

where N_i = total normal force in any member,

SN_i = normal force in any member due to loading,

α_i = normal force in any member due to $H_o = 1$,

β_i = normal force in any member due to $V_o = 1$,

γ_i = normal force in any member due to $M_o = 1$.

2.3.2 Least Work

From the Principle of Minimum Potential Energy:

$$U_i = U_e \quad (3)$$

where U_i = the internal work

and U_e = the external work.

The internal work:

$$U_i = U_s + U_v \quad (3a)$$

where U_s = the strain energy of the structure

and U_v = the strain energy due to volume change.

Once again, the energy due to any change in volume will be neglected and only the energy of normal forces will be considered, therefore:

$$U_i = U_s = \sum_{A}^{B} \frac{N_i^2 L_i}{2A_i E}$$

where L_i = length of any member,

A_i = cross-sectional area of any member,

E = modulus of elasticity,

or in its simpler form when $\lambda_i = \frac{L_i}{A_i E}$:

$$U_i = U_s = \sum_A^B \frac{N_i^2}{2} \lambda_i. \qquad (3b)$$

The external work,

$$U_e = U_l + U_r \qquad (3c)$$

where $U_l = \sum_A^B W + \sum_A^B W\Theta$ = the work due to applied loads

and $U_r = \sum_A^B R + \sum_A^B M\Theta$ = the work due to reactions.

The work of the supports in terms of reactive forces, and displacements is:

$$U_r = R_{AX}\Delta_{AX} + R_{AY}\Delta_{AY} + M_{AB}\Theta_A + R_{BX}\Delta_{BX} + R_{BA}\Delta_{BY} + M_{BA}\Theta_B$$

substituting from Equations (1):

$$U_r = H_o\Delta_{AX} + (W_1 + V_o)\Delta_{AY} - H_o\Delta_{BX} + (W_2 - V_o)\Delta_{BY} + (- M_o + cH_o + aV_o + CM_{AC})\Theta_A + (M_o - cH_o + aV_o - CM_{BC})\Theta_B \qquad (3d)$$

From Castigliano's theorems, assuming unyielding supports, the first partial derivative of the strain energy of a structural system with respect to a redundant is equal to zero. Allowing displacement of the supports we have again:

$$\frac{\partial U_s}{\partial H_o} = \frac{\partial U_r}{\partial H_o},$$
$$\frac{\partial U_s}{\partial V_o} = \frac{\partial U_r}{\partial V_o}, \qquad (4)$$
$$\frac{\partial U_s}{\partial M_o} = \frac{\partial U_r}{\partial M_o}.$$

The partial derivatives of Equation (3b) with respect to each redundant are:

$$\frac{\partial U_s}{\partial H_o} = \sum_A^B N_i \frac{\partial N_i \lambda_i}{\partial H_o} = \sum_A^B N_i \alpha_i \lambda_i,$$
$$\frac{\partial U_s}{\partial V_o} = \sum_A^B N_i \frac{\partial N_i \lambda_i}{\partial V_o} = \sum_A^B N_i \alpha_i \lambda, \qquad (4a)$$
$$\frac{\partial U_s}{\partial M_o} = \sum_A^B N_i \frac{\partial N_i \lambda_i}{\partial M_o} = \sum_A^B N_i \gamma_i \lambda_i$$

The partial derivatives of Equation (3d) with respect to each redundant are:

$$\frac{\partial U_r}{\partial H_o} = \Delta_{AX} - \Delta_{BX} + c(\Theta_A - \Theta_B),$$

$$\frac{\partial U_r}{\partial V_o} = \Delta_{AY} - \Delta_{BY} + a(\Theta_A + \Theta_B), \qquad (4b)$$

$$\frac{\partial U_r}{\partial M_o} = - \Theta_A + \Theta_B$$

If we let

$$\Delta_X = \Delta_{AX} - \Delta_{BX}$$

$$\Delta_Y = \Delta_{AY} - \Delta_{BY}$$

then Equations (4b) may be rewritten as:

$$\frac{\partial U_r}{\partial H_o} = \Delta_X + c(\Theta_A - \Theta_B)$$

$$\frac{\partial U_r}{\partial V_o} = \Delta_Y + a(\Theta_A + \Theta_B) \qquad (4c)$$

$$\frac{\partial U_r}{\partial M_o} = \Theta_B - \Theta_A$$

2.3.3 Deformation Equations

The deformation equations are obtained by substituting Equations (2), (4a), and (4c) into Equation (4). The deformation equations are:

$$\Delta_X + c(\Theta_A - \Theta_B) = \sum_A^B SN_i\alpha_i\lambda_i + H_o \sum_A^B \alpha_i^2\lambda_i$$

$$+ V_o \sum_A^B \alpha_i\beta_i\lambda_i + M_o \sum_A^B \alpha_i\gamma_i\lambda_i, \qquad (5a)$$

$$\Delta_Y + a(\Theta_A + \Theta_B) = \sum_A^B SN_i\beta_i\lambda_i + H_o \sum_A^B \beta_i\alpha_i\lambda_i$$

$$+ V_o \sum_A^B \beta_i^2\lambda_i + M_o \sum_A^B \beta_i\gamma_i\lambda_i, \qquad (5b)$$

$$- \Theta_A + \Theta_B = \sum_A^B SN_i\gamma_i\lambda_i + H_o \sum_A^B \gamma_i\alpha_i\lambda_i$$

$$+ V_o \sum_A^B \gamma_i\beta_i\lambda_i + M_o \sum_A^B \gamma_i^2\lambda_i. \qquad (5c)$$

Equations (5a), (5b), and (5c) may now be solved simultaneously to determine H_o, V_o and M_o. These equations may be reduced if H_o, V_o, and M_o are applied at the elastic center, as shown in Figure 2.3.2. When the three redundant forces are applied at the elastic center, the terms:

$$\sum_A^B \alpha_i\beta_i\lambda_i$$

$$\sum_A^B \alpha_i\gamma_i\lambda_i = \sum_A^B \gamma_i\alpha_i\lambda_i \qquad (5d)$$

$$\sum_A^B \beta_i\gamma_i\lambda_i = \sum_A^B \gamma_i\beta_i\lambda_i$$

will equal zero in the case of a symmetrical truss.

For a symmetrical truss, the lateral $\overline{X}$ location of the elastic center is always at the mid span.

The vertical, $\overline{Y}_o$, location of the elastic center is computed by:

$$\overline{Y}_o = \frac{\sum_A^B \overline{Y}_i \, d\overline{A}}{\sum_A^B d\overline{A}}$$

where $\overline{Y}_o$ = distance from the crown of the truss to the elastic center

$\overline{Y}_i$ = distance from the crown of the trust to the centroid of the ith bar

$$d\overline{A} = \frac{l_i}{A_i E} = \lambda_i$$

There are other variations on the method for locating the elastic center of a truss or frame that will yield the same location, however, this is the method used in this analysis.

Taking advantage of the relationships given in Equation (5d) resulting from the use of the elastic center, Equations (5a, 5b, and 5c) reduce to:

$$\Delta_X + c(\Theta_A - \Theta_B) = \sum_A^B SN_i \alpha_i \lambda_i + H_o \sum_A^B \alpha_i^2 \lambda_i,$$

$$\Delta_Y + a(\Theta_A + \Theta_B) = \sum_A^B SN_i \beta_i \lambda_i + V_o \sum_A^B \beta_i^2 \lambda_i, \qquad (5e)$$

$$- \Theta_A + \Theta_B = \sum_A^B SN_i \gamma_i \lambda_i + M_o \sum_A^B \gamma_i^2 \lambda_i.$$

If we denote:

$$D_X = \sum_A^B SN_i \alpha_i \lambda_i , \qquad C_X = \sum_A^B \alpha_i^2 \lambda_i ,$$

$$D_Y = \sum_A^B SN_i \beta_i \lambda_i , \qquad C_Y = \sum_A^B \beta_i^2 \lambda_i ,$$

$$D\Theta = \sum_A^B SN_i \gamma_i \lambda_i , \qquad D_\Theta = \sum_A^B \gamma_i^2 \lambda_i ,$$

We may then rewrite the deformation equations as:

$$\begin{aligned} \Delta_X + c(\Theta_A - \Theta_B) &= D_X + H_o C_X , \\ \Delta_Y + a(\Theta_A + \Theta_B) &= D_Y + V_o C_Y , \\ -\Theta_A + \Theta_B &= D_\Theta + M_o C_\Theta . \end{aligned} \tag{6}$$

Solving for the three redundants:

$$\begin{aligned} H_o &= \frac{\Delta_X}{C_X} + \frac{c}{C_X} (\Theta_A - \Theta_B) - \frac{D_X}{C_X} , \\ V_o &= \frac{\Delta_Y}{C_Y} + \frac{a}{C_Y} (\Theta_A + \Theta_B) - \frac{D_Y}{C_Y} , \\ M_o &= \frac{-\Theta_A + \Theta_B}{C_\Theta} - \frac{D_\Theta}{C_\Theta} . \end{aligned} \tag{7}$$

Substituting the results of Equation (7) into Equations (1),

$$R_{AX} = W_{1X} + \frac{\Delta_X}{C_X} + \frac{c}{C_X} (\Theta_A - \Theta_B) - \frac{D_Y}{C_X} \tag{8}$$

$$R_{AY} = W_{1Y} + \frac{\Delta_Y}{C_Y} + \frac{a}{C_Y} (\Theta_A + \Theta_B) - \frac{D_Y}{C_Y} \tag{9}$$

$$M_{AB} = \frac{\Theta_A - \Theta_B}{C_\Theta} + \frac{D_\Theta}{C_\Theta} + \frac{c}{C_X}\Delta_X + \frac{c^2}{C_X}(\Theta_A - \Theta_B) - c\frac{D_X}{C_X} + \frac{a}{C_Y}\Delta_Y + \frac{a^2}{C_Y}(\Theta_A + \Theta_B) - a\frac{D_Y}{C_Y} + CM_{AC} \quad (10)$$

$$R_{BX} = W_{2X} - \frac{\Delta_X}{C_X} - \frac{c}{C_X}(\Theta_A - \Theta_B) + \frac{D_X}{C_X} \quad (11)$$

$$R_{BY} = W_{2Y} - \frac{\Delta_Y}{C_Y} - \frac{a}{C_Y}(\Theta_A + \Theta_B) + \frac{D_Y}{C_Y} \quad (12)$$

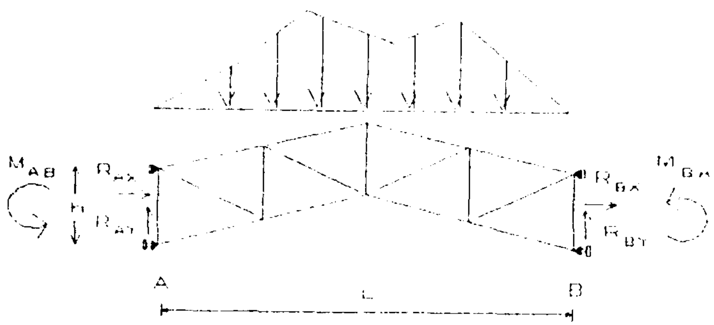
$$M_{BA} = -\frac{\Theta_A - \Theta_B}{C_\Theta} - \frac{D_\Theta}{C_\Theta} - \frac{c}{C_X}\Delta_X - \frac{c^2}{C_X}(\Theta_A - \Theta_B) + c\frac{D_X}{C_X} + \frac{a}{C_Y}\Delta_Y + \frac{a^2}{C_Y}(\Theta_A + \Theta_B) - a\frac{D_Y}{C_Y} - CM_{CB} \quad (13)$$

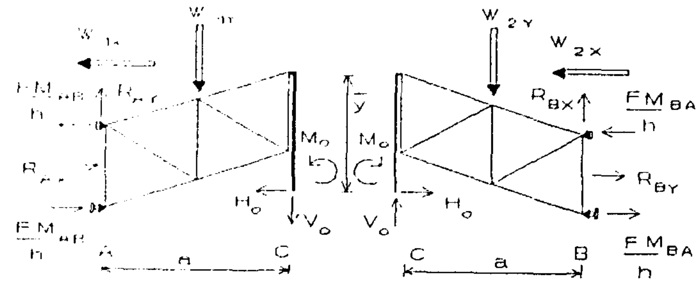
Equations (8) through (13) are the stiffness equations for the gabled truss.

Stiffness Matrix Coefficients:

$$Kt_{0,AB} = Kt_{0,BA} = \frac{1}{C_X}$$

$$Kt_{1,AB} = Kt_{1,BA} = \frac{c}{C_X}$$





a. Gabled truss structural system b. Gabled truss free-body diagram

Geometry: Symmetrical-gabled truss of variable h. Elastically restrained at A and B. Loaded by a coplanar system of forces.

Load Constants:	Frame Constants:	Elastic Center:
$D_X = \sum_{A}^{B} SN_i \alpha_i \lambda_i$	$C_X = \sum_{A}^{B} \alpha_i^2 \lambda_i$	$\overline{Y}_o = \dfrac{\Sigma\ \overline{Y}_i \quad d\overline{A}}{\Sigma\ dA}$
$D_Y = \sum_{A}^{B} SN_i \beta_i \lambda_i$	$C_Y = \sum_{A}^{B} \beta_i^2 \lambda_i$	
$D_\Theta = \sum_{A}^{B} SN_i \gamma_i \lambda_i$	$C_\Theta = \sum_{A}^{B} \gamma_i^2 \lambda_i$	$\overline{X}_o = a$ (mid span-symmetric truss)

Figure 2.3.3 Structural constants--gabled Pratt truss

$$Kt_{2,AB} = Kt_{2,BA} = \frac{1}{C_Y}$$

$$Kt_{3,AB} = Kt_{3,BA} = \frac{a}{C_Y}$$

$$Kt_{4,AB} = Kt_{4,BA} = \frac{1}{C_\Theta} + \frac{a^2}{C_Y} + \frac{C^2}{C_X}$$

$$Kt_{5,AB} = Kt_{5,BA} = - \frac{1}{C_\Theta} + \frac{a^2}{C_Y} - \frac{C^2}{C_X}$$

Load Functions:

$$FH_{AB} = W_{1X} - \frac{D_X}{C_X} \qquad FH_{BA} = W_{2X} + \frac{D_X}{C_X}$$

$$FV_{AB} = W_{1Y} - \frac{D_Y}{C_Y} \qquad FH_{BA} = W_{2Y} + \frac{D_Y}{C_Y}$$

$$FM_{AB} = - \frac{D_\Theta}{C_\Theta} - c \frac{D_X}{C_X} - a \frac{D_Y}{D_Y} + CM_{AC} \qquad FM_{BA} = - \frac{D_\Theta}{C_\Theta} + c \frac{D_X}{C_X} - a \frac{D_Y}{C_Y} - CM_{CB}$$

2.4 Column Stiffness Coefficients and Load Functions

As previously stated in Section 2.1, the column stiffness coefficients will not be derived in this paper as they are readily available in various undergraduate structural analysis text books. The column stiffness coefficients are the same as those for the standard beam element. The coefficients are shown in Section 2.4.1, accompanied by their orientation diagrams.

Four column load cases are shown in Section 2.4.2. These four load cases are taken from class notes given by Dr. J. Tuma during his lectures in the Theory of Structures course, taken by the writer while attending Arizona State University. These same four column load cases, along with numerous other variations of the load functions, may be found in Dr. J. Tuma's Handbook of Structural and Mechanical Matrices (1987). In using this handbook, the reader must remember to make the appropriate changes in the load applications orientation due to the 90 degree rotation of the member axis.

2.4.1 Column Stiffness Coefficients

$$Kc_0 = \frac{EA}{L} \qquad Kc_3 = \frac{4EI}{L}$$

$$Kc_1 = \frac{12EI}{L^3} \qquad Kc_4 = \frac{2EI}{L}$$

$$Kc_2 = \frac{6EI}{L^2}$$

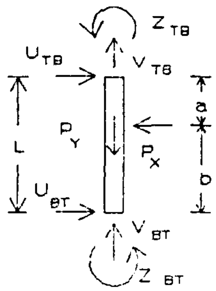
2.4.2 Column Fixed End Forces

P_X, P_Y = Concentrated loads $m = a/L$ $n = b/L$

$U_{TB} = + (1+2m)n^2 P_X$

$V_{TB} = + n\ P_Y$

$Z_{TB} = + mnb\ P_X$



$U_{BT} = +(1+2n)m^2 P_X$

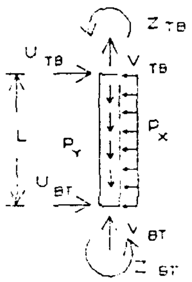
$V_{BT} = + m\ P_Y$

$Z_{BT} = - mna\ P_X$

$U_{TB} = + 1/2\ L\ P_X$

$V_{TB} = + 1/2\ L\ P_Y$

$Z_{TB} = + 1/12\ L^2 P_X$



$U_{BT} = + 1/2\ L\ P_X$

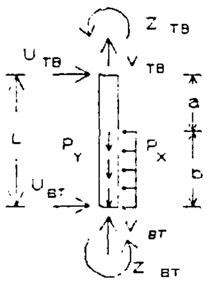
$V_{BT} = + 1/2\ L\ P_Y$

$Z_{BT} = - 1/12\ L^2 P_X$

$U_{TB} = + (2-n)n^2 b\ P_X/2$

$V_{TB} = + n\ b\ P_Y/2$

$Z_{TB} = + (4-3n)nb^2 P_X/12$



$U_{BT} = + (1+m+m^2 n)b\ P_X{}^2$

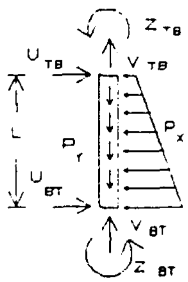
$V_{BT} = + 1/2\ (2-n)b\ P_Y/2$

$Z_{BT} = - (6-8n+3n^2)b^2 P_X/12$

$U_{TB} = + 3\ L\ P_Y/20$

$V_{TB} = + L\ P_X/6$

$Z_{TB} = + L^2 P_X/30$



$U_{BT} = 7\ L\ P_X/20$

$V_{BT} = + L\ P_Y/3$

$Z_{BT} = - L^2 P_X/20$

CHAPTER 3

Method of Analysis

3.1 Construction of the Structural Stiffness Matrix

The structural stiffness matrix is formed by merging the elemental stiffness matrices for the specified truss configuration and the elemental column stiffness matrix together to form a single stiffness matrix (also referred to as the stiffness matrix equations).

Up to this point, the reaction and deformation notation used in the derivation of the truss stiffness equations have been R_x, R_y, M, Δ_x, Δ_y, and Θ. We now wish to change this notation in order to conform with that used by Dr. Tuma (1987). The modified nomenclature is:

$R_{AX} = U_{AB}$	$R_{BX} = U_{BA}$	$\Delta_{AX} = U_A$	$\Delta_{BX} = U_B$
$R_{AY} = V_{AB}$	$R_{BY} = V_{BA}$	$\Delta_{AY} = V_A$	$\Delta_{BY} = V_B$
$M_{AB} = Z_{AB}$	$M_{BA} = Z_{BA}$	$\Theta_A = \Theta_A$	$\Theta_B = \Theta_B$

In the case where two subscripts are used, the first denotes the local joint of interest and the second identifies the terminal end of the member.

In a more general form, the subscripts L and R are used to define the left and right end of a structural element since the actual subscripts used in the analysis of a truss frame depend on the manner in which the joints are identified.

In the case of the nomenclature used above, U_{AB} is the same as U_{LR} and U_{BA} is the same as U_{RL} in more general terms.

3.1.1 Procedure of Analysis

The following is a step by step procedure for the analysis of a truss frame using the stiffness matrix method.

1. Determine the geometry of the truss-frame:
   a. label all joints alphabetically and number all members.
   b. identify all external dimensions, ensuring the truss frame is symmetrical as for the cases derived in Sections 2.2 and 2.3.
   c. identify the length, cross-sectional area, moment of inertia (I) of the columns, and the modulus of elasticity for all members.
2. Calculate the elastic center of the truss:
   a. for a parallel truss the elastic center is located at the mid-depth of the truss, h/2, and at the center of the truss span for a symmetrical truss-frame.
   b. for a gabled truss the elastic center must be calculated from the equation:

$$\bar{y}_o = \frac{\Sigma \bar{y}_i \; d\bar{A}}{\Sigma d\bar{A}}$$

3. Based on the truss shape given in either Sections 2.2 or 2.3, calculate:
   a. truss constants C_X, C_Y, and C_Θ
   b. load constants D_X, D_Y, and D_Θ
   c. cantilever moments at C due to loads applied only to the truss.
4. Using the load and truss constants from Step 3, calculate for the specific type of truss shape:
   a. truss stiffness coefficients
   b. fixed end forces and moments
5. Substitute the stiffness coefficients and fixed end forces and moments for the given truss into the appropriate elemental truss stiffness matrix, Kt.
6. Calculate the stiffness coefficients and the fixed end forces and moments of the elemental column stiffness matrix and substitute them into the elemental into the matrix, Kc.
7. Identify the unknown displacements for the truss frame in terms of u's, v's, and Θ's. Establish displacement relations. These exist only in the case of the symmetrically loaded truss frame.

8. Write the joint equilibrium equations for each joint in the truss-frame.

9. Using the joint equilibrium equations, assemble the structural stiffness matrix, K', by algebraically adding the elemental stiffness matrices for the truss and columns.

10. Solve for the unknown displacements in terms of u's, v's, and Θ's.

11. Substitute the values of the now known u's, v's, and Θ's into the elemental stiffness matrices and solve for the end forces and moments of each of the members.

It should be noted that at this point all of the bar forces in the truss were previously calculated in Step 3 above, therefore, their analysis has been already completed.

3.1.2 We begin the construction of the structural stiffness matrix by first defining the elemental stiffness matrices for each type of truss identified in Section 2 and for the columns, all in terms of the redefined nomenclature.

All of the elemental stiffness matrices are presented in terms of the global X-Y axis system (X axis horizontal, Y axis vertical), therefore, no rotational transformation of the elemental matrices are required in the analysis. In the end, the solutions for the unknown displacements in the global system are also the same in the local axis system.

3.1.3 Parallel Truss Elemental Stiffness Matrix

Before assembling the elemental stiffness matrix for the parallel truss, we will first rewrite the stiffness equations, Equations 6 through 9 of Section 2.2, in terms of the new nomenclature and the stiffness factors given by Kt_0, Kt_1, etc. We will also introduce the horizontal reactions U_{AB} and U_{BA} which were initially left out of the derivation by neglecting any axial deformation.

The slope deflection equations may be rewritten as:

$$U_{AB} = FH_{AB}$$

$$V_{AB} = Kt_2\ (v_{AB} - v_{BA}) + Kt_3\ (\Theta_A + \Theta_B) + FV_{BA}$$

$$Z_{AB} = Kt_3\ (v_{AB} - v_{BA}) + Kt_4\ \Theta_A + Kt_5\ \Theta_B + FM_{AB}$$

$$U_{BA} = FH_{BA}$$

$$V_{BA} = Kt_2\ (v_{AB} - v_{BA}) - Kt_3\ (\Theta_A + \Theta_B) + FV_{BA}$$

$$Z_{BA} = Kt_3\ (v_{AB} - v_{BA}) + Kt_5\ \Theta_A + Kt_4\ \Theta_B + FM_{BA}$$

Writing these six equations in matrix form becomes:

$$\begin{bmatrix} U_{AB} \\ V_{AB} \\ Z_{AB} \\ \hline U_{BA} \\ V_{BA} \\ Z_{BA} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Kt_2 & Kt_3 & 0 & -Kt_2 & Kt_3 \\ 0 & Kt_3 & Kt_4 & 0 & -Kt_3 & Kt_5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -Kt_2 & -Kt_3 & 0 & Kt_2 & -Kt_3 \\ 0 & Kt_3 & Kt_5 & 0 & -Kt_3 & Kt_4 \end{array}\right] \cdot \begin{bmatrix} u_{AB} \\ v_{AB} \\ \Theta_A \\ \hline u_{BA} \\ v_{BA} \\ \Theta_B \end{bmatrix} + \begin{bmatrix} FH_{AB} \\ FV_{AB} \\ FM_{AB} \\ FH_{BA} \\ FV_{BA} \\ FM_{BA} \end{bmatrix}$$

3.1.4 Gabled Truss Elemental Stiffness Matrix

Rewriting the stiffness equation for the gabled truss in terms of the new notation and K's:

$$U_{AB} = Kt_0 \ (U_{AB} - U_{BA}) + Kt_1 \ (\Theta_A - \Theta_B) + FH_{AB}$$

$$V_{AB} = Kt_2 \ (v_{AB} - v_{BA}) + Kt_3 \ (\Theta_A + \Theta_B) + FV_{AB}$$

$$Z_{AB} = Kt_1 \ (U_{AB} - U_{BA}) + Kt_3 \ (v_{AB} - v_{BA}) + Kt_4 \ \Theta_A + Kt_5 \ \Theta_B + FM_{AB}$$

$$U_{BA} = -Kt_0 \ (U_{AB} - U_{BA}) - Kt_1 \ (\Theta_A - \Theta_B) + FH_{BA}$$

$$V_{BA} = Kt_2 \ (v_{AB} - v_{BA}) - Kt_3 \ (\Theta_A + \Theta_B) + FV_{BA}$$

$$Z_{BA} = Kt_1 \ (U_{AB} - U_{BA}) + Kt_3 \ (v_{AB} - v_{BA}) + Kt_5 \ \Theta_A + Kt_4 \ \Theta_B + FM_{BA}$$

Writing these six equations in terms of its matrix form becomes:

$$
\begin{bmatrix} U_{AB} \\ V_{AB} \\ Z_{AB} \\ \hline U_{BA} \\ V_{BA} \\ Z_{BA} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
Kt_0 & 0 & Kt_1 & -Kt_0 & 0 & -Kt_1 \\
0 & Kt_2 & Kt_3 & 0 & -Kt_2 & Kt_3 \\
Kt_1 & Kt_3 & Kt_4 & -Kt_1 & -Kt_3 & Kt_5 \\
\hline
-Kt_0 & 0 & -Kt_1 & Kt_0 & 0 & Kt_1 \\
0 & -Kt_2 & -Kt_3 & 0 & Kt_2 & -Kt_3 \\
-Kt_1 & Kt_3 & Kt_5 & Kt_1 & -Kt_3 & Kt_4
\end{array}\right]
\cdot
\begin{bmatrix} u_{AB} \\ v_{AB} \\ \Theta_A \\ \hline u_{BA} \\ v_{BA} \\ \Theta_B \end{bmatrix}
+
\begin{bmatrix} FH_{AB} \\ FV_{AB} \\ FM_{AB} \\ FH_{BA} \\ FV_{BA} \\ FM_{BA} \end{bmatrix}
$$

3.1.5 Column Stiffness Matrix

The column stiffness matrix will not be derived in this paper. Simply stated, the column stiffness matrix may be obtained by applying the rotational transformation matrix to the general case of the stiffness matrix equation for the straight-horizontal bar. The general stiffness matrix for a horizontal bar may be found in Tuma (1987), along with an explanation of the transformation matrices. The column stiffness matrix is:

$$
\begin{bmatrix} U_{TB} \\ V_{TB} \\ Z_{TB} \\ \hline U_{BT} \\ V_{BT} \\ Z_{BT} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
Kc_1 & 0 & Kc_2 & -Kc_1 & 0 & Kc_2 \\
0 & Kc_0 & 0 & 0 & -Kc_0 & \\
Kc_2 & 0 & Kc_3 & -Kc_2 & 0 & Kc_4 \\
\hline
-Kc_1 & 0 & -Kc_2 & Kc_1 & 0 & -Kc_2 \\
0 & -Kc_0 & 0 & 0 & Kc_0 & 0 \\
Kc_2 & 0 & Kc_4 & -Kc_2 & 0 & Kc_3
\end{array}\right]
\cdot
\begin{bmatrix} u_{TB} \\ v_{TB} \\ \Theta_{TB} \\ \hline u_{BT} \\ v_{BT} \\ \Theta_{BT} \end{bmatrix}
+
\begin{bmatrix} FH_{TB} \\ FV_{TB} \\ FM_{TB} \\ FH_{BT} \\ FV_{BT} \\ FM_{BT} \end{bmatrix}
$$

where:

1. Kc represents the specific column stiffness coefficients as opposed to the previously defined truss stiffness factors, Kt.
2. U, V, Z, u, v, and Θ are in the global axis system of the truss frame.
3. The subscripts, TB and BT refer respectively to the near end and far end of the column with regards to the member and forces or displacements we are working with. More specifically, the top or bottom.

The fixed end forces and moments for a column are presented in Section 2.4 along with the column stiffness coefficients.

3.1.6 Construction of the Structural Stiffness Matrix

The overall construction of the structural stiffness matrix for a single truss frame is contained in Steps 1 through 9 of the Procedure of Analysis (Section 3.1.5).

Steps 1 through 6 deal with the calculation of the truss and column stiffness coefficients and load factors, and assembly of the elemental stiffness matrices. Steps 7 and 8 are key to the actual construction of the structural stiffness matrix.

Step 7 identifies the unknown displacements for a given truss frame in terms of u's, v's, and Θ's and establishes the displacement relationships. From the symmetry inherent in our particular truss frame, the displacements of a joint on the right half of the truss frame can be related to the displacement of the corresponding joint in the left half of the truss.

The joint equilibrium equations in Step 8 form the basis for the structural stiffness matrix.

The joint equilibrium equations are the summation of the forces acting on each of the members connected to a specific joint plus the fixed end forces and moments due to any loads applied to those same members or in simple format, for any given joint in a truss frame:

$$\Sigma U_i + \Sigma FH_i = 0$$

$$\Sigma V_i + \Sigma FV_i = 0$$

$$\Sigma Z_i + \Sigma FM_i = 0$$

In the case of a single bay truss frame, the term ΣU_i will equal the sum of U_T (due to the truss) and U_c (due to the column) at the joint where the truss and column are connected. Likewise, ΣFH_i is the sum of the horizontal fixed end force acting on both the column and the truss at the joint in question.

Once the joint equilibrium equations have been written for all joints in the truss frame (in terms of U, V, and Z), they are rewritten in terms of their corresponding elemental stiffness equations given by the truss/column stiffness coefficients, u's, v's, and Θ's.

Next, we make use of the displacement relationships and the symmetry of the truss frame previously found in Step 7. Selecting either the left or right half of the structure to work with, substitute the displacement equivalents into the joint equilibrium equations such that all displacements contained in the equilibrium equations represent only the left or right half truss frame displacements.

The structural stiffness matrix can now be assembled by placing the rewritten joint equilibrium equations into matrix format. For a general truss frame, symmetrical about the center line, as shown in Figure 3.1.1, the center line, as shown in Figure 3.1.1,

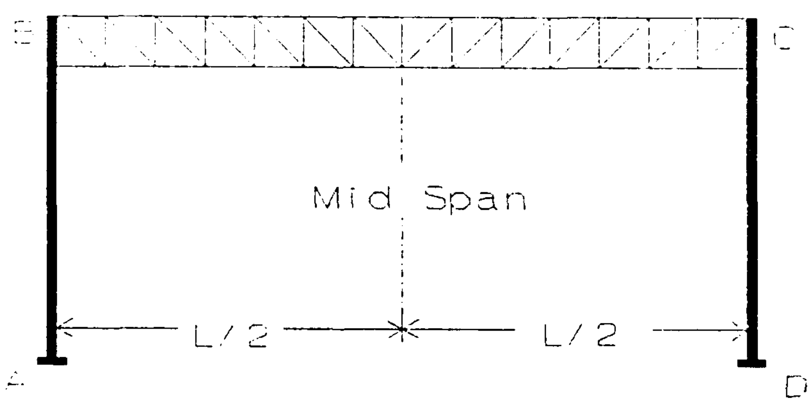


Figure 3.1.1 Symmetrical truss frame

the structural stiffness matrix becomes:

$$\begin{bmatrix} U_A \\ V_A \\ Z_A \\ \hline U_B \\ V_B \\ Z_B \end{bmatrix} = \left[\begin{array}{c|c} K_{LL} & K_{LR} \\ \hline K_{RL} & K_{RR} \end{array}\right] \cdot \begin{bmatrix} u_A \\ v_A \\ \Theta_A \\ \hline u_B \\ v_B \\ \Theta_B \end{bmatrix} + \begin{bmatrix} \Sigma\ FH_{AB} \\ \Sigma\ FV_{AB} \\ \Sigma\ FM_{AB} \\ \Sigma\ FH_B \\ \Sigma\ FV_B \\ \Sigma\ FM_B \end{bmatrix} \qquad (1)$$

where K_{XX} represents a 3 x 3 coefficient matrix.

3.2 Solution for Unknown Displacements

The unknown displacements may be found by using the structural stiffness matrix developed in Section 3.1.6:

$$\begin{bmatrix} U_A \\ V_A \\ Z_A \\ \hline U_B \\ V_B \\ Z_B \end{bmatrix} = \left[\begin{array}{c|c} K_{LL} & K_{LR} \\ \hline K_{RL} & K_{RR} \end{array}\right] \cdot \begin{bmatrix} u_A \\ v_A \\ \Theta_A \\ \hline u_B \\ v_B \\ \Theta_B \end{bmatrix} + \begin{bmatrix} \Sigma\ FH_A \\ \Sigma\ FV_A \\ \Sigma\ FM_A \\ \Sigma\ FH_B \\ \Sigma\ FV_B \\ \Sigma\ FM_B \end{bmatrix} \qquad (1)$$

or

$$(S) = (K') \times (\Delta) + (FEF) \qquad (2)$$

The unknown displacements are found by considering displacements caused by applied loads or fixed end forces. As such, Equation 1 may be rewritten as:

$$\begin{bmatrix} K_{LL} & K_{LR} \\ \hline K_{RL} & K_{RR} \end{bmatrix} \cdot \begin{bmatrix} u_A \\ v_A \\ \theta_A \\ \hline u_B \\ v_B \\ \theta_B \end{bmatrix} = \begin{bmatrix} -FH_A \\ -FV_A \\ -FM_A \\ -FH_B \\ -FV_B \\ -FM_B \end{bmatrix} \quad (3)$$

or

$$(K') \quad x \quad (\Delta) \quad = \quad (-FEF) \quad (4)$$

The unknown displacements may be found by solving Equation 4 in which:

$$\Delta = K')^{-1} \ (-FEF)$$

The solution may be to obtain by Gauss elimination, hand held calculator capable of matrix operations, or by means of a computer program for matrix algebra.

After the unknown displacements have been calculated, the remaining unknown displacements of the truss frame are found using the displacement relationships defined in Step 7 of the procedure of analysis if the displacement relationships exist.

3.3 Calculation of End Forces and Moments

Calculation of the end forces and moments, Step 11 of the procedure of analysis, is accomplished by inserting the solutions for the unknown displacements (Step 10) and the known (or zero) displacements into the elemental stiffness matrices created in Steps 5 and 6, earlier in the analysis.

The end forces and moments are the algebraic sum of the appropriate stiffness coefficients multiplied by the displacements plus the fixed end force or moment.

After obtaining the end forces and moments for each member of the truss frame, an equilibrium check should be made at each joint to ensure equilibrium conditions do in fact exist. A free body diagram is most useful to accomplish this.

3.4 Calculation of Bar Forces

The forces acting in the bars making up the truss were previously calculated during the calculation of the truss stiffness coefficients and load functions. The designer is strongly encouraged to make use of a table format during the calculation of the truss stiffness coefficients and load functions in order to better organize and retain the calculation of the bar forces, SN_i.

CHAPTER 4

Examples

4.1 Example 1--Parallel Truss Frame

A single span flat Pratt truss frame with dimensions and loads shown in Figure 4.1.1 is considered. The modulus of elasticity is constant for all members.

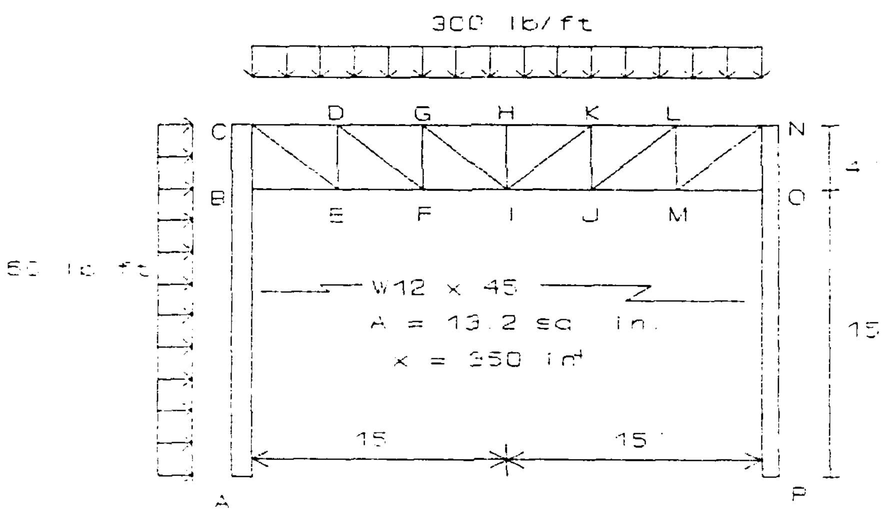


Figure 4.1.1 Flat Pratt truss frame

This flat Pratt truss frame is analyzed by the procedure given in Section 3.

1. Figure 4.1.2 shows the left half of the truss girder. Since the truss is symmetrical, only one-half of it must be evaluated.

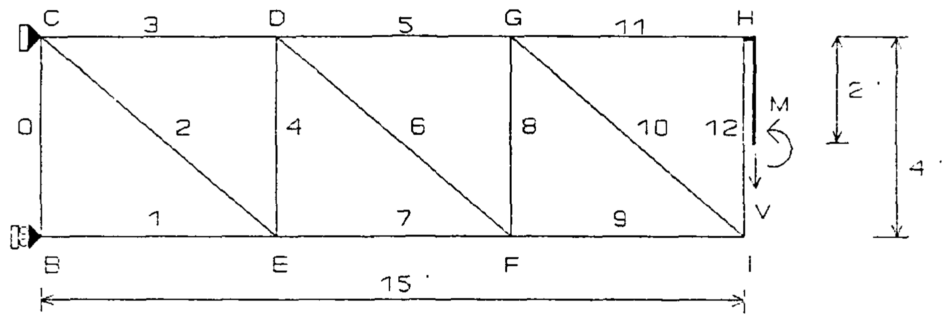


Figure 4.1.2 Left half--flat truss girder

Top Chords/Bottom Chords

$A = 7.5$ in^2 $\qquad$ $L = 30$ ft.

2 L 4 x 4 x ½ $\qquad$ $a = 15$ ft.

$E = 29$ E6 psi

Vertical and Diagonal Members

$A = 2.3$ in^2

2L 2 x 2 x ¼

2. The elastic center of a flat Pratt truss is located at the center of the truss span at a depth of h/2, or in this case 2 feet below the top chord.

3a. Truss Constants. The truss constants are given by:

$$C_X = 0$$

$$C_Y = \sum_{0}^{N} \beta_i^2 \lambda_i$$

$$C_\Theta = \sum_{0}^{N} \gamma_i^2 \lambda_i$$

Table 4.1.1 shows the properties for each member and the evaluation of the truss constants.

Since these values represent the truss constants for only half of the frame, they must be multiplied by 2 for the entire frame. Therefore the truss constants are:

$$C_X = 0$$

$$C_Y = \frac{1071.6}{E}$$

$$C_\Theta = \frac{6}{E}$$

3b. Load Constants. The forces acting on the truss are shown in Figure 4.1.3 below. The original distributed load is shown as a series of equivalent concentrated load acting at the joints.

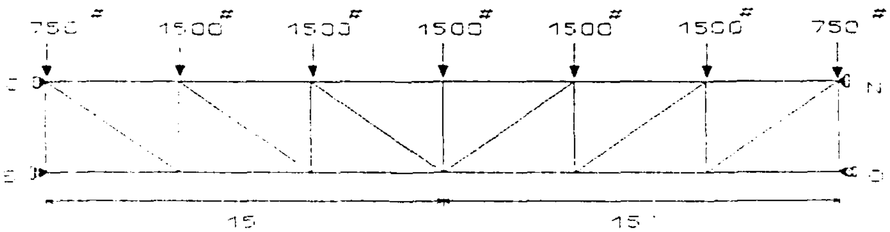


Figure 4.1.3 Forces acting on the flat truss

The load constants are given by:

$$D_X = 0$$

$$D_Y = \sum_{0}^{N} SN_i\beta_i\lambda_i$$

Table 4.1.1

Evaluation of Truss Constants

Member	$A_i (in^2)$	L_i (in)	λ_i	β_i	γ_i	$\beta_i^2 \lambda_i$	$\gamma_i^2 \lambda_i$
0	8.25	48	5.82	0	0	0	0
1	7.5	60	8	-3.75	0.25	112.50	0.50
2	2.3	76.8	33.4	1.60	0	85.50	0
3	7.5	60	8	2.50	-0.25	50.00	0.50
4	2.3	48	20.9	-1.00	0	20.9	0
5	7.5	60	8	1.25	-0.25	12.50	0.50
6	2.3	76.8	33.4	1.60	0	85.50	0
7	7.5	60	8	-2.50	0.25	50.00	0.50
8	2.3	48	20.9	-1.00	0	20.9	0
9	7.5	60	8	-1.25	0.25	12.50	0.50
10	2.3	76.8	33.4	1.60	0	85.50	0
11	7.5	60	8	0	-0.25	0	0.50
12	2.3	48	20.9	0	0	0	0
					Σ =	535.8	3.0

$$D_\Theta = \sum_{0}^{N} SN_i \gamma_i \lambda_i$$

Table 4.1.2 shows the elemental properties, influence factors, normal force in the truss member due to loading, and the load constants.

3c. The cantilever moments acting at the center of the truss span are:

CM_{IC} = [750(15) + 1500(10) + 1500(5)] 12

= 405000 in.-lbs.

CM_{IN} = [750(15) + 1500(10) + 1500(5)] 12

= 405000 in.-lbs.

The forces acting on the basic structure are:

W_{CIy} = 300 lb./ft. (15 ft.) = 4500 lbs.

$W_{CIx} = 0$

W_{INy} = 300 lb./ft. (15 ft.) = 4500 lbs.

$W_{INx} = 0$

4a. The truss stiffness coefficients are:

$Kt_0 = 0$

$Kt_1 = 0$

$Kt_2 = 1/C_y \quad = 1/(1071.6/E)$

$Kt_3 = a/C_y \quad = 180/(1071.6/E)$

$Kt_4 = 1/C_\Theta + a^2/C_y \quad = 1/(6/E) + (180)^2/(1071.6/E)$

$Kt_5 = -1/C_\Theta + a^2/C_y \quad = -1/(6/E) + (180)^2/(1071.6/E)$

Therefore, if we use E = 29E6 psi,

Table 4.1.2

Evaluation of Load Constants for Truss

Member	λ_i	β_i	γ_i	SN_i	$SN_i\beta_i\lambda_i$	$SN_i\gamma_i\lambda_i$
		Left Half				
0	5.82	0	0	0	0	0
1	8	-3.75	0.25	-8460	253800	-16920
2	33.4	1.60	0	6010	321174	0
3	8	2.50	-0.25	3760	75200	- 7520
4	20.9	-1.00	0	-3750	78375	0
5	8	1.25	-0.25	940	9400	- 1880
6	33.4	1.60	0	3606	192705	0
7	8	-2.50	0.25	-3760	75200	- 7520
8	20.9	-1.00	0	-2250	47025	0
9	8	-1.25	0.25	- 940	9400	- 1880
10	33.4	1.60	0	1202	64235	0
11	8	0	-0.25	0	0	0
12	20.9	0	0	- 750	0	0

Table 4.1.2 Continued

Member	λ_i	β_i	γ_i	SN_i	$SN_i\beta_i\lambda_i$	$SN_i\gamma_i\lambda_i$
		Right Half				
0	5.82	0	0	0	0	0
1	8	3.75	0.25	-8460	-253800	-16920
2	33.4	-1.60	0	6010	-321174	0
3	8	-2.50	-0.25	3760	- 75200	- 7520
4	20.9	1.00	0	-3750	- 78375	0
5	8	-1.25	-0.25	940	- 9400	- 1880
6	33.4	-1.60	0	3606	-192705	0
7	8	2.50	0.25	-3760	- 75200	- 7520
8	20.9	1.00	0	-2250	- 47025	0
9	8	1.25	0.25	- 940	- 9400	- 1880
10	33.4	-1.60	0	1202	- 64235	0
11	8	0	-0.25	0	0	0
12	20.9	0	0	- 750	0	0
				Σ =	0	-71440

The load constants are:

$D_X = 0$

$D_Y = 0$

$$D_\Theta = \frac{-71440}{E}$$

$K_{t0} = 0$

$K_{t1} = 0$

$K_{t2} = 27062$

$K_{t3} = 4871220$

$K_{t4} = 881653042$

$K_{t5} = 871986375$

4b. The fixed end forces and moments are:

$FH_{CN} = W_X = 0$

$FH_{NC} = W_X = 0$

$$FV_{CN} = W_Y - \frac{D_Y}{C_Y} = 4500 - \frac{0}{(1071.6/E)} = 4500 \text{ lbs.}$$

$$FV_{NC} = W_Y - \frac{D_Y}{C_Y} = 4500 + \frac{0}{(1071.6/E)}$$

$$FM_{CN} = \frac{D_\Theta}{C_\Theta} - a\frac{D_Y}{C_Y} + CM_{IC}$$

$$FM_{CN} = \frac{(71440/E)}{(6/E)} - 180(0) + 405000 = 416906$$

$$FM_{NC} = -\frac{D_\Theta}{C_\Theta} - a\frac{D_Y}{C_Y} + CM_{IN}$$

$$FM_{NC} = -\frac{(71440/E)}{(6/E)} - 180(0) + 405000 = -416906$$

5. The elemental stiffness matrix for a parallel Pratt truss is given in section 3.1.3. The elemental stiffness matrix is given in Figure 4.1.4.

$$\begin{bmatrix} U_{CN} \\ V_{CN} \\ Z_{CN} \\ \\ U_{NC} \\ V_{NC} \\ Z_{NC} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 27062 & 4871220 & 0 & -27062 & 4871220 \\ 0 & 4871220 & 881653042 & 0 & -4871220 & 871986375 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -27062 & -4871220 & 0 & 27062 & -4871220 \\ 0 & 4871220 & 871986375 & 0 & -4871220 & 881653042 \end{array}\right] \begin{bmatrix} u_{Cn} \\ v_{CN} \\ \theta_{CN} \\ \\ u_{NC} \\ v_{NC} \\ \theta_{NC} \end{bmatrix} + \begin{bmatrix} 0 \\ 4500 \\ 416906 \\ \\ 0 \\ 4500 \\ -416906 \end{bmatrix}$$

Figure 4.1.4 Parallel truss elemental stiffness matrix

6. The stiffness coefficients for the columns are:

$$K_{C0} = \frac{EA}{L} = \frac{29E6(13.2)}{19(12)} = 1678947$$

$$K_{C1} = \frac{12EI}{L^3} = \frac{12(29E6)(350)}{19(12)} = 10276$$

$$K_{C2} = \frac{6EI}{L^2} = \frac{6(29E6)(350)}{(19(12))^2} = 1171514$$

$$K_{C3} = \frac{4EI}{L} = \frac{4(29E6)(350)}{19(12)} = 178070175$$

$$K_{C4} = \frac{2EI}{L} = \frac{2(29E6)(350)}{19(12)} = 89035087$$

The fixed end forces for column AC are calculated using the equations given in section 2.4.2 for a distributed load acting perpendicular to the length of the column:

$$U_{TB} = 1/2\ L\ P_X = 1/2\ (19)(12)(60/12) = -\ 570 \text{ lbs.}$$

$$V_{TB} = 1/2\ L\ P_Y = 1/2\ (19)(12)(0) = 0 \text{ lbs.}$$

$$Z_{TB} = 1/12\ L^2\ P_X = 1/12\ ((19)(12))^2\ (-60/12)$$

$$= -21660 \text{ in-lbs.}$$

$$U_{BT} = 1/2\ L\ P_X = 1/2\ (19)(12)(-60) = -\ 570 \text{ lbs.}$$

$$V_{BT} = 1/2\ L\ P_Y = 1/2\ (19)(12)(0) = 0 \text{ lbs.}$$

$$Z_{BT} = -1/12\ L^2\ P_X = -1/12\ ((19)(12))^2\ (-60/12)$$

$$= 21660 \text{ in-lbs.}$$

The elemental stiffness matrices for columns AC and PN are shown in Figures 4.1.5 and 4.1.6, respectively.

$$
\begin{bmatrix} U_{CA} \\ V_{CA} \\ Z_{CA} \\ U_{AC} \\ V_{AC} \\ Z_{AC} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
10276 & 0 & 1171514 & -10276 & 0 & 1171514 \\
0 & 1678947 & 0 & 0 & -1678947 & 0 \\
1171514 & 0 & 178070175 & -1171514 & 0 & 89035087 \\
\hline
-10276 & 0 & -1171514 & 10276 & 0 & -1171514 \\
0 & -1678947 & 0 & 0 & 1678947 & 0 \\
1171514 & 0 & 89035087 & -1171514 & 0 & 178070175
\end{array}\right]
\begin{bmatrix} u_{CA} \\ v_{CA} \\ \Theta_{CA} \\ u_{AC} \\ v_{AC} \\ \Theta_{AC} \end{bmatrix}
+
\begin{bmatrix} -570 \\ 0 \\ -21660 \\ -570 \\ 0 \\ 21660 \end{bmatrix}
$$

<u>Figure 4.1.5</u> Column AC elemental stiffness matrix

$$
\begin{bmatrix} U_{NP} \\ V_{NP} \\ Z_{NP} \\ U_{PN} \\ V_{PN} \\ Z_{PN} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
10276 & 0 & 1171514 & -10276 & 0 & 1171514 \\
0 & 1678947 & 0 & 0 & -1678947 & 0 \\
1171514 & 0 & 178070175 & -1171514 & 0 & 89035087 \\
\hline
-10276 & 0 & -1171514 & 10276 & 0 & -1171514 \\
0 & -1678947 & 0 & 0 & 16978947 & 0 \\
1171514 & 0 & 89035087 & -1171514 & 0 & 178070175
\end{array}\right]
\begin{bmatrix} u_{NP} \\ v_{NP} \\ \Theta_{NP} \\ u_{PN} \\ v_{PN} \\ \Theta_{PN} \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Figure 4.1.6 Column PN elemental stiffness matrix

In the case of column PN, since no loads are applied on the column, all the fixed end moments are equal to zero.

7. The unknown displacements for the truss frame are:

$u_C = ?$	$u_N = ?$	$u_A = 0$	$u_P = 0$
$v_C = ?$	$v_N = ?$	$v_A = 0$	$v_P = 0$
$\Theta_C = ?$	$\Theta_N = ?$	$\Theta_A = 0$	$\Theta_P = 0$

Since the truss frame is loaded asymmetrically, there are no continuity relationships between the unknown displacements.

8. The joint equilibrium equations are:

Joint C

$$U_{CA} + U_{CN} - 570 = 0$$

$$V_{CA} + V_{CN} - 4500 = 0$$

$$Z_{CA} + Z_{CN} + 3[illegible]5246 = 0$$

Joint N

$$U_{NC} + U_{NP} = 0$$

$$V_{NC} + V_{NP} + 4500 = 0$$

$$Z_{NC} + Z_{NP} - 4[illegible]6906 = 0$$

9. Rewriting the joint equilibrium equations in terms of their corresponding elemental stiffness coefficients:

Joint C:

U:

$$10276\, u_{CA} + 0v_{CA} + 1171514\, \Theta_{CA} - 10276\, u_{AC} + 0v_{AC}$$
$$+ 1171514\, \Theta_{AC} + 0u_{CA} + 0v_{CN} + 0\Theta_{CN} + 0u_{NC}$$
$$+ 0v_{NC} + 0\Theta_{NC} - 570 = 0$$

V:

$$0\,u_{CA} + 1678947\,v_{CA} + 0\Theta_{CA} + 0v_{AC} - 1678947\,v_{AC} + 0\Theta_{AC}$$
$$+ 0u_{CA} + 27062\,v_{CM} + 4871220\,\Theta_{CN} + 0\Theta_{NC}$$
$$- 27062\,v_{NC} + 4871220\,\Theta_{NC} - 4500 = 0$$

Z:

$$1171514\,u_{CA} + 0v_{CA} + 178070175\,\Theta_{CA} - 1171514\,u_{AC} + 0v_{AC}$$
$$+ 89035087\,\Theta_{AC} + 0u_{CA} + 4871220\,v_{CN}$$
$$+ 881653042\,\Theta_{CN} + 0u_{NC} - 4871220\,v_{NC}$$
$$+ 871986375\,\Theta_{NC} + 395246 = 0$$

Joint N

U:

$$10276\,u_{NP} + 0v_{NP} + 1171514\,\Theta_{NP} - 10276\,u_{PN} + 0v_{PN}$$
$$+ 1171514\,\Theta_{PN} + 0u_{CA} + 0v_{CN} + 0\Theta_{CN} + 0u_{NC}$$
$$+ 0v_{NC} + 0\Theta_{NC} = 0$$

V:

$$0u_{NP} + 1678947\,v_{NP} + 0\Theta_{NP} + 0u_{PN} - 1678947\,v_{PN} + 0\Theta_{PN}$$
$$+ 0u_{CN} - 27062\,v_{CN} - 4871220\,\Theta_{CN} + 0u_{NC} + 27062\,v_{NC}$$
$$- 4871220\,\Theta_{NC} + 4500 = 0$$

Z:

$$1171514\,u_{NP} + 0v_{NP} + 178070175\,\Theta_{NP} - 1171514\,u_{PN} + 0v_{PN}$$
$$+ 89035087\,\Theta_{PN} + 0u_{CN} + 4871220\,v_{CN}$$
$$+ 871986375\,\Theta_{CN} + 0u_{NC} - 4871220\,v_{NC}$$
$$+ 881653042\,\Theta_{NC} - 416906 = 0$$

Making use of the known displacements determined in step 7 above, namely, $u_A = v_A = \Theta_A = u_P = v_P = \Theta_P = 0$, the joint equilibrium equations become:

Joint C

1. $10276\ u_C + 1171514\ \Theta_C - 570 = 0$
2. $1706010\ v_C + 4871220\ \Theta_C - 27062\ v_N + 4871220\ \Theta_N + 4500 = 0$
3. $1171514\ u_C + 4871220\ v_C + 1059723217\ \Theta_C - 4871220\ v_N + 871986375\ \Theta_N + 395246 = 0$

Joint N

4. $10276\ u_N + 1171514\ \Theta_N = 0$
5. $-27062\ v_C - 4871220\ \Theta_C + 1706010\ v_N - 4871220\ \Theta_N + 4500 = 0$
6. $4871220\ v_C + 871220\ \Theta_C + 1171514\ u_N - 4871220\ v_N + 1059723217\ \Theta_N - 416906 = 0$

10. Placing these six equations into matrix format as shown in Figure 4.1.7 and solving for the six unknown displacements:

$u_C = 0.979738$ in.	$u_N = -0.921436$ in.
$v_C = -0.002610$ in.	$v_N = -0.002750$ in.
$\Theta_C = -0.008107$ Rads	$\Theta_N = 0.008082$ Rads

$$
\begin{array}{c}
 \\ u_C \\ v_C \\ \Theta_C \\ \\ u_N \\ v_N \\ \Theta_N
\end{array}
\begin{array}{c}
\begin{array}{cccccc} u_C & v_C & \Theta_C & u_{AC} & v_{AC} & \Theta_{AC} \end{array} \\
\left[\begin{array}{rrr|rrr}
10276 & 0 & 1171514 & 0 & 0 & 0 \\
0 & 1706010 & 4871220 & 0 & -\ 27062 & 4871220 \\
1171514 & 4871220 & 1059723217 & 0 & -4871220 & 871986375 \\
\hline
0 & 0 & 0 & 10276 & 0 & 1171514 \\
0 & -\ 27062 & -\ 4871220 & 0 & 1706010 & -\ 4871220 \\
0 & 4871220 & 871986375 & 1171514 & -4871220 & 1059723217
\end{array}\right]
\end{array}
\cdot
\left[\begin{array}{c} u_C \\ v_C \\ \Theta_C \\ \\ u_N \\ v_N \\ \Theta_N \end{array}\right]
=
\left[\begin{array}{r} +\ \ 570 \\ -\ \ 4500 \\ -395246 \\ \\ 0 \\ -\ \ 4500 \\ +416906 \end{array}\right]
$$

$u_C = 0.979738$ in. $\quad u_N = -0.921436$ in.

$v_C = -0.002610$ in. $\quad v_N = -0.002750$ in.

$\Theta_C = -0.008107$ Rads $\quad \Theta_N = 0.008082$ Rads

Figure 4.1.7 Truss frame structural stiffness matrix

11. Having solved for the unknown displacements, the end forces and moments for each truss frame element may be found by substituting the displacements into the elemental stiffness matrices given in Figures 4.1.4, 4.1.5, and 4.1.6. Doing so, the end forces and moments become:

Member AC

U_{CA} = 0 U_{AC} = -1140 lbs.

V_{CA} = -4382.7 lbs. V_{AC} = 4382.7 lbs.

Z_{CA} = -317547 in.-lbs. Z_{AC} = 447604 in.-lbs.

Member CN

U_{CN} = 0 U_{NC} = 0

V_{CN} = 4382.7 lbs. V_{NC} = 4617.3 lbs.

Z_{CN} = 317547.4 in.-lbs. Z_{NC} = -359764.1 in.-lbs.

Member NP

U_{NP} = -0.5 lbs. U_{PN} = 0.5 lbs.

V_{NP} = -4617 lbs. V_{PN} = 4617 lbs.

Z_{NP} = 359687 in.-lbs. Z_{PN} = -359893 in.-lbs.

12. The free body diagram of the truss frame analyzed in Example 4.1 is shown in Figure 4.1.8.

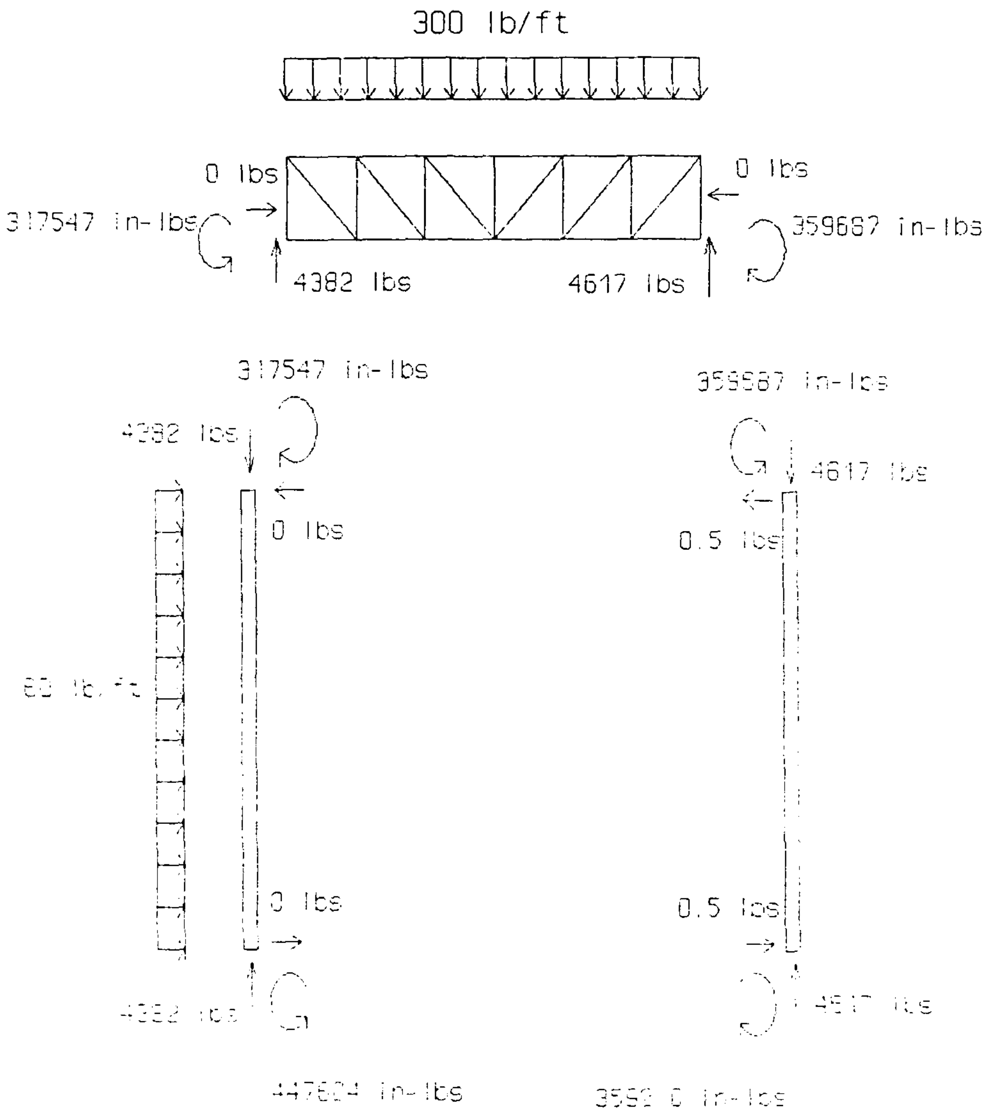


Figure 4.1.8 Free-body diagram--example 4.1

4.2 Example 2--Gabled Truss Frame

A single span gabled truss-frame with dimensions and loads shown in Figure 4.2.1 is considered. The modulus of elasticity is constants for all members.

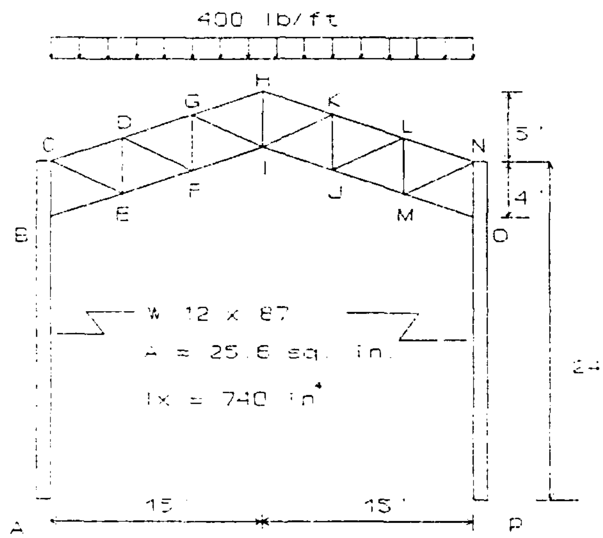


Figure 4.2.1 Gabled Pratt truss frame

This truss frame is analyzed by the procedure given in Section 3.

1. Figure 4.2.2 shows the left half of the gabled truss girder. Since the truss is symmetrical, only one-half of it must be evaluated.

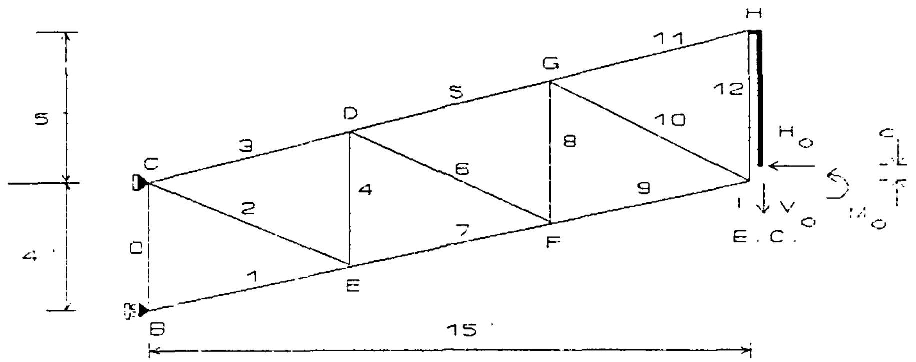


Figure 4.2.2 Left half--gabled truss girder

Top Chords/Bottom Chords

$A = 7.5\ in^2$ $\qquad$ $L = 30$ ft.

2 L 4 x 4 x ½ $\qquad$ $a = 15$ ft.

$c = 0.5$ ft.

$E = 29$ E6 psi

Vertical and Diagonal Members

$A = 2.3\ in^2$

2L 2 x 2 x ¼

2. The elastic center of a gabled Pratt truss is located by:

$$\bar{y}_o = \frac{\Sigma\ \bar{y}_i\ dA}{\Sigma\ dA}$$

Where dA may be taken as $\lambda_i = L_i/A_i$

$$\bar{y}_o = \frac{11904\text{ in.}}{220.4} = 4.5\text{ ft.}$$

The elastic center is located 4.5 feet below the crown of the truss girder.

3a. Truss Constants. The truss constants are given by:

$C_X = \Sigma\ \alpha_i^{\,2}\lambda_i$

$C_Y = \Sigma\ \beta_i^{\,2}\lambda_i$

$C_\Theta = \Sigma\ \gamma_i^{\,2}\lambda_i$

Table 4.2.1 shows the properties for each member and the evaluation of the truss constants.

Since these values represent the truss constants for only half of the frame, they must be multiplied by 2 for the entire frame. Therefore the truss constants are:

$C_X = 70.92/E$

$C_Y = 1015.12/E$

$C_\Theta = 8.58/E$

3b. Load Constants. The forces acting on the truss are shown in Figure 4.2.3 below. The original distributed load is shown as a series of equivalent concentrated loads acting at the joints.

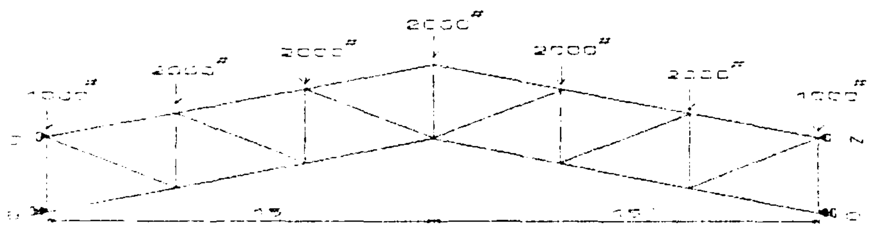


Figure 4.2.3 Forces acting on the gabled truss

Table 4.2.1

Evaluation of Truss Constants

Member	$A_i (in^2)$	L_i (in)	λ_i	α_i	β_i	γ_i	$\alpha_i^2 \lambda_i$	$\beta_i^2 \lambda_i$	$\gamma_i^2 \lambda_i$
0	2.3	48	20.87	-0.042	-1.25	0	0.037	32.609	0
1	7.5	63.24	8.43	0.132	-3.953	0.264	0.147	131.729	0.588
2	2.3	66.24	28.8	-0.460	1.379	0	6.094	54.767	0
3	7.5	63.24	8.43	-0.815	2.635	-0.316	5.599	58.531	0.842
4	2.3	48	20.87	0.333	-1.0	0	2.314	20.87	0
5	7.5	63.24	8.43	-0.376	1.317	-0.316	1.192	14.622	0.842
6	2.3	66.24	28.8	-0.460	1.379	0	6.094	54.767	0
7	7.5	63.24	8.43	-0.308	-2.635	0.264	0.800	58.531	0.588
8	2.3	48	20.87	0.333	-1.0	0	2.314	20.87	0
9	7.5	63.24	8.43	-0.747	-1.317	0.264	4.704	14.622	0.588
10	2.3	66.24	28.8	-0.460	1.379	0	6.094	54.767	0
11	7.5	63.24	8.43	0.064	0	-0.316	0.035	0	0.842
12	2.3	48	20.87	0.042	-1.0	0	0.037	20.87	0
						Σ =	35.46	507.56	4.29

The load constants are given by:

$$D_X = \sum_{0}^{N} SN_i \alpha_i \lambda_i$$

$$D_Y = \sum_{0}^{N} SN_i \beta_i \lambda_i$$

$$D_\Theta = \sum_{0}^{N} SN_i \gamma_i \lambda_i$$

Table 4.2.2 shows the elemental properties, influence factors, normal force in the truss members due to loading, and the load constants.

The load constants are:

D_X = -511748/E

D_Y = 0

D_Θ = -117244/E

3c. The cantilever moments acting at the center of the truss span are:

CM_{IC} = [1000(15) + 2000(15)]12 = 540000 in.-lbs.

CM_{IN} = [1000(15) + 2000(15)] 12 = 540000 in.-lbs.

The forces acting on the basic structure are:

W_{CIx} = 0

W_{CIy} = 400 lb./ft. (15 ft.) = 6000 lbs.

W_{NIx} = 0

W_{NIy} = 400 lb./ft. (15 ft.) = 6000 lbs.

Table 4.2.2

Evaluation of Load Constants for Truss

Member	λ_i	α_i	β_i	γ_i	SN_i	$SN_i\alpha_i\lambda_i$	$SN_i\beta_i\lambda_i$	$SN_i\gamma_i\lambda_i$
Left Half of Truss								
0	20.87	-0.042	-1.25	0	3752.33	- 3289	- 97888	0
1	8.43	0.132	-3.953	0.264	-11862.75	- 13200	395311	-26401
2	28.8	-0.460	1.379	0	6898.36	- 91389	273970	0
3	8.43	-0.815	2.635	-0.316	5272.33	- 36223	117114	-14044
4	20.87	0.333	-1.0	0	- 5000.00	- 34748	104350	0
5	8.43	-0.376	1.317	-0.316	1318.08	- 4178	14634	- 3511
6	28.8	-0.460	1.379	0	4139.02	- 54834	164382	0
7	8.43	-0.308	-2.635	0.264	- 5272.33	13689	117114	-11733
8	20.87	0.333	-1.0	0	- 3000.0	- 20849	62610	0
9	8.43	-0.747	-1.317	0.264	- 1318.08	8300	14633	- 2933
10	28.8	-0.460	1.379	0	1379.67	- 18277	54794	0
11	8.43	0.064	0	-0.316	0	0	0	0
12	20.87	0.042	-1.0	0	- 1000.00	- 876	20870	0

Table 4.2.2 Continued

Member	λ_i	α_i	β_i	γ_i	SN_i	$SN_i\alpha_i\lambda_i$	$SN_i\beta_i\lambda_i$	$SN_i\gamma_i\lambda_i$
			Right Half of Truss					
0	20.87	-0.042	1.25	0	3752.33	- 3289	97888	0
1	8.43	0.132	3.953	0.264	-11862.75	- 13200	-395311	-26401
2	28.8	-0.460	-1.379	0	6898.75	- 91389	-273970	0
3	8.43	-0.815	-2.635	-0.316	5272.33	- 36223	-117114	-14044
4	20.87	0.333	1.0	0	- 5000.00	- 34748	-104350	0
5	8.43	-0.376	-1.317	-0.316	1318.08	- 4178	- 14634	- 3511
6	28.8	-0.460	-1.379	0	4139.02	- 54834	-164382	0
7	8.43	-0.308	2.635	0.264	- 5272.33	13689	-117114	-11733
8	20.87	0.333	1.0	0	- 3000.0	- 20849	- 62610	0
9	8.43	-0.747	1.317	0.264	- 1318.08	8300	- 14633	- 2933
10	28.8	-0.460	-1.379	0	1379.67	- 18277	- 54794	0
11	8.43	0.064	0	-0.316	0	0	0	0
12	20.87	0.042	1.0	0	- 1000.00	- 876	- 20870	0
					Σ =	-511748	0	-117244

4a. The truss stiffness coefficients are:

$Kt_0 = 1/C_x \qquad = 1/(70.92/E)$

$Kt_1 = C/C_x \qquad = 6/(70.92/E)$

$Kt_2 = 1/C_y \qquad = 1/(1015.12/E)$

$Kt_3 = a/C_y \qquad = 60/(1015.12/E)$

$Kt_4 = 1/C_\theta + a^2/C_y + C^2/C_x \qquad = 1/(8.58/E)$

$+ (60)^2/(1015.12/E) + (6)^2/(70.92/E)$

$Kt_5 = - 1/C_\theta + a^2/C_y - C^2/C_x = -1/(8.58/E)$

$+ (60)^2/(1011.12/E) - (6)^2/(70.92/E)$

If we take E = 29E6 psi,

$K_{t0} = 408911$

$K_{t1} = 2453468$

$K_{t2} = 28568$

$K_{t3} = 1714083$

$K_{t4} = 120945750$

$K_{t5} = 84744220$

4b. The fixed end forces and moments are:

$$FH_{CN} = W_{CIx} - \frac{D_x}{C_x} = - \frac{(-511748/E)}{70.92/E} = 7215 \text{ lbs.}$$

$$FH_{NC} = W_{NIx} + \frac{D_x}{C_x} = \frac{-511748/E}{70.92/E} = -7215 \text{ lbs.}$$

$$FV_{CN} = W_{CIy} - \frac{D_y}{C_y} = 6000 - \frac{0}{1015.12/E} = 6000$$

$$FV_{NC} = W_{NIy} + \frac{D_y}{C_y} = 6000 + 0 = 6000$$

$$FM_{CN} = -\frac{D_\Theta}{C_\Theta} - c\frac{D_x}{C_x} - a\frac{D_y}{C_y} + CM_{CI}$$

$$= \frac{-(-117244/E)}{8.58/E} - \frac{6(-511748/E)}{70.92/E} - 180\frac{0}{1015.12/E}$$

$$+ 540000 = 596960$$

$$FM_{NC} = -\frac{D_\Theta}{C_\Theta} + c\frac{D_x}{C_x} - a\frac{D_y}{C_y} - CM_{NI}$$

$$= \frac{-(-117244/E)}{8.58/E} + \frac{6(-511748/E)}{70.92/E} - 180(0)$$

$$- 540000 = 596960$$

5. The elemental stiffness matrix for a gabled Pratt truss is given in section 3.1.4. The elemental stiffness matrix is shown in Figure 4.2.3.

6. The stiffness coefficients for the columns are:

$$K_{C0} = \frac{EA}{L} = \frac{29E6(25.6)}{24(12)} = 2577780$$

$$K_{C1} = \frac{12EI}{L^3} = \frac{12(29E6)(740)}{((24(12))^3} = 10780$$

$$K_{C2} = \frac{6EI}{L^2} = \frac{6(29E6)(740)}{((24(12))^2} = 1552370$$

$$K_{C3} = \frac{4EI}{L} = \frac{4(29E6)(740)}{24(12)} = 298055560$$

$$K_{C4} = \frac{2EI}{L} = \frac{2(29E6)(740)}{24(12)} = 149027780$$

$$
\begin{bmatrix} U_{CN} \\ V_{CN} \\ Z_{CN} \\ U_{NC} \\ V_{NC} \\ Z_{NC} \end{bmatrix}
=
\left[\begin{array}{rrr|rrr}
408911 & 0 & 2453468 & -408911 & 0 & -2453468 \\
0 & 28568 & 1714083 & 0 & -28568 & 1714083 \\
2453468 & 1714083 & 120945750 & -2453468 & -1714083 & 84744220 \\
\hline
-408911 & 0 & -2453468 & 408911 & 0 & 2453468 \\
0 & -28568 & -1714083 & 0 & 28568 & -1714083 \\
-2453468 & 1714083 & 84744220 & 2453468 & -1714083 & 12094575
\end{array}\right]
\begin{bmatrix} u_{Cn} \\ v_{CN} \\ \Theta_{CN} \\ u_{NC} \\ v_{NC} \\ \Theta_{NC} \end{bmatrix}
+
\begin{bmatrix} 7215 \\ 6000 \\ 596960 \\ -7215 \\ 6000 \\ -596960 \end{bmatrix}
$$

<u>Figure 4.2.4</u> Gabled truss elemental stiffness matrix

The fixed end forces for columns AC and PN are equal to zero since no load is acting on either column.

The elemental stiffness matrices for columns AC and PN are the same and shown below in Figure 4.2.4.

In the case of column PN, the subscripts CA are replaced with NP and similarly AC are replaced with PN.

7. The unknown displacements for the truss frame are:

$u_C = ?$	$u_N = ?$	$u_A = 0$	$u_P = 0$
$v_C = ?$	$v_N = ?$	$v_A = 0$	$v_P = 0$
$\Theta_C = ?$	$\Theta_N = ?$	$\Theta_A = 0$	$\Theta_P = 0$

Since the truss frame in this case is loaded symmetrically, the following displacement relationships exist in the truss frame:

$$u_C = -u_N$$

$$v_C = v_N$$

$$\Theta_C = -\Theta_N$$

8. The joint equilibrium equations are:

Joint C

$$U_{CA} + W_{CN} + 7215 = 0$$

$$V_{CA} + V_{CN} + 6000 = 0$$

$$Z_{CA} + Z_{CN} + 596960 = 0$$

$$\begin{bmatrix} U_{CA} \\ V_{CA} \\ Z_{CA} \\ U_{AC} \\ V_{AC} \\ Z_{AC} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 10780 & 0 & 1552370 & -10780 & 0 & 1552370 \\ 0 & 2577780 & 0 & 0 & -2577780 & 0 \\ 1552370 & 0 & 298055560 & -1552370 & 0 & 149027780 \\ \hline -10780 & 0 & -1552370 & 10780 & 0 & -1552370 \\ 0 & -2577780 & 0 & 0 & 2577780 & 0 \\ 1552370 & 0 & 149027780 & -1552370 & 0 & 298055560 \end{array}\right] \times \begin{bmatrix} u_{CA} \\ v_{CA} \\ \Theta_{CA} \\ u_{AC} \\ v_{AC} \\ \Theta_{AC} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

Figure 4.2.5 Column AC and PN elemental stiffness matrix

Joint N

$$U_{NC} + W_{NP} - 7215 = 0$$

$$V_{NC} + V_{NP} + 6000 = 0$$

$$Z_{NC} + Z_{NP} - 596960 = 0$$

9. Rewriting the joint equilibrium equations in terms of their corresponding elemental stiffness coefficients:

Joint C

U:

$$10780\ u_{CA} + 0v_{CA} + 1552370\ \Theta_{CA} - 10780\ u_{AC} + 0v_{AC}$$
$$+ 1552370\ \Theta_{AC} + 408911\ u_{CN} + 0v_{CN}$$
$$+ 2453468\ \Theta_{CN} - 408911\ u_{NC} + 0v_{NC}$$
$$- 2453468\ \Theta_{NC} + 7215 = 0$$

V:

$$0u_{CA} + 2577780\ v_{CA} + 0\Theta_{CA} + 0u_{AC} - 2577780\ v_{AC} + 0\Theta_{AC}$$
$$+ 0u_{CN} + 28568\ v_{CN} + 1714083\ \Theta_{CN} + 0u_{NC} - 28568\ v_{NC}$$
$$+ 1714083\ \Theta_{NC} + 6000 = 0$$

Z:

$$1552370\ u_{CA} + 0v_{CA} + 298055560\ \Theta_{CA} - 1552370\ u_{AC} + 0v_{AC}$$
$$+ 149027780\ \Theta_{AC} + 2453468\ u_{CN} + 1714083\ v_{CN}$$
$$+ 120945750\ \Theta_{CN} - 2453468\ u_{NC} - 1714083\ v_{NC}$$
$$+ 84744220\ \Theta_{NC} + 596960 = 0$$

Joint N

U:

$$- 408911\ u_{CN} + 0v_{CN} - 2453468\ \Theta_{CN} + 408911\ u_{NC}$$
$$+ 0v_{NC} + 2453468\ \Theta_{NC} + 10780\ u_{NP} + 0v_{NP}$$
$$+ 1552370\ \Theta_{NP} - 10780\ u_{PN} + 0v_{PN}$$
$$+ 1552370\ \Theta_{PN} - 7215 = 0$$

V:

$$0u_{CN} - 28568\ v_{CN} - 1714083\ \Theta_{CN} + 0u_{NC} + 28568\ v_{NC}$$
$$- 1714083\ \Theta_{NC} + 0u_{NP} + 2577780\ v_{NP}$$
$$+ 0_{NP} + 0u_{PN} - 2577780\ v_{PN} + 0\Theta_{PN} + 6000 = 0$$

Z:

$$- 2453468\ u_{CN} + 1714083\ v_{CN} + 84744220\ \Theta_{CN} + 2453468\ u_{NC}$$
$$- 1714083\ v_{NC} + 120945750\ \Theta_{NC}$$
$$+ 1552370\ u_{NP} + 0v_{NP} + 289055560\ \Theta_{NP}$$
$$- 1552370\ u_{PN} + 0v_{PN} + 149027780\ \Theta_{PN}$$
$$- 596960 = 0$$

Making use of the known displacements determined in step 7 above and the displacement relationships:

Joint C

U:

$$828602\ u_C + 6459306\ \Theta_C = -7215$$

V:

$$2577780\ v_C = -6000$$

Z:

$$6459306\ u_C + 334257090\ \Theta_C = -596960$$

Joint N

Since we are making use of the symmetrical properties of the truss frame, we do not need to complete the joint equations for joint N.

10. Placing the three equations from Joint C into matrix format as shown in Figure 4.2.5 and solving for the three unknowns:

$$\begin{bmatrix} 828602 & 0 & 6459306 \\ 0 & 2577780 & 0 \\ 6459306 & 0 & 334257090 \end{bmatrix} \begin{bmatrix} u_C \\ v_C \\ \Theta_C \end{bmatrix} = \begin{bmatrix} -7215 \\ -6000 \\ -596960 \end{bmatrix}$$

Figure 4.2.6 Truss frame structural stiffness matrix

u_C = 0.00614 in.

v_C = -0.00233 in.

Θ_C = -0.0019 Rads

Having solved for the unknown displacements at Joint C, the displacements at Joint N may be assigned by the displacement relationships given in step 7. Therefore,

u_C = 0.00614 in.	u_N = -0.00614 in.
v_C = -0.00233 in.	v_N = -0.00233 in.
Θ_C = -0.0019 Rads	Θ_N = 0.0019 Rads

11. With the unknown displacements now known, the end forces and moments for each truss frame element may be found by substituting the displacements into the elemental

stiffness matrices given in Figures 4.2.3 and 4.2.4. Doing so, the end forces and moments become:

Member AC

U_{CA} = -2883 lbs.	U_{AC} = 2883 lbs.
V_{CA} = -6006 lbs.	V_{AC} = 6006 lbs.
Z_{CA} = -556774 in.-lbs.	Z_{AC} = -273621 in.-lbs.

Member CN

U_{CN} = 2913 lbs.	U_{NC} = -2913 lbs.
V_{CN} = 6000 lbs.	V_{NC} = 6000 lbs.
Z_{CN} = 556774 in.-lbs.	Z_{NC} = -558305 in.-lbs.

Member NP

U_{NP} = 2883 lbs.	U_{PN} = -2883 lbs.
V_{NP} = -6006 lbs.	V_{PN} = 6006 lbs.
Z_{NP} = 556774 in.-lbs.	Z_{PN} = 273621 in.-lbs.

12. The free body diagram of the truss frame analyzed in Example 4.2 is shown in Figure 4.2.7.

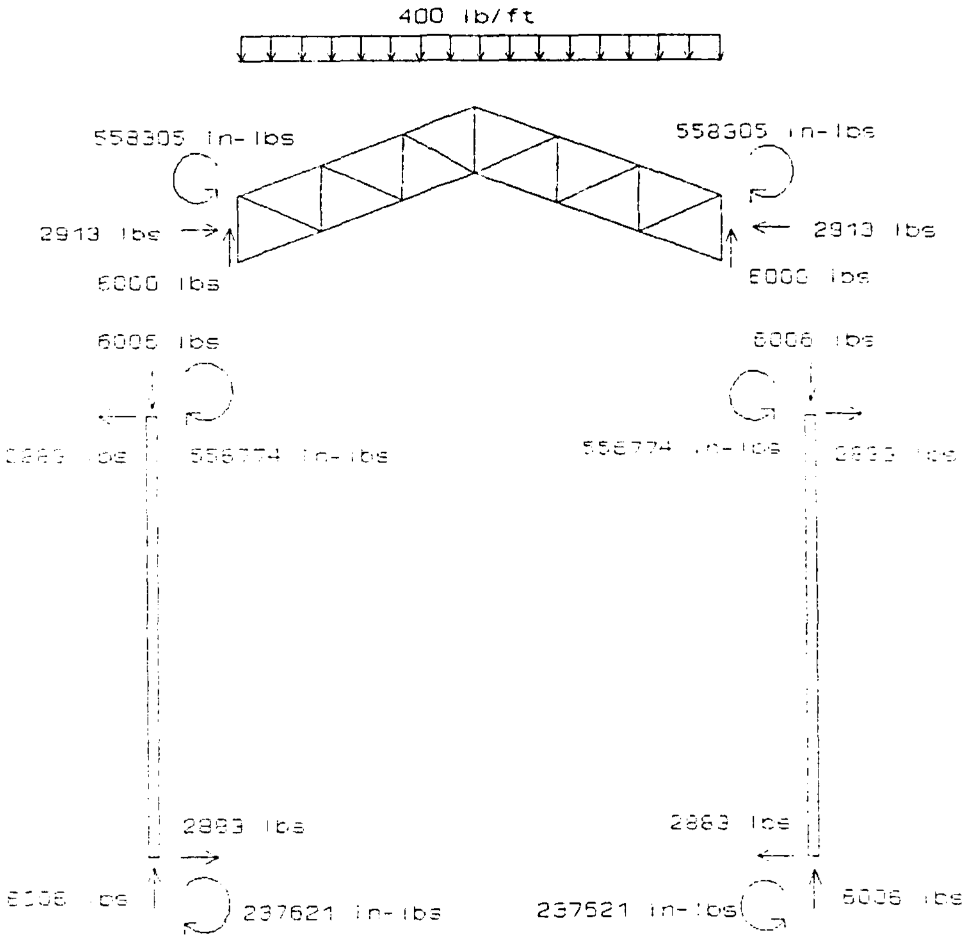


Figure 4.2.7 Free-body diagram--example 4.2

CHAPTER 5

Conclusions and Recommendations

5.1 Conclusions

The derivation of the stiffness coefficients, load functions, and stiffness matrices for the flat Pratt and gabled Pratt truss frames are presented. The stiffness coefficients and load functions are developed through the application of the theorem of least work for a general flat and gabled Pratt truss. The standard sign convention used in the development of the elemental stiffness coefficients, load functions, and stiffness matrices are taken from Tuma (1987). The column stiffness coefficients are taken from the Theory of Structures lectures given by Dr. Tuma at Arizona State University. The application of these two truss stiffness matrices and the column stiffness matrix in the solution of the truss frame problem are illustrated by two numerical examples.

5.2 Recommendations

This methodology can be automated intc a computer method of analysis.

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