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OPTIMAL HANKEL-NORM APPROXIMATION AND RATIONAL FUNCTION MODELS USING IMPULSE RESPONSE DATA

Charles K. Chui

Center for Approximation Theory
Texas A&M University
College Station, TX 77843

for

Underwater Sound Reference Detachment
Naval Research Laboratory
P.O. Box 568337
Orlando, FL 32856-8337

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Charles K. Chui

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Center for Approximation Theory
Texas A&M University
College Station, TX 77843

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The motivation of this work is rational function modeling of transfer functions using impulse response data from underwater acoustic transducers. However, since such data does not, in general, correspond to a rational model even in the absence of noise, an effective rational function approximation criterion is necessary. In this report, the approximation error measure used is the Hankel-norm. Approximation using the Hankel-norm is a linear algebra method with an assured global minimum while the usual least-squares approximation is a nonlinear method satisfying only local minimum criteria. There are three outstanding features of this Hankel-norm approach: (1) the rational model is uniquely determined by the optimality criterion; (2) this rational model is guaranteed to be stable; and (3) the exact error approximation is given by a singular value of the corresponding Hankel operator and can be easily computed. The main contribution of this report is to extend Kung's model reduction algorithm to approximating systems not exactly described by rational function models of finite order and to requiring only the solution of simple instead of generalized eigenvalue problems required in Kung's method.

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OPTIMAL HANKEL-NORM APPROXIMATION AND RATIONAL FUNCTION MODELS USING IMPULSE RESPONSE DATA

INTRODUCTION

It is well known that the underlying mathematical formulations in systems theory and signal processing are quite similar [1,2]. In particular, the transfer function $H(z)$ with the sequence of unit impulse responses $\{h_n\}$, plays the central role of these two important areas of research. In the ideal situation when the sequence $\{h_n\}$ is noise-free, then it is very easy to obtain the rational function model $H(z)$. One method is the use of Padé approximants. This method, being linear, is particularly simple. In systems theory, however, the dimension of the system, which agrees with the degree of the denominator of the rational function $H(z)$, although finite, may be unreasonably high. Hence, many model reduction methods have been introduced in the literature. All these methods amount to approximation of the rational function $H(z)$ by another rational function with a much lower degree (or lower system dimension). However, only one method stands out to be the most desirable. This method, known as *optimal Hankel-norm approximation*, has three outstanding features: first, the lower-degree rational function approximant is uniquely determined by an optimal criterion; second, this rational function approximant is guaranteed to be stable; and third, the exact measurement of the error of approximation can be determined. The mathematical description and derivation of the optimal Hankel-norm approximation is especially intriguing. Based on the infinite-dimensional theory due to Adamjan, Arov, and Krein [3], better known as AAK, it relates best rational approximation in the supremum norm and optimal approximation of the corresponding Hankel operator by Hankel operators with specified finite rank. The intimate relation between rational functions and finite-rank Hankel operators is governed by the beautiful classical result due to Kronecker (cf. Ref. 4); and the relation between best rational approximation in the supremum norm and optimal Hankel approximation in the operator norm is initiated by a fundamental result due to Nehari [5]. The AAK theory may be considered as a generalization of Nehari's theorem in that the singular values are used to give the exact measurement of the error of approximation. Note that the first singular value agrees with the spectral radius, which is the same as the distance in the supremum norm of the transfer function from the Hardy space H^∞ of bounded analytic functions in the open unit disc. This is the main theorem due to Nehari.

Unfortunately, the AAK theory is very deep, and there seems to be no easy way to find the AAK optimal approximants. Nevertheless, when the transfer function is already a rational function, S.Y. Kung [6] gave an algorithm to compute the AAK optimal approximation. This method is called H^∞-control (or systems reduction via Hankel approximation) in systems theory. However, even in the absence of noise, impulse response data from underwater acoustic transducers do not yield a rational model; and in general, data obtained from a rational model are contaminated with noise. Hence, for all practical purposes, the transfer function to be identified is *not* in the form of a rational function, and a rational approximation criterion is necessary. Recently, Chui, Li, and Ward [7,8] proved that if the truncated Hankel operators are used in place of the original Hankel operator (which represents the transfer function to be identified), then the AAK approximants of the truncated operators converge to the AAK approximant of the desired transfer function. We will derive a computational procedure based on this convergent result. In other words, if a sequence $\{h_n\}$ of unit impulse responses is given, we will derive a computational scheme to find the stable rational models that are optimal in the sense of best Hankel-norm approximation.

KRONECKER'S THEOREM

Let $\{h_n\}$, $n = 0, 1, \ldots$, be a causal sequence of unit impulse responses. Corresponding to this sequence, we associate two quantities: the z-transform, or symbol,

$$H(z) = \sum_{n=0}^{\infty} h_n z^{-n}, \tag{1}$$

and the infinite Hankel matrix $\Gamma_H = [h_{i+\ell-1}]$, $1 \le i, \ell < \infty$, or

$$\Gamma_H = \begin{bmatrix} h_1 & h_2 & h_3 & \cdot^{\cdot^{\cdot}} \\ h_2 & h_3 & \cdot^{\cdot^{\cdot}} & \\ h_3 & \cdot^{\cdot^{\cdot}} & & \end{bmatrix}. \tag{2}$$

Note that we do not include h_0 in the definition of Γ_H in (2) simply because of the standard convention in systems theory. The constant h_0 is easy to determine since it is the limit of $H(z)$ as z tends to infinity.

The rank of the matrix Γ_H is defined by the number of linearly independent columns of Γ_H; or equivalently, it is the dimension of the range of Γ_H on ℓ^2, where ℓ^2 denotes the space of square-summable sequences. The following classical result due to Kronecker can be found in [4]. A proof is included in the Appendix for easy reference.

Theorem 1. *(Kronecker)*

The infinite matrix Γ_H in (2) has finite rank if and only if the symbol

$$\sum_{n=1}^{\infty} h_n z^{-n} \tag{3}$$

is a strictly proper rational function in z; that is,

$$\sum_{n=1}^{\infty} h_n z^{-n} = \frac{P(z)}{Q(z)} \tag{4}$$

where $P(z)$ and $Q(z)$ are relatively prime polynomials in z with degree $(P) <$ degree (Q). Furthermore, in this situation,

$$\operatorname{rank}(\Gamma_H) = \text{ degree } (Q). \tag{5}$$

In fact, if Γ_H has finite rank k, say, then the first k columns of Γ_H are linearly independent, and there exist k constants $c_1, \ldots, c_k$, such that

$$\gamma_{k+\ell} = \sum_{i=1}^{k} c_i \gamma_{i+\ell-1}, \quad \ell = 1, 2, \ldots, \tag{6}$$

where γ_i denotes the i^{th} column of Γ_H. Furthermore, the finite square Hankel matrix

$$H_k = \begin{bmatrix} h_1 & h_2 & \ldots & h_k \\ h_2 & \ldots & \ldots & h_{k-1} \\ \ldots & \ldots & \ldots & \ldots \\ h_k & \ldots & \ldots & h_{2k-1} \end{bmatrix}, \tag{7}$$

which is a finite square truncation of Γ_H, is nonsingular. For more details, see Lemma 2.1 and its proof in [2] or the Appendix in this report.

THE ARMA MODEL

A sequence $\{h_n\}$, $n = 0, 1, \ldots$, is said to represent an ARMA model if its z-transform $H(z)$ defined in (1) is a proper rational function in z with all its poles lying in the open unit disc $|z| < 1$, namely:

$$\begin{aligned} H(z) = \sum_{n=0}^{\infty} h_n z^{-n} &= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\ &= \frac{b_0 z^p + b_1 z^{p-1} + \cdots + b_p}{z^p + a_1 z^{p-1} + \cdots + a_p} = \frac{\widetilde{P}(z)}{Q(z)} \end{aligned} \tag{8}$$

for arbitrary $M, N \geq 0$, where $p = \max(M, N)$, $b_{M+1} = \cdots = b_p = 0$, $a_{N+1} = \cdots = a_p = 0$,

$$\text{degree}(\widetilde{P}) \leq \text{degree}(Q) = p, \tag{9}$$

and

$$H(z) \text{ is analytic in } |z| \geq 1. \tag{10}$$

If $\widetilde{P}(z)$ and $Q(z)$ have no common factors, we say that the representation (8) of $H(z)$ is in *coprime form.* If (8) is in coprime form, then (10) is equivalent to

$$Q(z) \neq 0 \text{ for all } |z| \geq 1. \tag{11}$$

We remark that (10) is also equivalent to

$$\sum_{n=0}^{\infty} |h_n| < \infty. \tag{12}$$

In the following, we derive an algorithm to yield the *feedback sequence* $\{a_n\}$ and *feedforward sequence* $\{b_n\}$ in (8) from the sequence of unit impulse responses $\{h_n\}$, assuming that its z-transform $H(z)$ is known to be an ARMA model.

Algorithm I. (ARMA realization)

(1°) *Solve for $a_1, \ldots, a_p$ in the linear matrix equation:*

$$\begin{bmatrix} h_1 & h_2 & \ldots & h_p \\ h_2 & & \ldots & h_{p+1} \\ \cdots & \cdots & \cdots & \cdots \\ h_p & \cdots & \cdots & h_{2p-1} \end{bmatrix} \begin{bmatrix} a_p \\ a_{p-1} \\ \vdots \\ a_1 \end{bmatrix} + \begin{bmatrix} h_{p+1} \\ h_{p+2} \\ \vdots \\ h_{2p} \end{bmatrix} = 0. \tag{13}$$

(2°) *Compute $b_0, \ldots, b_p$ by matrix-vector multiplication in:*

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ a_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ a_p & \cdots & \cdots & a_1 & 1 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_p \end{bmatrix}, \tag{14}$$

where $a_1, \ldots, a_p$ have been computed in (13).

Remarks: From (7), with $k = p$, we see that the coefficient matrix of the matrix equation (13) (which is a finite Hankel matrix) is nonsingular. The coefficient matrix in (14) (which is a finite Toeplitz matrix) is a lower triangular matrix with unit diagonal.

Proof: Multiplying $Q(z)$ to $H(z)$ in (8), we have

$$(h_0 + h_1 z^{-1} + \cdots)(z^p + a_1 z^{p-1} + \cdots + a_p) = b_0 z^p + \cdots + b_p. \tag{15}$$

Hence, equating the coefficients of $z^p, z^{p-1}, \ldots, z, 1$, we have (14); and equating the coefficients of $z^{-1}, \ldots, z^{-p}$, we have (13).

THE HANKEL NORM

As mentioned in the Introduction, we must assume for all practical purposes that $H(z)$ is *not* a rational function. Hence, we usually start with a sequence $\{h_n\}$ of unit impulse responses which satisfies $\sum |h_n| < \infty$. In fact, in most situations, we only have input/output information $\{u_n\}/\{v_n\}$. In this case, we write

$$H(e^{j\omega}) = \frac{\sum_m v_m e^{-jm\omega}}{\sum_m u_m e^{-jm\omega}}; \tag{16}$$

and numerically compute the unit impulse responses h_n by applying DFT to the integral

$$h_n = \frac{1}{2\pi} \int_0^{2\pi} H(e^{j\omega}) e^{jn\omega} d\omega. \tag{17}$$

We do not dwell on this, since it is expected that other more efficient procedures will be introduced in our forthcoming report [9], but only remark here that any numerical approximation method for determining $\{h_n\}$ from (17) produces an error, which again necessarily results in a non-rational model $H(z)$, $z = e^{j\omega}$.

Now, as a result of the condition $\sum |h_n| < \infty$, we have a bounded function $H(z)$ on the unit circle $|z| = 1$. Hence, we may write

$$H(z) = H_a(z) + H_s(z), \tag{18}$$

where

$$H_a(z) = \left(\sum_{n=-\infty}^{\infty} h_n z^{-n} \right)_a = \sum_{n=-\infty}^{0} h_n z^{-n} \tag{19}$$

is called the *analytic part* of $H(z)$ and

$$H_s(z) = \left(\sum_{n=-\infty}^{\infty} h_n z^{-n} \right)_s = \sum_{n=1}^{\infty} h_n z^{-n} \tag{20}$$

is called the *singular part* of $H(z)$. Hence, the Hankel matrix Γ_H corresponding to $H(z)$, as defined in (2), is uniquely determined by the singular part $H_s(z)$ of $H(z)$, while the analytic part $H_a(z)$ of $H(z)$ does not influence Γ_H at all. Now, under the hypothesis $\sum |h_n| < \infty$, which is equivalent to BIBO (bounded input – bounded output) stability, we may conclude that Γ_H is a *bounded* linear operator on the space ℓ^2 of all square-summable sequences, with the operator norm defined by

$$\|\Gamma_H\| = \sup\{\|\Gamma_H \mathbf{x}\|_{\ell^2} \colon \|\mathbf{x}\|_{\ell^2} = 1\}, \tag{21}$$

where, for $\mathbf{x} = (x_1, x_2, \ldots)$,

$$\|\mathbf{x}\|_{\ell^2} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}. \tag{22}$$

The finiteness of $\|\Gamma_H\|$ follows from Nehari's theorem to be stated below; and it will be seen that, in fact, we have

$$\left(\sum_{n=1}^{\infty} |h_n|^2 \right)^{1/2} \leq \|\Gamma_H\| \leq \sum_{n=1}^{\infty} |h_n|. \tag{23}$$

We will verify (23) after we state Nehari's theorem; but at this point it should be noted that $\|\Gamma_H\|$ is the spectral radius of the operator Γ_H, or equivalently, the largest singular value of Γ_H.

The operator norm of Γ_H in (21) is also called the ***Hankel norm*** of the function $H(z)$ and will be denoted by

$$\|H(z)\|_\Gamma = \|\Gamma_H\|. \tag{24}$$

Actually, this is not a true norm but only a semi-norm since

$$\|H(z)\|_\Gamma = \|H(z) - g(z)\|_\Gamma \tag{25}$$

for any function $g(z) \in H^\infty$. Here and throughout, H^∞ denotes the space of all bounded analytic functions in $|z| < 1$. It is also called a Hardy space. The importance of this observation is that it provides a motivation of the celebrated theorem due to Nehari [5].

Theorem 2. *(Nehari)*

Let $H(z)$ be a bounded function defined on the unit circle $|z| = 1$. Then

$$\|H(z)\|_\Gamma = \inf_{g \in H^\infty} \|H(z) - g(z)\|_{L^\infty(|z|=1)}. \tag{26}$$

We now derive the inequalities in (23). First, note that $\Gamma_H = \Gamma_{H_s}$. By (26), with $g(z) = 0$, which is certainly in H^∞, we have

$$\begin{aligned}\|\Gamma_H\| = \|\Gamma_{H_s}\| = \|H_s(z)\|_\Gamma &\le \|H_s(z)\|_{L^\infty(|z|=1)} \\ &= \sup_{|z|=1} \left| \sum_{n=1}^{\infty} h_n z^{-n} \right| \\ &\le \sum_{n=1}^{\infty} |h_n|.\end{aligned} \tag{27}$$

This verifies the upper bound in (23). To verify the lower bound, let $\mathbf{e}_1 = (1, 0, 0, \ldots)$, which has unit ℓ^2 norm so that from the definition (21), we have

$$\begin{aligned}\|\Gamma_H\| &\ge \|\Gamma_H \mathbf{e}_1\|_{\ell^2} \\ &= \|(h_1, h_2, \ldots)\|_{\ell^2} = \left(\sum_{n=1}^{\infty} |h_n|^2 \right)^{1/2}.\end{aligned} \tag{28}$$

In the above formulation, we have also established that the Hankel norm lies between the $L^\infty(|z| = 1)$ and $L^2(|z| = 1)$ norms. Recall that

$$\|f\|_{L^\infty(|z|=1)} = \sup_{|z|=1} |f(z)|, \tag{29}$$

which has already been used in (26) and (27), and that

$$\|f\|_{L^2(|z|=1)} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{j\omega})|^2 d\omega \right)^{\frac{1}{2}}. \tag{30}$$

Indeed, by the isometry between ℓ^2 and $L^2(|z| = 1)$ we have the lower bound, and by the application of Nehari's theorem as in (27) we have the upper bound; that is,

$$\|H_s\|_{L^2(|z|=1)} \le \|H_s\|_\Gamma \le \|H_s\|_{L^\infty(|z|=1)}. \tag{31}$$

SINGULAR VALUES AND SINGULAR VECTORS

Let Γ_H be an infinite Hankel matrix as defined in (2). Under the assumption that $\sum |h_n| < \infty$, we have seen that Γ_H can be considered as a *bounded linear operator* on ℓ^2. For simplicity, we will call Γ_H a ***Hankel operator***. Since all Hankel operators are symmetric, the adjoint of Γ_H is its complex conjugate

$$\overline{\Gamma}_H = \begin{bmatrix} \bar{h}_1 & \bar{h}_2 & \bar{h}_3 & \\ \bar{h}_2 & \bar{h}_3 & & \cdot^{\cdot^{\cdot}} \\ \bar{h}_3 & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & \end{bmatrix}. \tag{32}$$

In particular, for a sequence of real-valued unit impulse responses, Γ_H is self-adjoint. In any case, $\Gamma_H\overline{\Gamma}_H$ is non-negative definite and has non-negative eigenvalues $s_1^2, s_2^2, \ldots$, with $s_1 \geq s_2 \geq \cdots \geq 0$, which are arranged in nonincreasing order, listing all multiple occurrences. Let $|\Gamma_H|$ denote the non-negative square-root of $\Gamma_H\overline{\Gamma}_H$. Note that although $|\Gamma_H|$ is Hermitian, it may *not* be Hankel; but this is not important in the following discussions. Clearly, the eigenvalues of $|\Gamma_H|$ are $s_1, s_2, \ldots$. Corresponding to each s_n, let $\mathbf{x}_n$ be an eigenvector of $|\Gamma_H|$; that is, $\mathbf{x}_n \neq 0$, $\mathbf{x}_n \in \ell^2$, and

$$|\Gamma_H|\mathbf{x}_n = s_n\mathbf{x}_n. \tag{33}$$

Let U be an *unitary operator* on ℓ^2 such that

$$\Gamma_H = U|\Gamma_H| \tag{34}$$

(cf. Ref. 10). Then by defining

$$\begin{cases} \xi_n = \mathbf{x}_n \\ \eta_n = U\mathbf{x}_n, \end{cases} \tag{35}$$

we see that

$$\begin{cases} \Gamma_H\xi_n = s_n\eta_n \\ \overline{\Gamma}_H\eta_n = s_n\xi_n. \end{cases} \tag{36}$$

Indeed, it is clear that

$$\begin{aligned} \Gamma_H\xi_n &= U|\Gamma_H|\xi_n \\ &= s_nU\xi_n = s_n\eta_n \end{aligned} \tag{37}$$

and

$$\begin{aligned} \overline{\Gamma}_H\eta_n &= \overline{\Gamma}_H U\xi_n \\ &= (\overline{\overline{U}^T\Gamma_H})^T\xi_n \\ &= |\Gamma_N|^T\xi_n = |\Gamma_H|\mathbf{x}_n \\ &= s_n\mathbf{x}_n = s_n\xi_n. \end{aligned} \tag{38}$$

We will call s_n a *singular va'ue* (or s-number) of Γ_H and will call the pair (ξ_n, η_n) a *singular vector pair* (or Schmidt pair) of Γ_H relative to s_n. Let us assume, without loss of generality, that the eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ of $|\Gamma_H|$ corresponding to the eigenvalues $\{s_1, s_2, \ldots\}$ are orthonormalized, so that the infinite matrix

$$V = [\mathbf{x}_1 \mathbf{x}_2 \ldots], \tag{39}$$

with $\mathbf{x}_n$ as its n^{th} column, is a unitary operator in ℓ^2. Then, by the definition of eigenvalues and eigenvectors, we have

$$|\Gamma_H| V = V \Sigma, \tag{40}$$

where

$$\Sigma = \operatorname{diag}(s_1, s_2, \ldots,) = \begin{bmatrix} s_1 & & \bigcirc \\ & s_2 & \\ \bigcirc & & \ddots \end{bmatrix} \tag{41}$$

is the diagonal matrix of the singular values of Γ_H. Hence, it follows from (34) that

$$\Gamma_H = (UV) \Sigma V^* \tag{42}$$

where V^* is the adjoint (or complex conjugate of the transpose) of V, and both (UV) and V^* are unitary operators. The decomposition (42) is called a *sinɡ 'ᵓr value decomposition* of Γ_H. (For finite matrices, see Ref. 11.) Another formulation ot (42) is the so-called *Schmidt series* representation

$$\Gamma_H = \sum_{n=1}^{\infty} s_n \eta_n \xi_n^*, \tag{43}$$

which follows from (42) and (35); where again, ξ_n^* denotes the complex conjugate of the transpose of the vector ξ_n, so that $\eta_n \xi_n^*$ is a rank-1 operator on ℓ^2. That is, Γ_H is the (strong) limit of the sequence of finite sums

$$\sum_{n=1}^{p} s_n \eta_n \xi_n^* \tag{44}$$

(which are rank p operators), as $p \to \infty$. "Strongness" here means convergence in the operator norm defined in (26). However, although the operators in (44) are finite-rank operators that approximate Γ_H, they are *not* Hankel operators, and consequently, are useless for our purpose of finding rational models for $H(z)$ (see Theorem 1).

AAK THEORY

Let $H(z)$ and Γ_H be defined as in (1) and (2) where the sequer. e $\{h_n\}$ of unit impulse responses is assumed to be in ℓ^1; that is,

$$\sum_{n=1}^{\infty} |h_n| < \infty. \tag{45}$$

Let $s_1 \geq s_2 \geq \cdots \geq 0$ denote the singular values of Γ_H with corresponding singular vector pairs $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$, respectively. Since s_1 is the largest singular value, it is the spectral radius of Γ, so that we have $s_1 = \|\Gamma_H\|$. Hence, as a consequence of Nehari's theorem (cf. Theorem 2), we have

$$s_1 = \|\Gamma_H\| = \inf_{g \in H^\infty} \|H(z) - g(z)\|_{L^\infty(|z|=1)}. \tag{46}$$

That is, the first singular value s_1 yields the exact measurement of the error of uniform approximation on $|z| = 1$ of $H(z)$ from the Hardy space H^∞. That is why Nehari's result is considered to be the starting point of the area of H^∞-control (cf. Ref. 12). To yield rational models, we introduce the notation:

$$\mathcal{R}_p^s = \left\{ \frac{P(z)}{Q(z)} : \ \deg P(z) < \deg Q(z) \leq p, \quad Q(z) \neq 0 \quad \text{for} \quad |z| \geq 1 \right\} \tag{47}$$

where P, Q are polynomials. That is, $\mathcal{R}_p^s$ is the collection of all stable proper rational functions of degree at most p. Also, let

$$H_p^\infty = H^\infty + \mathcal{R}_p^s \tag{48}$$

denote the collection of all functions $f(z) = g(z) + r(z)$ where $g(z) \in H^\infty$ and $r(z) \in \mathcal{R}_p^s$. To unify notations, we set

$$H^\infty = H_0^\infty, \quad \mathcal{R}_0^s = \{0\}. \tag{49}$$

To pose the approximation problem in terms of operators, let G_p denote the collection of all bounded Hankel operators with rank $\leq p$. Hence, G_0 consists only of the zero matrix. By Kronecker's theorem (cf. Theorem 1) and the equivalence of (11) to the boundedness of the corresponding Hankel operators, we may identify $\mathcal{R}_p^s$ with G_p.

Hence, Nehari's theorem (cf. Theorem 2) formulated as in (46) can be written as:

$$s_1 = \inf_{\Lambda \in G_0} \|\Gamma_H - \Lambda\| = \inf_{g \in H_0^\infty} \|H(z) - g(z)\|_{L^\infty(|z|=1)}. \tag{50}$$

Here, since $G_0 = \{0\}$, the first equality only says $s_1 = \|\Gamma_H\|$. The celebrated result of AAK in [3] is to generalize (50) from $p = 0$ to an arbitrary positive integer p.

Theorem 3. *(AAK)*

Let $H(z)$ be any function in $L^\infty(|z| = 1)$ and p be any non-negative integer. Then

$$s_{p+1} = \inf_{\Lambda \in G_p} \|\Gamma_H - \Lambda\| = \inf_{g \in H_p^\infty} \|H(z) - g(z)\|_{L^\infty(|z|=1)}. \tag{51}$$

Hence, the $(p+1)^{st}$ singular value of Γ_H gives the exact measurement of the distance in uniform norm on $|z| = 1$ of $H(z)$ from H_p^∞. In addition, since Γ_H is a compact operator under the assumption (45), we have

$$s_{p+1} \longrightarrow 0 \quad \text{as} \quad p \to \infty \tag{52}$$

(cf. Ref. 13). That is, we can approximate $H(z)$ as close as we wish from H_p^∞. So, if the Hankel norm defined in (24) is used, then in view of (25), the H^∞ or analytic part of the approximant is irrelevant. Thus, by taking the singular part $[\ ]_s$, we obtain a ***stable rational model*** of $H(z)$

To be more specific, we exhibit the formulation of the best Hankel-norm approximant. The following notations are needed. Let

$$\xi_{p+1} = (u_1^{(p+1)}, u_2^{(p+1)}, \ldots), \quad \eta_{p+1} = (v_1^{(p+1)}, v_2^{(p+1)}, \ldots) \tag{53}$$

be a singular vector pair of Γ_H corresponding to the $(p+1)^{\mathrm{st}}$ singular value s_{p+1} of Γ_H. Define the analytic and singular functions:

$$\begin{cases} \xi_{p+1}^{+}(z) = \displaystyle\sum_{i=1}^{\infty} u_i^{(p+1)} z^{i-1} \\ \eta_{p+1}^{-}(z) = \displaystyle\sum_{i=1}^{\infty} v_i^{(p+1)} z^{-i} \end{cases} \tag{54}$$

respectively, and set

$$\hat{g}_p(z) = H(z) - s_{p+1} \frac{\eta_{p+1}^{-}(z)}{\xi_{p+1}^{+}(z)}. \tag{55}$$

In the AAK paper [3], it is proved that $\hat{g}_p(z)$ is in H_p^∞ and solves the optimal approximation problem (51) in the sense that

$$s_{p+1} = \|\Gamma_H - \Gamma_{\hat{g}_p}\| = \|H(z) - \hat{g}_p(z)\|_{L^\infty(|z|=1)}. \tag{56}$$

As we remarked above, since

$$\Gamma_{\hat{g}_p} = \Gamma_{[\hat{g}_p]_s} \tag{57}$$

where $[\hat{g}_p]_s$ denotes the singular part of $\hat{g}_p$, we may conclude that

$$s_{p+1} = \|H - [\hat{g}_p]_s\|_\Gamma = \inf_{g \in H_p^\infty} \|H(z) - g(z)\|_\Gamma; \tag{58}$$

and this means that the singular part of $\hat{g}_p(z)$, defined by

$$\widehat{R}_p(z) = [\hat{g}_p(z)]_s, \tag{59}$$

provides an optimal ***stable rational approximation*** of $H(z)$ from $\mathcal{R}_p^s$. The proof of this result is very deep; but if $H(z)$ itself is also a rational model, a simpler derivation is accessible. We will outline this simpler version in the next section. The reason for doing so is that S.Y. Kung's model reduction algorithm, which will be discussed in this report, is based on this derivation.

DISCUSSION OF AAK'S THEOREM

We will outline the proof of AAK's theorem (cf. Theorem 3) for the special case where $H(z) = \sum h_n z^{-n}$ has real coefficients and the singular part $[H(z)]_s = \sum_{n=1}^{\infty} h_n z^{-n}$ of $H(z)$ is a strictly proper rational function, namely:

$$[H(z)]_s = H_s(z) = \frac{P_{M-1}(z)}{Q_M(z)}, \tag{60}$$

with degree (P_{M-1}) < degree $(Q_M) = M$, and $P_{M-1}(z), Q_M(z)$ being relatively prime. Note that $H_s(z) \in \mathcal{R}_M^s$ since Γ_H is a bounded Hankel operator.

Let $\{\lambda_i\}_{i=1}^M$ be the (necessarily non-zero) eigenvalues of the real rank-M Hankel matrix Γ_H. We arrange these eigenvalues in such a way that $|\lambda_i| = s_i$; hence,

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_M| > 0, \tag{61}$$

where s_i is the i^{th} singular value of Γ_H. (Note that since Γ_H is real, $\overline{\Gamma}_H \Gamma_H = \Gamma_H^2$; and this gives $s_i^2 = \lambda_i^2$.) Let $\mathbf{x}_i$ be the eigenvector of Γ_H corresponding to λ_i, $i = 1, \ldots, M$. Then since $s_i = |\lambda_i| = \lambda_i(\text{sgn } \lambda_i)$, we have

$$\begin{cases} \Gamma_H \xi_i = s_i \eta_i \\ \overline{\Gamma}_H \eta_i = s_i \xi_i \end{cases}, \tag{62}$$

where $\xi_i = \mathbf{x}_i$ and $\eta_i = (\text{sgn } \lambda_i)\mathbf{x}_i$. That is, $(\mathbf{x}_i, (\text{sgn } \lambda_i)\mathbf{x}_i) = (\xi_i, \eta_i)$ is a singular vector pair of Γ_H relative to the singular value $s_i = |\lambda_i|$.

Let $p < M$, and our goal be to replace $H_s(z) = P_{M-1}(z)/Q_M(z)$ by a rational function in $\mathcal{R}_p^s$. The notations introduced in (53) and (54) are now used.

Step 1. We claim that

$$\widetilde{H}(z) = H_s(z)\xi_{p+1}^+(z) - s_{p+1}\eta_{p+1}^-(z) \tag{63}$$

is analytic in $|z| < 1$; that is $[\widetilde{H}(z)]_s = 0$. To see this, first observe that

$$s_{p+1}\eta_{p+1} = \lambda_{p+1}\xi_{p+1} \tag{64}$$

so that

$$s_{p+1}\eta_{p+1}^-(z) = \lambda_{p+1} \sum_{i=1}^{\infty} u_i^{(p+1)} z^{-i}. \tag{65}$$

Second, it is clear that

$$\begin{aligned} &\sum_{i=1}^{\infty}\sum_{\ell=1}^{\infty} h_i u_\ell^{(p+1)} z^{\ell-i-1} \\ &= \sum_{i=1}^{\infty}\sum_{\ell=1}^{\infty} h_{i+\ell-1} u_\ell^{(p+1)} z^{-i} + \sum_{i=1}^{\infty}\sum_{\ell=1}^{\infty} h_{\ell-i+1} u_{\ell+1}^{(p+1)} z^{i-1} \end{aligned} \tag{66}$$

where $h_\ell = 0$ for $\ell \leq 0$. Finally, from $\Gamma_H \xi_{p+1} = \lambda_{p+1}\xi_{p+1}$, we have

$$\sum_{\ell=1}^{\infty} h_{i+\ell-1} u_\ell^{(p+1)} = \lambda_{p+1} u_i^{(p+1)}, \quad i = 1, 2, \ldots\ . \tag{67}$$

Hence, we may conclude, by applying (64), (65), (66), and (67) consecutively, that

$$
\begin{aligned}
[\widetilde{H}(z)]_s &= [H_s(z)\xi_{p+1}^{+}(z) - s_{p+1}\eta_{p+1}^{-}(z)]_s \\
&= [H_s(z)\xi_{p+1}^{+}(z) - \lambda_{p+1}\sum_{i=1}^{\infty} u_i^{(p+1)} z^{-i}]_s \\
&= \left[\sum_{i=1}^{\infty}\sum_{\ell=1}^{\infty} h_{i+\ell-1}u_\ell^{(p+1)} z^{-i} + \sum_{i=1}^{\infty}\sum_{\ell=1}^{\infty} h_{\ell-i+1}u_{\ell+1}^{(p+1)} z^{i-1} - \sum_{i=1}^{\infty}\lambda_{p+1}u_i^{(p+1)} z^{-i}\right]_s \\
&= \left[\sum_{i=1}^{\infty}\sum_{\ell=i}^{\infty} h_{\ell-i+1}u_{\ell+1}^{(p+1)} z^{i-1}\right]_s \\
&= \left[\sum_{i=1}^{\infty}\sum_{\ell=i+1}^{\infty} h_i u_\ell^{(p+1)} z^{\ell-i-1}\right]_s = 0.
\end{aligned}
\tag{68}
$$

Step 2. We claim that the function

$$B(z) = Q_M(z)\eta_{p+1}^{-}(z) \tag{69}$$

is a polynomial with real coefficients and of degree $\leq M-1$.

It is clear that $B(z)$ has real coefficients. We first observe that since $P_{M-1}(z)$ and $\xi_{p+1}^{+}(z)$ have only positive powers, we have

$$[P_{M-1}(z)\xi_{p+1}^{+}(z)]_s = 0. \tag{70}$$

Hence, it follows from Step 1 (cf. Eq. (63)) that

$$
\begin{aligned}
[B(z)]_s &= [Q_M(z)\eta_{p+1}^{-}(z)]_s \\
&= -\frac{1}{s_{p+1}}[P_{M-1}(z)\xi_{p+1}^{+}(z) - s_{p+1}Q_M(z)\eta_{p+1}^{-}(z)]_s \\
&= -\frac{1}{s_{p+1}}[Q_M(z)\{H_s(z)\xi_{p+1}^{+}(z) - s_{p+1}\eta_{p+1}^{-}(z)\}]_s \\
&= -\frac{1}{s_{p+1}}[Q_M(z)\widetilde{H}(z)]_s = 0.
\end{aligned}
\tag{71}
$$

That is, $B(z)$ is analytic and by the definition (69), it must be a polynomial. Now, since

$$\lim_{|z|\to\infty}\frac{B(z)}{Q_M(z)} = \lim_{z\to\infty}\eta_{p+1}^{-}(z) = 0, \tag{72}$$

we have degree $(B) <$ degree (Q_M), so that the degree of $B(z)$ is at most $M-1$.

Step 3. Let $B^*(z)$ and $Q_M^*(z)$ be the reciprocal polynomials relative to $B(z)$ and $Q_M(z)$, respectively; that is,

$$B^*(z) = z^{M-1}\overline{B\left(\frac{1}{\bar{z}}\right)} \quad \text{and} \quad Q_M^*(z) = z^M \overline{Q_N\left(\frac{1}{\bar{z}}\right)}. \tag{73}$$

We claim that the function

$$C(z) = \frac{P_{M-1}(z)B^*(z) - \lambda_{p+1}Q_M^*(z)B(z)}{Q_M(z)} \tag{74}$$

is a polynomial with real coefficients and of degree $\leq M-1$.

Indeed, since $P_{M-1}(z)B^*(z) - \lambda_{p+1}Q_M^*(z)B(z)$ is a polynomial of degree $\leq 2M-1$, we may use partial fractions to write

$$\frac{P_{M-1}(z)B^*(z) - \lambda_{p+1}Q_M^*(z)B(z)}{Q_M(z)Q_M^*(z)} = \frac{C(z)}{Q_M^*(z)} + \frac{D(z)}{Q_M(z)} \tag{75}$$

for some polynomials $C(z)$ and $D(z)$ of degree $\leq M-1$. However, since $Q_M(z)$ has all its zeros in $|z| < 1$, its reciprocal polynomial $Q_M^*(z)$ has all its zeros in $|z| > 1$, so that

$$\frac{1}{Q_M(z)} = z^{-M}\frac{1}{\overline{Q_M^*(\frac{1}{\bar{z}})}} = z^{-M}\sum_{i=0}^{\infty} q_i z^{-i} \tag{76}$$

for some $q_0, q_1, \ldots$. Hence, if degree $(D) \leq M-1$, we have

$$\begin{aligned}\left[\frac{D(z)}{Q_M(z)}\right]_s &= \left[z^{-M}D(z)\sum_{i=0}^{\infty} q_i z^{-i}\right]_s \\ &= z^{-M}D(z)\sum_{i=0}^{\infty} q_i z^{-i} = \frac{D(z)}{Q_M(z)}.\end{aligned} \tag{77}$$

On the other hand, since $Q_M^*(z)$ is analytic in $|z| < 1$, we have

$$\left[\frac{C(z)}{Q^*(z)}\right]_s = 0. \tag{78}$$

Now, in view of Step 1, namely: $[\widetilde{H}(z)]_s = 0$, we may apply (69) in Step 2 to verify that

$$\left[\frac{P_{M-1}(z)B^*(z) - \lambda_{p+1}Q_M^*(z)B(z)}{Q_M(z)Q_M^*(z)}\right]_s = 0, \tag{79}$$

so that by combining (77), (78), and (79) in (75), we have $D(z) = 0$. This verifies (74).

Step 4. Let $\widehat{R}_p(z)$ be defined as in (59). We claim that

$$\widehat{R}_p(z) = \left[\frac{C(z)}{B^*(z)}\right]_s . \tag{80}$$

Indeed, from (74), we have

$$\frac{C(z)}{B^*(z)} = H(z) - \lambda_{p+1}\frac{B(z)/Q_M(z)}{B^*(z)/Q_M^*(z)}. \tag{81}$$

Here, from the relations (69) and (64), it can be shown that

$$\left[\lambda_{p+1}\frac{B(z)/Q_M(z)}{B^*(z)/Q_M^*(z)}\right]_s = s_{p+1}\left[\frac{\eta_{p+1}^-(z)}{\xi_{p+1}^+(z)}\right]_s . \tag{82}$$

Therefore, in view of the definitions in (55) and (59), we have obtained (80).

Step 5. We can now complete the proof of the AAK theorem (cf. Theorem 3), for the special case of real and finite-rank Hankel operators Γ_H, by verifying (56), or equivalently:

$$s_{p+1} = \|\Gamma_H - \Gamma_{\widehat{R}_p}\|. \tag{83}$$

First, we remark that $\eta_{p+1}^-(z)/\xi_{p+1}^+(z)$ is a constant c multiple of a Blaschke product with $|c| = 1$, so that

$$\left\|\frac{\eta_{p+1}^-(z)}{\xi_{p+1}^+(z)}\right\|_{L^\infty(|z|=1)} = 1. \tag{84}$$

Therefore, from (55) and (31), we have

$$\begin{aligned} \|\Gamma_H - \Gamma_{\widehat{R}_p}\| &= \|\Gamma_H - \Gamma_{\hat{g}_p}\| \\ &= \|H - \hat{g}_p\|_\Gamma = s_{p+1}\left\|\frac{\eta_{p+1}^-}{\xi_{p+1}^+}\right\|_\Gamma \\ &\le s_{p+1}\left\|\frac{\eta_{p+1}^-(z)}{\xi_{p+1}^+(z)}\right\|_{L^\infty(|z|=1)} = s_{p+1}. \end{aligned} \tag{85}$$

On the other hand, since

$$\begin{aligned} [(H_s(z) - \widehat{R}_p(s))\xi_{p+1}^+(z)]_s &= s_{p+1}\left[\left[\frac{\eta_{p+1}^-(z)}{\xi_{p+1}^+(z)}\right]_s \xi_{p+1}^+(z)\right]_s \\ &= s_{p+1}\left[\frac{\eta_{p+1}^-(z)}{\xi_{p+1}^+(z)}\xi_{p+1}^+(z)\right]_s = s_{p+1}[\eta_{p+1}^-(z)]_s \\ &= s_{p+1}\eta_{p+1}^-(z), \end{aligned} \tag{86}$$

we have

$$\Gamma_{H-\widehat{R}_p}\xi_{p+1} = (\Gamma_h - \Gamma_{\widehat{R}_p})\xi_{p+1} = s_{p+1}\eta_{p+1}, \tag{87}$$

so that

$$\overline{\Gamma}_{H-\widehat{R}_p}\eta_{p+1} = s_{p+1}\xi_{p+1}. \tag{88}$$

That is, $(s_{p+1},(\xi_{p+1},\eta_{p+1}))$ is also a (singular value, singular vectors) pair of $\Gamma_{H-\widehat{R}_p}$. Hence, s_{p+1} does not exceed its spectral radius, namely:

$$s_{p+1} \leq \|\Gamma_{H-\widehat{R}_p}\| = \|\Gamma_H - \Gamma_{\widehat{R}_p}\|. \tag{89}$$

Therefore, by combining (85) and (89), we have proved (83).

Remark. From (80), it seems that $\widehat{R}_p(z)$ is a rational function in $\mathcal{R}^s_{M-1}$. Actually, we have now proved that $\widehat{R}_p(z)$ is in $\mathcal{R}^s_p$, where $p \leq M-1$. This means that only the p zeros of $B^*(z)$ that lie in $|z| < 1$ are used to yield $\widehat{R}_p$. The results in Steps 2, 3, and 4 are useful in the following discussion of Kung's model reduction algorithm.

KUNG'S MODEL REDUCTION ALGORITHM

Let $H_s(z) = P_{M-1}(z)/Q_M(z)$ be defined as in (60) where $P_{M-1}(z)$ and $Q_M(z)$ are coprime polynomials with degree $(P_{M-1}) <$ degree $(Q_M) = M$. Let $Q^*_M(z)$ be the reciprocal polynomial relative to $Q_M(z)$ as defined in (73). Since $H_s(z)$ is in $\mathcal{R}^s_M$, all the M zeros of $Q_M(z)$ lie in $|z| < 1$, so that all the M zeros of $Q^*_M(z)$ lie in $|z| > 1$. That is,

$$\frac{Q^*_M(z)}{Q_M(z)} = q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots \tag{90}$$

is the reciprocal of a finite Blaschke product.

We need the following notations:

$$H_M = \begin{bmatrix} h_1 & h_2 & \dots & h_M \\ h_2 & \dots & \dots & h_{M+1} \\ \dots & \dots & \dots & \dots \\ h_M & \dots & \dots & h_{2M-1} \end{bmatrix} \tag{91}$$

$$K_M = \begin{bmatrix} q_M & q_{M-1} & \cdots & q_1 \\ q_{M+1} & q_M & \cdots & q_2 \\ \dots & \dots & \dots & \dots \\ q_{2M-1} & q_{2M-2} & \cdots & q_M \end{bmatrix} \tag{92}$$

$$H_\Delta = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & h_1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & h_1 & \dots & h_{M-1} \end{bmatrix} \tag{93}$$

$$K_\Delta = \begin{bmatrix} q_0 & 0 & \dots & 0 \\ q_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ q_{M-1} & \dots & q_1 & q_0 \end{bmatrix}, \tag{94}$$

where H_M, H_Δ are finite Hankel matrices and K_M, K_Δ are finite Toeplitz matrices. (Note that the notations in (90) and (7) agree.)

Algorithm II. *(Kung)*

(1°) *Solve the generalized eigenvalue problem*

$$(H_M - \lambda K_M)\mathbf{q} = \mathbf{0} \tag{95}$$

for $\{\lambda_i, \mathbf{q}^i\}$, *where* $\{\mathbf{q}^1, \dots, \mathbf{q}^M\}$ *are linearly independent and*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_M|. \tag{96}$$

(2°) *For* $1 \leq p < M$, *perform the matrix-vector multiplication:*

$$\mathbf{r}^{p+1} = (H_\Delta - \lambda_{p+1} K_\Delta)\mathbf{q}^{p+1}. \tag{97}$$

(3°) *Set*

$$\begin{cases} \mathbf{r}^{p+1} = (b_0, \dots, b_{M-1}) \\ \mathbf{q}^{p+1} = (a_0, \dots, a_{M-1}) \end{cases}, \tag{98}$$

and compute the singular part

$$\widehat{R}_p(z) = \left[\frac{b_0 z^{M-1} + \cdots + b_{M-1}}{a_0 + a_1 z + \cdots + a_{M-1} z^{M-1}} \right]_s. \tag{99}$$

(In this computation, factorize $a_0 + a_1 z + \cdots + a_{M-1} z^{M-1}$ into $M-1$ linear factors and discard those terms whose roots do not lie in $|z| < 1$. We note that exactly p linear factors, counting multiplicities, are retained. Now use partial fraction expansion to determine $\widehat{R}_p(z)$.)

Proof. Let

$$H^M = \begin{bmatrix} h_{M+1} & h_{M+2} & \dots & h_{2M} \\ h_{M+2} & \dots & \dots & h_{2M+1} \\ \dots & \dots & \dots & \dots \\ h_{2M} & \dots & \dots & h_{3M-1} \end{bmatrix}, \tag{100}$$

and

$$K^M = \begin{bmatrix} q_{2M} & q_{2M-1} & \dots & q_{M+1} \\ q_{2M+1} & q_{2M} & \dots & q_{M+2} \\ \dots & \dots & \dots & \dots \\ q_{3M-1} & q_{3M-2} & \dots & q_{2M} \end{bmatrix}, \tag{101}$$

where H^M is a finite Hankel matrix and K^M a finite Toeplitz matrix. Set

$$\begin{cases} C(z) = b_0 z^{M-1} + \cdots + b_{M-1} \\ B(z) = a_0 z^{M-1} + \cdots + a_{M-1} \end{cases}, \tag{102}$$

so that

$$B^*(z) = a_0 + a_1 z + \cdots + a_{M-1} z^{M-1}. \tag{103}$$

Then it is straightforward to check that the identity (74) in Step 3 in the proof of AAK's theorem is equivalent to the matrix system

$$\begin{bmatrix} H_\Delta \\ \cdots \\ H_M \\ \cdots \\ H^M \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_{M-1} \end{bmatrix} - \lambda_{p+1} \begin{bmatrix} K_\Delta \\ \cdots \\ K_M \\ \cdots \\ K^M \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_{M-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ \vdots \\ b_{M-1} \\ \cdots \\ 0 \\ \vdots \\ 0 \\ \cdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{104}$$

Also, (104) is equivalent to

$$\begin{cases} [H_\Delta - \lambda_{p+1} K_\Delta]\mathbf{q}^{(p+1)} = \mathbf{r}^{(p+1)} \\ [H_M - \lambda_{p+1} K_M]\mathbf{q}^{(p+1)} = \mathbf{0} \\ [H^M - \lambda_{p+1} K^M]\mathbf{q}^{(p+1)} = \mathbf{0} \end{cases}, \tag{105}$$

where the notations in (98) are used. The second equation in (105) is Step (1°) in Algorithm II, while the first equation in (105) is Step (2°) in the algorithm. Note that the third equation in (105) is a consequence of the second equation by the Caley-Hamilton theorem in matrix theory. Now, the result $\widehat{R}_p(z)$ computed in Step (3°) of the algorithm agrees with $\widehat{R}_p(z)$ in (80); and hence, by AAK's theorem (cf. argument in Step 5 in the above section), we may conclude that $\widehat{R}_p(z)$ satisfies (83).

REFORMULATION OF OPTIMAL HANKEL-NORM RATIONAL MODEL

We have seen that Toeplitz matrices also occur in the computation of the rational model $\widehat{R}_p(z)$ in Kung's algorithm. In fact, a Toeplitz matrix associated with the transfer function $H(z)$ can be used to describe $\widehat{R}_p(z)$. Let

$$T_H = \begin{bmatrix} 0 & h_1 & h_2 & h_3 & \dots \\ 0 & 0 & h_1 & h_2 & \dots \\ 0 & 0 & 0 & h_1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{106}$$

be an upper triangular infinite Toeplitz matrix with zero main diagonal. As in (61), let $\{\lambda_i\}_{i=1}^M$ denote the (necessarily non-zero) eigenvalues of the rank-M Hankel operator Γ_H such that $|\lambda_i| = s_i$ is the i^{th} singular value of Γ_H, $i = 1, \dots, M$. As before, let

$$\mathbf{x}_i = \xi_i = (u_1^{(i)}, u_2^{(i)}, \dots) \tag{107}$$

denote an eigenvector of Γ_H corresponding to the eigenvalue λ_i, so that (ξ_i, η_i), with $\eta_i = (\operatorname{sgn} \lambda_i)\mathbf{x}_i$, is a singular vector pair of Γ_H corresponding to the singular value $s_i = |\lambda_i|$ (cf. Eq. (62)). Also, let $\hat{g}_p(z)$ be as in (55) and $\widehat{R}_p(z) = [\hat{g}_p(z)]_s$ be the optimal solution in best Hankel-norm approximation of $H(z)$ from $\mathcal{R}_p^s$. Then we have the following result.

Theorem 4. *Let $H(z)$ be a transfer function with real coefficients such that its singular part $H_s(z)$ is given as in (60). Then*

$$\widehat{R}_p(z) = \left[\frac{(1, z, z^2 \dots) \cdot T_H \mathbf{x}_{p+1}}{(1, z, z^2, \dots) \cdot \mathbf{x}_{p+1}} \right]_s . \tag{108}$$

Proof. From the equation $\Gamma_H \mathbf{x}_{p+1} = \lambda_{p+1} \mathbf{x}_{p+1}$, we have

$$\sum_{i=1}^{\infty} h_{i+\ell-1} u_i^{(p+1)} = \lambda_{p+1} u_\ell^{(p+1)}, \quad \ell = 1, 2, \dots; \tag{109}$$

and hence, by (65), it follows that

$$s_{p+1} \eta_{p+1}^-(z) = \lambda_{p+1} \sum_{i=1}^{\infty} u_i^{(p+1)} z^{-i} = \sum_{i=1}^{\infty} \left(\sum_{\ell=1}^{\infty} h_{i+\ell-1} u_\ell^{(p+1)} \right) z^{-i}. \tag{110}$$

This yields:

$$\begin{aligned} & H_s(z)\xi_{p+1}^+(z) - s_{p+1}\eta_{p+1}^-(z) \\ =& \left(\sum_{i=1}^{\infty} h_i z^{-i} \right) \left(\sum_{\ell=1}^{\infty} u_\ell^{(p+1)} z^{\ell-1} \right) - \sum_{i=1}^{\infty} \left(\sum_{\ell=1}^{\infty} h_{i+\ell-1} u_\ell^{(p+1)} \right) z^{-i} \\ =& \sum_{i=1}^{\infty} \left(\sum_{\ell=i+1}^{\infty} h_{\ell-i} u_{\ell+1}^{(p+1)} \right) z^i \\ =& (1, z, z^2, \dots) \cdot T_f \mathbf{x}_{p+1}. \end{aligned} \tag{111}$$

Since $\xi_{p+1}^+(z) = (1, z, z^2, \dots) \cdot \mathbf{x}_{p+1}$, we have

$$H_s(z) - s_{p+1}\frac{\eta_{p+1}^{-}(z)}{\xi_{p+1}^{+}(z)} = \frac{(1, z, z^2, \ldots) \cdot T_f \mathbf{x}_{p+1}}{(1, z, z^2, \ldots) \cdot \mathbf{x}_{p+1}},$$

so that by (59) and (55), which was proved by AAK [3] in general and in this report for our special case, we have obtained (108).

TRUNCATED HANKEL OPERATORS

To apply Kung's algorithm, it is essential to start with an ARMA model. However, as we mentioned before, in all practical purposes in underwater acoustic signal processing, the sequence $\{h_n\}$ of unit impulse responses which may be computed from input/output measurements (cf. Eqs. (16) and (17), for instance), is both physically and numerically noisy. Hence, the measured transfer function $H(z)$ cannot be ARMA or the Hankel operator Γ_H has infinite rank. In a recent work [7], the truncated (infinite) Hankel operators

$$\Gamma_H^n = \begin{bmatrix} h_1 & h_2 & \ldots & h_n & 0 \\ h_2 & & \iddots & \iddots & \\ \vdots & & & & \\ \vdots & \iddots & \iddots & & \\ h_n & & & & \\ & \iddots & & & \\ 0 & & & & \bigcirc \end{bmatrix} \tag{112}$$

are introduced and optimal Hankel-norm approximants $\widehat{R}_p^n(z)$ of

$$H_n(z) = \sum_{i=-\infty}^{n} h_i z^{-i} \tag{113}$$

are used to replace the optimal Hankel-norm approximant $\widehat{R}_p(z)$ of

$$H(z) = \sum_{i=-\infty}^{\infty} h_i z^{-i}. \tag{114}$$

Recall that this means:

$$\begin{cases} s_{p+1}^{(n)} = \|H_n(z) - \widehat{R}_p^n(z)\|_\Gamma = \inf\limits_{R(z)\in\mathcal{R}_p^s} \|H_n(z) - R(z)\|_\Gamma \\ s_{p+1} = \|H(z) - \widehat{R}_p(z)\|_\Gamma = \inf\limits_{R(z)\in\mathcal{R}_p^s} \|H(z) - R(z)\|_\Gamma \end{cases}, \tag{115}$$

where $s_1^{(n)} \geq s_2^{(n)} \geq \cdots$ are the singular values of $\Gamma_{H_n} = \Gamma_H^n$. It is at least intuitively clear that

$$s_{p+1}^{(n)} \longrightarrow s_{p+1} \quad \text{as} \quad n \to \infty; \tag{116}$$

and in [8], we have proved much more.

Theorem 5. *For each p, there exists a positive constant C_p, such that*

$$\|\widehat{R}_p^n(z) - \widehat{R}_p(z)\|_{L^\infty(|z|=1)} \le C_p \left\{ \sum_{k=n+1}^{\infty} |h_k| \right. \tag{117}$$

$$\left. + (n+1) \left(\sum_{k=n+1}^{\infty} |h_k|^2 \right)^{1/2} + \sum_{k=n+2}^{\infty} \left(\sum_{j=k}^{\infty} |h_j|^2 \right)^{\frac{1}{2}} \right\}.$$

Hence, if $\{h_n\}$ decays to zero very rapidly (as is the case in most underwater acoustic experiments), we know that $\widehat{R}_p^n(z)$ converges uniformly on $|z| = 1$ very rapidly to the optimal Hankel-norm rational model $\widehat{R}_p(z)$. In the next section, we will give an efficient algorithm for computing $\widehat{R}_p^n(z)$.

ALGORITHM FOR OPTIMAL RATIONAL MODEL

Let $\{h_n\}$, $n = 1, 2, \ldots$, be a sequence of unit impulse responses that satisfies

$$\sum_{n=1}^{\infty} |h_n| < \infty. \tag{118}$$

(Note that $H(z) = \sum h_n z^{-n}$ is not necessarily a rational model.) Given a tolerance $\varepsilon > 0$, we now give an algorithm for computing a stable (strictly proper) rational function $R_p(z)$ such that

$$\|H(z) - R_p(z)\|_\Gamma \le \varepsilon. \tag{119}$$

The degree p will be chosen to be the smallest possible, under the limitation of this method.

Algorithm III.

(1°) *Choose an ε_1, $0 < \varepsilon_1 < \varepsilon$ (e.g. $\varepsilon_1 = \frac{1}{2}\varepsilon$), and the smallest positive integer M_1, such that*

$$\sum_{n=M_1+1}^{\infty} |h_n| \le \varepsilon_1, \tag{120}$$

where $h_{M_1} \neq 0$.

(2°) *Find the singular values*

$$s_1^{(1)} \ge s_2^{(1)} \ge \cdots \ge s_{M_1}^{(1)} > 0 \tag{121}$$

of the finite Hankel matrix

$$\Gamma_H^{(1)} = \begin{bmatrix} h_1 & h_2 & \dots & h_{M_1} \\ h_2 & & \iddots & \\ \vdots & \iddots & & \bigcirc \\ h_{M_1} & & & \end{bmatrix}, \tag{122}$$

and choose the smallest positive integer $p_1 \leq M_1$ *such that*

$$s_{p_1}^{(1)} \leq \varepsilon - \varepsilon_1. \tag{123}$$

(One can find the eigenvalues of $\Gamma_H^{(1)}$ *and then take their absolute values to determine the singular values.)*

(3°) *If* p_1 *does not exist (i.e., if* $s_{M_1}^{(1)} > \varepsilon - \varepsilon_1$*), return to* (1°) *with* ε_1 *replaced by*

$$\varepsilon_2 = \frac{\varepsilon_1}{2} \tag{124}$$

and M_1 *replaced by* M_2*, so chosen that*

$$\sum_{n=M_2+1}^{\infty} |h_n| \leq \varepsilon_2 = \frac{\varepsilon_1}{2} \tag{125}$$

and $h_{M_2} \neq 0. M_2 > M_1$.

Repeat (2°) *by finding the singular values*

$$s_1^{(2)} \geq s_2^{(2)} \geq \cdots \geq s_{M_2}^{(2)} > 0 \tag{126}$$

of the Hankel matrix

$$\Gamma_H^{(2)} = \begin{bmatrix} h_1 & h_2 & \dots & h_{M_2} \\ h_2 & & \iddots & \\ \vdots & \iddots & & \\ h_{M_2} & & & \bigcirc \end{bmatrix} \tag{127}$$

and choosing the smallest $p_2 \leq M_2$ *such that*

$$s_{p_2}^{(2)} \leq \varepsilon - \varepsilon_2 \tag{128}$$

(4°) *If this fails (i.e. if* $s_{M_2}^{(2)} > \varepsilon - \varepsilon_2$*), repeat* (3°) *with*

$$\varepsilon_3 = \frac{\varepsilon_2}{2}, \tag{129}$$

etc. Suppose that the first time this procedure succeeds is at the k^{th} *iteration. That is,* k *is the smallest integer such that*

$$s_{M_k}^{(k)} \leq \varepsilon - \varepsilon_k = \varepsilon - \frac{1}{2^{k-1}}\varepsilon_1, \tag{130}$$

where

$$s_1^{(k)} \geq \cdots \geq s_{M_k}^{(k)} > 0 \tag{131}$$

are the singular values of the M_k*-dimensional square Hankel matrix*

$$\Gamma_H^{(k)} = \begin{bmatrix} h_1 & h_2 & \dots & h_{M_k} \\ h_2 & & \iddots & \\ \vdots & \iddots & & \\ h_{M_k} & & & \bigcirc \end{bmatrix}, \tag{132}$$

with $h_{M_k} \neq 0$, *and*

$$\sum_{n=M_k+1}^{\infty} |h_n| \leq \varepsilon_k = \frac{1}{2^{k-1}}\varepsilon_1. \tag{133}$$

Let p_k *be the smallest integer such that* $s_{p_k}^{(k)} \leq \varepsilon - \varepsilon_k$, *and set*

$$p = p_k - 1; \tag{134}$$

so that

$$0 < s_{M_k}^{(k)} \leq s_{p+1}^{(k)} = s_{p_k}^{(k)} \leq \varepsilon - \frac{1}{2^{k-1}}\varepsilon_1. \tag{135}$$

(5°) *Compute the eigenvalue-eigenvector pair*

$$(\lambda_{p+1}, \mathbf{q}^{p+1}) \tag{136}$$

of $\Gamma_H^{(k)}$, *where* $|\lambda_{p+1}| = s_{p+1}^{(k)}$.

(6°) *Perform the matrix-vector multiplication*

$$\mathbf{r}^{p+1} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & \iddots & h_1 \\ & \iddots & \iddots & \vdots \\ 0 & h_1 & \dots & h_{M_k-1} \end{bmatrix} \mathbf{q}^{p+1}. \tag{137}$$

(7°) *Write:*

$$\begin{cases} \mathbf{r}^{p+1} = (b_0, \dots, b_{M_k-1}) \\ \mathbf{q}^{p+1} = (a_0, \dots, a_{M_k-1}) \end{cases}; \tag{138}$$

and compute the singular part

$$R_p(z) = \left[\frac{b_0 z^{M_k-1} + \cdots + b_{M_k-1}}{a_0 + a_1 z + \cdots + a_{M_k-1} z^{M_k-1}}\right]_s \tag{139}$$

by using partial fractions, retaining only the p poles in $|z| < 1$, counting multiplicities. Then $R_p(z) \in \mathcal{R}_p^s$ and satisfies

$$\|H(z) - R_p(z)\|_\Gamma \leq \varepsilon. \tag{140}$$

Proof. We first remark that $\det \Gamma_H^{(k)} = (-1)^{M_k} h_{M_k}^{M_k} \neq 0$ so that all the singular values $s_i^{(k)}$, $i = 1, \ldots, M_k$, are positive (nonzero). From a well-known result in operator theory (cf. [13]), we have

$$\max_{1 \leq i \leq M_k} |s_i - s_i^{(k)}| \leq \|\Gamma_H - \Gamma_H^{(k)}\|. \tag{141}$$

On the other hand, it follows from the definition of $\Gamma_H^{(k)}$ and the second inequality in (23) that

$$\|\Gamma_H - \Gamma_H^{(k)}\| \leq \sum_{n=M_k+1}^{\infty} |h_n|. \tag{142}$$

Hence, by combining (141), (142), and (133), we obtain

$$\max_{1 \leq i \leq M_k} |s_i - s_i^{(k)}| \leq \frac{1}{2^{k-1}} \varepsilon_1. \tag{143}$$

Now, since $\{h_n\}$ satisfies (118), we have $s_n \to 0$ by (52); so that in view of (143) (where M_k necessarily tends to infinity, as $k \to \infty$), it follows that (130) is satisfied for all sufficiently large values of k. This shows that the iteration procedure (1°)-(4°) converges.

Next, from the definition of $\Gamma_H^{(k)}$ in (132), we note that the corresponding rational symbol of $\Gamma_H^{(k)}$ is

$$H_k(z) = \sum_{i=1}^{M_k} h_i z^{-i} = \frac{h_1 z^{M_k-1} + \cdots + h_{M_k}}{z^{M_k}} = \frac{P(z)}{Q(z)}, \tag{144}$$

where degree $(P) <$ degree $(Q) = M_k$ and $Q(z) = z^{M_k}$. Since the reciprocal polynomial of $Q(z)$ is $Q^*(z) = 1$, we have

$$\frac{Q^*(z)}{Q(z)} = z^{-M_k}. \tag{145}$$

Hence, using the notations from (92) and (94) with M replaced by M_k, we have:

$$K_{M_k} = \begin{bmatrix} 1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & 1 \end{bmatrix}, K_\Delta = [\bigcirc]. \tag{146}$$

Also, again from the definition of $\Gamma_H^{(k)}$ in (132), using the notations from (91) and (93) with $M = M_k$, we have:

$$H_{M_k} = \Gamma_H^{(k)} \tag{147}$$

and

$$H_\Delta = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & h_1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & h_1 & \cdots & h_{M_k-1} \end{bmatrix}. \tag{148}$$

This yields

$$\begin{cases} (H_{M_k} - \lambda K_{M_k})\mathbf{q} = \Gamma_H^{(k)}\mathbf{q} - \lambda\mathbf{q} \\ (H_\Delta - \lambda K_\Delta)\mathbf{q} = H_\Delta\mathbf{q}. \end{cases} \tag{149}$$

Hence, Step (1°) in Kung's algorithm (i.e. Algorithm II) is equivalent to

$$\Gamma_H^{(k)}\mathbf{q}^{p+1} = \lambda_{p+1}\mathbf{q}^{p+1}, \tag{150}$$

and Step (2°) in Kung's algorithm (i.e. Algorithm II) is equivalent to

$$\mathbf{r}^{p+1} = H_\Delta\mathbf{q}^{p+1}. \tag{151}$$

Since (150) is the same as Step (5°) and (151) is the same as Step (6°) in Algorithm III, we have proved that $R_p(z)$, as defined in (139), satisfies

$$\|R_p(z) - H_k(z)\|_\Gamma = s_{p+1}^{(k)}, \tag{152}$$

where $H_k(z)$ is defined in (144). Hence, by applying the triangle inequality and the information from (142), (133), (152), and (135) consecutively, we have

$$\begin{aligned} \|H(z) - R_p(z)\|_\Gamma &= \|\Gamma_H - \Gamma_{R_p}\| \\ &\le \|\Gamma_H - \Gamma_H^{(k)}\| + \|\Gamma_H^{(k)} - \Gamma_{R_p}\| \\ &\le \sum_{n=M_k+1}^{\infty} |h_n| + \|R_p(z) - H_k(z)\|_\Gamma \\ &\le \frac{1}{2^{k-1}}\varepsilon_1 + s_{p+1}^{(k)} \\ &\le \frac{1}{2^{k-1}}\varepsilon_1 + \left(\varepsilon - \frac{1}{2^{k-1}}\varepsilon_1\right) = \varepsilon. \end{aligned} \tag{153}$$

This establishes (140).

Remarks. Algorithm III can be easily adapted and modified to produce a possibly lower degree rational model $R_p(z)$ that satisfies the design criterion (119). In fact, as a consequence of Theorem 5, we can theoretically obtain the lowest degree optimal Hankel-norm rational model $\widehat{R}_p(z)$ in the limiting case. One suggestion is to repeat this algorithm for various values of ε_1, such that

$$\varepsilon_1 = \frac{\ell}{L}\varepsilon, \quad \ell = 1, \ldots, L-1, \tag{154}$$

for a suitably large integer L. As to the modification of the algorithm itself, one may wish to try various values of ε_i, $i = 2, 3, \ldots, k$, instead of $\varepsilon_2 = \frac{1}{2}\varepsilon_1, \varepsilon_3 = \frac{1}{2}\varepsilon_2 = \frac{1}{2^2}\varepsilon_1, \ldots, \varepsilon_k = \frac{1}{2}\varepsilon_{k-1} = \frac{1}{2^{k-1}}\varepsilon_1$ as suggested in (124), (129), and (130). The only restriction is that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > \varepsilon_k$. The smaller the values of $\varepsilon_\ell - \varepsilon_{\ell-1}$, $\ell = 2, \ldots, k$, chosen, the better the chance is to find the lowest degree rational model $\widehat{R}_p(z)$. Of course, more computing time is required.

We also remark that although the Hankel-norm specification in (119) is not as desirable as the supremum (or uniform) norm specification, it is very close to it, in view of Nehari's theorem (i.e. Theorem 2), since the only thing that can go wrong is an H^∞ (or analytic) additive factor which only contributes to noncausal information. In addition, the Hankel-norm specification is more desirable than the L^2 norm (or RMS) specification, since it follows from (31) that

$$\|H(z) - R_p(z)\|_{L^2} \le \|H(z) - R_p(z)\|_\Gamma, \tag{155}$$

where $L^2 = L^2(|z| = 1)$ and $H(z)$ is assumed to be causal, in the sense that $H(z) = H_s(z)$ or

$$H(z) = \sum_{n=1}^{\infty} h_n z^n, \tag{156}$$

where $\{h_n\}$ satisfies (118). Hence, as a consequence of (154), if $R_p(z)$ satisfies the design criterion $\|H(z) - R_p(z)\|_\Gamma \le \varepsilon$, it also satisfies the design criterion $\|H(z) - R_p(z)\|_{L^2} \le \varepsilon$.

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APPENDIX

Since it is important to be able to relate a rational model

$$H(z) = \sum_{n=1}^{\infty} h_n z^{-n} = \frac{P(z)}{Q(z)}, \tag{157}$$

where $P(z)$ and $Q(z)$ are coprime polynomials satisfying

$$\text{degree}(P) < \text{degree}(Q) = M, \tag{158}$$

with the infinite Hankel matrix

$$\Gamma_H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_2 & h_3 & \iddots \\ h_3 & \iddots & \iddots \\ \iddots & \iddots & \end{bmatrix}, \tag{159}$$

we include in this Appendix a proof of Kronecker's theorem given as Theorem 1 on page 2. To be consistent with the notation in (6), let

$$\gamma_i = (h_i, h_{i+1}, \ldots) \tag{160}$$

denote the i^{th} column vector of Γ_H. We also need the notation

$$H_M^i = \begin{bmatrix} h_i & \ldots & h_{i+M-1} \\ \vdots & \iddots & \vdots \\ h_{i+M-1} & \ldots & h_{i+2M-2} \end{bmatrix} \tag{161}$$

for the M-dimensional cofactors of Γ_H with leading entry h_i. Then

$$H_M^1 = H_M \tag{162}$$

is the principle cofactor of Γ_H of dimension M introduced in (7) and (91). We have the following preliminary result.

Lemma 1. *The infinite Hankel matrix Γ_H has finite rank $= k$ if and only if the first k column vectors $\gamma_1, \ldots, \gamma_k$ of Γ_H are linearly independent and there exist k numbers $c_1, \ldots, c_k$ such that*

$$\gamma_{k+\ell} = \sum_{i=1}^{k} c_i \gamma_{i+\ell-1}, \quad \ell = 1, 2, \ldots \,. \tag{163}$$

Furthermore, if rank $\Gamma_H = k < \infty$, then the principal cofactor $H_k^1 = H_k$ is a nonsingular square matrix.

Proof. By definition, rank $\Gamma_H < \infty$ if and only if Γ_H has only a finite number of linearly independent columns. Suppose rank $\Gamma_H < \infty$. Let $\gamma_1, \ldots, \gamma_r$ be linearly dependent; and let k be the largest integer, $1 \leq k < r$, such that $\gamma_1, \ldots, \gamma_k$ are linearly independent. Then since $\gamma_1, \ldots, \gamma_{k+1}$ are linearly dependent, there exist constants $c_1, \ldots, c_k$ such that

$$\gamma_{k+1} = \sum_{i=1}^{k} c_i \gamma_i \,. \tag{164}$$

This is (163) for $\ell = 1$. Note that both sides of (164) are infinite dimensional vectors. By deleting the first $(\ell - 1)$ entries of these vectors, (164) becomes (163) for $\ell = 2, 3, \ldots$. That is, each i^{th} column of Γ_H, for $i > k$, is a linear combination of its previous k columns; and hence, by using (163) repeatedly, each γ_i, $i > k$, is a linear combination of $\gamma_1, \ldots, \gamma_k$. So, for finite rank Γ_H, its rank is given by the largest k for which $\gamma_1, \ldots, \gamma_k$ are linearly independent. Of course, rank $\Gamma_H = \infty$ if and only if such a k does not exist. This proves the first statement in the lemma. Now, suppose that rank $\Gamma_H = k < \infty$. By repeated applications of (163), it is clear that every minor $\det H_k^i$, $i = 2, 3, \ldots$, is a constant multiple of the principle minor $\det H_k^1$. Recall that if rank $\Gamma_H = k$, then Γ_H has some k dimensional cofactor with nonzero determinant, (i.e. $\det H_k^i \neq 0$ for some i), so that $\det H_k = \det H_k^1 \neq 0$.

We are now ready to prove Theorem 1. Suppose that $H(z)$ in (157) satisfies (158). Then by dividing both the numerator and denominator by the leading coefficient of $Q(z)$, we may write

$$\begin{cases} P(z) = b_1 z^{M-1} + \cdots + b_M \\ Q(z) = z^M + a_1 z^{M-1} + \cdots + a_M \end{cases} \tag{165}$$

so that (157) is equivalent to

$$(h_1 z^{-1} + h_2 z^{-2} + \cdots)(z^M + a_1 z^{M-1} + \cdots + a_M) = b_1 z^{M-1} + \cdots + b_M. \tag{166}$$

Hence, by equating the coefficients of $z^{M-1}, \ldots, z, 1$, we have

$$\begin{cases} b_1 = h_1 \\ b_2 = h_2 + h_1 a_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_M = h_M + h_{M-1} a_1 + \cdots + h_1 a_{M-1} \end{cases}; \tag{167}$$

and by equating the coefficients of $z^{-1}, z^{-2}, \ldots$, we have

$$\begin{cases} h_{M+1} + \sum_{i=1}^{M} a_i h_{M-i+1} = 0 \\ h_{M+2} + \sum_{i=1}^{M} a_i h_{M-i+2} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{168}$$

Now, if we define

$$c_i = -a_{M-i+1}, \quad i = 1, \ldots, M, \tag{169}$$

then we observe that (168) is equivalent to

$$\begin{aligned} \gamma_{M+1} &= \sum_{i=1}^{M} -a_i \gamma_{M-i+1} \\ &= \sum_{i=1}^{M} c_i \gamma_i \,, \end{aligned} \tag{170}$$

which, by the same argument as the proof of Lemma 1, yields

$$\gamma_{M+\ell} = \sum_{i=1}^{M} c_i \gamma_{i+\ell-1}, \quad \ell = 1, 2, \dots \,. \tag{171}$$

Since $P(z)$ and $Q(z)$ are coprime and the leading coefficient of $Q(z)$ is normalized to be 1, the set of coefficients $\{a_i\}$, and hence $\{c_i\}$, in (170) is unique. That is, $\gamma_1, \dots, \gamma_M$ are linearly independent. Hence, by Lemma 1, rank $\Gamma_H = M$.

Conversely, suppose that rank $\Gamma_H = M$. Then by Lemma 1, we can find coefficients $c_1, \dots, c_M$, such that (171) holds. Hence, defining $a_1, \dots, a_M$ and $b_1, \dots, b_M$ by (169) and (167), respectively, we have both (167) and (168), which yields (166); or equivalently, we have $H(z) = P(z)/Q(z)$ by using (165) to define $P(z)$ and $Q(z)$. The linear independence of $\gamma_1, \dots, \gamma_M$ is equivalent to the uniqueness of $\{a_i\}$ and $\{b_i\}$, $i = 1, \dots, M$, which, in turn, implies that $P(z)$ and $Q(z)$ are coprime.