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CLASSIFICATION OF INTERMITTENT
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Each of J items has a characteristic Signature which varies in time. At time 0, the value of a Signature and the identity of the corresponding item are known. No further values of Signatures are observed until a later time $t>0$. At time t a Signature associated with an unknown item is observed. The problem is to estimate the identity of the item whose Signature is observed at time t. The estimation procedure studied is to estimate the identity of the item that is associated with the Signature at time t to be that one which maximizes the posterior probability of being associated with the observed Signature. Univariate and multivariate Gaussian and univariate Cauchy autoregressive processes are considered as models for the Signatures. The robustness of the univariate Gaussian, (respectively Cauchy), procedure when applied to Cauchy, (respectively Gaussian) data is studied. The results suggest that the Gaussian classification procedure is biased towards classifying the Signature observed at time t as being associated with the *same* item that is associated with the Signature at time 0. The Cauchy procedure is biased towards classifying a Signature observed at time t as being associated with a *different* item than the one associated with the Signature at time 0.

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CLASSIFICATION OF INTERMITTENT DEPENDENT OBSERVATIONS

by
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1. THE PROBLEM

Consider the following classification problem. Suppose there are J items (e.g., diseases) each of which has a characteristic Signature which varies in time; the Signature of Item i is

$$\mathbf{Y}_i(t) = \boldsymbol{\theta}_i + \mathbf{X}_i(t) \qquad i = 1,2,\ldots,J \quad t = 0,1,2,\ldots \tag{1.1}$$

For the moment $\{\mathbf{X}_i(t)\}$ is an unspecified multivariate (or univariate) stochastic process, but one that stays near θ_i in finite time and has some stationary or steady-state behavior. In many cases, paths of $\mathbf{X}_i(t)$ will appear somewhat "continuous," so successive $\mathbf{X}_i(t)$'s are not well-modeled as iid random variables. One could think of $\mathbf{Y}_i(t)$ as physical indices characteristic of a particular disease, e.g., blood pressure, heart-beat pattern, cholesterol levels. Examples from equipment reliability are also of interest; here physical indices might be vibration, variations in heat level, oil leakage, and even fuel consumption in the case of engines.

In many circumstances $\mathbf{Y}_i(t)$ is only observable occasionally, at times unrelated to the value of $\mathbf{Y}_i(t)$ but driven by other forces such as the scheduling of a routine physical exam or system inspection. Suppose that the Signature and the identity of the item associated with the Signature are both observed at time t = 0, on such an occasion. Suppose that, later on, however, only the Signature of an item is observed. The first question is: What is the probability that, given the Signature value observed, its originating item is any particular one of the J candidates?

In Gaver and Jacobs [1989], the processes $\{X_i(t)\}$ are assumed to be univariate Gaussian and a Bayesian classification procedure is studied. In this paper, Section 2 assumes $\{X_i(t)\}$ are multivariate normal autoregressive processes. In Section 3, $\{X_i(t)\}$ is a univariate Cauchy autoregressive process whose marginal distribution has longer tails than the Gaussian. A Bayesian classification procedure for the Cauchy data is studied. In Section 4, we study the behavior of the univariate Cauchy and Gaussian classification procedures when autoregressive data having the wrong marginal distribution are presented to them. The results suggest that the Gaussian classification procedure is biased towards classifying a Signature produced at time t as being associated with the *same* item that produced the Signature at time 0. The Cauchy classification procedure is biased towards classifying a Signature produced at time t as being associated with a *different* item than the one producing the Signature at time 0. These effects are strongest for small times t. The largest number of misclassifications occur for small times t when the Gaussian classification procedure is presented with Cauchy data and a *different item* is associated with the Signature at time t than the item associated with the Signature at time 0; in this situation the Gaussian procedure is relatively less sensitive to the change in the item associated with the Signature. Misclassifications by the Cauchy classification procedure are modest in comparison to this extreme case.

In summary, it is important to realize that the performance of a Bayesian classification procedure can be influenced by its underlying distributional assumptions. A classification procedure based on Gaussian distributional assumptions can be reluctant to classify a new observation coming from a different item as being associated with a new item. A classification procedure based on Cauchy distributional assumptions can be reluctant to classify a new observation which comes from the same item as that being associated with the same item. Hence, if there is uncertainty about the underlying distribution of the data, it might be better to combine results of several classification procedures based on different distributional assumptions.

2. THE MULTIVARIATE NORMAL CASE

2.1 The Classification Question

Assume for illustration that the Signature of Item j is multivariate AR(1):

$$\mathbf{Y}_j(t) = \boldsymbol{\theta}_j + \mathbf{X}_j(t) \tag{2.1}$$

where $\boldsymbol{\theta}_j$, $\mathbf{Y}_j(t)$, and $\mathbf{X}_j(t)$ are d-dimensional column vectors. The process $\{\mathbf{X}_j(t)\}$ is a vector AR(1) process

$$\mathbf{X}_j(t) = \mathbf{A}_j\mathbf{X}_j(t-1) + \mathbf{E}_j(t) \tag{2.2}$$

where $\mathbf{A}_j$ is a d×d matrix and $\{\mathbf{E}_j(t)\}$ is a sequence of d-dimensional column vectors which are independent multivariate normal with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Lambda}_j$. The variance-covariance matrix for $\mathbf{X}_j(t+1)$ is

$$\boldsymbol{\Gamma}_j(t+1) = E\left[\mathbf{X}(t+1)\mathbf{X}^T(t+1)\right] = \mathbf{A}_j\boldsymbol{\Gamma}_j(t)\mathbf{A}_j^T + \boldsymbol{\Lambda}_j. \tag{2.3}$$

We will assume $\mathbf{A}_j$ and $\boldsymbol{\Lambda}_j$ are such that there is a finite unique solution to the equation

$$\boldsymbol{\Gamma}_j = \mathbf{A}_j\boldsymbol{\Gamma}_j\mathbf{A}_j^T + \boldsymbol{\Lambda}_j. \tag{2.4}$$

Assume $\mathbf{X}_j(0)$ has a normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Gamma}_j$. It follows that $\{\mathbf{X}_j(t)\}$ is a stationary sequence with mean $\mathbf{0}$ and variance–covariance matrix $\boldsymbol{\Gamma}_j$.

The conditional distribution of $\mathbf{X}_j(t)$ given $\mathbf{X}_j(0) = \mathbf{x}$ is multivariate normal with mean $\mathbf{A}_j^t\mathbf{x}$ and variance–covariance matrix

$$\Lambda_j(t) = \sum_{n=0}^{t-1} \mathbf{A}_j^n \Lambda_j \left(\mathbf{A}_j^n\right)^T. \tag{2.5}$$

Thus, $\Gamma_j = \lim_{t\to\infty} \Lambda_j(t)$.

The conditional distribution of the actually observable $\mathbf{Y}_j(t)$ given $\mathbf{Y}_j(0)=\mathbf{y}(0)$ is multivariate normal with mean $\boldsymbol{\theta}_j+\mathbf{A}_j^t(\mathbf{y}(0)-\boldsymbol{\theta}_j)$ and variance-covariance matrix $\Lambda_j(t)$.

Operational Scenario: There are, potentially, J items. Let C(t) be the identity of the item whose Signature is observed at time t. Put $p_j(t) \equiv P\{C(t)=j\}$. Assume that it is known that the Signature observed at time 0 comes from Item i; that is, C(0) = i and $\mathbf{Y}(0) = \mathbf{Y}_i(0) = \mathbf{y}(0)$. If it has been a long time since a Signature from item i has been observed, it is reasonable to suppose that

$$P\{C(0) = i, \mathbf{Y}(0) = \mathbf{y}(0)\}$$
$$= p_i(0)\left((2\pi)^d|\Gamma_i|\right)^{-0.5} \exp\left\{-\frac{1}{2}(\mathbf{y}(0)-\boldsymbol{\theta}_i)^T \Gamma_i^{-1}(\mathbf{y}(0)-\boldsymbol{\theta}_i)\right\} \tag{2.6}$$

the long-run or steady-state distribution. Further,

$$P\{C(t) = i, \mathbf{Y}(t) = \mathbf{y}(t)|C(0) = i, \mathbf{Y}(0) = \mathbf{y}(0)\}$$
$$= p_i(t)\left[(2\pi)^d|\Sigma_i(t)|\right]^{-0.5} \exp\left\{-\frac{1}{2}(\mathbf{y}(t)-\mathbf{m}_i(t))^T \Sigma_i(t)^{-1}(\mathbf{y}(t)-\mathbf{m}_i(t))\right\} \tag{2.7}$$

where

$$\mathbf{m}_i(t) = \boldsymbol{\theta}_i + \mathbf{A}_i^t(\mathbf{y}(0)-\boldsymbol{\theta}_i) \tag{2.8}$$

and

$$\Sigma_i(t) = \Lambda_i(t). \tag{2.9}$$

For $j \neq i$, we will assume the conditional distribution of $\mathbf{Y}(t)$ given $C(0) = i$, $\mathbf{Y}(0) = \mathbf{y}(0)$, $C(t) = j$ is multivariate normal with mean $\mathbf{m}_j(t) = \boldsymbol{\theta}_j$ and variance-covariance matrix $\boldsymbol{\Gamma}_j \equiv \boldsymbol{\Sigma}_j(t)$, since it is still a long time since a Signature from Item j is observed.

It now follows that

$$P\{\mathbf{Y}(t) \in d\mathbf{y}(t) | C(0) = i, \mathbf{Y}(0) = \mathbf{y}(0)\}$$
$$= \sum_{j=1}^{J} p_j(t) \left[(2\pi)^d |\boldsymbol{\Sigma}_j(t)|\right]^{-0.5} \exp\left\{-\frac{1}{2}(\mathbf{y}(t) - \mathbf{m}_j(t))^T \boldsymbol{\Sigma}_j(t)^{-1} (\mathbf{y}(t) - \mathbf{m}_j(t))\right\}. \quad (2.10)$$

Thus, the posterior probability of the identity of the item associated with the Signature is

$$P\{C(t) = j | C(0) = i, \mathbf{Y}(0) = \mathbf{y}(0), Y(t) = y(t)\} =$$
$$p_j(t) |\boldsymbol{\Sigma}_j(t)|^{-0.5} \exp\left\{-\frac{1}{2}(\mathbf{y}(t) - \mathbf{m}_j(t))^T \boldsymbol{\Sigma}_j(t)^{-1} (\mathbf{y}(t) - \mathbf{m}_j(t))\right\}$$
$$\times \left[\sum_{k=1}^{J} p_k(t) |\boldsymbol{\Sigma}_k(t)|^{-0.5} \exp\left\{-\frac{1}{2}(\mathbf{y}(t) - \mathbf{m}_k(t))^T \boldsymbol{\Sigma}_k(t)^{-1} (\mathbf{y}(t) - \mathbf{m}_k(t))\right\}\right]^{-1}. \quad (2.11)$$

2.2 The Probability of an Incorrect Classification

In this section we assume that the item that is associated with the Signature at time t given the last complete observation at time 0 will be estimated to be that one which maximizes the posterior probability (2.11).

For a simple illustration we will suppose that there are only J=2 possible items with known parameters $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$.

Given $\mathbf{Y}(0)=\mathbf{y}(0)$, $C(0)=1$, and $C(t) = 1$, the conditional distribution of $\mathbf{Y}(t)$ is multivariate normal with mean

$$\mathbf{m}_1(t) = \boldsymbol{\theta}_1 + \mathbf{A}_1^t(\mathbf{y}(0) - \boldsymbol{\theta}_1) \tag{2.12}$$

and variance-covariance matrix

$$\boldsymbol{\Sigma}_1(t) = \sum_{k=0}^{t-1} \left(\mathbf{A}_1^k\right)^T \boldsymbol{\Lambda}_1 \mathbf{A}_1^k. \tag{2.13}$$

Let the matrices $H_i(t)$ and H_i be such that

$$\mathbf{H}_i(t)\mathbf{H}_i(t)^T = \boldsymbol{\Sigma}_i(t) \tag{2.14}$$

and

$$\mathbf{H}_i\mathbf{H}_i^T = \boldsymbol{\Gamma}_i. \tag{2.15}$$

It follows that

$$\mathbf{Y}(t) \stackrel{d}{=} \mathbf{m}_1(t) + \mathbf{H}_1(t)\mathbf{U} \tag{2.16}$$

where $\mathbf{U}$ is a d-dimensional column vector each of whose components are independent standard *normal random variables; the notation* $\stackrel{d}{=}$ means equal in distribution. Thus, given $Y(0) = y(0)$, $C(0) = 1$, $C(t) = 1$,

$$\left(\mathbf{Y}(t) - \mathbf{m}_1(t)\right)^T \boldsymbol{\Sigma}_1(t)^{-1}\left(\mathbf{Y}(t) - \mathbf{m}_1(t)\right) \stackrel{d}{=} \mathbf{U}^T\mathbf{U} \tag{2.17}$$

and

$$\begin{aligned}&\left(\mathbf{Y}(t) - \mathbf{m}_2(t)\right)^T \boldsymbol{\Sigma}_2(t)^{-1}\left(\mathbf{Y}(t) - \mathbf{m}_2(t)\right) \\ &\qquad \stackrel{d}{=} \left(\mathbf{m}_1(t) - \mathbf{m}_2(t) + \mathbf{H}_1(t)\mathbf{U}\right)^T \boldsymbol{\Sigma}_2(t)^{-1}\left(\mathbf{m}_1(t) - \mathbf{m}_2(t) + \mathbf{H}_1(t)\mathbf{U}\right)\end{aligned} \tag{2.18}$$

where $\mathbf{m}_2(t) = \boldsymbol{\theta}_2$ and $\boldsymbol{\Sigma}_2(t) = \Gamma_2$. Thus, the probability of a misclassification is

$$P\{\text{classify the item as } 2|C(0) = 1, \mathbf{Y}(0) = \mathbf{y}(0), C(t) = 1\}$$

$$= P\Big\{p_2(t)|\mathbf{\Sigma}_2(t)|^{-0.5}\exp\Big\{-\frac{1}{2}(\mathbf{m}_1(t)-\mathbf{m}_2(t)+\mathbf{H}_1(t)\mathbf{U})^T\mathbf{\Sigma}_2(t)^{-1}(\mathbf{m}_1(t)-\mathbf{m}_2(t)+\mathbf{H}_1(t)\mathbf{U})\Big\}$$

$$> p_1(t)|\mathbf{\Sigma}_1(t)|^{-0.5}\exp\Big\{-\frac{1}{2}\mathbf{U}^T\mathbf{U}\Big\}\Big\}$$

$$= P\left\{\frac{p_2(t)}{p_1(t)}\left(\frac{|\mathbf{\Lambda}_1(t)|}{|\mathbf{\Gamma}_2|}\right)\right\}^{\frac{1}{2}} > \exp\Big\{-\frac{1}{2}\mathbf{U}^T\mathbf{U}+\frac{1}{2}(\mathbf{a}(t)+\mathbf{H}_1(t)\mathbf{U})^T\mathbf{\Gamma}_2^{-1}(\mathbf{a}(t)+\mathbf{H}_1(t)\mathbf{U})\Big\} \quad (2.19)$$

where

$$\mathbf{a}(t) = \mathbf{\theta}_1 + \mathbf{A}_1^t(\mathbf{y}(0)-\mathbf{\theta}_1)-\mathbf{\theta}_2. \quad (2.20)$$

Example: Assume $\mathbf{\Lambda}_i = \mathbf{\Lambda}, \mathbf{A}_i = \mathbf{A}$ for i = 1, 2, and $p_1(t) = p_2(t)$; then

$$P\{\text{wrong classification } |C(0) = 1, \mathbf{Y}(0) = \mathbf{y}(0), C(t) = 1\}$$

$$= P\left\{\left(\frac{|\mathbf{\Lambda}(t)|}{|\mathbf{\Gamma}|}\right)^{\frac{1}{2}} > \exp\Big\{-\frac{1}{2}\mathbf{U}^T\mathbf{U}+\frac{1}{2}(\mathbf{a}(t)+\mathbf{H}(t)\mathbf{U})^T\mathbf{\Gamma}^{-1}(\mathbf{a}(t)+\mathbf{H}(t)\mathbf{U})\Big\}\right\} \quad (2.21)$$

where

$$\mathbf{\Lambda}(t) = \sum_{k=0}^{t-1}\mathbf{A}^k\mathbf{\Lambda}(\mathbf{A}^k)^T \equiv \mathbf{H}(t)\mathbf{H}(t)^T;$$

$\mathbf{\Gamma} = \lim_{t\to\infty}\mathbf{\Lambda}(t) \equiv \mathbf{H}\mathbf{H}^T$ is the solution to the equation

$$\mathbf{\Gamma} = \mathbf{A}\mathbf{\Gamma}\mathbf{A}^T + \mathbf{\Lambda};$$

and

$$\mathbf{a}(t) = \mathbf{\theta}_1 + \mathbf{A}^t(\mathbf{y}(0)-\mathbf{\theta}_1)-\mathbf{\theta}_2.$$

Note that as $t\to\infty$

$$P\{\text{wrong classification } | C(0) = 1, \mathbf{Y}(0) = \mathbf{y}(0), C(t) = 1\}$$

$$\to P\left\{1 > \exp\left\{-\frac{1}{2}\mathbf{U}^T\mathbf{U} + \frac{1}{2}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 + \mathbf{H}\mathbf{U})^T \boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 + \mathbf{H}\mathbf{U})\right\}\right\}$$

$$= P\left\{1 > \exp\left\{\frac{1}{2}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)^T\left(\mathbf{H}\mathbf{H}^T\right)^{-1}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) + (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)^T\left(\mathbf{H}^T\right)^{-1}\mathbf{U}\right\}\right\}$$

$$= P\left\{-\frac{1}{2}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)^T\left(\mathbf{H}\mathbf{H}^T\right)^{-1}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) > (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)^T\left(\mathbf{H}^T\right)^{-1}\mathbf{U}\right\}. \tag{2.22}$$

A Simulation

Table 1 gives the results of a simulation experiment for the case $\boldsymbol{\theta}_1 = (1,1)^T$

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.5 \end{bmatrix}$$

and

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case,

$$\boldsymbol{\Gamma} = \begin{bmatrix} 5.4 & 3.8 \\ 3.8 & 4.5 \end{bmatrix}.$$

Figure 1 shows contours from a bivariate normal distribution having mean $\boldsymbol{\theta}_1$ *and variance-covariance matrix* $\boldsymbol{\Gamma}$.

In each replication two independent vector random variables are generated; one is $\mathbf{Y}(0)$ which has a normal distribution with mean $\boldsymbol{\theta}_1$ and variance-covariance matrix $\boldsymbol{\Gamma}$; the other is $\mathbf{U}$, whose components are two independent standard normal random variables. For each time $t = 1, 2, \ldots, 40$, $\mathbf{Y}(t)$ is calculated as

$$\mathbf{Y}(t) = \mathbf{m}(t) + \mathbf{H}(t)\mathbf{U}. \tag{2.23}$$

with $\mathbf{m}(t) = \boldsymbol{\theta}_1 + \mathbf{A}^t\,(\mathbf{y}(0) - \boldsymbol{\theta}_1)$; $\mathbf{Y}(t)$ has the same distribution as a Signature from Item 1 when the Signature at time 0 is also from Item 1. There are 1000 replications. Table 1 presents the fraction of replications for which the incorrect classification is made of Item 2 being the one producing the Signature at time t; that is those replications for which

$$\left(\frac{|\Lambda(t)|}{|\Gamma|}\right)^{\frac{1}{2}} > \exp\left\{\frac{1}{2}(\mathbf{y}(t)-\boldsymbol{\theta}_2)^T\Gamma^{-1}(\mathbf{y}(t)-\boldsymbol{\theta}_2)-\frac{1}{2}(\mathbf{y}(t)-\boldsymbol{\theta}_1)^T\Lambda(t)^{-1}(\mathbf{y}(t)-\boldsymbol{\theta}_1)\right\}. \tag{2.24}$$

Note that the fractions are not independent since common random numbers are used.

The contours of the distribution in Figure 1 suggest that it is more lik:ly to make a misclassification if $\boldsymbol{\theta}_2 = (2,2)^T$ than if $\boldsymbol{\theta}_2 = (-2,2)^T$; the fractions in Table A support this. The fractions in Table A also suggest that the probability of misclassification is an increasing function of t. This observation is supported by the fact that the variances of the components of $\mathbf{Y}(t)$ increase as t increases.

TABLE A. FRACTION OF MISCLASSIFICATION

$\boldsymbol{\theta}_1 = (1,1)^T$

Time:	**1**	**2**	**3**	**4**	**5**	**10**	**20**	**30**	**40**
$\boldsymbol{\theta}_2 = (2,2)^T$	0.10	0.13	0.16	0.20	0.21	0.30	0.39	0.41	0.41
$\boldsymbol{\theta}_2 = (-2,2)^T$	0.04	0.06	0.06	0.07	0.07	0.09	0.09	0.09	0.09

3. CAUCHY UNIVARIATE MODEL

In this section we consider Bayesian classification for a time series model having marginal distributions with a longer tail than the Gaussian distribution.

We assume that

$$Y_i(t) = \theta_i + X_i(t)$$

with

$$X_i(t) = \rho_i X_i(t-1) + \varepsilon_i(t)$$

where $|\rho_i| < 1$; $\{\varepsilon_i(t)\}$ are independent sequences of independent identically distributed Cauchy random variables with location parameter 0 and precisions $\left[(1-|\rho_i|)\alpha_i\right]^{-0.5}$; and $X_i(0)$ has a Cauchy distribution with parameters 0 and $\alpha_i^{-0.5}$. Under these assumptions $\{X_i(t);\ t = 0, 1, 2, \ldots\}$ is a stationary sequence of random variables with marginal Cauchy distribution having parameters 0 and $\alpha_i^{-0.5}$.

It follows that

$$P\{Y_i(0) \in dy(0), Y_i(t) \in dy_i(t)\}$$

$$= \frac{1}{\pi}\alpha_i\left[\alpha_i^2 + (y(0) - \theta_i)^2\right]^{-1} \frac{1}{\pi}\alpha_i\left(1-|\rho_i|^t\right)\left[\left(\alpha_i\left(1-|\rho_i|^t\right)\right)^2 + \left(y(t) - \theta_i - \rho_i^t(y(0) - \theta_i)\right)^2\right]^{-1}. \tag{3.1}$$

Let C(t) denote the identity of the item associated with the Signature at time t and put $P\{C(t)=i\} = p_i(t)$; then

$$P\{Y(t) \in dy(t), C(t) = i | C(0) = i, Y(0) = y(0)\}$$

$$= p_i(t)\frac{1}{\pi}\alpha_i\left(1-|\rho_i|^t\right)\left[\left(\alpha_i\left(1-|\rho_i|^t\right)\right)^2 + \left(y(t) - \theta_i - \rho_i^t(y(0) - \theta_i)\right)^2\right]^{-1}$$

$$\equiv p_i(t)\frac{1}{\pi}\alpha_i(t)\left[\alpha_i(t)^2+\left(y(t)-m_i(t)\right)^2\right]^{-1}. \qquad (3.2)$$

Thus,

$$P\{Y(t)\in dy(t)|C(0)=i,Y(0)=y(0)\}=\sum_{j=1}^{J}p_j(t)\frac{1}{\pi}\alpha_j(t)\left[\alpha_j(t)^2+\left(y(t)-m_j(t)\right)^2\right]^{-1}$$

where $\alpha_i(t)$ and $m_i(t)$ are defined in (3.2) and it is natural to define $m_j(t)=\theta_j$ and $\alpha_j(t)=\alpha_j$ for $j\neq i$. Hence, given item i is associated with the Signature at time 0, the posterior probability that item j is associated with the Signature observed at time t is

$$\begin{aligned}
&P\{C(t)=j|C(0)=i,Y(0)=y(0),Y(t)=y(t)\} \qquad (3.3)\\
&\quad=p_j(t)\alpha_j(t)\left[\alpha_j(t)^2+\left(y(t)-m_j(t)\right)^2\right]^{-1}\\
&\quad\times\left\{\sum_{k=1}^{J}p_k(t)\alpha_k(t)\left[\alpha_k(t)^2+\left(y(t)-m_k(t)\right)^2\right]^{-1}\right\}^{-1}.
\end{aligned}$$

3.2 The Probability of Making an Incorrect Classification

In this section we assume that the item associated with the Signature at time t given the last complete observation at time 0 is estimated to be that one which maximizes the posterior probability (3.3). For simplicity we will suppose there are J=2 possible items with known parameters θ_1 and θ_2.

First

$$\begin{aligned}
&P\{Y(t)\in dy(t)|Y(0)=y(0),C(0)=1,C(t)=j\}\\
&\quad=\frac{1}{\pi}\alpha_j(t)\left[\left(\alpha_j(t)\right)^2+\left(y(t)-m_j(t)\right)^2\right]^{-1} \qquad (3.4)
\end{aligned}$$

where

$$\alpha_1(t) = \alpha_1\left[1 - |\rho_1|^t\right]; \alpha_2(t) = \alpha_2 \tag{3.5}$$

$$m_1(t) = \theta_1 + \rho_1^t\left(y(0) - \theta_1\right); m_2(t) = \theta_2. \tag{3.6}$$

Note that given Y(0) = y(0), C(0) = 1 and C(t) = 1,

$$Y(t) \stackrel{d}{=} \left[\theta_1 + \rho_1^t\left(y(0) - \theta_1\right)\right] + \left(1 - |\rho_1|^t\right)\alpha_1 W \equiv m_1(t) + \alpha_1(t) W$$

where W is a Cauchy random variable with location parameter 0 and precision 1. Hence, the probability of making the incorrect classification of estimating Item 2 as being associated with the Signature at time t given Item 1 is responsible for Signatures at time 0 and time t and Y(0) = y(0) is

$$P\left\{\text{Classify as Item } 2 \middle| C(0) = 1, Y(0) = y(0), C(t) = 1\right\}$$
$$= P\left\{\frac{p_2(t)}{p_1(t)}\alpha_2(t)\left[\alpha_2(t)^2 + \left(Y(t) - \theta_2\right)^2\right]^{-1} > \alpha_1(t)\left[\alpha_1(t)^2 + \left(Y(t) - m_1(t)\right)^2\right]^{-1} \middle| C(0) = 1, Y(0) = y(0), C(t) = 1\right\}$$
$$= P\left\{\frac{p_2(t)}{p_1(t)}\frac{\alpha_2(t)}{\alpha_1(t)}\left[\alpha_2^2 + \left(m_1(t) + \alpha_1(t)W - \theta_2\right)^2\right]^{-1} > \left[\alpha_1(t)^2 + \left(\alpha_1(t)W\right)^2\right]^{-1} \middle| C(0) = 1, Y(0) = y(0), C(t) = 1\right\}$$
$$= P\left\{\frac{p_2(t)}{p_1(t)}\alpha_2(t)\alpha_1(t) > \frac{\left[\alpha_2^2 + \left(m_1(t) + \alpha_1(t)W - \theta_2\right)^2\right]}{1 + W^2} \middle| C(0) = 1, Y(0) = y(0), C(t) = 1\right\}.$$

(3.7)

Note that as $t \to 0$, $\alpha_1(t) \to 0$, and $m_1(t) \to y(0)$. Hence, the conditional probability of a wrong classification tends to

$$P\left\{0 > \frac{\left[\alpha_2^2 + \left(y(0) - \theta_2\right)^2\right]}{1 + W^2}\right\} = 0. \tag{3.8}$$

As $t \to \infty$, $\alpha_1(t) \to \alpha_1$, $m_1(t) \to \theta_1$ and the conditional probability of a wrong classification tends to

$$P\left\{\frac{p_2(\infty)}{p_1(\infty)}\alpha_2\alpha_1\left(1+W^2\right) > \left[\alpha_2^2 + (\alpha_1 W + \theta_1 - \theta_2)^2\right]\right\}. \tag{3.9}$$

If $\alpha_2 = \alpha_1 = \alpha$ and $p_2(\infty) = p_1(\infty)$, then as $t \to \infty$

$$P\{\text{incorrect classification} \mid Y(0) = y(0), C(0) = 1, C(t) = 1\}$$

$$= P\left\{\alpha^2\left(1+W^2\right) > \alpha^2\left(1+\left[W+\frac{(\theta_1-\theta_2)}{\alpha}\right]^2\right)\right\}$$

$$= P\left\{W^2 > \left[W+\frac{(\theta_1-\theta_2)}{\alpha}\right]^2\right\}$$

$$= P\left\{W > \left|\frac{\theta_1-\theta_2}{2\alpha}\right|\right\} \tag{3.10}$$

which increases as α increases and decreases as $|\theta_1 - \theta_2|$ increases.

4. ARE BAYESIAN CLASSIFICATION PROCEDURES ROBUST?

In this section the robustness of the univariate Cauchy and Gaussian classification procedures against misspecification of the form of the marginal distribution will be studied.

4.1 Gaussian Data.

In this subsection we assume that the Signatures of the Items form Gaussian time series. In particular we assume that

$$Y_i(t) = \theta_i + X_i(t) \tag{4.1}$$

with

$$X_i(t+1) = \rho_i X_i(t) + \varepsilon_i(t) \tag{4.2}$$

where $\{\varepsilon_i(t)\}$ are independent identically distributed normal random variables with mean 0 and variance σ_i^2 and $|\rho_i| < 1$. The independent random variable $X_i(0)$ has a normal distribution with mean 0 and variance $\sigma_i(\infty)^2 = \sigma_i^2 / \left(1-\rho_i^2\right)$. Thus $\{X_i(t), t \geq 0\}$ is a stationary sequence of normal random variables with mean 0 and variance $\sigma_i(\infty)^2$. Let C(t) be the identity of the Item associated with the Signature at time t.

As was shown in Gaver and Jacobs (1989), the conditional distribution of Y(t) given Y(0)=y(0), C(0)=i, C(t)=i is normal with mean

$$m_i(t) = \theta_i + \left(y(0) - \theta_i\right)\rho_i^t \tag{4.3}$$

and standard deviation

$$\sigma_i(t) = \sigma_i(\infty)\sqrt{1-\rho_i^{2t}}. \tag{4.4}$$

For simplicity we will assume $P\{C(t) = i\} = p_i(t) \equiv p(t)$ and there are 2 Items with parameters θ_1 and θ_2; thus, $p(t) = \frac{1}{2}$

Suppose the Cauchy procedure is used to estimate the identity of the Item associated with the Signature at time t; that is, the Item which maximizes the posterior probability (3.3) is the estimate of the Item associated with the Signature. Hence, the probability of an incorrect classification is

$$P\{\text{Classify as Item } 2 | C(0) = 1, Y(0) = y(0), C(t) = 1\}$$

$$= P\left\{\frac{\alpha_2(t)}{\alpha_1(t)}\left[\alpha_2(t)^2(m_1(t) + \sigma_1(t)Z - \theta_2)^2\right]^{-1} > \left[\alpha_1(t)^2 + (\sigma_1(t)Z)^2\right]^{-1}\right\}$$

$$= P\left\{\alpha_2(t)\alpha_1(t)\left[\alpha_2(t)^2 + (m_1(t) + \sigma_1(t)Z - \theta_2)^2\right]^{-1} > \left[1 + \frac{(\sigma_1(t)Z)^2}{\alpha_1(t)}\right]^{-1}\right\}$$

$$= P\left\{\alpha_2\alpha_1\left(1 - |\rho_1|^t\right)\left[\alpha_2(t)^2 + (m_1(t) + \sigma_1(t)Z - \theta_2)^2\right]^{-1} > \left[1 + \left(\frac{\sigma_1(\infty)\sqrt{1 - \rho_1^{2t}}}{\alpha_1\left(1 - |\rho_1|^t\right)} Z\right)^2\right]^{-1}\right\} \tag{4.5}$$

where Z is a standard normal random variable.

Note that as $t \to 0$

$$P\{\text{Classify as Item } 2 | C(0) = 1, Y(0) = y(0), C(t) = 1\}$$

$$= P\left\{\alpha_2\alpha_1\left(1 - |\rho_1|^t\right)\left[\alpha_2^2 + (m_1(t) + \sigma_1(t)Z - \theta_2)^2\right]^{-1} > \left[1 + \left[\frac{\sigma_1(\infty)}{\alpha_1}\right]^2 \frac{1 - \rho_1^{2t}}{\left(1 - |\rho_1|^t\right)^2} Z^2\right]^{-1}\right\}$$

$$= P\left\{\alpha_2\alpha_1\left(1-|\rho_1|^t\right)\left[\alpha_2^2+\left(m_1(t)+\sigma_1(t)Z-\theta_2\right)^2\right]^{-1} > \left[1+\left[\frac{\sigma_1(\infty)}{\alpha_1}\right]^2\frac{1+|\rho_1|^t}{1-|\rho_1|^t}Z^2\right]^{-1}\right\}$$

$$= P\left\{\alpha_1\alpha_2\left(1-|\rho_1|^t\right)\left[\alpha_2^2+\left(m_1(t)+\sigma_1(t)Z-\theta_2\right)^2\right]^{-1} > \left(1-|\rho_1|^t\right)\left[\left(1-|\rho_1|^t\right)+\left[\frac{\sigma_1(\infty)}{\alpha_1}\right]^2\left(1+|\rho_1|^t\right)Z^2\right]^{-1}\right\}$$

$$= P\left\{\left[\alpha_2^2+\left(m_1(t)+\sigma_1(t)Z-\theta_2\right)^2\right] < \alpha_1\alpha_2\left[\left(1-|\rho_1|^t\right)+\left[\frac{\sigma_1(\infty)}{\alpha_1}\right]^2\left(1+|\rho_1|^t\right)Z^2\right]\right\}$$

$$\to P\left\{\frac{1}{\alpha_1\alpha_2}\left[\alpha_2^2+\left(y(0)-\theta_2\right)^2\right] < \left[\frac{\sigma_1(\infty)}{\alpha_1}\right]^2 2Z^2\right\}$$

$$= P\left\{\frac{\alpha_1}{2\alpha_2\sigma_1(\infty)^2}\left[\alpha_2^2+\left(y(0)-\theta_2\right)^2\right] < Z^2\right\}. \qquad (4.6)$$

Thus, the conditional probability of an incorrect classification does not tend to 0 as $t\to\infty$ as it would if the correct model were used; see Gaver and Jacobs (1989) (3.6).

Note that as $t\to\infty$

$P\{\text{Classify as Item } 2 | C(0)=1, Y(0)=y(0), C(t)=1\}$

$$= P\left\{\alpha_2\alpha_1\left[\alpha_2^2+\left(\theta_1+\sigma_1(\infty)Z-\theta_2\right)^2\right]^{-1} > \left[1+\left(\frac{\sigma_1(\infty)}{\alpha_1}Z\right)^2\right]^{-1}\right\}. \qquad (4.7)$$

If $\alpha_1 = \alpha_2 = 1$, then the above equals

$$P\left\{\left[1+\left(\sigma_1(\infty)Z+\theta_1-\theta_2\right)^2\right]^{-1} > \left[1+\left(\sigma_1(\infty)Z\right)^2\right]^{-1}\right\}$$

$$= P\left\{(\sigma_1(\infty)Z)^2 > (\sigma_1(\infty)Z + \theta_1 - \theta_2)^2\right\}$$

$$= P\left\{Z^2 > \left(Z + \frac{(\theta_1 - \theta_2)}{\sigma_1(\infty)}\right)^2\right\}$$

$$= P\left\{Z > \frac{|\theta_1 - \theta_2|}{2\sigma_1(\infty)}\right\} \tag{4.8}$$

which is the same as if the correct model had been used to make the decision; see (3.9) of Gaver and Jacobs (1989).

Now we consider the case in which a different Item is associated with the Signature at time t than the one associated with the Signature at time 0. Once again for simplicity we assume $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$ with $|\rho| < 1$ and for the Cauchy model $\alpha_1 = \alpha_2 = \alpha$. Let $\sigma(\infty) = \sigma/\sqrt{1-\rho^2}$ and $\sigma(t) = \sigma(\infty)\sqrt{1-\rho^{2t}}$. We will assume Item 1 is associated with the Signature at time 0 and Item 2 is associated with the Signature at time t.

For the Gaussian classification procedure of Gaver and Jacobs (1989), the probability of an incorrect classification is

$$
\begin{aligned}
&P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\} \\
&\quad = P\{(1-\rho^{2t})^{-1/2} \exp\{-\tfrac{1}{2}[\theta_2 + \sigma(\infty)Z - (\theta_1 + (y(0)-\theta_1)\rho^t)]^2/\sigma(\infty)^2(1-\rho^{2t})\} \\
&\qquad\qquad > \exp\{-\tfrac{1}{2}(\theta_2 + \sigma(\infty)Z - \theta_2)^2/\sigma(\infty)^2\}\} \\
&\quad = P\{-\tfrac{1}{2}\ln(1-\rho^{2t}) - \tfrac{1}{2}[\theta_2 + \sigma(\infty)Z - (\theta_1 + (y(0)-\theta_1)\rho^t)]^2/\{\sigma(\infty)^2(1-\rho^{2t})\} \\
&\qquad\qquad > -\tfrac{1}{2}(\sigma(\infty)Z)^2/\sigma(\infty)^2\} \\
&\quad = P\{(1-\rho^{2t})\sigma(\infty)^2 \ln(1-\rho^{2t}) + [\theta_2 + \sigma(\infty)Z - (\theta_1 + (y(0)-\theta_1)\rho^t)]^2 \\
&\qquad\qquad < (1-\rho^{2t})(\sigma(\infty)Z)^2\}
\end{aligned} \tag{4.9}
$$

where Z is a standard normal random variable. As $t \to 0$

$$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\} \to P\left\{\left(Z + \frac{\theta_2 - y(0)}{\sigma(\infty)}\right)^2 < 0\right\} = 0; \quad (4.10)$$

that is, if the Item associated with the Signature at time t is different than the one associated with the Signature at time 0, then as $t \to 0$, the probability of an incorrect classification using the Gaussian procedure on Gaussian data tends to 0.

As $t \to \infty$, the probability of an incorrect classification,

$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$\to P\left\{\left(Z + \frac{\theta_2 - \theta_1}{\sigma(\infty)}\right)^2 < Z^2\right\} = P\left\{Z < \frac{-|\theta_2 - \theta_1|}{2\sigma(\infty)}\right\}. \quad (4.11)$$

Suppose now the Cauchy classification procedure is used on the Gaussian data with Item 2 associated with the Signature at time t and Item 1 associated with the Signature at time 0. The probability of an incorrect classification

$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$= P\{\alpha(1-|\rho|^t)[[\alpha(1-|\rho|^t)]^2 + [\theta_2 + \sigma(\infty)Z - [\theta_1 + (y(0)-\theta_1)\rho^t]]^2]^{-1} > \alpha[\alpha^2 + (\theta_2 + \sigma(\infty)Z - \theta_2)^2]^{-1}\}$$

$$= P\{(1-|\rho|^t))\,[\alpha^2 + (\sigma(\infty)Z)^2] > [(\alpha(1-|\rho|^t))^2 + [\theta_2 + \sigma(\infty)Z - [\theta_1 + (y(0)-\theta_1)\rho^t]]^2]\}. \quad (4.12)$$

As $t \to 0$

$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$\to P\{0 > [\theta_2 + \sigma(\infty)Z - y(0)]^2\} = 0; \quad (4.13)$$

that is, as $t \to 0$, the probability of a correct classification for the Cauchy procedure tends to 1 for the case in which the Item associated with the Signature at time t is different from the one associated with the Signature at time 0, even though the data are Gaussian.

As $t \to \infty$

P{classify as 1 | Y(0) = y(0), C(0) = 1, C(t) = 2}

$$\to P\left\{Z^2 > \left(Z + \frac{\theta_2 - \theta_1}{\sigma(\infty)}\right)^2\right\} = P\left\{Z < -\frac{|\theta_2 - \theta_1|}{2\sigma(\infty)}\right\}. \tag{4.14}$$

Hence, as $t \to \infty$ the probability of an incorrect identification tends to the same normal tail probability for both the Cauchy and Gaussian classification procedures.

Thus, for the two limiting cases $t \to 0$ and $t \to \infty$, both the Cauchy and Gaussian procedures have the same misclassification probabilities for the scenario in which the Item associated with the Signature at time t is different than the one associated with the Signature at time 0. Note that these are theoretical limiting results with all parameters known.

To investigate further the behavior of the two classification procedures on Gaussian data when Item 1 is associated with the Signature at time 0 and Item 2 is associated with the Signature at time t, let

$$g_t(y(0), Z) = (\theta_2 + \sigma(\infty)Z - [\theta_1 + (y(0) - \theta_1)\rho^t])^2.$$

The conditional probability of an incorrect classification by the Gaussian procedure is from (4.9)

$P\{\text{Classify as Item } 1 | Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$=P\{(1-\rho^{2t})\sigma(\infty)^2 \ln(1-\rho^{2t}) < (1-\rho^{2t})\,(\sigma(\infty)Z)^2 - g_t(y(0),Z)\}$$

$$\geq P\{(1-\rho^{2t})\sigma(\infty)^2 \ln(1-\rho^{2t}) < (1-|\rho|^t)\,(\sigma(\infty)Z)^2 - g_t(y(0),Z)\}$$

$$\geq P\{-\alpha^2(1-|\rho|^t)\ |\rho|^t < (1-|\rho|^t)\,(\sigma(\infty)Z)^2 - g_t(y(0),Z)\}$$

for t sufficiently close to 0. From (4.12) it follows that the conditional probability of misclassification for the Cauchy procedure is

$$P\{(1-|\rho|^t)(\sigma(\infty)Z)^2 - g_t(y(0),Z) > \alpha^2(1-|\rho|^t)\,[1-|\rho|^{t}-1]\}.$$

Hence for t sufficiently small, the incorrect Cauchy procedure will tend to have fewer misclassifications than the Gaussian procedure applied to Gaussian data in the scenario in which different Items are producing the Signatures at time 0 and t.

4.2 Cauchy Data

In this subsection we assume the Signatures form time series with Cauchy marginal distributions as in Section 3. In particular, we assume that

$$Y_i(t) = \theta_i + X_i(t) \tag{4.15}$$

with

$$X_i(t+1) = \rho_i X_i(t) + \varepsilon_i(t) \tag{4.16}$$

where $\{\varepsilon_i(t)\}$ are independent identically distributed Cauchy random variables with location 0 and precision $[(1-|\rho_i|)\alpha_i]^{-0.5}$ with $|\rho_i| < 1$. The independent random variable $X_i(0)$ has a Cauchy distribution with location 0 and precision $\alpha_i^{-1/2}$. Under these assumptions $\{X_i(t)\}$ is a stationary sequence of random variables with marginal Cauchy distribution having parameters 0 and $\alpha_i^{-1/2}$. Further, the conditional distribution of Y(t) given Y(0) = y(0), C(0) = i, C(t) = i is Cauchy with location parameter

$$m_i(t) = \theta_i + \rho_i^t(y(0) - \theta_i) \tag{4.17}$$

and precision parameter $\alpha_i(t)^{-1/2}$ with

$$\alpha_i(t) = \alpha_i(1 - |\rho_i|^t). \tag{4.18}$$

Let C(t) be the identity of the Item associated with the Signature at time t.

For simplicity we will assume there are two items with parameters θ_1 and θ_2. Further $P\{C(t) = i\} = p_i(t) \equiv p(t)$.

Suppose the Gaussian procedure of Gaver and Jacobs (1989) is used to estimate the identity of the item associated with the Signature at time t; that is, the item which maximizes (2.12) of Gaver and Jacobs (1989) is the estimate of the Item associated with the Signature at time t. Hence, the probability of an incorrect classification is

$$P\{\text{Classify as Item } 2 | C(0) = 1, Y(0) = y(0), C(t) = 1\}$$

$$= P\left\{ \frac{1}{\sigma_2(\infty)} \exp\left\{ -\frac{1}{2}\left[\frac{(Y(t) - \theta_2)}{\sigma_2(\infty)}\right]^2 \right\} > \frac{1}{\sigma_1(t)} \exp\left\{ -\frac{1}{2}\left[\frac{Y(t) - m_1(t)}{\sigma_1(t)}\right]^2 \right\} \middle| \begin{matrix} C(0) = 1, \\ Y(0) = y(0), \\ C(t) = 1 \end{matrix} \right\} \tag{4.19}$$

where

$$m_1(t) = \theta_1 + \rho_1^t(y(0) - \theta_1) \tag{4.20}$$

$$\sigma_1(t) = \sigma_1(\infty)\sqrt{1 - \rho_1^{2t}} \tag{4.21}$$

$$\sigma_1(\infty) = \sigma_1 / \sqrt{(1 - \rho_1^2)}; \sigma_2(\infty) = \sigma_2 / \sqrt{(1 - \rho_2^2)} \tag{4.22}$$

and σ_i, i = 1, 2 are the assumed standard deviations of the normal distributions. We will assume $\sigma_1 = \sigma_2 = \sigma$; $\rho_1 = \rho_2 = \rho$; $\alpha_1 = \alpha_2$. Hence,

$$P\{\text{Classify as Item } 2|C(0)=1, Y(0)=y(0), C(t)=1\}$$

$$= P\left\{\frac{1}{\sigma(\infty)}\exp\left\{-\frac{1}{2}\left(\frac{m_1(t)+\alpha(t)W-\theta_2}{\sigma(\infty)}\right)^2\right\} > \frac{1}{\sigma(t)}\exp\left\{-\frac{1}{2}\left(\frac{\alpha(t)W}{\sigma(t)}\right)^2\right\}\right\} \quad (4.23)$$

where W is a Cauchy random variable with location parameter 0 and precision 1.

$$P\{\text{Classify as Item } 2|C(0)=1, Y(0)=y(0), C(t)=1\}$$

$$= P\left\{\sqrt{1-\rho^{2t}} > \exp\left\{-\frac{1}{2}\frac{\left(\alpha\left[1-|\rho|^t\right]\right)^2}{\left(1-\rho^{2t}\right)\sigma(\infty)^2}W^2 + \frac{1}{2}\frac{\left(\rho^t(y(0)-\theta_1)+(\theta_1-\theta_2)+\alpha(t)W\right)^2}{\sigma(\infty)^2}\right\}\right\}$$

$$= P\left\{\sqrt{1-\rho^{2t}} > \exp\left\{-\frac{1}{2}\frac{\alpha^2\left[1-|\rho|^t\right]}{\left(1+|\rho|^t\right)\sigma(\infty)^2}W^2 + \frac{1}{2}\frac{\left(\rho^t(y(0)-\theta_1)+(\theta_1-\theta_2)+\alpha(t)W\right)^2}{\sigma(\infty)^2}\right\}\right\} \quad (4.24)$$

$$\xrightarrow{t\to 0} P\left\{0 > \exp\left\{\frac{1}{2}\frac{(y(0)-\theta_2)^2}{\sigma(\infty)^2}\right\}\right\} = 0;$$

thus, the probability of misclassification tends to zero as $t\to 0$ even though the incorrect model is being used; the correct Cauchy procedure also has a probability of misclassification tending to zero as $t\to 0$. As $t\to\infty$

$$P\{\text{Classify as Item } 2|C(0)=1, Y(0)=y(0), C(t)=1\}$$

$$= P\left\{1 > \exp\left\{-\frac{1}{2}\frac{\alpha^2 W^2}{\sigma(\infty)^2} + \frac{1}{2}\frac{\left[\theta_1-\theta_2+\alpha W\right]^2}{\sigma(\infty)^2}\right\}\right\}$$

$$= P\left\{1 > \exp\left\{\frac{1}{2}\frac{1}{\sigma(\infty)^2}\left[(\theta_1-\theta_2)^2 + 2(\theta_1-\theta_2)\alpha W\right]\right\}\right\}$$

$$P\left\{0 > \frac{1}{2}(\theta_1-\theta_2)^2 + (\theta_1+\theta_2)\alpha W\right\}$$

$$= P\left\{W < -\frac{1}{2}\frac{1}{\alpha}|\theta_1-\theta_2|\right\}$$

which is the same as (3.10) the corresponding probability when the correct Cauchy procedure is used.

To further explore the behavior as $t \to 0$, let

$$g_t(W,y(0)) = [\rho^t(y(0)-\theta_1) + (\theta_1-\theta_2) + \alpha(1-|\rho|^t)W]^2$$

and

$$B(t) = \alpha^2(1-|\rho|^t)W^2.$$

For t small (4.24) becomes

$$P\{\text{Classify as Item } 2|Y(0) = y(0), C(0) = 1, C(t) = 1\}$$

$$=P\{\sigma(\infty)^2(1+|\rho|^t)\ln(1-\rho^{2t}) + B(t) > (1+|\rho|^t)\, g_t(W,y(0))\}$$

$$\le P\{-\alpha^2|\rho|^t+B(t) > (1+|\rho|^t)g_t(W,y(0))\}$$

$$\le P\{-\alpha^2|\rho|^t+B(t) > g_t(W,y(0))\}$$

$$=P\{\alpha^2(1-|\rho|^t) +B(t) > \alpha^2+g_t(W,y(0))\}$$

$$=P\{(\alpha(1-|\rho|^t))^2 + (1-|\rho|^t)\, B(t) > (1-|\rho|^t)\, [\alpha^2+g_t(W,y(0))]\}$$

$$=P\{\alpha[\alpha^2+g_t(W,y(0))]^{-1} > \alpha(1-|\rho|^t)\, [[\alpha(1-|\rho|^t)]^2 + (\alpha(1-|\rho|^t)W)^2]^{-1}\}$$

which is the conditional probability of misclassification for the Cauchy procedure on Cauchy data. Hence for small t, the incorrect Gaussian procedure will tend to have fewer misclassifications than the correct Cauchy procedure for the scenario in which the same item is associated with the Signature at both times.

Now we consider the case in which the Item associated with the Signature at time t is different than the one associated with the Signature at time 0. Once again for simplicity we assume $\alpha_1 = \alpha_2 = \alpha$, $\rho_1 = \rho_2 = \rho$ with $|\rho|<1$ and for the Gaussian model $\sigma_1 = \sigma_2 = \sigma$. Let $\sigma(\infty) = \sigma/(1-\rho^2)^{0.5}$ and $\sigma(t) = \sigma(\infty)\,(1-\rho^{2t})^{0.5}$. We will assume Item 1 is associated with the Signature at time 0 and Item 2 is associated with the Signature at time t.

For the Gaussian classification procedure of Gaver and Jacobs (1989) the probability of an incorrect classification is

$$
\begin{aligned}
&P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\} \\
&=P\{(1-\rho^{2t})^{-0.5} \exp\{-\tfrac{1}{2}[\theta_2 + \alpha W - (\theta_1 + \rho^t(y(0)-\theta_1))]^2/\sigma(\infty)^2(1-\rho^{2t})\} \\
&\qquad >\exp\{-\tfrac{1}{2}(\theta_2+\alpha W-\theta_2)^2/\sigma(\infty)^2\}\} \\
&=P\{-\tfrac{1}{2}\ln(1-\rho^{2t}) - \tfrac{1}{2}[\theta_2 + \alpha W - (\theta_1 + \rho^t(y(0)-\theta_1))]^2/\{\sigma(\infty)^2(1-\rho^{2t})\} \\
&\qquad >-\tfrac{1}{2}(\alpha W)^2/\sigma(\infty)^2\} \\
&=P\{\sigma(\infty)^2(1-\rho^{2t})\ln(1-\rho^{2t}) +(\theta_2 + \alpha W-(\theta_1 + \rho^t(y(0)-\theta_1)))^2 \\
&\qquad <(\alpha W)^2(1-\rho^{2t})\}
\end{aligned}
\tag{4.26}
$$

where W is a standard Cauchy random variable.

As $t \to 0$, the probability of an incorrect classification,

$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$\to P\{(\theta_2+\alpha W-y(0))^2 < 0\} = 0.$$

Hence as $t\to 0$, the probability of an incorrect classification tends to 0 for the Gaussian procedure on Cauchy data.

As $t\to\infty$, the probability of an incorrect decision

$$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}\to P\{(\theta_2 + \alpha W-\theta_1)^2 <(\alpha W)^2\}$$

$$=P\{(W+\frac{\theta_2-\theta_1}{\alpha})^2 < W^2\} = P\{W>\frac{|\theta_2-\theta_1|}{2\alpha}\}.$$

Suppose now the Cauchy classification procedure is used on the Cauchy data with Item 2 associated with the Signature at time t and Item 1 associated with the Signature at time 0. The probability of an incorrect classification is $P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$=P\{\alpha(1-|\rho|^t)[\alpha^2(1-|\rho|^t)^2 + [\theta_2+\alpha W-(\theta_1+\rho^t(y(0)-\theta_1))]^2]^{-1}$$

$$>\alpha[\alpha^2 + (\theta_2 +\alpha W-\theta_2)^2]^{-1}\}$$

$$=P\{(1-|\rho|^t)[\alpha^2 + (\alpha W)^2]$$

$$>[\alpha^2(1-|\rho|^t)^2 + [\theta_2+\alpha W-(\theta_1+\rho^t(y(0)-\theta_1))]^2]\}. \tag{4.27}$$

As $t\to 0$, the probability of an incorrect classification

$$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\} \to P\{0>(\theta_2+\alpha W - y(0))^2\} = 0.$$

Thus, the probability of an incorrect identification using the Cauchy procedure tends to 0 as $t\to 0$ for the case in which the Item associated with the

Signature at time t is different from the Item associated with the Signature at time 0.

As $t \to \infty$, the probability of an incorrect identification

$$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$$

$$\to P\{W^2 > (W + \frac{(\theta_2 - \theta_1)}{\alpha})^2\} = P\{W > \frac{|\theta_2 - \theta_1|}{2\alpha}\}$$

the same as for the Gaussian procedure.

Hence, for the two limiting cases $t \to 0$ and $t \to \infty$ both the Cauchy and Gaussian procedures have the same misclassification probabilities for the case in which the Item associated with the Signature at time t is different than the one associated with the Signature at time 0. Note these are theoretical limiting results with all parameters known.

To further explore the differences between the Gaussian and Cauchy procedures for the scenario of different Items associated with Signatures and Cauchy data, let

$$g_t(W, y(0)) = [\alpha W + \theta_2 - \theta_1 - \rho^t (y(0) - \theta_1)]^2. \tag{4.28}$$

From (4.26) for the Gaussian procedure, the probability of an incorrect classification

$P\{\text{classify as } 1 \mid Y(0) = y(0), C(0) = 1, C(t) = 2\}$

$$= P\{\sigma(\infty)^2 (1 - \rho^{2t}) \ln (1 - \rho^{2t}) < (\alpha W)^2 (1 - \rho^{2t}) - g_t(W, y(0))\}.$$

For the Cauchy procedure, the probability of an incorrect classification

P{classify as 1 | Y(0) = y(0), C(0) = 1, C(t) = 2}

$$= P\{\alpha^2(1-|\rho|^t)[1-|\rho|^{t-1}] < (1-|\rho|^t)\,(\alpha W)^2 - g_t(W,y(0))\}$$

$$= P\{\alpha^2(1-|\rho|^t)|\rho|^t < (1-|\rho|^t)\,(\alpha W)^2 - g_t(W,y(0))\}$$

$$\leq P\{\sigma(\infty)^2(1-\rho^{2t})\ln(1-\rho^{2t}) < (1-|\rho|^t)\,(\alpha W)^2 - g_t(W,y(0))\}$$

$$\leq P\{\sigma(\infty)^2(1-\rho^{2t})\ln(1-\rho^{2t}) < (1-\rho^{2t})\,(\alpha W)^2 - g_t(W,y(0))\}$$

for t sufficiently small. Thus, for small t the Gaussian procedure will tend to have more incorrect classifications than the Cauchy procedure for the scenario of Cauchy data with the Item associated with the Signature at time t being different than the one associated with the signature at time 0. This effect is made stronger by the fact that if the Gaussian procedure is used then an estimate of $\sigma(\infty)^2$ will be needed. An estimate of $\sigma(\infty)^2$ for Cauchy data will tend to be very large since the Cauchy distribution does not have a finite variance. This effect will be seen in the simulations of the next subsection.

4.3 Results of simulation experiments

This subsection reports on results of simulation experiments to assess the behavior of the Gaussian and Cauchy classification procedures when they are confronted with data from the other distribution. For simplicity we assume there are two Items. In the first subsection the autoregressive process producing the data is Gaussian. In the second subsection the autoregressive process producing the data is Cauchy. In both subsections classification procedures using both the Cauchy and Gaussian distributional assumptions are assessed. In all cases $\rho_1 = \rho_2 = 0.5$, $\theta_1 = 1$, $\theta_2 = 2$. The simulations use the LLRANDOM random number generator; cf. Lewis and Uribe [1981].

a. Gaussian Data

The simulation in this subsection uses data from a Gaussian autoregressive process. We will assume the means of the two Signatures, θ_1 and θ_2, are known and $\rho_1 = \rho_2 = \rho$ is also known. It remains to assess values for the (presumed known) scale parameters of the two classification procedures. In particular what should the scale parameter $\alpha = \alpha_1 = \alpha_2$ of the Cauchy procedure be when it is applied to Gaussian data? To obtain reasonable values for $\sigma_1 = \sigma_2 = \sigma$ for the Gaussian classification procedure and $\alpha = \alpha_1 = \alpha_2$ for the Cauchy classification procedure, the following simulation experiment was performed. The experiment has 100 replications. In each replication 100 independent, standard normals are generated. For each replication, the standard deviation of the data is computed and the maximum likelihood estimate of α is obtained numerically assuming a Cauchy density function of the form

$$f(x) = \frac{1}{\pi}\frac{\alpha}{\alpha^2+x^2}\ , \quad -\infty < x < \infty.$$

The medians of the 100 estimates of α and the 100 standard deviations are calculated. The values obtained are $\hat{\sigma}_M = 1.0$ $\hat{\alpha}_M = 0.607$. Note that the estimates of α are using the incorrect model assumption of Cauchy for the Gaussian data. The value of $\hat{\sigma}_M$ is used in the Gaussian procedure to classify observations. The value of $\hat{\alpha}_M$ is used in the Cauchy procedure.

Tables 1 and 2 show results for simulation experiments with 500 replications. In each replication Y(0) is generated from a normal distribution with mean θ_1 and standard deviation $\sigma(\infty) = \sigma/\sqrt{1-\rho^2}$ with σ=1 and ρ = 0.5. For Table 1 Y(t) is generated from a normal distribution with mean

$$m(t) = \theta_1 + \rho^t(Y(0)-\theta_1)$$

and standard deviation $\sigma(t) = \sigma(\infty)\sqrt{1-\rho^t}$; namely the Signature observed at time t is from Item 1. For Table 2 Y(t) is generated from a normal distribution with mean θ_2 and standard deviation $\sigma(\infty)$; namely the Signature at time t is from Item 2.

In both Tables the Gaussian classification procedure assumes $\hat{\sigma}_M = 1.0$ is the correct standard deviation. The Cauchy classification procedure assumes $\hat{\alpha}_M = 0.607$ is the correct value for α.

The values in Table 1 suggest that when the same Item is producing the Signature at time 0 and t, then the Gaussian procedure produces more correct classifications for small time t. However, the number of correct classifications is the same for both procedures for larger t.

The values of Table 2 suggest that if a different Item is producing the Signature at time t, then the Cauchy classification procedure has more correct classifications at time t for small t even though the data are Gaussian. For larger t, both procedures have the same number of correct identifications.

b. Cauchy Data

In this subsection the data arise from a Cauchy autoregressive process. The mean Signatures of the two Items, θ_1 and θ_2, are assumed known and $\rho = \rho_1 = \rho_2$ is also assumed known. It remains to assess values for the scale parameters of the Gaussian and Cauchy classification procedures. In particular, what should the scale parameter $\sigma = \sigma_1 = \sigma_2$ of the Gaussian procedure be when it is applied to Cauchy data?

TABLE 1. GAUSSIAN DATA

Item 1 Produces the Signature at time 0
Item 1 Produces the Signature at time t

Time	Fraction Correct Identifications		Number of times	
t	Gaussian Proc.	Cauchy Proc.	Gaussian Correct /Cauchy Incorrect	Gaussian Incorrect/Cauchy Correct
1	0.77	0.65	50	0
2	0.68	0.67	5	0
5	0.67	0.67	0	0
10	0.70	0.70	0	0

TABLE 2. GAUSSIAN DATA

Item 1 Produces the Signature at time 0

Item 2 Produces the Signature at time t

Time	Fraction Correct Identifications		Number of times	
	Gaussian Proc.	Cauchy Proc.	Gaussian Correct /Cauchy Incorrect	Gaussian Incorrect/Cauchy Correct
1	0.64	0.71	0	38
2	0.69	0.71	0	6
5	0.64	0.64	1	0
10	0.68	0.68	0	0

To obtain reasonable values for σ for the Gaussian classification procedure and α for the Cauchy classification procedure, the following simulation experiment was performed. The experiment has 100 replications. Each replication generates 100 standard Cauchy random numbers. For each replication the standard deviation of the data is computed and the maximum likelihood estimate of α is obtained numerically. The medians of the 100 estimates of α and the 100 standard deviations are computed. The values obtained are

$$\hat{\sigma}_M = 13.23 \text{ and } \hat{\alpha}_M = 1.03.$$

Note the high value of the standard deviation.

Tables 3 and 4 present results of simulation experiments in which the data are from a Cauchy autoregressive process. All experiments have 500 independent replications. For each replication Y(0) is generated from a Cauchy distribution with location parameter θ_1, and scale parameter 1; that is, Item 1 is producing the Signature at time 0. For replications reported in Table 4, Y(t) is generated from a Cauchy distribution with location parameter θ_2 and scale parameter 1; that is, Item 2 is producing the Signature at time t. For replications reported in Table 3, Y(t) is generated from a Cauchy distribution having density function

$$f(x) = \frac{1}{\pi} \frac{\alpha(t)}{\alpha(t)^2 + (x-m(t))^2}$$

with

$$m(t) = \theta_1 + \rho^t(y(0)-\theta_1)$$

and

$$\alpha(t) = (1-|\rho|^t);$$

that is, Item 1 is also producing the Signature at time t.

In both Tables 3 and 4, the Gaussian classification procedure assumes a standard deviations $\sigma_1 = \sigma_2 = \hat{\sigma}_M$. The Cauchy classification procedure assumes the α-parameters $\alpha_1 = \alpha_2 = \hat{\alpha}_M$.

The results of Table 3 indicate that for small times t, if the same Item is producing the Signature at time 0 and time t, then the Gaussian classification procedure has more correct classifications even though the data are Cauchy. For larger times t, the number of correct classifications is the same for both procedures. On the other hand, the results of Table 4 indicate that if a different item is producing the Signature at time t, then the Cauchy classification procedure has many more correct classifications than the Gaussian procedure for small times t. Once again the number of correct identifications is the same for both procedures as t becomes larger.

TABLE 3. CAUCHY DATA

Item 1 Produces the Signature at time 0

Item 1 Produces the Signature at time t

Time	Fraction Correct Identifications		Number of times	
	Gaussian Proc.	Cauchy Proc.	Gaussian Correct /Cauchy Incorrect	Gaussian Incorrect/Cauchy Correct
1	0.98	0.73	124	0
5	0.72	0.69	18	0
10	0.66	0.66	0	1

TABLE 4. CAUCHY DATA

Item 1 Produces the Signature at time 0

Item 2 Produces the Signature at time t

Time	Fraction Correct Identifications		Number of times	
	Normal Proc.	Cauchy Proc.	Normal Correct /Cauchy Incorrect	Normal Incorrect/Cauchy Correct
1	0.09	0.74	0	316
2	0.14	0.70	0	282
5	0.67	0.61	0	29
10	0.64	0.64	0	0

c. Summary

The differences in performance of the two classification procedures appear for small time t. If the same Item is producing Signatures at both 0 and t, then the Gaussian classification procedure has more correct classifications for small times t for both Gaussian and Cauchy data. If a different Item is producing the Signature at time t, then the Cauchy classification procedure has more correct classifications for both Gaussian and Cauchy data. The effect is strongest if the data are from a Cauchy autoregressive process; in this case the Gaussian procedure does very poorly when different Items are producing the Signatures.

In summary, it is important to realize that the performance of a Bayesian classification procedure can be influenced by its underlying distributional assumptions. A classification procedure based on Gaussian distributional assumptions can be reluctant to classify a new observation coming from a different item as being associated with a new item. A classification procedure based on Cauchy distributional assumptions can be reluctant to classify a new observation which comes from the same item as that being associated with the same item. Hence, if there is uncertainty about the underlying distribution of the data, it might be better to combine results of several classification procedures based on different distributional assumptions.

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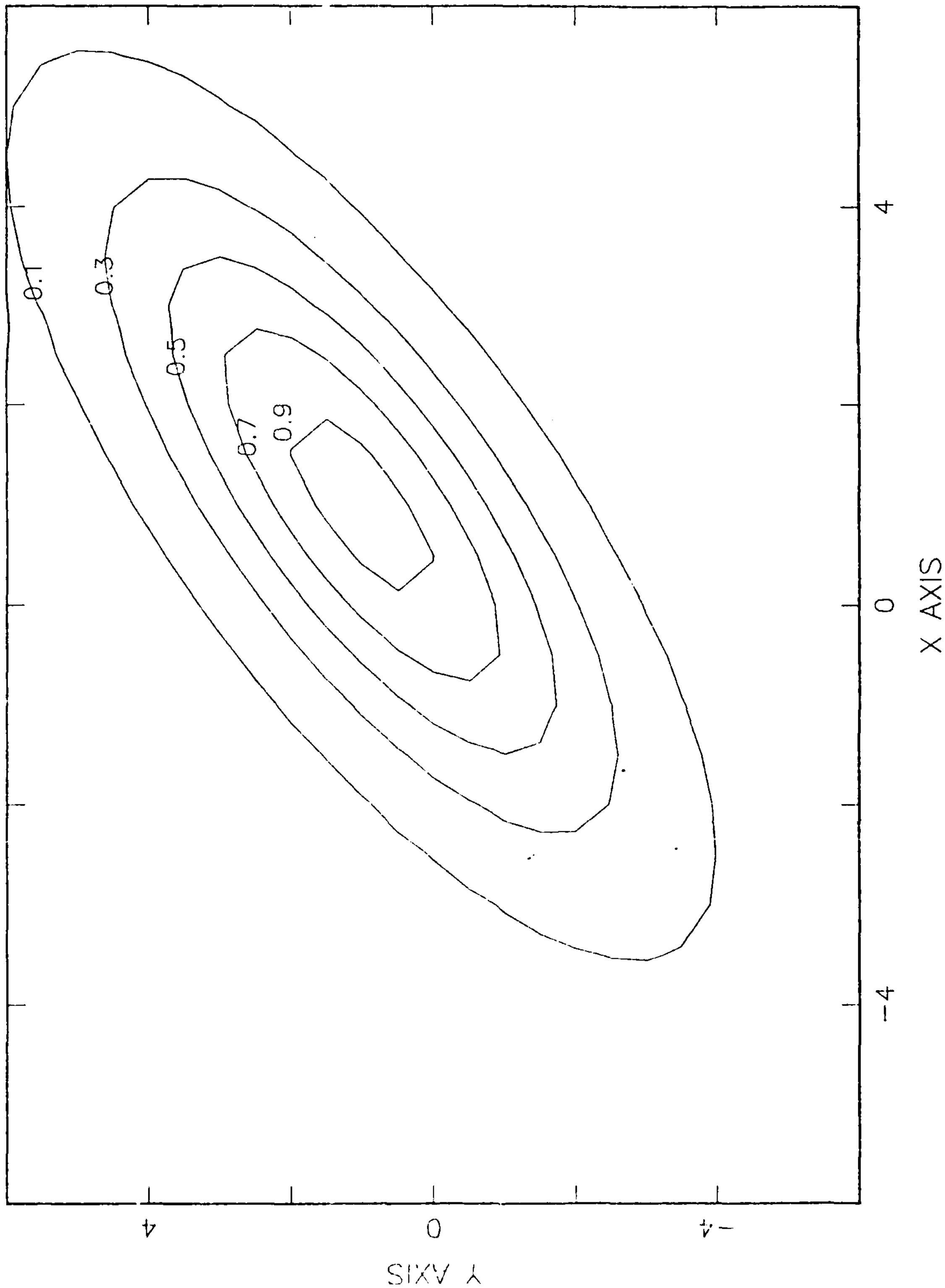

Figure 1

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