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POWER SPECTRA OF INTERNAL GRAVITY WAVES

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13. ABSTRACT (Maximum 200 words) The OH layer located in the region of 85 km altitude emits strong infrared radiation. Gravity waves can modulate the brightness of this layer over a wide range of spatial scales. Such fluctuations constitute, in effect, a form of IR clutter which could potentially degrade Air Force surveillance systems in certain situations. For this reason there is an interest in the spatial and temporal variations of atmospheric internal gravity waves.

A physical, similitude model of internal gravity waves assumes saturation of the waves and control by cascade processes of the temporal and horizontal scales of the waves. This model contains all the power spectral densities (PSD's) (sometimes merely called spectra) to be found in the formalism of Garrett and Munk. The latter is a purely empirical model for internal gravity waves applicable to the atmosphere and ocean. The main new predictions of the present model are that the dissipation rate, ε, controls the amplitudes of the frequency and horizontal wave number spectra. The validity of the proposed model is unknown at this time, and will depend upon the future experimental tests. It will be shown, however, that based on "typical" parametric values, results from the model are encouraging.

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Preface

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Power Spectra of Internal Gravity Waves

1. INTRODUCTION

This report answers a question implied in three earlier publications (Dewan, 1979; Dewan and Good, 1984, 1986). The question is "Can the so-called universal spectral model of Garrett and Munk (1972, 1975, 1979) be explained by physical similitude arguments?" The apparent success of the similitude argument in Dewan and Good (1984, 1986) suggests that it, indeed, can be done. The original models of Garrett and Munk (GM) cited above were based on a combination of empirical observations of oceanic internal gravity waves plus the general properties of these waves that can be deduced from the equations of motion such as dispersion relations and so-called "wave functions". Subsequently, VanZandt (1982) showed that the universal spectral model also applies to the internal waves in the atmosphere. The term "universal" in both contexts, however, must be understood as somewhat of an exaggeration. For example, the amplitude of the spectra fall "mostly within a factor of 2"* (GM, 1972, GM, 1975) and the log-log slopes of certain wave number spectra are between 5/3 and 2 (GM, 1972) or 2.5 (GM, 1975) or, in the case of the atmosphere, nearly 3 (Dewan et al, 1984).

In an earlier paper (Dewan, 1979) it was suggested that the GM spectral model might be applicable to the atmosphere; but, the situation studied in that paper involved exceptionally

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* In Section 6 it will be mentioned that, in the atmosphere, a change of altitude can cause the amplitude of the temporal PSD to change by a factor of 200.

vigorous waves in the stratosphere. No attention was given there to the "universal" amplitude property; but it was suggested that the waves involved were interacting sufficiently to provide a "cascade effect" to explain spectral slope. An analogy to turbulent cascades was made there.

Historically speaking, the turbulent cascade was first described in a poem by Richardson (1922): "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls, and so on to viscosity (in the molecular sense)." In Dewan (1979) a detailed argument was presented for the case that a similar thing could be occurring for atmospheric gravity waves and that "Big waves have little waves that feed on deformation, and little waves have lesser waves to turbulent dissipation (in the eddy sense)." Kolmogorov deduced his inertial range spectrum for velocity fluctuations, $\phi = c_1 \varepsilon^{2/3} k^{-5/3}$ (ε is the dissipation rate, k the wave number and c_1 a universal constant of order unity) by means of a similitude argument based on the Richardson poem. In Dewan (1979) $\phi = c_2 \varepsilon^{2/3} k_x^{-5/3}$ (k_x is horizontal wave number and $c_2 \neq c_1$) was derived in exactly the same manner via the second "poem". In this report I wish to extend the argument in Dewan (1979) in order to attempt to find a physical explanation of the remainder of the GM spectral model.

In apparent contradiction to the above approach, Dewan and Good (1984, 1986) explained the universal spectrum, $\Psi_u(k_z) = c_3 N^2 k_z^{-3}$ (where $\Psi_u(k_z)$ is the PSD for horizontal velocity fluctuations, N is the buoyancy frequency and k_z the vertical wave number) which is observed in the stratosphere (Dewan et al, 1984), and troposphere (Daniels, 1982, and Endlich and Singleton, 1969) by means of an argument based on "saturation" rather than on "cascade" effects. Again, a similitude argument was basic to this approach in spite of the fact that several other, more detailed arguments were also presented. This work was subsequently augmented by Smith et al (1987) who, among other things, deduced the value c_3 above and, most importantly, predicted the altitude dependence of the dominant vertical wave number which they called m_*.

In subsequent years there have been further tests of the "saturation hypothesis" (in the form of the $\Psi_u(k_z) = N^2 k_z^{-3}$ dependence). For example see Fritts et al (1988), Gardner et al (1989), Gardner and Voelz (1987), Gardner (1989), Hass and Meyer (1987) and Wu and Widdel (1989). More recently Tsuda et al (1989) wrote: "The fact that the argument of the saturated gravity wave spectral model of Dewan and Good (1986) and Smith et al (1987) improves as the observed spectra are improved suggests that the model is essentially correct, in spite of the heuristic nature of some of its assumptions."

Most recently, Hines (1990)† re-examined these "heuristic assumptions" and seems to have put the entire program on a more extended physical footing. In addition, he also raised the question that the saturation picture may give the correct answer for the wrong reason. While this seems not to have disturbed the similitude argument in Dewan and Good (1984, 1986), Hines' new physical picture is very stimulating to further progress with this problem. The

† A detailed response to the criticisms of Hines (1990) is postponed to a future publication.

reader is encouraged to examine Hine's paper for the details; but, his approach involves strongly interacting waves and the "compliant wave number" regime first discussed by Munk (1981). In this regime the small waves ($m > m_*$) are doppler shifted and strongly modified by critical layer processes due to the convective effects of the large waves. In a word, Hines' new picture has made one thing very clear for the present purpose, and this is that the success of the $\Psi_u(k_z) = c_3 N^2 k_z^{-3}$ spectrum no longer argues against cascade processes which is ours.† In fact the opposite is true; that is to say, the cascade effects in waves, as suggested in Dewan (1979), appear more valid when viewed from the perspective of Hines' "wave-interaction" suggestion. Usually one regards "saturation" as implying that, as waves grow, they eventually reach a "saturated amplitude" above which they "dump energy" directly from their large scales into small turbulent scales. In contrast, Hines' picture of strongly interacting waves is consistent with cascade processes of energy flowing from large to small scale via intermediate scales. A second benefit here from the Hines picture is its agreement with the long discussed and disturbing fact that gravity waves are not the weakly interacting waves that theorists would prefer them to be for calculational convenience. Rather they are unavoidably strongly interacting. The latter point has been emphasized in the oceanic internal wave context by Holloway (1980, 1981).

For completeness it should be mentioned that certain previous authors have raised the question of whether or not the GM model or the saturated spectrum is related to buoyancy range (BR) turbulence. Weinstock (1985), for example, has suggested that the saturated spectrum may be the (BR) turbulence as described by Lumley †† (1964), which also has N^2k^{-3} dependence. This view was also suggested by Holloway (1983) and Gargett (1981). Also, Sidi et al (1987) showed, on the basis of saturation, that the temporal part of the GM spectrum would have a ω^{-1} dependence and he considered this as a form of BR. In what follows I will, in contrast, get a ω^{-2} dependence, but in general, I am at this point somewhat in overall agreement with the general idea that the GM spectrum may be of "BR turbulence". On the other hand, the word "turbulence" in this context is very misleading in the sense that one usually assumes, by definition, that turbulence does not obey dispersion relations. As discussed in Dewan (1985), there may exist fluctuations that are between the extremes of turbulence and waves which continue to retain dispersion relations and limited propagation properties, and yet have very active nonlinear mode interactions. Perhaps this is what physically happens in BR "turbulence" as well as in the "universal spectrum range of gravity waves"; and, it would explain why similitude arguments based on cascades are successful (if indeed the main predictions I make here are found to be valid in future experiments).

In order to relate the GM formalism to similitude arguments I will first give a brief review of their formalism in Section 2 and then present the derivation of the temporal part of their formalism in Section 3. In Section 4 a number of testable PSD's are derived which are then

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† On the other hand, since it refers to interaction between a single large scale region and a single small scale region, it does not necessarily imply that a cascade is involved.

†† Compare Phillips, p. 293 (1980) whose discussion is highly to be recommended especially in conjunction with what he says on p. 232 regarding "promiscuous" mode interaction.

collected in Section 5 where predictions for future tests are described. In Section 6 a limited comparison between theory and experiment is made which demonstrates that the predictions are not unreasonable in the sense that they appear to be of the right order of magnitude. Finally, still further experimental support is obtained by comparing the altitude dependence of a temporal PSD of velocity fluctuations with the altitude dependence of dissipation rate, ε, since the theory says that there is a direct relationship between them.

2. A BRIEF REVIEW OF THE GARRETT-MUNK (GM) FORMALISM

The GM (1972, 1975)† formalism for the spectral model of internal oceanic gravity waves begins with the spectrum for scalar energy per unit of mass which they designate by $E(k_z, \omega)$ where k_z is the vertical wave number and ω is the radial frequency (k_x, the horizontal wave number is also used). They represent this spectrum as being factorable:

$$E\left(k_z, \omega\right) \equiv E' B(\omega) \frac{A(\lambda)}{k_{z_*}} \tag{2.1}$$

where k_{z_*} represents the vertical wave number for the scale where the energy begins to decrease rapidly with k_z. It also is the "dominant" or "characteristic" wave number in the sense that, as Smith et al (1987) have shown, "most of the energy lies within a factor of 3 of the mode at $(k_{z_*}/2^{1/3})$." The λ is a non-dimensional wave number:

$$\lambda \equiv \left(k_x/k_{x_*}\right) \text{ or } \left(k_z/k_{z_*}\right). \tag{2.2}$$

Also $A(\lambda) \equiv (t-1)\,(1+\lambda)^{-t}$.

In GM (1975) the value $t = 2.5$ was chosen. In their formalism non-dimensional quantities were used by making use of normalization with representative scales of time and distance. For example, all frequencies were normalized by $\hat{N}$ where $\hat{N}$ is the "extrapolated Väisälä frequency" in their depth dependent buoyancy frequency profile given by $\hat{n} = \hat{N}e^{-\hat{y}/\hat{b}}$ where $\hat{n}$ is the dimensional local frequency, $\hat{y}$ the depth and $\hat{b}$ the characteristic length for this depth dependence.

† More recent versions appeared in GM (1979) and Munk (1981).

The "hats" over the letters were used by GM to designate dimensionality.[†] Wave numbers were normalized by $\widehat{M} \equiv \hat{b}^{-1} 2\pi$. In the following I will try to avoid use of their non-dimensional approach because (a) in the case of the atmosphere it is known that this would give an incorrect scaling with respect to "$\widehat{N}$", and (b) the dimensional approach is more convenient for the experimental comparisons that will be made.

Both $B(\omega)$ and $A(\lambda)$ are defined so they integrate to unity. This will be equally valid even when dimensional terms are used because the integrals themselves are dimensionless. Thus we have

$$\int_{\omega_I}^{N} B(\omega)\, d\omega = 1; \text{ and} \int_{0}^{\lambda_{max}} A(\lambda)\, d\lambda = 1 \tag{2.3}$$

where ω_I is the inertial frequency. In order to obtain the spectra ("power spectral densities" to be more exact, henceforth denoted by "PSD's") of velocity components, GM use what they call "wave functions" in a manner to be shown below. In addition, they integrate over λ or ω in order to obtain the one dimensional PSD's. For example, in order to obtain the PSD with respect to k_z of the horizontal velocity fluctuations, u, they use

$$\Psi_u(k_z) = \int_{\omega_I}^{N} \frac{E'\, B(\omega)\, A(k_z/k_{z_*})}{k_{z_*}} \tilde{U}^2(\omega)\, d\omega \tag{2.4}$$

where

$$\tilde{U}^2(\omega) \equiv \frac{n\left(\omega^2 + \omega_I^2\right)}{\omega^2} \tag{2.5}$$

and where $\tilde{U}^2(\omega)$ is the wave function for horizontal velocity, again, ω_I is the inertial frequency, E′ is a dimensionless energy which has been empirically determined, and n is a dimensionless parameter relating to buoyancy frequency which I shall set = 1 for the atmospheric model. The correct dependence on N is, from Dewan and Good (1984, 1986),

† In this paper I shall use this notation exclusively to refer to the GM model. The new model being presented assumes that all quantities have dimensions.

$\alpha\, N^2 k_z^{-3}$ for the universal part (i.e. high wave number) of the spectrum. Smith et al (1987) brought this into the context of the GM formalism by writing it as

$$\Psi_u(k_z) = \frac{\alpha N^2}{k_{z_*}^3}\left(1 + \frac{k_z}{k_{z_*}}\right)^{-3}. \tag{2.6}$$ †

Notice that it is here that the dependence on N differs from the GM model, but is supported by atmospheric observations (Dewan and Good, 1986). Smith et al (1987) found a value for α, as has been mentioned, and it is $\alpha = 1/6$ (compare Hines, 1990).

The vertical velocity, v, wavefunction is given by GM (1972, 1975) as (n = 1)

$$\tilde{V}^2(\omega) = \frac{\omega^2 - \omega_i^2}{N^2}. \tag{2.7}$$

Thus

$$\Psi_v(k_z) = \int_{\omega_i}^{N} \frac{E'\,B(\omega)\,A\left(k_z/k_{z_*}\right)}{k_{z_*}} \tilde{V}^2(\omega)\, d\omega. \tag{2.8}$$

Similar relations hold for the k_x type PSD's where in that case one uses $\lambda \equiv k_x/k_{x_*}$.

The ω dependence, for example, is given by

$$\Psi_u(\omega) = \tilde{U}^2(\omega)\, E'\, B(\omega) \int_0^{\lambda_{max}} A(\lambda)\, d\lambda \tag{2.9}$$

$$= \tilde{U}^2(\omega)\, E'\, B(\omega)$$

from Eq. (2.3).

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† In actual fact, Smith et al (1987) used a slight modification of this for convenience.

In order to obtain PSD's for displacements one employs the relevant velocity component wave function and divides it by ω^2. For example, the PSD for vertical displacement, ζ, as a function of ω, is given by

$$\Psi_\zeta(\omega) = \frac{\tilde{V}^2(\omega)}{\omega^2} E' B(\omega) \tag{2.10}$$

and so on.

It should be mentioned that Garrett and Munk (GM) (1972, 1975) obtained these wave functions in a complicated way; but, they can be obtained directly from the polarization relations when suitable approximations are made.

The values of $\hat{E}$ and $\hat{B}(\omega)$ in the GM models are

$$\hat{B}(\omega) = \frac{2}{\pi} \frac{\omega_i \hat{N}}{\omega^2} \frac{1}{\left(1 - \frac{\omega_i^2}{\omega^2}\right)^{1/2}} \tag{2.11}^\dagger$$

and

$$\hat{E} = 2\pi \times 10^{-5} \hat{N}^2 / \hat{M}^2 . \tag{2.12}$$

(i.e. $E' = 2\pi \times 10^{-5}$).

In the next section I will deduce $\hat{E}\,\hat{B}(\omega)$ from a similitude argument. The above model is strictly an "empirical fit". In both cases, it should be pointed out, a "non-equatorial" model is assumed. This is because something unphysical happens when one sets $\omega_i = 0$.

$\dagger$ Here ω_i and ω are dimensional.

3. A SIMILITUDE ARGUMENT FOR E $B(\omega)$††

As was mentioned previously, a rationale has been given in Dewan (1979) for an atmospheric gravity wave cascade and that this leads to the observed $k_x^{-5/3}$ dependence. In the present section the same strong mode-interaction cascade picture will be used to derive E $B(\omega)$. The cascade assumption made here will also entail an additional premise, that is, the wave cascade is in steady state. In other words, I shall assume, as in Dewan (1979) and as in the case of inertial range turbulence, that the input and output of the cascade are both equal to the dissipation rate, ε, and remain approximately constant over time.

These assumptions make it easy to obtain E $B(\omega)$, for, the latter energy spectrum can only depend upon two parameters, namely ω and ε. These have the dimensions of T^{-1} and $L^2\,T^{-3}$ respectively. The dimensions of E $B(\omega)$ are of velocity variance per frequency bandwidth or L^2T^{-1}. Thus

$$\frac{L^2}{T} = \alpha' \, \varepsilon^a \, \omega^b = \alpha' \left(\frac{L^2}{T^3}\right)^a \left(\frac{1}{T}\right)^b. \tag{3.1}$$

For equality, a = 1, b = −2 thus

$$E\,B(\omega) = \alpha' \frac{\varepsilon}{\omega^2} \qquad \text{QED.} \tag{3.2}$$

where α' is a universal constant of order unity. [Here $B(\omega) \equiv (\omega_i / \omega^2)$ and $E \equiv (\alpha'\varepsilon / \omega_i)$.]

It is not surprising that this is identical to the result found in Tennekes and Lumley (1972) for the inertial range frequency spectrum. More interesting is the fact that, in the turbulence case, Eq. (3.2) does not agree with experiment. In view of this, Tennekes (1975) subsequently did a more careful analysis of isotropic inertial range turbulence in which he took into account the advective doppler effects of large eddies upon the measured frequencies of the small eddies when the latter are measured in the Eulerian frame. He obtained

$$\phi(\omega) = \beta\, \varepsilon^{2/3}\, q^{2/3}\, \omega^{-5/3} \tag{3.3}$$

†† The hat notation is dropped.

(where $1/2\ q^2$ is the mean kinetic energy per unit mass) which does agree with experiment. As will be discussed below, in the case of oceanic gravity waves the ω^{-2} dependence seems to be correct, whereas, from VanZandt (1982), in the case of the atmosphere there is evidence for $\omega^{-5/3}$. The Tennekes argument offers a way to explain the difference, and it would entail verifiable predictions regarding q. For the details see the Appendix.

In Dewan and Good (1986) physical interpretations of the similitude results were given. As was mentioned, these interpretations have been questioned by Hines (1990). Similar physical pictures for the cascade are available. For example, one can imagine that each band-width around a given frequency, ω, has an associated velocity variance which I will call $v^2(\omega)$. The latter is related to ε and $\tau(\omega)$, the "mode interaction time" of the ω mode. In other words, $\tau(\omega)$ is the time needed for the "ω-band" in question to give up a significant fraction of its "kinetic energy per unit mass", $v^2(\omega)$. We are here assuming strong interactions. In this case τ begins to become somewhat "comparable" to the period of the mode in question (compare Dewan, 1985). I therefore let $\tau \sim \omega^{-1}$. Thus $v^2(\omega)$ would scale as ε/ω. Next I make an assumption similar to one found in Dewan and Good (1986) and let the band-width, $\Delta\omega$, scale as ω. Then, I find from this that

$$E\,B(\omega) \sim \frac{v^2(\omega)}{\Delta\omega} = (\text{const.})\,\frac{\varepsilon}{\omega^2} \tag{3.4}$$

as in Eq. (3.2). But this argument of $\Delta\omega \sim \omega$ is exactly the sort of thing Hines (1990) finds inadequate because, left unanswered are some questions: (a) why is $\Delta\omega \sim \omega$? If the answer is that you have wave-packets of frequency, say, ω, and that they all contain a comparable number of oscillations, then the next questions are (b) why should that be true, and (c) why should all the wave-packet scales get "equal representation" in time? To avoid such questions we must, for now at least, let the similitude argument stand on its own feet. This is regrettable because it will be shown below that in this model, $E = \dfrac{\alpha'\varepsilon}{\omega_1}$ (which makes a lot of physical sense in my cascade picture since most of the energy is at frequency ω_1 and hence $v^2(\omega_1) \sim \varepsilon/\omega_1$) represents most of the "E" to be found. Is it really impossible to answer the embarrassing questions raised by Hines?

A way out of this dilemma was suggested, in private communication, by D. Fritts. His suggestion is that a "dynamical similarity" exists at all relevant scales "like fractals". Fritts' suggestion, it should be pointed out, is totally consistent with the cascade picture espoused in the present paper. In fact, such fractal scaling should be directly caused by the cascade process. Put another way, the relationship between self-similar cascades in turbulence and self-similar fractals is well known. See for example Mandelbrot (1975 and 1983). The point here is that it is postulated that waves too can have self-similar cascades (Dewan, 1979 and the discussion above) and this gives rise to the self-similarity conditions $\Delta\omega \sim \omega$ and $\Delta k \sim k$, and the "equal representation" assumptions which were previously left unexplained in Dewan and

Good (1986) and Smith et al (1987). The cascade hypothesis, therefore, answers at least the main questions and objections raised by Hines (1990).

Experiments suggested below involve simultaneous measurements of PSD's and ε as well as k_{z_*}, etc. If the predictions of the cascade model hold up, then the present "theory" will have been "verified". It would be desirable, then, to go beyond similitude arguments; however, as in the case of inertial range turbulence it is not entirely clear how this could be done rigorously. Dimensional analysis, or similitude, can sometimes be surprisingly powerful and, at the same time, very simple. It would constitute a grave error to mistake this simplicity for lack of rigor or substance in my view. The inertial range turbulence model of Kolmogorov, which is based entirely on similitude, is the most rigorous and experimentally valid model for this type of turbulence yet published. It has stood the test of time for one-half century.

4. THE STRONG-MODE-INTERACTION MODEL FOR THE "UNIVERSAL SPECTRUM"

A comprehensive PSD model can be obtained from the results of Section 2 (the GM formalism) and Section 3 (the similarity argument for E B(ω)). From these I will obtain $\Psi_v(k_z)$, $\Psi_u(\omega)$, $\Psi_v(\omega)$, $\Psi_u(k_x)$ and $\Psi_v(k_x)$. In addition it will be possible to obtain the constants of order unity such as α and α' from certain consistency requirements and, in addition, the quantities k_{z_*} and k_{x_*} will be obtained from dimensional and physical considerations.

4.1 The Vertical Wavenumber PSD's of Horizontal Velocity $\Psi_u(k_z)$ and Vertical Velocity $\Psi_v(k_z)$

The equation for $\Psi_u(k_z)$ was given as Eq. (2.6). Following Smith et al (1987) the value of α in this equation can be obtained from the relation

$$N^2 = \int_0^{k_{max}} k_z^2\, \Psi(k_z)\, dk_z = \int_0^{k_{max}} \frac{\alpha N^2}{k_{z_*}^3}\left(1 + \frac{k_z}{k_{z_*}}\right)^{-3} k_z^2\, dk_z. \tag{4.1}$$

This represents (Dewan and Good, 1986 and Smith et al, 1987) an overall saturation condition (cf. alternate interpretation by Hines, 1990).

When the operations are carried out is is possible to obtain (in a manner similar to Smith et al, 1987)

$$\alpha = \left[\ln\left(k_{z_{max}}/k_{z_*}\right) - \frac{3}{2}\right]^{-1}. \tag{4.2}$$

In Dewan and Good (1986) and Smith et al (1987) the theoretical value for $k_{z_{max}}$ was not discussed. An indication is however to be found in Dewan (1985) where it is argued that the largest possible wave-number for a "wave" is given by $k_{max} = 2\pi\left(N^3/\varepsilon\right)^{1/2}$. It was pointed out that this wave number would correspond to the case where the shortest period wave would lose a substantial amount of energy through mode interaction in one period, thus rendering it close to "turbulence". "k_{max}" can also be recognized as the well known "buoyancy wave number". I have chosen the 2π following Woods (1974). This "buoyancy wavenumber" choice for k_{max} has already been mentioned in passing by Munk (1981); but, he did not state his rationale. In contrast, Hines (1990) suggested that Eq. (4.1) be solved in order to determine k_{max}.

The value of α' in Eq. (3.2) for E B(ω) can be obtained with the help of the GM formalism in the following way. From Sections 2 and 3 we have

$$\Psi_u\left(k_z\right) = \int_{\omega_1}^{N} E\,B(\omega)\,\frac{A\left(\lambda_z\right)}{k_{z_*}}\,\tilde{U}^2(\omega)\,d\omega \tag{4.3}$$

$$\lambda_z \equiv k_z/k_{z_*} \tag{4.4}$$

$$\tilde{U}^2(\omega) \equiv \left(\omega^2 + \omega_1^2\right)/\omega^2 \tag{4.5}$$

$$A\left(\lambda_z\right) \equiv 2\left(1 + \lambda_z\right)^{-3} \tag{4.6}$$

and

$$E\,B(\omega) = \frac{\alpha'\,\varepsilon}{\omega^2}. \tag{4.7}$$

If Eqs. (4.4) to (4.7) are inserted into Eq. (4.3) and the integral performed and if the result is equated to the expression for $\Psi_u (k_z)$ in Eq. (2.6), then one can obtain a relation between α and α':

$$\frac{\alpha N^2}{k_{z_*}^2} = \left(\frac{8}{3}\right) \frac{\alpha' \varepsilon}{\omega_i} . \tag{4.8}$$

Solving for α'

$$\alpha' = \frac{3}{8} \left[\frac{N^2 \omega_i}{k_{z_*}^2 \varepsilon} \right] \alpha . \tag{4.9}$$

Notice the dimensionless combination in the square brackets. Let us set it equal to a dimensionless constant and solve for k_{z_*} to obtain

$$k_{z_*} = C_z \sqrt{\frac{N^2 \omega_i}{\varepsilon}} \tag{4.10}$$

where C_z is a dimensionless universal constant to be obtained from experiment. The physical meaning behind Eq. (4.10) is important. Both Hines (1990) and Munk (1981) have pointed out that the large scale waves can doppler shift the small scale waves and that for a certain critical value of the vertical wave number, m_c (to use their notation), the waves can be forced to approach their critical layer. Hines gives $m_c = N/(2\ v_{rms})$ where v_{rms} represents the root mean square velocity due to the large waves. Munk refers to his m_c as the boundary for "compliant waves".

In Eq. (4.10), k_{z_*} can be seen to represent the quantity m_c above provided that the "cascade" picture be kept in mind. The reason is as follows. It has already been mentioned in Section 3 that $v^2 (\omega_i) \sim \varepsilon/\omega_i$ and since E $B(\omega) \sim \omega^{-2}$ it follows that the main contribution to v^2 is $v^2 (\omega_i)$ and hence it is approximately true that $v_{rms}^2 \sim \varepsilon/\omega_i$. Hence, $k_{z_*} \sim N/v_{rms}$ in Eq. (4.10) which is to say that k_{z_*} indeed separates free from compliant waves as suggested by Hines (I am ignoring constants of order unity in this statement).

Next, consider $\Psi_v (k_z)$, the PSD for vertic l velocity. The GM formalism gives

$$\Psi_v(k_z) = \int_{\omega_i}^{N} E\,B(\omega)\,\widetilde{V}^2(\omega)\,\frac{A(\lambda_z)}{k_{z_*}}\,d\omega \tag{4.11}$$

$$\text{where } \widetilde{V}^2(\omega) = \frac{\left(\omega^2 - \omega_i^2\right)}{N^2}. \tag{4.12}$$

Using Eq. (4.7) for E B(ω), Eq. (4.6) for $A(\lambda_z)$ and Eq. (4.9) for α' and the fact that $N >> \omega_i$,

$$\Psi_v(k_z) = \frac{3}{4}\,\frac{\omega_i N\alpha}{k_{z_*}^3}\left(1 + \lambda_z\right)^{-3}. \tag{4.13}$$

Comparing Eq. (4.13) to $\Psi_u(k_z) = \frac{\alpha N^2}{k_{z_*}^3}\left(1 + \lambda_z\right)^{-3}$

$$\frac{\Psi_v(k_z)}{\Psi_u(k_z)} = \frac{3}{4}\,\frac{\omega_i}{N}. \tag{4.14}$$

This ratio, it is easy to show, also holds for $\Psi_u(k_x)$ and $\Psi_v(k_x)$, and since a typical value of (ω_i/N) is 10^{-2} it is clear that the vertical wave motions are usually about two orders of magnitude smaller than the horizontal motions. The value of α in Eq. (4.13) is given by Eq. (4.2) and hence depends weakly upon $(k_{z_{max}}/k_{z_*}) \sim (N/\omega_i)^{1/2}$ $\left(\text{since } k_{max} \cong 2\pi\left(N^3/\varepsilon\right)^{1/2}\right.$ and $\left.k_{z_*} \cong C_z\left(N^2\,\omega_i/\varepsilon\right)^{1/2}\right)$.

Notice that, when one sets $\omega_i = 0$ in some of the above equations, some results are obtained which are not experimentally correct. For example $\Psi_v(k_z)$ is presumably not zero at the equator. Thus this model is not applicable to regions close to the equator.

4.2 The Temporal Frequency, ω, PSD's: $\Psi_u(\omega)$, $\Psi_v(\omega)$

Using the GM formalism we can obtain the horizontal velocity PSD from

$$\Psi_u(\omega) = \int_0^{k_{z_{max}}/k_{z_*}} E\,B(\omega)\,A(\lambda_z)\,\widetilde{U}^2(\omega)\,d\lambda_z. \tag{4.15}$$

Next we use Eq. (2.3), Eq. (4.6) (for $A(\lambda_z)$), Eq. (4.7) (for $E\,B(\omega)$), Eq. (4.5) (for $\widetilde{U}^2(\omega)$), Eq. (4.2) (for α) and Eqs. (4.9) and (4.10) (for α'), to obtain

$$\Psi_u(\omega) = \left(\frac{3}{8}\right)\left(\frac{\alpha\,\varepsilon}{\omega^2\,C_z^2}\right)\left(\frac{\omega^2 + \omega_i^2}{\omega^2}\right). \tag{4.16}$$

The quantities in Eq. (4.16) are given in units of radians (eg radians/sec) and Ψ_u is in (m^2/s^2) /(radians/sec). In comparing Eq. (4.16) with certain data these units must be changed to (m^2/s^2) /(cycles/sec). To make the conversion one multiplies the right side by 2π.

It is important to note that, in Eq. (4.16), when $\omega >> \omega_i$, the frequency dependence goes as ω^{-2}. This is similar to the frequency dependence in the GM model. The frequency PSD for vertical velocity fluctuations is given by

$$\Psi_v(\omega) = \int_0^{k_{z_{max}}/k_{z_*}} A(\lambda_z)\,E\,B(\omega)\,\widetilde{V}^2(\omega)\,d\lambda_z \tag{4.17}$$

where $\widetilde{V}^2(\omega)$ is given by Eq. (4.14). In a manner exactly similar to Eq. (4.16) the result is

$$\Psi_v(\omega) = \left(\frac{3}{8}\right)\left(\frac{\alpha\,\varepsilon}{C_z^2\,\omega^2}\right)\left[\frac{\omega^2 - \omega_i^2}{N^2}\right]^{\dagger} \tag{4.18}$$

again with the previous 2π option. Also the dependence when $\omega >> \omega_i$ is similar to the GM model in that the dependence goes as ω^0; that is to say, it is constant.

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† Note that this implies that $\Psi_v = \Psi_u \dfrac{\omega^2}{N^2}$ when $\omega >> \omega_i$.

4.3 The Horizontal Wavenumber, k_x, PSD's: $\Psi_u(k_x)$, $\Psi_v(k_x)$

As has been mentioned the form $\Psi(k_z) = (\mathrm{CONST.})\,\varepsilon^{2/3} k_x^{-5/3}$ was derived in Dewan (1979) for both velocity components (u and v). In that paper, however, certain things were left out of account. For example the following items were omitted: (a) the wave function effects on velocity component PSD's, (b) the roll off effect seen in the GM formalism that takes place at the dominant wave number k_{x_*}, and (c) the numerical value of the constant of order unity. The purpose of the present section is to finish this task.

From the above mentioned form for $\Psi(k_x)$ one is led to the assumption, in the context of the GM formalism, that

$$\Psi_u(k_x) = \frac{\alpha'' \varepsilon^{2/3}}{k_{x_*}^{5/3}}\left(1+\lambda_x\right)^{-5/3} \tag{4.19}$$

where $\lambda_x \equiv k_x/k_{x_*}$. I have chosen to commence with the horizontal velocity component because the latter is involved in the Hines-Munk critical layer mechanism. From the GM formalism

$$\Psi_u(k_x) = \int_{\omega_1}^{N} E\,B(\omega)\frac{A(\lambda_x)}{k_{x_*}}\tilde{U}^2(\omega)\,d\omega. \tag{4.20}$$

Again, $A(\lambda) = (s-1)(1+\lambda)^{-s}$ and setting $s = 5/3$, $A(\lambda_x) = \left(\frac{2}{3}\right)\left(1+\lambda_x\right)^{-5/3}$. Again using $E\,B(\omega) = \alpha'\,\varepsilon/\omega^2$, and $\tilde{U}^2(\omega) = \left(\omega^2+\omega_1^2\right)/\omega^2$ and performing the integral,

$$\Psi_u(k_x) = \frac{\alpha'\,\varepsilon(8/9)}{k_{x_*}\,\omega_1}\left(1+\lambda_x\right)^{-5/3}. \tag{4.21}$$

The value of α'' is obtained by equating Eq. (4.21) to Eq. (4.19)

$$\alpha'' = \left(\frac{\varepsilon^{1/3} k_{x_*}^{2/3}}{\omega_1}\right)\left(\frac{8}{9}\right)\alpha' \tag{4.22}$$

$$\alpha'' = \left(\frac{\alpha}{3} \frac{C_x^{2/3}}{C_z^2} \right)$$

from Eqs. (4.9) and (4.10) and Eq. (4.23) below.†

Again we note the dimensionless combination in Eq. (4.22) and in a manner similar to Eq. (4.10) we obtain

$$k_{x_*} = C_x \sqrt{\frac{\omega_l^3}{\varepsilon}} \tag{4.23}$$

which has an important physical interpretation in the context of the Hines-Munk critical layer picture. The horizontal phase trace velocity is given by ω/k_x. For a critical layer to occur, $(\omega/\bar{k}_x) = u_{rms}$, where $\bar{k}_x$ is the critical wave number at that frequency, ω. Unlike the case for k_{z_*} where only one k_z is critical, one has $\bar{k}_x(\omega)$ which is to say it is frequency dependent. From the cascade assumption $(\varepsilon/\omega_l) \sim u_{rms}^2$ hence

$$\bar{k}_x^2 \sim \frac{\omega^2 \omega_l}{\varepsilon} \tag{4.24}$$

Thus the physical meaning of k_{x_*} is that it is the value of $\bar{k}_x$ for the lowest wave frequency,i.e. $\omega = \omega_l$.

It should be noted that since $\bar{k}_x(\omega)$ depends on ω, this may provide a clue to the interpretation of the ω dependence found by Pinkel (1981, 1984). That is to say, when a $\Psi_u(\omega, k_x)$ spectrum is obtained, our approach would indicate that at frequency ω, $k_{x_*} = C_x \sqrt{\frac{\omega^2 \omega_l}{\varepsilon}}$ and that the dominant wave number is frequency dependent. That is to say the roll off point in k_x dependence would change with frequency.

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† As has been mentioned, when Eq. (3.2) or (4.7) is integrated between ω_l and N, and the normalization of B (ω), see (2.3), is used, one obtains $E = \alpha' \varepsilon \omega_l^{-1}$. But $E = q^2$, the velocity variance; thus, to within a constant of order unity, α', we have $q^2 = \varepsilon/\omega_l$.

In order to obtain $\Psi_v(k_x)$ one makes use of Eq. (4.14) which also holds for $\Psi_{u,v}$† (k_x). That is we use $\Psi_v(k_x) = \frac{3}{4}\frac{\omega_i}{N}\Psi_u(k_x)$ obtaining

$$\Psi_v(k_x) = \left[\frac{\alpha}{4}\frac{\omega_i}{N}\frac{C_x^{2/3}}{C_z^2}\right]\frac{\varepsilon^{2/3}}{k_{x_*}^{5/3}}\left(1+\lambda_x\right)^{-5/3}. \tag{4.25}$$

A final caution is in order here. In the case mentioned in Dewan (1979) the fluctuations were exceptionally severe and were, in fact, previously interpreted as ordinary turbulence. In addition, the velocity components were comparable in size. The present model, however, is not intended for unusually high energy situations. It is intended to represent the usual "background" "universal" case. In this sense, the experimental data discussed in Dewan (1979) do not represent "confirmation" of the present model. The latter awaits future experiments.

A similar case of exceptionally high energy gravity waves is discussed in VanZandt et al (1990) where $\Psi_v(\omega)$ falls, not as ω^0, as in the less excited case, but as $\omega^{-5/3}$ [as would be expected for $\Psi_u(\omega)$]. In both of these situations, [that is, Dewan (1979) and VanZandt (1990)] the observed departure from the predicted PSD may be due to the cause suggested by Van Zandt (1990). This is, namely, that the vertical velocity component measurements are contaminated by the effects of the very large horizontal components of the velocities.

5. COLLECTION OF PREDICTIONS

5.1 Model Equations

$$\left.\begin{array}{l}
\Psi_u(k_z) = \dfrac{\alpha N^2}{k_{z_*}^3}\left(1+\lambda_z\right)^{-3}\ (\mathrm{m^2/s^2})/(\mathrm{rad/m}) \\[2ex]
\lambda_z \equiv \left(k_z/k_{z_*}\right), \qquad k_{z_*} \equiv C_z\sqrt{\left(N^2\,\omega_i/\varepsilon\right)} \\[2ex]
\alpha \doteq \left[\ln\left(k_{z_{max}}/k_{z_*}\right) - \left(\frac{3}{2}\right)\right]^{-1}
\end{array}\right\} \tag{5.1}$$

---

† This means Ψ_u <u>and</u> Ψ_v.

$$\Psi_v(k_z) = \left(\frac{3}{4}\right) \frac{\omega_i N \alpha}{k_{z_*}^3} (1 + \lambda_z)^{-3} \; (m^2/s^2)/(rad/m) \tag{5.2}$$

$$\left.\begin{aligned} \Psi_u(k_x) &= \frac{\alpha'' \varepsilon^{2/3}}{k_{x_*}^{5/3}} (1 + \lambda_x)^{-5/3} \; (m^2/s^2)/(rad/m) \\ \alpha'' &= \frac{\alpha C_x^{2/3}}{3 C_z^2}, \qquad \lambda_x \equiv k_x / k_{x_*} \\ k_{x_*} &= C_x \sqrt{(\omega_i^3/\varepsilon)} \end{aligned}\right\} \tag{5.3}$$

$$\Psi_v(k_x) = \frac{3}{4}\left(\frac{\omega_i}{N}\right) \Psi_u(k_x) \tag{5.4}$$

$$\left.\begin{aligned} \Psi_u(\omega) &= \frac{\alpha' \varepsilon}{\omega^2} \left[\frac{\omega^2 + \omega_i^2}{\omega^2}\right] (2\pi)(m^2/s^2)/\,(cycles/sec) \\ \alpha' &= \left(\frac{3}{8}\right) \frac{\alpha}{C_z^2} \end{aligned}\right\} \tag{5.5}$$

$$\Psi_v(\omega) = \frac{\alpha' \varepsilon}{\omega^2} \left[\frac{\omega^2 - \omega_i^2}{N^2}\right] (2\pi) \, (m^2/s^2)/\,(cycles/sec) \tag{5.6}$$

5.2 Suggestions for Future Tests of This Model

Clearly, future tests would involve measurements of ε simultaneously with power spectral measurements of velocity fluctuations. In situ and remote measurements of ε have been performed by Barat (1975), Cadet (1977), and Hocking (1988a, 1988b, 1985) for example.

The values of k_{z_*} and k_{x_*} could be obtained directly from the relevant PSD's (from the "roll off" characteristics) and, by using known values of N, ε, and ω_i, the values of C_x and C_z can be obtained from the equations for k_{x_*} and k_{z_*} above. The dependence of these and the PSD's on ω_i could be checked by conducting experiments at various latitudes. Such dependences are all, of course, entirely new predictions.

It seems that the most convenient tests of the predicted ε dependence of these gravity waves would involve the frequency PSD's (Eqs. (5.5) and (5.6)). If these obtain verification it would then be appropriate to test the more difficult to measure $\Psi(k_x)$ PSD's for ε dependence.

Oceanic measurements of ε together with appropriate PSD's would also play an important role in the verification of this model in spite of the difference of N dependence upon depth. Sonar would then replace radar as the instrument of choice.

6. ORDER OF MAGNITUDE COMPARISON WITH EXPERIMENT

If simultaneous measurements of PSD's and ε in the atmosphere or ocean were presently available it would be possible to put the present model to a test immediately. Unfortunately, no such data seem to exist yet. It is nevertheless possible to find out if the theory predicts quantities "within the bounds" of observation. In carrying out this exercise I learned that the model is rather elaborately "interconnected" and hence "vulnerable" to quantitative observation. For example, k_{z_*} and C_z enter into the PSD coefficient in both a direct and (through α) indirect way.

In the following, "reasonable" values for the unknowns were used and the relevant papers cited.

6.1 Atmospheric Comparison

As mentioned in Section 4.1, the $\Psi_u\,(k_z)$ PSD for the atmosphere has been found to be valid. We therefore turn to a frequency PSD namely $\Psi_u\,(\omega)$ and compare it with experiment.

For convenience I list, on the basis of Section (5.1), the following useful equations:

$$k_{z_*} = C_z\sqrt{\frac{N^2\,\omega_1}{\varepsilon}}, \qquad \lambda_{z_*} = 2\pi/k_{z_*}, \qquad k_{z_{max}} = 2\pi\sqrt{\frac{N^3}{\varepsilon}}, \qquad \lambda_{z_{max}} = 2\pi/k_{z_{max}} \tag{6.1}$$

$$\Psi_u(\omega_1) = \left(\frac{3}{8}\right) C_z^{-2}\,\varepsilon\left(2/\omega_1^2\right) 2\pi\alpha \tag{6.2}$$

$$\Psi_u(N) = \left(\frac{3}{8}\right) C_z^{-2}\,\varepsilon\left(1/N^2\right) 2\pi\alpha \tag{6.3}$$

$$\alpha = \left[\ln\left(k_{z_{max}}/k_{z_*}\right) - \left(\tfrac{3}{2}\right)\right]^{-1}. \tag{6.4}$$

The "typical" values for the parameters will be chosen to be

ε	=	$5 \times 10^{-4}\ m^2/s^3$	Cadet (1977)
N	=	10^{-2} rad/s	VanZandt (1982), his Fig. 1
ω_i	=	10^{-4} rad/s (~40°N)	VanZandt (1982), his Fig. 1
C_z	=	0.2	(chosen because of experimental agreement)
α	=	0.235	[from (6.4)]

From these the following results can be obtained: $k_{z_*} = 8.9 \times 10^{-4}$ rad/m, $\lambda_{z_*} = 7$ km, $k_{z_{max}} = 2.8 \times 10^{-1}$ rad/m, $\lambda_{z_{min}} = 22$ m. A comparison between these numbers and those found in Smith et al (1987) (λ_{z_*} in stratosphere ≈ 5 km), and Dewan and Good (1986) ($\lambda_{z_{min}}$ ~ 40 m) shows that these length scales are acceptable. The PSD's $\Psi_u(\omega_i)$ and Ψ (N) are

$$\Psi_u(\omega_i) = 1.4 \times 10^6\ (m^2/s^2)/(\text{cycles/sec});\ \left[\text{cf. } 10^5 - 10^{6.3}\right]$$

$$\Psi_u(N) = 69\ (m^2/s^2)/(\text{cycles/sec});\ \left[\text{cf. } 10^{1.8} \text{ or } 63 - 10^{3.5}\right].$$

The values indicated by "cf." are the experimental ranges taken from VanZandt (1982), Fig. 1. Thus, I conclude that if I choose $C_z = 0.2$ I can get "good" agreement with experiment. Is the value for C_z, "a universal constant of order unity", a reasonable value? The answer is yes as can be shown by considering Bond (1929).

It has been mentioned that VanZandt (1982) observed that $\Psi_u(\omega)$ had a $\omega^{-5/3}$ rather than a ω^{-2} dependence. In this case one would be forced to alter the present model along the lines of Tennekes (1975) as was mentioned earlier in Section 3. See Eq. (3.3). This modification is very simple. E B(ω) would then become proportional† to $\left(\frac{\varepsilon}{(\omega_i)^{1/3}}\right)\omega^{-5/3}$. I feel that such a modification would be indicated once the present model has been at least partially validated by direct measure-ments of ε when $\Psi_u(\omega)$ or $\Psi_v(\omega)$ were simultaneously measured, and if the $\omega^{-5/3}$ continued to be observed.

† A detailed argument for why $\omega^{-5/3}$ should be correct for the atmosphere while ω^{-2} applies to the ocean will be found in the appendix.

It is possible to put the present model to one further test at this time. According to Eq. (5.5) the $\Psi_u(\omega)$ PSD is linearly dependent on ε. Measurements of ε in the mesosphere have been made by Hocking (1988) and found to be of order $0.1 m^2 s^{-3}$ at 86-88 km altitude. Comparing this to $\varepsilon = 5 \times 10^{-4} m^2 s^{-3}$ found in the stratosphere by Cadet (1977) (11 km) it is found that the present theory predicts that $\Psi_u(\omega)$ at the higher altitude can be 200 times as large as it is at the lower altitude. Has this been observed? According to Balsley and Carter (1982) "In the $\omega^{-5/3}$ portions of the spectra, the spectral energy density/unit mass is about 250 times smaller in the troposphere (8 km) than in the mesosphere (86 km)". This is encouraging. Of course a valid test would require simultaneous measurements of ε and $\Psi_u(\omega)$ at such altitudes. For example, Balsley and Garello (1985) found a ratio of about 50 in $\Psi_u(\omega)$ between 13 km and 87 km.

Between 50 and 250 we have a factor of 5; but the GM model claim is that these PSD's are generally within a factor of 2. This might imply that, in this sense, the atmosphere is more variable. An interesting discussion about ε and universality will be found in Munk (1981), sections 9.9.6 and 9.9.7.

It can be seen from the above that in the case of atmospheric spectra such as $\Psi_{u,v}(\omega)$ and $\Psi_{u,v}(k_x)$, the present model's dependence on ε makes it desirable to abandon the use of the word "universal" for such PSD's. Since ε varies over a large range in the ocean it is curious that these PSD's have not been reported as being very variable there.

6.2 Oceanic Comparison

The appropriate values for "typical" oceanic values are as follows:

ε	=	10^{-8} (m^2/sec^3)	Woods (1974) and GM (1972), p. 251 also Monin and Ozmidov (1985)
$\hat{N}$	=	3.5×10^{-3} (rad/sec)	GM (1972), 2 cycles/hv
$\hat{\omega}_i$	=	7×10^{-5} (rad/sec)	GM (1972), 0.04 cycles/hv, p. 261
$\hat{M}$	=	7.7×10^{-4} (rad/m)	GM (1972), 0.122 cycles/km
E'	=	$2\pi \times 10^{-5}$	GM (1972), p. 252
C_z	=	(0.2)	(from previous section)

Next to each quantity at least one reference is given for support. Perhaps the most variable parameter is ε which Monin and Ozmidov (1985) indicate can vary over a very wide range depending upon depth, location, and weather conditions. Values listed by them (pp. 124-129) go from 10^{-10} to 10^{-5} m^2/sec^3. The "hat" notation is, as was mentioned above, from GM who use it to mean "dimensional" to distinguish the quantities in their model which are primarily non-dimensional. The quantity C_z was set equal to 0.2 as it was in the previous atmospheric comparison since C_z is presumed to be a universal dimensionless constant.

As has already been mentioned the GM model gives a scaling with respect to N which differs from the present one. This matter cannot be further investigated here. In what follows one can only determine if the present model is, in some sense, reasonable. The following are presented for comparison:

$$\text{GM model: } \Psi_u(\omega) = \left\{ \frac{2E'}{\pi} \left(\frac{\hat{N}}{\hat{M}}\right)^2 \hat{\omega}_i \right\} \left[\frac{\hat{\omega}^2 + \hat{\omega}_i^2}{\hat{\omega}^3 \sqrt{\hat{\omega}^2 - \hat{\omega}_i^2}} \right] \tag{6.5}$$

$$\text{Present model: } \Psi_u(\omega) = \left\{ (2\pi) \left(\frac{3}{8}\right) \frac{E\alpha}{C_z^2} \right\} \left[\frac{\omega^2 + \omega_i^2}{\omega^4} \right] \tag{6.6}$$

$$\alpha = \left[\ln\left(k_{z_{max}}/k_{z_*}\right) - 1.5\right]^{-1}$$

and

$$\text{GM model: } \Psi_u(k_z) = \left\{ \frac{\hat{N}^2}{\hat{M}^2} \frac{E'}{\hat{k}_{z_*}} (1.5)(3/2) \right\} \left[1 + \lambda_z\right]^{-2.5} \tag{6.7}$$

$$\text{Present model: } \Psi_u(k_z) = \left\{ \frac{\alpha N^2}{k_{z_*}^3} \right\} \left[1 + \lambda_z\right]^{-3} \tag{6.8}$$

The values k_{z_*}, λ_{z_*}, k_{max} and $\lambda_{z_{max}}$ are obtained from Eq. (6.1). Table 6.1 shows the comparisons between the models.

Table 6.1 Comparison of Present Model with "Observations".

Present Model	"Observation"
$\lambda_{z_{min}} = 0.48$ m $\lambda_{z_*} = 107$ m $\alpha = 0.256$ $k_{z_*} = 5.9 \times 10^{-2}$ rad/m^{-1} $k_{z_{max}} = 13$ m^{-1}	$\lambda_{z_{min}} =\sim 1$ meter Woods (1974) $\lambda_{z_*} = 170$ m Holloway (1981), p. 58
$\left[(2\pi)\left(\frac{3}{8}\right)\frac{\alpha\varepsilon^{\dagger}}{C_z^2}\right] = 1.51 \times 10^{-7}\,(m^2/s^2)\left(\frac{rad}{s}\right)$ = part of $\Psi_u(\omega)$ [Eq. (6.6)]	$\left[\frac{2E'}{\pi}\left(\frac{\hat{N}}{\hat{M}}\right)^2 \hat{\omega}_i\right] = 5.8 \times 10^{-8}\,(m^2/s^2)\left(\frac{rad}{s}\right)$ = part of $\Psi_u(\omega)_{GM}$ [Eq. (6.5)]
$\frac{\alpha N^2}{k_{z_*}^3} = 1.6 \times 10^{-2}\,\frac{m^2/s^2}{(rad/m)}$ = part of $\Psi_u(k_z)$ [Eq. (6.8)]	$\frac{\hat{N}^2}{\hat{M}^2}\frac{E'}{k_{z_*}} = 5 \times 10^{-2}\,\frac{m^2/s^2}{(rad/m)}$ = part of $\Psi_u(k_z)_{GM}$ [Eq. (6.7)]
† In order to avoid problems with dimensional inconsistency, it would appear necessary to ascribe the units $\left(\frac{m^2}{s^2}\frac{rad}{s}\right)$ to ε in the present context (pointed out to me by T. Van Zandt in a private communication).	
$\Psi(\omega_i) = 62\,(m^2/s^2)/(\text{cycles/sec})$ [Eq. (6.6)]	$56\,(m^2/s^2)/(\text{cycles/sec})$
$\Psi(N) = 1.2 \times 10^{-2}\,(m^2/s^2)/(\text{cycles/sec})$ [Eq. (6.6)]	$1.7 \times 10^{-2}\,(m^2/s^2)/(\text{cycles/sec})$ Both from GM (1972), Fig. 4. See text.

Note that the $\Psi_u(\omega)$ comparisons were performed in three ways. First the values of the "curly brackets" in Eqs. (6.5) and (6.6) were compared, then $\Psi_u(\omega_i)$ from Eq. (6.6) was compared with the value derived from Fig. 4 of GM (1972). The same was done for $\Psi_u(N)$ from Eq. (6.6).

To make the comparisons, the graphical values had to be multiplied by $\frac{3.5 \times 10^{-3}\ \text{rad/s}}{2\pi\ \text{rad/cy}} = \hat{N}$ in cycles/sec.

Regarding $\Psi_u\ (k_z)$ there was a difference in slope. The present model gives $(1 + \lambda_z)^{-3}$ in comparison to $(1 + \lambda_z)^{-2.5}$ inGM (1975). We do not know whether or not the difference is real or due to a difference in data processing. For example, omission of "pre-whitening" techniques could explain the differences. As can be seen, the values of the curly brackets are comparable. In general, the numerical values in Table 6.1 agree to within a factor of about 3 or better.

What has been proven in Sections (6.1) and (6.2)? All that was accomplished was that the model appears to give the correct order of magnitude. In some circles this is called a "sanity test". The model is not"obviously wrong", in other words.

7. CONCLUSIONS

A similitude model, based mainly on a wave interaction cascade, has been given to "explain" the GM model from physical considerations. By necessity, the model has experimental consequences that remain to be tested. These, at present, are its main virtue. If future experiments validate the model then it will have demonstrated predictive value. The main experiments suggested include simultaneous measurements of ε and certain PSD's as given in Section 6.

The exact roles of "saturation" due to "instability" and "cascade" due to "nonlinear mode-interaction" are not entirely clear at present; but it appears as if both are taking place. Holloway (1980) has pointed out that nonlinear mode-interaction can, in and of itself, represent "saturation" of a type with critical Richardson number ≈ 4. Both types of "saturation" might take place and be related to each other and bring about the observed spectra. Obviously the situation is not completely understood yet however.

As has been mentioned previously there has been some question of whether or not the waves discussed in this paper should be called turbulence of some kind (e.g. "buoyancy turbulence", or "2-dimensional turbulence") - see Gage (1979), Trefethen and Panton (1989), Sidi et al (1988), Weinstock (1985), Holloway (1983) and Gargett et al (1981). The above discussion shows that wave properties exist (like propogation, dispersion relations, wave functions, and polarization relations) and yet, at the same time there are some of the properties of turbulence such as self-similar cascades in these velocity fluctuations. We seem to be dealing with a hybrid dynamics for which no name yet exists. Does this mean that we need a new word like "waveulence" or "turbundulence", or will "buoyancy turbulence" be altered in meaning to stand for the above?

Regarding the term "universal spectrum", the predicted dependence of $\Psi_{u,v}\ (\omega)$ and $\Psi_{u,v}\ (k_x)$ upon ε indicates that, if the model proposed here is valid, these spectra would be much "less" universal than the $\Psi_{u,v}\ (k_z)$ spectra. It was also mentioned that the slope of the $\Psi_u\ (\omega)$ spectrum may be –2 or –5/3 depending upon the conditions (see the Appendix).

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Appendix

Convective Effects upon the Eulerian Spectrum for Energy as a Function of ω: EB(ω)

In the text it was mentioned that the frequency spectrum of horizontal velocity fluctuations had an ω^{-2} dependence for internal gravity waves in the ocean whereas in the atmosphere van Zandt showed evidence for an $\omega^{-5/3}$ dependence. It was also mentioned in the text that Tennekes (1975) gave an argument in the context of turbulence that might possibly be adapted to the case of waves and might therefore be useful in explaining the difference in ω dependence in oceanic and atmospheric observations. The purpose of this appendix is to carry out this suggestion.

Tennekes (1975) pointed out that the Kolmogorov dimensional arguments for inertial range, or cascade, turbulence lead to

$$\phi_L(\omega) = \frac{\beta_L \varepsilon}{\omega^2} \qquad \text{(A-1)}$$

where $\phi_L(\omega)$ is the velocity fluctuation PSD, the subscript L designates "Lagrangian description", and β_L is a constant of order unity. He then pointed out that experimental evidence contradicts (A-1). He next showed that for sufficiently large Reynolds number based on "outer length", the inertial range would turn into an "inertial advective" range which includes the advective effects of large eddies upon the small eddies. This changes the Eulerian spectrum which now would no longer resemble the Lagrangian spectrum due to the doppler effects. More specifically he showed that advective doppler effects cause the Eulerian frequency PSD for velocity fluctuations to have the following form:

$$\phi_E(\omega) = \beta_E \frac{\left(q^{2/3}\,\varepsilon^{2/3}\right)}{\omega^{5/3}} \qquad \text{(A-2)}$$

where β_E is a constant of order unity, the subscript E designates Eulerian frame, and q is the root mean square velocity fluction due to the turbulent field. This relation will be derived below.

In order to ascertain the conditions necessary for advective effects to become important, Tennekes made use of the fact that such effects are most prominent at the highest frequencies. In inertial range turbulence the highest frequency is known to be

$$\omega_{c.o.} = \sqrt{\frac{\varepsilon}{\nu}} \qquad \text{(A-3)}$$

where the subscript c.o. refers to "cut off". The kinetic energy (per unit mass) in the Lagrangian and Eulerian frames at $\omega_{c.o.}$ are designated $\frac{1}{2}u_L^2(\omega_{c.o.})$ and $\frac{1}{2}u_E^2(\omega_{c.o.})$ respectively. Note that $\omega_{c.o.}$ is the highest frequency for the L frame but not for the E frame since doppler effects can make frequencies shift upwards. Tennekes compared these energies and found

$$\frac{\frac{1}{2}u_E^2(\omega_{c.o.})}{\frac{1}{2}u_L^2(\omega_{c.o.})} \doteq Re_l^{1/6} = \left(\frac{q\,l}{\nu}\right)^{1/6} \qquad \text{(A-4)}$$

where l is the outer scale of the cascade and Re is the Reynolds number. Thus for large enough Re, the Eulerian energy at the Lagrangian frequency cut off can be significantly larger than the corresponding energy in the Lagrangian frame due to advection.

In order to deduce the PSD in the Eulerian (E) frame, that is (A-2), Tennekes used

$$\omega \sim \frac{q}{r} \tag{A-5}$$

which is the frequency in E due to q advective doppler effects on eddies of scale r caused by the largest scale eddies. From Kolmogorov's theory he used

$$\frac{1}{2} u_L^2(\omega) \sim \varepsilon^{2/3} r^{2/3}. \tag{A-6}$$

Inserting (A-5) into (A-6), by elimination of r,

$$\frac{1}{2} u_E^2(\omega) \sim \varepsilon^{2/3} \left(\frac{q}{\omega}\right)^{2/3}. \tag{A-7}$$

Using the fact that the PSD is the "kinetic energy", i.e. $\frac{1}{2} u^2$, per unit frequency he obtained

$$\phi_E(\omega) = \frac{\beta_E (\varepsilon q)^{2/3}}{\omega^{5/3}}, \qquad \text{QED.} \tag{A-8}$$

This differs from (A-1) in the same way that atmospheric waves seem to differ from oceanic waves. Thus in the case of cascade turbulence (A-2) is valid when Re in (A-4) is sufficiently large.

Now we turn to the question of what happens to the frequency spectra of waves. As will be shown, a "frozen waves" hypothesis which is analogous to the "frozen turbulence" implied by (A-5) appears to be justified. I shall start by assuming it is valid and then subsequently present evidence that it is indeed justified.

We start by deriving a counterpart to (A-4) for waves with the idea that the result will possibly indicate a difference between ocean and atmosphere. The highest or "cut off" frequency for internal waves is the buoyancy frequency N. This replaces the $\omega_{c.o.}$ in the Tennekes argument. The first question to answer, then, is what is the value of $\frac{1}{2} u_L^2(N)$? From the main text we have

$$\Phi_L(\omega) \equiv EB(\omega) = \frac{\alpha' \varepsilon}{\omega^2} \tag{A-9}$$

Setting $\omega = N$ in (A-8) and $\Delta\omega = \omega$ as discussed in the text in the context of fractal self-similarity, we obtain

$$\frac{1}{2} u_L^2 (N) \sim \Phi_L (N) \bullet N = \frac{\alpha' \varepsilon}{N} . \tag{A-10}$$

In order to obtain $\frac{1}{2} u_E^2 (N)$ we follow Tennekes by assuming the frozen hypothesis and commence with Lagrangian spatial dependence. From the main text we have

$$\Psi_u (k_z) = \alpha'' \varepsilon^{2/3} k_x^{-5/3} . \tag{A-11}$$

Notice here that I have chosen the horizontal velocity components only. This is because the vertical components at any wave number are smaller by a factor of $\left(\frac{\omega_1}{N}\right)$ as shown in Equ.'s (4.14) and (5.4). Next, the frozen assumption gives

$$\omega_E = q\, k_x \tag{A-12}$$

which is our counterpart to Tenneke's (A-5) above. This assumes that for the high Eulerian frequencies, ω_E, one senses only the convective effects due to spatial scales which are caused by the wave field as a whole. The latter effect is of course due to q, the r.m.s. velocity (compare Hines, 1991). Solving (A-12) for k_x and substituting the result into (A-11) we obtain (under our assumption) an estimate for $\Phi_E (\omega_E)$, the Eulerian PSD fcr velocity fluctuations:

$$\Psi_u (\omega_E) \sim \alpha'' \varepsilon^{2/3} \left(\frac{\omega_E}{q}\right)^{-5/3} . \tag{A-13}$$

In order to obtain $\frac{1}{2} u_E^2 (N)$ we again multiply by the appropriate band width. But since

(Eq. A-13) was obtained from (A-11) which is a density, not in ω but in $\underline{k}$, we must multiply ψ_u by Δk. Also, $\omega_E = N$ is the relevant frequencry. What then does one use for Δk? The answer is $\Delta k = \frac{N}{q}$ (using A-12 with $\omega_E = N$ and using $\Delta k = k$ as before). Thus we have

$$\frac{1}{2} u_E^2(N) = \psi_u(N) \bullet \Delta k = \alpha'' \varepsilon^{2/3} \left(\frac{N}{q}\right)^{-5/3} \frac{N}{q} \tag{A-14}$$

$$= \alpha'' \varepsilon^{2/3} \left(\frac{N}{q}\right)^{-2/3}.$$

We finally arrive at the ratio [in analogy with (A-4)]

$$\frac{\frac{1}{2} u_E^2(N)}{\frac{1}{2} u_L^2(N)} = \frac{\alpha'' \varepsilon^{2/3} (N/q)^{-2/3}}{\alpha' \varepsilon N^{-1}} = \left(\frac{\alpha''}{\alpha'}\right) \frac{N^{1/3} q^{2/3}}{\varepsilon^{1/3}} \tag{A-15}$$

where (A-10) was used. But, as shown in the main text,

$$q^2 = \int_{\omega_1}^{N} EB(\omega)\, d(\omega) = \frac{\alpha' \varepsilon}{\omega_1} \tag{A-16}$$

Solving (A-16) for ε and inserting into (A-15),

$$\frac{\frac{1}{2} u_E^2(N)}{\frac{1}{2} u_L^2(N)} \sim \left(\frac{N}{\omega_1}\right)^{1/3} \tag{A-17}$$

to within constants of order unity.

This equation states that as the ratio N/ω_1 increases, the convective effects upon the higher frequencies increases and, as a consequence of the earlier arguments of Tennekes that were described above and which continue to hold under the present assumptions, the PSD for energy changes from

$$EB(\omega_L) \sim \frac{\varepsilon}{\omega^2} \qquad \text{(A-18)}$$

to

$$EB(\omega_E) \sim \varepsilon^{2/3}\, q^{2/3}\, \omega^{-5/3} = \frac{\varepsilon}{\omega_i^{1/3}}\, \omega^{-5/3} \qquad \text{(A-19)}$$

where I have omitted constants of order unity. In Eq. (A-19) I used (A-14) with ω_E replacing N. The Eulerian PSD was then obtained by then dividing $\frac{1}{2} u_E^2 (\omega_E)$ by the bandwidth ω_E in the usual way, and thus I obtained the middle term of (A-19).

The next question is whether or not (A-17) can explain the difference between the slopes observed between oceanic and atmospheric temporal frequency spectra. In the main text, when comparisons were made between observations and theory, approximately the following values were employed.

Atmospheric: $N = 2.2 \times 10^{-2}$ rad/sec*, $\omega_i = 10^{-4}$ rad/sec† (50°N)

Oceanic: $N = 3.5 \times 10^{-3}$ rad/sec, $\omega_i = 7 \times 10^{-5}$ rad/s (29°N)
[9×10^{-5} rad/s (39°N)]

A closer look at Table 1 of GM (1972) p. 237 shows that his Fig. 1 is really based on data such that 39°N is a more appropriate latitude [this is based on a weighted average of ω_i] thus we will replace ω_i above, for the ocean, by 9×10^{-5} rad/s (in brackets). This would not change any conclusions in the main text. Thus, from (A-16) we have

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† These values are not exactly the same as those in Section 6.1 of the main text. There I used $N = 10^{-2}$ rad/s and $\omega_i = 10^{-4}$ rad/s (40°N). The present values here I believe to be more accurate. They change no conclusions of the main text however. The new value of $\lambda_{z_*} = 3.5$ km instead of 7 km, $\lambda_{min} = 7.9$ m instead of 22 m and $\psi_u(N) = 14\ (m^2/s^2)/(c/s)$ instead of $69 (m^2/s^2)/(c/s)$. This last value takes it lower than the experimental minimum of 63; but, remember that the point of the present appendix is to show that the -5/3 slope is valid and this would change $\psi_u(N)$ to a higher value. For these reasons the conclusions of the main text were left unchanged.

Atmospheric: $\left(\frac{N}{\omega_i}\right)^{1/3} = \frac{u_E^2}{u_L^2} = 6.0$

Oceanic: $\left(\frac{N}{\omega_i}\right)^{1/3} = \frac{u_E^2}{u_L^2} = 3.4$

These are in the ratio of (6/3.4) = 1.8 which, as can easily be shown, is just sufficient to convert the atmospheric PSD slope from $\omega^{-5/3}$ to ω^{-2}. In other words, reducing the PSD at $\omega = N$ in van Zandt's Figure 1 by 80% steepens his slope from –5/3 to –2 on the log-log graph. In conclusion it appears that we have a quantitative explanation of the difference between atmospheric and oceanic measurements of $\psi_u(\omega)$. One interesting consequence of this is the following prediction. When $(N/\omega_i)^{1/3} = 6.0$ in the ocean, then one should also see an $\omega^{-5/3}$ dependence! What latitude gives the appropriate ω_i? Using $N = 3.5 \times 10^{-3}$ rad/s, we are led to

$$\omega_i = \frac{N}{6^3} = 1.6 \times 10^{-5} \text{ rad/s} \tag{A-20}$$

But since

$$\left.\begin{aligned} \omega_i &= 2\Omega \sin\phi = (1.46 \times 10^{-4}) \sin\phi \\ \phi &= \text{arc sin} \frac{(1.6 \times 10^{-5})}{(1.46 \times 10^{-4})} = 6.3^\circ\text{N} \end{aligned}\right\} \tag{A-21}$$

This latitude, unfortunately is very close to the equator. As was mentioned in the text, the present model is not applicable at $\phi = 0^\circ$. This therefore raises the question of whether or not it is valid to predict the above result from the model.

An alternative test would be to see if there were an extreme northern latitude where the atmospheric PSD would change to an ω^{-2} dependence. That would require $\left(\frac{N}{\omega_i}\right)^{1/3} = 3.4$ and for $N = 2.2 \times 10^{-2}$, ω_i would have to be 5.6×10^{-4} rad/s which is not possible. Perhaps with sufficiently accurate PSD's the implied changes in PSD slope with latitude could be detected but such accuracy may be too expensive to be immediately practical. The other tests in the text are much more practical to perform. If these tests turn out to be positive, then possibly, a latitude test may be tempting to perform.

We now turn to the question of the validity of the "frozen wave assumption" made above. The relation between the Eulerian and Lagrangian frequencies where the Lagrangian frame has a velocity q (presumed horizontal for previously given reasons) is

$$\omega_E = \omega_L \pm q k_x \tag{A-22}$$

In order to ascertain the value of $u_E^2(N)$ we may use

$$u_E^2(N) = \int_{\omega_i}^{N} \psi_u(k_x, \omega_L)\, d\omega_L \bullet \Delta k_x \tag{A-23}$$

where

$$\psi_u(k_x, \omega_L) \equiv \frac{\alpha'\varepsilon}{\omega_L^2}\left[\frac{\omega_L^2 + \omega_i^2}{\omega_L^2}\right]\frac{(2/3)}{k_{x*}}\left(1 + \frac{k_x}{k_{x*}}\right)^{-5/3} \tag{A-24}$$

which represents $EB(\omega)\,\tilde{U}(\omega)\,\frac{A(\lambda_x)}{k_{x*}}$ from the main text. When (A-22) is used to eliminate k_x from (A-24) one can obtain, for the case of $\omega_E = N$ and hence $k_x = \left(\frac{N - \omega_L}{8}\right)$ from (A-22),

$$u_E^2(N) = \int_{\omega_i}^{N} q^3\,\frac{\left(\omega_L^2 + \omega_i^2\right)}{\omega_L^2}\,(2/3)\left(\frac{\omega_i}{\omega_i + N - \omega_L}\right)^{5/3} d\omega_L \bullet \Delta k_x \tag{A-25}$$

where the approximation

$$k_{x_*} = \sqrt{\frac{\omega_i^3}{\varepsilon}} \sim \frac{\omega_i}{q} \tag{A-26}$$

was used. Notice that the relative sign between k_x and q was chosen as positive, and therefore, a positive change of ω_L in the integration of (A-24) will result in larger rather than smaller contributions from what was the k_x term in (A-23). We regard the contributions of the opposite case to be comparatively negligible. Now Eq. (A-24) can be written

$$u_E^2(N) = \Delta k_x\, q^3\, \omega_i^{5/3} \bullet (\text{INT}) \tag{A-27}$$

where

$$(\text{INT}) \equiv \int_{\omega_i}^{N} \frac{\left(\omega_L^2 + \omega_i^2\right)}{\omega_L^4} \frac{1}{\left(\omega_i + N - \omega_L\right)^{5/3}} \bullet d\omega_L \tag{A-28}$$

where the (2/3) factor is now dropped since C_z, and α' have already been set equal to unity. Using the usual order of magnitude values, $\omega_i = 10^{-4}$ and $N = 10^{-2}$ (rad/sec) a numerical integration of INT is found to give $(3.7 \pm .4) \times 10^7$.

But we seek a convenient estimate of (A-27) which gives the dependence on ω_i and N explicitly. It is clear, by inspection, that the greatest contribution of the integrand would be for values of ω_L near ω_i. For example, if the integrand of (A-27) is called f (ω_L) [letting ω_i and N assume the above values] and plotting $\omega_L = n\, \omega_i$, i.e. as multiples of ω_i, we would then have Table A-1.

TABLE A1	
n	$f(n\ \omega_l)$
1	4.3×10^{11}
2	6.8×10^{10}
3	2.7×10^{10}
10	2.6×10^{9}
20	7.7×10^{8}
30	4.2×10^{8}
40	3.1×10^{8}
50	2.7×10^{8}
60	2.7×10^{8}
70	3.1×10^{8}
80	4.5×10^{8}
90	1.1×10^{9}
100(= N)	4.6×10^{10}

From this table we see (a) there is an order of magnitude drop from $n = 1$ to $n = 2$ in $f(n\ \omega_l)$. (b) the integrand bounces back but is always an order of magnitude (or two or three) below the initial value. We are therefore led to compare, numerically, the value of INT, given above, namely 3.7×10^7, to the approximation

$$\left.\begin{aligned} \psi(\omega_l)\,\omega_l &= \frac{2\,\omega_l^2}{\omega_l^4}\,\frac{1}{(\omega_l + N - \omega_l)^{5/3}}\,\omega_l \\ &= \left[\frac{2}{\omega_l\, N^{5/3}}\right] = 4.3 \times 10^7. \end{aligned}\right\} \qquad \text{(A-29)}$$

This is acceptable agreement and thus we consider it appropriate to accept the approximation. When $\omega_L = \omega_l$, $k_x = N/q$ from

$$N - \omega_l = qk_x \approx N \qquad \text{(A-30)}$$

calling $\Delta k_x = k_x = N/q$ in (A-27) and using INT $= \psi(\omega_l)\,\omega_l$ above, we obtain

$$\frac{u_E^2(N)}{q^2} = \frac{N}{q} \bullet \frac{q^3}{q^2} \bullet \omega_i^{5/3} \bullet \left[\frac{2}{\omega_i N^{5/3}}\right] = 2\left(\frac{\omega_i}{N}\right)^{2/3} \tag{A-31}$$

where the (A-29) bracket term was used. Thus

$$\frac{1}{2}\, u_E^2(N) = 2q^2\left(\frac{\omega_i}{N}\right)^{2/3} \tag{A-32}$$

and, from (A-15) $\frac{1}{2} u_L^2(N) = \alpha' \varepsilon N^{-1} = \left(\omega_i q^2\right)/N$ (from $\varepsilon = \omega_i q^2$ and $\alpha' \sim 1$), the ratio

$$\frac{\frac{1}{2} u_E^2(N)}{\frac{1}{2} u_L^2(N)} = 2\left(\frac{N}{\omega_i}\right)^{1/3} \tag{A-33}$$

which is the same as the frozen case (A-17) within constants of order unity.

This proves that, as far as our purposes are concerned, the "frozen waves" hypothesis is valid. Finally it will be noted that our conclusions were never based on the absolute value of (A-17) but rather the ratio of this quantity taken from the atmosphere and ocean. For this reason, plus the fact that the constant of order unity involved would be universal, the conclusion is independent of the actual value of that constant.

Notes Added in Proof

1 - The unusual history behind the theory of Kolmogorov and Obukhov regarding inertial range turbulence is given in "Statistical Fluid Mechanics" by A.S. Monin and A.M. Yaglom, MIT Press, 1965, vol. 1, pp. 12-17. In addition to the rhyme of L.F. Richardson mentioned in the text Kolmogorov was also aware of the "4/3" length scaling law of the diffusion by turbulence of clouds which, as was later realized, follows dimensionally from the cascade hypothesis.

3 - The present report does not discuss the problems arising from variable N and variable shears with altitude, their filtering effects, etc. In regard to saturation see Van Zandt, T., Fritts, D., *Pageoph.*, **130**, p. 399 (1989) who examine some of such effects. The growth of wave amplitude with altitude is a crucial factor.

4 - The agreement between the model and the ε measurements of Hocking and $\psi_u(\omega)$ measurements of Balsley and Carter that is mentioned on p. 21 was not used during my construction of the present model. It was learnt by the author only at the last stage of writing.

In other words I claim that it was really a "prediction" that was consistent with an unexpected observation. This is the reason that I think that simultaneous measurements of $\psi_u(\omega)$ and ε are both justified and worth the effort.

5 - Regarding the simultaneous presence of cascade and saturation it should be clearly stated that this theory assumes (a) cascade processes control ω and k_x dependent PSD's, and (b) saturation controls the k_z PSD of horizontal velocity fluctuations. A planned future report will describe the simultaneous presence of these two processes in detail and will explain why (b) does not significantly interfere with (a).