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AN ALTERNATIVE FORM OF THE MODIFIED POINT-MASS EQUATION OF MOTION

JAMES W. BRADLEY

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James W. Bradley

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The point-mass equation of motion is usually given in the form of an explicit expression for the acceleration $\dot{\vec{U}}$. The modified point-mass equation is obtained by adding yaw-of-repose terms to the right-hand side. These new terms are functions of $\dot{\vec{U}}$, so that $\dot{\vec{U}}$ now occurs on both sides of the equation. In this report, the modified point-mass equation is rewritten in the form of an explicit expression for the acceleration.

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I. Introduction

Early in our attempt to create a program for fitting the modified point-mass (MPM) equation to radar data, we noted a peculiar feature of this equation in its present form: it is not given as an explicit expression for the highest order derivative (in this case, the acceleration vector $\dot{\vec{U}}$). Instead, it has the form

$$\dot{\vec{U}} = f(t, P, \vec{U}, \dot{\vec{U}}) \tag{1}$$

where P is a parameter set.

This feature of $\dot{\vec{U}}$ co-existing on both sides of (1) is not an intolerable burden when the aim is merely to solve the equation numerically, using a specified set of parameter values. Indeed, for a given set P, our Firing Tables Branch routinely solves (1) by a predictor-corrector technique ideally suited to handling that ubiquitous $\dot{\vec{U}}$ (although additional calculations are required at the start of the integration).

In our fitting problem, however, the trajectory is known and we want to determine the parameter values that yield the best fit (say, in some least-squares sense) to that observed trajectory. In order to use one of the packaged routines for accomplishing this curve fitting, we decided that it was–if not absolutely essential–then at least extremely desirable to work with an explicit expression for the acceleration.

In the hope that such a re-formulation may be of interest to a wider audience than the one enthralled by our ongoing trajectory fitting problem (the latter group being a very small one indeed), we decided to release the transformation as an independent result.

II. Modifying a Modified Equation

The point-mass equation of motion can be written in the form

$$\dot{\vec{U}} = -A\vec{V} + \vec{G} \tag{2}$$

where

$\vec{U}$ = projectile velocity with respect to the earth

$\vec{V} = \vec{U} - \vec{W}$ = projectile velocity with respect to the air

$\vec{W}$ = wind velocity with respect to the earth

$\vec{G}$ = the sum of the gravity and Coriolis accelerations

and where

$$A = \left(\frac{\rho S \ell}{2m} C_D\right)\left(\frac{V}{\ell}\right).$$

(All symbols are defined in the List of Symbols.)

The MPM equation - the modified version of Eq.(2) - is derived in Reference 1.[1] This MPM equation differs from the point-mass equation in that the modified version contains additional terms involving the yaw of repose, $\vec{\alpha}_e$. (The yaw of repose is the steady-state angle of attack resulting from the gravity-induced curvature of the trajectory.) In the expression given in Ref. 1 for $\vec{\alpha}_e$, we will ignore the Magnus moment coefficient; that is, we set

$$C_{M_{p\alpha}} = 0. \tag{3}$$

Under this minor restriction, the expression for $\vec{\alpha}_e$ [Eq.(3.3) of Ref. 1] reduces to

$$\vec{\alpha}_e = B\,\dot{\vec{U}} \times \vec{V} \tag{4}$$

where

$$B = \left(\frac{k_a^2}{\frac{\rho S \ell}{2m} C_{M_\alpha}}\right) \left(\frac{\dot{\phi}\ell}{V}\right) \left(\frac{\ell}{V^3}\right)$$

and where $\dot{\phi}$ is the axial spin rate.

Then the MPM equation [Eq.(3.1) of Ref. 1] can be written in the form

$$\dot{\vec{U}} = -A\vec{V} + \vec{G} + \frac{h_L \vec{\beta}}{V} - \frac{h_M \vec{\beta} \times \vec{V}}{V^2} \tag{5}$$

where

$$\vec{\beta} = \vec{\alpha}_e / B = \dot{\vec{U}} \times \vec{V}$$

and where h_L and h_M are dimensionless coefficients related to the lift and Magnus forces, respectively:

$$\begin{aligned} h_L &= k_a^2 \left(\frac{C_{L_\alpha}}{C_{M_\alpha}}\right) \left(\frac{\dot{\phi}\ell}{V}\right) \\ h_M &= k_a^2 \left(\frac{C_{N_{p\alpha}}}{C_{M_\alpha}}\right) \left(\frac{\dot{\phi}\ell}{V}\right)^2 . \end{aligned}$$

For spin-stabilized artillery rounds, MPM trajectories based on Eq.(5) – compared with point-mass trajectories – are in much closer agreement with complete 6DOF rigid-body computer runs.

Eq.(5) differs from the point-mass equation (2) not only in the addition of the h_L and h_M terms, but also in the fact that the drag coefficient C_D contained in coefficient A is now allowed to vary linearly with $|\vec{\alpha}_e|^2$:

$$C_D = C_{D_0} + C_{D_2} |\vec{\alpha}_e|^2. \tag{6}$$

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[1] "Equations of Motion for a Modified Point Mass Trajectory," Robert F. Lieske and Mary L. Reiter, US Army Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland, Report R1314, March 1966, AD 485869.

Thus we see that in the MPM equation (5), $\dot{\vec{U}}$ appears not only on the left-hand side but *in every term except* $\vec{G}$ *on the right-hand side.* The purpose of this report is to present the following explicit expression for $\dot{\vec{U}}$:

$$\boxed{\dot{\vec{U}} = (C - A)\vec{V} + \vec{G}_A} \tag{7}$$

where

$$\begin{aligned} C &= \frac{h_a(\vec{G} \bullet \vec{V})}{(1 + h_a)V^2} \\ \vec{G}_A &= \frac{1}{1 + h_a}\left[\vec{G} + \frac{h_L(\vec{G} \times \vec{V})}{(1 - h_M)V}\right] \\ h_a &= \frac{h_L^2}{1 - h_M} - h_M. \end{aligned}$$

The derivation of this form of the MPM equation is given in the Appendix.

The attentive reader may be uneasy about that coefficient A in Eq.(7). It does, after all, contain C_D, which in turn contains $|\vec{\alpha}_e|$, which would seem to involve $\dot{\vec{U}}$. However, a $\dot{\vec{U}}$-free expression for $\vec{\alpha}_e$ can now be obtained by substituting for $\dot{\vec{U}}$ from Eq.(7) in the definition of $\vec{\alpha}_e$, Eq.(4). We obtain

$$\boxed{\vec{\alpha}_e = B\vec{G}_A \times \vec{V}} \tag{8}$$

– an expression devoid of $\dot{\vec{U}}$.

III. Concluding Remarks

Note that the appearance of $\dot{\vec{U}}$ on the right-hand side of the original MPM equation (5) is, in a sense, illusory. If we use the re-defined yaw of repose from Eq.(8), then (5), like (7), represents an explicit expression for $\dot{\vec{U}}$.

The fact that the drag term vanishes in the cross-product $\dot{\vec{U}} \times V$ is what allows us to declare C_D a function of $\vec{\alpha}_e$ and still obtain an explicit expression for $\dot{\vec{U}}$. If an unkind ballistician insisted that one or both of the aerodynamic parameters h_L and h_M must depend on the yaw of repose then no such explicit expression could be obtained.

Appendix : Derivation of Eq.(7)

To obtain Eq.(7) of the text, we need a series of preliminary relationships. First, we note that Eq.(4) implies that $\vec{\alpha}_\epsilon$ – and hence $\vec{\beta}$ – are perpendicular to $\vec{V}$. Therefore

$$\vec{\beta}\bullet\vec{V} = 0. \tag{A1}$$

Next we note that by Eq.(5),

$$\dot{\vec{U}}\bullet\vec{V} = \left(-A\vec{V} + \vec{G} + \frac{h_L\vec{\beta}}{V} - \frac{h_M\vec{\beta}\times\vec{V}}{V^2}\right)\bullet\vec{V}.$$

But the h_L term above is zero by Eq.(A1) and the h_M term is zero by the triple scalar product rule. Hence

$$\dot{\vec{U}}\bullet\vec{V} = -AV^2 + \vec{G}\bullet\vec{V} \equiv h_b \tag{A2}$$

where the symbol h_b is introduced as a temporary convenience. Then

$$\begin{aligned}\vec{\beta}\times\vec{V} &= (\dot{\vec{U}}\times\vec{V})\times\vec{V}\\ &= h_b\vec{V} - V^2\dot{\vec{U}}\end{aligned} \tag{A3}$$

by the triple vector product rule. This same rule gives us

$$\begin{aligned}(\vec{\beta}\times\vec{V})\times\vec{V} &= (\vec{\beta}\bullet\vec{V})\vec{V} - V^2\vec{\beta}\\ &= -V^2\vec{\beta}.\end{aligned} \tag{A4}$$

We can now obtain a convenient expression for $\vec{\beta}$:

$$\begin{aligned}\vec{\beta} &= \dot{\vec{U}}\times\vec{V}\\ &= \left(-A\vec{V} + \vec{G} + \frac{h_L\vec{\beta}}{V} - \frac{h_M\vec{\beta}\times\vec{V}}{V^2}\right)\times\vec{V}\\ &= \vec{G}\times\vec{V} + \frac{h_L(\vec{\beta}\times\vec{V})}{V} - \frac{h_M(\vec{\beta}\times\vec{V})\times\vec{V}}{V^2}\end{aligned}$$

so that by Eqs.(A3) and (A4),

$$\vec{\beta} = \vec{G}\times\vec{V} + \frac{h_L(h_b\vec{V} - V^2\dot{\vec{U}})}{V} + h_M\vec{\beta}.$$

Combining the $\vec{\beta}$ terms, we obtain

$$\vec{\beta} = \frac{\vec{G}\times\vec{V} + (h_L h_b/V)\vec{V} - (h_L V)\dot{\vec{U}}}{1 - h_M}. \tag{A5}$$

Substituting (A3) and (A5) in Eq.(5) of the text, we obtain

$$\dot{\vec{U}} = -A\vec{V} + \vec{G} + \frac{h_L}{(1-h_M)V}\left[\vec{G}\times\vec{V} + \left(\frac{h_L h_b}{V}\right)\vec{V} - (h_L V)\dot{\vec{U}}\right] - \frac{h_M}{V^2}\left[h_b\vec{V} - V^2\dot{\vec{U}}\right].$$

Combining similar terms, we have

$$\left(1 + \frac{h_L^2}{1 - h_M} - h_M\right)\dot{\vec{U}} = \left[-A + \frac{h_b}{V^2}\left(\frac{h_L^2}{1 - h_M} - h_M\right)\right]\vec{V} + \vec{G} + \frac{h_L \vec{G} \times \vec{V}}{(1 - h_M)V}$$

or

$$(1 + h_a)\dot{\vec{U}} = \left(-A + \frac{h_a h_b}{V^2}\right)\vec{V} + \vec{G} + \frac{h_L \vec{G} \times \vec{V}}{(1 - h_M)V}. \tag{A6}$$

But from Eq.(A2),

$$\begin{aligned} -A + \frac{h_a h_b}{V^2} &= -A + h_a\left(-A + \frac{\vec{G} \bullet \vec{V}}{V^2}\right) \\ &= -(1 + h_a)A + \frac{h_a \vec{G} \bullet \vec{V}}{V^2}. \end{aligned} \tag{A7}$$

Substituting (A7) in (A6), we have

$$(1 + h_a)\dot{\vec{U}} = -(1 + h_a)A\vec{V} + \left(\frac{h_a \vec{G} \bullet \vec{V}}{V^2}\right)\vec{V} + \vec{G} + \frac{h_L \vec{G} \times \vec{V}}{(1 - h_M)V}.$$

from which Eq.(7) of the text follows at once.

List of Symbols

A	$\left(\frac{\rho S\ell}{2m}C_D\right)\left(\frac{V}{\ell}\right)$
B	$\left(\frac{k_a^2}{\frac{\rho S\ell}{2m}C_{M_\alpha}}\right)\left(\frac{\dot{\phi}\ell}{V}\right)\left(\frac{\ell}{V^3}\right)$
C	$\frac{h_a(\vec{G}\bullet\vec{V})}{(1+h_a)V^2}$
C_D	drag coefficient, $\lvert drag\ force\rvert/(\rho V^2 S/2)$
C_{Do}, C_{D2}	zero-yaw and yaw-drag coefficients: $C_D = C_{Do} + C_{D2}\lvert\vec{\alpha}_e\rvert^2$
C_{ℓ_p}	roll damping moment coefficient: $\lvert roll\ damping\ moment\rvert = \pm(\rho V^2 S\ell/2)(\dot{\phi}\ell/V)C_{\ell_p}$
C_{L_α}	lift force coefficient: $\lvert lift\ force\rvert = \pm(\rho V^2 S/2)\lvert\vec{\alpha}_e\rvert C_{L_\alpha}$
$C_{M_{p\alpha}}$	Magnus moment coefficient: $\lvert Magnus\ moment\rvert = \pm(\rho V^2 S\ell/2)(\dot{\phi}\ell/V)\lvert\vec{\alpha}_e\rvert C_{M_{p\alpha}}$
C_{M_α}	static moment coefficient: $\lvert static\ moment\rvert = \pm(\rho V^2 S\ell/2)\lvert\vec{\alpha}_e\rvert C_{M_\alpha}$
$C_{N_{p\alpha}}$	Magnus force coefficient: $\lvert Magnus\ force\rvert = \pm(\rho V^2 S/2)(\dot{\phi}\ell/V)\lvert\vec{\alpha}_e\rvert C_{N_{p\alpha}}$
$\vec{G}$	the sum of gravity and Coriolis accelerations
$\vec{G}_A$	$\frac{1}{1+h_a}\left[\vec{G} + \frac{h_L(\vec{G}\times\vec{V})}{(1-h_M)V}\right]$
h_a	$\frac{h_l^2}{1-h_M} - h_M$
h_b	$\dot{\vec{U}}\bullet\vec{V} = -AV^2 + \vec{G}\bullet\vec{V}$
h_l	$k_a^2\left(\frac{C_{L\alpha}}{C_{M_\alpha}}\right)\left(\frac{\dot{\phi}\ell}{V}\right)$
h_M	$k_a^2\left(\frac{C_{N_{p\alpha}}}{C_{M_\alpha}}\right)\left(\frac{\phi\ell}{V}\right)^2$
I_x	axial moment of inertia

k_a^2	$I_x/m\ell^2$
ℓ	reference length
m	projectile mass
MPM	Modified Point Mass
R	effective radius of the earth (6 356 766 m)
S	reference area, $\pi\ell^2/4$
t	time
$\vec{U}$	projectile velocity with respect to the earth
V	$\vert\vec{V}\vert$
$\vec{V}$	$\vec{U} - \vec{W}$, projectile velocity with respect to the air
$\vec{W}$	wind velocity with respect to the earth
$\vec{\alpha}_e$	the yaw of repose, $B\,\dot{\vec{U}} \times \vec{V} = B\,\vec{G}_A \times \vec{V}$
$\vec{\beta}$	$\vec{\alpha}_e/B$
ρ	air density
$\dot{\phi}$	axial spin rate
$\dot{(\ )}$	$d(\ )/dt$

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