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Efficient Approaches For Report/Cluster Correlation in Multitarget Tracking Systems

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An efficient approach for correlating sensor reports and target clusters is described as a special case of a more general algorithm developed by the authors. A variety of practical considerations are discussed in the context of the TRC tracking and correlation system developed at the Naval Research Laboratory for use in the Strategic Defense Initiative (SDI) National Testbed.

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EFFICIENT APPROACHES FOR REPORT/CLUSTER CORRELATION IN MULTITARGET TRACKING SYSTEMS

INTRODUCTION

Gating is an important component of most multi-object tracking systems. Its function is to identify sensor reports, e.g., radar or infrared (IR) returns from missiles, planes, etc., that correlate highly with current state estimates (i.e., tracks). For small numbers of objects, it is feasible to calculate a probability of correlation for every track/report pair and reject those whose probabilities fall below some threshold. For large numbers of objects, however, the quadratic growth in the number of pairs for which correlation probabilities are computed by this "brute force" approach represents an enormous bottleneck. This combinatorial problem is of particular concern in Strategic Defense Initiative (SDI) tracking and correlation for which numbers of objects on the order of 100,000 must be processed in real time. This report discusses an approach that significantly reduces the computational complexity of the correlation process in the TRC tracking and correlation system developed at the Naval Research Laboratory.

TRC is a multihypothesis tracker/correlator that was developed to conduct experiments in multiple-target tracking. Unfortunately, early tests of the TRC revealed that combinatorial problems severely limited the size of the scenarios that could be examined. Subsequent analysis demonstrated that these limitations were the result of a correlation (gating) algorithm that scaled in time quadratically in the size of the scenario. Research into approaches for reducing this computational complexity identified two primary difficulties:

1. The correlation threshold, or gating criterion, depends on error covariances that are generally unique to each track and report.

2. The measurement times of the reports are generally distributed over some non-zero time interval, yet the correlation measurement function is defined only for track/report pairs that are valid at the same time.

These two factors appear to demand the comparison of every track to every report. However, in the case of report/cluster gating in which clusters are defined by a spatial separation threshold, the correlation process can be performed with a computational complexity that is significantly better than quadratic. The approach described is a special case of a more general gating algorithm developed by the authors [1].

The gating problem can be stated technically as follows: given a motion model and a set of tracks consisting of current state estimates with associated error covariances and a set of sensor

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reports consisting of measurement timestamps and position measurements with associated error covariances, determine in real time which pairs (i,j) satisfy the gating criterion:

$$S_{ij}(dX_{ij},\Gamma_j) = \frac{\exp(-dX_{ij}^t\Gamma_j^{-1}dX_{ij}/2)}{(2\pi)^{d/2}\sqrt{|\Gamma_j|}}, \tag{1}$$

and

$$S_{ij} \leq S_j, \tag{2}$$

where d is the measurement dimension, dX_{ij} is the residual vector difference of report i and track j (projected to the time of the report), Γ_j is the residual covariance of the track, and S_j is the gating threshold selected for the track.

Pairs that satisfy the gate in Eq. (2) can be efficiently identified by deriving from the gating threshold a search volume for each report, thus limiting the number of correlation candidates to be examined. This transforms the problem from probability space into Euclidean space where efficient computational geometric methods can be applied. The calculation of the volume depends on the report and track covariances, the gating threshold, and the maximal time differentials between the current states of the tracks and the observation times of the reports. Methods for calculating this search volume are developed in Refs. 1 and 2. In the TRC system, however, this step is partially obviated by the necessary maintenance of assumed causally independent (in terms of the above correlation measure) clusters of tracks. The approach described in this report exploits this fact and thus is less general than the approaches described in Refs. 1 and 2. However, because cluster information is often maintained by systems used for tracking multiple-warhead missiles, squadrons of aircraft, and other targets capable of dispersive maneuvers, the results in this paper are widely applicable.

TRC CLUSTERING

Spatial clusters are maintained by TRC to reduce the multiplicity of hypotheses (tentative track/report pairings) generated by its multihypothesis tracking (MHT) algorithm. The clusters are defined by a minimum separation criterion that requires an object to be a member of a given cluster if and only if it is within the minimum separation distance (MSD) of another member of the cluster. This minimum separation distance is determined by the correlation measure, the motion characteristics of the targets, and the resolution of the sensor(s) and is intended to impose a causal partition of the object set. Since the MHT algorithm requires a correlation measure to be computed for every track/report pair associated with a particular cluster, the role of the gating algorithm is to assign incoming reports to their appropriate track clusters. Given N_R reports and N_C clusters, a brute force approach would scale as $N_R N_C$ and thus would be appropriate only for small N_C. However, because N_C is purely data dependent and in general approaches N_R as the tracking process converges, a more sophisticated approach is necessary to reduce the upper bound computational complexity.

In the TRC model, track clusters are represented by pseudotracks. A pseudotrack is a track structure constructed by averaging over tracks within a cluster. Specifically, the pseudotrack position is the mean of the cluster, and its covariance is computed to approximate the covariance distribution of the tracks within the cluster. The Gaussian density with mean μ and error covariance

Σ for a cluster **C** is defined [3] as:

$$\mu = \sum_{j=1}^{N} w_j \mu_j \tag{3}$$

$$\Sigma = \sum_{j=1}^{N} w_j \Sigma_j + (\mu_j - \mu)^T (\mu_j - \mu), \tag{4}$$

where each w_j represents a weighting factor reflecting the likelihood of association of the jth track/report pair based on the feasible track/report matchings in which it appears. Since the dynamics of objects within the same cluster are assumed strongly correlated, pseudotracks permit the treatment of clusters as if they were single-track objects. Use of these pseudotracks can achieve the brute force $O(N_R N_C)$ scaling already mentioned. To improve this scaling, a method is required that avoids the projection of every pseudotrack to the time of every report. This can be accomplished by using the already assumed MSD threshold. For example, a search radius can be computed for each cluster by projecting the tracks through the scan period and determining the maximum distance any track reaches from the centroid of the cluster* and adding the MSD. This defines a search volume for each cluster within which every correlated target should be found. Since the search volumes are not in general disjoint, a secondary test must be applied to resolve ambiguous cases. In the TRC, this is done by computing a correlation score for each such report with the pseudotracks of the clusters with which it gates.

To treat correlation as a point enclosure problem requires that either the tracks or the reports be point objects. However, both sets consist of volumetric objects since each report is generally associated with a thresholded covariance volume. A simplistic solution to this problem is to *deflate* one of the sets by adding the maximum radius of its elements to the radius of each element of the other set. In cases where the distribution in radii of the elements of the two sets is broad, this approach may introduce a large degree of inefficiency and the more sophisticated strategy described in Ref. 2 may be required.

EFFICIENT SEARCHING

To efficiently identify the tracks within the gating radius of each report, the tracks must be placed into a search structure from which the desired set can be retrieved without having to examine every track. (Actually, if the number of reports greatly exceeds the number of tracks, it may be more efficient to construct the search structure from the set of reports. This consideration is discussed more fully in the next section.) Data structures with storage requirements proportional to the number of tracks, N_T, are known which provide this capability [4]. They require only $O(N_T \log N_T)$ setup time and between $O(d \log N_T + k)$ and $O(N_T^{1-1/d} + k)$ average retrieval time, where k is the average number of tracks per report and d is the number of dimensions of the search space. However, investigations by the authors have revealed that a variation of one of these data structures [5] provides a small linear improvement in the average retrieval time when the computed radius is small relative to the average interobject separation.

The degree to which efficient search algorithms can be applied depends heavily on the difference in dimensionality between the state estimates, or tracks, and the sensor measurements, or

*More precisely, the maximum distance any point within any member track's thresholded covariance volume must be determined.

reports. Often tracks will maintain estimates of position and one or more of its derivatives (and possibly some number of target attributes such as temperature or size). If one or more of these parameters must be derived from multiple reports, the reports are said to be subdimensional. For example, bearing-only measurements from IR sensors are subdimensional with respect to position. When measurements are of full positional dimensionality, correlation of tracks and reports requires satisfaction of *orthogonal range queries*. Satisfaction of a range query determines the elements of a point set that fall within an isothetic (i.e., coordinate-aligned) hyper-rectangle defined by ranges in each of the measurement dimensions. If the measurement dimensions are not orthogonal, an appropriate coordinate transformation or projection is required. If the measurements are subdimensional, however, no such transformation may exist, and the use of efficient search algorithms may be precluded.

Because the TRC system is designed to process both IR and radar reports, the issue of subdimensional correlation becomes important. Specifically, the correlation of tracks and bearing-only reports defines a class of query volumes. In principle, these volumes extend infinitely along the sensor line of sight. A simple limitation on the maximum distance any target can be assumed from the sensor results in finite search volumes. Unfortunately, the approximation of such regions by isothetic volumes may be inadequate for efficient search. Another option is to transform the tracks to the spherical coordinate system of the sensor and define two-dimensional range queries in the measurement angles. This approach is very efficient in the case of one sensor. In the case of multiple sensors, however, the transformation steps scale as the product of the number of tracks and the number of sensors. If the ratio of sensors to tracks is small, this may not be an important consideration. If the ratio is not small, however, a different approach is required. In the case of satellite-based IR sensors, knowledge that the line-of-sight vectors tend to be tangent to the Earth can be exploited to good advantage. Specifically, the transformation of all tracks to an Earth-centered spherical coordinate system allows the sensor measurement regions to be relatively well approximated by ranges in the two angular coordinates. The scaling of this approach is then relatively independent of the number of sensors.

THE MULTIPLE TIMESTAMP PROBLEM

The fact that the computation of the gating radius must consider the maximal time differential between the report measurements and the latest track updates leads to a source of possible inefficiency. In particular, the search volumes scale approximately as the cube (in three dimensions) of the time differential. Thus, if the report measurements are made over a very long scan period, the gating radius may become impractically large. To alleviate this problem, a technique has been devised that projects copies the complete set of tracks to time intervals throughout the scan period. This ensures that the maximum time differential used for any report is no more than the length of the scan period divided by twice the number of projections. Furthermore, a function has been derived that computes the number of projections required to optimize the tradeoff between the computational cost incurred by a large time differential and the cost of making multiple track projections. A thorough complexity analysis (see the Appendix) reveals that this geometric dilution strategy permits better than quadratic scaling even for scenarios involving very long scan periods.

An important consideration when minimizing the average track/report time differential by subdividing the scan period is whether the searching operations required for correlation should be performed on the track or report datasets. Searching on the track dataset results in the following scaling for the overall gating process for reasonable* target density and scan length:

$$\begin{aligned} \text{Setup time} &= O(mN_T \log N_T) \\ \text{Search time} &= O(N_R(\log N_T + k)), \end{aligned} \tag{5}$$

where m is the number of subdivisions of the scan period and k is the average number of objects found per report. This scaling includes cost of constructing m search structures and the performing of N_R searches on those structures. The resultant scaling for searches on the report dataset for similar scenarios is approximately:

$$\begin{aligned} \text{Setup time} &= O(N_R \log(N_R/m)) \\ \text{Search time} &= O(mN_T(\log(N_R/m) + k)), \end{aligned} \tag{6}$$

where the values m and k are not in general the same as in Eq. (5). This case involves binning the reports into m bins of N_R/m reports according to their timestamps and constructing a search structure for each bin. (Note that this binning process for unordered reports can be performed in worst-case linear time by using a simple variation of standard median-finding algorithms [6].) Scaling of the search process consists of the cost of projecting each track to the middle of the time interval associated with each of the m bins and performing a search.

A cursory examination of the relative scaling behavior of the two approaches reveals that the former should provide better performance when N_T greatly exceeds N_R, while the latter may be preferred when N_R greatly exceeds N_T. Thus, the former approach might be preferred in an MHT system (that does not use clustering) in which the number of hypotheses is many times the number of reports, and the latter approach would be preferred when tracking is performed in heavily cluttered environments where the number of tracked objects is much less than the number of reports. In cases where N_T and N_R are expected to be roughly comparable, the former approach may be preferred because it avoids the m factor in the computationally more expensive search step.

In MHT systems such as the TRC, where the assignment algorithm requires a correlation measurement for every track/report pair associated with the same cluster, only the pseudotracks must be considered by the gating algorithm. As far as the gating algorithm is concerned, $N_T = N_C$. When N_C is much smaller than N_R, the construction of search structures from the report datasets is probably more appropriate for this type of report/cluster correlation. In the case of the TRC, however, the choice was made to construct search structures from the track datasets for two reasons:

1. N_C should approach N_R as the tracking process converges, and
2. communications constraints in some proposed SDI battle management environments effectively require that reports be processed as they are received, thus precluding the batch processing required to construct search structures from reports.

DISCUSSION

A module developed from this study of the gating problem has been incorporated into a version of the TRC tracking and correlation system used in the SDI National Testbed. Results of tests on SDI-type scenarios reveal that the new gating approach scales approximately as $N_R \log N_C$ and

*The term *reasonable* in this context can be rigorously defined by using results presented in the Appendix.

provides significant performance improvements over brute force even for small numbers (40 to 50) of clusters in the small scan length case. Additional tests of the standalone module demonstrate that the multiple-projection strategy can maintain this scaling for scan lengths of at least 10 seconds. Even for scan lengths an order of magnitude larger, however, subquadratic scaling may be possible. (Actual computation times that indicate the magnitude of the scaling coefficient may be found in Refs. 7 and 8.)

To summarize the complexity analysis provided in the Appendix, the scaling of the proposed algorithm is given by

$$O(N_R + M_{D0} N_T \log N_T),$$
$$M_{D0} \propto (\rho N_R \tau^{\ell} / N_T (\log N_T + c))^{1/(\ell+1)},$$

where ρ is the target density, τ is the scan length, and ℓ is the report dimensionality. For the report/cluster case, letting $N_T = N_C$ yields the appropriate scaling.

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Appendix

ALGORITHMIC SCALING ANALYSIS

This appendix describes the computational problems associated with gating and provides a detailed complexity analysis of the proposed solution. Th[illegible] term "track" is used here to refer to the mean position and radial extent of a cluster [illegible]d in the body of the paper. This permits the treatment of the report/cluster correlati[illegible] in a more general context.

BACKGROUND

In part because Eq. (1) is very expensive to evaluate, much of the previous work on gating has emphasized the use of intermediate or "coarse" gating criteria that replace the calculation of Eq. (1) with a function that is computationally cheaper to evaluate. The result is the identification of a superset of candidate pairs that includes the pairs that satisfy Eq. (2). The subset satisfying Eq. (2) is denoted as either the pairs that correlate at gating or the pairs that satisfy the final gate. Typically this preprocessing includes defining a gating volume V_G based on the numerator of the right-hand side (RHS) of Eq. (1) so that a coarse correlation measure is evaluated by assuming a Gaussian distribution as in Eq. (1). For example, the pairs pass one gate if $\gamma V_G < (dX^t \Gamma_j^{-1} dX)$, where γV_G is a threshold that may be obtained from a table or an error function integration for a certain probability of correlation. These pairs might also be preprocessed by coarser gating criteria that have larger V_G but are cheaper to evaluate. Ideally, one completes the gating calculation by computing S_{ij} for the set of candidate pairs and performing the comparison in Eq. (2). Overall processing work is then reduced because only the pairs with sufficiently high coarse correlation values must be reevaluated by using the numerically expensive function of Eq. (1). These techniques address the coefficient of the scaling but not the scaling itself because they explicitly apply a coarse correlation function to all $N_T N_R$ possible pairs. If the number of pairs explicitly evaluated by coarse gating are of order $(N_T N_R)$, then for sufficiently large N_T and N_R, real-time processing can be precluded on any computer even with the use of numerically simple coarse correlation functions. The objective is to describe an efficient approach to gating and to analyze its scaling. The overall algorithm will be shown to scale significantly better than quadratic even when reports have unequal timestamps within a scan. The methods described are compatible with virtually all of the previous work on auxiliary gating criteria and coarse gates.

TERMINOLOGY

A d-dimensional report, or observation, is defined to be a set of d elements measured simultaneously at some specified time. This time is the validity time of the observation or the timestamp of the observation. The timestamps fall within a period of time of length τ, called a scan, where the times of the reception of the first and the last observations fix the beginning and the end of the scan. Associated with each track and report are ℓ position components, where $\ell \leq d$.

Given a set of N_T tracks and a set of N_R reports, at most $N_T N_R$ scores S_{ij} can be formed. Of these, a fraction q of them will fall above the thresholds and satisfy Eq. (2), where q could be as low as $1/N_T$ or $1/N_R$ or smaller. Ideally, only the $qN_T N_R$ scores would be calculated; at worst, all $N_T N_R$ score calculations would be made. The following brute force approach is an example of an algorithm which scales quadratically: integrate the equations of motion of each of the N_T tracks to the times of each of the N_R reports and compute the scores. For each report, keep those scores that are above the desired threshold. The dominant cost of this is the $O(N_R N_T)$ score calculations and integrations. Of course, even if each score calculation is replaced by a coarse gate calculation, the scaling is still quadratic.

THE GEOMETRY OF GATING

When tracks and reports are valid at the same time, track/report pairs that are close in position tend to be correlated. The basis for this intuition is reflected by the appearance of the mean position difference in the exponential of Eq. (1). The covariances will in part determine gate volumes around the mean positions. Thus, gating can be conceptually related to geometry by saying that reports and tracks that gate with each other are those pairs with intersecting gate volumes. Let N_n be the number of tracks per report that should gate, as determined by Eqs. (1) and (2). Let the gating volumes be determined ideally in the sense that the set of pairs that should gate by Eqs. (1) and (2) is identical to the set of pairs having intersecting gate volumes. Let ρ be the object density and V_{IG} the ideal gating volume per report. Let the average of a quantity X over all the reports be given by $\overline{X}$. Then the total number of gating pairs is $\overline{N_n} N_R = \overline{\rho V_{IG}} N_R = qN_T N_R$.

The prescription for calculating the required $qN_T N_R$ scores involves in part using estimates of V_{IG} (e.g., V_G) in a search structure for identifying the pairs. A spherical search volume can be assumed, although it is not necessarily optimal. A search radius R_G specifies the search volume V_G. R_G is denoted as R_0 when a given report has the same timestamp as the track file to be searched. When the number of correlations that should be made is small, i.e., when $qN_T N_R$ is not comparable to $N_T N_R$, then $\overline{\rho V_{IG}} N_R$ is also small. Assume there is an R_0 per report such that: (a) the actual search neighborhood per report, V_G, includes V_{IG}, and (b) $\overline{\rho V_G}$ is comparable to $\overline{\rho V_{IG}}$.

When there is a distribution in time of tracks and reports throughout a scan, the required search radius R_G might define a search neighborhood so much larger than V_{IG} that the number of candidate pairs found is no longer comparable to $qN_T N_R$. It is insufficient to superimpose the tracks and reports to determine which error ellipses intersect, because evaluation of Eq. (1) requires that the function arguments correspond to the same time. Thus, the gating volume must take into account bounds on the location possibilities of the objects due to dynamics and time differences. In this case, the estimate of the gate volume is also time dependent, i.e., $V_G = V_G(\delta T, R_0)$, and its search radius can be modeled as

$$R_G = R_0 + \alpha|\delta T| , \tag{A1}$$

where α is some upper bound on the velocity and δT is the maximum time difference possible between any track and a report within the scan length τ and possibly equal to τ itself. Thus scaling could depend on the two parameters on the right-hand side (RHS) of Eq. (A1). Two limiting cases of Eq. (A1) are considered:

$$\overline{\alpha|\delta T|} \quad << \overline{R}_0 \tag{A2}$$

and

$$\overline{\alpha|\delta T|} \quad >> \overline{R}_0. \tag{A3}$$

The former describes the case in the limit of small scan length and the latter describes the case in the limit of large scan length.

In the ideal case of zero scan length, all reports from a given scan have identical timestamps. To perform gating, then, the track file is projected to the time of the reports. If a distribution exists in the measurement timestamps of the reports, however, the problem is much more complicated because the projection of the track file to the time of each report is explicitly an $O(N_T N_R)$ process. Fortunately, this difficulty can be addressed by subdividing the scan into a small number of intervals (i.e., not of order N_T or N_R). If these intervals are sufficiently small, the difference in timestamps of reports in the same interval will be small enough that object dynamics do not contribute significantly to the gating volume. Specifically, if M_D sequential track data structures (TDSs) are integrated to M_D equally spaced times within the scan of length τ, any report would be at most $|\delta T| = \tau/(2M_D)$ time units away from a TDS The average radius for the search is then decreased by M_D as compared to the case having one TDS copy at the middle of the scantime. Therefore, the volume extent as well as the average number of candidates returned $(\overline{N}_G)$ is smaller by $(1/M_D)^l$ in the isotropic dense limit l-dimensional case. More precisely, assume that the density ρ of objects in space is constant and uniform. Then the average number of candidate tracks found for each report depends on the average search volume $V_G = \gamma(\ell)\langle R_G^\ell\rangle$, where $\gamma(\ell)$ is a geometric factor depending on the dimension ℓ of the report state vector. The brackets $\langle\cdot\rangle$ denote the average over the temporal range $(t_i - \delta T)$, to the report. Assuming that the time distribution of reports within the scan interval is uniform,* using Eq. (A1) gives

$$V_G = \frac{\gamma(\ell)}{(\ell+1)\alpha\delta T}\left[(R_0 + \alpha|\delta T|)^{\ell+1} - (R_0)^{\ell+1}\right], \tag{A4}$$

$$V_G \approx \frac{\gamma(\ell)}{\ell+1}\Big(\alpha|\delta T|\Big)^{\ell}, \text{ and} \tag{A5}$$

$$N_G \approx \rho\frac{\gamma(\ell)}{\ell+1}(\alpha|\delta T|)^{\ell} = \rho\frac{\gamma(\ell)}{\ell+1}\left(\frac{\alpha\tau}{2}\right)^{\ell} M_D^{-\ell} \quad . \tag{A6}$$

The RHS of Eq. (A5) assumes the large scan length search condition of Eq. (A3), i.e.,

$$\alpha|\delta T| = \alpha\tau/2M_D \gg R_0. \tag{A7}$$

Because the searching time and the scoring time depend on N_G (the scoring time being directly proportional to N_G) the use of multiple extrapolated track files ($M_D > 1$) to cover the scan interval can reduce the cost of the gating process by reducing the search volume. The question that must be answered, then, is whether the improved scaling compensates for the cost of the multiple projections.

Let $\overline{N}_G$ be the number of gating candidates per report returned in the search step. Then the total cost in time can be modeled as

$$\begin{aligned} C(N_T, N_R, M_D, \overline{N}_G) = {} & C_e M_D N_T + C_d M_D N_T \log N_T \\ & + C_{se} N_R(\log N_T + \overline{N}_G) + C_{sc}\overline{N}_G N_R. \end{aligned} \tag{A8}$$

*An overestimate of the worst case is when the reports are $\delta T = \tau/(2M_D)$ time units away from a TDS, i.e., at the largest possible time difference, where the reported average case is small by $2^\ell/(\ell+1)$.

The terms on the RHS of Eq. (A8) give, respectively, the cost for integrating the tracks to the desired time of the data structures, the cost of making the tree data structures, the cost of searching the appropriate tree data structure for each report, and the cost of scoring the pairs.

Equation (A8) modeled using Eq. (A6) has a minimum value that occurs for the optimal M_D:

$$M_{D0} = (\ell \frac{\kappa_2}{\kappa_1})^{1/(\ell+1)}, \tag{A9}$$

where

$$\kappa_1 \equiv N_T(C_e + C_d \log N_T), \tag{A10}$$

$$\kappa_2 \equiv N_R(C_{sc} + C_{se}) \frac{\gamma(\ell)}{\ell+1} \rho (\frac{\alpha\tau}{2})^\ell, \tag{A11}$$

and the total cost is

$$C_{\min} \approx C_{se} N_R \log N_T + M_{D0} \frac{\ell+1}{\ell}(C_e N_T + C_d N_T \log N_T). \tag{A12}$$

Equation (A12) defines cost in terms of the important scaling parameters for multiple-target tracking except that it does not consider combinations of $\overline{R}_0$ with $\alpha\tau$ because of the approximation in Eq. (A3).

ANALYSIS OF THE GENERAL CASE

Let $\xi \equiv \alpha\tau/(2R_0\mathcal{M}_D)$, where the symbol for the number of TDSs is now $\mathcal{M}_D$ to make a distinction for the limit of Eq. (A3). Instead of taking the approximation in Eq. (A4) leading to Eq. (A5), use the binomial expansion and $N_G = \rho V_G$ to obtain from Eq. (A4):

$$N_G = \rho \frac{\gamma(\ell)}{\ell+1}(R_0)^\ell \sum_{i=1}^{\ell+1} \binom{\ell+1}{i} \xi^{i-1}. \tag{A13}$$

To find C_{min}, it is useful to find the partial derivative of Eq. (A13) with respect to $\mathcal{M}_D$:

$$\partial N_G / \partial \mathcal{M}_D = -\rho \frac{\gamma(\ell)}{\ell+1} \frac{(R_0)^\ell}{\mathcal{M}_D} \sum_{i=1}^{\ell} i \binom{\ell+1}{i+1} \xi^i. \tag{A14}$$

This yields an equation for the number of TDSs that minimize the cost of Eq. (A8):

$$\mathcal{M}_{D0}^{\ell+1} = M_{D0}^{\ell+1} \sum_{i=1}^{\ell} \frac{i}{\ell} \left(\text{[illegible]} \frac{1}{\xi} \right)^{\ell-i}. \tag{A15}$$

Equation (A12) is a polynomial equation for $\mathcal{M}_{D0}$ of ℓ terms and of degree $\ell+1$. For $l = 1$, $\mathcal{M}_{D0} = M_{D0}$.

To show that C_{min} of Eq. (A12) is an overestimate in τ when it is not true that $\alpha\tau >> 2\overline{R}_0$, $\mathcal{M}_{D0}$ and M_{D0} can be compared for a given N_T and N_R and a fixed estimate of $\overline{\rho V}_{IG}$ through $\overline{\rho V}_G(\overline{R}_0)$. Specifically, let τ_L be some value of a scan length for which $\alpha\tau_L >> 2\overline{R}_0$ and for which

Eq. (A9) was evaluated to be $M_{D0}(\tau_L)$. Then, using the ratio of Eq. (A15) at τ and at τ_L and using Eq. (A9) for $\mathcal{M}_{D0}$ at τ_L, the value of $\mathcal{M}_{D0}$ at some arbitrary scan length is

$$\mathcal{M}_{D0}^{\ell+1} \approx (\frac{\tau}{\tau_L})^{\ell}[M_{D0}(\tau_L)]^{\ell+1} \sum_{i=1}^{\ell} \frac{i}{\ell} \binom{\ell+1}{i+1} (\frac{2\overline{R}_0 \mathcal{M}_D}{\alpha\tau})^{\ell-i}. \tag{A16}$$

Equation (A16) yields $\mathcal{M}_{D0}$ for an arbitrary scan length given τ_L. Equations (A8), (A13), and (A16) give the cost of the gating process in terms of relevant parameters N_T, N_R, ρ, τ, R_0 (and therefore $\overline{R}_0$) and their combinations. Each of the terms in Eqs. (A12) and (A16) has its contribution in τ in the form of τ^i, where i is some positive integer. Thus, N_G (and $\overline{N}_G$) and $\mathcal{M}_{D0}^{\ell+1}$ decrease as τ decreases on some interval $(0, \tau_L)$. Notice also that as $\overline{N}_G$ and $\mathcal{M}_{D0}$ decrease, the cost as given by Eq. (A8) decreases. And since, for the case of Eq. (A3) and for $\tau = \tau_L$, $\mathcal{M}_{D0}$ approaches M_{D0} and the large scan length case is therefore an overestimate of the cost for the general case with a smaller scan length and with the other parameters held fixed.