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AFOSR Grant No. AFOSR-87-0368

Final Report

October, 1990

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1. Introduction.

Of primary interest in this project are systems that are modelled as beams or thin plates which may undergo either small or large deformations. Hence, linear, so called Kirchhoff, and nonlinear, so called von Karman, are amoung the model equations to be studied. Linear models are derived under assumptions of pure bending with deformations that are small compared to thickness which in turn is assumed to be small. Larger deformations give rise to in plane forces from the stretching of the middle surface and lead to nonlinear models, [22,23]. For beams linear models with various linear and nonlinear damping terms are among those of interest.

An identification problem seeks to determine parameters within a mathematical model from observed data. The central issue is how to utilize data to determine the desired parameters with in the context of the model. Thus, available is a model equation

$$L(q)u(q) = f(q) \tag{1.1}$$

in which the parameters q to be estimated belong to a specified admissible set Q_{ad} of a Banach or Hilbert space Q. The set Q_{ad} should be physically meaningful, such that the mapping $q \mapsto u(q)$ is well-defined from the state space X, and such that $q \mapsto u(q)$ is continuous with respect to suitable topologies on Q and X. Available are data z which are in the form of measurements on the system. We view these data as belonging to an observation space Z (Hilbert space). One seeks to determine a parameter q such that Cu(q) "matches" z in a suitable sense. Here C represents an observation operator that maps X into Z.

One approach to this problem, sometimes called the "regularized" output least squares method, cf[1-6,9-19,25-35], is formulated as a minimization problem:

Find $q_0 \in Q_{ad}$ such that

(1.2) $$J(q_0) = \text{infimum}\{J(q): q \in Q_{ad}\}$$

where

(1.3) $$J(q) = \|Cu(q) - z\|_Z^2 + \beta \|q\|_Q^2$$

with $\beta \geq 0$. This formulation is given with Q and Z Hilbert spaces. Solving this problem often requires constraints on Q_{ad} to provide (i) existence of a solution to the state equation (1.1) and (ii) sufficient compactness to obtain existence of a solution to (1.2)-(1.3). Issues of interest for (1.2)-(1.3) include existence, regularity of the solution, approximation, stability with respect to data or constraints, and uniqueness or identifiabilty.

If one expects to use estimation techniques to identify parameters from measurements, then one must first decide upon the basic form of the mathematical model (1.1). Hence, one must decide whether the mathematical model is appropriate for the physical system being observed. The mathematical model embodies the pertinent principles and assumptions from physics and continuum mechanics in addition to those made on the basis of geometry, etc. Certainly, one must be cautious about using models to fit data that do not satisfy the assumptions upon which the models are based.

Having chosen the mathematical model, one must address how to use the data within the framework it imposes. The data for systems of interest in this work are assumed to be obtained as pointwise measurements of deformation, velocity, or acceleration or their Fourier transforms. These data typically should be processed in some manner and perhaps smoothed before their use. Also, data should be in a form consistent with the mathematical model having the proper units, etc. The mathematical analysis of identification problems such as (1.1)-(1.3) guides in how best to formulate estimation to use these data. Of direct interest to this analysis is to determine how to design and collect data to utilize desirable theoretical properties. Conversely if data is available from a system, an important issue is to determine how much one may reasonably expect to determine from the data.

In Section 2 we indicate results that we have obtained on the estimation of elastic coefficients in structures composed of beams. In Section 3 we report our results on the estimation of elastic parameters in nonlinear static models of thin plates undergoing large deformations. In Section 4 we consider the estimation of electrical conductivity from laboratory measurements of electrical potential. This is of interest since in this study we have available laboratory measurements with which to test our algorithms. Finally, in Section 5 we indicate some results on the stability of sets of optimal estimators with respect to data as viewed as set-valued mappings.

2. Estimation in a connected beam model.

During this project we have initiated an investigation concerning the estimation of coefficients or the design of beam systems. In particular, we have studied the following simple problem. Suppose two beams, 1 and 2, of length ℓ_1 and ℓ_2, respectively, are joined at right angles to one another. The other end of beam 1 is clamped and the remaining end of beam 2 is free. In response to a force that is perpendicular to the plane containing the beams are deformations w_1 and w_2 of beams 1 and 2, respectively, and a rotation θ of twist along beam 1. For the static problem we study the following system of elliptic equations with coupled boundary conditions in which x and y represent local coordinates for beams 1 and 2 respectively.

$$(a\ w_{xx})_{xx} = f_1 \text{ in } (0,\ \ell_1)$$

$$\text{(2.1)(i)} \qquad -(b\ \theta_x)_x = g \text{ in } (0,\ \ell_1)$$

$$(c\ w_{2yy})_{yy} = f_2 \text{ in } (0,\ \ell_2)$$

with boundary conditions

$$w_1(0) = w_{1x}(0) = 0$$

$$\theta(0) = 0$$

at the clamped end,

(2.1)(ii) $$w_{1xx}(\ell_1) = 0$$

(2.1)(iii) $$w_1(\ell_1) = w_2(0)$$

(2.1)(iv) $$\theta(\ell_1) = w_{2y}(0)$$

(2.1)(v) $$(b\,\theta_x)(\ell_1) = (c\,w_{2yy})(0)$$

(2.1)(vi) $$(a\,w_{1xx})_x(\ell_1) = (c\,w_{2yy})_y(0)$$

at the junction, and

$$w_{2yy}(\ell_2) = (c\,w_{2yy})_y(\ell_2) = 0$$

at the free end. Equations (2.1) are the Euler equations for the potential energy functional

(2.2) $$\mathbb{P}(w_1, \theta, w_2) = \int_0^{\ell_1} [a(x)\, w_{1xx}^2(x) + b(x)\, \theta_x^2(x)]\, dx + \int_0^{\ell_2} c(y)\, w_{2yy}^2(y)\, dy - 2\int_0^{\ell_1} [f_1(x)\, w_1(x) + g(x)\, \theta(x)]\, dx - 2\int_0^{\ell_2} f_2(y)\, w_2(y)\, dy.$$

The solution of (2.1) may be obtained by minimizing the functional (2.2) over the Hilbert space

$$V = \{v = (w_1,\theta,w_2) \in H^2(0,\ell_1) \times H^1(0,\ell_1) \times H^2(0,\ell_2) : w_1(0) = w_{1x}(0) = 0,\ \theta(0) = 0,\ \theta(\ell_1) = w_{2y}(0),\ w_1(\ell_1) = w_2(0)\}.$$

To solve (2.1) numerically, the space of basis functions should satisfy the conditions at the junction. However, it is much more difficult to enforce the essential boundary conditions at the junction on the basis elements that are used to construct the finite dimensional approximating space as one does for example with the clamped boundary conditions. We take two approaches to this problem. The first is to use basis functions that are non-conforming at the junction and hence do not satisfy the essential boundary conditions at the junction. The conditions are enforced as constraints on the discrete version of the problem to minimize the potential energy functional. We use a penalization or an augmented Lagrangian method to impose the conditions. The second approach involves directly changing variables by using the junction constraints to reduce the number of variables of the problem. In this method we solve for certain variables in terms of the others by using the junction conditions. In this way the minimization problem for the potential energy becomes an unconstrained problem in the new variables. In effect these new variables are coefficients of basis functions that satisfy the junction conditions.

We are confident at this point that our model is reasonable. This is based on experimental spectral data that we have obtained with the aid of Prof. D.L. Russell at MIPAC. Specifically, we measured the eigenfrequencies for an aluminum carpenter's square that was clamped at one end. We obtained the first six frequencies as

3.75, 10.62, 39.37, 75, 126.87, and 198.75 Hz.

Based on our model using a direct method for the treatment of the boundary condition at the junction, we obtained using a cubic spline based scheme frequencies of

2.28, 14.28, 39.69, 78.45, 129.93, and 194.82 Hz.

Considering experimental and discretization error, we feel that these numbers compare favorably. A new set of experiments will be run again this spring with improved clamping apparatus to obtain a new set of data. In addition to comparing with experimental data, we have also compared our results with the NISA engineering package for static deflection problems with point loads. Predictions for the model equations (2.1) agree to within 5 to 6 decimal places to those from the NISA package.

We have considered estimation problems for this structure. A report of results is in the Proceedings of the 1989 IEEE Conference on Decision and Control [33]. In that work we formulate a sequence of estimation problems where in each case the underlying system is one obtained from the augmented Lagrangian functional associated with the minimization of the potential energy functional. Hence, a sequence of estimation problems is defined on the approximating systems obtained from the augmented Lagrangian and penalty approximating scheme. A manuscript describing further results for beams at different angles is in preparation.

3. Estimation of parameters in plate models.

In work funded under this grant we studied the estimation of both damping and elastic parameters in linear equations that modeled small deformations of thin plates [25-33]. In this work we obtained results concerning the properties and regularity of solutions, the approximation of solutions, and the numerical treatment of sample problems. In addition we studied the stability of regularized output-least-squares optimal estimators with respect to data, second order sufficient conditions, and the error of solutions in terms of sampling density of the data [34].

We wish to indicate in more detail the investigations we have been conducting on model equations modelling thin plates but with large deformations. We describe our work on time independent problems. Currently, we are working on time dependent problems. Our approach currently is to use weak solutions. However, we are also considering classical solutions as in [21,24] with observations before blow-up times. In the derivation of approximate plate theories, linear mathematical models are obtained by including bending terms only. The retention of only these terms embodies the assumption that deformation is in fact small when compared to the thickness of the plate which in turn is assumed to be small. For larger deformations it is necessary to include terms that model the stretching of the middle plane [7,8]. These models give rise to the so called von Karman equations.

$$\left.\begin{aligned} Au &= \varepsilon[\phi, u] + f \\ B\phi &= -[u, u] \end{aligned}\right\} \text{ in } \Omega \tag{3.1}$$

where the bracket term is given by

$$[\varphi, \psi] = \varphi_{xx}\psi_{yy} + \varphi_{yy}\psi_{xx} - 2\varphi_{xy}\psi_{xy}.$$

The operators A and B are typically fourth order linear elliptic operators. The function u represents the deformation of the plate from an equilibium position and ϕ is commonly called the Airy stress function. Equations (3.1) are accompanied with boundary conditions, for example, homogenous Dirichlet boundary conditions

$$u = \frac{du}{dn} = \phi = \frac{d\phi}{dn} = 0 \text{ on } \Gamma.$$

We have recently considered the estimation of the parameter a =

$a(x,y)$ in the specific cases of

$$A\varphi = \Delta(a\,\Delta\varphi)$$

and

$$A\varphi = \nu\,(\Delta\,(a\,\Delta\varphi)) + $$
$$+\ (1-\nu)((a\varphi_{xx})_{xx} + (a\varphi_{yy})_{yy} + 2\,(a\varphi_{xy})_{xy})$$

with

$$B\varphi = \Delta^2\varphi.$$

Such operators arise with variable coefficients for thin plates of uniform thickness but with variable Young's modulus. It turns out that solutions u of (3.1) are stationary points of a quartic functional. Moreover, it is possible that equation (3.1) has more than one solution [7,8.20], and therefore the parameter to state mapping may not be well-defined. Even so we have demonstrated existence of an optimal parameter for the following regularized output-least squares problem.

(3.2) Find $a_0 \in Q_{ad}$ such that $J(a_0) = \inf\{J(a): a \in Q_{ad}\}$

where

$$J(a) = \|u(a) - z\|^2_{L^2(\Omega)} + \beta\,\|a\|^2_{H^2(\Omega)}$$

and

(3.3) $$Q_{ad} = \{a \in H^2(\Omega): a \geq \delta > 0\}.$$

If there may exist multiple solutions to the equation (3.1), the fit-to-data functional above is not well-defined. However, it can be shown that the solution set $U(a)$ of equation (3.1) is closed and compact in the weak topology of $H^2_0(\Omega)$. Since the embedding of $H^2_0(\Omega)$ into $L^2(\Omega)$ is compact when Ω is an open domain in $\mathbb{R}^2$ with a Lipschitz boundary, we denote by

$$\|u(a) - z\|_{L^2(\Omega)} = \min\{\|u - z\|_{L^2(\Omega)}: u \in U(a) \text{ for } a \in Q_{ad}\}.$$

The minimimum exists by the continuity of the L^2-norm with respect to the weak topology on $H^2_0(\Omega)$. It can be shown that there exists a solution to the problem (3.2)-(3.3).

Let us set

$$\|\varphi\|_0 = \|\varphi\|_{L^2} \text{ and } \|\varphi\| = \|\Delta\varphi\|_0.$$

It is well-known that $\|\cdot\|$ thus defined is a norm on $H^2_0(\Omega)$. Moreover, there is a constant k such that for any $\varphi \in H^2_0(\Omega)$, $\|\varphi\|_0 \leq k\,\|\varphi\|$. In addition there is a constant K such that for u, v, and w in $H_0^2(\Omega)$

$$\left| \int_\Omega [u, v]\, w\, dx \right| \le K \|u\| \|v\| \|w\|.$$

Set $\rho = (k \|f\|_0)/\nu$ and $r = \varepsilon K^2/\nu$.

It is natural to consider linear approximations to (3.1). By imposing conditions on ε, ν, and f we may obtain by the contraction mapping principle approximating linear systems for which convergence may be established. Based on this approach we obtain.

Theorem. Let M be chosen such that $\frac{2M}{3} < \rho < M$ and assume r satisfies $r < (M - \rho)/(M^3)$. Then for any $a \in Q_{ad}$ there is a unique solution u(a) of (3.1) with $\|u(a)\| \le M$.

In [31] we consider the following approximating linear systems.

(3.4) $$u_{-1} = \phi_0 = 0$$

for i = 0, 1,...

$$\Delta(a\, \Delta u_i) = \varepsilon\, [\phi_i, u_{i-1}] + f \text{ in } \Omega$$

$$u_i = \frac{du_i}{dn} = 0 \text{ on } \Gamma$$

(3.5) $$\Delta^2 \phi_{i+1} = -[u_i, u_i] \text{ in } \Omega$$

$$\phi_i = \frac{d\phi_i}{dn} = 0 \text{ on } \Gamma.$$

Theorem. With M, ρ, and r defined as above, the sequence u_i generated from (3.4)-(3.5) converges uniformly to u(a) in $H_0^2(\Omega)$ for $a \in Q_{ad}$ where $\|u(a)\| \le M$ and

$$\|u_i - u(a)\| \le M(3\, r^2 M)^i/(1-3r^2 M).$$

The assumptions that $2M/3 < \rho$ and $r < (M - \rho)/M^3$ imply that $3r^2M < 1$. Hence, if we consider the mapping $a \mapsto u(a)$ of Q_{ad} into the ball $\{u \in H_0^2(\Omega): \|u\| \le M\}$, then it is well-defined. We may demonstrate differentiability of solutions of (3.1) with respect to a under the assumption that $3r^2M < 1$. Therefore, regularity results for optimal estimators may be obtained under the above conditions.

Theorem. Let $3r^2M < 1$. If a is a solution of (3.2)-(3.3), then $a \in H^{2+\alpha}(\Omega)$ for $\alpha \in (0,1)$.

These results enable us to provide an approximation theory for these problems.

We base an estimation algorithm on the above approximating problems in which we estimate the parameter a in each linear approximating system and use the computed state for the update in the next linear problem. Details and the results of numerical experiments are reported in [31].

The above formulation imposes rather strong conditions on f, Ξ and ν in order to apply the contraction mapping princple. These conditions also imply differentiability of the mapping $a \mapsto u(a)$. We note however that existence of optimal estimators does not depend upon the these conditions. We consider the following weak model error formulation. Define $w : H^2(\Omega) \times V \mapsto V$ as follows

(3.6) $$\Delta^2 w = \Delta(a \Delta u) - \Xi [\phi, u] - f$$
$$\Delta^2 \phi = - [u, u] \qquad \text{in } \Omega$$

with boundary conditions

$$w = \frac{dw}{dn} = 0 \text{ on } \Gamma$$

It is easy to see that

(3.7) $$w(a,u) = 0$$

if and only if equation (3.7) is satisfied. We can show that the equation (3.7) implicitly defines a function $a \mapsto u(a)$ if

(3.8) $$\nu - K \left(\frac{\Xi}{\nu}\right)^{\frac{1}{2}} \|f\|_{H^{-2}(\Omega)} > 0$$

and this function is differentiable. Thus, we may weaken conditions to obtain local existence and local differentiability of the parameter-to-state mapping.

The optimization problem in this formulation is given as Find (a_0,u_0) such that

$$J(a_0,u_0) = \inf\{J(a,u): a \in Q_{ad} \text{ and } w(a,u) = 0\}$$

where

$$J(a,u) = \|u - z\|^2 + \beta \|a\|^2_{H^2}.$$

Again under the condition (3.8), we may obtain the existence of Lagrange multipliers associated with the constraint (3.3). This Lagrange multiplier may be estimated in terms of $\|u_0-z\|_V$. Thus we may find conditions for the choice of β and the positive definiteness of the second derivative of the Lagrangian functional. With these estimates we may obtain local uniqueness and stability results and show that the augmented Lagrangian method converges [32]. These results have been submitted for publication. They have also been presented at the Midwest Differential Equations Conference in November 1989.

Without further assumptions such as those in (3.7) we may penalize the constraint (3.6). This general formulation is given by means of the minimization problem

$$\text{Minimize } \|u - z\|_V^2 + \beta \|a\|_{H^2}^2 + \frac{K}{2} \|w(a,u)\|_{H^2}^2.$$

We have analyzed this problem and have presented initial results in a colloquium at the Division of Applied Mathematics at Brown University last spring.

We include some preliminary results of numerical tests using both the penalty method and the augmented Lagrangian for this problem. In this case we consider the equation

$$\Delta^2 w(a,u) = \Delta(a \Delta u) + \varepsilon [B(u,u),u] - f$$

with $w = w(a,u) \in H_0^2(\Omega)$ and

$$\Delta^2 B(u,u) = [u, u].$$

We consider the minimization problem:

$$\text{Minimize } L(a,u;\lambda) \text{ subject to } u \in V \text{ and } a \in H^2$$

for

$$L(a,u;\lambda) = \frac{1}{2} \|u - z\|_{L^2}^2 + \frac{\beta}{2} \|a\|_{H^2}^2 + (\lambda, w(a,u))_V + \frac{K}{2} \|w(a,u)\|_V^2$$

where $V = H_0^2$. We generate f by determining f from

$$f = \Delta (a_T \Delta u_T) + \varepsilon [B(u_T,u_T),u_T]$$

where $u_T = 256\ x^2\ y^2\ (1-x)^2\ (1-y)^2$. We used discretizations with 8 subintervals in both x and y directions for approximating u and 3 subintervals in both x and y directions for approximating a. Accordingly, the mesh for the example that we give is very coarse. Approximating functions were tensor products of cubic B-splines adjusted for boundary conditions. We used the conjugate gradient method for the optimization steps. Our initial guesses were off by relative L^2 errors of 87% and 45% for u and a, respectively. Below we give values of the exact and computed functions for the state u and the coefficient a. As an example consider a problem with a discontinuous elliptic coefficient

$$a_T = \begin{cases} 1.5, & (x,y) \text{ such that } 0.25 \leq y \leq 0.75 \\ 1.0, & \text{otherwise} \end{cases}$$

with initial relative L^2 errors of 79.4% and 231%, for u and a, respectively. After 19 iterations we obtained the following with the augmented Lagrangian method.

	(0,0)	(0.2,0.2)	(0.4,0.4)	(0.6,0.6)	(0.8,0.8)
u_T	0	0.168	0.849	0.849	0.168
u_{calc}	0	0.111	0.989	0.849	0.260

For the coefficient a we have

	(0,0)	(0.2,0.2)	(0.4,0.4)	(0.6,0.6)	(0.8,0.8)
a_T	1.00	1.00	1.50	1.50	1.00
a_{calc}	0.649	0.948	1.67	1.67	0.925

$$\|w(a,u)\|_{L^2} = 0.0884.$$

For the penalty method we have for the state

	(0,0)	(0.2,0.2)	(0.4,0.4)	(0.6,0.6)	(0.8,0.8)
u_T	0	0.168	0.849	0.849	0.168
u_{calc}	0	0.171	0.844	0.794	0.0893

For the coefficient we obtain the following.

	(0,0)	(0.2,0.2)	(0.4,0.4)	(0.6,0.6)	(0.8,0.8)
a_T	1.00	1.00	1.50	1.50	1.50
a_{calc}	0.533	0.603	1.43	1.51	0.619

$$\|w(a,u)\|_{L^2} = 0.140.$$

4. Estimation with experimental data.

In the previous proposal we expressed one goal as that of considering estimation problems using data that has been obtained experimentally. We feel that this is the essential next step in producing useful estimation algorithms. We report our efforts here for an experiment to obtain electrical conductivity from potential measurements in a circular reservoir. The situation is the same as that of determining elasticity coefficients for a membrane from displacement measurements. It is shown in [8] that such a membrane equation may be obtained as a limiting case of von Karman equations.

The experimental setup was as follows. A cylindrical tank with circular cross-section Ω is filled with sand in which there is a salt water solution. There are eleven locations in Ω at which a source electrode may be located or potential may be measured. There is a ground located at an additional position. The walls of the tank are nonconducting. The electrodes that are used are such that the current is focused so as to be two dimensional. This allows us to consider a two dimensional problem. At a certain point in time, a volume of salt water is extracted and replaced by injecting fresh water. Consequentially, the electical conductivity is a spacially dependent function. An electrical source of current is placed at a position and measurements are then made at the remaining locations. These measurements are repeated with the source at various locations. The diffusion of the salt into the fresh region is sufficiently slow as compared with the time it takes to make the measurements that the problem may be considered time independent.

The governing model is given by a two dimensional elliptic Neumann problem

$$- \nabla (a \nabla u) = f \text{ in } \Omega$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

where u is the electrical potential function, the coefficient a is the electric conductivity, and f is the source term. Here for the purpose of mathematical formulation we consider f to be given by

$$f = \mp \Phi_s$$

where Φ_s is a function of small support (in a ball centered at the source location x_s) such that

$$\int_\Omega \Phi_s(x)\, dx = 1.$$

Measurements of the electrical potential are made at N_{obs} locations over the domain Ω. Thus the observation operator C takes u into $\mathbb{R}^{N_{obs}}$ is given by

$$C\,u(a) = \mathrm{col}\{\langle u(a),\phi_i\rangle : i=1,\ldots,N_{obs}\}$$

where ϕ_i are continuous linear functionals on $H^1(\Omega)$(or possibly $H^2(\Omega)$ depending on the regularity). For example, we can take

$$\langle u,\phi_i\rangle = \frac{1}{mD_i}\int_{D_i} u(x)\,dx$$

where D_i is a disk centered at the observation location x_i to be an observation functional. The measurements are given as an N_{obs}-vector z of real numbers.

To treat the equation, we consider the weak model error $w = w(a,u)$ defined as the solution of

$$-\Delta w + w = -\nabla\cdot(a\nabla u) - f \text{ in } \Omega$$

with

boundary condition

$$\frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega.$$

The estimation problem is now given in terms of the optimization problem

Find $(a_0,u_0) \in Q \times H^1(\Omega)$ such that

$$J(a_0,u_0) = \inf\{J(a,u): a \in Q \text{ and } u \in H^1(\Omega) \text{ satisfy } w(a,u) = 0\}$$

where

$$J(a,u) = \gamma\,|Cu - z|^2 + \beta\,\|a\|_Q^2,$$

$Q = H^2(\Omega)$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{N_{obs}}$.

To solve this problem numerically, we approximate the disk centered at the origin with quadrilaterals obtained by partitioning the y-direction with an odd number of N_y equally spaced horizontal lines where the $(N_y - 1)/2 + 1$ is line $y = 0$. Each horizontal segment we partitioned into N_x subintervals. Connecting the corresponding points forms the system of finite elements. This discretization may be mapped to a square region that is partitioned similarly by defining transformations that map each subquadrilateral to the corresponding subrectangle. We use a system of basis functions that are bicubic functions made up of tensor products of cubic B-splines as basis functions for the square. The composition of the transformations with these functions then determines basis functions on the disk. We used 64 basis functions to approximate the state and 49 to approximate the coefficient a.

The estimation problem is solved by means of the augmented Lagrangian method. Thus we form the functional

$$F(a,u;\lambda,K) = J(a,u) + \langle\lambda, w(a,u)\rangle + K \|w(a,u)\|^2_{H^1}$$

and use the conjugate gradient method for minimization at each step. We use γ = 1.0, $\varepsilon = 10^{-8}$, and K = 10.0. The coefficient a is calculated for two different sets of data to compare the estimated a's. We obtain the following.

CASE 1.

x	y	values of a (est)
-53.2	-109.8	2.87
-17.7	-109.8	3.09
17.7	-109.8	2.99
-116.4	-36.6	2.83
-38.8	-36.6	3.04
38.8	-36.6	2.82
-116.4	36.6	2.48
-38.8	36.6	2.80
38.8	36.6	2.75

The relative error for $u = \mathrm{sqrt}\left(\dfrac{\sum_{i=1}^{N_{obs}} (u(x_i,y_i) - z_i)^2}{\sum_{i=1}^{N_{obs}} z_i^2} \right)$

is 4.9% while $\|w\|_{H^1}$ =0.071. In the second case we have the following

CASE 2.

x	y	values of a (est)
-53.2	-109.8	2.91
-17.7	-109.8	3.11
17.7	-109.8	3.00
-116.4	-36.6	2.87
-38.8	-36.6	3.05
38.8	-36.6	2.83
-116.4	36.6	2.51
-38.8	36.6	2.82
38.8	36.6	2.78

with relative error for u = 4.4% while $\|w\|_{H^1}$ =0.038. Thus, we are able to estimate the coefficient a so that, for different sets of data, the values of the estimated function agree to within the accuracy of the data. We feel that this is as much as one can reasonably expect and is a successful application of our methods. This work will appear as an invited paper in a special issue on parameter identification in the Journal on Advances in Water Resouces [35].

5. Stability properties with respect to data.

We have studied stability properties of optimal estimators with respect to data. Except for results in [13] in which there is strong regularization and it is assumed that the data is in the attainable set solutions of output least squares estimation problems are not unique. In [9,31,34] conditions are given for regularization under which there is a certain degree of stability. These results do not require that the data belong to the attainable set and do not provide for uniqueness. We have thus begun a study of the continuity properties of the set of optimal estimators as a set-valued function. To be more specific, we study the following sample problem. Let Ω be a bounded open domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ. We consider the following sample boundary value problem

$$(5.1)\qquad -\nabla\cdot(a\nabla u) = f \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$

with $f \in W^{-1,2}(\Omega)$. The coefficient a belongs to $W^{k,2}(\Omega)$ for k=1,2 and satisfies

$$(5.2)\qquad 0 < \mu_0 \le a \le \mu_1 \text{ a.e. in } \Omega$$

For a fit-to-data functional we use

$$(5.3)\qquad J(a;z) = \|u(a) - z\|^2_{L^2} + \beta\, |a|^2_k$$

where we denote the dependence of u on a by u(a) and the dependence of J on z by $J(\cdot;z)$. The regularizing parameter β is required to be positive. Finally, $|\cdot|_k$ is the seminorm

$$(5.4)\qquad |\varphi|^2_k = \int_\Omega \sum_{0<|\alpha|\le k}^{n} |D^\alpha\varphi|^2\,dx.$$

The estimation problem is given as

$$(5.5)\qquad \text{Find } a_0(z) \in Q_{ad}$$
$$\text{such that } J(a_0(z);z) = \{J(a;z) : a \in Q_{ad}\}$$

where

$$Q_{ad} = \{a \in W^{k,2} : a \text{ satisfies } (5.2)\}.$$

There exists a solution to (5.5) and we denote the set of solutions to (5.5) by $Q(z)$ and study the continuity properties of the set-valued mapping $z \mapsto Q(z)$. We show that in general this mapping is upper semicontinuous with the weak topology on Q_{ad} and the strong L^2 topology on the domain space as a set-valued mapping in the following sense:

Given a neighborhood N of $Q(z_0)$ there exists a neighborhood M of z_0 such that $Q(M) \subseteq N$.

By introducing stronger regularization, for example with $n=2$ taking $k=2$ or as in the cases in [9,31] or taking Q_{ad} to be in a finite dimensional space one obtains the stability results of [9,31]. These results amount to weak lower semicontinuity of the mapping $z \mapsto Q(z)$ in the following sense:

Given an element $a_0 \in Q(z_0)$ and a neigh- borhood N of a_0 there exists a neighborhood M of z_0 such that if $z \in M$ then $Q(z) \cap N \neq \emptyset$. With the two results together we obtain that the mapping $z \mapsto Q(z)$ is continuous as a set-valued mapping with respect to the Hausdorff metric on the collection of closed sets in the weak topology on bounded sets of Q_{ad}.

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Invited talks during Grant Period

"Estimation of elastic parameters using transform data," MIPAC Seminar, Dept.of Mathematics, University of Wisconsin-Madison, October 1987.

"Estimation of damping coefficients in Kelvin-Voight models of plates," IEEE 1987 Conf. on Decision and Control.

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"Estimation of damping coefficients in beam and plate models," Technische Universitat, Graz, Austria, June 1988.

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"Estimation of a boundary heat transfer coefficient," Department of Aerospace and Mechanical Engineering, University of Oklahoma, October 1988.

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