AD-A229 403

(2)

REPORT DOCUMENTATION PAGE

a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release Distribution unlimited
b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 90 1128
6a. NAME OF PERFORMING ORGANIZATION: E.N.S. de CACHAN / Laboratoire de Mécanique et Technologie	6b. OFFICE SYMBOL (If applicable)
6c. ADDRESS (City, State, and ZIP Code): 61, Avenue du Président Wilson 94235 CACHAN CEDEX - FRANCE	7a. NAME OF MONITORING ORGANIZATION: EOARD
	7b. ADDRESS (City, State, and ZIP Code): Box 14 FPO N.Y. 09510-0200
8a. NAME OF FUNDING / SPONSORING ORGANIZATION: AFOSR	8b. OFFICE SYMBOL (If applicable): NA
8c. ADDRESS (City, State, and ZIP Code): Bolling AFB, DC 20332 - 6448 BKI 410	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR-87-0329
	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. 61 102 F; PROJECT NO. 2 301; TASK NO. D1; WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

MICROMECHANICS OF FATIGUE (UNCLASSIFIED)

12. PERSONAL AUTHOR(S) Prof. Jean LEMAITRE, Dr. René BILLARDON

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Final	FROM 890315 TO 900314	90 May 13	

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Brittle and ductile isotropic damage mechanisms are studied from a meso-mechanical view point. Relationships between crack density and void volume fraction defined at meso-scale on one hand, and a scalar internal variable characterizing damage on the other hand, are given. A general form for the evolution law for this damage variable is also derived from the same approach. A threshold which defines the onset of this evolution is derived from thermodynamical considerations. The ultimate stage of this evolution, i.e. the local failure is derived from localization criteria.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: Lt Col. George HARITOS	22b. TELEPHONE (Include Area Code): 202 767 0463 / 22c. OFFICE SYMBOL: NA

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE: UNCLASSIFIED

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Grant n° AFOSR - 89 - 0329

Final scientific report

May 1990

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ABSTRACT

In section 1., brittle and ductile isotropic damage mechanisms are studied from a meso-mechanical viewpoint. Relationships between crack density and void volume fraction defined at meso-scale on one hand, and a scalar internal variable characterizing damage on the other hand, are given.

In section 2., a general form for the evolution law for this damage variable is derived. A threshold which defines the onset of this evolution is derived from thermodynamical considerations.

In section 3., it is proposed to relate the ultimate stage of continuum damage evolution, i.e. the local failure, to localization phenomena. The corresponding criteria are studied in details.

TABLE OF CONTENTS

1. MICROMECHANICS OF DAMAGE

Micromechanics consists in deriving the behaviour of materials at meso-scale from the study of specific mechanisms at micro-scale. The micro-mechanisms must be precisely defined from physical observations for both the geometries and the kinematics. Their mechanical modelling is performed with elementary usual constitutive equations established at meso- or macro-scale for strain, crack growth and fracture. When compared to the direct analysis of the macroscopic properties, the additional power of this approach comes from a better modelling of the possible interactions between different mechanisms and from the homogenization that bridges the gap between micro- and meso-scales.

When using classical continuum thermodynamics concepts, the effects at meso-scale of material degradation at micro-scale are characterized by an internal variable called damage. Hereafter, the effects of the material degradation are assumed to be isotropic at meso-scale and the damage variable to be a scalar denoted by D. In this section, definitions for this scalar variable are derived from the study of two different micro-mechanisms.

1.1. Brittle isotropic damage

1.1.1. Microcracks and scalar damage variable

The main mechanism of brittle damage is the nucleation, growth and coalescence of microcracks up to the initiation at meso-scale of a crack. Hereafter, a relationship between the micro-crack pattern and the damage variable D is established.

Let us consider a Representative Volume Element at meso-scale as a cube of dimension (l * l * l). This RVE is assumed to be constituted at microscale of cubic cells of dimension (d * d * d) in which may lie a microcrack of any area s_i and any orientation (see Fig. 1). The number of cells is $m=l^3/d^3$ and the number of cracks $n \leq m$.

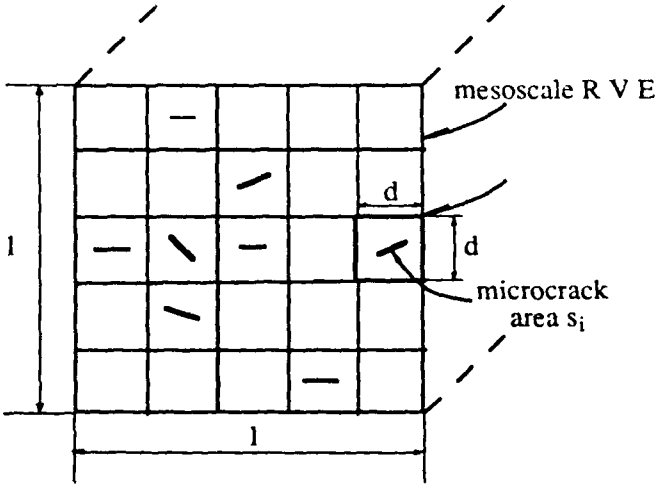

Fig. 1. Micro- and meso-models for brittle damage

The geometry being defined, the modelling consists in writing the balance of the dissipated energy calculated by classical fracture mechanics one one hand, and calculated by continuum damage mechanics on the other hand.

For a cracked cell i, subjected to a given state of stress, if G_i denotes the strain energy release rate corresponding to a crack of area s_i, D_i the equivalent damage of the cell and Y_i the strain energy density release rate, the balance of dissipated energy can be written as :

$$G_i \dot{s}_i = Y_i \dot{D}_i d^3$$

For the n cracked cells of the meso-cube, the previous relation becomes :

$$\sum_1^n G_i \dot{s}_i = \sum_1^n Y_i \dot{D}_i d^3$$

Assuming that brittle growth of microcracks occurs at $G = G_c$ = constant, corresponding to $Y = Y_c$ = constant, it can be written that :

$$G_c \sum_1^n \dot{s}_i = Y_c d^3 \sum_1^n \dot{D}_i$$

Furthermore, if it is assumed that $s_i = 0$ corresponds to $D_i = 0$, integration of the previous relation yields :

$$G_c \sum_1^n s_i = Y_c d^3 \sum_1^n D_i$$

The simplest homogenization consists in defining the damage D at macro-scale by the mean value of the damages D_i of all the l^3/d^3 micro-cells, so that

$$D = \frac{1}{m} \sum_1^m D_i = \frac{1}{n} \sum_1^n D_i = \frac{d^3}{l^3} \sum_1^n D_i$$

or $$D = \frac{G_c}{Y_c l^3} \sum_1^n s_i$$

The term $\frac{G_c}{n Y_c l^3}$ may be evaluated from a rupture criterion defining the initiation of a meso-crack. Because of the localization of the damage phenomenon, it can be assumed that the meso-crack initiation occurs when only part of the flat volume ($l * l * d$) is micro-cracked, the other microcraks being neglected.

In other words, if it is assumed that the mesocrack initiates, i.e. $D = D_c$ at meso-scale, when

$$\sum_1^n s_i = k l^2$$

then,

$$D_c = \frac{G_c}{Y_c l^3} k l^2 \qquad \frac{G_c}{Y_c l^3} = \frac{D_c}{k l^2}$$

Hence,

$$D = \frac{\sum_{1}^{n} s_i}{l^2} \frac{D_c}{k}$$

In this case, the damage variable D appears as the micro-cracks surface density ($\sum_{1}^{n} s_i / l^2$) corrected by a factor (here D_c/k).

If the following simplest fracture criterion is considered

$$\sum_{1}^{n} s_i = l^2 \rightarrow k = 1 \rightarrow D_c = 1 \qquad \text{then,} \quad D = \frac{\sum_{1}^{n} s_i}{l^2}$$

By the way, this calculation gives an order of magnitude of a characteristic length which allows for the matching between fracture mechanics and damage mechanics namely l, the size of the Representation Volume Element. Since,

$$\frac{D}{\sum_{1}^{n} s_i} = \frac{G_c}{Y_c\, l^3} = \frac{D_c}{k\, l^2}$$

$$l = k\, D_c \frac{G_c}{Y_c}$$

Hence, for the simple fracture criterion $k = 1$, $D_c = 1$,

$$l \cong \frac{G_c}{Y_c}$$

For most metallic materials

light alloys		Steel and high alloys	
0.005	$\leq G_c \leq$	0.05	MPa m
2.	$\leq Y_c \leq$	10.	MPa
0.0025	$\leq l \leq$	0.005	m

and for concrete in tension $G_c \cong 3\ 10^{-5}$ Mpa.m, $Y_c \cong 1.5.10^{-4}$ MPa, so that $l \cong 2.10^{-1}$. This shows that the size of the physical RVE must be of the order of the millimeter for metals and of the order of the decimeter for concrete.

1.1.2. Brittle damage growth

As a specific example, let us derive the kinetic damage evolution law at meso-scale which corresponds to the fatigue microcracks growth at micro-scale of Fig.1. For simplicity sake, the analysis is restricted to a two dimensional problem, for which e denotes the uniform thickness of the whole RVE.

With $D_c = k$, it has been established that :

$$D = \frac{\sum_{1}^{n} s_i}{l^2} \quad \text{and} \quad \dot{D} = \frac{\sum_{1}^{n} \dot{s}_i}{l^2}$$

and, for each cracked cell, if 2a denotes the crack length, the following relations hold :

$$K_i = (EG_i)^{1/2} \qquad \dot{K}_i = \frac{E^{1/2}}{2} G_i^{-1/2} \dot{G}_i \qquad \dot{s}_i = 2e \dot{a}_i$$

The surface growth rate $\dot{s}_i$ of each crack can be expressed as a function of the strain energy release rate G_i of the corresponding cell, by means of the Paris' law of fatigue crack growth. If N denotes the number of cycles of loading in mode I, and $K_M = K_{Max}$ the amplitude of the stress intensity factor (with $K_{min} = 0$), then

$$\frac{\delta a}{\delta N} = C K_M^{\eta}$$

where C and η are two material constants, with $\eta \cong 4$ for many metallic materials.
If one assumes that this Paris' law corresponds to the integration over one cycle of :

$$\dot{a} = \eta C K^{\eta-1} \dot{K} \ ,$$

then

$$\dot{s}_i = C e E^{\eta/2} G_i^{\frac{\eta}{2}-1} \dot{G}_i$$

A relationship between G_i and Y_i can be found through their definition from the elastic energy. If w denotes the elastic strain energy density and W_i the elastic strain energy of the elementary cell, then :

$$G_i = - \frac{\partial W_i}{\partial s_i} , \qquad \text{whereas} \qquad Y_i = - \frac{\partial w_i}{\partial D_i}$$

Since

$$W_i = w_i\, d^2\, e,$$

$$G_i = -\frac{\partial(w_i\, d^2\, e)}{\partial D}\,\frac{dD}{ds}$$

If

$$D_i \equiv \frac{s_i}{d_i^2} = \frac{2\, a_i\, e}{d_i\, e}\ ,\ G_i = Y_i\, d_i\ ,\ \dot{G}_i = \dot{Y}_i\, d_i$$

then

$$\dot{s}_i = \eta\, C\, e\, E^{\eta/2}\, d^{\eta/2}\, Y_i^{\frac{\eta}{2}-1}\, \dot{Y}_i$$

Hence, the damage rate is :

$$\dot{D}_i = \frac{\sum \dot{s}_i}{l^2} = \frac{\eta\, C\, E^{\eta/2}\, d^{\eta/2}}{l^2} \sum_{1}^{n} e\, Y_i^{\frac{\eta}{2}-1}\, \dot{Y}_i$$

Assuming that all the n cracked cells have the same strain energy density release rate $Y_i = Y_n$, the homogenized strain energy density release rate for the meso-RVE is :

$$Y = n\, Y_n \quad \text{and} \quad \dot{Y} = n\, \dot{Y}_n$$

Consequently

$$\sum_{1}^{n} e_i\, Y_i^{\frac{\eta}{2}-1}\, \dot{Y}_i = n\, e\, Y_n^{\frac{\eta}{2}-1}\, \dot{Y}_n = e\, n^{1-\frac{\eta}{2}}\, Y_i^{\frac{\eta}{2}-1}\, \dot{Y}$$

$$\boxed{\dot{D} = \frac{\eta\, C\, E^{\eta/2}\, d^{\eta/2}\, e}{l^2\, n^{\frac{\eta}{2}-1}}\, Y^{\frac{\eta}{2}-1}\, \dot{Y}}$$

In this example, the damage rate is an increasing function of the strain energy density release rate ; for most materials since $\eta \equiv 4$, the damage rate is quasi-proportional to the strain energy density release rate. The damage rate is also proportional to the rate $\dot{Y}$. This will be used in section 2.1 as a guideline to derive a general kinetic law for damage evolution.

1.2. Ductile isotropic damage

1.2.1. Microcavities and scalar damage variable

The main mechanism of ductile damage is the nucleation, the growth and the coalescence of microcavities by large local plastic deformations. Hereafter, a relationship between the density of micro voids and the damage variable D is established.

Let us consider again a Representative Volume Element at meso-scale as a cube of dimension (l * l * l). This RVE is assumed to be constituted at microscale of cubic cells of dimension (d * d * d) in which may lie a void of volume d^3 (Fig.2.).

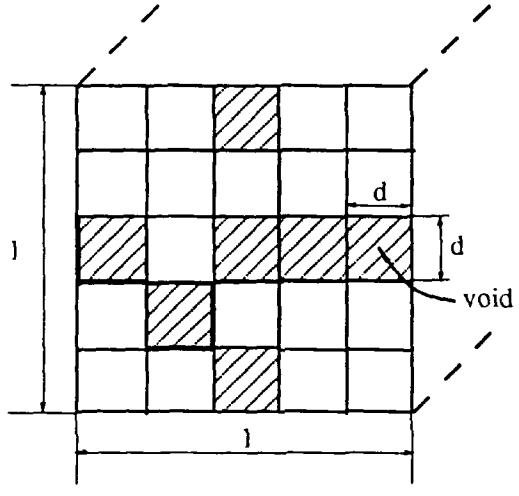

Fig. 2. Micro-meso element for ductile damage

On this very simple geometry, the modelling consists in writing the balance of the dissipated energy calculated from the growth of the cavities on one hand, and calculated by continuum damage mechanics on the other hand.

For the geometrical model under consideration, the porosity P can be defined as :

$$P = 1 - \frac{\rho}{\rho_0} = \frac{n\, d^3}{l^3}$$

where ρ and ρ_0 are the current and initial porosity respectively, and n the number of cavities.

According to Gurson's model, the porosity P at meso-scale is equal to the hydrostatic part $\varepsilon_H^p = \varepsilon_{kk}^p$ of the plastic strain due to the growth of voids.

Here, this assumption leads to the following equality written at meso-scale :

$$\dot{P} = \dot{\varepsilon}_H^p$$

At meso-scale, given an homogenized stress σ_{ij} and a plastic strain rate $\dot{\varepsilon}^p_{ij}$, the total power dissipated is

$$\mathcal{P} = \sigma_{ij} \dot{\varepsilon}^p_{ij}$$

This can be split in two parts by means of the deviatoric and hydrostatic quantities i.e.

$$\mathcal{P} = (\sigma_{ij}^D + \sigma_H \delta_{ij}) (\dot{\varepsilon}^{pD}_{ij} + \dot{\varepsilon}^p_H \delta_{ij})$$

$$= \sigma_{ij}^D \dot{\varepsilon}^{pD}_{ij} + 3 \sigma_H \dot{\varepsilon}^p_H + 0$$

The first term is the power dissipated in pure plasticity by slips. The second term which corresponds to the irreversible change of volume may be interpreted as the power dissipated in the RVE to increase the material discontinuities by growth of the cavities. This latter part must balance the damage dissipation i.e. :

$$3 \sigma_H \dot{\varepsilon}^p_H l^3 = Y \dot{D} l^3$$

so that

$$\dot{D} = \frac{3 \sigma_H}{Y} \dot{P}$$

Assuming for simplicity, proportional loading, perfect plasticity, $\frac{3 \sigma_H}{Y}$ = const, and the initial condition $P = 0 \rightarrow D = 0$, the integration yields :

$$D = \frac{3 \sigma_H}{Y} n \frac{d^3}{l^3}$$

As for brittle damage, because of the localization of the damage phenomenon, it can be assumed that the meso-crack initiation occurs when a set of cavities occupies only part of the flat volume ($l * l * d$), the other cavities in the RVE being neglected.

In other words, the critical value of the porosity corresponding to $D = 1$ is assumed as

$$P_c = n \frac{d^3}{l^3} = k \frac{l^2 d}{l^3} = k \frac{d}{l}$$

This allows for the calculation of the term $\frac{3 \sigma_H}{Y}$ in the damage equation :

$$D = 1 = \frac{3 \sigma_H}{Y} k \frac{d}{l}$$

Hence,

$$\boxed{D = \frac{n d^2}{k l^2}}$$

1.2.2. Ductile damage growth

As a specific example, let us derive the kinetic damage evolution law at meso-scale which corresponds to the voids growth at micro-scale. For simplicity sake, the analysis is restricted to the particular case k = 1.

The kinetic law for damage evolution can be directly derived from the expression for D established in the previous section, so that

$$\dot{D} = \frac{d^2}{l^2}\,\dot{n} + 2\,n\,\frac{d\,\dot{d}}{l^2}$$

The first term accounts for the increase of the number of cavities and $\dot{n}$ denotes the number of cavities nucleated per second. The second term accounts for the cavity growth. In the Gurson model, the porosity rate is also the sum of two terms accounting for nucleation and growth.

a) *Damage growth by nucleation of cavities*

To model nucleation, Tvergaard proposed the following kinetic law for porosity

$$\dot{P} = A\,\dot{\sigma}_{eq} + B\,\dot{\sigma}_H$$

where A and B are material parameters.

Assuming for simplicity sake a sudden nucleation of cavities of a fixed size d

$$\dot{P} = \dot{n}\,\frac{d^3}{l^3}$$

and

$$\dot{D} = \dot{P}\,\frac{l}{d} = \frac{l}{d}\,(A\,\dot{\sigma}_{eq} + B\,\dot{\sigma}_H)$$

so that,

$$\dot{D} = \frac{l}{d}\,\dot{\sigma}_{eq}\,\left(A + B\,\frac{\dot{\sigma}_H}{\dot{\sigma}_{eq}}\right)$$

Damage can be expressed as a function of the accumulated plastic strain rate $\dot{p}$ written in terms of the plastic tangent modulus E_T. Assuming proportional loading, i.e.

$$\frac{\dot{\sigma}_H}{\dot{\sigma}_{eq}} = \frac{\sigma_H}{\sigma_{eq}}$$

$$\dot{p} = \frac{2}{3}\,\dot{\varepsilon}^p_{ij}\,\dot{\varepsilon}^p_{ij} = \frac{\dot{\sigma}_{eq}}{E_T}$$

so that,

$$\boxed{\dot{D} = \frac{l}{d}\,E_T\left(A + B\,\frac{\sigma_H}{\sigma_{eq}}\right)\dot{p}}$$

b) *Damage growth by enlargement of a fixed number n of cavities.*

The problem of void growth has received much attention in the past 20 years. Essential results are the McClintock and Rice & Tracey analyses which derive the rate of growth of a cylindrical or spherical cavity of volume V in a perfectly plastic infinite body as a function of the accumulated plastic strain rate $\dot{p}$ and triaxiality ratio σ_H/σ_{eq} , i.e.

$$\dot{V} = 0.85\ V\ \dot{p}\ \exp\left(\frac{3}{2}\ \frac{\sigma_H}{\sigma_{eq}}\right)$$

Taking

$$V = d^3$$

leads to

$$3\ d^2\ \dot{d} = 0.85\ d^3\ \dot{p}\ \exp\left(\frac{3}{2}\ \frac{\sigma_H}{\sigma_{eq}}\right)$$

Since

$$\dot{D} = 2\ n\ \frac{d\ \dot{d}}{l^2}\ ,\ D = n\ \frac{d^2}{l^2}\ ,\ \dot{D} = 2\ D\ \frac{\dot{d}}{d}$$

$$\boxed{\dot{D} = 0.57\ D\ \dot{p}\ \exp\left(\frac{3}{2}\ \frac{\sigma_H}{\sigma_{eq}}\right)}$$

c) *Conclusions*

In this example, the damage rate by nucleation and growth of voids appears as

- proportional to the accumulated plastic strain rate,
- an increasing function of the triaxiality ratio $\frac{\sigma_H}{\sigma_{eq}}$,
- through E_T or D, a function of the current state of the material.

This will be used in section 2.1 as a guideline to derive a general kinetic law for damage evolution.

2. GENERAL PROPERTIES AND FORMULATION

2.1. A general form for continuum damage evolution law

a) *General formalism*

Within the framework of classical continuum thermodynamics, the thermo-mechanical state of a material is described by the following set of independent state variables :

$$\mathbb{V} \equiv (\varepsilon, T, \mathbf{V}) \qquad \text{with} \qquad \mathbf{V} \equiv (D, \varepsilon^p, \mathbf{V}^p),$$

where ε, ε^p denote the total and plastic strain tensors respectively, T the temperature, $\mathbf{V}$ the set of the internal variables, and D the damage variable. For simplicity sake, hereafter only isothermal situations will be considered.

The reversible behaviour is described by the Helmholtz specific free energy

$$\Psi = \Psi(\varepsilon, \mathbf{V}),$$

chosen as

$$\Psi = \Psi^e(\varepsilon - \varepsilon^p, D) + \Psi^p(\varepsilon^p, \mathbf{V}^p)$$

with

$$\Psi^e = \frac{1}{2} (\varepsilon - \varepsilon^p) : (1 - D)\, \Xi_0 : (\varepsilon - \varepsilon^p)$$

where Ξ_0 denotes the elasticity matrix of the undamaged material. The thermodynamic forces

$$\mathbb{A} \equiv (\sigma, \mathbf{A})$$

are defined from the following state laws :

$$\sigma = \rho \frac{\partial \Psi}{\partial \varepsilon}, \qquad \mathbf{A} \equiv (-Y, -\sigma, \mathbf{A}^p) = \rho \frac{\partial \Psi}{\partial \mathbf{V}},$$

where ρ denotes the mass density, σ the stress tensor, and Y the strain energy density release rate or damage energy release rate.

The irreversible behaviour is described by a dissipation potential

$$\Phi = \Phi(\sigma, \mathbf{A} ; \mathbb{V}),$$

from which the following evolution laws are derived :

$$\dot{\varepsilon}^p = -\frac{\partial \Phi}{\partial \sigma}, \qquad \dot{\mathbf{V}} = -\frac{\partial \Phi}{\partial \mathbf{A}}$$

In particular, the damage evolution law can be chosen such that

$$\dot{D} = \dot{\lambda} \frac{\partial F_D}{\partial Y}$$

b) *damage vs (micro-)plasticity*

In section 1., it has been established that damage is always related to some irreversible strain either at micro- or meso-level. This property can be taken into account in the evolution law for the damage variable by assuming that the factor $\dot{\lambda}$ is proportional to the accumulated plastic strain so that

$$\dot{D} = \frac{\partial F_D}{\partial Y} \dot{p}$$

The irreversible nature of damage is directly taken into account by the fact that the variable $\dot{p}$ is always positive or null.

In most materials, a certain amount of plasticity must be accumulated before damage at meso-level appears. In metals, this corresponds to the accumulation of micro-stresses in the vicinity of initial defects, of dislocations, ..., prior to the nucleation of micro-cracks or micro-voids. To model this phenomenon, since the damage evolution is governed by the accumulated plastic strain rate, it is natural to introduce a threshold p_D on the variable p, such that

$$\dot{D} = \frac{\partial F_D}{\partial Y} \dot{p} \qquad \text{if } p \geq p_D$$

$$\dot{D} = 0 \qquad \text{if } p < p_D$$

or

$$\dot{D} = \frac{\partial F_D}{\partial Y} \dot{p}\, H(p - p_D)$$

where H() is the Heavyside step function.

In monotonic loading, p_D can be identified as the uniaxial damage threshold ε_D^p, whereas for fatigue or creep processes p_D is a function of the applied stress, as it will be discussed in section 2.2.

c) *driving force for damage*

From the thermodynamical analysis, it has been deduced that the driving force for damage is the strain energy density release rate Y. Hence, F_D must be a function of Y :

$$F_D = F_D\,(Y, \ldots)$$

d) *influence of the triaxiality ratio*

Another important feature of fracture mechanisms is the influence of the triaxiality ratio (σ_H/σ_{eq}), where it is recalled that σ_H denotes the hydrostatic stress and σ_{eq} the Von Mises' equivalent stress. This effect is directly taken into account through the damage energy release rate Y which is a function of the triaxiality factor R_v :

$$Y = \frac{\sigma_{eq}^2 R_v}{2 E (1 - D)^2}$$

with

$$R_v = \frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2$$

e) *a general form*

In order to choose the proper and simplest expression for F_D, let us recall the kinetic damage laws obtained by micromechanics for particular mechanisms in section 1. :

Brittle damage by fatigue growth of micro-cracks :

$$\dot{D} = \frac{\eta\, C\, E^{\eta/2}\, d^{\eta/2}\, e}{1^{\frac{\eta}{2}}\, n^{\frac{\eta}{2}-1}}\; Y^{\frac{\eta}{2}-1}\, \dot{Y}$$

η being of the order of 4

$$\dot{D} = (\text{const}) * Y\, \dot{Y}$$

Although no plasticity has been introduced in the analysis, it always exists at micro-scale at the crack tips of the micro-cracks and it is possible, at least formally to relate $\dot{Y}$ to $\dot{p}$ through a plasticity constitutive equation :

$$Y = Y(\sigma_{eq}) \rightarrow \dot{Y}(\dot{\sigma}_{eq})$$

and

$$\sigma_{eq}(p) \rightarrow \dot{\sigma}_{eq}(\dot{p})$$

Ductile damage by nucleation of micro-cavities :

$$\dot{D} = \frac{1}{d} E_T \left(A + B \frac{\sigma_H}{\sigma_{eq}}\right) \dot{p}$$

Ductile damage by enlargement of micro-cavities :

$$\dot{D} = 0.57\, D\, \dot{p} \exp\left(\frac{3}{2} \frac{\sigma_H}{\sigma_{eq}}\right)$$

The qualitative conclusion which can be drawn from these three results is that the damage rate $\dot{D}$ can be considered as proportional to Y, which is a function of (σ_H/σ_{eq}), and $\dot{p}$:

$$\dot{D} \sim Y \dot{p}$$

or

$$F_D \sim Y^2$$

As in any realistic constitutive equation, a material dependent scale factor, such as (const), $\frac{1}{d} E_T$, or 0.57, must be introduced. Let us denote by S this material constant so that

$$F_D \sim \frac{Y^2}{S}$$

Finally, according to the qualitative properties listed above, the damage potential is naturally written as :

$$F_D(Y; (p.D)) = \frac{Y^2}{2S} \; H(p - p_D)$$

where the factor 2 has been introduced to compensate for the factor (1/2) coming from the derivation. Hence, the proposed general continuum damage evolution law is the following :

$$\boxed{\dot{D} = \frac{Y}{S} \dot{p} \; H(p - p_D)}$$

where two material dependent parameters are introduced, viz. S and p_D which characterize the energetic resistance against the damage process and the damage threshold, respectively. The effects of the temperature T are taken into account through the variation of these coefficients with T and through the accumulated plastic strain rate $\dot{p}$ which is also a function of T.

Several important properties, though not directly introduced in the formulation, are also naturally exhibited by this general evolution law, i.e. :

- the non linear accumulation of damage,
- the effect of mean stress in fatigue,
- the non linear interaction of different kinds of damage.

2.2. Damage threshold

Under monotonic loading, the damage threshold p_D can be identified with the uniaxial damage threshold ε_D^p, whereas in the case of fatigue or creep loadings it is a function of the applied stress. It corresponds to the critical level of plasticity which induces the nucleation of microcracks without any consequence on the mechanical properties and can be related to the energy stored in the material.

Experiments in fatigue have shown that the total plastic strain energy dissipated may reach tremendous values before failure although the stored energy remains constant at microcrack initiation. This stored energy is the result of microstress concentrations which develop in the neighbouring of dislocation networks in metals and of inhomogeneities in other materials. For a unit volume, it is equal to the difference between the total plastic strain energy ($\int_0^t \sigma_{ij}\ \dot{\varepsilon}_{ij}^p\ dt$) and the energy dissipated in heat as given by the Clausius-Duhem inequality of the second principle of thermodynamics.

For instance, in the case of a material exhibiting kinematic $\mathbb{X}$ and isotropic R strain hardenings and no damage, under an isothermal transformation the rate of energy dissipated in heat is :

$$\emptyset = \sigma_{ij}\ \dot{\varepsilon}_{ij}^p - R\ \dot{p} - X_{ij}\ \dot{\alpha}_{ij} \geq 0$$

This expression may be calculated from :

- the potential of dissipation e.g.

$$F = (\mathbb{s}^D - \mathbb{X})_{eq} - R - \sigma_y - \frac{3}{4\ X_\infty} X_{ij}\ X_{ij}$$

- its associated normality flow rule

$$\dot{\varepsilon}_{ij}^p = \frac{\partial F}{\partial \sigma_{ij}} \dot{\lambda}\ ,\quad \dot{p} = -\frac{\partial F}{\partial R}\ \dot{\lambda} = \dot{\lambda},\quad \dot{\alpha}_{ij} = -\frac{\partial F}{\partial X_{ij}} \dot{\lambda}$$

- and the yield criterion

$$f = (\mathbb{s}^D - \mathbb{X})_{eq} - R - \sigma_y = 0$$

so that

$$\emptyset = \left(\sigma_y + \frac{3}{2\ X_\infty}\ X_{ij}\ X_{ij}\right)\dot{p}$$

Hence, the stored energy W_s as a function of time t is :

$$\boxed{W_s(t) = \int_0^t \sigma_{ij}\ \dot{\varepsilon}_{ij}^p\ dt - \int_0^t \left(\sigma_y + \frac{3}{2\ X_\infty}\ X_{ij}\ X_{ij}\right)\dot{p}\ dt}$$

This formula can be simplified if the following assumptions are made :

- the effect of the kinematic hardening is neglected :

$$W_s = \int_0^p (\sigma_{eq} - \sigma_y)\, dp$$

- the variation of σ_{eq} is neglected as for a quasi perfectly plastic material

$$W_s = \left[Sup\ (\sigma_{eq}) - \sigma_y \right] p$$

If this stored energy is considered as a constant for the damage threshold p_D, its value can be identified in the one-dimensional monotonic case used as a reference with $p_D = \varepsilon_{p_D}$.

In the particular case considered above, and with the crude approximations made, the onset for the damage process, or damage threshold, corresponds in the case of a monotonic loading to the ultimate stress σ_u :

$$\left[Sup\ (\sigma_{eq}) - \sigma_y \right] p_D = (\sigma_u - \sigma_y)\ \varepsilon_{p_D}$$

so that

$$\boxed{p_D = \varepsilon_{p_D} \frac{\sigma_u - \sigma_y}{Sup(\sigma_{eq}) - \sigma_y}}$$

As a summary, in the particular case considered in this section, the whole set of equations that governs the damage evolution is :

$$\boxed{\dot{D} = \frac{Y}{S}\ \dot{p}\ H\left(p - \varepsilon_{p_D} \frac{\sigma_u - \sigma_y}{Sup(\sigma_{eq}) - \sigma_y} \right)}$$

3. LOCAL FAILURE CRITERIA

The ultimate stage of continuum damage evolution corresponds to meso-crack initiation. As a first approximation, this critical damage state can be characterized by a critical value D_c of the damage variable, for instance $D_c = 1$.

In fact, the value of D_c is not solely material dependent. As discussed below, it also depends on the local stress state .

It is proposed to relate the conditions for the meso-crack initiation to the conditions for the localization of the deformation in the material. In this section, these latter conditions are studied in details, inside, at the boundaries or at the interfaces of rate-independent solids for both the linear and the non-linear case. Physical interpretations of these conditions are also given.

3.1. Introduction

Since the pioneering works of Hadamard, Hill, Mandel and Rice, the localization of the deformation in rate-independent materials is treated as the bifurcation of the rate problem.

Here, we give a general view of the various bifurcation and localization phenomena for possibly heterogeneous solids made of rate-independent materials. Under the small strain assumption, the behaviour of these materials is described by the following piece-wise linear rate constitutive laws :

$$\text{(1)} \qquad \dot{\underline{\underline{\sigma}}} = \underline{\underline{\underline{\underline{L}}}} : \underline{\underline{\varepsilon}}(\mathbf{v}) \quad \text{with} \quad \begin{cases} \underline{\underline{\underline{\underline{L}}}} = \underline{\underline{\underline{\underline{E}}}} & \text{when } f < 0, \text{ or } \quad f = 0 \text{ and } \underline{\underline{\underline{\underline{b}}}} : \underline{\underline{\underline{\underline{E}}}} : \underline{\underline{\varepsilon}}(\mathbf{v}) < 0 \\ \underline{\underline{\underline{\underline{L}}}} = \underline{\underline{\underline{\underline{H}}}} & \text{when } \quad f = 0 \text{ and } \underline{\underline{\underline{\underline{b}}}} : \underline{\underline{\underline{\underline{E}}}} : \underline{\underline{\varepsilon}}(\mathbf{v}) \geq 0 \end{cases}$$

where $\dot{\underline{\underline{\sigma}}}$ and $\dot{\underline{\underline{\varepsilon}}}(\mathbf{v}) = \underline{\underline{\varepsilon}}$ respectively denote the stress and strain rates, $\mathbf{v}$ the velocity, and f the yield function.

The general class of materials modelled by relations (1) includes for instance the elasto-plastic damageable solids the behaviour of which is described by the constitutive equations discussed in section 2.1.a.

It is shown that, in general, different types of localization phenomena may occur, depending on the failure of one of the three conditions which are described in section 3.2. Their physical interpretation is the following :

- the ellipticity condition is very classical. Its failure is the condition for localization given by Rice and linked to the appearance of deformation modes involving discontinuities of the velocity gradient. It has also been related to stationary acceleration waves ;

- the boundary complementing condition governs instabilities at the boundary of the solid. Its failure leads to deformation modes localized at the boundary and is related to stationary surface waves (for instance Rayleigh waves) ;

- the interfacial complementing condition governs instabilities at interfaces. Its failure leads to deformation modes localized at each side of the interface and is related to stationary interfacial waves (Stonely waves).

3.2. Rate problem analysis : the linear case

Let us consider for instance the body sketched in Fig.3.

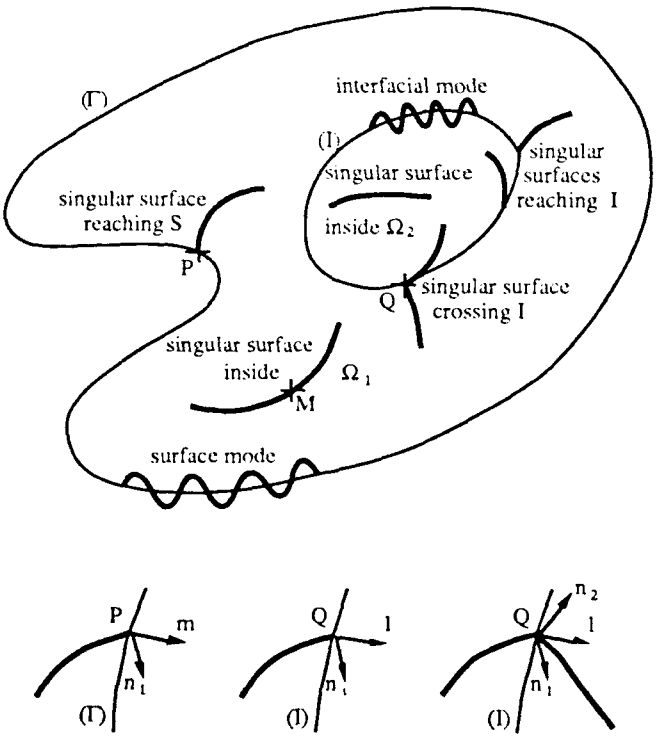

Fig. 3. Different types of localization modes

Qualitative results can be exhibited from the analysis of the rate problem for the so-called linear comparison solid (see Hill). In this case, this linear problem is well-posed if and only if the following conditions are met :

- *the ellipticity condition* : the rate equilibrium equations must be elliptic in the closure of the body Ω, i.e.

$\det(\mathbf{n} \,.\, \Xi \,.\, \mathbf{n}) \neq 0$ for any vector $\mathbf{n} \neq \mathbf{0}$, and any point $M \in \Omega$.

- *the boundary complementing condition* : this relation between the coefficients of the field and boundary operators must be satisfied at every point P belonging to the boundary Γ where the boundary conditions are formally written as $\Xi(\mathbf{v}) = \mathbf{g}$. This condition is easily phrased in terms of an associated problem on a half space defined by $z > 0$. It requires for every vector $\mathbf{k} = (k_1, k_2, 0) \neq \mathbf{0}$, that the only solution to the rate equilibrium equations with constant coefficients (equal to those of the operator at point P), in the form

$$\mathbf{v}(x, y, z) = \mathbf{w}(z) \exp[i\,(k_1\, x + k_2\, y)]$$

with bounded $\mathbf{w}$ and satisfying the homogeneous boundary conditions $\Xi(\mathbf{v}) = \mathbf{0}$, is the identically zero solution $\mathbf{v} \equiv \mathbf{0}$.

- *the interfacial complementing condition* : this relation between the coefficients of the field operators in Ω_1 and Ω_2 must be satisfied at every point Q of the interface I between Ω_1 and Ω_2. This condition is again easily phrased in terms of an associated problem on the whole space divided by the plane interface $z = 0$. It requires for every vector $\mathbf{k} = (k_1, k_2, 0) \neq \mathbf{0}$, that the only solution to the rate equilibrium equations with constant coefficients (equal to those of the operators at point Q, in Ω_1 for $z < 0$ and in Ω_2 for $z > 0$), in the form

$$(\mathbf{v}_1(x, y, z), \mathbf{v}_2(x, y, z)) = (\mathbf{w}_1(z), \mathbf{w}_2(z)) \exp[i\,(k_1\, x + k_2\, y)]$$

with bounded $(\mathbf{w}_1, \mathbf{w}_2)$ and satisfying the continuity requirements (continuity of the velocity and the traction rates) across the interface $z = 0$, is the identically zero solution $(\mathbf{v}_1(z), \mathbf{v}_2(z)) \equiv (\mathbf{0}, \mathbf{0})$; (where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the solutions, respectively for $z < 0$ and for $z > 0$).

When these three conditions are fulfilled, the rate boundary problem admits a finite number of linearly independent solutions, which depend continuously on the data, and which constitute diffuse modes of defonnation.

Remarks :

- these three conditions are local, and this is particulally important when considering their numerical implementation ;

- the above-given results remain valid for an arbitrary number of non-intersecting interfaces, an interfacial cor dition being written for each interface ;

- the failure of these conditions can be interpreted as localization criteria as recalled in section 3.1. These localization criteria can also be used as indicators of the local failure of the material ;

- both boundary and interfacial complementing conditions fail in the elliptic regime of the equilibrium equations, or at the latest, when the ellipticity condition fails. Thus, localized modes of deformation at the boundary or at the interface generally occur before the onset of so-called shear banding modes.

3.3. The non-linear case : some results

Although the complete analysis of the non-linear problem is not yet available, some results can be given for the possibility of emergence of deformation modes involving jumps of the velocity gradient for the bi-linear rate constitutive laws (1).

The necessary and sufficient conditions for the onset of such modes inside the body have been given by Borré & Maier who extended the results given by Rice, and Rudnicki & Rice for so-called continuous and discontinuous localizations. We have amplified these results by seeking necessary and sufficient conditions for which a discontinuity surface for the velocity gradient appears at, or reaches the boundary of the solid. These conditions are given below for the constitutive laws (1) with

$$\mathbb{H} = \mathbb{E} - \frac{(\mathbb{E} : \mathbf{a}) \otimes (\mathbf{b} : \mathbb{E})}{h}$$

where it is assumed that $h > 0$, and $\mathbb{E}$ is strictly positive definite.

At a point P of the boundary Γ where only surface traction rates $\dot{\mathbf{F}}$ are applied, the necessary and sufficient conditions for continuous localization [i.e. the material is in loading (L = H) on each side of the singular surface] are

(2a)
- i) there exists $\dot{\boldsymbol{\varepsilon}}_0$ such that $\mathbf{m} \cdot \mathbb{H} : \dot{\boldsymbol{\varepsilon}}_0 = \dot{\mathbf{F}}$
- ii) $\det (\mathbf{n} \cdot \mathbb{H} \cdot \mathbf{n}) = 0$
- iii) $(\mathbf{m} \cdot \mathbb{E} \cdot \mathbf{n}) \cdot (\mathbf{n} \cdot \mathbb{E} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbb{E} : \mathbf{a}) = \mathbf{m} \cdot \mathbb{E} : \mathbf{a}$

At a point P of the boundary Γ where only surface traction rates $\dot{\mathbf{F}}$ are applied, the necessary and sufficient conditions for discontinuous localization [i.e. the material is in loading ($\mathbb{L} = \mathbb{H}$) on one side and in unloading ($\mathbb{L} = \mathbb{E}$) on the other side of the singular surface] are

i) there exists $\dot{\boldsymbol{\varepsilon}}_0$ such that $\mathbf{m} \cdot \mathbb{H} : \dot{\boldsymbol{\varepsilon}}_0 = \dot{\mathbf{F}}$ and $\mathbb{b} : \mathbb{E} : \dot{\boldsymbol{\varepsilon}}_0 > 0$

(2b) ii) $\det (\mathbf{n} \cdot \mathbb{H} \cdot \mathbf{n}) < 0$

iii) $(\mathbf{m} \cdot \mathbb{E} \cdot \mathbf{n}) \cdot (\mathbf{n} \cdot \mathbb{E} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbb{E} : \boldsymbol{\varepsilon}) = \mathbf{m} \cdot \mathbb{E} : \boldsymbol{\varepsilon}$

In these conditions, $\mathbf{m}$ denotes the unit outward normal to the boundary of the solid in P, whereas $\mathbf{n}$ denotes the unit normal to the singular surface in P.

At a point Q of an interface I, two types of singular surface may occur. The singular surface can either stop at, or cross the interface (see Fig.2). In the latter case, the singular surface can meet the interface at different angles on each side. Analogous conditions to conditions (2) can be exhibited.

Remarks :

- conditions (2) are to be compared with the corresponding conditions inside the solid given by Borré & Maier and stated as

(3) $\det (\mathbf{n} \cdot \mathbb{H} \cdot \mathbf{n}) \leq 0$

with equality corresponding to continuous, and inequality corresponding to discontinuous localization ;

- given conditions (2) and (3), singular surfaces of the type discussed here generally appear first inside the body ;

- similar conditions to conditions (2) can be exhibited for general boundary conditions. (Note that for displacement boundary conditions, relation (3) applies both inside the body and at the boundary.) ;

- conditions (2) and (3) are a priori unrelated to the boundary complementing condition.