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The list of references in this report cites all our major contributions to the literature of rotational mechanics. Two papers in particular—"Rotational Elastic Dynamics" and "Equilibrium Mechanics of Rotating Systems" form the principal basis for the present report. Indeed, these papers are reproduced here with occasional remarks and comments inserted regarding very recent developments in the field.

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THE NONLINEAR CONTROL THEORY OF COMPLEX MECHANICAL SYSTEMS

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ABSTRACT

This report summarizes a body of research dealing with the nonlinear control theory of complex mechanical systems. The principal focus is on the dynamics of rotating systems with uncontrolled degrees of freedom, and we treat specific model problems of pointing, shape, and orientation control of complex spacecraft in a zero gravity environment. We also examine the dynamics of rotating kinematic chains.

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THE NONLINEAR CONTROL THEORY OF COMPLEX MECHANICAL SYSTEMS

Contents:

ROTATIONAL ELASTIC DYNAMICS

by

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Abstract

The combined dynamical effects of elasticity and a rotating reference frame are explored for structures in a zero gravity environment. A simple yet general approach to modeling is presented, and this approach is applied to analyze in detail the dynamics of a specific prototypical structure. Energy dissipation is included and its effects are studied in detail in a model problem. Bifurcations and stability are analyzed as well.

1. Introduction

There is now a fairly general awareness among aerospace engineers that the dynamics of the complex spacecraft currently in production and on the drawing boards will be greatly influenced by continuum mechanical effects such as elasticity. Indeed as the designs being contemplated increase in size and complexity ([13]) the dynamic effects of flexible members become more important, and recognizing this, many researchers over the last decade have focused their efforts on obtaining new methods for the design, analysis, and control of flexible mechanisms. Space does not permit (nor would it be in keeping with the main purpose of this report) to survey the vast literature on flexible space structures; the interested reader can get some idea of research activity in this area by referring to any one of a number of collections of papers and conference proceedings, such as [13] or the more recent volume [11].

The purpose of this report is to describe recent research which has been aimed at developing a mathematical theory of the rotational dynamics of complex mechanical systems which include articulated and elastic components. Our objective in this research has been to carry out a study of the global qualitative dynamics of such systems in sufficient depth as to allow predictions regarding the stability and asymptotic behavior of spacecraft due to a variety of energy dissipation mechanisms such as viscoelastic material damping of vibrations of elastic parts. We believe that historical evidence points to the value of developing a fairly complete global asymptotic stability theory of this type since there are numerous examples of missions in space which did not achieve their stated objectives because certain long term mechanical effects were never adequately taken into account in the mission planning. Explorer I, the first successfully launched American satellite, provides the best known example of such untoward behavior. Upon achieving earth orbit, the pencil-shaped satellite was supposed to rotate about its major axis of inertia. Before it had completed one orbital revolution, however, radio signals indicated that a tumbling motion had developed and was increasing in amplitude.

While explanations of Explorer's errant behavior have been offered by a number of researchers (see e.g. references to this problem in [7]), we are aware of no attemps to obtain a rigorous mathematical analysis of such occurences, with the exception of [3], where some of the results of the present report were announced. We also mention a recent paper [8] which deals with a somewhat different, although related aspect of a similar problem and several more recent efforts reported in [11] which develop results similar to those described in Section 5 below.

Our report is organized as follows. In Section 2 we derive equations describing the rotational dynamics of complex structures. These include the general effects of inertial forces created by rotation of the reference frame. Section 3 focuses on a general theoretical framework for Lagrangian mechanics with damping. In Section 4, a simple structure consisting of a rigid body with an elastic beam appendage is studied, and we present what we believe is the simplest reasonable continuum mechanical model of such a system undergoing three degree of freedom rotations. The asymptotic steady state dynamics for this system are studied in Section 5, and in Section 6 we present a detailed analysis of the qualitative dynamics of a closely related model having only one rotational degree of freedom. A second part of the report treats "Equilibrium Mechanics of Rotating Systems," dealing principally with rotating kinematic chains.

2. The rotational dynamics of complex structures.

In this section we will derive equations of motion for a class of structures consisting of elastic, fluid or rigid components. While these equations are completely general, they are most useful in describing any structure whose configuration is conveniently specified by the position of the structure with respect to a moving coordinate frame together with the position of that frame in space, i.e. relative to some fixed inertial system.

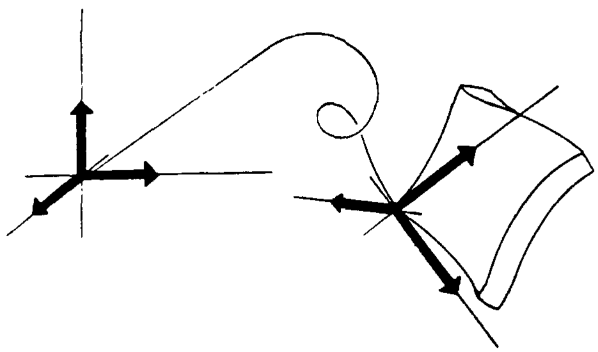

Figure 2.1: ***The body frame is translated and rotated with respect to the inertial frame.***

We fix an orthonormal basis forming an inertial frame ("space frame"), and choose a "body" coordinate system designated by a set of orthonormal vectors $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$. Choice of the $\mathbf{g}_i$'s will depend on the particular problem at hand, and it affects the simplicity of resulting equations. In section 4, where we treat an example of a rigid body with an elastic appendage, we affix the body frame $\mathbf{g}_i$ to the rigid body, although other choices are possible [6].

The position and orientation of the body frame (with respect to the chosen inertial frame) may be described at each time t by an element of the special Euclidean group,

$SE(3,\mathbf{R})$, represented by a 4x4 matrix

$$X(t) = \begin{pmatrix} Y(t) & y(t) \\ 0 & 1 \end{pmatrix}$$

(see figure 2.1), where $Y(t) \in SO(3)$ is an orthogonal matrix describing the orientation of the body frame and $y \in \mathbf{R}^3$ is the position of the origin of that frame in space. (cf. [5]).

Therefore, if a point P is given by the vector u in the body frame and by vector U in the space frame, then u and U are related via

$$\hat{U} = X\hat{u},$$

where $\hat{U} = \binom{U}{1}$ and $\hat{u} = \binom{u}{1}$.

The positions of various elements of our flexible structure relative to the body frame will be described by a vector function $u(z,t) = u(z_1, z_2, z_3, t)$ denoting the position at time t of each particle whose "unperturbed" position is at z. Here "unperturbed" can mean either initial or undeformed, depending on one's choice. In section 4, z will denote the neutral position of a particle of the flexible structure.

In summary, we consider systems whose configuration space is given by
$\{q\} = \{(Y,u,y)\} = SE(3) \times C \equiv SO(3) \times \mathbf{R}^3 \times C$, where $C = \{u(\cdot,t)\}$ is a suitably defined function space whose elements are functions $u(z)$ describing the configuration of the body relative to the body frame.

We describe now the kinematics of the system and give an expression for the kinetic energy. The evolution of the matrix $X(t) \in SE(3)$ can be described by a differential equation

$$\dot{X}(t) = \begin{pmatrix} Y(t)\Omega(t) & \dot{y}(t) \\ 0 & 0 \end{pmatrix},$$

where $\Omega(t)$ is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

of angular velocities ω_i about the corresponding body axes $\mathbf{g}_i$, $i = 1, 2, 3$.

The inertial coordinates $U(z,t)$ of a point P are related to its body coordinates $u(z,t)$ via

$$\hat{U}(z,t) = X\hat{u}(z,t)$$

and the corresponding velocities are given by

$$\begin{aligned} \frac{d}{dt}\hat{U} &= \dot{X}\hat{u} + X\hat{u}_t = \\ &= \begin{pmatrix} Y(\Omega u + u_t) + \dot{y} \\ 0 \end{pmatrix}, \end{aligned}$$

where u_t denotes the partial derivative with respect to t.

The kinetic energy of the body is then given by

$$T(q,\dot{q}) = \frac{1}{2}\int_B \|\dot{U}\|^2 dm = \frac{1}{2}\int_B \|Y(\Omega u + u_t) + \dot{y}\|^2 dm, \tag{2.1}$$

where B denotes the point set comprising the body (at time t) described in the body coordinate system, and dm is the mass distribution in this coordinate system.

The kinetic energy of almost any rotating structure will be of this form, as it does not depend on the constitutive relations governing the structure itself. More refined dynamical models, as treated in subsequent sections, will embody structural information in the expression for *potential* energy

$$V(q) = V(Y, u, y)$$

(which we assume to be independent of $\dot{q}$). Without specifying the form of V at this point, we prove the main theorem of this section.

Theorem 2.1. *Let A be defined by*

$$A(t) = \int_B ((\Omega u + u_t)u^T)_a dm,$$

where $M_a = \frac{1}{2}(M - M^T)$ *denotes the anti-symmetric part of a matrix* M. *Equations of motion of any system whose configuration space is* $\{q\} = \{(Y,u,y)\} = SO(3) \times C \times \mathbf{R}^3$ *as above are given by*

$$\dot{A}(t) + [\Omega(t), A(t)] - \int_B (Y^T \ddot{y} u^T)_a dm = \mathcal{T} - (Y^{-1} V_Y)_a \tag{2.2}$$

$$u_{tt} + \Omega^2 u + \dot{\Omega} u + 2\Omega u_t + Y^T \ddot{y} = F - \frac{\delta V}{\delta u} \tag{2.3}$$

$$\int_B Y(\Omega^2 u + \dot{\Omega} u + 2\Omega u_t + u_{tt}) + \ddot{y} dm = f - \frac{\partial V}{\partial y} \tag{2.4}$$

$$\dot{Y} = Y\Omega \tag{2.4$'$}$$

where V_Y *is the matrix of partial derivatives* $\frac{\partial V}{\partial Y_{ij}}$, *and* $\frac{\delta V}{\delta u}$ *is the Frechet derivative of* V *with respect to* u.

In the skew-symmetric matrix

$$\mathcal{T} = \begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0, \end{pmatrix}$$

τ_i *is the net nonconservative torque applied to the system about the body axis* $\mathbf{g}_i$, $F = F(z)$ *is the distributed nonconservative force density acting on the particle positioned at* $u(z,t)$ *expressed in the body coordinate system, and* f *is the net nonconservative exogenous force.*

Remark: In subsequent sections nonconservative forces will arise due to viscoelastic damping.

Before giving the proof, we rewrite these equations so as to provide a clearer picture of the physical situation. Let $\omega = (\omega_1, \omega_2, \omega_3)^T$ be the angular velocity of the body frame *expressed in that frame*; ω is related to the angular velocity matrix $\Omega = Y^{-1}\dot{Y}$ as follows:

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \ \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Denote by S the operator taking a skew-symmetric matrix Ω into vector $\omega : S\Omega = \omega$. One easily verifies the following:

Lemma 2.1. *Given any pair A, B of skew-symmetric 3×3 matrices and any pair u, v of 3-vectors, the following identities hold*

(i) $S([A,B]) = S(A) \times S(B)$

(ii) $S(uv^T - vu^T) = v \times u$

(iii) $Au = S(A) \times u$

Applying the operator S to equation (2.2) and using Lemma 2.1 on equations (2.2)-(2.4) we obtain

Corollary 2.1 *Equations of motion (2.2)-(2.4) are equivalent to*

$$\dot{a}(t) + \omega(t) \times a(t) + \int_B u(z,t) \times Y^{-1}\ddot{y}(t)dm = S(\mathcal{T} - (Y^{-1}V_Y)_a) \tag{2.5}$$

$$u_{tt} + \omega \times (\omega \times u) + \dot{\omega} \times u + 2\omega \times u_t + Y^{-1}\ddot{y} = F - \frac{\delta V}{\delta u} \tag{2.6}$$

$$\int_B [Y(\omega \times (\omega \times u) + \dot{\omega} \times u + 2\omega \times u_t + u_{tt}) + \ddot{y}]dm = f - \frac{\partial V}{\partial y} \tag{2.7}$$

where a is defined by

$$a(t) = \int_B u \times (u_t + \omega \times u)dm.$$

Written in this way equations (2.5)-(2.7) give an explicit description of the inertial forces on the mechanical system viewed in the moving body frame. Introducing the derivation $D = \frac{d}{dt}(\cdot) + \omega \times (\cdot)$, we obtain yet another rendering of these equations.

Corollary 2.2 *Equation of motion (2.2)-(2.4) are equivalent to*

$$\int_B u \times (D^2 u + Y^{-1}\ddot{y})dm = S(\mathcal{T} - (Y^{-1}V_Y)_a) \tag{2.5$'$}$$

$$D^2u + Y^{-1}\ddot{y} = F - \frac{\delta V}{\delta u} \tag{2.6$'$}$$

$$\int_B (YD^2u + \ddot{y})dm = f - \frac{\partial V}{\partial y}. \tag{2.7$'$}$$

The motions of any complex structure undergoing free or forced rotation are described by equations (2.5)-(2.7). This formulation is thus fairly general, and it can incorporate external forces and torques (due, for example, to gravitational and magnetic fields) and internal forces (due, say, to actuation of joints or the constitutive properties of the material). In the next section 3 we will incorporate dissipative effects in this formulation, and in section 4, we shall develop a complete dynamical model of a system where the constitutive relations are those of a simple damped beam.

Proof of theorem 2.1: The proof will be given in two parts: First, the treatment of the inclusion $Y \in SO(3)$ as the holonomic *constraint* onto $S0(3)$ as a submanifold of $GL(3)$, with an appropriate modification of the Lagrange equations (Lemma 2.2), and second, the utilization of the *left-invariance* of (the rigid path of) kinetic energy to simplify the resulting equations (Lemma 2.4).

We would like to write the equations of motion of the structure in the Lagrangian form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q},\ q = (Y, u, y).$$

An appropriate modification of the Lagrange equations is needed, however, to account for the fact that Y is constrained to the submanifold $S0(3) \subset Gl(3)$. Such a modification is unnecessary if the Lagrangian equations are expressed in terms of a local coordinate system on the constraint manifold–cf. [1], page 77. This approach ignores the symmetry in our system of equations, however. The following Lemmas 2.2 and 2.4 are key to our proof of Theorem 2.1

Lemma 2.2 *Any extremal $q(t)$ of the action $\int L(q,\dot{q})dt$ with q constrained to a submanifold M_0 of a Riemannian manifold M satisfies the differential equation*

$$\pi_q\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) = 0,$$

where π_q is the normal projection from T_qM onto T_qM_0.

We omit a straightforward proof. Corresponding to the three components of $q = (Y, u, y)$ we obtain three components for the equations of motion:

$$P_Y\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{Y}} - \frac{\partial T}{\partial Y} + \frac{\partial V}{\partial Y}\right) = 0, \tag{2.8}$$

$$\frac{d}{dt}\frac{\delta T}{\delta \dot{u}} - \frac{\delta T}{\delta u} = -\frac{\delta V}{\delta u}, \tag{2.9}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} = -\frac{\partial V}{\partial y}, \tag{2.10}$$

where P_Y is the orthogonal projection from $T_Y Gl(3)$ onto $T_Y SO(3)$ in the trace norm $\langle A, B\rangle = tr\ A^T B$. Here $\frac{\partial T}{\partial \dot{Y}}, \frac{\partial T}{\partial Y}$ are the derivatives with respect to the standard Riemannian structure on $TGL(3)$; they can be represented as matrices of partial derivatives: $\frac{\partial T}{\partial \dot{Y}} = (\frac{\partial T}{\partial \dot{Y}_{ij}})$, and $\frac{\partial T}{\partial Y} = (\frac{\partial T}{\partial Y_{ij}})$, $\frac{\delta T}{\delta u}$ denotes the Frechet derivative of T with respect to the distributed parameter u. The expression for the projection P_Y is provided by

Lemma 2.3. *For any $Y \in SO(3)$ and $A \subset T_Y Gl(3)$*

$$P_Y A = Y(Y^{-1}A)_a, \tag{2.11}$$

where $$X_a = \frac{1}{2}(X - X^T)$$

is the anti-symmetric part of X.

Proof. This follows from decomposing the Lie algebra of $gl(3)$ as the orthogonal direct sum of symmetric and skew symmetric matrices. ($gl(3) = so(3) \oplus so(3)^{\perp}$.) ∎

Lemma 2.4 *Suppose T is a left-invariant function on $TGl(n)$, i.e. suppose there is a function K defined on $gl(n)$ (= the Lie algebra of $Gl(n)$ = space of real $n \times n$ matrices) such that $T(Y, \dot{Y}) = K(\Omega)$ where $\Omega = Y^{-1}\dot{Y}$. Then for $(Y, \dot{Y}) \in TSO(n)$*

$$P_Y\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{Y}} - \frac{\partial T}{\partial Y}\right) = Y\left(\frac{d}{dt}M + [\Omega, M]\right)$$

where $M = (\frac{\partial K}{\partial \Omega})_a$, and $\frac{\partial K}{\partial \Omega}$ is the derivative of K with respect to Ω evaluated at $\Omega = Y^{-1}\dot{Y}$.

Remark. There is an orthogonal direct sum decomposition: $gl(n) = so(n) \oplus so(n)^{\perp}$, where $so(n)$ = the Lie algebra of $n \times n$ skew-symmetric metrices, and where orthogonality is defined in terms of the trace form inner product $[A, B] = tr\ AB^T$ defined on $gl(n)$. If the function K appearing in the statement of Lemma 2.4 can be decomposed as $K = K_1 + K_2$ where K_i depends only on the i-th component ($i = 1, 2$) in this orthogonal direct sum, then $M = K_1'(\Omega)$, where by K_1' we mean the derivative of K_1 with respect to the natural differentiable structure on $so(n)$ defined in terms of the Killing form.

Proof. $\frac{d}{dt}\frac{\partial T}{\partial \dot{Y}}$ and $\frac{\partial T}{\partial Y}$ may be thought of as elements in the cotangent bundle $T^*Gl(n)$. Making the usual identifications, the standard Riemannian structure on $Gl(n)$ may be prescribed explictly in terms of the trace form, and we may write $\frac{d}{dt}\frac{\partial T}{\partial \dot{Y}}$, $\frac{\partial T}{\partial Y}$ and $\frac{\partial K}{\partial \Omega}$ all as $n \times n$ matrices

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{Y}} = (Y^{-1})^T\left[\frac{d}{dt}(\frac{\partial K}{\partial \Omega}) - \Omega^T\frac{\partial K}{\partial \Omega}\right]$$

$$\frac{\partial T}{\partial Y} = -(Y^{-1})^T\Omega^T\frac{\partial K}{\partial \Omega}.$$

For $(Y, \dot{Y}) \in T\ SO(n)$, we have $(Y^{-1})^T = Y$ and $\Omega^T = -\Omega$, and the result follows from Lemma 2.3. ∎

The proof of Theorem 2.1 proceeds as follows. Using left-invariance of the first term in the expression for kinetic energy (cf. (2.1))

$$T = \frac{1}{2}\int_B \|\Omega u + u_t\|^2 + 2(\Omega u + u_t, Y^{-1}\dot{y}) + \|\dot{y}\|^2\ dm,$$

we obtain equation (2.2) as a consequence of Lemma 2.4. The remaining equations (2.3) and (2.4) follow by direct computation from equation (2.9) and (2.10). We omit the details.

3. Lagrangian Mechanics with Damping

A major advantage of Lagrangian versus Newtonian mechanics is the invariance of Lagrange's equations with respect to coordinate changes; it is this invariance that facilitates significantly the derivation of our equations of a motion. It is thus desirable to have the extension to the dissipative case. Such a modification is described in [9], [12]; we reproduce it here in a slightly more general form.

Let $D(q, \dot{q})$ be the so-called dissipation function defined as follows:

$$\dot{q} D_{\dot{q}} = \textit{rate of dissipation of energy per second.}$$

(One can think of $D_{\dot{q}}$ as the generalized dissipation force, and $\dot{q}$ =velocity. The above simply says: *velocity* $\cdot$ *force* = *power*).

Let $L(q, \dot{q})$ be the Lagrangian of the system. The equations governing the system are

$$\frac{d}{dt} L_{\dot{q}} - Lq + D_{\dot{q}} = 0. \tag{3.1}$$

Remark 3.1. If D is quadratic in $\dot{q}$ (as will be the case in the application presented in the next section), then

$$D = \frac{1}{2} \dot{q} D_{\dot{q}} = \frac{1}{2} (\text{rate of dissipation}).$$

Equations (3.1) are consistent with the definition of D, as shown by

Theorem 3.1. *If $E(q, \dot{q})$ is the total energy of the system, then*

$$\frac{d}{dt} E(q, \dot{q}) = -\dot{q} D_{\dot{q}}.$$

Proof. E is given by the Legendre transform* in $\dot{q}$ of L:

$$E = \dot{q} \frac{\partial L}{\partial \dot{q}} - L.$$

* Usually E is expressed as a function of $q, p = \frac{\partial L}{\partial \dot{q}}$; we keep $\dot{q}$ rather than p here.

Differentiation by time gives

$$\dot{E} = \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - L_q\dot{q} - L_{\dot{q}}\ddot{q} = -\dot{q}D_{\dot{q}};$$

we used equation (3.1) in the last step. ∎

Equation (3.1) has the same invariance property as the conservative Lagrange equations.

Theorem 3.2.[12] *The dissipative Lagrangian system (3.1) is invariant under the change of variables $q = q(Q)$. More precisely, if $q(t)$ satisfies (3.1), then $Q(t)$ satisfies equations of the same form:*

$$\frac{d}{dt}\mathcal{L}_{\dot{Q}} - \mathcal{L}_Q + \mathcal{D}_{\dot{Q}} = 0, \tag{3.1$'$}$$

where $\mathcal{L}(Q,\dot{Q}) = L(q(Q), q'(Q)\dot{Q})$ and $\mathcal{D}(Q,\dot{Q}) = D(q(Q), q'(Q)\dot{Q})$.

Proof: A simple calculation shows:

$$\frac{d}{dt}\mathcal{L}_{\dot{Q}} - \mathcal{L}_Q + \mathcal{D}_{\dot{Q}} = q'(Q)^T\left(\frac{d}{dt}L_{\dot{q}} - L_q + D_{\dot{q}}\right)$$

∎

4. A Rotating Rigid Body with a Beam Attachment

Consider the spacecraft depicted in Figure 4.1. The key features of this structure are a rigid body to which a flexible cantilevered beam-like appendage of length ℓ is attached.

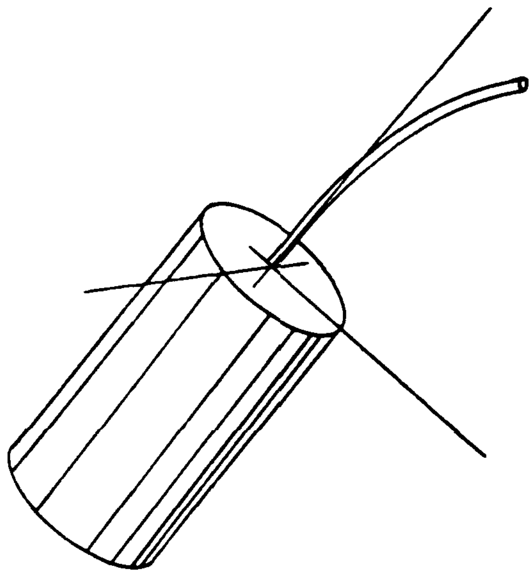

Figure 4.1: *A rigid body with cantilevered beam attachment.*

As we have mentioned in section 2, we affix the moving frame to the rigid body. More specifically, we place the origin of the frame at the point of attachment of the beam and align the $z_3 \equiv z$-axis along the undeflected beam. Viewing this cantilevered beam as essentially a one-dimensional object, we describe the elastic deformations $u(z_3, t) \equiv u(z, t)$ with respect to the coordinate axes $(z_1, z_2, z_3) \equiv (x, y, z)$ depicted in Figure 4.1. More precisely, $u(z, t)$ is the position of the particle whose neutral position is at $(0, 0, z)$. The decomposition of the system into the rigid part and the elastic beam corresponds to the decomposition of kinetic energy (2.1) into the sum of rotational and translational energies of the rigid part and the energy of the beam. (Note that $u(\cdot, t)$ restricted to the rigid component is just the identity mapping for all t.) We have

$$T = \frac{1}{2}\omega^T I\omega + \mathbf{m}_b \dot{y}^T Y \Omega \bar{c} + \frac{1}{2}\mathbf{m}_b \|\dot{y}\|^2$$

$$+\frac{1}{2}\int_0^\ell \|Y(\Omega u + u_t) + \dot{y}\|^2 dz$$

where $S(\Omega) = \omega$, $\bar{c}$ is the center of mass of the rigid body in the body frame, $\mathbf{m}_b$ is the mass of the rigid body, we have scaled the linear mass density of the beam to be one, and the inertia tensor with respect to the body frame is given by

$$I = \begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{pmatrix},$$

where I_x is the moment of inertia with respect to the x-axis, etc. (cf. [1]).

Since the beam is clamped at the origin of the (x, y, z)–*coordinate system* and free at its other end, the following boundary conditions are assumed:

$$u_i(0,t) = \frac{\partial u_i}{\partial z}(0,t) = \frac{\partial^2 u_i}{\partial z^2}(l,t) = \frac{\partial^3 u_i}{\partial z^3}(l,t) = 0,\ i = 1,2,$$

and $u_3(0,t) = \frac{\partial u_3}{\partial z}(\ell, t) = 0$. These boundary conditions are standard in the theory of clamped-free beams. Here u_1, u_2, u_3 are the deflections: $u = (u_1, u_2, z + u_3)$. Note that u_3 is *not* the z-coordinate of u.

The equations of motion for our rotating satellite are obtained by incorporating (2.5)-(2.7) into the formalism of Lagrangian mechanics, as discussed in the previous section. Thus we look for extremals of the Lagrangian $L = T - V$ with kinetic energy T given above and potential V given by the *strain energy*

$$V(u) = \frac{1}{2}\int_0^\ell [\mu_1(u_1'')^2 + \mu_2(u_2'')^2 + \mu_3(u_3')^2] dz, \quad ' = \frac{\partial}{\partial z}, \tag{4.2}$$

where only quadratic terms were retained and the material is assumed to obey a linear Hooke's law. Here μ_1 (respectively μ_2) gives the bending elasticity within the xz-plane (yz-plane respectively), and μ_3 is the Hooke's constant giving the beam's stretching elasticity. Unless the beam is abnormally thick, $\mu_3 >> \mu_1,\ \mu_2$. The dissipation function is given by

$$D = D(u, \dot{u}) = \frac{1}{2}\int_0^\ell k_1(\dot{u}_1'')^2 + k_2(\dot{u}_2'')^2 + k_3(\dot{u}_3')^2 dz, \quad \dot{} = \frac{\partial}{\partial t}, \quad ' = \frac{\partial}{\partial z}. \tag{4.3}$$

where $\dot{u}_1'', \dot{u}_2''$ can be thought of as the rates of change of appropriate curvatures, while $\dot{u}_3'$ is the rate of change of the contraction coefficient u_3'. The k_i's are positive constants reflecting the rates of energy dissipation due to deformation of material in the beam.

Remark 4.1. It might seem at a first glance that the first two terms in the integral (4.2) should be replaced by a quadratic form (in u_1'', u_2'')

$$a(u_1'')^2 + 2bu_1''u_2'' + c(u_2'')^2;$$

however, by properly turning the body coordinate system around the z-axis, we can diagonalize this form. In Figure 4.2, the x-axis is chosen along the direction in which the beam bends most easily ($\mu_1 \leq \mu_2$); consequently, the beam offers the stiffest resistance to bending within the yz-plane.

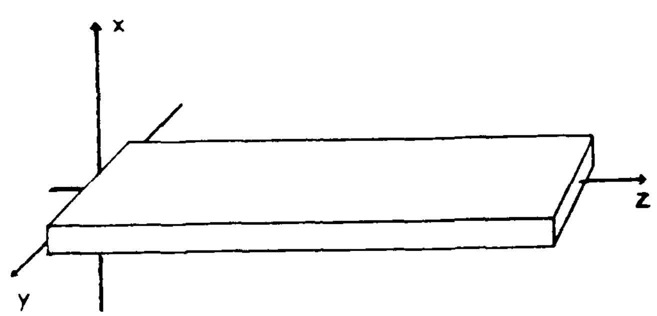

Figure 4.2: *A beam with elastic coefficients $\mu_1 < \mu_2$.*

Remark 4.2. We must point out that the beam was assumed to be of uniform crossection. Expression (4.2) would have to be modified to include the beams with variable crossections like the ones show in Figure 4.3.

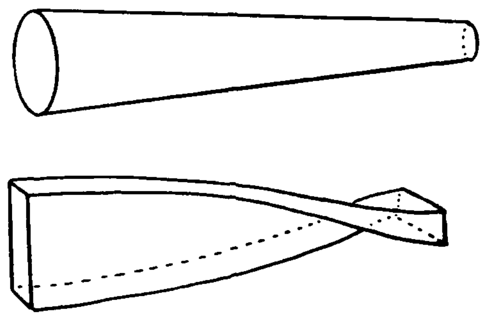

Figure 4.3: *Beams with variable crossection.*

The modification is in fact quite simple: one would only have to replace the first terms in (4.2) by a quadratic form with z-dependent coefficients. In the case of a helical beam we would take the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \equiv A \equiv A(z) = R^{-1} \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} R$$

where $R(z)$ is a rotation matrix $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ by z-dependent angle $\alpha = \text{const} \cdot z$

Remark 4.3. To incorporate torsional deformations of the beam, we introduce the torsion angle $\alpha(z;t)$, which is the angle formed between the x-axis and the projection onto the xy-plane of the segment rigidly connected to the beam so that in the unperturbed position of the beam it is attached at $(0,0,z)^T$ and is parallel to the x-axis. Thus the variable $\alpha(z;t)$ describes a normal bundle of the beam. Potential energy of the beam is given by

$$V = \frac{1}{2}\int_0^\ell (< R^{-1}\mu R v'', v'' > + \mu_3(u_3')^2 + \mu_4(\alpha')^2)dz,$$

where $\mu = diag(\mu_1, \mu_2)$, μ_4 is the torsional elasticity coefficient, and $v = (u_1, u_2)$.

A slightly more subtle remark regarding our model with potential energy defined by (4.2) is that it does not involve potential terms incorporating tensile forces into the model. Our model is thus somewhat different from what one would obtain using a so-called "geometrically exact" beam theory. It is perhaps surprising that the rotational equilibria described in the next section seem to differ little from what one would expect in the case of a geometrically exact model. We refer the reader to the recent paper [36] for a more complete discussion of the effects of constitutive restrictions on the qualitative dynamics of rotating systems.

The following theorem is a straightforward consequence of the results presented in the previous two sections.

Theorem 4.1. *Given the system depicted in Figure 4.1 and described above with kinetic energy (4.1), potential energy (4.2) and dissipation function (4.3), the equations of motion are given by*

$$Da + (\mathbf{m}_b \bar{c} + \int_0^\ell u) \times Y^{-1}\ddot{y} = 0 \tag{4.4}$$

$$D^2 u + \mu \partial u + k \partial \dot{u} + Y^{-1}\ddot{y} = 0 \tag{4.5}$$

$$\mathbf{m}_b (y + Y\bar{c})^{\cdot\cdot} + \int_0^\ell Y D^2 u \, dz = 0 \tag{4.6}$$

where the quantities in these equations are given as follows. $D(\ ) = \frac{d}{dt}(\ ) + \omega \times (\ )$, $a(t) = I\omega(t) + \int_0^\ell u \times (Du)dz$, *$I$ is the inertia tensor in the body frame defined above, $\mathbf{m}_b$ is the mass of the rigid body component,* $\dot{Y} = Y S^{-1}(\omega)$ *(with $S(\ \cdot)$ is as defined as in Lemma 2.1), and ∂ is the differential operator defined by* $\partial = (\partial_z^4, \partial_z^4, -\partial_z^2)$, $\mu = diag(\mu_1, \mu_2, \mu_3)$ and $k = diag(k_1, k_2, k_3)$.

Equations (4.4)-(4.6) are equivalent to

$$I\dot{\omega} + \omega \times I\omega + \int_0^{\ell} u \times [u_{tt} + \dot{\omega} \times u + 2\omega \times u_t + \omega \times (\omega \times u) + Y^{-1}\ddot{y}]dz \tag{4.7}$$

$$+\mathbf{m}_b\bar{c} \times Y^{-1}\ddot{y} = 0$$

$$u_{tt} + \dot{\omega} \times u + 2\omega \times u_t + \omega \times (\omega \times u) + \mu\partial u + k\partial u_t + Y^{-1}\ddot{y} = 0, \tag{4.8}$$

$$\mathbf{m}y + Y(\int_0^{\ell} u dz + \mathbf{m}_b\bar{c}) = c' + c''t, \tag{4.9}$$

where $\mathbf{m}_b$ denotes the mass of the rigid body component, and $\mathbf{m}$ denotes the total mass of the body-beam system. Since the mass density of the beam has been normalized to be 1, we find $\mathbf{m} = \ell + \mathbf{m}_b$. (4.7) can be further rewritten, using (4.8), as

$$I\dot{\omega} + \omega \times I\omega - \int_0^{\ell} u \times [\mu\partial u + c\partial\dot{u}]dz + \mathbf{m}_b\bar{c} \times Y^{-1}\ddot{y} = 0. \tag{4.7$'$}$$

One easily checks that the total angular momentum $\int (y+Yu) \times \frac{d}{dt}(y+Yu)dm$ is conserved, either using Noether's theorem or by direct computation.

Remark 4.4. If the center of mass of the system is at rest with respect to the space frame, we may assume that $c' = c'' = 0$.

We indicate the physical meaning of the terms in the equations (4.7)- (4.9). The sum of the first two terms in (4.7) is interpreted as the rate of change of the rigid body's angular momentum. The second term in the brackets gives the inertial force on the beam due to the body's angular acceleration, the third term is the Coriolis force, the fourth is the centrifugal force and the last term is the D'Alembert force. Thus equation (4.7) expresses the conervation of the total angular momentum in space expressed in the body's coordinate system. Equation (4.8) is just Newton's law for the beam expressed in the non-inertial body frame (the D'Alembert principle)—it accounts for various inertial forces.

Equation (4.9) expresses the conservation of the linear momentum of the whole system. It is important at this point to make the following remark.

Remark 4.5 There is an apparent paradox associated with eq. (4.7)′: one might expect to be able to express the integral term in terms of u and its derivatives at $z = 0$, since *the body feels the beam only through the attachment point*. As it turns out, this expectation is not met. In formulating our continuum mechanical model of the beam, we have neglected certain effects such as torsional deformations. The implicit rigidity in our model leads to this nonlocal character in the equation.

5. Asymptotic Dynamics of a Rotating Elastic Structure

In this section we begin an analysis of the asymptotic behavior of the body-beam structure described in the preceding section. For finite dimensional dissipative mechanical systems, LaSalle's invariance principle [10] can be used to show that states asymptotically approach a minimal invariant subset of the zero set of the dissipation function discussed in Section 3. In Section 6, it is shown that this type of analysis may be extended to certain infinite dimensional systems with features in common with our body-beam model described by equations (4.4)-(4.6). While we do not prove that all solutions to (4.4)-(4.6) tend to the zero set of the dissipation function D as defined by equation (4.3), we do offer a more or less complete characterization of the set of asymptotic equilibrium states (by which we mean the set of solutions to (4.4)-(4.6) for which the equality $D = 0$ also holds). We prove, in particular, that in asymptotic steady state the beam displacement function $u(\cdot\,,\cdot)$ does not depend on t. Moreover, it is shown that asymptotic equilibrium angular velocities are constant vectors parallel to the principal axes of the steady state inertia tensor. This means that the motion of the system is a pure rotation with no precession. This is the content of the following theorem.

Theorem 5.1: *Relative equilibrium solutions of (4.7)-(4.9) (by which we mean solutions which also satisfy $D = 0$ with D as defined in (4.3)) have the following properties:*

(i) there is no dependence on the time variable t in the beam function: $u(z,t) = u_\infty(z)$;

(ii) the angular velocity ω is a constant ω_∞;

(iii) the equilibrium angular momentum is a constant, $a_\infty = J_\infty \omega_\infty$, with the equilibrium inertia tensor of the combined body-beam system given by

$$J_\infty = I + \int_0^\ell u^T u E - u u^T dz - \mathbf{m}(C_m^T C_m E - C_m C_m^T) \tag{5.1}$$

where E = the identity matrix and $C_m = \frac{1}{\mathbf{m}}(\mathbf{m}_b\bar{c} + \int_0^\ell u dz)$ is the center of mass of the body-beam system (expressed in the body frame).

(iv) equilibrium rotations are aligned with a principal axis of the equilibrium inertia tensor, and thus

$$J_\infty \omega_\infty = \lambda \omega_\infty \tag{5.2}$$

Proof: To prove (i), note that for equilibria, $\int k_1(\dot{u}_1'')^2 + k_2(\dot{u}_2'')^2 + k_3(\dot{u}_3')^2 dz = 0$. Hence $\dot{u}_1'' = \dot{u}_2'' = \dot{u}_3' \equiv 0$, and the result follows as a consequence of the boundary conditions.

Using the time-independence of u, we show that corresponding values of ω are constant. Since $u_t \equiv 0$, equation (4.8) may be rewritten

$$\dot{\omega} \times u + \omega \times (\omega \times u) + \mu \partial u + Y^{-1}\ddot{y} = 0.$$

Differentiating with respect to z at $z = 0$ and using the boundary condition $\frac{\partial u}{\partial z} = \vec{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ at $t = 0$, we obtain

$$\dot{\omega} \times \vec{\mathbf{k}} + \omega \times (\omega \times \vec{\mathbf{k}}) + \mu \partial u_z(0) = 0$$

Denoting the last term by $(c_1, c_2, c_3)^T$, rewrite this as

$$\begin{pmatrix} \dot{\omega}_1 - \omega_2\omega_3 \\ \dot{\omega}_2 + \omega_1\omega_3 \\ \omega_1^2 + \omega_2^2 \end{pmatrix} = \begin{pmatrix} c_2 \\ -c_1 \\ c_3 \end{pmatrix}. \tag{5.3}$$

Multiplying the first two components by ω_1 and ω_2 respectively and adding we obtain

$$c_2\omega_1 - c_1\omega_2 = \dot{\omega}_1\omega_1 + \dot{\omega}_2\omega_2. \tag{5.4}$$

From the last component of (5.3) it follows that the right hand side is zero, and then (5.4) together with the last component of (5.3) implies ω_1 and ω_2 are constant, if c_1 and c_2 are not both zero.

To establish (ii) in the case that $c_1 = c_2 = 0$, we show that there is no possible choice of body coordinates for any rotating rigid body in which the relationships $\dot{\omega}_1 = \omega_2\omega_3$ and $\dot{\omega}_2 = -\omega_1\omega_3$ hold as in (5.3) unless all ω_i's are constant. We proceed by noting there is an orthogonal change of basis $x = B\omega$ ($B \in SO(3)$) such that the angular velocity vector x is expressed in a principal axis coordinate system with components satisfying the system of differential equations

$$\dot{x}_i = \alpha_i x_j x_k$$

where $j = \sigma(i)$, $k = \sigma^2(i)$, $i = 1,2,3$, and σ is any permutation on the three symbols $i = 1,2,3$. It may be shown that the definition of the coefficients α_i in terms of the principal moments of inertia implies $|\alpha_i| < 1$, and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$. The form of the equations $\dot{\omega}_1 = \omega_2\omega_3$, $\dot{\omega} = -\omega_1\omega_3$ imposes (polynomial) conditions on the α_i's and the entries of the orthogonal matrix B. A straightforward analysis shows that if C is any orthogonal matrix and if the transformed vector $y = C^T x$ has its first two entries obeying equations of the form $\dot{y}_1 = \beta_1 y_2 y_3$, $\dot{y}_2 = \beta_2 y_1 y_3$, then in fact C must be a permutation matrix (a nonsingular matrix in which each column has a single 1 and all other entries equal 0). The β_i's therefore must be equal, modulo a permutation, to the α_i's, and in particular we must have $|\beta_i| < 1$, which is not consistent with the form of the equations for ω_1, ω_2 unless all ω_i's are constant. This completes the proof of statement (ii) in the theorem.

(iii) follows from a direct computation using (4.4) and (4.6). It may also be obtained as a direct consequence of the parallel axis theorem.

(iv) follows since ω_∞ must satisfy $\omega_\infty \times J_\infty \omega_\infty = 0$. ∎

Corollary 5.1: *The equilibrium beam function $u_\infty(\cdot)$ and the equilibrium angular velocity ω_∞ are related by equations (5.1),(5.2) together with the fourth order system of* *ordinary differential equations*

$$\mu \partial u = -\Omega_\infty^2 (u - C_m), \tag{5.8}$$

where c_m *is the center of mass of the body-beam system in the body frame,* $u = (u_1, u_2, u_3 + z)$, and $S\Omega_\infty = \omega_\infty$, *and where the boundary conditions are as prescribed in Section 4.*

Proof: In light of Theorem 5.1, equation (4.8) may be rewritten as

$$\mu \partial u = -\Omega_\infty^2 u - Y^{-1} \ddot{y}.$$

From equation (4.9) (in which $c' = c'' = 0$) we see that $y(t) = -Y(t)C_m$. Moreover, since Y satisfies $\dot{Y}(t) = Y(t)\Omega_\infty$, we have that $Y(t) = Y_0 e^{\Omega_\infty t}$. Then $\ddot{y} = -Y(t)\Omega_\infty^2 C_m$ and $Y^{-1}\ddot{y} = -\Omega_\infty^2 C_m$, proving the Corollary.

Remark 5.1: (5.8) prescribes a nonlinear boundary value problem since Ω_∞ depends on $u_\infty(\cdot)$ through equations (5.1) and (5.2). Some idea of the complexity involved in explicitly determining $u_\infty(\cdot)$ may be gleaned from our solution to the model problem described by equations (6.1)-(6.3) in the following section.

Remark 5.2: A detailed analysis of the way in which the dissipative dynamical system (4.4)-(4.6) evolves toward steady state is beyond the scope of this report. Nevertheless, a heuristic description which makes contact with the classical theory of rigid bodies may be given as follows. For a rotating rigid body there are two well known conserved quantities: the kinetic energy E, and the magnitude of angular momentum $|M|$. If M represents the angular momentum vector with respect to the body frame, then the energy $E = \frac{1}{2} M^T I^{-1} M$ is a quadratic form in M with constant coefficients and the conservation of energy confines .

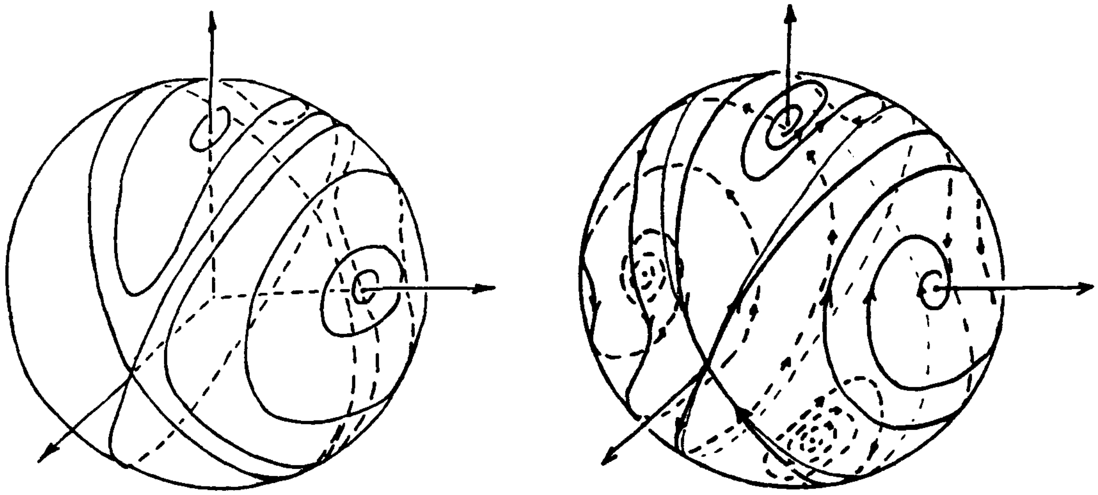

Figure 5.2: *Momentum spheres in the conservative and dissipative cases.*

M to an ellipsoid, resulting in figure 5.2a. If a dissipative elastic appendage is present then M, while still confined to a sphere, will move to smaller and smaller energy ellipsoids, as indicated in figure 5.2b. For the purpose of this heuristic discussion we ignore the infinite-dimensionality introduced by the beam.

It should be noted that the *basins of the two sinks S_1 and S_2 in figure 5.2b are interlaced*, and *it is difficult to predict which sink of M will appear if $M(0)$ is near the maximum energy point N*. This somewhat delicate phenomenon is said to have been overlooked in the design of the Explorer II mission. The satellite oriented itself along the proper axis, but in the direction opposite to the desired one.

It must also be noted that the picture in figure 5.2 is valid for moderate $|M|$ only, and it undergoes bifurcations as $|M|$ is increased. These bifurcations are due to buckling of the beam at higher angular velocities. Such a phenomenon is analyzed in complete detail in a model body-beam problem involving a 1 d.o.f. rotation in in the next section and in a model problem involving a mass-spring attachment to a spatially rotating body in [28].

6. Bifurcations, Stability and Dissipation in a Model Problem

In this section we illustrate two interesting phenomena in a model rotating body-beam system—bifurcations and the existence of finitely many predetermined angular velocities which are the asymptotic limits of all motions. We will see that the example considered below is an infinite-dimensional Morse system. It is similar to the general body-beam system considered in the preceding sections. We also point out an unexpected connection between this problem and that of studying bound states of a nonlinear Schrödinger equation, c.f. [15].

Figure 6.1 shows our model consisting of a disk with a beam attached to its center and perpendicular to the disk's plane.

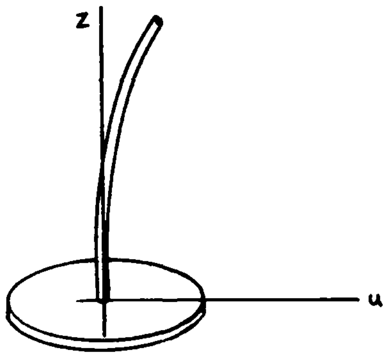

Figure 6.1: *A model problem with 1 rotational d.o.f.*

The disk is constrained to rotate around the z-axis without friction, so that the angular momentum of the system is conserved. The beam is constrained to the z-u-plane, and all the deflections are parallel to the u-axis. Equations of motion of the beam, including internal damping, are

$$\rho u_{tt} + \mu u_{zzzz} + k u_{zzzzt} = \omega^2 u, \tag{6.1}$$

$$\omega\left(\int_0^1 \mu u^2 dz + I\right) = M, \tag{6.2}$$

$$u(0,t) = u_z(0,t) = u_{zz}(1,t) = u_{zzz}(1,t) = 0, \tag{6.3}$$

where μ, k charactrize elasticity and damping respectively, and equation (6.2) expresses conservation of angular momentum, I being the disk's moment of inertia. From now on we set $\rho = \mu = 1$ to simplify notation. ***Steady-state solutions*** will satisfy the ordinary differential equation

$$-u_{zzzz} + \omega^2 u = 0. \tag{6.4}$$

(6.4)-(6.3) define a nonlinear eigenvalue problem with M as a parameter, the nonlinearity lying in the u-dependence of ω in (6.2).

Remark 6.1. The stationary problem (6.4)-(6.3) admits a revealing variational formulation: its solutions are the critical points of total energy

$$E = \frac{1}{2} I\omega^2 + \int_0^1 (u_{zz}^2 + \omega^2 u + u_t^2) dz \tag{6.5}$$

when the angular momentum M is fixed, as one easily checks. This has a simple physical explanation: As the beam vibrates energy is dissipated. Ultimately the virbatioins are damped out, and the total energy of the system is extremized. Actually, for some exceptional (in a sense which we make precise in Theorem 6.1) initial conditions the limiting energy value will be not minimal, but rather critical. (See Theorem 6.1 and Figure 6.4.) We note also that

$$\dot{E} = -\int_0^1 (u_{zzt})^2 dz \tag{6.6}$$

Remark 6.2 on Stability. The same variational formulation suggests a stability criterion for the dynamical system (6.1)-(6.3). Namely, consider the space S of all triples $(u, \dot{u}, \omega)$ with angular momentum M, i.e. satisfying (6.2). If the stationary solution $x_0 = (u, 0, \omega)$ minimizes total energy (6.5) on S, then this stationary solution is stable, even in the undamped case. This means that for all initial data $(u, u_t, \omega)_{t=0}$ in S sufficiently close to the minimal $(u, 0, \omega)$ (in the energy metric) the solution will stay close to the minimizing solution x_0 for all time. This is quite similar to the standard minimal energy criterion of Lagrange; here, however, the energy is minimal only subject to the angular momentum being constrained. This minimality is guaranteed if the following condition holds: $E_{xx} + \lambda M_{xx} > 0$ on the tangent space to $M = const.$ at the point x_0, where λ is the Lagrange multiplier: $M_x = \lambda E_x$.

This is the idea behind the Energy-Casimir method, which is the second derivative test in the presence of the constraints and which is applicable in the infinite-dimensional settings. This method has been used to establish stability in a variety of problems (see, e.g., the work and references in Krishnaprasad and Marsden [8]).

A complete picture of the behavior of our model system, in particular of the global phase portrait and its bifurcations, is given by Theorems 6.1 and 6.2 below. A perhaps surprising consequence of this result is that *the system selects a discrete set of angular velocities for its stationary rotations independent of* the angular momentum M! This phenomenon is observed in a number of model problems, but it is by no means universal. See [36] on this point. Another aspect of this result is that our model is an *infinite-dimensional Morse system.*

To state our first result we combine (6.1) and (6.2) into a single nonlinear equation

$$u_{tt} + u_{zzzz} + k u_{zzzzt} = \omega^2(u) u$$

where $\omega(u) = M/(\|u\|^2 + I)$, $\|u\|^2 = \int_0^1 u^2 dz$. This can be rewritten as a system

$$\dot{w} = \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\partial^4 u - \partial^4 v + \omega^2(u) u \end{pmatrix} \equiv F(w) \tag{6.7}$$

Theorem 6.1. Qualitative behavior of the system. *Assume that $k > 0$, i.e. the damping is present.*

There exists an infinite sequence $0 < \omega_1 < \omega_2 < \ldots \to \infty$ of preferred angular velocities associated with the problem (6.7) with boundary conditions (6.8) such that if the angular momentum M lies in the interval $(M_k, M_{k+1}) = (I\omega_k, I\omega_{k+1})$, the problem (6.7),(6.8) has k distinct nontrivial stationary solutions $(u,v) = (u_i(z), 0)$, $1 \le i \le k$, and a trivial solution $(u,v) = (0,0)$, with angular velocities $\omega(u_i) \equiv \frac{M}{I + \int_0^1 u_i^2 dz} = \omega_i$, $1 \le i \le k$ and $\omega(0) = \frac{M}{I}$. These solutions are depicted in Figure 6.2. The solution with the smallest angular velocity is dynamically stable, i.e. any solution of (6.7),(6.8) with nearby initial conditions remains close to it for all time, in the L^2-norm given by

$$\|(u,\dot{u})\| = \|(u,v)\| = \int_0^1 (u^2 + v^2) dz.$$

Stationary solutions with higher angular velocities are unstable, and furthermore, i-th solution (in the order of increasing angular velocity) $(u,\dot{v}) \equiv (u_i, 0)$ has $(i-1)$-dimensional unstable manifold; this dimension coincides with the number of zeroes of u_i. In particular, if $M < M_1 = I\omega_1$, only one mode $u \equiv 0$ is present and is stable. As M crosses the bifurcation values M_i, the system undergoes a series of pitchfork bifurcations shown in Figure 6.3.

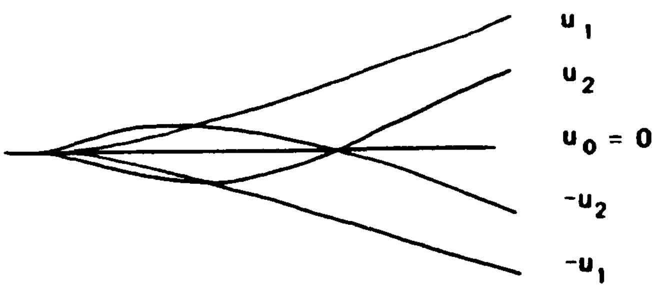

Figure 6.2: *Solutions to (6.1)-(6.3)*

There have been several recent studies of the evolution of solutions to infinite dimensional systems toward equilibria (see Dafermos [4] and references therein). Theorem 6.2

below shows that under mild (physically reasonable) assumptions solutions of our systems approach the equilibrium states described above.

Theorem 6.2 *Fix $M > 0$ and let $k > 0$, say $k = 1$. Any solution $w(t)$ of (6.7) with $u_0(z), v_0(z) \in H^{(12)}(0,1)$ tends in the L^2-norm as $t \to \infty$ to one of the stationary solutions from Theorem 6.1 (See Figure 6.4).*

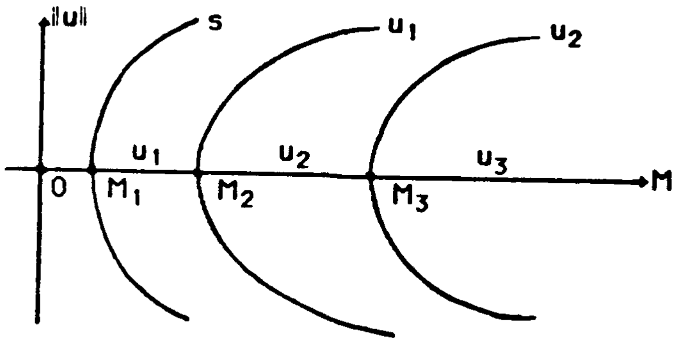

Figure 6.3 *s denotes stable equilibria, and $u_1, u_2, \dots$ denote unstable equilibria with 1, 2, ... dimensional unstable manifolds. The numbers and L^2-norm of equilibria depend on the parameter M.*

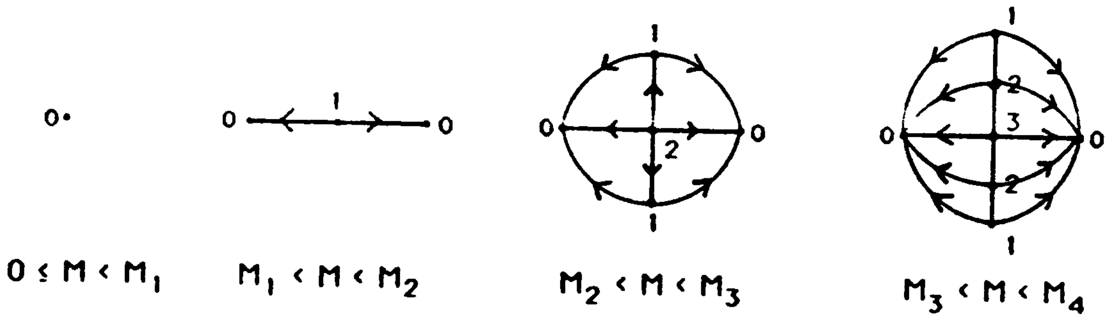

Figure 6.4. *Global phase portrait of the system (6.7),(6.8). Indices show the dimension of the unstable manifolds, which are depicted by arrows.*

Proof: of Theorem 6.1.

1. *Existence of $k+1$ solutions for $M_k < M < M_{k+1}$.* Stationary solutions satisfy the nonlinear boundary value problem (6.3)-(6.4). There exists a sequence $0 < \omega_1 < \omega_2 < \ldots$ of angular velocities for which this problem has nontrivial solutions (the eigenfunctions of $\frac{\partial^4}{\partial z^4}$ with boundary conditions (6.3)) which we denote by $c_i e_i$, c_i arbitrary and $\|e_i(z)\|_{L^2} = 1$. Define $M_j = I\omega_j$.* Freedom of choice of c_i is used to satisfy the angular momentum constraint (6.2), which becomes

$$\omega_i(c_i^2 + I) = M.$$

This can be solved for c_i if and only if $M > I\omega_i \equiv M_i$; setting $u_i = c_i e_i$ proves the existence of k solutions if $M > M_k$. One additional solution is obtained by setting $u_{k+1} \equiv 0$, $\omega_{k+1} = \frac{M}{I}$.

2. *Stability of stationary modes.* We restrict ourselves to the undamped ($k = 0$) case, as the dissipation does not affect stability of *stationary* modes. The governing equations can be written as

$$u_{tt} = -u_{zzzz} + \left(\frac{M}{\|u\|^2 + I}\right)^2 u \overset{def}{=} A(u). \tag{6.8}$$

The dimension of the unstable manifold of the mode $u_i(z)$ is given by the number of positive eigenvalues of the linearization of the operator $A(u)$ at $u = u_i$. This linearization is given by

$$A'(u)v = -\partial^4 v + \left(\frac{M}{\|u\|^2 + I}\right)^2 v - \frac{4M^2}{(\|u\|^2 + I)^3}(u, v)u,$$

* M_j is the total angular momentum when the beam is undeflected, since the straight beam's ($u \equiv 0$) contribution to the angular momentum is zero.

where $\partial = \frac{\partial}{\partial z}$ and $(u,v) = \int_0^1 u(z)v(z)dz$.

At $u = u_i$ we have

$$Bv \equiv A'(u_i)v = -\partial^4 v + \omega_i^2 v - 4\omega_i^3 M^{-1}(u_i, v)u_i$$

The last term in this expression is a scalar multiple of the orthogonal projection of v onto u_i. Comparing B with the operator C given by

$$Cv = -\partial^4 v + \omega_i^2 v,$$

we see that the spectrum of B can be obtained from that of C by replacing the eigenvalue 0 of C corresponding to the eigenfunction u_i by $-4\omega_i^3 M^{-1}\|u_i\|^2$. Thus the eigenvalues of B are

$$\omega_i^2 - \omega_1^2, \omega_i^2 - \omega_2^2, \ldots, \omega_i^2 - \omega_{i-1}^2, -4\omega_i^3 M^{-1}\|u_i\|^2, \omega_i^2 - \omega_{i+1}^2, \ldots,$$

of which precisely $i-1$ are positive. This shows that the dimension of the unstable manifold at the stationary solution u_i is $(i-1)$.

Proof of Theorem 6.2 consists of two steps:

(1) Showing that for any $\epsilon > 0$ the solution $w(t)$ enters and stays in an ϵ-neighborhood (in L^2 norm) of the zero set of the dissipation function. This set is $\{(u,v) : \int_0^1 v_{xx}^2 dx = 0\} \equiv \{(u,v) : (u(x), 0),\ u \in H^{(12)}\}$, see (6.6).

(2) Showing that if $w(t)$ stays in an ϵ-neighborhood of the zero set of $\{v \equiv 0\}$, then w is ϵ-close to one of the stationary points of the flow, figure 6.5.

Figure 6.5 *Solutions which remain ϵ-close to $\{v = 0\}$ must be ϵ-close to a stationary point.*

Proof of Step 1. Pick $\epsilon > 0$, and consider two neighborhoods of $\{v \equiv 0\}$: $\mathcal{N}_\epsilon = \{(u, v) : \|v\|_{L^2} < \epsilon\}$ and $\mathcal{N}_{\epsilon/2}$. Our solution $w(t)$ enters $\mathcal{N}_{\epsilon/2}$ for some $t > 0$ with necessity: otherwise for all t large enough we would have

$$\dot{E} = -\int_0^1 u_{zzt}^2 dz \ = -\int_0^1 v_{zz}^2 dz \le -\lambda_o \int_0^1 v^2 dz \le \lambda_o (\frac{\epsilon}{2})^2, \tag{6.9}$$

resulting in $E(t) \to -\infty$, which is a contradiction. Here λ_o is the smallest eigenvalue of $\partial^4 = (\partial^4/\partial z^4)$ with boundary conditions (6.3).

Thus to prove that $w(t)$ stays in $\mathcal{N}_\epsilon$ forever after some $T > 0$ it remains only to exclude the possibility of infinitely many trips between $\mathcal{N}_{\epsilon/2}$ and the exterior of $\mathcal{N}_\epsilon$, see figure 6.5. We will do so by showing that each trip results in a loss of energy at least $\Delta E > 0$ depending only on ϵ and $w(0)$ at each crossing, so that $w(t)$ can afford only finite number of trips.

Let t_1, t_2 be two consecutive crossing times of the boundaries of $\mathcal{N}_\epsilon$ and $\mathcal{N}_{\epsilon/2}$. We have

$$|E(t_2) - E(t_1)| = |\int_{t_1}^{t_2} \int_0^1 v_{zz}^2 dz| \ge |t_2 - t_1| \lambda_o (\frac{\epsilon}{2})^2, \tag{6.10}$$

where we have used the fact that $\int_0^1 v^2 dz \ge (\frac{\epsilon}{2})^2$ and the Poincaré type estimate as in (6.9). It remains only to provide a lower bound on the trip time $|t_2 - t_1|$; it is provided by the upper bound on the velocity $\dot{w}(t)$ which is the result of the smoothness assumption.

Lemma 6.1. *Any solution $w(t) = (u(z,t), v(z,t))$ of (6.7) with $u_o(z), v_o(z) \in H^{(12)}(0,1)$ has velocity bounded in the L^2 norm: i.e. there exists $C = C(u_o, v_o) > 0$ such that for all $t \ge 0$*

$$\|F(w(t))\|^2 = \|v\|_{L^2}^2 + \| -\partial^4 u - \partial^4 v + \omega^2(u) u\|_{L^2}^2 < C^2.$$

Proof of this lemma is given in the Appendix.

By the choice of t_1, t_2 and using Lemma 6.1, we obtain

$$\frac{\epsilon}{2} \leq \|w(t_2) - w(t_1)\| = \| \int_{t_1}^{t_2} F(w(s))ds\| \leq |t_2 - t_1| \sup_{t \geq 0} \|F\| \leq C|t_2 - t_1|,$$

implying

$$|t_2 - t_1| \geq \frac{\epsilon}{2C},$$

which together with (6.10) proves that $w(t)$ stays in $\mathcal{N}_\epsilon$ for all $t \geq T(\epsilon)$.

Proof of step 2. We show now that if $w(t)$ stays in $\mathcal{N}_\epsilon$ for all $t \geq T(\epsilon)$ then it must tend to an equilibrium point of the flow (6.7) –more precisely, we will show that for every $\delta > 0$ there exists an $\epsilon > 0$ such that if $\|v\| \leq \epsilon$ for all $t \geq T(\epsilon)$ then for some equilibrium solution $w_e = (u_e, 0)$ we have $\|u_e - u\|_{L^2} \leq \delta$ for all $t \geq T(\epsilon)$. Our strategy is to show that if a solution is $v(t)$ *not* close to an equilibrium, then its velocity $v(t)$ must grow thus taking it outside the neighborhood $\mathcal{N}_\epsilon$, leading to a contradiction.

Assume the contrary: There is some $\delta > 0$ such that for every $\epsilon > 0$ there is a $t_o = t_o(\epsilon) \geq T(\epsilon)$ such that although

$$\|v(t)\| \leq \epsilon \quad \text{for all } t \geq T(\epsilon) \tag{6.11}$$

we have

$$\|u(t_o) - u_e\| \geq \delta, \tag{6.12}$$

for every equilibrium u_e. The idea is to show that being far from any equilibrium causes an increase in velocity thus causing $\|v\| > \epsilon$ (a contradiction). The details are as follows.

(6.12) implies for $f(u) \equiv -\partial^4 u + \omega^2(u)u$:

$$\|f(u(t_o))\| \geq \lambda \|u(t_o) - u_e\| \geq \lambda\delta, \tag{6.13}$$

for some $\lambda > 0$ independent of δ, if we choose δ small enough (which we do).

We have: (cf. (6.7))

$$\|v(t) - v(t_o)\|_{L^2} \geq \| \int_{t_o}^{t} f(u(s))ds\| - \| \int_{t_o}^{t} \partial^4 v(s)ds\|. \tag{6.14}$$

From the proof of Lemma 6.1 it is clear that there exists $C > 0$ such that $\|\partial^4 v\|_{L^2} \leq C\epsilon$ for t large enough; without loss of generality we assume that this already holds for $t \geq t_o$.

We have from (6.14):

$$\|v(t)\| \geq \| \int_{t_o}^{t} f(u(s))ds\| - \epsilon(1 + C)(t - t_o) \text{ for all } t \geq t_o \tag{6.15}$$

Furthermore, if t', t'' are sufficiently large, and if

$$\|u(t'') - u(t')\| \leq \sqrt{\epsilon}$$

holds, then

$$\|f(u(t'')) - f(u(t'))\| \leq \frac{1}{2}\|f(u(t'))\|$$

This can be seen from the proof of Lemma 6.1 which shows that all the harmonics of sufficiently high order decay exponentially, with $w(t)$ tending to a finite-dimensional subspace of $L^2 \times L^2$. Applying this remark to (6.15) with $t' = t_o, t'' = t$, we obtain the final lower bound on v, using (6.13):

$$\|v(t)\| \geq \frac{1}{2}(t - t_o)\|f(u(t_o))\| - \epsilon(1 + C)(t - t_o) \geq \tag{6.16}$$

$$(t - t_o)[\frac{1}{2}\lambda\delta - \epsilon(1 + C)],$$

as long as

$$\|u(t) - u(t_o)\| \leq \sqrt{\epsilon}.$$

Using this estimate on the velocity $v(t)$, we will show that $w(t)$ leaves the box $\|v\| \leq \epsilon$, $\|u - u(t_o)\| \leq \sqrt{\epsilon}$ through the "horizontal" boundary, i.e. $\|v(t^*)\| = \epsilon$ for some first exit time $t^* > t_o$, see figure 6.5. This will result in the desired contradiction.

To see that $\|v\| = \epsilon$ is reached first, we note that otherwise

$$\|u(t^*) - u(t_o)\| = \sqrt{\epsilon},$$

and

$$\epsilon(t^* - t_o) \geq \| \int_{t_o}^{t^*} v(t) dt \| = \|u(t^*) - u(t_o)\| = \sqrt{\epsilon},$$

i.e. the time required to reach the *vertical* boundary is large:

$$t^* - t_o \geq \frac{1}{\sqrt{\epsilon}}.$$

Using this in (6.16)—which is valid for $t_o \leq t \leq t^*$—we obtain

$$\|v(t_o + \frac{1}{\sqrt{\epsilon}})\| \geq \frac{1}{\sqrt{\epsilon}} \left[\frac{1}{2} \lambda \delta - \epsilon(1 + C) \right] > \epsilon,$$

which is a desired contradiction, if we pick ϵ sufficiently small. $\blacksquare$

This shows that given any $\delta > 0$ there exists an $\epsilon > 0$ such that if w stays in $\mathcal{N}_\epsilon$, then $u(z,t)$ is less than δ (in $L^2[0,1]$) from an equilibrium solution u_e of (6.8). Since the first step in this proof showed that for ϵ-neighborhood of the zero set $\{v \equiv 0\}$ of the dissipation function, this proves Theorem 6.2.

APPENDIX

Proof of **Lemma 6.1.**

Expand $u(x,t)$ in the orthonormal basis $\{e_j(x)\}$ of the eigenfunctions of ∂^4 with the boundary conditions (6.3):

$$u(x,t) = \sum_{k=0}^{\infty} a_k(t) e_k(x);$$

system (6.1) is equivalent to a series of ODE's for the amplitudes:

$$\ddot{a}_k + \lambda_k \dot{a}_k + \lambda_k a_k = \omega^2 a_k,$$

$\{\lambda_k\}$ are the eigenvalues of ∂^4, or

$$\begin{cases} \dot{a} = b \\ \dot{b} = -\lambda b - \lambda a + \omega^2 a, \end{cases} \tag{A.1}$$

where we have dropped the subscript k for the sake of brevity.

It suffices to prove that there exists $C > 0$ such that

$$\|\partial^4 u\|^2 + \|\partial^4 v\|^2 = \sum_{k=0}^{\infty} \lambda_k^2 (a_k^2(t) + b_k^2(t)) < C \text{ for all } t \geq 0. \tag{4}$$

This would show that $\|F(w)\|_{L^2}$ is bounded for all t, since the boundedness of the last term $\omega^2(u)u$ in L^2 is obtained as follows:

$$\|\omega^2(u)u\|_{L^2}^2 = \omega^4(u)\|u\|^2 \leq (\frac{M}{I})^4 \|u\|^2 \leq (\frac{M}{I})^4 \frac{1}{\lambda_0^2} \|\partial^4 u\|_{L^2}^2 \leq$$

$$(\frac{M}{I})^4 \lambda_0^{-2} C.$$

To estimate $a_k(t)$, $b_k(t)$ we use the smoothness of the initial conditions to get an upper bound on $a_k(0), b_k(0)$: estimates

$$\|\partial^{12} u_0\|_{L^2} = \sum a_k^2(0) \lambda_k^6 < c$$

$$\|\partial^{12} v_0\| = \sum b_k^2(0)\lambda_k^6 < c$$

imply

$$|a_k(0)|,\ |b_k(0)| < c\lambda_k^{-3}.$$

We will show that for all $t \geq 0$ the estimates

$$|a_k(t)|,\ |b_k(t)| < c\lambda_k^{-2} \qquad (A.3)$$

hold, for k large enough, thus implying (A.2):

$$\sum \lambda_k^2 a_k^2 \leq \sum \lambda_k^2 c^2 \lambda_k^{-4} = c^2 \sum \lambda_k^{-2} < \infty$$

since $\lambda_k \sim k^4$.

To show (A.3) we rewrite (A.1) as

$$\dot{z} = Az + Rz$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\lambda & -\lambda \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0 & 0 \\ \omega^2 & 0 \end{pmatrix}.$$

Introducing the Lyapunov matrix

$$B = \int_0^\infty e^{A^T s} e^{As} ds \text{ satisfying } A^T B + BA = -I,$$

we obtain

$$\frac{d}{dt}(Bz, z) = -(z, z) + ((R^T B + BR)z, z) \leq$$

$$-(1 - \|R^T B + BR\|)(z, z).$$

An explicit computation gives

$$B = \begin{pmatrix} 1 + \frac{1}{2\lambda} & \frac{1}{2\lambda} \\ \frac{1}{2\lambda} & \frac{1}{2\lambda} + \frac{1}{2\lambda^3} \end{pmatrix}$$

and thus $\|R^T B + BR\| \leq \frac{\text{“const.”}}{\lambda}$, since $\omega < \frac{M}{I}$.

We have

$$\frac{d}{dt}(Bz, z) \leq -(1 - \frac{\text{“const.”}}{\lambda})(z, z) \leq -\frac{1}{2}(z, z),$$

the latter inequality holding for all $\lambda = \lambda_k$ large enough.

Now, the minimal eigenvalue ρ of the positive definite matrix B is estimated from below as

$$\rho > \frac{1}{2\lambda} \qquad \text{(for } \lambda \text{ large enough).}$$

We have for all $t > 0$ and for k large enough

$$\frac{1}{2\lambda}(z(t), z(t)) < (Bz(t), z(t)) \leq (Bz_o, z_o) \leq (z_0, z_0),$$

or

$$a_k^2(t) + b_k^2(t) \leq 2\lambda_k(a_k^2(0) + b_k^2(0)) \leq$$

$$\leq 2\lambda_k c^2 \lambda_k^{-6} = 2c^2 \lambda_k^{-5}.$$

This implies (A.3), q.e.d.

EQUILIBRIUM MECHANICS OF ROTATING SYSTEMS

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ABSTRACT

Equilibrium rotations of a planar kinematic chain undergoing a single degree of freedom (out of plane) rotation in a gravitational field are studied. Sharp bounds on the number of dynamic equilibria are given, and the dependence of this number on physical parameters is discussed for the special case of two-link chains and chains with identical links. The theory indicates that there will be a richer set of dynamically stable equilibria for carefully designed chains with unequal links. This is born out by laboratory experiments.

1. INTRODUCTION

As a prelude to a comprehensive study of the dynamic stability of rotating mechanical systems, it is useful to study the equilibrium dynamics of various classes of prototypes. Indeed, a number of participants at this conference have carried out such studies involving rotating mechanisms featuring various different types of elastic and articulated appendages. In our own recent work (Baillieul and Levi, 1987), the rotational dynamics of a simple structure consisting of an elastic rod attached to a rigid body were studied. The transient dynamics for structures of this type involve vibrations of the attachment which are subject to inertial forcing and viscoelastic damping. We have shown that in the absence of external forcing, the damping will have a dominant influence on the long term dynamics, and ultimately all movement of the appendage with respect to the rigid body coordinate frame will disappear. Moreover, we have shown that in steady state, rotations always tend to a constant velocity, and for the three rotational degree of freedom problem, this angular velocity vector is aligned with a principal axis of the steady state inertia tensor of the body beam system. While we do not yet have a complete understanding of the nonlinear relationship between steady state angular velocity and steady state elastic deformation of the beam appendage, it has been possible to carry out a detailed analysis of the qualitative dynamics of a closely related model having only one rotational degree of freedom. In this system, the rotation is restricted to be about the neutral axis of the beam. All steady state solutions and corresponding stability characteristics have been determined, and an explicit description of the system's dependence on the angular momentum has been developed. It has been shown that as angular momentum is increased, new equilibria appear as a result of a sequence of pitchfork bifurcations. No matter how rich the equilibrium set becomes, however, there is at most one stable equilibrium configuration if we assume the beam is uniform. (Below a certain threshold angular momentum, the neutral or undeflected state of the beam is stable, while above this threshold the undeflected state ceases to be stable, and the first mode shape becomes the stable configuration for all higher values of angular momentum.)

The bifurcations displayed in the rotating beam model are similar to phenomena described in classical works on whirling shafts (see e.g. Dick, 1948). In order to obtain a more detailed understanding of this type of buckling in the context of the modern qualitative theory of non-linear dynamics, we have turned our investigations to the analysis of rotating planar kinematic chains. In the next section, we provide a precise statement of the problem under investigation and summarize recent results (detailed in Baillieul, 1987) regarding the numbers of equilibria. In Section 3, we present new results which suggest that chains which are uniform (i.e. have nearly identical links) will have a less rich set of interesting dynamically stable equilibrium configurations than chains in which the links have carefully prescribed unequal masses and lengths.

2. EQUILIBRIA IN A ROTATING CHAIN

In this section we consider the rotational dynamics of the rotating simple kinematic chain depicted in Figure 2.1. Here a certain number, n, of links of various lengths are connected to form a single-strand planar kinematic chain. We suppose this chain is suspended in such a way that the force of gravity tends to extend the chain to its maximum total length. We further assume there is (360°) free rotation (with no actuation or friction) about all joints in the planar chain, but that torque may be applied about the vertical axis (which passes through all links and joint axes when the linkage is in the neutral (fully extended) configuration). We suppose the links are massless, but each joint in the planar chain has mass m_{i-1} $(i = 2, \ldots, n)$, and there is a mass m_n located at the tip of the final link. The angles ψ_i depicted in Figure 2.1 measure deviation of each link from its neutral vertical position. We also let ψ_0 denote the angle of rotation of the mechanism about the vertical axis. (ψ_0 is measured with respect to an arbitrarily chosen reference.) Depending on the mass distribution in the links of an actual physical chain, the mathematical idealization we have chosen to study may or may not provide a model which will predict detailed dynamical behavior. The advantage of working in this simplified setting, however, is that we are able to display a wide range of qualitative features and to understand the way in which these depend on parameters.

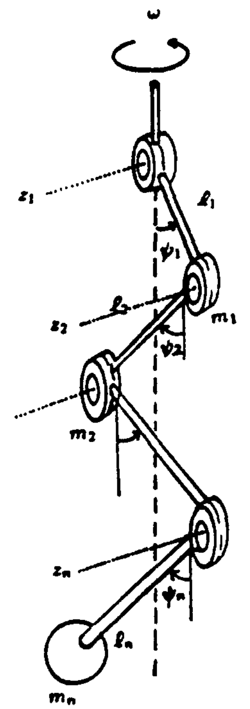

Figure 2.1: ***An n-degree of freedom planar kinematic chain in a gravitational field rotated about its neutral axis.***

Following a geometric formalism along lines similar to those detailed in Baillieul and Levi (1987), it is strightforward to write down the dynamical equations of this rotating chain. At present, however, we are simply interested in the special case where $\dot{\psi}_0 = \omega$ (a constant) and $\dot{\psi}_j = 0$ for $j = 1, \ldots, n$. Such equilibrium solutions to the dynamical equations are also given as the zeros of the gradient system

$$\frac{\partial L}{\partial \psi} = 0$$

where

$$\begin{aligned} L &= L(\omega; \psi_1, \ldots, \psi_n) \\ &= -\frac{1}{2}[m_1(\ell_1 \sin\psi_1)^2 + m_2(\ell_1 \sin\psi_1 + \ell_2 \sin\psi_2)^2 + \\ &\qquad \cdots + m_n(\ell_1 \sin\psi_1 + \ldots + \ell_n \sin\psi_n)^2]\omega^2 \\ &\quad -[m_1\ell_1 \cos\psi_1 + m_2(\ell_1 \cos\psi_1 + \ell_2 \cos\psi_2) + \\ &\qquad \cdots + m_n(\ell_1 \cos\psi_1 + \ldots + \ell_n \cos\psi_n)]g \end{aligned}$$

with g designating acceleration due to gravity. This system of equations may be explicitly written out

$$[(m_1+\ldots+m_n)\ell_1 s_1+(m_2+\ldots+m_n)\ell_2 s_2+\ \ldots\ +m_n\ell_n s_n]c_1\omega^2$$

$$-(m_1+\ldots+m_n)s_1 g=0$$

$$[(m_2+\ldots+m_n)\ell_1 s_1+(m_2+\ldots+m_n)\ell_2 s_2+\ \ldots\ +m_n\ell_n s_n]c_2\omega^2$$

$$-(m_2+\ldots+m_n)s_2 g=0$$

$$\ldots$$

$$[m_n\ell_1 s_1+m_n\ell_2 s_2+\ldots+m_n\ell_n s_n]c_n\omega^2-m_n s_n g=0. \tag{2.1}$$

where as usual $s_i=\sin\psi_i$ and $c_i=\cos\psi_i$.

Frictional forces will oppose motions in the joints of these kinematic chains. and the stability of each equilibrium will be characterized by the signs of the eigenvalues of the Hessian $\frac{\partial^2 L}{\partial\psi^2}$. Before discussing the stability characteristics of various equilibria. we establish upper and lower bounds on the numbers of solutions to (2.1). The exact number of solutions in any particular case will be seen to depend on the values of the parameters ℓ_i, m_i, and ω. It will be shown in the next section that for the case $n=2$ the bounds we have obtained are tight.

Theorem 2.1: *For small values of $|\omega|$, there are precisely 2^n distinct solutions to the system (3.1). These solutions are given explicitly by $\{(\psi_1,\ldots,\psi_n):\psi_i=0 \text{ or } \pi\}$. The index (= number of negative eigenvalues of the Hessian ($\frac{\partial^2 L}{\partial\psi^2}$)) of any such solution is the number of vector entries, ψ_j, which are equal to π.*

This theorem is proved in Baillieul (1987). Physical intuition suggests that there will be more than 2^n equilibrium solutions as the parameter ω (or equivalently ℓ) is increased in (2.1). To obtain a sharper picture of the way in which bifurcations occur. we simplify the mathematics by means of some assumptions and a change of variables. First. note that for the purposes of establishing a qualitative theory of solutions to (2.1). there is no loss of generality in normalizing g to be 1 and only allowing ω and the ℓ_i, m_i parameters to vary. Then letting $x_i = \cos\psi_i$, $y_i = \sin\psi_i$, (2.1) is transformed into a system of algebraic (polynomial) equations:

$$
\begin{aligned}
&[(m_1 + \cdots + m_n)\ell_1 y_1 + (m_2 + \cdots + m_n)\ell_2 y_2 + \\
&\cdots + m_n \ell_n y_n] x_1 \omega^2 - (m_1 + \cdots + m_n) y_1 = 0 \\
&\qquad \vdots \\
&[\ell_1 y_1 + \ell_2 y_2 + \cdots + \ell_n y_n] x_n \omega^2 - y_n = 0 \\
&x_1^2 + y_1^2 - 1 = 0 \\
&\qquad \vdots \\
&x_n^2 + y_n^2 - 1 = 0
\end{aligned} \tag{2.2}
$$

We shall call the system (2.2) the *algebraic equilibrium equations* for the rotating planar chain. The precise connection between solutions to (2.1) and (2.2) is given in the following easily proved lemma.

Lemma 2.1: *Real solutions to (2.2) define the solutions to (2.1).*

While there are $2n$ equations in $2n$ unknowns in (2.2)—a seeming increase in complexity over (2.1)—we may appeal to classical algebraic geometry and intersection theory in determining the number of solutions to (2.2). Indeed, since there are $2n$ quadratic equations, Bezout's theorem suggests there will be as many as $2^{2n} = 4^n$ (possibly complex) solutions. The precise number is given by the following theorem, whose proof is given in Baillieul (1987) without reference to the classical theorem of Bezout.

Theorem 2.2: *For every choice of angular velocity ω and mass and link-length parameters $m_1, \ldots, m_n$, $\ell_1, \ldots, \ell_n$, there are 4^n (some possibly complex) solutions to the system (2.2).*

The proof here depends on the following lemma (also proved in Baillieul, 1987) which is of interest in its own right.

Lemma 2.2: *As $\omega \to \infty$, solutions to (2.2) remain finite.*

Because there is a one-to-one correspondence between solutions to (2.1) and real solutions to the corresponding algebraic equilibrium equations (2.2), Theorems 2.1 and 2.2 taken together imply the following result.

Corollary 2.1: *Given any values for the parameters $\ell_1, \ldots, \ell_n, m_1, \ldots, m_n, \omega^2$, there are at least 2^n but no more than 4^n solutions to the equilibrium equations (2.1).*

3. THE ROTATING DOUBLE PENDULUM AND THE CASE OF UNIFORM CHAINS

The analysis of the preceding section shows that for a two link chain there are between 4 and 16 equilibrium states. To facilitate a detailed analysis of the dynamics of this mechanism, we reformulate equations (2.1) by letting $g = 1$, $\ell = \ell_1$, $\ell_2 = r\ell$, $(0 < r < \infty)$, $\alpha = m_2/(m_1 + m_2)$. Under these assumptions (2.1) may be rewritten

$$(\ell \sin\psi_1 + \alpha r \ell \sin\psi_2)\cos\psi_1 \omega^2 - \sin\psi_1 = 0$$

$$(\ell \sin\psi_1 + r\ell \sin\psi_2)\cos\psi_2 \omega^2 - \sin\psi_2 = 0. \tag{3.1}$$

The following theorem gives a substantial answer to the question of parametric dependence in the case $n = 2$.

Theorem 3.1: *Let α and r be fixed in (3.1), and suppose $r > 1$.*

(i) As ω^2 increases from 0, four pitchfork bifurcations occur involving the fixed equilibrium points $(\psi_1, \psi_2) = (0,0)$, $(\pi, 0)$, $(0, \pi)$. For sufficiently large values of ω^2, these bifurcated equilibria account for twelve of the possible solutions to (3.1).

(ii) If a bifurcation other than accounted for in (i) occurs, it involves the appearnce of a symmetric pair of saddle nodes.

(iii) If $\alpha r > 1$, then for sufficiently large values of ω^2 there are only twelve solutions to (3.1). Two are stable equilibria (the index of the Hessian of L is 0), six are index 1 critical points of L, and four are index 2 critical points. If $\alpha r \leq 1$, then for all sufficiently large values of ω^2 there are sixteen solutions (counting multiplicities) to (3.1). Four are stable, eight have index 1, and four have index 2.

Proof: (i) It is an elementary calculation to show that the Hessian of L is represented by the 2×2 matrix

$$(m_1 + m_2)\begin{pmatrix} \ell^2[(s_1 + \alpha r s_2)s_1 - c_1^2]\omega^2 + \ell c_1 & -\alpha r \ell^2 c_1 c_2 \omega^2 \\ -\alpha r \ell^2 c_1 c_2 \omega^2 & \alpha r \ell^2[(s_1 + r s_2)s_2 - r c_2^2]\omega^2 + \alpha r \ell c_2 \end{pmatrix}$$

Bifurcation of solutions to (3.1) occur when this matrix becomes singular. For the equilibrium points $(0,0)$, $(\pi,0)$, and $(0,\pi)$, the Hessian has repspective singular loci satisfying

$$r(1-\alpha)\omega^4 - (1+r)\omega^2 + 1 = 0$$

$$r(1-\alpha)\omega^4 + (r-1)\omega^2 - 1 = 0$$

$$r(1-\alpha)\omega^4 + (1-r)\omega^2 - 1 = 0.$$

(The matrix is never singular for the unstable equilibrium (π,π).) Noting that only real solutions of these equations are of interest, we find they may be rewritten to provide formulae expressing ω^2 as a function of r and α:

$$(a) \quad \omega^2 = \frac{r + 1 - \sqrt{(r+1)^2 - 4r(1-\alpha)}}{2r(1-\alpha)}$$

$$(b) \quad \omega^2 = \frac{1 - r + \sqrt{(1-r)^2 + 4r(1-\alpha)}}{2r(1-\alpha)}$$

$$(c) \quad \omega^2 = \frac{r - 1 + \sqrt{(r-1)^2 + 4r(1-\alpha)}}{2r(1-\alpha)}$$

$$(d) \quad \omega^2 = \frac{r + 1 + \sqrt{(r+1)^2 - 4r(1-\alpha)}}{2r(1-\alpha)}.$$

These expressions for ω^2 are listed in order of increasing magnitude. When ω^2 increases through the value given by (a), a pitchfork bifurcation occurs at $(\psi_1, \psi_2) = (0,0)$ wherein this equilibrium is transformed from being an index zero critical point of L to an index one critical point. In the process, two index zero critical points are spawned. When ω^2 increases through the value given by (b) (resp. (c)), the equilibrium $(\pi,0)$ (resp. $(0,\pi)$) is transformed from having index one to index two. Two new eqilibria of index one are spawned at this bifurcation. Finally when ω^2 increases through the value given in (d), the equilibirum $(0,0)$ goes from index one to index two, giving rise to two new index one equilibria in the process.

(ii) The generic bifurcations in a gradient system such as this depending on three parameters r, α, ω^2 involve saddle nodes. (See e.g. Guckenheimer and Holmes, 1983.) Each time a bifurcation occurs for particular parameter values and $(\psi_1, \psi_2) = (\psi_1^*, \psi_2^*)$, simple physical considerations dictate that a symmetric bifurcation occurs at the same parameter values and $(\psi_1, \psi_2) = (-\psi_1^*, -\psi_2^*)$.

As $\omega^2 \to \infty$, it follows as in lemma 2.2 that solutions to (3.1) approach solutions to

$$(\sin\psi_1 + \alpha r \sin\psi_2)\cos\psi_1 = 0$$

$$(\sin\psi_1 + r \sin\psi_2)\cos\psi_2 = 0. \tag{3.2}$$

The solutions to this system, together with the index of the Hessian evaluated at each solution are given in Table 3.1. The conclusion of the theorem is a straightforward consequence of these facts.

c_1	s_1	c_2	s_2	Index
0	1	0	1	0
0	-1	0	1	0
0	1	0	-1	0
0	-1	0	-1	0
0	1	$\sqrt{1-1/r^2}$	$-1/r$	1
0	1	$-\sqrt{1-1/r^2}$	$-1/r$	1
0	-1	$\sqrt{1-1/r^2}$	$1/r$	1
0	-1	$-\sqrt{1-1/r^2}$	$1/r$	1
$\sqrt{1-\alpha^2r^2}$	$-\alpha r$	0	1	1
$-\sqrt{1-\alpha^2r^2}$	$-\alpha r$	0	1	1
$\sqrt{1-\alpha^2r^2}$	$-\alpha r$	0	-1	1
$-\sqrt{1-\alpha^2r^2}$	$-\alpha r$	0	-1	1
1	0	1	0	2
1	0	-1	0	2
-1	0	1	0	2
-1	0	-1	0	2

Table 3.1(a): *The 16 solutions to (3.2) in case* $\alpha r < 1$.

c_1	s_1	c_2	s_2	Index
0	1	0	1	0
0	-1	0	1	0
0	1	0	-1	1
0	-1	0	-1	1
0	1	$\sqrt{1-1/r^2}$	$-1/r$	1
0	1	$-\sqrt{1-1/r^2}$	$-1/r$	1
0	-1	$\sqrt{1-1/r^2}$	$1/r$	1
0	-1	$\sqrt{1-1/r^2}$	$1/r$	1
1	0	1	0	2
1	0	-1	0	2
-1	0	1	0	2
-1	0	-1	0	2

Table 3.1(b): *The 12 solutions to (3.2) in case* $\alpha r > 1$.

We conclude by considering the special case $r = 3/2$. Using MACSYMA we are able to determine curves in the (α, ω)-parameter space where bifurcations occur. These are depicted in Figure 3.1 where we have normalized ℓ to be 1. Table 3.2 shows how the distribution of critical points of each index is related to the total number of solutions to (3.1).

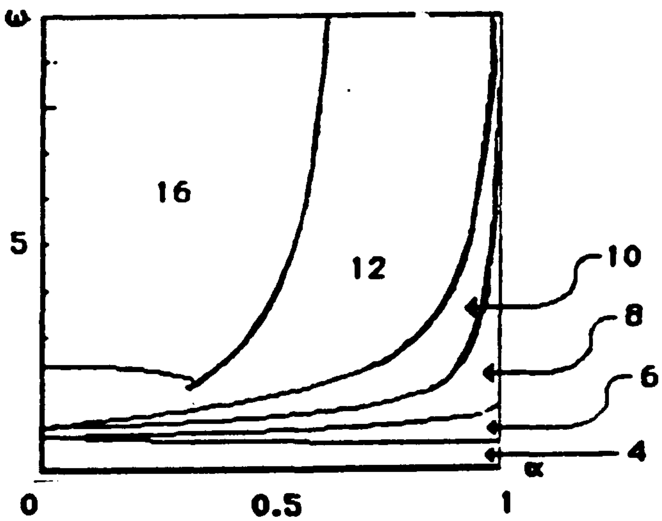

Figure 3.1: *Bifurcation locus of equations (3.1) in the case* $r = 1.5$. *The top curve is cusped as approximately* $\alpha = .33$.

i	c_i	c_i	c_i	c_i	c_i
0	1	2	2	2	2
1	2	3	4	5	6
2	1	1	2	3	4
total	4	6	8	10	12

Table 3.2(a): *When $\alpha r > 1$ the number of critical points of L (= solutions to (3.1)) increases with $|\omega|$ from 4 to 12. The table gives the number of critical points of each index.*

i	c_i	c_i	c_i	c_i	c_i	c_i
0	1	2	2	2	2	4
1	2	3	4	5	6	8
2	1	1	2	3	4	4
total	4	6	8	10	12	16

Table 3.2(b): *When $\alpha r \leq 1$ the number of critical points of L also depends on $|\omega|$. This is no longer a monotonic dependence, however, as indicated by the cusped bifurcation locus illustrated in Fig. 3.1. The table gives the number of critical points of each index cooresponding to each possible number of solutions to (3.1).*

This example and the approach we have used to study it lead us to several observations about the dynamics of more general chains. We remark that for the two link chains under investigation there appear to be a less interesting set of bifurcations when $r < 1$. (The possibility of 16 equilibria as $\omega \to \infty$ is ruled out.) For $r = 1$, a bifurcation occurs at $\omega = \infty$, as illustrated in Table 3.1 where several solutions to the equilibrium equations coalesce. In general, at $\omega = \infty$ we may in principle explicitly display the 4^n solutions to (2.2). In proving Theorem 3.2 in Baillieul (1987), it was established that for $\omega = \infty$ and generic choices of the mass and link length parameters, there are $\binom{n}{k} \cdot 2^n$ solutions to (2.2) with $x_i \neq 0$ for exactly k values of i between 1 and n. Certain of these solutions must coalesce when all link lengths, ℓ_i, and mass parameters, m_i, are equal. To see this, observe that when $\ell_1 = \ell_2 = \cdots = \ell_n = \ell$ and $m_1 = m_2 = \cdots = m_n = m$, the system (2.2) at $\omega = \infty$ is equivalent to:

$$(ny_1 + (n-1)y_2 + \cdots + y_n)x_1 = 0$$

$$((n-1)y_1 + (n-1)y_2 + \cdots + y_n)x_2 = 0$$

$$\vdots$$

$$(y_1 + y_2 + \cdots + y_n)x_n = 0$$

$$x_i^2 + y_i^2 - 1 = 0 \quad (i = 1, \ldots, n).$$

Consider solutions of the form $x_1 = 0$ and $x_i \neq 0$ for $i = 2, \ldots, n$. For this case, $y_1 = \pm 1$ and the above system of equations reduces to

$$(n-1)y_2 + (n-2)y_3 + \cdots + y_n = \mp(n-1)$$

$$(n-2)y_2 + (n-2)y_3 + \cdots + y_n = \mp(n-2)$$

$$\vdots$$

$$y_2 + y_3 + \cdots + y_n = \mp 1.$$

Subtracting the second equation from the first, we obtain $y_2 = \mp 1$ and $x_2 = 0$, showing that this solution has coalesced with the solution in which we a priori stipulate $x_2 = 0$.

That there are fewer than the maximum possible number of distinct equilibria when we have identical link lengths and joint masses suggests that there may also be fewer stable equilibria. We may obtain insight into the situation for large values of ω from the following.

Theorem 3.2: *For an n-link chain of the form under discussion with equilibrium dynamics given by (2.2), the only stable equilibria at $\omega = \infty$ are those for which each ψ_i is equal either to $\frac{\pi}{2}$ or $\frac{-\pi}{2}$.*

The proof of this theorem will not be given here, but the key idea is that if some ψ_i's take values other than $\pm\frac{\pi}{2}$, the linkage will be equivalent in equilibrium to a chain with fewer links which is in an unstable equilibrium.

Theorem 3.3: *For an n-link chain (as depicted in Fig. 2.1) with equal link lengths ($\ell_1 = \cdots = \ell_n = \ell$) and equal joint masses ($m_1 = \cdots = m_n = m$), the only stable equilibria at $\omega = \infty$ are those for which all $\psi_i = \frac{\pi}{2}$ (or symmetrically all $\psi_i = -\frac{\pi}{2}$.*

Proof: By theorem 3.2 we know that at $\omega = \infty$ all ψ_i are $\pm\frac{\pi}{2}$. Thus we may assume without loss of generality that for some integer $k \leq n$, $\psi_{n-k} = \frac{\pi}{2}$ and $\psi_{n-k+1} = \psi_{n-k+2} = \cdots = \psi_n = -\frac{\pi}{2}$. At $\omega = \infty$, when each $\psi_i = \pm\frac{\pi}{2}$, the Hessian

$$\frac{\partial^2 L}{\partial \psi^2} = diag[(ns_1 + (n-1)s_2 + \cdots + s_n)m\ell^2 s_1,$$

$$((n-1)s_1 + (n-1)s_2 + \cdots + s_n)m\ell^2 s_2,$$

$$\cdots, (s_1 + s_2 + \cdots + s_n)m\ell^2 s_n].$$

For stable equilibria, each of these diagonal entries will be positive. Hence we have

$$(k+1)(s_1 + \cdots + s_{n-k}) + ks_{n-k-1} + \cdots + s_n > 0$$

while

$$k(s_1 + \cdots + s_{n-k}) + ks_{n-k-1} + \cdots + s_n < 0.$$

Let $\alpha = s_1 + \cdots + s_{n-k}$. These inequalities may be rewritten as

$$(k+1)\alpha > \frac{k(k+1)}{2}$$

$$k\alpha < \frac{k(k+1)}{2}$$

which, taken together, are equivalent to

$$\frac{k}{2} < \alpha < \frac{k+1}{2}.$$

But this combined inequality can never be satisfied for integral values of α. Because of this, we conclude that the only stable configurations occur when either all $\psi_i = \frac{\pi}{2}$ or all $\psi_i = -\frac{\pi}{2}$ for $(i = 1, \ldots, n)$. This proves the theorem.

4. CONCLUSION

Theorem 3.3 suggests that for all sufficiently large values of slew rates ω, chains with equal link lengths and joint masses will have only one (symmetric pair) stable equilibrium. This will occur with the chain being flared out with all angles ψ_i having the same sign. As was indicated in the special case $n = 2$, in order for there to be a richer set of dynamically stable equilibria, uneven mass sizes and unequal link lengths are called for. While the models we have studied are idealizations of physical chains, our laboratory experiments with the controlled slewing of actual chains seem to validate the theoretical predictions.

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