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Multivariate Generalizations of Student's t-Distribution

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ABSTRACT

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(The authors)

1 Introduction

If the random variables $x_1, \dots, x_p$ follow the multivariate normal (MVN) distribution with means zero, common variance σ^2, and a correlation matrix $\boldsymbol{\Omega} = [\rho_{ij}]$, and if $(\nu s^2)/\sigma^2$ is an independent χ^2 variable with ν degrees of freedom, then the random vector $\boldsymbol{t} = (t_1, \dots, t_p)$, where $t_i = x_i/s$ (for $i = 1, \dots, p$) is said to have the p-variate t-distribution with ν degrees of freedom. This distribution is the multivariate analogue of Student's t with density function:

$$f(t_1, t_2, \dots, t_p) = \frac{\Gamma(\frac{p+\nu}{2})\,|\boldsymbol{\Omega}^{-1}|^{1/2}}{(\nu\pi)^{p/2}\,\Gamma(\nu/2)}(1 + \frac{1}{\nu}\boldsymbol{t}'\boldsymbol{\Omega}^{-1}\boldsymbol{t})^{-(p+\nu)/2}. \tag{1}$$

The distribution has applications in a number of statistical problems, most notably in the multiple comparison of several treatments with a control (Dunnett, 1955), and as John (1961) has noted, in the construction of simultaneous confidence bounds for the parameters in a linear model. We will discuss these applications of the multivariate-t distribution and suggest a numerical method for evaluating the probabilities associated with this distribution.

2 Dunnett's Test

Consider the problem of comparing each of p treatments with a control in respect to their means $\mu_0, \mu_1, \mu_2, \dots, \mu_p$, where μ_0 designates the control and x_i, $i = 1, 2, \dots, p$, the treatments. Assume that the observations are normally and independently distributed with common within-group standard deviation σ. In this case, Dunnett (1955) has provided a procedure for making confidence statements about the p differences $\mu_i - \mu_0$, such that the probability of *all* p statements being simultaneously correct is equal to a specified P level. Dunnett's procedure and the associated tables are available for the case of equal sample sizes in all groups. Here, we will expand the procedure to the case where the sample sizes are not equal, and to an even more general class of problems involving simultaneous statistical inference.

Suppose that there are n_0 observations for the control, n_1 observations for the first treatment, ... , n_p observations for the p-th treatment, and denote these observations by X_{ij} ($i = 0, 1, \dots, p$; $j = 1, 2, \dots, n_i$) and the corresponding i-th treatment mean as $\bar{X}_i$. Assume that there is an estimate of σ^2 available (denoted s^2) based on ν degrees of freedom, which is independent of the estimator of the mean. Now let

$$z_i = \frac{\bar{X}_i - \bar{X}_0 - (\mu_i - \mu_0)}{\sqrt{\frac{n_0+n_i}{n_0 n_i}}}. \tag{2}$$

and let $t_i = z_i/s$ for $i = 1, 2, \ldots, p$. As Dunnett (1955) notes, the lower confidence limits with joint confidence coefficient P for the p treatment effects $\mu_i - \mu_0$ are given by

$$\bar{X}_i - \bar{X}_0 - d_i s \sqrt{\frac{n_0 + n_i}{n_0 n_i}}, \tag{3}$$

if the p constants d_i are chosen so that

$$\text{Prob}(t_1 < d_1, t_2 < d_2, \ldots, t_p < d_p) = P. \tag{4}$$

To find the p constants d_i that satisfy these equations, the joint distribution of the t_i is required, which is the multivariate analogue of Student's t-distribution defined by Dunnett and Sobel (1955). Dunnett (1955) has shown how the problem of evaluating the multivariate t-distribution can be reduced to the problem of evaluating the corresponding MVN distribution. For the latter, notice that the joint distribution of the z_i is a MVN distribution with means 0 and variances σ^2. The correlation between z_i and z_j is given by:

$$\rho_{ij} = 1/\sqrt{\left(\frac{n_0}{n_i} + 1\right)\left(\frac{n_0}{n_j} + 1\right)}. \tag{5}$$

which for the special case of equal sample sizes equals 1/2 for all i and j. Dunnett and Sobel note that the joint probability statement given above can be written in the following way:

$$\begin{aligned} P &= \text{Prob}(t_1 < d_1, t_2 < d_2, \ldots, t_p < d_p) \\ &= \text{Prob}(z_1 < d_1 s, z_2 < d_2 s, \ldots, z_p < d_p s) \\ &= \int_{-\infty}^{+\infty} F(d_1 s, d_2 s, \ldots, d_p s) \, f(s) \, ds. \end{aligned} \tag{6}$$

where $F(d_1 s, d_2 s, \ldots, d_p s)$ is the MVN cdf of the z_i and $f(s)$ is the one-dimensional density function of s. Thus, with probability values for $F(\cdot)$, the above equation can be evaluated using numerical integration over the distribution of s. For this, note that the density function of s is given by Pearson and Hartley (1976) as:

$$f(s) = \frac{\nu^{\nu/2}}{\Gamma(\frac{\nu}{2}) 2^{(\nu/2)-1}} \sigma^{-\nu} s^{\nu-1} \exp(-\nu s^2/2\sigma^2). \tag{7}$$

Since $s^2/\sigma^2 = \chi^2/\nu$ we can rewrite the equation for P in terms of integration over the distribution of $u = s/\sigma$ (which is defined on 0 to $+\infty$) as:

$$
\begin{aligned}
P &= \int_0^{+\infty} F(d_1 u, d_2 u, \ldots, d_p u)\, f(u) \frac{ds}{du}\, du \\
&= \int_0^{+\infty} F(d_1 u, d_2 u, \ldots, d_p u)\, \frac{\nu^{\nu/2}}{\Gamma(\frac{\nu}{2}) 2^{(\nu/2)-1}} u^{\nu-1} \exp(-\nu u^2/2)\, du. \quad (8)
\end{aligned}
$$

Numerical integration over the distribution of c can then be performed to yield the associated probability P for selected values of d, p, and ν.

3 Some Special Cases

Direct evaluation of the MVN cdf is not possible for $p > 3$. In the following, we note some special cases for which reduction formulae are available.

3.1 Case 1: $n_0 = n_i = n$. $(i = 1, \ldots, p)$

When all sample sizes are equal, the correlation in (5) is 1/2 for all possible pairings of the treatment groups and the control. Dunnett (1955) has given tables for the critical values of this distribution. In this case, the MVN probability in (8) is simply

$$F_p(0, 0, \ldots, 0; \{.5\}) = \frac{1}{p+1}. \quad (9)$$

3.2 Case 2: $n_0 = n$ and $n_i = m$. $(i = 1, \ldots, p)$

When the p treatment groups are each of size m, but the control group is of size n, where $n \neq m$, then from (5), $\rho_{ij} = \rho$ for all i, j, and the the probability in (8) is

$$F_p(ds, ds, \ldots, ds; \{\rho\}) = \int_{-\infty}^{\infty} \left[F^p \left(\frac{ds + \rho^{1/2} y}{\sqrt{1-\rho}} \right) \right] f(y) d(y), \quad (10)$$

where $f(t) = exp(-\frac{1}{2}t^2)/(2\pi)^{1/2}$ and $F(t) = \int_{-\infty}^{ds} f(t)dt$. see Gupta (1963).

3.3 Case 3: $n_0 = n$ and n_i unequal

When the treatment group sample sizes are unequal, the correlation matrix $\{\rho_{ij}\}$ has the special form

$$
\begin{aligned}
\rho_{ij} &= \alpha_i \alpha_j \\
&= \left(\frac{n_0}{n_i}+1\right)^{-1/2}\left(\frac{n_0}{n_j}+1\right)^{-1/2},
\end{aligned}
$$

for $(i \neq j)$, where $-1 \leq \alpha_i \leq +1$. In this case, the MVN cdf is:

$$
\begin{aligned}
F_p(d_1 s, d_2 s, \ldots, d_p s; \{\rho_{ij}\}) &= \int_{-\infty}^{\infty} Prob\left\{X_i < \left(\frac{d_i s - \alpha_i y}{\sqrt{1-\alpha_i^2}}\right);\ \text{all i}\right\} f(y)d(y) \\
&= \int_{-\infty}^{\infty}\left[\prod_{i=1}^{p} F\left(\frac{d_i s - \alpha_i y}{\sqrt{1-\alpha_i^2}}\right)\right] f(y)d(y), \qquad (11)
\end{aligned}
$$

(see Dunnett and Sobel, 1955). This MVN integral can be approximated to any practical degree of accuracy using Gauss-Hermite qaudrature (Stroud and Sechrest, 1966).

4 The General Case

The special cases in the previous sections provide methods for evaluating the MVN integral in (8), that cover all possible applications of the Dunnett type multiple comparison with control procedure, regardless of the sample sizes of the various groups. Nevertheless, there are still situations in which a completely general solution is required. Of course, for the general case, a more general method for evaluating the probabilities of the MVN cdf $F(\cdot)$ is needed. For example, in regression analysis the $(b_1, b_2, \ldots, b_p)$ are MVN with means $(\beta_1, \beta_2, \ldots, \beta_p)$ and variance covariance matrix $\{c_{ij}\}\sigma^2 = S^{-1}\sigma^2$, where $S_{ij} = \sum_{r=1}^{N}(x_{ir} - \bar{x}_i)(x_{jr} - \bar{x}_j)$ for $(i, j = 1, 2, \ldots, p)$. In this case, $\{\rho_{ij}\} = (c_{ii}c_{jj})^{-1/2}c_{ij}$ and none of the previous reduction formulae apply. One computationally tractable possibility is to use Clark's (1961) formulae for the moments of the maximum of p correlated normal variables as applied by Gibbons *et. al.*, (1987) to the problem of approximating MVN orthant probabilities. A brief description of this approximation is now provided, and we will show that these probabilities are sufficiently accurate for practical purposes.

4.1 The Clark Algorithm

We begin by noting that the MVN cdf $F_p(d_1 s, d_2 s, \ldots, d_p s; \{\rho_{ij}\})$ can be written as:

$$
F_p = Pr(x_1 \leq h_1, x_2 \leq h_2, \ldots x_n \leq h_n), \qquad (12)
$$

where $h_i = d_i s$. If $h_1 \ldots h_n = h = 0$, and the x_i follow a standardized MVN distribution, F_n^0 is a so-called "orthant" probability. However, note that we can also write this MVN probability as:

$$F_p^0 = Pr\left\{max(x_1, \ldots, x_n) \leq 0\right\}. \tag{13}$$

If $max(x_1, \ldots, x_n)$ were normally distributed, which it clearly is not, with mean $E[max(x_1, \ldots, \quad ,)]$ and variance $V[max(x_1, \ldots, x_n)]$, then,

$$F_p^0 = F\left[\frac{E(max(x_1, \ldots, x_n) - h}{\sqrt{V(max(x_1, \ldots, x_n)}}\right]. \tag{14}$$

where in this case $h = 0$. For general h_i, we would set $h = 0$ and subtract h_i from the mean of x_i.

In order to proceed, we need the first two moments of $max(x_1, \ldots x_n)$ where the x_i have a joint MVN distribution with general correlation $\{\rho_{ij}\}$, and some bound on the error introduced by assuming that $max(x_1, \ldots x_n)$ has a normal distribution. Clark (1961), has provided an approximation for the first four moments of the maximum of p jointly normal correlated random variables, and Gibbons *et. al.*, (1990), have shown that the accuracy of the approximation is approximately 10^{-3} in problems of this kind. An overview of the approximation is provided in the following.

Let any three successive components from an p-variate vector, $\boldsymbol{y}_1$, be distributed:

$$\begin{bmatrix} y_i \\ y_{i+1} \\ y_{i+2} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_i \\ \mu_{i+1} \\ \mu_{i+2} \end{bmatrix}, \begin{bmatrix} \sigma_i^2 & & \\ \sigma_i \sigma_{i+1} \rho_{i,i+1} & \sigma_{i+1}^2 & \\ \sigma_i \sigma_{i+2} \rho_{i,i+2} & \sigma_{i+1} \sigma_{i+2} \rho_{i+1,i+2} & \sigma_{i+2}^2 \end{bmatrix}\right)$$

Let $\tilde{y}_i = \max(y_i) = y_i$, and compute the probability that $y_{i+1} > \tilde{y}_i$ as follows:

$$\begin{aligned} \text{set} \qquad & z_{i+1} && = (\mu_i - \mu_{i+1})/\zeta_{i+1}, \\ \text{where} \qquad & \zeta_{i+1}^2 && = \sigma_i^2 + \sigma_{i+1}^2 - 2\sigma_i \sigma_{i+1} \rho_{i,i+1}. \\ \text{Then} \qquad & P(y_{i+1} > \tilde{y}) && = P(y_{i+1} - \tilde{y} > 0) \\ & && = \Phi(-z_{i+1}) \end{aligned}$$

the value of the univariate normal distribution function at the standard deviate $-z_{i+1}$.

Now let $\tilde{y}_{i+1} = \max(y_i, y_{i+1})$ and assume (as an approximation) that $(y_{i+2}, \tilde{y}_{i+1})$ is bivariate normal with means,

$$\begin{aligned} \mu(y_{i+2}) &= \mathcal{E}(y_{i+2}) &=& \mu_{i+2} \\ \mu(\tilde{y}_{i+1}) &= \mathcal{E}(\tilde{y}_{i+1}) &=& \mu_i\Phi(z_{i+1}) + \mu_{i+1}\Phi(-z_{i+1}) + \zeta_{i+1}\phi(z_{i+1}), \end{aligned} \tag{15}$$

variances

$$\begin{aligned} \sigma^2(y_{i+2}) &= \mathcal{E}(y_{i+2}^2) - \mathcal{E}^2(y_{i+2}) = \sigma_{i+2}^2, \\ \sigma^2(\tilde{y}_{i+1}) &= \mathcal{E}(\tilde{y}_{i+1}^2) - \mathcal{E}^2(\tilde{y}_{i+1}), \end{aligned} \tag{16}$$

where

$$\mathcal{E}(\tilde{y}_{i+1}^2) = (\mu_i^2 + \sigma_i^2)\Phi(z_{i+1}) + (\mu_{i+1}^2 + \sigma_{i+1}^2)\Phi(-z_{i+1}) + (\mu_i + \mu_{i+1})\zeta_{i+1}\phi(z_{i+1}), \tag{17}$$

and correlation

$$\rho(\tilde{y}_{i+1}, y_{i+2}) = \frac{\sigma_i\rho_{i,i+2}\Phi(z_{i+1}) + \sigma_{i+1}\rho_{i+1,i+2}\Phi(-z_{i+1})}{\sigma(\tilde{y}_{i+1})}. \tag{18}$$

Then,

$$P\left(y_{i+2} = \max(y_i, y_{i+1}, y_{i+2})\right) = P\left((y_{i+2} - y_{i+1} > 0) \cap (y_{i+2} - y_i > 0)\right) \tag{19}$$

is approximated by

$$\begin{aligned} P(y_{i+2} > \tilde{y}_{i+1}) &= P(y_{i+2} - \tilde{y}_{i+1} > 0) \\ &= \Phi\left(\frac{\mu_{i+2} - \mu(\tilde{y}_{i+1})}{\sqrt{\sigma_{i+2}^2 + \sigma^2(\tilde{y}_{i+1}) - 2\sigma_{i+2}\sigma(\tilde{y}_{i+1})\rho(\tilde{y}_{i+1}, y_{i+2})}}\right). \end{aligned} \tag{20}$$

Assuming as a working approximation that $\tilde{y}_{i+1}$ is normally distributed with the above mean and variance, we may therefore proceed, recursively from $i = 1$ to $i = p - 1$, where y_{p+1} is an independent dummy variate with mean zero and variance zero (i.e. $y_{p+1} = 0$). Then, for example,

$$\begin{aligned} & P[y_{p+1} = \max(y_1, y_2, \ldots, y_{p+1})] \\ &= P\left[(y_{p+1} - y_1 > 0) \cap (y_{p+1} - y_2 > 0) \cap \ldots \cap (y_{p+1} - y_p > 0)\right] \\ &= P\left[(-y_1 > 0) \cap (-y_2 > 0) \cap \ldots \cap (-y_p > 0)\right] \end{aligned} \tag{21}$$

approximates the negative orthant. The probability of any other orthant can be obtained by reversing the signs of the variates corresponding to 1's in the orthant pattern.

More generally, to compute any MVN orthant probability, for example,

$$\int_{-\infty}^{h} \int_{-\infty}^{h} \cdots \int_{-\infty}^{h} f(x_1, x_2, \ldots, x_p; \{\rho_{ij}\}) dx_1 \ldots dx_p \tag{22}$$

we compute the negative orthant setting $\mu_{p+1} = h$. Finally, to approximate the integral for general h_i, we compute the negative orthant by setting $\mu_{p+1} = 0$ and $\mu_i = \mu_i - h_i$. In the present context $h_i = d_i s$

4.2 Applications

To illustrate the use of the general approximation, consider the following two multivariate prediction problems.

4.2.1 Confidence Bounds for Means

Simultaneous confidence bounds for the means of correlated normal variables can now be found using the general method. Suppose $x_1, \ldots, x_p$ are MVN with mean vector $\mu_1, \ldots, \mu_p$ and dispersion matrix $\sigma^2\{\rho_{ij}\}$, where $\{\rho_{ij}\}$ is the correlation matrix. The Clark algorithm can be used to satisfy the inequality,

$$\bar{x}_i - N^{-\frac{1}{2}} ds < \mu_i < \bar{x}_i + N^{-\frac{1}{2}} ds \tag{23}$$

for $(i = 1, \ldots, p)$.

4.2.2 Confidence Bounds for a Future Observation

Similarly, simultaneous confidence bounds for a future p-variate observation may also be found in this way. Suppose $x_1, \ldots, x_p$ represent a future observation vector from a MVN population with equal variances and correlation matrix $\{\rho_{ij}\}$. A previous sample of size N is available from which the estimates $\bar{x}$ and s are obtained. The Clark algorithm can be used to satisfy the inequality,

$$\bar{x}_i - (1 + 1/N)^{-\frac{1}{2}} ds < x_i < \bar{x}_i + (1 + 1/N)^{-\frac{1}{2}} ds \tag{24}$$

for $(i = 1, \ldots, p)$. The value $h = ds$ is selected, such that the desired confidence level P in (6) is obtained.

Table 1

95% Critical Values for Various Modifications of Dunnett's Test

						I	II	III	IV
Case	n_0	n_1	n_2	n_3	n_4	$(\rho_{ij} = \rho = .5)$	$(\rho_{ij} = \rho)$	$(\rho_{ij} = \alpha_i\alpha_j)$	$\{\rho_{ij}\}$
1	10	10	10	10	10	2.22	2.22	2.22	2.20
1	20	20	20	20	20	2.19	2.19	2.19	2.17
2	10	20	20	20	20	2.19	2.13	2.13	2.12
2	10	30	30	30	30	2.18	2.09	2.09	2.07
2	20	10	10	10	10	2.20	2.25	2.25	2.22
3	10	30	50	20	10	2.19	2.10	2.11	2.10
3	10	5	50	10	50	2.18	2.09	2.13	2.13

I Dunnett's original test (correct for case 1 only)

II n_0 controls and $m = n_1 =, \cdots, = n_4$ treatments (correct for cases 1 & 2)

III All n_i potentially different (1 dimensional quadrature; correct for cases 1,2,3)

IV General $\{\rho_{ij}\}$ (Clark approximate for all cases)

5 . Illustrations

Table 1 presents comparisons of various modifications of Dunnett's test for various sample size combinations for a 5 group study.

Inspection of Table 1 reveals that all three of the reduction formulae work exactly as anticipated. The general solution based on the Clark approximation performs quite well, and if anything, it's accuracy is best in those cases when it is most needed, *i.e.*, when the correlations are heterogeneous. Dunnett's original tabled values (*i.e.*, case I), appear to overestimate the true values when $n_0 < n_i$ and underestimate the true values when $n_0 < n_i$. In general, the case II solution (*i.e.*, $\{\rho_{ij}\} = \rho$) works reasonably well under all conditions; however, it is somewhat biased in the final example in which the sample sizes are quite variable.

As a second numerical example, let us return to the problem of obtaining simultaneous confidence bounds for regression coefficients. Mosteller and Tukey (1977, pages 549-551) recovered demographic transition data on fertility rates and five socioeconomic indicators from 47 Swiss provinces in 1888. The socioeconomic indicators were:

1. Proportion of population involved in agriculture as an occupation.
2. Proportion of draftees receiving highest mark on army examination.

3. **Proportion of population whose education is beyond primary school.**
4. **Proportion of the population who are catholic.**
5. **Infant mortality**: proportion of live births who live less than 1 year.

The common standardized fertility measure I_g was used as the dependent measure and the socioeconomic indicators $x_1, \ldots, x_5$ were the predictors. The least squares estimated regression equation was:

$$\hat{I}_g = .645 - .203x_1 - .295x_2 - .896x_3 + .001x_4 + 1.316x_5$$

This regression equation reveals that fertility is inversely related to socioeconomic status, which is consistent with the fact that at the time, fertility was beginning to fall from the high level generally found in underdeveloped countries to the lower level that it has today. The correlation matrix $\{\rho_{ij}\} = (c_{ii}c_{jj})^{-1/2}c_{ij}$ of the $(b_1, b_2, \ldots b_5)$ was:

$$\{\rho_{ij}\} = \begin{bmatrix} 1.00 & & & & \\ .21 & 1.00 & & & \\ .39 & -.59 & 1.00 & & \\ -.26 & .55 & -.47 & 1.00 & \\ .17 & -.03 & .15 & -.17 & 1.00 \end{bmatrix}$$

and the unbiased moment estimator s^2 of σ^2,

$$s^2 = \frac{1}{N - p - 1}(T_{yy} - b_1T_{y1} - b_2T_{y2} - \ldots - b_5T_{y5}) \tag{25}$$

where

$$T_{yi} = \sum_{k=1}^{N} y_k(x_{ik} - \bar{x}_i) \quad (i = 1, 2, \ldots, p) \tag{26}$$

and

$$T_{yy} = \sum_{k=1}^{N}(y_i - \bar{y})^2 \tag{27}$$

was $s^2 = .0045$. The elements $c_{ii}, i = 1, 2, \ldots, p$ of S^{-1} were $c_{11} = .96$, $c_{22} = 11.89$, $c_{33} = 6.55$, $c_{44} = .000024$, and $c_{55} = 30.58$. Using the general approximation, we find that the inequalities

$$b_i - dc_{ii}^{\frac{1}{2}}s \leq \beta_i \leq b_i + dc_{ii}^{\frac{1}{2}}s \quad (i = 1, 2, \ldots, p) \tag{28}$$

are simultaneously satisfied for $P = .95$ when $d = 2.32$, which yields the confidence limits:

$$\begin{array}{rcccl}
-.36 & \leq & \beta_1 & \leq & -.051 \\
-.83 & \leq & \beta_2 & \leq & .24 \\
-1.29 & \leq & \beta_3 & \leq & -.50 \\
.00024 & \leq & \beta_4 & \leq & .0018 \\
.46 & \leq & \beta_5 & \leq & 2.18
\end{array}$$

The confidence limit for b_2 (*i.e.*, proportion of draftees receiving highest marks on army examination), was the only interval that included $\beta = 0$. For a single interval $d = t_{41,.05} = 2.02$, which is considerably smaller than the simultaneous value of $d = 2.32$ used here. Had we used a simple Bonferroni type adjustment (*i.e.*, $\alpha = .05/5 = .01$), then $d = t_{41,.01} = 2.70$, which would clearly have been overly conservative.

6 Summary

In this paper we have provided methods for evaluating the multivariate t-distribution with and without restrictions on the form of the correlation matrix $\{\rho_{ij}\}$. Using these results, Dunnett's test for multiple treatments compared to a single control was then generalized to various unbalanced cases. In the more general case, in which $\{\rho_{ij}\}$ does not have a simple unidimensional form, we have applied Clark's approximation to the moments of the maximum of n correlated random normal variables to the problem of approximating the required MVN cdf. This approach appears to work well, and is the only computationally tractable solution for the case of general $\{\rho_{ij}\}$. Application to the problem of obtaining simultaneous confidence limits for regression coefficients, clearly illustrates the importance of this approach, given that repeated use of limits designed for a single comparison yield inadequate coverage, and simplistic adjustments that do not take the correlational structure into consideration, (*e.g.*, Bonferroni adjusted $\alpha^* = \alpha/p$), yield limits that are overly conservative.

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