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DEVELOPMENT OF AN ADVANCED CONTINUUM THEORY FOR COMPOSITE LAMINATES

Phase I Final Report

June 27, 1990

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A continuum theory for laminated composite materials, referred to as "Cosserat Composite Theory", was developed. The theory was represented by a set of well defined conservation laws that within the context of purely mechanical theory exhibits the following features: i) it accounts for the effect of microstructures, ii) it accounts for the effect of geometric nonlinearity, iii) it accounts for the interlaminar stresses and therefore delamination can be considered, iv) it is capable of incorporating the effect of material nonlinearity, v) it accounts for the effect of curvature, vi) it possesses a continuum character, and finally vii) it is applicable to both static and dynamic problems. The composite laminate was modeled as a series of Cosserat surfaces which were considered as microstructures. Various quantities associated with the microstructure were defined and the corresponding quantities for composite laminates were derived. The nonlinear constitutive equations for an elastic composite laminate were presented. Using a systematic linearization procedure, the linear Cosserat Composite Theory was derived. Finally the application of the theory to flat and cylindrical laminates was considered.

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Table of Contents

Introduction 1

A. Preliminaries 8

2. Coordinate systems 8
3. General background 10
4. Basic equations of classical continuum mechanics in general curvilinear coordinates 14

B. Introduction to theory of two-dimensional directed continuum, i.e., a Cosserat theory 23

5. Definition of a shell-like body 25
6. General kinematical results for shells 28
7. Superposed rigid body motions 33
8. Stress resultants, stress couple and other related definitions 35
9. Basic field equations for a shell-like body 41

C. Modeling of a composite laminate as a series of Cosserat (directed) surfaces 47

10. Coordinate systems for a composite laminate 50
11. Definition of a shell-like representative element (micro-structure) 52
12. Kinematics of micro- and macro-structures 58
13. Superposed rigid body motion 67
14. Stress-resultants, stress-couples and other definitions 69
15. Basic field equations for a shell-like representative element (micro-structure) 73
16. Conservation laws for a shell-like representative element (micro-structure) 79
17. Conservation laws for composite laminates 87
18. Summary of basic principles for composite laminates 103
19. Considerations on composite contact force and composite contact couple 108
20. Further consideration of the composite conservation laws 115
21. Further consideration of composite contact couple 121
22. Basic field equations of composite laminates 129

D. Elastic composite laminates (nonlinear theory) 140

23. Constitutive equations for nonlinear elastic composite laminates. Direct approach 140
24. The complete theory 149
25. A constrained theory of composite laminates 152
26. Constitutive equations of an elastic composite laminate: Derivation from three-dimensional classical continuum theory 161

E. Linearized theory 169

27. Linearized kinematics 169
28. Linearized field equations 175
29. Linear constitutive relations for elastic composite laminates 177

BASE

F. Application and comparison with the available theories 180
30. Preliminaries: Part I 181
31. Preliminaries: Part II 187
32. Linear constitutive relations for composite laminates: An alternative procedure 193
33. Some results for the case of a normal director 197
34. Theory of initially flat composite laminates 201
35. Theory of initially flat cylindrical composite laminates 207
36. Comparison with the available theories 222
References 236
Figures 252

BASE

Introduction

Composite materials have fully established themselves as workable engineering materials. Early military application during World War II led to large-scale commercial and aerospace utilization. Today, industries such as aircraft, automobiles, sporting goods, electronics, and appliances are quite dependent on composite materials. In particular, advance composite materials for aerospace, structural, power and propulsion application offer significant advantages in terms of efficiency and cost. A widespread and efficient application of composite materials requires detailed and reliable knowledge of their physical properties and, in turn, of their behavior under applied loads. Because of potentially diverse structural and physical variety of reinforced composites, it is neither practical nor economical to rely solely on experimental determination of their properties. Therefore, similar to any other branch of physical sciences, it is desirable to develop a theory (or theories) so that we can analyze, explain, and predict the behavior of composite materials under various in-use loading conditions.

Generally speaking, composite materials are based on the concept of compounding reinforcing elements and matrix materials such that they form a reinforced composite. The mechanical behavior of such materials is termed mechanics of composite materials. More specifically, a composite material is one in which two or more constituents are combined to produce a new material with mechanical properties different from those of the individual constituents. It is assumed that the constituents of a composite material retain their individual chemical and mechanical integrity and characteristics. A typical composite material consists of a bounding, or matrix material containing a second reinforcing material in the form of continuous or discontinuous filaments or laminations. Major parameters involved in mechanics of composite materials are: volume fractions of reinforcing elements and matrix, direction of reinforcement, geometry of reinforcing elements and position of reinforcing elements relative to each other. Additional variety stems from the physical properties of the constituents. Altogether, the variation of the geometrical and physical parameters can lead to an enormous number of possibilities. It is, therefore, desirable to have theories that can describe the physical behavior of the composite in terms of the known geometrical layout of the composite and the known physical properties of the consituents.

An appropriate classification of the mechanics of composite materials may be brought about by the definition of two areas of composite material behavior as follows:

a) Macromechanics: The study of composite material behavior wherein the material is presumed homogeneous and the effect of the constituent materials are detected only as averaged apparent properties of the composite.

b) Micromechanics: The study of composite material behavior wherein the interaction of the constituent materials is examined in detail as part of the behavior of the heterogeneous composite material.

The properties of a lamina can be experimentally determined in the "as made" state or can be mathematically derived on the basis of the properties of the constituent materials. That is, we can predict lamina properties by the procedures of micromechanics and we can measure lamina properties by physical means and use the properties in a micromechanical analysis of the structure. Knowledge of how to predict properties is essential in constructing composites that have certain apparent or macroscopic properties. Thus, micromechanics is a natural compliment to macromechanics, and the formulation of an adequate (continuum) theory that could describe the mechanical behavior considering the micro-structure of of composite materials is highly desirable and of major concern in material engineering, especially in relation to aerospace industries, due to many advantages that composite materials offer in terms of cost, weight and peformance.

In the last three decades several continuum theories have been proposed as models of elastostatics or elastodynamics of composite materials. In general these theories may be divided into two major categories as follows:

1) theories that do not account for the effect of microstructure.

2) theories that consider the behavior of microstructure and try to account for its effect in continuum.

The so-called "effective modulus" theories replace the actual composite by a homogeneous, generally anisotropic medium whose material constants are a geometrically weighted average of the properties of the constituents. While yielding satisfactory results for certain geometries under static loads, such an approach exhibits serious deficiencies for virtually all geometries when applied to dynamic problems such as impact and wave propagation. Specifically, effective modules theories are incapable of reproducing the dispersion and attenuation observed in composite materials. Such a behavior is a well known phenomena in composites and is a result of the microstructure of the particular composite. The dynamic behavior of a composite material (or a continuum in general) is of great importance when the material is subjected to high-rate loads such as the ones that are generated by impact or explosive charges. We briefly elaborate on this point.

The dynamic response of deformable heterogeneous materials may be broadly classified into two groups as follows.

i) The wave length of the characteristic response of the material is very long compared with the scale of the inhomogeneity. Then the material response is governed by the effective properties of the equivalent homogeneous medium. In this case the structural response and wave propagation are identical to those of homogeneous materials.

ii) The wave length of the response is not ideally long with respect to the characteristic dimension of the inhomogeneity. In this situation very complicated dynamic effects can occur. The interfaces between material phases cause wave reflection and refraction. This phenomena is due to the existence of microstructure in the composite.

Considering (i) and (ii) above, it is clear that any continuum theory designed to account for the dynamic response of a composite must, in some fashion, reflect the effect of microstructure in the composite. In addition to dynamic response of the composite laminates the issue of interlaminar behavior of composite laminates is of great importance. This issue is directly related to delamination and edge effects in composites. In recent years, delamination has become one of the most feared failure modes in laminated composite structures. This problem has also initiated a great deal of research in the field of composite laminates. What began in 1970 as somewhat of an academic curiosity turned into a beehive of research activity in recent years. This in turn indicates the desire for having an adequate theory that can account for the effect of interlaminar stresses.

A review of the literature on continuum theories developed for composite laminates reveals that most of the theories that, in some fashion, account for the effect of micro-structure are linear in nature. Consequently, these theories are not capable to model the behavior of composite laminates undergoing large deformation. Moreover, all continuum theories, with the exception of one, are proposed for composite laminates with initially flat configurations. Hence, these theories are not appropriate for curved geometries. In addition the available continuum theories are mainly developed to predict only the dynamic response of the composite laminates. Therefore, they do not seem to be adequate for problems involving static response of the laminated composites with specified boundary conditions. Indeed the literature on the proposed continuum theories of composite laminates may be divided into two groups. One group is concerned with the formulation of theories that are adequate for dynamic response of composite laminates and another group that is involved with the formulation of theories that are appropriate for intelaminar response of composite materials. In fact, there exists no theory that is adequate for treating static and dynamic problems at the same time.

Considering the above restrictions of existing theories, the specific objectives of this investigation were to develop a continuum theory for laminated composite materials that could account for the effect of i) microstructure, ii) nonlinearity, and iii) curved geometry. As will become apparent, we have not only met the aforementioned objectives, but we have also achieved a complete theory which will have a widespread use in related industries. We would like to mention that we are very encouraged with the results of Phase I of the work; in particular, we have to state that the theory presented here is a coherent continuum theory represented by a set of well-defined conservation laws predicated on physical observations which are physically sound and mathematically accurate. Within the context of purely mechanical theory, the developed theory exhibits the following features:

a) It accounts for the effect of micro-structure

b) It accounts for the effect of geometric nonlinearity

c) It accounts for the effect of material nonlinearity

d) It accounts for the effect of curvature

e) It accounts for the effect of interlaminar stresses

f) It has a continuum character

g) It is applicable to both static and dynamic analysis.

The material presented here is divided into six parts (Part A through Part F) which contain 37 sections numbered consecutively.

Part A is concerned with some preliminary materials needed for subsequent developments. This part contains section (2) through section (4). Section (2) discusses the coordinate systems and other relevant notations. Section (3) provides some general background on spatial and surface base vectors and related matters while section (4) presents the relevant results from classical continuum mechanics in general curvilinear coordinates.

Part B contains an introductory account of the theory of Cosserat surfaces. This part includes section (5) through section (9). Section (5) presents a concise definition of a shell-like body; section (6) deals with the kinematics of a Cosserat surface while rigid body motions is discussed in section (7). The definition of various stress resultants are given in section (8) and the basic field equations of the Cosserat surface are derived in section (9).

Part C deals with the modeling of a composite as a series of Cosserat surfaces and contains sections (10) through section (22). Section (10) introduces the coordinate systems appropriate for composite laminate while section (11) gives a precise definition of a shell-like micro-structure. Sections (12) and (13) are concerned with the kinematics and rigid body motion, respectively. Section (14) provides definitions of the various quantities associated with micro-structure. Section (15) contains the basic field equations for the micro-structure while the associated conservation laws are given in section (16). Section (17) includes the derivation of the composite conservation laws. A summary of basic principles for composite laminates is given in section (18). Sections (19), (20) and (21) contain some results concerning composite contact force, composite contact couple, composite conservation laws, and composite stress and couple stress tensors. Basic field equations for composite laminates are derived in section (22).

Part D pertains to elastic composite laminates and includes sections (23) through (26). In section (23) the nonlinear constitutive equations of an elastic composite laminate are derived using the direct approach. Section (24) presents the complete theory while section (25) considers the question of constraints in composite laminates. Section (26) contains the three dimensional approach to the derivation of the constitutive relations.

Part E deals with the linearization of the Cosserat composite theory. In section (27) the linearized kinematics are derived, while the linearization of the basic field equations and constitutive relations are performed in sections (28) and (29).

Part F contains the application of the theory to flat and cylindrical laminates and also comparisons with the available theories. Sections (30) and (31) provide some preliminaries. Section (32) presents a practical approach to the derivation of explicit constitutive relations. Section

(33) deals with the case of a normal director while theories of initially flat and initially cylindrical composites are discussed in sections (34) and (35). Section (36) presents comparisons between the Cosserat composite theory and other available theories. Section (37) contains some relevant conclusions regarding composite Cosserat theory.

As mentioned earlier, the results of this phase of research were very promising. In fact, at this point, we have at hand a complete theory for mechanics of composite laminates which is based on sound frame-invariant conservation laws and in a rigorous mathematical framework without any ad hoc assumptions. We plan to continue this development toward explicit derivations of field equations and constitutive laws for various composite structures and reinforcement configurations. Examination of delamination phenomena and edge effects, which is a natural outcome of the present theory, is another line of activity which will be followed. For this purpose we plan to simulate some recent experiments [Pagano, 1989] by using the Composite Cosserat Theory and compare the results with experimental data and those of available theories.

Recent developments in computational mechanics [Simo, J. C., et al., 1989, 1990] have proved that classical Cosserat shell theory can be cast into an efficient and accurate numerical framework suitable for nonlinear finite element analysis. The present composite Cosserat theory was developed based on a systematic extension of classical Cosserat shell theory. We plan to extend the present theoretical developments to a numerical framework which is based on mathematical principles that their applicability has been demonstrated in the course of previously mentioned research of J. C. Simo and co-workers. Following the numerical developments, various shell elements for a wide range of composite materials with different reinforcing configuration and multi-constituent structures, will be designed. The details of these activities were presented in Phase II proposal for the subject project.

Part A. Preliminaries

In this part we introduce the coordinate systems, and the corresponding notations which will be used in the subsequent development. We also record some relevant results from classical three-dimensional continuum mechanics. We will not provide proofs, as they are available elsewhere, and only make reference to appropriate literature on the subject.

2. Coordinate Systems

Let the points of a region $\mathcal{R}$ in a three dimensional Euclidean space be referred to a fixed right-handed rectangular Cartesian coordinate system x^i $(i = 1,2,3)$ and let η^i $(i = 1,2,3)$ be a general *convected* curvilinear system defined by the transformation

$$x^i = x^i(\eta^1,\eta^2,\eta^3) \tag{2.1}$$

We assume the above transformation is nonsingular in $\mathcal{R}$ and has a unique inverse

$$\eta^i = \eta^i(x^1,x^2,x^3) \tag{2.2}$$

The existence of the unique inverse implies

$$\det\left(\frac{\partial x^i}{\partial \eta^i}\right) \neq 0 \tag{2.3}$$

We recall that a convected coordinate system is normally defined in relation to a continuous body and moves continuously with the body throughout the motion of the body from one configuration to another[1].

Throughout this work, all Latin indices (subscripts or superscripts) take the values 1,2,3; all Greek indices (subscripts or superscripts) take the values 1,2 and the usual summation

[1] The subject of convected coordinate system has been discussed in references [Oldroyd, 1950] and [Lodge, 1974].

convention is employed. We will use a comma for partial differentiation with respect to either space or surface coordinates such as η^i or η^α and a superposed dot for material time derivative, i.e., differentiation with respect to time holding the material coordinates, such as η^i or η^α, fixed. Also, we use a vertical bar (|) or a double vertical bar (||) for covariant differentiation in 2 and 3 dimensional spaces, respectively. In the course of derivation of various results for the composite laminate we will encounter covariant differentiation with respect to a coordinate system which corresponds to composite continuum[2]. To denote this we will use a single boldfaced vertical bar (**|**). Also, for later convenience, often we set $\eta^3 = \xi$ and adopt the notation

$$\eta^i = (\eta^\alpha, \xi) \tag{2.4}$$

In what follows, when there is a possibility of confusion, quantities which represent the same physical/geometrical concepts will be denoted by the same symbol but with an added asterisk (*) for classical three dimensional continuum mechanics or an added hat (ˆ) for the Cosserat surface and no addition for composite laminate. For example, the mass densities of a body in the contexts of the classical continuum mechanics, the Cosserat surface and the composite laminate will be denoted by ρ^*, $\hat{\rho}$ and ρ, respectively.

---

[2] As it will become clear later, in order to adequately represent the effect of micro-structure in a continuum composite laminate, we need to introduce an additional dimension (or coordinate) in the direction of ply lay-up.

3. General Background

Consider a three-dimensional body $\mathcal{B}$, embedded in a region $\mathcal{R}$ of the Euclidean 3-space, and let the particles (material points) of $\mathcal{B}$ be identified by a convected coordinate system (2.2). Let **P** denote the position vector, relative to a fixed origin, say O, of a typical particle of $\mathcal{B}$ in a reference configuration. Then, we have

$$\mathbf{P} = \mathbf{P}(\eta^\alpha, \xi) \tag{3.1}$$

This, in view of (2.2) of section (2), may also be expressed as a function of x^i. We recall that, in general, the numerical values of the coordinates associated with each material point of a continuum varies from one configuration to another. However, when the particles of a continuum are referred to a convected coordinate system, the numerical values of the coordinates of a particle remain the same for all time. The position vector of a typical particle of $\mathcal{B}$ in the deformed configuration at time t, relative to the same fixed origin will be denoted by

$$\mathbf{p} = \mathbf{p}(\eta^\alpha, \xi, t) \tag{3.2}$$

We note that equation (3.1) specifies the place occupied by the material point η^i in a reference configuration, while the place occupied by the same material point η^i in the present (deformed) configuration is specified by (3.2). We assume that the vector function **p** in (3.2), which describes the motion of the body $\mathcal{B}$ is differentiable with respect to η^α, ξ and t as many times as may be required. We recall the formulae

$$\begin{gathered} \mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial \eta^i} \;,\; g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \;,\; g = \det(g_{ij}) \;,\; g^{1/2} = [\mathbf{g}_1\, \mathbf{g}_2\, \mathbf{g}_3] \neq 0 \;, \\ \mathbf{g}^i = g^{ij}\mathbf{g}_j \;,\; \mathbf{g}^i \cdot \mathbf{g}^j = g^{ij} \;,\; \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i{}_j \end{gathered} \tag{3.3}$$

where $\mathbf{g}_i$ and $\mathbf{g}^i$ are the covariant and the contravariant base vectors at time t, g_{ij} is the metric tensor, g^{ij} is its conjugate, $\delta^i{}_j$ is the Kronecker symbol in 3-dimensional space, and [ ] denotes scalar triple product.

Formulae analogous to (3.3), valid in a reference configuration are given by

$$G_i = \frac{\partial \mathbf{P}}{\partial \eta^i} \ , \ G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \ , \ G = \det(G_{ij}) \ , \ G^{1/2} = [\mathbf{G}_1 \, \mathbf{G}_2 \, \mathbf{G}_3] \neq 0 \ ,$$

$$\mathbf{G}^i = G^{ij} \mathbf{G}_j \ , \ \mathbf{G}^i \cdot \mathbf{G}^j = G^{ij} \ , \ \mathbf{G}^i \cdot \mathbf{G}_j = \delta^i{}_j \tag{3.4}$$

A material surface in $\mathcal{B}$ can be defined by the equation $\xi = \xi(\eta^\alpha)$. The equations resulting from (3.1) and (3.2) with $\xi = \xi(\eta^\alpha)$ represent the parametric forms of this surface in the reference and present configuration. In particular, with reference to (3.2)

$$\xi = \xi(\eta^\alpha) = \text{constant} \tag{3.5}$$

defines a one parameter family of surfaces in space each of which is assumed to be smooth and non-intersecting. Let the surface $\xi = 0$ in the present (deformed) configuration at time t be denoted by $\mathcal{s}$. Any point of this surface is specified by the position vector $\mathbf{r}$, relative to the same fixed origin 0 in the 3-dimensional space, and we have

$$\mathbf{r} = \mathbf{r}(\eta^\alpha, t) = \mathbf{p}(\eta^\alpha, 0, t) \tag{3.6}$$

Let $\mathbf{a}_\alpha$ denote the base vectors along the η^α-curves on the surface $\mathcal{s}$. Moreover, let $\mathbf{a}_3 = \mathbf{a}_3(\eta^\alpha, t)$ be the unit normal to $\mathcal{s}$. We recall the results

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \eta^\alpha} = \mathbf{g}_\alpha(\eta^\gamma, 0, t) \ , \tag{3.7}$$

$$\mathbf{a}^\alpha \cdot \mathbf{a}_3 = 0 \ , \ \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \ , \ \mathbf{a}_3 = \mathbf{a}^3 \ , \ [\mathbf{a}_1 \, \mathbf{a}_2 \, \mathbf{a}_3] \neq 0 \ . \tag{3.8}$$

We also recall the formulae

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \ , \ a = \det(a_{\alpha\beta}) \ , \ \mathbf{a}^\alpha = a^{\alpha\beta} \mathbf{a}_\beta \ ,$$

$$\mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\alpha\beta} \ , \ a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha{}_\beta \ , \tag{3.9}$$

$$b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} \tag{3.10}$$

BASE

$$a_{\alpha|\beta} = b_{\alpha\beta}a_3 \ , \ a_{3,\alpha} = -b^{\gamma}{}_{\alpha}a_{\gamma} \ , \ b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \tag{3.11}$$

where $a_{\alpha\beta}$ is the metric tensor of the surface and $b_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface. We recall that the three equations given by (3.11) are the formulae of Gauss, Weingarten and the Mainardi-Codazzi, respectively.

Considering expression (3.6), we recall that **r** is the position vector of a typical point of the surface $\mathcal{s}$, i.e., the material surface $\xi = 0$ in the present configuration of the body $\mathcal{B}$ at time t. Let the corresponding surface (i.e., $\xi = 0$) in the reference configuration be denoted by $\mathcal{S}$. Any point of this surface in the reference configuration, is specified by:

$$\mathbf{R} = \mathbf{R}(\eta^{\alpha}) = \mathbf{P}(\eta^{\alpha},0) \tag{3.12}$$

It should be clear that if the reference configuration of $\mathcal{B}$ is chosen to be the initial configuration at time t = 0, then we will have

$$\mathbf{R} = \mathbf{R}(\eta^{\alpha}) = \mathbf{r}(\eta^{\alpha},0) \tag{3.13}$$

Let A_{α} be the base vectors along the coordinate curves on the surface $\mathcal{S}$. Then by (3.4) and (3.12) we obtain

$$A_{\alpha} = \frac{\partial \mathbf{R}}{\partial \eta^{\alpha}} = \mathbf{G}_{\alpha}(\eta^{\gamma},0) \tag{3.14}$$

and

$$A_{\alpha} \cdot A_3 = 0 \ , \ A_3 \cdot A_3 = 1 \ , \ A_3 = A^3 \ , \ [A_1 \ A_2 \ A_3] \neq 0 \tag{3.15}$$

where A_3 is the unit normal to $\mathcal{S}$. The duals of the relations (3.9) to (3.11) are given by

$$\begin{gathered} A_{\alpha\beta} = A_{\alpha} \cdot A_{\beta} \ , \ A = \det(A_{\alpha\beta}) \ , \ A^{\alpha} = A^{\alpha\beta}A_{\beta} \ , \\ A^{\alpha} \cdot A^{\beta} = A^{\alpha\beta} \ , \ A^{\alpha\gamma}A_{\gamma\beta} = \delta^{\alpha}{}_{\beta} \end{gathered} \tag{3.16}$$

BASE

$$B_{\alpha\beta} = B_{\alpha\beta} = -A_\alpha \cdot A_{3,\beta} = A_3 \cdot A_{\alpha,\beta} \tag{3.17}$$

$$A_{\alpha|\beta} = B_{\alpha\beta}A_3 \ , \ A_{3,\alpha} = -B^\gamma{}_\alpha A_\gamma \ , \ B_{\alpha\beta|\gamma} = B_{\alpha\gamma|\beta} \tag{3.18}$$

4. Basic equations of classical continuum mechanics in general curvilinear coordinates

In this section we summarize some preliminary results from the three-dimensional theory for non-polar media in terms of general curvilinear coordinates.

We define a body, designated by $\mathcal{B}^*$, as a set of particles (material points)[3]. We designate the particles of the body by P^* and assume that the body is smooth and can be put into correspondence with a domain of the three-dimensional Euclidean space. Thus, by assumption, a particle P^* of the body can be put into a one-to-one correspondence with the triples or real numbers P_1,P_2,P_3 in a region of Euclidean 3-space. We assume the mapping from the body manifold to the domain of a Euclidean 3-space is one-to-one, invertible, and differentiable as many times as desired.

Consider a body $\mathcal{B}^*$ with its particles P^* and let the boundary of $\mathcal{B}^*$ (a closed surface) be designated by $\partial\mathcal{B}^*$. We define a configuration of the body $\mathcal{B}^*$ to be a mapping onto a domain in the three dimensional Euclidean space, E^3 which assigns a position vector $\mathbf{p}^*$ to each particle (material point) of the body. Thus, the configuration of the body at time t is the region of Euclidean 3-space which is occupied by the particles of the body at the instant t of time $(-\infty < t < +\infty)$. We define a motion of the body as a time sequence of configurations. Often it is convenient to select one particular configuration and refer everything concerning the body and its motion to this configuration. In what follows we shall identify the particle P^* of the body with its position vector in a configuration (e.g., present or initial).

Let η^i $(i = 1,2,3)$ be a general convected curvilinear coordinates. Consider a body $\mathcal{B}^*$ and let its boundary be a closed surface and be denoted by $\partial\mathcal{B}^*$. Let

$$\mathbf{p}^* = \mathbf{p}^*(\eta^i,t) \tag{4.1}$$

[3] Note that from now on when we refer to a body in the sense of classical continuum mechanics, we will denote it by an added asterisk (*). The same will be true for the quantities associated with the body.

denote the position vector of a material point in the present configuration of the body $\mathcal{B}^*$ at time t. Then we may write

$$\mathbf{g}_i^* = \frac{\partial \mathbf{p}^*}{\partial \eta^i} \ , \ g_{ij}^* = \mathbf{g}_i^* \cdot \mathbf{g}_j^* \tag{4.2}$$

and

$$ds^2 = d\mathbf{p}^* \cdot d\mathbf{p}^* = g_{ij}^* \, d\eta^i d\eta^j \tag{4.3}$$

where $(4.2)_{1,2}$ and (4.3) are the covariant base vectors, the metric tensor, and the square of a line element in the present configuration at time t, respectively. In the same manner we denote the position vector, the covariant base vectors, the metric tensor and the square of a line element in a reference configuration as follows

$$\mathbf{P} = \mathbf{P}^*(\eta^i) \tag{4.4}$$

$$\mathbf{G}_i^* = \frac{\partial \mathbf{P}^*}{\partial \eta^i} \ , \ G_{ij}^* = \mathbf{G}_i^* \cdot \mathbf{G}_j^* \tag{4.5}$$

$$dS^2 = d\mathbf{P}^* \cdot d\mathbf{P}^* = G_{ij}^* \, d\eta^i d\eta^i \tag{4.6}$$

We define a strain measure through

$$ds^2 - dS^2 = 2\gamma_{ij}^* \, d\eta^i d\eta^j \tag{4.7}$$

$$\gamma_{ij}^* = \tfrac{1}{2}(g_{ij}^* - G_{ij}^*) \tag{4.8}$$

where γ_{ij}^* are the covariant components of the symmetric strain tensor. Moreover, the velocity is given by

$$\mathbf{v}^* = \dot{\mathbf{p}}^* = \frac{\partial \mathbf{p}^*}{\partial t} \tag{4.9}$$

Under a superposed rigid body motion, the position vector $\mathbf{p}^*$ will be displaced to the position $\mathbf{p}^{*+}$ given by

$$\mathbf{p}^{*+} = \mathbf{p}^{*+}(\eta^i,t) = \mathbf{p}_o^{*+}(t') + Q(t)[\mathbf{p}^*(\eta^i,t) - \mathbf{p}_o^*(t)] \tag{4.10}$$

where $t' = t + a'$ and a' is an arbitrary constant and the second order space tensor Q is a proper orthogonal tensor function of t which satisfies

$$Q = Q(t) \ , \ QQ^T = Q^TQ = I \ , \ \det Q = 1 \ . \tag{4.11}$$

We denote the counterpart of γ_{ij}^*, after superposed rigid body motion (4.10), by γ_{ij}^{*+} and we recall that under the motion (4.10) the strain γ_{ij}^* remains unaltered, i.e.,

$$\gamma_{ij}^* = \gamma_{ij}^{*+} \ . \tag{4.12}$$

It is clear from (4.8) that γ_{ij}^* vanishes for a rigid deformation, i.e.,

$$\gamma_{ij}^* = 0 \ . \tag{4.13}$$

Let $\mathcal{P}^*$, bounded by a closed surface $\partial\mathcal{P}^*$, refer to an arbitrary part of the body $\mathcal{B}^*$ in the present configuration. Then within the scope of the classical (nonpolar) continuum mechanics, the system of forces acting over any part $\mathcal{P}^*$ of the body $\mathcal{B}^*$ in motion consists of the sum of the two types of forces, $\mathbf{F}_b^*$ and $\mathbf{F}_c^*$, as described below:

Let $\mathbf{b}^* = \mathbf{b}^*(\eta^i,t)$ be a vector field, per unit mass ρ^*, defined for material points in the region of the Euclidean space, occuped by $\mathcal{B}^*$ at time t. This vector field is called the *body force*. The *resultant body force* acting on the part $\mathcal{P}^*$ in the present configuration at time t is defined by

$$\mathbf{F}_b^* = \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* dv^* \tag{4.14}$$

where dv^* denotes the element of volume. In addition, let the outward unit normal vector at a material point on the boundary $\partial\mathcal{P}^*$ of the part $\mathcal{P}^*$ at time t be denoted by $\mathbf{n}^*$ and be given by

$$\mathbf{n}^* = n_i^* \mathbf{g}^{*i} = n^{*i} \mathbf{g}_i^* \tag{4.15}$$

Let $\mathbf{t}^* = \mathbf{t}^*(\eta^i,t;\mathbf{n}^*)$ be defined for the material points on the boundary $\partial\mathcal{P}^*$ at time t. The vector $\mathbf{t}^*$ is called the *contact force* or the *stress vector* acting on the part $\mathcal{P}^*$ of $\mathcal{B}^*$. The *resultant contact* force exerted on the part $\mathcal{P}^*$ at time t is then defined by

$$\mathbf{F}_c^* = \int_{\partial\mathcal{P}^*} \mathbf{t}^*(\eta^i,t;\mathbf{n}^*)da^* \tag{4.16}$$

where da^* is the element of area whose outward unit normal is $\mathbf{n}^*$. Moreover, we assume the existence of a strain energy density $\varepsilon^* = \varepsilon^*(\eta^i,t)$ per unit mass ρ^*.

In terms of the above definitions of the various field quantities, with reference to the present configuration and within the context of the classical (nonpolar) continuum mechanics, the conservation laws in the purely mechanical theory are given by

$$\begin{aligned}
&a : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^* dv^* = 0 \\
&b : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^*\mathbf{v}^* dv^* = \int_{\mathcal{P}^*} \rho^*\mathbf{b}^* dv^* + \int_{\partial\mathcal{P}^*} \mathbf{t}^* da^* \\
&c : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^*\mathbf{p}^* \times \mathbf{v}^*\, dv^* = \int_{\mathcal{P}^*} \rho^*\mathbf{p}^* \times \mathbf{b}^* dv^* + \int_{\partial\mathcal{P}^*} \mathbf{p}^* \times \mathbf{t}^*\, da^* \\
&d : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^*(\varepsilon^*+\kappa^*)dv = \int_{\mathcal{P}^*} \rho^*(\mathbf{b}^* \times \mathbf{v}^* dv^*) + \int_{\partial\mathcal{P}^*} \mathbf{t}^* \cdot \mathbf{v}^* da^*
\end{aligned} \tag{4.17}$$

where ε^* is the specific internal energy per unit mass ρ^* and κ^* denotes the kinetic energy per unit mass ρ^* and has the form

$$\kappa^* = \tfrac{1}{2}\, \mathbf{v}^* \cdot \mathbf{v}^* \tag{4.18}$$

Equations $(4.17)_a$ to $(4.17)_d$ represent mathematical statements of conservation of mass, conservation of linear momentum, conservation of moment of momentum, and conservation of energy, respectively.

Under suitable continuity assumptions, the principle of linear momentum and that of moment of momentum imply the existence of a tensor field $\tau^{*ij} = \tau^{*ij}(\eta^k,t)$ such that

$$\mathbf{t}^* = \frac{\mathbf{T}^{*i}\, n_i^*}{g^{*1/2}} = \tau^{*ij}\, n_i^*\, \mathbf{g}_j^*$$

$$\mathbf{T}^{*i} = g^{*1/2}\, \tau^{*ij}\, \mathbf{g}_j^* = g^{*1/2}\, \tau_j^{*i}\, \mathbf{g}^{*j} \tag{4.19}$$

Moreover, with the help of (4.19), the transport theorem, and the divergence theorem, the balance laws $(4.17)_a$ and $(4.17)_b$ can be reduced to the Cauchy equations of motion, i.e.,

$$\mathbf{T}^i{}_{,i} + \rho^*\, \mathbf{b}^*\, g^{*1/2} = \rho^*\, \mathbf{c}^*\, g^{*1/2}$$

$$\mathbf{g}_i \times \mathbf{T}^i = 0 \tag{4.20}$$

where $\mathbf{c}^*$ is the acceleration vector. In (4.19) and (4.20) τ^{*ij} and $\tau^{*i}{}_j$ are the contravariant and mixed components of the stress tensor and a comma denotes partial differentiation with respect to η^i. It can be shown that the equations of motion (4.20) are equivalent to

$$\tau^{*ij}{}_{\|j} + \rho^* b^{*i} = \rho^* c^{*i} \;\;,\;\; \tau^{*ij} = \tau^{*ji} \tag{4.21}$$

where the double vertical bar (| |) stands for covariant differentiation with respect to g_{ij}^* and c^{*i} are the contravariant components of the acceleration

$$\mathbf{c}^* = \dot{\mathbf{v}}^* \tag{4.22}$$

and where a superposed dot is the material time derivative with respect to t holding η^i fixed. Moreover, with the use of the divergence theorem, i.e.,[4]

$$\int_{\mathcal{P}^*} \operatorname{div} \mathbf{v}^* dv^* = \int_{\partial\mathcal{P}} \mathbf{v}^* \cdot \mathbf{n}^* da^*$$

or (4.23)

$$\int_{\mathcal{P}^*} v^{*i}{}_{\|i}\, dv^* = \int_{\mathcal{P}^*} \frac{1}{\sqrt{g^*}} (v^{*i} \sqrt{g^*})_{,i} \cdot dv^* = \int_{\partial\mathcal{P}} v^{*i}\, n_i^*\, da^*$$

[4] See [Green and Zerna, 1968], page 31.

and the equations of motion, it can be shown that $(4.17)_d$ reduces to

$$\rho\dot{\varepsilon}^* = \tau^{*ij}\,\dot{\gamma}^*_{ij} \tag{4.24}$$

For an elastic body we make the constitutive assumption that

$$\varepsilon^* = \varepsilon^*(\gamma^*_{ij}) \tag{4.25}$$

together with a similar assumption for the stress tensor τ^{*ij}. In (4.25), the dependence of ε^* on the reference metric tensor is understood, although this is not shown explicitly. Making use of (4.24) and (4.25), we obtain the results

$$\tau^{ij} = \rho^* \frac{\partial\varepsilon^*}{\partial\gamma^*_{ij}} \tag{4.26}$$

In the last expression, the partial derivative is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial\varepsilon^*}{\partial\gamma^*_{ij}} + \frac{\partial\varepsilon^*}{\partial\gamma^*_{ji}}\right)$$

Before closing this section we discuss basic jump conditions in the context of three-dimensional classical continuum mechanics. Thus far, all kinematic and kinetical variables occurring in the conservation laws have been assumed to be continuous throughtout the body $\mathcal{B}^*$. Sometimes we encounter circumstances in which some kinematic/kinetical variables are discontinuous across a surface which moves through the body; the surface is called a *surface of discontinuity.*

Suppose that at time t an arbitrary material volume of the body occupies a part $\mathcal{P}^*$ bounded by a closed surface $\partial\mathcal{P}^*$. Let $\mathcal{P}^*$ be divided into two regions $\mathcal{P}^*_1$, $\mathcal{P}^*_2$ (see figure 1) separated by a moving surface $\sigma(t)$, and let $\partial\mathcal{P}^{*\prime}, \partial\mathcal{P}^{*\prime\prime}$ denote the portions of the surface $\partial\mathcal{P}^*$ which form parts of the boundaries $\partial\mathcal{P}^*_1$ and $\partial\mathcal{P}^*_2$ such that

$$\partial \mathcal{P}^{*\prime} = \partial \mathcal{P}_1^* \cap \partial \mathcal{P}^* \quad , \quad \partial \mathcal{P}^{*\prime\prime}\, \partial \mathcal{P}_2^* \cap \partial \mathcal{P}^*$$

$$\mathcal{P}^* = \mathcal{P}_1^* \cup \mathcal{P}_2^* \quad , \quad \partial \mathcal{P}^* = \partial \mathcal{P}^{*\prime} \cup \partial \mathcal{P}^{*\prime\prime} \tag{4.27}$$

$$\partial \mathcal{P}_1^* = \partial \mathcal{P}^{*\prime} \cup \sigma(t) \quad , \quad \partial \mathcal{P}_2^* = \partial \mathcal{P}^{*\prime\prime} \cup \sigma(t)$$

Let the velocity of the surface $\sigma(t)$ along its outward normal, when σ is regarded as part of the boundary $\partial \mathcal{P}_1^*$, be denoted by u_n. Then, $-u_n$ is the normal velocity of σ when this surface is regarded as part of the boundary of $\mathcal{P}_2^*$. Let ψ be any function which takes different values ψ_1 and ψ_2 on either side of σ in the regions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$, respectively. We adapt the notation $[\;\;]$ to indicate the difference of ψ_2 and ψ_1 and write

$$[\psi] = \psi_2 - \psi_1 \tag{4.28}$$

We also adopt the notations

$$w_{1n} = v_{1n} - u_n \;\; , \;\; w_{2n} = v_{2n} - u_n \tag{4.29}$$

where v_{1n} and v_{2n} are the velocities of the material points in the regions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ along the normal to σ, respectively. In accordance with the notation in (4.28), we can write

$$[w_n] = w_{2n} - w_{1n} \tag{4.30}$$

Recall that the transport theorem for a part $\mathcal{P}^*$ can be written as

$$\frac{d}{dt}\int_{\mathcal{P}^*} \phi \, dv^* = \int_{\mathcal{P}^*} (\dot{\phi} + \phi \operatorname{div} \mathbf{v}^*) dv^* = \int_{\mathcal{P}^*} \frac{\partial}{\partial t} \phi \, dv^* + \int_{\partial \mathcal{P}^*} \phi \mathbf{v}^* \cdot \mathbf{n}^* \, da^* \tag{4.31}$$

where in writing the above the divergence theorem has been used. We now proceed to obtain the counterpart of (4.31) for the region under consideration which includes the surface $\sigma(t)$. To this end we apply (4.31) to regions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ for a function $\rho^* \psi$ as follows:

$$\frac{d}{dt}\int_{\mathcal{P}_1^*} \rho^* \psi \, dv^* = \int_{\mathcal{P}_1^*} \frac{\partial}{\partial t}(\rho^*\psi)dv^* + \int_{\partial\mathcal{P}_1^*}(\rho^*\psi)\mathbf{v}^* \cdot \mathbf{n}^* \, da^*$$

$$= \int_{\mathcal{P}_1^*} \frac{\partial}{\partial t}(\rho^*\psi)dv^* + \int_{\partial\mathcal{P}^{*}}(\rho^*\psi)\mathbf{v}^* \cdot \mathbf{n}^* da^* + \int_\sigma(\rho^*\psi)\mathbf{v}^* \cdot \mathbf{n}^* da^* \tag{4.32}$$

and

$$\frac{d}{dt}\int_{\mathcal{P}_2^*} \rho^* \psi dv^* = \int_{\mathcal{P}_2^*} \frac{\partial}{\partial t}(\rho^*\psi)dv^* + \int_{\partial\mathcal{P}_2^*}(\rho^*\psi)\mathbf{v}^* \cdot \mathbf{n}^* da^*$$

$$= \int_{\mathcal{P}_2^*} \frac{\partial}{\partial t}(\rho^*\psi)dv^* + \int_{\partial\mathcal{P}^{*}}(\rho^*\psi)\mathbf{v}^* \cdot \mathbf{n}^* da^* + \int_\sigma(\rho^*\psi)\mathbf{v}^* \cdot \mathbf{n}^* da^* \tag{4.33}$$

Adding both sides of (4.32) and (4.33) we obtain

$$\frac{d}{dt}\int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*}(\rho^*\psi)dv^* = \int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*}(\rho^*\dot{\psi})dv^* + \int_\sigma [\rho^*\psi w_n^*] \, da^* \tag{4.34}$$

where in obtaining (4.34) we have also made use of the divergence theorem and the equation of continuity. Making use of (4.34) and with reference to the present configuration, we obtain the conservation laws for the part $\mathcal{P}_1^* \cup \mathcal{P}_2^*$ in the form:

$$a: \frac{d}{dt}\int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* dv^* = 0$$

$$b: \frac{d}{dt}\int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* \mathbf{v}^* dv^* = \int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* \mathbf{b}^* dv^* + \int_{\partial\mathcal{P}_1^* \cup \partial\mathcal{P}_2^*} \mathbf{t}^* da^* + \int_\sigma [\mathbf{t}^*] da^*$$

$$(4.35)$$

$$c: \frac{d}{dt}\int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* (\mathbf{p}^* \times \mathbf{v}^*) dv^* = \int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* (\mathbf{p}^* \times \mathbf{v}^*) dv^* + \int_{\partial\mathcal{P}_1^* \cup \partial\mathcal{P}_2^*} \mathbf{p}^* \times \mathbf{t}^* da^* + \int_\sigma [\mathbf{p}^* \times \mathbf{t}^*] da^*$$

$$d: \frac{d}{dt}\int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* (\varepsilon^* + k^*) dv^* = \int_{\mathcal{P}_1^* \cup \mathcal{P}_2^*} \rho^* \mathbf{b}^* \cdot \mathbf{v}^* dv^* + \int_{\partial\mathcal{P}_1^* \cup \partial\mathcal{P}_2^*} \mathbf{t}^* \cdot \mathbf{v}^* da^* + \int_\sigma [\mathbf{t}^* \cdot \mathbf{v}^*] da^*$$

Application of $(4.17)_{a,b,c,d}$ to parts $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ and subtraction of the results from corresponding equations in (4.35) yields

$$
\begin{aligned}
&a : [\![\rho^* w_n^*]\!] = 0 \\
&b : [\![\rho^* v^* w_n^* - \mathbf{t}^*]\!] = 0 \\
&c : \text{no new equation} \\
&d : [\![\rho^* (\varepsilon^* + k^*) w_n^* - \mathbf{t}^* \cdot v^*]\!] = 0
\end{aligned}
\qquad (4.36)
$$

BASE

B. Introduction to theory of two-dimensional directed continuum, i.e., a Cosserat theory

We introduce in this section the main concepts and ingredients of a theory of two-dimensional continuum, namely, a Cosserat surface. The concept of *oriented* or *directed* media originated in the work of Duhem in 1893. The first systematic study and development of theories of oriented media in one, two and three dimensions was conducted by the brothers Eugene and Francois Cosserat in 1909. Further study on the subject was carried out by Ericksen and Truesdell in 1958 who introduced the terminology of *directors*. A complete general theory of a Cosserat surface with a single director in the context of thermomechanics was developed by Green, Naghdi and Wainwright in 1965. A thorough study of the theory of directed surfaces was conducted by Naghdi in 1972 which in addition to the basic theory, includes certain general considerations regarding the construction of nonlinear constitutive equations for elastic shells. An account of recent developments on one and two dimensional Cosserat continuums with special attention towards elastic rods and shells was also given by Naghdi in 1982. Our exposition of the two-dimensional Cosserat (directed) surfaces in this part will closely follow the developments given by [Naghdi, 1972].

In general, two different approaches may be adapted for the construction of two-dimensional mechanical theories such as those for shells or fluid sheets. One approach starts with the 3-dimensional equations of classical continuum mechanics and by applying approximation procedures obtains a set of two-dimensional field equations and constitutive equations for the continuum under consideration. In the other approach the continuum is modelled as a two-dimensional directed continuum, called a Cosserat surface; and then the field equations and the appropriate constitutive equations are developed. It should be emphasized that in the latter approach, namely the introduction of an alternative model and formulation of the theory by the direct approach, the nature of the field equations in the 3-dimensional theory is not ignored. In fact, some of the developments of the field equations by the direct approach are motivated and aided by available information obtained from the 3-dimensional theory. It can be shown that the two foregoing approaches may be put into a one-to-one correspondence. As discussed in [Naghdi 1972, 1982], most of the difficulties that occur in the derivation from three-dimensional theory are related to the construction of relevant constitutive equations. These difficulties, however, do not occur in the direct approach and in this sense the direct approach offers a clear advantage over the three-dimensional one. The entire development by the direct approach is exact in the sense that it rests on 2-dimensional postulates valid for nonlinear behavior of materials. However, a theory of this kind cannot be expected to represent all the features that could only be predicted by the relevant full 3-dimensional equations.

As mentioned previously the ingredients of the two different approaches can be put into a one-to-one correspondence. For the purpose of this study we make use of the three-dimensional approach as it is more appropriate to our later development. It is to be emphasized that the relevant equations to be obtained at the end of this section will be the same whether we use the direct approach or the three-dimensional approach.

5. Definition of a shell-like body

Consider a body $\mathcal{B}^*$ in the present configuration and let its boundary be a closed surface, denoted by $\partial\mathcal{B}^*$, and composed of three material surfaces as follows.

a) The material surfaces

$$\begin{aligned} &\mathscr{s}_1 : \ \xi = \xi_1(\eta^\alpha) \\ &\qquad\qquad\qquad\qquad \xi_1 < 0 < \xi_2 \\ &\mathscr{s}_2 : \ \xi = \xi_2(\eta^\alpha) \end{aligned} \tag{5.1}$$

with the material surface

$$\mathscr{s}_0 : \ \xi = 0 \tag{5.2}$$

lying entirely between them.

b) The material surface

$$\mathscr{s}_l : \ f(\eta^\alpha) = 0 \tag{5.3}$$

such that ξ = const. are closed smooth curves on the surface (5.3).

The surfaces $(5.1)_{1,2}$ are called the *major surfaces* or the *bottom face* and the *top face*, respectively. Since $\eta^i = \{\eta^\alpha, \xi\}$ are defined by (2.2) as convected (material) coordinates, the material surfaces $(5.1)_{1,2}$ will have the same parametric representation in all configurations. In general, ξ_1 and ξ_2 are functions of the surface coordinates η^α but in special cases they may be constants. We assume the surfaces $\mathscr{s}_0$, $\mathscr{s}_1$ and $\mathscr{s}_2$ do not intersect themselves, and each other. This implies the condition $(5.1)_3$ and $g^* \neq 0$. The surface $\mathscr{s}_0$ is not necessarily midway between the bounding surfaces $\mathscr{s}_1$ and $\mathscr{s}_2$. However, in a reference configuration of the body $\mathcal{B}^*$, a surface corresponding to $\mathscr{s}_0$ may be chosen midway between the surfaces corresponding to $\mathscr{s}_1$ and $\mathscr{s}_2$. Such a three dimensional body (i.e., the body $\mathcal{B}^*$ as characterized above) is called a *shell* if the dimension of the body along the normals to the surface $\mathscr{s}_0$, called the height, is small in comparison to its other dimensions. A shell is said to be *thin* if its thickness is much smaller than a characteristic length of the surface $\mathcal{S}_o$, for example the local minimum radius of curvature of $\mathcal{S}_o$. Figure (2) shows an element of a shell like body in the present configuration.

Let $\rho^*(\eta^\alpha,\xi,t)$ and $\rho_o^*(\eta^\alpha,\xi)$ be the mass densities of $\mathcal{B}^*$ in the deformed and reference configurations, respectively. Then the conservation of mass (in three dimensions) implies

$$\rho^* g^{*1/2} = \rho_o^* G^{*1/2} \tag{5.4}$$

We define the surface mass density (i.e., mass per unit area) $\hat{\rho}$, of $\mathcal{S}_o$ at time t in the present configuration by the expression

$$\hat{\rho}\, a^{1/2} = \int_{\xi_1}^{\xi_2} \rho^* g^{*1/2}\, d\xi \ , \ \hat{\rho} = \hat{\rho}(\eta^\alpha,t) \tag{5.5}$$

where a is $\det(a_{\alpha\beta})$ of the surface $\mathcal{S}_o$. Since the quantity $\rho^* g^{*1/2}$ is independent of time, it follows that $\hat{\rho}\, a^{1/2}$ is also independent of time, although both $\hat{\rho}$ and $a = \det(a_{\alpha\beta})$ may depend on t. The mass of an arbitrary part $\mathcal{P}^*$ of the body $\mathcal{B}^*$ bounded by the surfaces $(5.1)_{1,2}$ and a surface of the form (5.3) may be expressed

$$M_p = \int_{\mathcal{P}^*} \rho^*\, dv^* = \int_{\hat{\eta}_1}\int_{\hat{\eta}_2}\int_{\xi_1}^{\xi_2} \rho^* g^{*1/2}\, d\eta^1\, d\eta^2\, d\xi$$

$$= \int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \{\int_{\xi_1}^{\xi_2} \rho^* g^{*1/2}\, d\xi\}\, d\eta^1\, d\eta^2 = \int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho}\, a^{1/2}\, d\eta^1\, d\eta^2$$

$$= \int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} \tag{5.6}$$

where $\hat{\mathcal{P}}$ denotes an arbitrary part of the surface $\mathcal{S}_o$ which corresponds to $\mathcal{P}^*$ and $\hat{\eta}_1$ and $\hat{\eta}_2$ denote the applicable ranges of integration for the coordinates η^1 and η^2, respectively. Also, in obtaining (5.6) we have made use of (5.5) and the following expressions

$$dv^* = (\mathbf{g}_1^* \times \mathbf{g}_2^*) \cdot \mathbf{g}_3^*\, d\eta^1\, d\eta^2\, d\xi = g^{*1/2}\, d\eta^1\, d\eta^2\, d\xi \tag{5.7}$$

$$d\mathcal{a} = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{a}_3\, d\eta^1\, d\eta^2 = a^{1/2}\, d\eta^1\, d\eta^2 \tag{5.8}$$

The relation of the surface $s_o : \xi = 0$ to the boundary $s_1 : \xi = \xi_1$ and $s_2 : \xi = \xi_2$ can be fixed by imposing the condition

$$\int_{\xi_1}^{\xi_2} \rho^* g^{*1/2} \xi\, d\xi = \int_{\xi_1}^{\xi_2} k^* \xi\, d\xi = 0 \tag{5.9}$$

where

$$k^* = k^*(\eta^\alpha, \xi) = \rho^* g^{*1/2} = \rho_o^* G^{*1/2} \tag{5.10}$$

We notice that k^* is independent of time. Once the position of the surface $s_o : \xi = 0$ relative to the positions of the surfaces $s_1 : \xi = \xi_1$ and $s_2 : \xi = \xi_2$ is determined by (5.10) in a configuration (e.g., a reference configuration) it remains so determined. This completes our description of a *shell-like* body, namely a three dimensional body $\mathcal{B}^*$ bounded by the surfaces $(5.1)_1$, $(5.1)_3$ and (5.3).

We will refer to the duals of the surfaces s_o, s_1, s_2 in the reference configuration by $\mathcal{S}_o, \mathcal{S}_1, \mathcal{S}_2$, respectively. We also note that the dual of (5.5) in the reference configuration is given by

$$\hat{\rho}_o A^{1/2} = \int_{\xi_1}^{\xi_2} \rho_o^* G^{*1/2}\, d\xi \tag{5.11}$$

and

$$\hat{\rho}_o A^{1/2} = \hat{\rho}\, a^{1/2} \tag{5.12}$$

in view of (5.10).

6. General kinematical results for shells

We begin our development of the kinematical results by assuming that the position vector $\mathbf{p}^*(\eta^\alpha,\xi,t)$ of a material point in the deformed thin shell has the form

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,t) + \xi\, \mathbf{d}(\eta^\alpha,t) \tag{6.1}$$

The above is a special assumption which is regarded to be valid for thin shells[5].

The velocity vector $\mathbf{v}^*$ of the three-dimensional shell-like continuum at time t is given by

$$\mathbf{v}^* = \frac{\partial \mathbf{p}^*(\eta^\alpha,\xi,t)}{\partial t} = \dot{\mathbf{p}}^*(\eta^\alpha,\xi,t) \tag{6.4}$$

where a superposed dot denotes the material time derivative, holding $\eta^i = \{\eta^\alpha,\xi\}$ fixed. From (6.1) and (6.4) we obtain

$$\mathbf{v}^* = \mathbf{v} + \xi\, \mathbf{w} \tag{6.5}$$

where

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \tag{6.6}$$

From (6.1) and (4.2) we have

$$\mathbf{g}_\alpha^* = \mathbf{a}_\alpha + \xi \frac{\partial \mathbf{d}}{\partial \eta^\alpha} \quad , \quad \mathbf{g}_3^* = \mathbf{d} \tag{6.7}$$

where $\mathbf{a}_\alpha$ are the base vectors of the surface s_o. The base vectors $\mathbf{g}_i^*(\eta^\alpha,\xi,t)$ in (6.7) when evaluated on the surface s_o reduce to:

---

[5] In a more general approach, we may begin the kinematical development by assuming that $\mathbf{p}(\eta^\alpha,\xi,t)$ is an analytical function of ξ in the region $\xi_1 < \xi < \xi_2$ and can be represented as (see [Naghdi, 1975, section 7])

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,t) + \sum_{N=1}^{\infty} \xi^n\, \mathbf{d}_N(\theta^\alpha,t)$$

This generality is not needed for our present purposes and we therefore adhere to the assumption (6.1).

$$\mathbf{g}^*_\alpha(\eta^\gamma,0,t) = \mathbf{a}_\alpha(\eta^\gamma,t)$$

$$\mathbf{g}^*_3(\eta^\gamma,0,t) = \mathbf{d}(\eta^\gamma,t) \tag{6.8}$$

where $\mathbf{g}_i^*$ satisfy the condition

$$[\mathbf{g}_1^* \; \mathbf{g}_2^* \; \mathbf{g}_3^*] \neq 0 \tag{6.9}$$

This restriction holds at all times and for all values of $\eta^i = \{\eta^\alpha,\xi\}$. In particular, it is valid for $\xi = 0$ so that by (6.9) we also have

$$[\mathbf{a}_1 \; \mathbf{a}_2 \; \mathbf{d}] \neq 0 \tag{6.10}$$

This condition implies that the director **d** cannot be tangent to the surface s_o.

Let **v** be some three-dimensional vector field defined on s_o, and let v^i, v_i be the covariant and contravariant components of **v** referred to the base vectors $\mathbf{a}_i = \{\mathbf{a}_\alpha,\mathbf{a}_3\}$ or $\mathbf{a}^i$. We then have

$$\mathbf{v} = v^i\mathbf{a}_i = v^\alpha\mathbf{a}_\alpha + v^3\mathbf{a}_3 = v_i\mathbf{a}^i = v_\alpha\mathbf{a}^\alpha + v_3\mathbf{a}^3 \tag{6.11}$$

Recalling the expressions for the gradient of **v**, we have

$$\mathbf{v}_{,\alpha} \equiv \mathbf{v}_{|\alpha} = v_{i\alpha}\mathbf{a}^i = v^i{}_\alpha\mathbf{a}_i$$

$$v_{i\alpha} = \mathbf{a}_i \cdot \mathbf{v}_{,\alpha} \;,\; v^i{}_\alpha = \mathbf{a}^i \cdot \mathbf{v}_{,\alpha}$$

$$v_{\lambda\alpha} = v_{\lambda|\alpha} - b_{\alpha\lambda}v_3 \;,\; v_{3\alpha} = v_{3,\alpha} + b_\alpha^\lambda v_\lambda \tag{6.12}$$

$$v^\lambda{}_\alpha = v^\lambda{}_{|\alpha} - b^\lambda{}_\alpha v_3 \;,\; v^3{}_\alpha = v^3{}_{,\alpha} + b_{\lambda\alpha}v^\lambda$$

where a vertical bar (|) stands for covariant derivative with respect to $a_{\alpha\beta}$. The lowering and raising of indices of the tensor functions such as v^i in (6.11) and $v_{i\alpha}$ in (6.12) is accomplished by using a space metric tensor defined by

$$g^*_{\alpha\beta}(\eta^\alpha,0,t) = a_{\alpha\beta} \;,\; g^*_{\alpha 3}(\eta^\alpha,0,t) = 0 \;,\; g^*_{33} = 1 \tag{6.13}$$

Considering (6.1) and (6.7) and making use of the general formulae (6.11) and (6.12), we write

$$\mathbf{d} = d_i \mathbf{a}^i = d^i \mathbf{a}_i \ , \ d^\alpha = a^{\alpha\beta} d_\beta \ , \ d^3 = d_3 \tag{6.14}$$

$$\begin{aligned}
&\text{a)} \quad \frac{\partial \mathbf{d}}{\partial \eta^\alpha} \equiv \mathbf{d}_{,\alpha} \equiv d_{i|\alpha} \mathbf{a}^i = \lambda_{i\alpha} \mathbf{a}^i \\
&\text{b)} \quad \lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\beta\alpha} d_3 \ , \ \lambda_{3\alpha} = d_{3,\alpha} + b^\beta{}_\alpha d_\beta \\
&\text{c)} \quad \lambda^\gamma{}_\alpha = a^{\gamma\beta} \lambda_{\beta\alpha} \ , \ \lambda^3{}_\alpha = \lambda_{3\alpha}
\end{aligned} \tag{6.15}$$

We also introduce the notations

$$\begin{aligned}
&\text{a)} \quad \mathbf{d} \cdot \mathbf{d} = d^\beta d_\beta + \sigma \ , \ \sigma = (d_3)^2 \\
&\text{b)} \quad \mathbf{d} \cdot \mathbf{d}_{,\alpha} = d^\beta \lambda_{\beta\alpha} + \sigma_\alpha \ , \ \sigma_\alpha = d^3 \lambda_{3\alpha} \\
&\text{c)} \quad \mathbf{d}_{,\alpha} \cdot \mathbf{d}_{,\beta} = \lambda^\gamma{}_\alpha \lambda_{\gamma\beta} + \sigma_{\alpha\beta} \ , \ \sigma_{\alpha\beta} = \lambda^3{}_\alpha \lambda_{3\beta}
\end{aligned} \tag{6.16}$$

We may now write

$$\begin{aligned}
&\text{a)} \quad g^*_{\alpha\beta} = a_{\alpha\beta} + \xi\left(\mathbf{a}_\beta \cdot \frac{\partial \mathbf{d}}{\partial \eta^\alpha} + \mathbf{a}_\alpha \cdot \frac{\partial \mathbf{d}}{\partial \eta^\beta}\right) + \xi^2 \frac{\partial \mathbf{d}}{\partial \eta^\alpha} \cdot \frac{\partial \mathbf{d}}{\partial \eta^\beta} \\
&\qquad\quad = a_{\alpha\beta} + \xi(\lambda_{\beta\alpha} + \lambda_{\alpha\beta}) + \xi^2(\lambda^\gamma{}_\alpha \lambda_{\gamma\beta} + \sigma_{\alpha\beta}) \\
&\text{b)} \quad g^*_{\alpha 3} = \mathbf{a}_\alpha \cdot \mathbf{d} + \xi \mathbf{d} \cdot \frac{\partial \mathbf{d}}{\partial \eta^\alpha} = d_\alpha + \xi(d^\gamma \lambda_{\gamma\alpha} + \sigma_\alpha) \\
&\text{c)} \quad g^*_{33} = \mathbf{d} \cdot \mathbf{d} = d^\gamma d_\gamma + \sigma
\end{aligned} \tag{6.17}$$

The duals of (6.14) to (6.17) in the reference configuration are given by

$$\mathbf{D} = D_i \mathbf{A}^i = D^i \mathbf{A}_i \ , \ D^\alpha = A^{\alpha\beta} D_\beta \ , \ D^3 = D_3 \tag{6.18}$$

$$\text{a)}\quad \frac{\partial \mathbf{D}}{\partial \eta^\alpha} = D_{i|\alpha}\mathbf{A}^i = \Lambda_{i\alpha}\mathbf{A}^i$$

$$\text{b)}\quad \Lambda_{\beta\alpha} = D_{\beta|\alpha} - B_{\beta\alpha}D_3 \ , \ \Lambda_{3\alpha} = D_{3,\alpha} + B^\beta{}_\alpha D_\beta \tag{6.19}$$

$$\text{c)}\quad \Lambda^{\gamma}_{\cdot\alpha} = A^{\gamma\beta}\Lambda_{\beta\alpha} \ , \ \Lambda^{3}_{\cdot\alpha} = \Lambda_{3\alpha}$$

$$\text{a)}\quad \mathbf{D}\cdot\mathbf{D} = D^\beta D_\beta + \Sigma \ , \ \Sigma = (D_3)^2$$

$$\text{b)}\quad \mathbf{D}\cdot\mathbf{D}_{,\alpha} = D^\beta\lambda_{\beta\alpha} + \Sigma_\alpha \ , \ \Sigma_\alpha = D^3\Sigma_{3\alpha} \tag{6.20}$$

$$\text{c)}\quad \mathbf{D}_{,\alpha}\cdot\mathbf{D}_{,\beta} = \Lambda^{\gamma}_{\cdot\alpha}\Lambda_{\gamma\beta} + \Sigma_{\alpha\beta} \ , \ \Sigma_{\alpha\beta} = \Lambda^3{}_\alpha\Lambda_{3\beta}$$

and

$$\text{a)}\quad G^{*}_{\alpha\beta} = A_{\alpha\beta} \ [illegible] \ \mathbf{A}_\beta\cdot\frac{\partial \mathbf{D}}{\partial \eta^\alpha} + \mathbf{A}_\alpha\cdot\frac{\partial \mathbf{D}}{\partial \eta^\beta}) + \xi^2\,\frac{\partial \mathbf{D}}{\partial \eta^\alpha}\cdot\frac{\partial \mathbf{D}}{\partial \eta^\beta}$$

$$= A_{\alpha\beta} + \xi(\Lambda_{\beta\alpha} + \Lambda_{\alpha\beta}) + \xi^2(\Lambda^{\gamma}_{\cdot\alpha}\Lambda_{\gamma\beta} + \Sigma_{\alpha\beta})$$

$$\tag{6.21}$$

$$\text{b)}\quad G^{*}_{\alpha 3} = \mathbf{A}_\alpha\cdot\mathbf{D} + \xi\mathbf{D}\cdot\frac{\partial \mathbf{D}}{\partial \eta^\alpha} = D_\alpha + \xi(D^\gamma\Lambda_{\gamma\alpha} + \Sigma_\alpha)$$

$$\text{c)}\quad G^{*}_{33} = \mathbf{D}\cdot\mathbf{D} = D^\gamma D_\gamma + \Sigma$$

where $\mathbf{A}_i = \{\mathbf{A}_\alpha, \mathbf{A}_3\}$ are the base vectors and the unit normal of the surface $\mathcal{S}_o : \ \xi = 0$ in the reference configuration.

Recalling the expression for the strain γ^{*}_{ij}, i.e.,

$$\gamma^{*}_{ij} = \tfrac{1}{2}(\mathbf{g}^{*}_i\cdot\mathbf{g}^{*}_j - \mathbf{G}_i\cdot\mathbf{G}_j) = \tfrac{1}{2}(g^{*}_{ij} - G^{*}_{ij}) \tag{6.22}$$

with the help of (6.14) to (6.21) we can readily record the components of the strain γ^{*}_{ij} as follows:

$$2\gamma^*_{\alpha\beta} = 2e_{\alpha\beta} + \xi(\kappa_{\beta\alpha} + \kappa_{\alpha\beta}) + \xi^2[(\lambda^{\cdot\gamma}_{\cdot\alpha}\lambda_{\gamma\beta} - \Lambda^{\cdot\gamma}_{\cdot\alpha}\Lambda_{\gamma\beta}) + S_{\alpha\beta}]$$

$$2\gamma^*_{\alpha 3} = \gamma_\alpha + \xi[(d^\gamma\lambda_{\gamma\alpha} - (D^\gamma\Lambda_{\gamma\alpha}) + S_\alpha] \qquad (6.23)$$

$$2\gamma^*_{33} = (d^\gamma d_\gamma - D^\gamma D_\gamma) + S$$

where in obtaining (6.23) we have introduced the relative kinematic variables $e_{\alpha\beta}$, $\kappa_{i\alpha}$ and γ_i as follows

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) \ , \ \kappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha} \ , \ \gamma_i = d_i - D_i \qquad (6.24)$$

We have also made use of the following expressions and definitions:

$$\mathbf{d} \cdot \mathbf{d} - \mathbf{D} \cdot \mathbf{D} = (d^\gamma d_\gamma - D^\gamma D_\gamma) + S \ , \ S = (d_3)^2 - (D_3)^2$$

$$\mathbf{d} \cdot \mathbf{d}_{,\alpha} - \mathbf{D} \cdot \mathbf{D}_{,\alpha} = (d^\gamma\lambda_{\gamma\alpha} - D^\gamma\Lambda_{\gamma\alpha}) + S_\alpha \ , \ S_\alpha = d^3\lambda_{3\alpha} - D^3\Lambda_{3\alpha} \qquad (6.25)$$

$$\mathbf{d}_{,\alpha} \cdot \mathbf{d}_{,\beta} - \mathbf{D}_{,\alpha} \cdot \mathbf{D}_{,\beta} = (\lambda^\gamma{}_\alpha\lambda_{\gamma\beta} - \Lambda^\gamma{}_\alpha\Lambda_{\gamma\beta}) + S_{\alpha\beta} \ , \ S_{\alpha\beta} = \lambda^{\cdot 3}_{\cdot\alpha}\lambda_{3\beta} - \Lambda^3{}_\alpha\Lambda_{3\beta}$$

Before closing this section we make a remark that the kinematic variables (6.24) represent measures of surface strains, bending and rotation of normal to the surface, i.e.,

$e_{\alpha\beta}$ is called stretch and is a measure of strain

$\kappa_{i\alpha}$ is a measure of bending

γ_i is a measure of rotation of the normal.

7. Superposed rigid body motion

We recall that when the motion of $\mathcal{B}^*$ differs from the given motion by a rigid motion, the position vector $\mathbf{p}^{*+}$ has the form

$$\mathbf{p}^{*+} = \mathbf{p}^{*+}(\eta^i,t') = \mathbf{p}_o^{*+}(t') + Q(t)[\mathbf{p}^*(\eta^i,t) - \mathbf{p}_o^*(t)] \tag{7.1}$$

where Q(t) is a proper orthogonal tensor function of time. Also, under superposed rigid body motion, the position vector $\mathbf{r}$ of the surface s_0 of $\mathcal{B}$ is displaced to

$$\mathbf{r}^+ = \mathbf{r}^+(\eta^\alpha,t') = \mathbf{r}_o^+(t') + Q(t)[\mathbf{r}(\eta^\alpha,t) - \mathbf{r}_o(t)] \tag{7.2}$$

From (7.1) and (7.2) we obtain

$$\mathbf{p}^+ - \mathbf{r}^+ = \xi \mathbf{d}^+ \tag{7.3}$$

or

$$\xi\, \mathbf{d}^+ = \{\mathbf{p}_o^+(t') + Q(t)[\mathbf{p}(\eta^i,t) - \mathbf{p}_o(t)]\} - \{\mathbf{r}_o^+(t') + Q(t)[\mathbf{r}(\eta^\alpha,t) - (\mathbf{r}_o(t)]\}$$

$$= \{\mathbf{p}_o^+(t') - \mathbf{r}_o^+(t')\} + Q(t)\{\mathbf{p}(\eta^i,t) - \mathbf{r}(\eta^\alpha,t)\} - Q(t)\{\mathbf{p}_o(t) - \mathbf{r}_o(t)\}$$

Hence,

$$\mathbf{d}^+(\eta^\alpha,t) = Q(t)\mathbf{d}(\eta^\alpha,t) \tag{7.4}$$

where in obtaining (7.4) we have made use of the fact that

$$\mathbf{p}(\eta^i,t) - \mathbf{r}(\eta^\alpha,t) = \xi\, \mathbf{d}(\eta^\alpha,t) \tag{7.5}$$

and

$$\mathbf{p}_o^+(t') - \mathbf{r}_o^+(t') = Q(t)\{\mathbf{p}_o(t) - \mathbf{r}_o(t)\} \quad . \tag{7.6}$$

Considering (7.4) and the fact that Q(t) is a proper orthogonal tensor function, i.e.,

$$Q^T Q = Q^{-1} Q = I \tag{7.7}$$

we may write

$$\mathbf{d}^+ \cdot \mathbf{d}^+ = Q\,\mathbf{d} \cdot Q\,\mathbf{d} = Q^T Q\,\mathbf{d} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} \tag{7.8}$$

In obtaining (7.8) we have used the relation

$$\mathbf{U} \cdot Q\,\mathbf{V} = Q^T\,\mathbf{U} \cdot \mathbf{V} \tag{7.9}$$

for any two vectors **U** and **V**. It is clear from (7.8) that the magnitude of $\mathbf{d}(\eta^\alpha,t)$ under superposed rigid body motions remains unchanged. In the contemporary literature, any three dimensional vector field which transforms according to transformation (7.4) and possesses the property (7.8) under superposed rigid body motions is called a *director*.

8. Stress resultants, stress-couples and other related definitions

Preliminary to our derivation of equations of motion for a shell-like body, we need to define appropriate stress-resultants, stress-couples and resultant body forces. This will be accomplished in this section.

Consider a shell-like body $\mathcal{B}^*$ bounded by a closed surface $\partial\mathcal{B}^*$, as specified in section 5, which consists of the material surfaces

$$\begin{aligned} s_1 &: \ \xi = \xi_1(\eta^\alpha) \\ & \qquad\qquad\qquad \xi_1 < 0 < \xi_2 \\ s_2 &: \ \xi = \xi_2(\eta^\alpha) \end{aligned} \tag{8.1}$$

and a lateral material surface of the form

$$s_l = f(\eta^\alpha) = 0 \tag{8.2}$$

We recall that the relation of the material surface $s_o : \xi = 0$ to the bounding surfaces $(8.1)_{1,2}$ is fixed by the condition

$$\int_{\xi_1}^{\xi_2} \rho^* \, g^{*1/2} \, \xi \, d\xi = \int_{\xi_1}^{\xi_2} k^* \, \xi \, d\xi = 0 \tag{8.3}$$

where

$$k^* = k^*(\eta^\alpha,\xi) = \rho^* \, g^{*1/2} = \rho_o^* \, G^{*1/2} \tag{8.4}$$

Consider an arbitrary region of a material surface $s_o : \xi = 0$ in the present configuration, denoted by $\hat{\mathcal{P}}$, and let $\partial\hat{\mathcal{P}}$ be the boundary curve of $\hat{\mathcal{P}}$. Also, let $\mathcal{P}^*$, with boundary $\partial\mathcal{P}^*$, refer to an arbitrary part of the shell-like body $\mathcal{B}^*$ in the present configuration such that:

a) $\mathcal{P}^*$ contains $\hat{\mathcal{P}}$.

b) $\partial\mathcal{P}^*$ consists of portions of the surfaces $(8.1)_{1,2}$ and a surface of the form (8.2) at time t.

c) $\partial\mathcal{P}^*$ coincides with $\partial\hat{\mathcal{P}}$ on the surface $s_0 : \xi = 0$.

Moreover, let $\partial\mathcal{P}_l^*$ refer to the part of $\partial\mathcal{P}^*$ specified by a lateral surface of the form (8.2) such that

$$\partial\mathcal{P}_l^* = \partial\mathcal{P}^* = \partial\hat{\mathcal{P}} \text{ on } s_0 : \xi = 0 \tag{8.5}$$

Let the boundary $\partial\hat{\mathcal{P}}$ of $\hat{\mathcal{P}}$ in the present configuration be denoted by a closed curve c and defined by the position vector **r** in $\partial\mathcal{P}$. Let

$$\eta^\alpha = \eta^\alpha(s) \tag{8.6}$$

be the parametric equations of the curve c, with s as the arc parameter. Further, let λ and ν denote the unit tangent vector and the outward unit normal to c lying in the surface $s_0 : \xi = 0$. Then we have

$$\lambda = \frac{\partial \mathbf{r}(\eta^\alpha(s))}{\partial s} = \lambda^\alpha \mathbf{a}_\alpha \ , \ \lambda^\alpha = \frac{d\eta^\alpha(s)}{ds} \tag{8.7}$$

$$\nu = \lambda \times \mathbf{a}_3 = \nu^\alpha \mathbf{a}_\alpha = \nu_\alpha \mathbf{a}^\alpha = \varepsilon^\alpha{}_\beta \lambda^\beta \mathbf{a}_\alpha \tag{8.8}$$

$$\lambda = \mathbf{a}_3 \times \nu = \mathbf{a}_3 \times \nu_\alpha \mathbf{a}^\alpha = \varepsilon^{\alpha\beta} \nu_\alpha \mathbf{a}_\beta \tag{8.9}$$

where $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ are the ε-symbols in two-dimensional space;

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta 3} = a^{1/2}\, e_{\alpha\beta} \ , \ \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta 3} = a^{-1/2}\, \varepsilon_{\alpha\beta}$$

$$e_{11} = e_{22} = 0 \ , \ e^{11} = e^{22} = 0 \tag{8.10}$$

$$e_{12} = -e_{21} = 1 \ , \ e^{12} = -e^{21} = 1$$

We also recall that the elements of area on the surfaces

$$s_1 = \xi = \xi_1(\eta^\alpha) = \text{constant}$$

$$s_2 = \xi = \xi_2(\eta^\alpha) = \text{constant} \tag{8.11}$$

are given by[1]

$$da = (g^* g^{*33})^{1/2} d\eta^1\, d\eta^2 \ \text{ for } \xi_1, \xi_2 \text{ constants} \tag{8.11}$$

Moreover, the element of area on the lateral surface $\partial\mathcal{P}_n^*$ is

$$n_1^* d\mathit{a} = g^{*1/2}\, d\eta^2\, d\xi$$

$$\Rightarrow \ d\mathit{a} = (n^{*1} d\eta^2 - n^{*2} d\eta^1) g^{*1/2}\, d\xi \tag{8.12}$$

$$n_2^* d\mathit{a} = -g^{*1/2} d\eta^1 d\xi$$

where n_i are the components of the outward unit normal to the surface $\mathbf{n} = n^{*i}\mathbf{g}^{*i} = n^{*i}\mathbf{g}_i^*$.

Let $\mathbf{N} = \mathbf{N}(\eta^\alpha, t; \nu)$ and $\mathbf{M} = \mathbf{M}(\eta^\alpha, t; \nu)$, represent, respectively, the *resultant force*[2] and *resultant couple*[2] vectors, each per unit length of a curve c in the present configuration. We define these resultants as follows

$$\int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds = \int_{\partial\mathcal{P}_n^*} \mathbf{t}^*\, da \quad , \quad \int_{\partial\hat{\mathcal{P}}} \mathbf{M}\, ds = \int_{\partial\mathcal{P}_l^*} \mathbf{t}^*\, \xi\, da \tag{8.13}$$

The integration on the right-hand sides of $(8.13)_{1,2}$ is over the surface $\int_{\partial\mathcal{P}_l^*}$. The conditions

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1 See Appendix 1 for details.

2 To emphasize the dependence of N,M on ν we write $\mathbf{N}(\nu)$ and $\mathbf{M}(\nu)$, in place of $\mathbf{N}$ and $\mathbf{M}$ whenever it is appropriate.

$(7.13)_{1,2}$ stipulate that the action of **N** and **M** on a portion of a curve c is *equipolant* (i.e., equivalent in effectiveness) to the action of the stress vector $\mathbf{t}^*$ upon a corresponding portion of the normal surface $\int_{\partial \mathcal{P}_n^*}$, which coincides with $\partial\hat{\mathcal{P}}$ on the surface $\mathcal{S}_0 : \xi = 0$. We also define additional resultants

$$N^\alpha a^{1/2} = \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha}\, d\xi \quad , \quad M^\alpha a^{1/2} = \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha}\, \xi\, d\xi \tag{8.14}$$

and

$$\mathbf{m}\, a^{1/2} = \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}\, d\xi \tag{8.15}$$

Recall the relations between the stress vector $\mathbf{t}^*$, the stress tensor τ^{*ij} and the vector $\mathbf{T}^{*i}$ in classical continuum mechanics:

$$\mathbf{t}^* = \frac{\mathbf{T}^{*i} n_i}{g^{1/2*}} = \tau^{*ij}\, n_i^*\, \mathbf{g}_j^* \ , \ \mathbf{T}^{*i} = g^{*1/2}\, \tau^{*ij}\, \mathbf{g}_j^* = g^{*1/2}\, \tau_j^{*i}\, \mathbf{g}^{*j} \tag{8.16}$$

where $\mathbf{n} = n^{*i}\, \mathbf{g}_i^*$ is the outward unit normal to the surface on which $\mathbf{t}^*$ acts. Considering (8.13), and making use of (8.16), we obtain

$$\int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds = \int_{\partial\mathcal{P}_n^*} \mathbf{t}^*\, da = \int_{\partial\mathcal{P}_n^*} g^{*-1/2}\, \mathbf{T}^{*i}\, n_i^* da$$

$$= \int_{\partial\hat{\mathcal{P}}} \int_{\xi_1}^{\xi_2} (\mathbf{T}^{*1}\, d\eta^2 - \mathbf{T}^{*2}\, d\eta^1) d\xi$$

$$= \int_{\partial\hat{\mathcal{P}}} a^{1/2} (\mathbf{N}^1 d\eta^2 - \mathbf{N}^2 d\eta^1) = \int_{\partial\hat{\mathcal{P}}} \mathbf{N}^\alpha \nu_\alpha ds \tag{8.17}$$

Since (8.17) is valid for any arbitrary part $\hat{\mathcal{P}}$ with closed boundary $\partial\hat{\mathcal{P}}$, it follows that

$$\mathbf{N} = \mathbf{N}^\alpha \nu_\alpha \tag{8.18}$$

In a similar manner, from $(8.13)_2$ we can obtain

$$\mathbf{M} = \mathbf{M}^{\alpha} \nu_{\alpha} \tag{8.19}$$

With the help of (8.18) and (8.19) we can obtain results analogous to those of classical continuum mechanics, namely

$$\mathbf{N}(\nu) = -\mathbf{N}(-\nu) \ , \ \mathbf{M}(\nu) = -\mathbf{M}(-\nu) \tag{8.20}$$

According to expressions (8.20) the resultant force and the resultant couple both per unit length of c, acting on the opposite sides of the same curve at a given point are equal in magnitude and opposite in direction.

Next we define two-dimensional body forces as follows:

$$\hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\bar{\mathbf{t}}^* g^{*1/2}(g^{*33})^{1/2}]_{\xi=\xi_2} + [\bar{\mathbf{t}}^* g^{*1/2}(g^{*33})^{1/2}]_{\xi=\xi_1} \tag{8.21}$$

$$\hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} = \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi d\xi + [\bar{\mathbf{t}}^* (g^* g^{*33})^{1/2}\xi]_{\xi=\xi_2} + [\bar{\mathbf{t}}^* (g^* g^{*33})^{1/2}\xi]_{\xi=\xi_1} \tag{8.22}$$

where $\bar{\mathbf{t}}^*$ is the prescribed value of $\mathbf{t}^*$ on the boundary $\partial\mathcal{B}^*$ of $\mathcal{B}^*$. In the above expressions $\bar{\mathbf{t}}^*$ represents the prescribed surface loads on the surfaces $s_1 : \xi = \xi_1$ and $s_2 : \xi = \xi_2$. Making use of (8.16), we may reduce (8.21) and (8.22) to

$$\hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\mathbf{T}^{*3}]_{\xi_1}^{\xi_2} = \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\mathbf{T}^{*3}]_{\xi=\xi_2} - [\mathbf{T}^{*3}]_{\xi=\xi_1} \tag{8.23}$$

$$\hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} = \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi d\xi + [\mathbf{T}^{*3}\xi]_{\xi_1}^{\xi_2} = \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi d\xi + [\mathbf{T}^{*3}\xi]_{\xi=\xi_2} - [\mathbf{T}^{*3}\xi]_{\xi=\xi_1} \tag{8.24}$$

It is to be remembered that in obtaining the above formulae we have assumed ξ_1, ξ_2 in $\xi = \xi_1(\eta^\alpha)$, $\xi = \xi_2(\eta^\alpha)$ to be constants. Also, in obtaining (8.23) and (8.24) we have used

$$\mathbf{n}^* = (g^{*33})^{-1/2}[0,0,-\mathbf{g}^{*3}] \text{ on the surface } s_1 : \xi = \xi_1(\eta^\alpha)$$

$$\mathbf{n}^* = (g^{*33})^{-1/2}[0,0,+\mathbf{g}^{*3}] \text{ on the surface } s_2 : \xi = \xi_2(\eta^\alpha) \tag{8.25}$$

for the outward unit normals to the surfaces s_1 and s_2.

9. Basic field equations for a shell-like body

In this section we derive basic field equations of motion for a shell-like body. To this end we make use of the various resultants defined in section 8 and the three-dimensional equations of motion in classical continuum mechanics, namely

$$\mathbf{T}^{*i}{}_{,i} + \rho^* \, \mathbf{b}^* \, g^{*1/2} = \rho^* \, \dot{\mathbf{v}}^* \, g^{*1/2} \tag{9.1}$$

and

$$\mathbf{g}_i^* \times \mathbf{T}^{*i} = 0 \tag{9.2}$$

where

$$\mathbf{t}^* = g^{*-1/2} \, \mathbf{T}^{*i} \, n_i^* \ , \ \ \mathbf{T}^{*i} = g^{*1/2} \, \tau^{*ij} \, \mathbf{g}_j^* \tag{9.3}$$

The derivation is effected by

i) integration of each term in (9.1) and (9.2) with respect to ξ, and

ii) integration, after multiplication by ξ, of each term in (9.1) and (9.2) with respect to ξ.

Consider equation (9.1) and integrate both sides of the equation with respect to ξ between ξ_1 and ξ_2 to obtain

$$\int_{\xi_1}^{\xi_2} \mathbf{T}^{*i}{}_{,i} \, d\xi + \int_{\xi_1}^{\xi_2} \rho^* \, \mathbf{b}^* \, g^{*1/2} \, d\xi = \int_{\xi_1}^{\xi_2} \rho^* \, \dot{\mathbf{v}}^* \, g^{*1/2} \, d\xi \tag{9.4}$$

We now consider each term in (9.4) separately. Thus, we write

$$\int_{\xi_1}^{\xi_2} T^{*i}{}_{,i}\, d\xi = \int_{\xi_1}^{\xi_2} T^{*\alpha}{}_{,\alpha}\, d\xi + \int_{\xi_1}^{\xi_2} T^{*3}{}_{,3}\, d\xi = [\int_{\xi_1}^{\xi_2} T^{*\alpha}\, d\xi]_{,\alpha} + [T^{*3}]\ \begin{matrix}\xi=\xi_2\\ \xi=\xi_1\end{matrix}$$

$$= (N^{\alpha}\, a^{1/2})_{,\alpha} + [T^{*3}]\ \begin{matrix}\xi=\xi_2\\ \xi=\xi_1\end{matrix} \tag{9.5}$$

Also,

$$\int_{\xi_1}^{\xi_2} \rho^{*}\, \mathbf{b}^{*}\, g^{*1/2}\, d\xi = \hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} - [T^{*3}]\ \begin{matrix}\xi=\xi_2\\ \xi=\xi_1\end{matrix} \tag{9.6}$$

and

$$\int_{\xi_1}^{\xi_2} \rho \dot{\mathbf{v}}^{*}\, g^{*1/2}\, d\xi = \int_{\xi_1}^{\xi_2} \rho^{*}\, g^{*1/2}(\dot{\mathbf{v}} + \xi \dot{\mathbf{w}}) d\xi = \hat{\rho}\, a^{1/2}\, \dot{\mathbf{v}} \tag{9.7}$$

where in obtaining the last result we have made use of

$$\rho^{*}\, g^{*1/2} = k^{*}\ ,\ \hat{\rho}\, a^{1/2} = \int_{\xi_1}^{\xi_2} k^{*} d\xi\ ,\ \int_{\xi_1}^{\xi_2} k^{*}\, \xi\, d\xi = 0 \tag{9.8}$$

and

$$\mathbf{v}^{*} = \mathbf{v} + \xi\, \mathbf{w} \tag{9.9}$$

with $\mathbf{v}$ and $\mathbf{w}$ as functions of η^1 and η^2 only. Introducing (9.5) to (9.7) in (9.4), we obtain

$$(N^{\alpha}\, a^{1/2})_{,\alpha} + \hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \hat{\rho}\, \dot{\mathbf{v}}\, a^{1/2} \tag{9.10}$$

Recalling the tensor identity

$$(N^{\alpha}\, a^{1/2})_{,\alpha} = a^{1/2}\, N^{\alpha}{}_{|\alpha} \tag{9.11}$$

we can reduce (9.10) to

$$N^{\alpha}{}_{|\alpha} + \hat{\rho}\, \hat{\mathbf{f}} = \hat{\rho}\, \dot{\mathbf{v}} \tag{9.12}$$

Next, we multiply (9.1) by ξ and then integrate with respect to ξ, i.e.,

$$\int_{\xi_1}^{\xi_2} \mathbf{T}^{*i}{}_{,i}\, \xi\, d\xi + \int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2}\, \xi\, d\xi = \int_{\xi_1}^{\xi_2} \rho^* \dot{\mathbf{v}}^* g^{*1/2}\, \xi\, d\xi \tag{9.13}$$

Considering each term in (9.13), we write

$$\begin{aligned}
\int_{\xi_1}^{\xi_2} \mathbf{T}^{*i}{}_{,i}\, \xi\, d\xi &= \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha}{}_{,\alpha}\, \xi\, d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}{}_{,3}\, \xi d\xi \\
&= [\int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha}\, \xi\, d\xi]_{,\alpha} + \int_{\xi_1}^{\xi_2} [(\mathbf{T}^{*3}\, \xi)_{,3} - \mathbf{T}^{*3}] d\xi \\
&= (\mathbf{M}^{\alpha} a^{1/2})_{,\alpha} + [\xi\, \mathbf{T}^{*3}]\,\Big|_{\xi=\xi_1}^{\xi=\xi_2} - \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}\, d\xi \\
&= (\mathbf{M}^{\alpha} a^{1/2})_{,\alpha} + [\xi\, \mathbf{T}^{*3}]\,\Big|_{\xi=\xi_1}^{\xi=\xi_2} - \mathbf{m}\, a^{1/2}
\end{aligned} \tag{9.14}$$

Also,

$$\int_{\xi_1}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2}\, \xi\, d\xi = \hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} - [\xi\, \mathbf{T}^{*3}]\,\Big|_{\xi=\xi_1}^{\xi=\xi_2} \tag{9.15}$$

and

$$\int_{\xi_1}^{\xi_2} \rho^* g^{*1/2} \dot{\mathbf{v}}^*\, \xi\, d\xi = \int_{\xi_1}^{\xi_2} k^* (\xi \dot{\mathbf{v}} + \xi^2\, \dot{\mathbf{w}}) d\xi = \hat{\rho}\, y^2\, \dot{\mathbf{w}}\, a^{1/2} \tag{9.16}$$

where we have made use of $(9.8)_3$ and defined the coefficient y^{α} as follows[1]

$$\hat{\rho}\, y^{\alpha} a^{1/2} = \int_{\xi_1}^{\xi_2} k^*\, \xi^{\alpha}\, d\xi \qquad (\alpha = 1,2) \tag{9.17}$$

We notice that the coefficients y^{α} are independent of time but they may be functions of coordi-

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[1] Although in this section we do not need to define y^1 and y^2 (since $y^1 = 0$ by $(9.8)_3$), for later use and convenience it is preferable to adhere to the definition (9.17).

nates $\eta^i = \{\eta^\alpha, \xi\}$. Introducing (9.14) to (9.16) into (9.13), we obtain

$$(\mathbf{M}^\alpha a^{1/2})_{,\alpha} - \mathbf{m}\, a^{1/2} + \hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} = \rho\, y^2\, \dot{\mathbf{w}}\, a^{1/2} \tag{9.18}$$

Again, by making use of the tensor identity

$$(\mathbf{M}^\alpha a^{1/2})_{,\alpha} = a^{1/2}\, \mathbf{M}^\alpha{}_{|\alpha} \tag{9.19}$$

we can reduce (9.18) to

$$\mathbf{M}^\alpha{}_{|\alpha} - \mathbf{m} + \hat{\rho}\, \hat{\mathbf{l}} = \rho\, y^2\, \dot{\mathbf{w}} \tag{9.20}$$

Next, we consider (9.2) and integrate it with respect to ξ between ξ_1 and ξ_2 to obtain

$$\int_{\xi_1}^{\xi_2} (\mathbf{g}_i^* \times \mathbf{T}^{*i}) d\xi = 0 \tag{9.21}$$

Recalling that

$$\mathbf{g}_\alpha^* = \mathbf{a}_\alpha + \xi\, \mathbf{d}_{,\alpha} \ , \ \mathbf{g}_3 = \mathbf{d} \tag{9.22}$$

we can rewrite (9.21) as follows

$$\int_{\xi_1}^{\xi_2} (\mathbf{g}_i^* \times \mathbf{T}^{*i}) d\xi = \int_{\xi_1}^{\xi_2} [(\mathbf{a}_\alpha + \xi \mathbf{d}_{,\alpha}) \times \mathbf{T}^{*\alpha} + \mathbf{d} \times \mathbf{T}^{*3}] d\xi$$

$$= \int_{\xi_1}^{\xi_2} [(\mathbf{a}_\alpha + \xi\, \mathbf{d}_{,\alpha}) \times \mathbf{T}^{*\alpha}] d\xi + \int_{\xi_1}^{\xi_2} \mathbf{d} \times \mathbf{T}^{*3}\, d\xi$$

$$= \mathbf{a}_\alpha \times \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} d\xi + \mathbf{d}_{,\alpha} \times \int_{\xi_1}^{\xi_2} \mathbf{T}^\alpha\, \xi\, d\xi + \mathbf{d} \times \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}\, d\xi$$

$$= \mathbf{a}_\alpha \times (\mathbf{N}^\alpha a^{1/2}) + \mathbf{d}_{,\alpha} \times (\mathbf{M}^\alpha a^{1/2}) + \mathbf{d} \times (\mathbf{m}\, a^{1/2}) = 0 \tag{9.23}$$

Since $a \neq 0$, we obtain

$$\mathbf{a}_\alpha \times \mathbf{N}^\alpha + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha = 0 \tag{9.24}$$

We now proceed to obtain the equation of balance and energy. To this end and within the scope of the classical continuum mechanics, we recall the principle of balance of energy in purely mechanical theory, i.e.,

$$\int_{\partial\mathcal{P}^*} \mathbf{t}^* \cdot \mathbf{v}^* \, da^* + \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* \cdot \mathbf{v}^* \, dv^* = \frac{d}{dt}\int_{\mathcal{P}^*} \tfrac{1}{2} \rho^* \mathbf{v}^* \cdot \mathbf{v}^* \, dv^* + \frac{d}{dt}\int_{\mathcal{P}^*} \rho^* \varepsilon^* \, dv^* \quad (9.25)$$

where $\varepsilon^* = \varepsilon^*(\eta^i, t)$ is the specific internal energy. By making use of the transport theorem and the divergence theorem we can reduce (9.25) to

$$\int_{\partial\mathcal{P}^*} g^{*-1/2} \mathbf{T}^{*i} n_i^* \cdot \mathbf{v}^* \, da^* + \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* \cdot \mathbf{v}^* \, dv^* = \int_{\mathcal{P}^*} \rho^* \dot{\mathbf{v}}^* \cdot \mathbf{v}^* \, dv^* + \int_{\mathcal{P}^*} \rho^* \dot{\varepsilon}^* \, dv^*$$

or (9.26)

$$\int_{\mathcal{P}^*} (g^{*-1/2} \mathbf{T}^{*i} \cdot \mathbf{v}^*)_{|i} \, dv^* + \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* \cdot \mathbf{v}^* \, dv^* = \int_{\mathcal{P}^*} \rho^* \dot{\mathbf{v}}^* \cdot \mathbf{v}^* \, dv^* + \int_{\mathcal{P}^*} \rho^* \dot{\varepsilon}^* \, dv^*$$

or

$$\int_{\mathcal{P}^*} g^{*-1/2} (\mathbf{T}^{*i} \cdot \mathbf{v}^*)_{,i} \, dv^* + \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* \cdot \mathbf{v}^* \, dv^* = \int_{\mathcal{P}^*} \rho^* (\dot{\mathbf{v}}^* \cdot \mathbf{v}^* + \dot{\varepsilon}^*) dv^*$$

or (9.27)

$$\int_{\mathcal{P}^*} \{ g^{*-1/2} (\mathbf{T}^{*i} \cdot \mathbf{v}^*)_{,i} + \rho^* \mathbf{b}^* \cdot \mathbf{v}^* - \rho^* (\dot{\mathbf{v}}^* \cdot \mathbf{v}^* + \dot{\varepsilon}^*) \} dv^*$$

where in obtaining (9.27) we have also made use of the tensor identity

$$(g^{*1/2} v^*)_{,i} = g^{*1/2} v_{|i} \quad (9.28)$$

Since (9.27) must hold for any aribtrary part $\mathcal{P}^*$ of the body $\mathcal{B}^*$, we obtain

$$g^{*-1/2} (\mathbf{T}^{*i} \cdot \mathbf{v}^*)_{,i} + \rho^* \mathbf{b}^* \cdot \mathbf{v}^* - \rho^* (\dot{\mathbf{v}}^* \cdot \mathbf{v}^* + \dot{\varepsilon}^*) = 0 \quad (9.29)$$

This equation, with the help of equation of motion (9.1), may be reduced further

$$g^{*-1/2} \mathbf{T}^{*i}_{,i} \cdot \mathbf{v}^* + g^{*-1/2} \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i} + \rho^* \mathbf{b}^* \cdot \mathbf{v}^* - \rho^* \dot{\mathbf{v}}^* \cdot \mathbf{v}^* = \rho^* \dot{\varepsilon}^*$$

or

$$(g^{*-1/2} \mathbf{T}^{*i}_{,i} + \rho^* \mathbf{b}^* \cdot \mathbf{v}^* - \rho^* \dot{\mathbf{v}}^*) \cdot \mathbf{v}^* + g^{*-1/2} \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i} = \rho^* \dot{\varepsilon}^*$$

or

$$\rho^* \dot{\varepsilon}^* = g^{*-1/2}\, \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i}$$

or

$$\rho^* g^{*1/2}\, \dot{\varepsilon}^* = \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i} \tag{9.30}$$

Integrating both sides of (9.30) with respect to ξ between ξ_1 and ξ_2 we obtain

$$\int_{\xi_1}^{\xi_2} \rho^*\, g^{*1/2}\, \dot{\varepsilon}^*\, d\xi = \int_{\xi_1}^{\xi_2} \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i}\, d\xi = \int_{\xi_1}^{\xi_2} \mathbf{T}^{*i} \cdot (\mathbf{v} + \xi\, \mathbf{w})_{,i}\, d\xi$$

$$= \int_{\xi_1}^{\xi_2} \mathbf{T}^{*i} \cdot \mathbf{v}_{,i}\, d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*i} \cdot (\xi\, \mathbf{w})_{,i}\, d\xi$$

$$= \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} \cdot \mathbf{v}_{,\alpha}\, d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} \cdot (\xi \mathbf{w})_{,\alpha}\, d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3} \cdot (\xi \mathbf{w})_{,3}\, d\xi$$

$$= \mathbf{v}_{,\alpha} \cdot \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha}\, d\xi + \mathbf{w}_{,\alpha} \cdot \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha}\, \xi\, d\xi + \mathbf{w} \cdot \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}\, d\xi$$

$$= a^{1/2}\, \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + a^{1/2}\, \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + a^{1/2}\, \mathbf{m} \cdot \mathbf{w} \tag{9.31}$$

We now define a two-dimensional (surface) specific internal energy $\hat{\varepsilon}$ by the condition

$$a^{1/2}\, \hat{\rho}\, \hat{\varepsilon} = \int_{\xi_1}^{\xi_2} \rho^*\, g^{*1/2}\, \varepsilon^*\, d\xi \tag{9.32}$$

Hence, (9.31) can be reduced to

$$\hat{\rho}\, \dot{\hat{\varepsilon}} = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w} = P \tag{9.33}$$

where

$$P = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w} \tag{9.34}$$

is the mechanical power.

C. Modeling of a composite laminate as a series of Cosserat (directed) surfaces

To begin with, the continuum itself is a model representing an idealized body in some sense. We may recall that the continuum model (in classical mechanics) is intended to represent phenomena in nature which appear at a scale larger than the interatomic distances. From such intuitive notions the well defined classical field theories of mechanics have been constructed and the "macroscopic" behavior of the general medium in question has been successfully studied. In the context of classical continuum mechanics a body is thought of as a set of particles (material points), say x. Each material point has a distinct identity and occupies at each instant of time t an exclusive place in a Euclidean three-dimensional space, so that one can identfy each material point x with its place (i.e., the position vector from a fixed reference point) in the space. It is implied that no more interesting information would be perceived by a finer observation of material points. Hence, microscopic details, if any, are discarded.

For a large class of bodies, these preconceptions are justified, but there are also cases when a closer look at a material point reveals some microscopic order and that at least partial information of interest could be extracted by considering the effect of the microscopic order. It is therefore desirable to construct continuum theories that in some fashion incorporate the effect of the microstructure while enjoying, if possible, to some extent the level of generality available in the classical continuum mechanics. There are different types of materials that exhibit microstructural behavior. One class of such materials is composites, i.e., bodies in which two or more substances are combined in a specific geometrical fashion to produce a new material with mechanical properties different from those of the individual constituents. Roughly speaking, a continuum with microstructure is a continuum whose properties and behavior are affected by the local deformations of the material points in any of its volume elements.

The practical analysis of the mechanical response of composite bodies involves analytical studies on two levels of abstraction. These areas of investigation are known as micromechanics and macromechanics. In micromechanics, one attempts to recognize the fine details of the material structure, i.e., a heterogeneous body, consisting of reinforcing elements, such as fibers, plies, particles, etc., embedded in a matrix material. In other words, micromechanics establishes the relation between the properties of the constituents and those of the unit composite cell. In macromechanics, on the other hand, one attempts to consider the composite body as an assembly of interacting cells, and study the overall behavior of the composite. For clarity, we emphasize that the term micromechanics does not imply studies on the atomic scale. We also note that within the context of the present discussion, the physical dimensions involved at the microstructural level are much smaller than the physical dimension involved at the macrostructural level. In what follows we confine our attention to laminated composite bodies.

We define a composite laminate as a three-dimensional continuum consisting of multiple layers (two or more) of materials which act together as a single (integral) physical entity. Here we confine our attention to laminated composites composed of multiple layers of only two materials, each of which are considered to be homogeneous. The layers are not considered to be necessarily flat and could have any type of curvature (see figure 3). Thus the laminated medium under consideration is assumed to consist of alternating layers of two homogeneous materials. We assume the thickness of each layer (ply) is much smaller than its other two dimensions and also smaller than the dimensions of the composite laminate. For example if θ^α are curvilinear surface coordinates of a layer (ply) and θ^3 is the third out of surface coordinate of the layer and the layers alternate in the direction of θ^3, the dimension of one set of alternating layers (one of each material) is much smaller in comparison to the dimension of the composite in the direction of θ^3.

In order to construct a continuum theory, we should look for a (some) representative (repetitive) feature(s) within the body. For the laminated medium under consideration the most distinct representative feature is the alternating feature of the layers. Hence, we choose the combination of one layer of reinforcement and one layer of matrix as a representative element for the laminated composite. We then model this representative element as a Cosserat (directed) surface using the theory described in previous section. Next we assume the composite laminate is composed of infinitely many of such Cosserat surfaces adjacent to each other. We now proceed to formalize this idea. Consider a finite three-dimensional body $\mathcal{B}$ in a Euclidean 3-space and let a set of convected coordinates θ^i (i = 1,2,3) be assigned to each particle (material point) P of $\mathcal{B}$. Assume at each particle P there exists a Cosserat surface, $\mathcal{s}$ (i.e., a material surface together with a deformable vector field called the director) such that θ^α are the coordinates of the surface. If at each point P the Cosserat surface is now identified by a representative element (i.e., one layer of matrix together with one layer of reinforcement) of the laminated composite and if the body $\mathcal{B}$ is identified with the composite laminate itself, the model of a composite laminate with micro-structure is at hand. It is to be emphasized that in the present discussion each Cosserat surface is itself a three dimensional shell-like body $\mathcal{B}^*$ consisting of two layers of different homogeneous materials. We also notice that the material points within each representative element $\mathcal{B}^*$ are regular particles in the sense of classical continuum mechanics while the material points of $\mathcal{B}$ are endowed not only with an assigned mass density but also with a director. For clarity, we will refer to the body $\mathcal{B}$ as composite laminate, macro-continuum or macro-structure and to the body $\mathcal{B}^*$ as representative element, micro-continuum or micro-structure. Also, we will refer to particles of $\mathcal{B}$ as macro-particles or composite particles while the particles of the micro-structures will be referred to as micro-particles or simply particles (material points). Parameters or variables that represent similar physical quantities in micro-body, Cosserat surface and macro-body will be designated with the same symbol but with an additional asterisk (*) and an over hat (ˆ) for the micro-body and Cosserat surface, respectively. For example, the mass density of the composite laminate will be called composite mass (or macro-mass) density and will be denoted by ρ while the mass densities of the Cosserat surface and that of the micro-structure will be designated by $\hat{\rho}$ and ρ^*, respectively. We recall that each Cosserat surface represents a three-dimensional body in the sense of classical continuum mechanics and its boundary consists of a lateral (normal) surface and two major (upper and lower) surfaces. We assume that at each composite particle the Cosserat surface coincides with the lower surface of the micro-structure. Hence, each geometric point P of the body $\mathcal{B}$ is a point $\hat{\mathcal{P}}$ on a Cosserat surface and at the same time is considered to coincide with a point P^* on the lower surface of the shell-like micro-structure.

10. Coordinate systems for a composite laminate

At each point P of the macro-body $\mathcal{B}$ we introduce a set of convected coordinates θ^i (i = 1,2,3). Also, at each point P^* on the lower surface of the shell-like micro-structure which coincides with P we introduce another set of convected coordinates η^i (i = 1,2,3). We assume the transformation from θ^i to η^i exists, i.e.,

$$\theta^i = \theta^i(\eta^k) = \theta^i(\eta^1,\eta^2,\eta^3) \tag{10.1}$$

and

$$\det\left(\frac{\partial\theta^i}{\partial\eta^j}\right) \neq 0 \tag{10.2}$$

This implies the existence of a unique inverse for the above transformation. At this point we make the additional assumption that

$$\theta^\alpha = \eta^\alpha \quad (\alpha = 1,2)$$
$$\theta^3 = \frac{1}{\varepsilon}\eta^3 \quad , \quad \varepsilon \ll 1 \tag{10.3}$$

The first of the above assumptions is for convenience (not necessary) while the second one is needed since the thickness of a representative element (micro-structure) is considered to be much smaller than the dimension(s) of the composite laminate (macro-structure). We will return to this point later. As before, for convenience we set $\eta^3 = \xi$ and adopt the notation $\eta^i = \{\eta^\alpha,\xi\}$. Using this notation (10.1) and (10.3) reduce to

$$\theta^i = \theta^i(\eta^k) = \eta^i(\eta^1,\eta^2,\xi) \tag{10.4}$$

and

$$\theta^\alpha = \eta^\alpha \quad (\alpha = 1,2)$$
$$\theta^3 = \frac{1}{\varepsilon}\xi \ , \quad \varepsilon \ll 1 \qquad (10.5)$$

11. Definition of a shell-like representative element (micro-structure)

Within the context of three-dimensional classical continuum mechanics, consider a body $\mathcal{B}^*$ in the present configuration and let its boundary be a closed surface, denoted by $\partial\mathcal{B}^*$, and be composed of the following material surfaces:

a) The material surfaces

$$\begin{aligned} s_0 &: \ \xi = 0 \\ & \qquad\qquad\qquad 0 < \xi_2 \\ s_2 &: \ \xi = \xi_2(\eta^\alpha) \end{aligned} \tag{11.1}$$

b) The material surface

$$s_l : \ f(\eta^\alpha) = 0 \tag{11.2}$$

such that ξ = const. are closed smooth curves on the surface (11.2). We also consider a material surface of the form

$$s_1 : \ \xi = \xi_1(\eta^\alpha) \qquad 0 < \xi_1 < \xi_2 \tag{11.3}$$

lying entirely between s_0 and s_2. From now on we will refer to surfaces defined above as follows.

a) s_0 : bottom face (lower major surface) of the micro-structure (representative element).

b) s_1 : interface (middle major surface) of the micro-structure (representative element)

c) s_2 : top face (upper major surface) of the micro-structure (representative element).

d) s_l : lateral (major) surface or normal surface of the micro-structure (representative element).

We recall that since $\eta^i = \{\eta^\alpha, \xi\}$ are defined by (10.4) and (10.5) as convected coordinates, the material surfaces (11.1) and (11.3) will have the same parametric representation in all configurations. In general ξ_1 and ξ_2 are functions of the surface coordinate η^α but in special cases they may be constants. We assume the surfaces s_0, s_1 and s_2 do not intersect themselves, or each other. This implies the condition $(11.3)_2$ and $g^* \neq 0$. The surface s_1 is not necessarily midway between the bounding surfaces s_0 and s_2. Such a three dimensional body $\mathcal{B}^*$ as characterized above and depicted in figure 4, is called a shell-like representative element or a shell-like micro-structure if the dimension of the body along the normals to the surface s_0, called the *height* of the micro-structure, is much smaller in comparison to its other two dimensions or a characteristic length of the surface s_0.

Considering our description of the body $\mathcal{B}^*$, we may note that $\mathcal{B}^*$ consists of two distinct parts $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ as defined below.

a) Part $\mathcal{B}_1^*$, a shell-like body bounded by the major surfaces s_0 and s_1 and by a lateral surface s_{l_1} which is the portion of the surface s_l bounded by its intersections with s_0 and s_1.

b) Part $\mathcal{B}_2^*$, a shell-like body bounded by the major surfaces s_1 and s_2 and by a lateral surface s_{l_2} which is the portion of the surface s_l bounded by its intersections with s_1 and s_2.

Considering (a) and (b) above, we have

$$\mathcal{B}^* = \mathcal{B}_1^* \cup \mathrm{B}_2^* \qquad s_l = s_{l_1} \cup s_{l_2} \tag{11.4}$$

We assume that $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ consist of two different materials which are perfectly bonded at their interface surface, namely the surface $s_1 : \xi = \xi_1$. We will designate the physical quantities associated with $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ with subscripts 1 and 2, respectively. For example, the mass densities of $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ will be designated by ρ_1^* and ρ_2^*, respectively. It is clear that the physical quantities associated with the body $\mathcal{B}^*$ may have a jump across the surface s_1 : $\xi = \xi_1$.

Let $\rho^*(\eta^\alpha,\xi,t)$ and $\rho_o^*(\eta^\alpha,\xi)$ be the mass densities of $\mathcal{B}^*$ in the deformed and reference configurations, respectively. Then the conservation of mass (in three dimensions) implies

$$\rho_\alpha^* g^{*1/2} = \rho_{o\alpha}^* G^{*1/2} \quad (\alpha = 1,2) \tag{11.5}$$

We define the surface mass density or micro-structure mass density, defined per unit area of s_o at time t in the present configuration by the expression

$$\hat{\rho}\, a^{1/2} = \int_o^{\xi_2} \rho^* g^{*1/2} d\xi$$
$$\hat{\rho} = \hat{\rho}(\eta^\alpha, t) \tag{11.6}$$

where $\hat{\rho}$ denotes the mass density and a is $\det(a_{\alpha\beta})$ of the surface s_o. In view of our description of the body $\mathcal{B}^*$, we have

$$\hat{\rho}\, a^{1/2} = \int_o^{\xi_2} \rho^*\, g^{*1/2}\, d\xi = \int_o^{\xi_1} \rho_1\, g^{*1/2}\, d\xi + \int_{\xi_1}^{\xi_2} \rho_2^*\, g^{*1/2}\, d\xi \tag{11.7}$$

Since the quantities $\rho_1^*\, g^{*1/2}$ and $\rho_2^*\, g^{*1/2}$ are independent of time, it follows that $\hat{\rho}\, a^{1/2}$ is also independent of time, although both $\hat{\rho}$ and a may depend on t. The total mass of an arbitrary part $\mathcal{P}^*$ of the body $\mathcal{B}^*$ (composed of parts $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ of $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$, respectively) bounded by the surface $(11.1)_{1,2}$ and a surface of the form (11.2) may be expressed

$$\mathcal{M} = \int_{\mathcal{P}} \rho^* dv^* = \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \int_0^{\xi_2} \rho^* g^{*1/2}\, d\eta^1\, d\eta^2\, d\xi$$

$$= \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \{\int_0^{\xi_2} \rho^* g^{*1/2}\, d\xi\}\, d\eta^1 d\eta^2$$

$$= \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \{\int_0^{\xi_1} \rho^* g^{*1/2}\, d\xi + \int_{\xi_1}^{\xi_2} \rho^* g^{*1/2}\, d\xi\}\, d\eta^1 d\eta^2$$

or

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \hat{\rho}\, a^{1/2}\, d\eta^1 d\eta^2 = \int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} \tag{11.8}$$

where $\hat{\mathcal{P}}$ denotes an arbitrary part of the surface $s_0 : \xi = 0$ which corresponds to $\mathcal{P}^*$ and $\hat{\eta}_1, \hat{\eta}_2$ denote the applicable ranges of integration for the coordinates η^1 and η^2, respectively. Also, in obtaining (11.8) we have made use of (11.7) and the following formula:

$$dv^* = (\mathbf{g}_1^* \times \mathbf{g}_2^*) \cdot \mathbf{g}_3^*\, d\eta^1 d\eta^2 d\xi = g^{*1/2} d\eta^1 d\eta^2 d\xi \tag{11.9}$$

$$d\hat{a} = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{a}_3\, d\eta^1 d\eta^2 = a^{1/2}\, d\eta^1 d\eta^2 \tag{11.10}$$

For later use we define the following quantities

$$\hat{\rho} a^{1/2} = \hat{\lambda} = \int_0^{\xi_2} \lambda^*\, d\xi \ , \ \lambda^* = \rho^* g^{*1/2} \tag{11.11}$$

and

$$\hat{\rho} a^{1/2} y^\alpha = \hat{\lambda} y^\alpha = \int_0^{\xi_2} \lambda^* \xi^\alpha d\xi \ , \qquad (\alpha = 1,2) \tag{11.12}$$

In view of (11.7), we may rewrite (11.11) as

$$\hat{\rho} a^{1/2} = \hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_2 \tag{11.13}$$

where

$$\hat{\lambda}_1 = \int_0^{\xi_1} \lambda^* d\xi = \int_0^{\xi_1} \rho_1^*\, g^{*1/2} d\xi \tag{11.14}$$

$$\hat{\lambda}_2 = \int_{\xi_1}^{\xi_2} \lambda^* d\xi = \int_{\xi_1}^{\xi_2} \rho_2^* \, g^{*1/2} d\xi \tag{11.15}$$

Also, expression (11.12) may be rewritten as

$$\hat{\rho} a^{1/2} y^\alpha = \hat{\lambda} \, y^\alpha = \hat{\lambda}_1 y^\alpha + \hat{\lambda}_2 y^\alpha \tag{11.16}$$

where

$$\hat{\lambda}_1 y^\alpha = \int_o^{\xi_1} \lambda^* \xi^\alpha d\xi = \int_o^{\xi_1} \rho_1^* g^{*1/2} \xi^\alpha d\xi \tag{11.17}$$

and

$$\hat{\lambda}_2 y^\alpha = \int_{\xi_1}^{\xi_2} \lambda^* \xi^\alpha d\xi = \int_{\xi_1}^{\xi_2} \rho_2^* g^{*1/2} \xi^\alpha d\xi \tag{11.18}$$

This completes our description of a shell-like micro-structure (representative element), namely a three dimensional body $\mathcal{B}^*$ composed of two shell-like bodies $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ such that

$$\mathcal{B}^* = \mathcal{B}_1^* \cup \mathcal{B}_2^* \tag{11.19}$$

where $\mathcal{B}^*$ is bounded by the surfaces $(11.1)_{1,2}$ and (11.2), $\mathcal{B}_1^*$ is bounded by the surfaces $(11.1)_1$, (11.3) and (11.2), $\mathcal{B}_2^*$ is bounded by the surfaces (11.3), $(11.1)_2$ and (11.2) where $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ are perfectly bonded together at the surface (11.3).

We will refer to the duals of the surface s_o, s_1, s_2 in the reference configuration by $\mathcal{S}_o, \mathcal{S}_1, \mathcal{S}_2$, respectively. We also note that the duals of (11.6) and (11.7) in the reference configuration are as follows:

$$\hat{\rho}_o A^{1/2} = \int_o^{\xi_2} \rho_o^* \, G^{*1/2} \, d\xi \tag{11.20}$$

and we have

$$\hat{\rho}_o \, A^{1/2} = \hat{\rho} \, a^{1/2} \tag{11.21}$$

Also,

$$\hat{\rho}_o A^{1/2} = \int_o^{\xi_2} \rho_o^* G^{*1/2} d\xi = \int_o^{\xi_1} \rho_{o1}^* G^{*1/2} d\xi + \int_{\xi_1}^{\xi_2} \rho_{o2}^* G^{*1/2} d\xi \tag{11.22}$$

and we have

$$\hat{\rho}_{o1} A^{1/2} = \hat{\rho}_1 a^{1/2} \ , \ \hat{\rho}_{o2} A^{1/2} = \hat{\rho}_2 a^{1/2} \tag{11.23}$$

in view of (11.5).

12. Kinematics of micro- and macro-structures

We begin our development of the kinematical results by assuming that the position vector of a particle P^* of a representative element (micro-structure), i.e., $\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,t)$ in the present configuration has the form

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \xi(\theta^3)\mathbf{d}(\eta^\alpha,\theta^3,t) \tag{12.1}$$

The dual of (12.1) in a reference configuration is given by

$$\mathbf{P}^* = \mathbf{R}^*(\eta^\alpha,\theta^3) + \xi(\theta^3)\mathbf{D}(\eta^\alpha,\theta^3) \tag{12.2}$$

If the reference configuration is taken to be the initial configuration at time t = 0, we obtain

$$\begin{aligned}\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,0) &= \mathbf{r}(\eta^\alpha,\theta^3,0) + \xi\mathbf{d}(\eta^\alpha,\theta^3,0)\\ &= \mathbf{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3)\\ &= \mathbf{P}(\eta^\alpha,\xi,\theta^3)\end{aligned} \tag{12.3}$$

The velocity vector $\mathbf{v}^*$ of the three-dimensional shell-like micro-structure at time t is given by

$$\mathbf{v}^* = \frac{\partial \mathbf{p}^*(\eta^\alpha,\xi,\theta^3,t)}{\partial t} = \dot{\mathbf{p}}^*(\eta^\alpha,\xi,\theta^3,t) \tag{12.4}$$

where a superposed dot denotes the material time derivative, holding η^i and θ^i fixed. From (12.1) and (12.4) we obtain

$$\mathbf{v}^* = \mathbf{v} + \xi\mathbf{w} \tag{12.5}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}\ ,\quad \mathbf{w} = \dot{\mathbf{d}} \tag{12.6}$$

From (12.1) and (4.2) we have

BASE

$$\mathbf{g}^*_\alpha = \mathbf{a}_\alpha + \xi \frac{\partial \mathbf{d}_{,\alpha}}{\partial \eta^\alpha} \;\; , \;\; \mathbf{g}^*_3 = \mathbf{d} \tag{12.7}$$

where $\mathbf{a}_\alpha$ are the surface base vector of the surface s_0. The base vectors $g_i^*(\eta^\alpha,\xi,\theta^3,t)$ in (12.7) when evaluated on the surface $s_0 : \xi = 0$ reduce to

$$\mathbf{g}^*_\alpha(\eta^\gamma,0,\theta^3,t) = \mathbf{a}_\alpha(\eta^\gamma,\theta^3,t)$$
$$\mathbf{g}^*_3(\eta^\gamma,0,\theta^3,t) = \mathbf{d}(\eta^\gamma,\theta^3,t) \tag{12.8}$$

where g_i^* satisfy the condition

$$[\mathbf{g}^*_1 \; \mathbf{g}^*_2 \; \mathbf{g}^*_3] \neq 0 \tag{12.9}$$

This restriction holds for all time and values of $\eta^i = \{\eta^\alpha,\xi\}$ and θ^3. In particular, it is valid for $\xi = 0$ so that by (12.9) we also have

$$[\mathbf{a}_1 \; \mathbf{a}_2 \; \mathbf{d}] \neq 0 \tag{12.10}$$

this condition implies that the director $\mathbf{d}$ cannot be tangent to the surface s_0.

We reall that the director $\mathbf{d}$ is a three-dimensional vector and it can be written as

$$\mathbf{d} = d_i \mathbf{g}^i = d^i \mathbf{g}_i \;\; , \;\; d_i = \mathbf{g}_i \cdot \mathbf{d} \;\; , \;\; d^i = g^{ij} d_j \tag{12.11}$$

where d_i and d^i denote the covariant and contravariant components of $\mathbf{d}$ referred to $\mathbf{g}^i$ and $\mathbf{g}_i$, respectively. The gradient of the director $\mathbf{d}$ may be obtained as follows:

$$\begin{aligned} \mathbf{d}_{,i} = (d^j \mathbf{g}_j)_{,i} = d^j{}_{,i} \mathbf{g}_j + d^j \mathbf{g}_{j,i} &= d^j{}_{,i} \mathbf{g}_j + d^j \{ {}_i{}^k{}_j \} \mathbf{g}_k \\ &= d^j{}_{,i} \mathbf{g}_j + d^k \{ {}_i{}^j{}_k \} \mathbf{g}_j \\ &= (d^j{}_{,i} + \{ {}_i{}^j{}_k \} d^k) \mathbf{g}_j \\ &= d^j{}_{|i} \mathbf{g}_j \end{aligned} \tag{12.12}$$

where { } stands for the Christoffel symbol of the second kind and a vertical bar (|) denotes covariant differentiation with respect to g_{ij}. In obtaining (12.12) we have made use of the tensor identity

$$\mathbf{g}_{j,i} = \{_i{}^k{}_j\}\mathbf{g}_k \tag{12.13}$$

For convenience we introduce the notations

$$\begin{aligned} \lambda_{ij} &= \mathbf{g}_i \cdot \mathbf{d}_{,j} = d_{i|j} \\ \lambda^i{}_j &= \mathbf{g}^i \cdot \mathbf{d}_{,j} = d^i{}_{|j} \end{aligned} \tag{12.14}$$

From (12.14) it is clear that

$$\lambda^i{}_j = g^{ik}\lambda_{kj} \tag{12.15}$$

Making use of (12.14) we may rewrite (12.12) as

$$\mathbf{d}_{,i} = \lambda_{ji}\mathbf{g}^j = \lambda^j{}_i\mathbf{g}_j \tag{12.16}$$

Consider now the velocity vector **v** which can be written in the form

$$\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i \tag{12.17}$$

Since the coordinates θ^i are convected, it follows that

$$\mathbf{v}_{,i} = \mathbf{v}_{|i} = \dot{\mathbf{g}}_i \tag{12.18}$$

Following the same procedure used in (12.12), we can reduce (12.18) to

BASE

$$\mathbf{v}_{,i} = (v^j\mathbf{g}_j)_{,i} = v^j{}_{,i}\mathbf{g}_j + v^j\mathbf{g}_{j,i} = v^j{}_{,i}\mathbf{g}_j + \{{}_i{}^k{}_j\}v^j\mathbf{g}_k$$

$$= v^j{}_{,i}\mathbf{g}_j + \{{}_i{}^j{}_k\}v^k\mathbf{g}_j$$

$$= (v^j{}_{,i} + \{{}_i{}^j{}_k\}v^k)\mathbf{g}_j$$

$$= v^j{}_{|i}\mathbf{g}_j \tag{12.19}$$

where in obtaining (12.19) we have made use of (12.13) and (12.17). We now introduce the notations

$$v_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} = v_{i|j}$$
$$v^i{}_j = \mathbf{g}^i \cdot \mathbf{v}_{,j} = v^i{}_{|j} \tag{12.20}$$

From (12.20) it is clear that

$$v^i{}_j = g^{ik}v_{k|j} \tag{12.21}$$

Making use of (12.20), we may rewrite (12.19) as

$$\mathbf{v}_{,i} = v_{ji}\mathbf{g}^j = v^j{}_i\mathbf{g}_j \tag{12.22}$$

We observe that both λ_{ij} and v_{ij} represent the covariant derivative of vector components and hence transform as components of second order covariant tensors.

Since v_{ij} is a second order covariant tensor, we may decompose it into its symmetric and its skew-symmetric parts, i.e.,

$$v_{ij} = v_{(ij)} + v_{[ij]} = \eta_{ij} + \omega_{ij} \tag{12.23}$$

where

$$\eta_{ij} = v_{(ij)} = \frac{1}{2}(v_{ij} + v_{ji}) \tag{12.24}$$

and

$$\omega_{ij} = v_{[ij]} = \frac{1}{2}(v_{ij} - v_{ji}) \tag{12.25}$$

represent the symmetric and the skew-symmetric parts of v_{ij}, respectively. From (12.24) and (12.25), after making use of (12.18) and (12.20), we have

$$\eta_{ij} = \frac{1}{2}(v_{ij} + v_{ji}) = \frac{1}{2}(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \mathbf{g}_j \cdot \dot{\mathbf{g}}_i) = \frac{1}{2}\overline{(\mathbf{g}_i \cdot \mathbf{g}_j)}^{\cdot} = \frac{1}{2}\dot{g}_{ij} = \eta_{ji} \tag{12.26}$$

and

$$\omega_{ij} = \frac{1}{2}(v_{ij} - v_{ji}) = \frac{1}{2}(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j - \mathbf{g}_j \cdot \dot{\mathbf{g}}_i) = -\omega_{ji} \tag{12.27}$$

Also, in iew of (12.18) and (12.23), we may express $\dot{\mathbf{g}}_i$ in the form

$$\dot{\mathbf{g}}_i = \mathbf{v}_{,i} = (\eta_{ki} + \omega_{ki})\mathbf{g}^k \tag{12.28}$$

Moreover, the time rate of change of the determinant of g_{ij}, i.e., g is obtained as follows

$$\dot{g} = \overline{\det(g_{ij})}^{\cdot} = \frac{\partial}{\partial g_{kl}}(\det(g_{kj}))\dot{g}_{kl} = gg^{ij}\dot{g}_{ij} \tag{12.29}$$

where we have made use of the formula for the derivative of a determinant, namely

$$\frac{\partial}{\partial g_{kl}}(\det(g_{ij})) = gg^{kl} \tag{12.30}$$

Also, by making use of the relation

$$g^{ij}g_{kj} = \delta^i{}_j \tag{12.31}$$

we obtain an expression for $\dot{g}^{ij}$ as follows

$$\overline{(g^{ij}g_{kj})}^{\cdot} = \dot{\delta}^i{}_j = 0$$

or

$$\dot{g}^{ij} g_{ki} = - g^{ik} \dot{g}_{kj}$$

or

$$\dot{g}^{ij} g_{kj} g^{jl} = - g^{jl} g^{ik} \dot{g}_{kj}$$

or

$$\dot{g}^{ik} \delta_k^l = - g^{ik} g^{lj} \dot{g}_{kj}$$

or

$$\dot{g}^{ij} = - g^{ik} g^{jl} \dot{g}_{kl} \tag{12.32}$$

Next, we proceed to obtain an expression for the director velocity $\mathbf{w}$. Thus, we write

$$\begin{aligned}
\mathbf{w} = \dot{\mathbf{d}} &= w_k \mathbf{g}^k = w^k \mathbf{g}_k = \dot{\overline{(d_i \mathbf{g}^i)}} \\
&= \dot{d}_i \mathbf{g}^i + d_i \dot{\mathbf{g}}^i = \dot{d}_i \mathbf{g}^i + d_i \dot{\overline{(g^{ij} \mathbf{g}_j)}} \\
&= \dot{d}_k \mathbf{g}^k + d_i (\dot{g}^{ij} \mathbf{g}_j + g^{ij} \dot{\mathbf{g}}_j) \\
&= \dot{d}_k \mathbf{g}^k + d_i \{ - g^{ik} g^{jl} \dot{g}_{kl} \mathbf{g}_j + g^{ij} (\eta_{kj} + \omega_{kj}) \mathbf{g}^k \} \\
&= \dot{d}_k \mathbf{g}^k - d^k g^{jl} \dot{g}_{kl} \mathbf{g}_j + d^j \eta_{kj} \mathbf{g}^k + d^j \omega_{kj} \mathbf{g}^k \\
&= \dot{d}_k \mathbf{g}^k + d^i \omega_{ki} \mathbf{g}^k - d^i \dot{g}_{ik} \mathbf{g}^k + d^i \eta_{ki} \mathbf{g}^k \\
&= \dot{d}_k \mathbf{g}^k + d^i \omega_{ki} \mathbf{g}^k - d^i (2\eta_{ik}) \mathbf{g}^k + d^i \eta_{ki} \mathbf{g}^k \\
&= \dot{d}_k \mathbf{g}^k + d^i (\omega_{ki} - \eta_{ki}) \mathbf{g}^k
\end{aligned} \tag{12.33}$$

where in obtaining (12.33) we have made use of (12.28) and (12.32). The gradient of the director velocity is obtained in a similar manner:

$$\mathbf{w}_{,i} = \dot{\mathbf{d}}_{,i} = \overline{(\mathbf{d}_k\mathbf{g}^k)_{,i}}^{\cdot} = \overline{(\lambda_{ki}\mathbf{g}^k)}^{\cdot} = \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}\dot{\mathbf{g}}^k$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}\overline{(g^{kj}\mathbf{g}_j)}^{\cdot} = \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}(\dot{g}^{kj}\mathbf{g}_j + g^{kj}\dot{\mathbf{g}}_j)$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}(-g^{km}g^{jl}\dot{g}_{ml}\mathbf{g}_j) + \lambda_{ki}g^{kj}(\eta_{mj} + \omega_{mj})\mathbf{g}^m$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k - \lambda^m{}_i\dot{g}_{ml}\mathbf{g}^l + \lambda^j{}_i(\eta_{mj} + \omega_{mj})\mathbf{g}^m$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k - \lambda^m{}_i(2\eta_{ml})\mathbf{g}^l + \lambda^j{}_i\eta_{kj}\mathbf{g}^k + \lambda^j{}_i\omega_{kj}\mathbf{g}^k$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda^j{}_i\omega_{kj}\mathbf{g}^k - 2\lambda^j{}_i\eta_{jk}\mathbf{g}^k + \lambda^j{}_i\eta_{kj}\mathbf{g}^k$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda^j{}_i(\omega_{kj} - \eta_{kj})\mathbf{g}^k \tag{12.34}$$

The dual of expressions (12.7) to (12.16) in the reference configuration follows from (12.2) in a similar manner and is given by:

$$\mathbf{G}^*_\alpha = \mathbf{A}_\alpha + \xi\mathbf{D}_{,\alpha} \quad , \quad \mathbf{G}^* = \mathbf{D} \tag{12.35}$$

$$\begin{aligned} \mathbf{G}^*_\alpha(\eta^\gamma,0,\theta^3) &= \mathbf{A}_\alpha(\eta^\gamma,\theta^3,t) \\ \mathbf{G}^*_3(\eta^\gamma,0,\theta^3) &= \mathbf{D}(\eta^\gamma,\theta^3,t) \end{aligned} \tag{12.36}$$

where $\mathbf{G}_i^*$, $\mathbf{d}$ satisfy the conditions

$$[\mathbf{G}^*_1\ \mathbf{G}^*_2\ \mathbf{G}^*_3] \neq 0 \tag{12.37}$$

and

$$[\mathbf{G}^*_1\ \mathbf{G}^*_2\ \mathbf{D}] \neq 0 \tag{12.38}$$

Moreover,

$$\mathbf{D} = D_i \mathbf{G}^i = D^i \mathbf{G}_i \quad , \quad D_i = \mathbf{G}_i \cdot \mathbf{D} \quad , \quad D^i = G^{ij} D_i \tag{12.39}$$

$$\mathbf{D}_{,i} = D^j{}_{|i} \mathbf{G}_j = \Lambda^j{}_i \mathbf{G}_j = \Lambda_{ji} \mathbf{G}^j \tag{12.40}$$

where we have

$$\begin{aligned} \Lambda_{ij} &= \mathbf{G}_i \cdot \mathbf{D}_{,j} = D_{i|j} \\ \Lambda^i{}_j &= \mathbf{G}^i \cdot \mathbf{D}_{,j} = D^i{}_{|j} \end{aligned} \tag{12.41}$$

and

$$\Lambda^i{}_j = G^{ik} \Lambda_{kj} \tag{12.42}$$

We now introduce relative kinematical measures γ_{ij}, $\mathcal{K}_{ij}$ and γ_i such that

$$\gamma_{ij} = \frac{1}{2}(g_{ij} - G_{ij}) = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \gamma_{ji} \tag{12.43}$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{12.44}$$

and

$$\gamma_i = d_i - D_i \tag{12.45}$$

Making use of (12.7), (12.11), (12.16), (12.35), (12.39) and (12.40) we may obtain

$$\begin{aligned} \gamma^*_{\alpha\beta} = \gamma^*_{\beta\alpha} &= \frac{1}{2}\{(\mathbf{g}_\alpha + \xi \mathbf{d}_{,\alpha}) \cdot (\mathbf{g}_\beta + \xi \mathbf{D}_{,\beta}) - (\mathbf{G}_\alpha + \xi \mathbf{D}_{,\alpha}) \cdot (\mathbf{G}_\beta + \xi \mathbf{D}_{,\beta})\} \\ &= \frac{1}{2}\{(g_{\alpha\beta} - G_{\alpha\beta}) + \xi[(\mathbf{g}_\alpha \cdot \mathbf{d}_{,\beta} - \mathbf{G}_\alpha \cdot \mathbf{D}_{,\beta}) + (\mathbf{g}_\beta \cdot \mathbf{d}_{,\alpha} - \mathbf{G}_\beta \cdot \mathbf{D}_{,\alpha}) \\ &\quad + \xi^2(\mathbf{d}_{,\alpha} \cdot \mathbf{d}_{,\beta} - \mathbf{D}_{,\alpha} \cdot \mathbf{D}_{,\beta})\} \end{aligned}$$

or

$$\gamma^*_{\alpha\beta} = \gamma^*_{\beta\alpha} = \gamma_{\alpha\beta} = \frac{1}{2}\,\xi(\mathcal{K}_{\alpha\beta} + \mathcal{K}_{\beta\alpha}) + \frac{1}{2}\,\xi^2(\lambda^i{}_\alpha\lambda_{i\beta} - \Lambda^i{}_\alpha\Lambda_{i\beta}) \tag{12.46}$$

Also,

$$\gamma^*_{\alpha 3} = \gamma^*_{3\alpha} = \frac{1}{2}\{(\mathbf{g}_\alpha + x\mathbf{d}_{,\alpha}) \cdot \mathbf{d} - (\mathbf{G}_\alpha + \xi\mathbf{D}_{,\alpha}) \cdot \mathbf{D}\}$$

$$= \frac{1}{2}\,\{(\mathbf{g}_\alpha \cdot \mathbf{d} - \mathbf{G}_\alpha \cdot \mathbf{D}) + \xi(\mathbf{d} \cdot \mathbf{d}_{,\alpha} - \mathbf{D} \cdot \mathbf{D}_{,\alpha})\}$$

or

$$\gamma^*_{\alpha 3} = \gamma^*_{3\alpha} = \frac{1}{2}\,\{\gamma_\alpha + \xi(d^i\lambda_{i\alpha} - D^i\Lambda_{i\alpha})\} \tag{12.47}$$

and

$$\gamma^*_{33} = \frac{1}{2}\,(\mathbf{d} \cdot \mathbf{d} - \mathbf{D} \cdot \mathbf{D}) = \frac{1}{2}\,(d^i d_i - D^i D_i) \tag{12.48}$$

13. Superposed rigid body motion

We recall that when the motion of the body $\mathcal{B}^*$ differs from the given motion by a rigid motion, the position vector $\mathbf{p}^+$ has the form

$$\mathbf{p}^{*+} = \mathbf{p}^{*+}(\eta^i,t') = \mathbf{p}_o^{*+}(t') + Q(t)[\mathbf{p}^*(\eta^i,t) - \mathbf{p}_o^*(t)] \tag{13.1}$$

where Q(t) is a proper orthogonal tensor function of time. Also, under superposed rigid body motions, the position vector $\mathbf{r}$ of the surface s_0 of $\mathcal{B}^*$ changes to

$$\mathbf{r}^+ = \mathbf{r}^+(\theta^i,t') = \mathbf{r}_o^+(t') + Q(t)[\mathbf{r}(\eta^\alpha,t) - \mathbf{r}_o(t)] \tag{13.2}$$

Since

$$\mathbf{p}^* - \mathbf{r} = \xi\mathbf{d} \quad , \quad \mathbf{p}^{*+} - \mathbf{r}^+ = \xi\mathbf{d}^+ \tag{13.3}$$

with the help of (13.1) and (13.2) we obtain

$$\xi\mathbf{d}^+ = \mathbf{p}^+ - \mathbf{r}^+ = [\mathbf{p}_o^+(t') - \mathbf{r}_o^+(t')] + Q(t)\{(\mathbf{p}^* - \mathbf{r}) - (\mathbf{p}_o^* - \mathbf{r}_o)\} \tag{13.4}$$

For a rigid motion we have

$$(\mathbf{p}_o^{*+} - \mathbf{r}_o^+) = Q(t)(\mathbf{p}_o^* - \mathbf{r}_o) \tag{13.5}$$

Hence, it follows that the vector function $\mathbf{d}^+(\eta^\alpha,t)$ must transform according to

$$\mathbf{d}^+(\eta^\alpha,t) = Q(t)\mathbf{d}(\eta^\alpha,t) \tag{13.6}$$

under superposed rigid body motion. It is easily seen from (13.6) that the magnitude of $\mathbf{d}(\eta^\alpha,t)$ under superposed rigid body motions remains unchanged:

$$\mathbf{d}^+ \cdot \mathbf{d}^+ = (Q\mathbf{d}) \cdot (Q \cdot \mathbf{d}) = Q^T Q\mathbf{d} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} \tag{13.7}$$

since for a proper orthogonal tensor Q we have

$$QQ^T = Q^TQ = I \quad , \quad \det(Q) = 1 \tag{13.8}$$

and since for any two vectors **U** and **V** we have

$$\mathbf{U} \cdot Q\mathbf{V} = Q^T\mathbf{U} \cdot \mathbf{V} \tag{13.9}$$

BASE

14. Stress-resultants, stress-couples and other definitions

Consider a shell-like three-dimensional micro-structure $\mathcal{B}^*$ bounded by a closed surface $\partial\mathcal{B}^*$, as specified in section 11, which consists of the material surfaces

$$\begin{aligned} &s_0 : \xi = 0 \\ & & 0 < \xi_2 \\ &s_2 : \xi = \xi(\eta^\alpha) \end{aligned} \tag{14.1}$$

and a normal (lateral) material surface of the form

$$s_l : f(\eta^\alpha) = 0 \tag{14.2}$$

such that ξ = const. are closed smooth curves on the surface (14.2). Let s_1 be a material surface of the form

$$s_1 \quad : \quad \xi = \xi_1(\eta^\alpha) \qquad 0 < \xi_1 < \xi_2 \tag{14.3}$$

lying entirely between s_0 and s_2. Moreover, let $\mathcal{B}^*$ be composed of two shell-like bodies $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ with their lateral surfaces $\partial\mathcal{B}_1^*$ and $\partial\mathcal{B}_2^*$, respectively, as specified in section 11.

Consider an arbitrary part of the material surface $s_0 : \xi = 0$ in the present configuration and let it be denoted $\hat{\mathcal{P}}$. Also, let $\mathcal{P}^*$, with boundary $\partial\mathcal{P}^*$, refer to an arbitrary part of the shell-like body $\mathcal{B}^*$ in the present configuration such that:

a) $\mathcal{P}^*$ contains $\hat{\mathcal{P}}$;

b) $\partial\mathcal{P}^*$ consists of portions of the surfaces $(14.1)_{1,2}$ and a surface of the form (14.2) at time t;

c) $\partial\mathcal{P}^*$ coincides with $\partial\hat{\mathcal{P}}$ on the surface $s_0 : \xi = 0$.

Moreover, let $\partial \mathcal{P}_l^*$ refer to the part of $\partial \mathcal{P}^*$ specified by a lateral surface of the form (14.2) such that

$$\partial \mathcal{P}_l^* = \partial \mathcal{P}^* = \partial \hat{\mathcal{P}} \quad \text{on} \quad s_0 : \xi = 0 \tag{14.4}$$

Since $\mathcal{B}^*$ is composed of two shell-like bodies $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ separated by the surface $s_1 : \xi = \xi_1$, the part $\mathcal{P}^*$ is also composed of two parts $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ with their corresponding boundaries $\partial \mathcal{P}_1^*$ and $\partial \mathcal{P}_2^*$, respectively.

Let the boundary $\partial \hat{\mathcal{P}}$ of $\hat{\mathcal{P}}$, in the present configuration be denoted by a closed curve c and defined by the position vector $\mathbf{r}$ on $\partial \hat{\mathcal{P}}$. Let

$$\eta^\alpha = \eta^\alpha(s) \tag{14.5}$$

be the parametric equations of the curve c, with s as the arc parameter. Further, let λ and ν denote the unit tangent vector and the outward unit normal to c lying in the surface $s_o : \xi = 0$. Then we have

$$\lambda = \frac{\partial \mathbf{r}(\eta^\alpha(s))}{\partial s} = \lambda^\alpha \mathbf{a}_\alpha \ , \ \lambda^\alpha = \frac{d\eta^\alpha(s)}{ds} \tag{14.6}$$

$$\nu = \lambda \times \mathbf{a}_3 = \nu^\alpha \mathbf{a}_\alpha = \nu_\alpha \mathbf{a}^\alpha = \varepsilon^{\alpha\beta} \nu_\alpha \mathbf{a}_\beta \tag{14.7}$$

$$\lambda = \mathbf{a}_3 \times \nu = \mathbf{a}_3 \times \nu_\alpha \mathbf{a}^\alpha = \varepsilon^{\alpha\beta} \nu_\alpha \mathbf{a}_\beta \tag{14.8}$$

where $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ are the ε-symbols in two-dimensional space;

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta 3} = a^{1/2} e_{\alpha\beta} \ , \ \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta 3} = a^{-1/2} e^{\alpha\beta}$$

$$e_{11} = e_{22} = 0 \ , \ e^{11} = e^{22} = 0 \tag{14.9}$$

$$e_{12} = -e_{21} = 1 \ , \ e^{12} = -e^{21} = 1$$

We also recall that the elements of area on the surfaces

$$\mathcal{S}_0 : \xi = 0$$
$$\mathcal{S}_2 : \xi = \xi_2(\eta^\alpha) = \text{constant} \tag{14.10}$$

are given by

$$da = (g^* g^{*33})^{1/2} d\eta^1 d\eta^2 \quad \text{for } \xi_2 = \text{const.} \tag{14.11}$$

Moreover, the element of area on the lateral surface $\partial \mathcal{P}_l^*$ is

$$\left.\begin{array}{l} n^{*1} da = g^{*1/2} d\eta^2 d\xi \\ n^{*2} da = -g^{*1/2} d\eta^1 d\xi \end{array}\right\} \Rightarrow da = (n^{*1} d\eta^2 - n^{*2} d\eta^1) g^{*1/2} d\xi \tag{14.12}$$

where n_i^* are the components of the outward unit normal to the lateral surface.

Let $\mathbf{N} = \mathbf{N}(\eta^\alpha, t; \nu)$ and $\mathbf{M} = \mathbf{M}(\eta^\alpha, t; \nu)$ be, respectively, the *resultant force* and *resultant couple* vectors, each per unit length of a curve c in the present configuration. We define these resultants as follows:

$$\int_{\partial \mathcal{P}} \mathbf{N}\, ds = \int_{\partial \mathcal{P}_l^*} \mathbf{t}^*\, da \quad , \quad \int_{\partial \mathcal{P}} \mathbf{M}\, ds = \int_{\partial \mathcal{P}_l^*} \mathbf{t}^*\, \xi\, da \tag{14.13}$$

We also define additional resultants

$$\mathbf{N}^\alpha a^{1/2} = \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi = \int_0^{\xi_1} \mathbf{T}^{*\alpha} d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} d\xi \tag{14.14}$$

$$\mathbf{M}^\alpha = \int_0^{\xi_2} \mathbf{T}^{*\alpha} \xi\, d\xi = \int_0^{\xi_1} \mathbf{T}^{*\alpha} d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} d\xi \tag{14.15}$$

$$\mathbf{m}\, a^{1/2} = \int_0^{\xi_2} \mathbf{T}^{*3}\, d\xi = \int_0^{\xi_1} \mathbf{T}^{*3}\, d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}\, d\xi \tag{14.16}$$

Following the same procedure as in (8.17), we can show

$$\mathbf{N} = \mathbf{N}^{\alpha} \nu_{\alpha} \quad , \quad \mathbf{M} = \mathbf{M}^{\alpha} \nu_{\alpha} \tag{14.17}$$

and

$$\mathbf{N}(\nu) = -\mathbf{N}(-\nu) \quad , \quad \mathbf{M}(\nu) = -\mathbf{M}(-\nu) \tag{14.18}$$

We also need to define two-dimensional body forces, i.e.,

$$\hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\bar{\mathbf{t}}^* g^{*1/2}(g^{*33})^{1/2}]_{\xi=\xi_2} + [\bar{\mathbf{t}}^* g^{*1/2}(g^{*33})^{1/2}]_{\xi=0} \tag{14.19}$$

$$\hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi\, d\xi + [\bar{\mathbf{t}}^*(g^* g^{*33})^{1/2}\xi]_{\xi=\xi_2} + [\bar{\mathbf{t}}^*(g^* g^{*33})^{1/2}\xi]_{\xi=0} \tag{14.20}$$

where $\bar{\mathbf{t}}^*$ is the value of $\mathbf{t}$ on the boundary $\partial\mathcal{B}^*$ of $\mathcal{B}^*$. In the same manner as in section (8), by making use of (8.16), (14.19) and (14.20), we obtain

$$\hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\mathbf{T}^{*3}]_{o}^{\xi_2} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\mathbf{T}^{*3}]_{\xi=\xi_2} - [\mathbf{T}^{*3}]_{\xi=0} \tag{14.21}$$

and

$$\hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi\, d\xi + [\mathbf{T}^{*3}\xi]_{\xi_1}^{o} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi\, d\xi$$

$$+ [\mathbf{T}^{*3}\xi]_{\xi=\xi_2} - [\mathbf{T}^{*3}\xi]_{\xi=0} \tag{14.22}$$

where in obtaining the above formulae we have assumed ξ_2 in $\xi = \xi_2(\eta^{\alpha})$ to be constant. Also in obtaining (14.21) and (14.22) we have used

$$\begin{aligned} \mathbf{n}^* &= (g^{*33})^{-1/2}[0,0,-g^{*3}] \quad \text{on the surface} \quad \mathcal{S}_o : \xi = 0 \\ \mathbf{n}^* &= (g^{*33})^{-1/2}[0,0,+g^{*3}] \quad \text{on the surface} \quad \mathcal{S}_2 : \xi = \xi_2(\eta^{\alpha}) \end{aligned} \tag{14.23}$$

for the outward unit normal to the surfaces $\mathcal{S}_o$ and $\mathcal{S}_2$.

15. Basic field equations for a shell-like representative element (micro-structure)

We now proceed to derive basic field equations of motion for a shell-like representative element (micro-structure) as defined in section (11). To this end we make use of the various resultants defined in section (14) and procedures described in section (9). Recall the three-dimensional equations of motion in classical continuum mechanics, namely[1]

$$\mathbf{T}^{*i}{}_{,i} + \rho^* \mathbf{b}^* g^{*1/2} = \rho^* \dot{\mathbf{v}}^* g^{*1/2} \tag{15.1}$$

and

$$\mathbf{g}_i^* \times \mathbf{T}^{*i} = 0 \tag{15.2}$$

where

$$\mathbf{t}^* = g^{*-1/2} \mathbf{T}^{*i} n_i^* \ , \ \mathbf{T}^{*i} = g^{*1/2} \tau^{*ij} \mathbf{g}_j^* \tag{15.3}$$

The de-ivation is effected by

i) integration of each term in (15.1) and (15.2) with respect to ξ between $\xi = 0$ and $\xi = \xi_2$, and

ii) integration after multiplication by ξ, of each term in (15.1) and (15.2) with respect to ξ between $\xi = 0$ and $\xi = \xi_2$.

Consider equation (15.1) and integrate both sides of the equation with respect to ξ between $\xi = 0$ and $\xi = \xi_2$ to obtain

$$\int_0^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi + \int_0^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi = \int_0^{\xi_2} \rho^* \dot{\mathbf{v}}^* g^{*1/2} d\xi \tag{15.4}$$

[1] In the literature on continuum formulation of composite materials, it is customary to write two sets of equations of motion, i.e., one for the matrix and one for the reinforcing material. Moreover, to keep equations as simple as possible and since we have admitted to have jumps in various field quantities across the interface surface S_1 it suffices to write equations of motion as in (15.1).

Considering each term in (15.4) separately, we obtain

$$\int_o^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi = \int_o^{\xi_2} \mathbf{T}^{*\alpha}_{,\alpha} d\xi + \int_o^{\xi_2} \mathbf{T}^{*3}_{,3}\, d\xi = [\int_o^{\xi_2} \mathbf{T}^{*\alpha} d\xi]_{,\alpha} + [\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2}$$

$$= (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + [\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} \tag{15.5}$$

Also,

$$\int_o^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi = \hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} - [\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} \tag{15.6}$$

and

$$\int_o^{\xi_2} \rho^* \dot{\mathbf{v}}^* g^{*1/2} d\xi = \int_o^{\xi_2} \rho^* g^{*1/2}(\dot{\mathbf{v}} + \xi\, \dot{\mathbf{w}}) d\xi = \hat{\rho}\, a^{1/2}\, \dot{\mathbf{v}} + \hat{\rho} y^1 a^{1/2}\, \dot{\mathbf{w}} \tag{15.7}$$

where in obtaining the last result we have made use of (11.11), (11.12) and (12.1), i.e.,

$$\hat{\rho} a^{1/2} = \hat{\lambda} = \int_o^{\xi_2} \lambda^*\, d\xi \ , \ \lambda^* = \rho^* g^{*1/2} \tag{15.8}$$

$$\hat{\rho} a^{1/2} y^\alpha = \hat{\lambda} y^\alpha = \int_o^{\xi_2} \lambda^*\, \xi^\alpha d\xi \ , \ (\alpha = 1,2) \tag{15.9}$$

and

$$\mathbf{v}^* = \mathbf{v} + \xi \mathbf{w} \tag{15.10}$$

with $\mathbf{v}$ and $\mathbf{w}$ as functions of η^1 and η^2 only. Introducing (15.5) to (15.7) in (15.4), we obtain

$$(\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \hat{\rho}\hat{\mathbf{f}} a^{1/2} = \hat{\rho} a^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) \tag{15.11}$$

Making use of the tensor identity

$$(\mathbf{N}^\alpha a^{1/2})_{,\alpha} = a^{1/2} \mathbf{N}^\alpha{}_{|\alpha} \tag{15.12}$$

we can reduce (15.10) to

$$N^{\alpha}{}_{|\alpha} + \hat{\rho}\hat{\mathbf{f}} = \hat{\rho}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \tag{15.13}$$

Next, we multiply (15.1) by ξ and then integrate with respect to ξ between $\xi = 0$ and $\xi = \xi_2$ to obtain

$$\int_o^{\xi_2} \mathbf{T}^{*i}{}_{,i}\xi\, d\xi + \int_o^{\xi_2} \rho^*\mathbf{b}^* g^{*1/2}\, \xi d\xi = \int_o^{\xi_2} \rho^*\dot{\mathbf{v}}^* g^{*1/2}\, \xi d\xi \tag{15.14}$$

Considering each term in (15.14) separately, we write

$$\begin{aligned}\int_o^{\xi_2} \mathbf{T}^{*i}{}_{,i}\, \xi d\xi &= \int_o^{\xi_2} \mathbf{T}^{*\alpha}{}_{,\alpha}\, \xi d\xi + \int_o^{\xi_2} \mathbf{T}^{*3}{}_{,3}\, \xi d\xi \\ &= [\int_o^{\xi_2} \mathbf{T}^{*\alpha}\, \xi d\xi]_{,\alpha} + \int_o^{\xi_2} [(\mathbf{T}^{*3}\xi)_{,3} - \mathbf{T}^{*3}] d\xi \\ &= (\mathbf{M}^{\alpha} a^{1/2})_{,\alpha} + [\xi \mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} - \int_o^{\xi_2} \mathbf{T}^{*3}\, d\xi \\ &= (\mathbf{M}^{\alpha} a^{1/2})_{,\alpha} + [\xi \mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} - \mathbf{m} a^{1/2}\end{aligned} \tag{15.15}$$

and

$$\int_o^{\xi_2} \rho^*\mathbf{b}^* g^{*1/2}\, \xi d\xi = \hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} - [\xi \mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} \tag{15.16}$$

Also

$$\int_o^{\xi_2} \rho^* g^{*1/2} \dot{\mathbf{v}}^*\, \xi d\xi = \int_o^{\xi_2} \lambda^*(\xi\dot{\mathbf{v}} + \xi^2\dot{\mathbf{w}}) d\xi = \hat{\rho} a^{1/2} y^1 \dot{\mathbf{v}} + \hat{\rho} a^{1/2} y^2 \dot{\mathbf{w}}$$

where in obtaining the last result we have made use of (15.8), (15.9) and (15.10). Introducing (15.15) to (15.17) into (15.14), we obtain

$$(\mathbf{M}^{\alpha} a^{1/2})_{,\alpha} = \mathbf{m} a^{1/2} + \hat{\rho}\hat{\mathbf{l}} a^{1/2} = \hat{\rho} a^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \tag{15.18}$$

Making use of the tensor identity

$$(\mathbf{M}^{\alpha}a^{1/2})_{,\alpha} = a^{1/2}\mathbf{M}^{\alpha}{}_{|\alpha} \tag{15.19}$$

we reduce (15.18) to

$$\mathbf{M}^{\alpha}{}_{|\alpha} - \mathbf{m} + \hat{\rho}\hat{\mathbf{l}} = \hat{\rho}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \tag{15.20}$$

Next, we consider (15.2) and integrate with respect to ξ between $\xi = 0$ and $\xi = \xi_2$

$$\int_o^{\xi_2} (\mathbf{g}_i^* \times \mathbf{T}^{*i})d\xi = 0 \tag{15.21}$$

Recalling the expressions for $\mathbf{g}_i^*$, i.e.,

$$\mathbf{g}_\alpha^* = \mathbf{a}_\alpha + \xi\mathbf{d}_{,\alpha} \;,\;\; \mathbf{g}_3^* = \mathbf{d} \tag{15.22}$$

we proceed to reduce (15.21) as follows:

$$\int_o^{\xi_2} (\mathbf{g}_i^* \times \mathbf{T}^{*i})d\xi = \int_o^{\xi_2} [(\mathbf{a}_\alpha + \xi\mathbf{d}_{,\alpha}) \times \mathbf{T}^{*\alpha} + \mathbf{d} \times \mathbf{T}^{*3}]d\xi$$

$$= \int_o^{\xi_2} (\mathbf{a}_\alpha + \xi\mathbf{d}_{,\alpha}) \times \mathbf{T}^{*\alpha}d\xi + \int_o^{\xi_2} \mathbf{d} \times \mathbf{T}^{*3}d\xi$$

$$= \mathbf{a}_\alpha \times \int_o^{\xi_2} \mathbf{T}^{*\alpha}d\xi + \mathbf{d}_{,\alpha} \times \int_o^{\xi_2} \mathbf{T}^{*\alpha}\,\xi d\xi + \mathbf{d} \times \int_o^{\xi_2} \mathbf{T}^{*3}d\xi$$

$$= \mathbf{a}_\alpha \times (\mathbf{N}^{\alpha}a^{1/2}) + \mathbf{d}_{,\alpha} \times (\mathbf{M}^{\alpha}a^{1/2}) + \mathbf{d} \times (\mathbf{m}a^{1/2}) = 0 \tag{15.23}$$

Since $a \neq 0$, we obtain

$$\mathbf{a}_\alpha \times \mathbf{N}^{\alpha} + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^{\alpha} = 0 \tag{15.24}$$

We continue to obtain the equation of balance of energy. We recall that the conservation of energy can be reduced to (9.30), repeated here for convenience

$$\rho^* g^{*1/2}\dot{\varepsilon}^* = \mathbf{T}^{*i} \cdot \mathbf{v}_{,i}^* \tag{15.25}$$

We integrate both sides of (15.25) with respect to ξ between $\xi = 0$ and $\xi = \xi_2$ to obtain

$$\int_0^{\xi_2} \rho^* g^{*1/2} \dot{\varepsilon}^* d\xi = \int_0^{\xi_2} \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i} d\xi = \int_0^{\xi_2} \mathbf{T}^{*i} \cdot (\mathbf{v} + \xi \mathbf{w})_{,i} \cdot d\xi$$

$$= \int_0^{\xi_2} \mathbf{T}^{*\alpha} \cdot \mathbf{v}_{,\alpha} d\xi + \int_0^{\xi_2} \mathbf{T}^{*\alpha} \cdot (\xi \mathbf{w})_{,\alpha} d\xi + \int_0^{\xi_2} \mathbf{T}^{*3} \cdot (\xi \mathbf{w})_{,3} \, d\xi$$

$$= \mathbf{v}_{,\alpha} \cdot \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi + \mathbf{w}_{,\alpha} \cdot \int_0^{\xi_2} \mathbf{T}^{*\alpha} \xi d\xi + \mathbf{w} \cdot \int_0^{\xi_2} \mathbf{T}^{*3} \, d\xi$$

$$= a^{1/2} \, \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + a^{1/2} \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + a^{1/2} \mathbf{m} \cdot \mathbf{w} \tag{15.26}$$

We now define a specific internal energy for the representative element (microstructure) by

$$\hat{\rho} a^{1/2} \hat{\varepsilon} = \int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{15.27}$$

From (15.26) and (15.27) we obtain the equation of balance of energy for the micro-structure

$$\hat{\rho}(\dot{\hat{\varepsilon}}) = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w} = \hat{P} \tag{15.28}$$

where

$$\hat{P} = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w} \tag{15.29}$$

is the mechanical power of the micro-structure (representative element). This completes the derivation of the field equations for the shell-like micro-structure. These field equations are in their local forms. The global form of these equations will be derived and discussed in the next section.

Before closing this section we proceed to discuss the continuity of stress throughout the micro-structure, i.e., at the interface of the matrix and reinforcement. To this end we recall the jump conditions (4.35), i.e.,

$$[\![\rho^*]\!] = 0$$

$$[\![\rho^* \mathbf{v}^* w_n^* - \mathbf{t}^*]\!] = 0 \qquad (15.30)$$

$$[\![\rho^* (\varepsilon^* + k^*) w_n^* - \mathbf{t}^* \cdot \mathbf{v}^*]\!] = 0$$

In our case since the surface of discontinuity is a material surface we have $\mathbf{v}^* = \mathbf{u}^*$ and we obtain

$$w_n^* = 0 \qquad (15.31)$$

Hence, equation $(15.30)_1$ is identically satisfied and equation $(15.30)_2$ reduces to

$$[\![\mathbf{t}]\!] = 0 \qquad (15.32)$$

This shows that at a material surface of two media the stress vector is continuous. Since this result holds for any material surface of two media, we can conclude that within the shell-like body $\mathcal{B}^*$ and at the surface $s_1 : \xi = \xi_1$, the stress vector is continuous.

To ensure the continuity of displacement across the interface we must require for the director to be continuous (not to have jump) across the interface. However, at this point, to keep the formulation general, we do not impose such a condition. Moreover, in some cases (such as delamination) it may be appropriate to admit jump for displacement.

16. Conservation laws for a shell-like representative element (micro-structure)

This section is concerned with the derivation of the global field equations (conservation laws) for our shell-like representative element (micro-structure). The derivation is accomplished by integrating the basic field equations, derived in section 15, over an appropriate region of two dimensional space covered by η^1, η^2 coordinates. To this end we consider an arbitrary part $\mathcal{P}$ of the materials surface $s_0 : \xi = 0$ (see 14.1) in the present configuration and let $\partial\hat{\mathcal{P}}$ be the boundary (curve) of $\hat{\mathcal{P}}$. The basic field equations (in local form) for the part $\mathcal{P}$ were derived in section 15. For convenience, we rewrite these equations in the forms appropriate tc our development in this section as follows:

$$
\begin{aligned}
&a : \overline{(\hat{\rho} a^{1/2})}^{\cdot} = 0 \\
&b : \hat{\rho} a^{1/2}(\dot{\mathrm{v}} + y^1 \dot{\mathrm{w}}) = (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \hat{\rho}\hat{\mathbf{f}} a^{1/2} \\
&c : \hat{\rho} a^{1/2}(y^1 \dot{\mathrm{v}} + y^2 \dot{\mathrm{w}}) = (\mathbf{M}^\alpha a^{1/2})_{,\alpha} - \mathbf{m} a^{1/2} + \hat{\rho}\hat{\mathbf{l}} a^{1/2} \qquad (16.1) \\
&d : \mathbf{a}^\alpha \times \mathbf{N}^\alpha + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha = 0 \\
&e : \hat{\rho}(\dot{\hat{\varepsilon}}) = \mathbf{N}^\alpha \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^\alpha \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w}
\end{aligned}
$$

At this point we need to consider the kinematics of a surface integral and deduce an integral formula which will be utilized in the rest of the section. Consider a sufficiently smooth scalar-valued or vector-valued function of position and time, ϕ, and define the integral

$$
I = \int_{\hat{\mathcal{P}}} \phi \, da \qquad (16.2)
$$

over $\hat{\mathcal{P}}$ in the present configuration. Since the above integral is a function of time, its derivative with respect to t may be calculated as follows:

$$\dot{I} = \frac{d}{dt}\int_{\hat{\mathcal{P}}} \phi \, da = \frac{d}{dt}\int_{\mathcal{P}} J\phi d\mathcal{A} = \int_{\mathcal{P}} (\overline{J\phi}) d\mathcal{A}$$

$$= \int_{\mathcal{P}} (\dot{\phi} J + \phi \dot{J}) d\mathcal{A} = \int_{\mathcal{P}} J(\dot{\phi} + J^{-1}\dot{J}\phi) d\mathcal{A}$$

$$= \int_{\hat{\mathcal{P}}} (\dot{\phi} + J^{-1}\dot{J}\phi) da \tag{16.3}$$

where $d\mathcal{A}$ is the element of area in the reference configuration and where we have made use of

$$da = J d\mathcal{A} \ , \qquad J = (\frac{a}{A})^{1/2} \tag{16.4}$$

and the region of integration of the last integral in (16.3) is again over $\hat{\mathcal{P}}$. We now recall

$$\dot{a} = \overline{\det(a_{\alpha\beta})} = \frac{\partial}{\partial a_{\lambda\gamma}} [\det(a_{\alpha\beta})] \dot{a}_{\lambda\gamma} = a a^{\alpha\beta} \dot{a}_{\alpha\beta} = 2a\eta^{\alpha}_{\alpha} \tag{16.5}$$

where $\eta^{\alpha}{}_{\beta} = a^{\alpha\gamma} v_{\gamma\beta}$ and η_{ij} is given by (12.24).

By (16.4) and (16.5) we obtain

$$\dot{J} = (\frac{1}{A})^{1/2}\overline{(a^{1/2})} = \frac{1}{2}(\frac{1}{A})^{1/2} a^{-1/2}\dot{a} = \frac{1}{2}(\frac{1}{A})^{1/2} a^{-1/2}(2a\eta^{\alpha}_{\alpha})$$

$$= (\frac{a}{A})^{1/2}\eta^{\alpha}_{\alpha} = J\eta^{\alpha}_{\alpha} \tag{16.6}$$

Hence, from (16.3) and (16.6) we have

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \phi \, d\hat{a} = \int_{\hat{\mathcal{P}}} (\dot{\phi} + \eta^{\alpha}_{\alpha}\phi) d\hat{a} \tag{16.7}$$

Now consider $(16.1)_a$ and write

$$\overline{(\hat{\rho}a^{1/2})}^{\cdot} = [\dot{\hat{\rho}}\, a^{1/2} + \hat{\rho}(\dot{a^{1/2}})] = \dot{\hat{\rho}}\, a^{1/2} + \frac{1}{2}\,\hat{\rho}\, a^{-1/2}\,\dot{a}$$

$$= \dot{\hat{\rho}}\, a^{1/2} + \hat{\rho}\, a^{1/2}\, \eta^{\alpha}_{\alpha} = a^{1/2}(\dot{\hat{\rho}} + \eta^{\alpha}_{\alpha}\hat{\rho}) = 0 \qquad (16.8)$$

If we now integrate both sides of $(16.1)_a$ with respect to η^1, η^2 and make use of (16.7) and (16.8), we obtain

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \overline{(\hat{\rho}a^{1/2})}^{\cdot}\, d\eta^1 d\eta^2 = \int_{\hat{\eta}_1}\int_{\hat{\eta}_2} a^{1/2}(\dot{\hat{\rho}} + \eta^{\alpha}_{\alpha}\hat{\rho})d\eta^1 d\eta^2 = \int_{\hat{\mathcal{P}}} (\dot{\hat{\rho}} + \eta^{\alpha}_{\alpha}\hat{\rho})d\hat{a} = 0 \qquad (16.9)$$

or

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} a^{1/2}\hat{\rho}\, d\hat{a} = 0 \qquad (16.10)$$

where $d\hat{a}$ is the element of area of the shell-like micro structure. This is the conservation of mass for an arbitrary part $\hat{\mathcal{P}}$ of our shell-like micro-structure.

Next we consider $(16.1)_b$ and integrate with respect to η^1 and η^{α} to obtain

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho}a^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})d\eta^1 d\eta^2 = \int_{\hat{\eta}_1}\int_{\hat{\eta}_2} (\mathbf{N}^{\alpha}a^{1/2})_{,\alpha} d\eta^1 d\eta^2 + \int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho}\hat{\mathbf{f}}a^{1/2} d\eta^1 d\eta^2 \qquad (16.11)$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ denote appropriate ranges of integration for η^1 and η^2. We now utilize Stokes' theorem and (16.8) to reduce (16.11) as follows:

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \{\hat{\rho}a^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) + a^{1/2}(\dot{\hat{\rho}} + \eta^{\alpha}_{\alpha}\hat{\rho})(\mathbf{v} + y^1\mathbf{w})\}d\eta^1 d\eta^2 =$$

$$\int_{\partial\hat{\mathcal{P}}} \mathbf{N}^{\alpha}\nu_{\alpha}\, ds + \int_{\hat{\mathcal{P}}} \hat{\rho}\hat{\mathbf{f}}\, d\hat{a}$$

or

$$\int_{\hat{\mathcal{P}}} \{\overline{[\hat{\rho}(\mathbf{v} + y^1\mathbf{w})]}^{\cdot} + \eta^{\alpha}_{\alpha}[\hat{\rho}(\mathbf{v} + y^1\mathbf{w})]\}d\hat{a} + \int_{\hat{\mathcal{P}}} \hat{\rho}\hat{\mathbf{f}}\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds$$

or

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}}\hat{\rho}(\mathbf{v}+y^1\mathbf{w})d\hat{a}=\int_{\hat{\mathcal{P}}}\hat{\rho}\hat{\mathbf{f}}d\hat{a}+\int_{\partial\hat{\mathcal{P}}}\mathbf{N}ds \tag{16.12}$$

where in obtaining (16.12) we considered the fact that

$$\mathbf{N}=\mathbf{N}^{\alpha}v_{\alpha} \tag{16.13}$$

and $d\hat{a}$ is the element of area of the shell-like micro-structure. Equation (16.12) is the conservation of linear momentum for an arbitrary part of the shell-like micro-structure.

Following the same procedure we consider $(16.1)_c$ and integrate with respect to η^1, η^2

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2}\hat{\rho}a^{1/2}(y^1\dot{\mathbf{v}}+y^2\dot{\mathbf{w}})d\eta^1d\eta^2=$$

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2}(\mathbf{M}^{\alpha}a^{1/2})_{,\alpha}d\eta^1d\eta^2-\int_{\hat{\eta}_1}\int_{\hat{\eta}_2}\mathbf{m}a^{1/2}d\eta^1d\eta^2+\int_{\hat{\eta}_1}\int_{\hat{\eta}_2}\hat{\rho}\hat{\mathbf{l}}a^{1/2}d\eta^1d\eta^2$$

Again we make use of (16.7), (16.8) and Stokes' theorem and write

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2}\{\hat{\rho}a^{1/2}(y^1\dot{\mathbf{v}}+y^2\dot{\mathbf{w}})+a^{1/2}(\dot{\hat{\rho}}+\eta^{\alpha}_{\alpha}\hat{\rho})(y^1\mathbf{v}+y^2\mathbf{w}\}d\eta^1d\eta^2=$$

$$\int_{\partial\hat{\mathcal{P}}}\mathbf{M}^{\alpha}v_{\alpha}d\hat{a}-\int_{\hat{\mathcal{P}}}\mathbf{m}d\hat{a}+\int_{\hat{\mathcal{P}}}\hat{\rho}\,\hat{\mathbf{l}}\,d\hat{a}$$

or

$$\int_{\hat{\mathcal{P}}}\{[\overline{\hat{\rho}(y^1\mathbf{v}+y^2\mathbf{w}}]+\eta^{\alpha}_{\alpha}[\hat{\rho}(y^1\mathbf{v}+y^2\mathbf{w})]\}d\hat{a}=\int_{\hat{\mathcal{P}}}(\hat{\rho}\hat{\mathbf{l}}-\mathbf{m})d\hat{a}+\int_{\partial\hat{\mathcal{P}}}\mathbf{M}ds \tag{16.14}$$

where we have made use of the expression

$$\mathbf{M}=\mathbf{M}^{\alpha}v_{\alpha} \tag{16.15}$$

Equation (16.14) is the conservation of director momentum for an arbitrary part of the shell-like micro-structure.

We now consider $(16.1)_b$, $(16.1)_c$, $(16.1)_d$ and write

$$\hat{\rho} a^{1/2}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})] = \mathbf{r} \times (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \hat{\rho}\mathbf{r} \times \hat{\mathbf{f}} a^{1/2} \tag{16.16}$$

$$\hat{\rho} a^{1/2}\,[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] = \mathbf{d} \times (\mathbf{M}^\alpha a^{1/2})_{,\alpha} - \mathbf{d} \times \mathbf{m} a^{1/2} + \hat{\rho}\mathbf{d} \times \hat{\mathbf{l}} a^{1/2} \tag{16.17}$$

$$0 = a^{1/2}(\mathbf{a}_\alpha \times \mathbf{N}^\alpha) + a^{1/2}(\mathbf{d} \times \mathbf{m}) + a^{1/2}(\mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha) \tag{16.18}$$

Adding (16.16), (16.17) and (16.18), we obtain

$$\hat{\rho}\, a^{1/2}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})] + \hat{\rho}\, a^{1/2}[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] =$$

$$\mathbf{r} \times (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \mathbf{a}_\alpha \times (\mathbf{N}^\alpha a^{1/2}) + \hat{\rho}\, a^{1/2}\mathbf{r} \times \hat{\mathbf{f}} +$$

$$\mathbf{d} \times (\mathbf{M}^\alpha a^{1/2})_{,\alpha} + \mathbf{d}_{,\alpha} \times (\mathbf{M}^\alpha a^{1/2}) + \hat{\rho}\, a^{1/2}\, \mathbf{d} \times \hat{\mathbf{l}}$$

or

$$\hat{\rho}\, a^{1/2}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})] + \hat{\rho}\, a^{1/2}[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] =$$

$$(a^{1/2}\mathbf{r} \times \mathbf{N}^\alpha)_{,\alpha} + \hat{\rho}\, a^{1/2}\mathbf{r} \times \mathbf{f} + (a^{1/2}\mathbf{d} \times \mathbf{M}^\alpha)_{,\alpha} + \hat{\rho}\, a^{1/2}\mathbf{d} \times \hat{\mathbf{l}} \tag{16.19}$$

Integrating (16.19) with respect to η^1, η^2 and making use of (16.7), (16.8), (16.13), (16.15) and Stokes' theorem, we obtain

$$\int_{\dot{\eta}_1}\int_{\dot{\eta}_2} a^{1/2}\{\hat{\rho}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})] + (\dot{\hat{\rho}} + \eta^\alpha_\alpha\hat{\rho})[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})]\}d\eta^1 d\eta^2 +$$

$$\int_{\dot{\eta}_1}\int_{\dot{\eta}_2} a^{1/2}\{\hat{\rho}[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] + (\dot{\hat{\rho}} + \eta^\alpha_\alpha\hat{\rho})[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})]\}d\eta^1 d\eta^2 =$$

$$\int_{\dot{\eta}_1}\int_{\dot{\eta}_2} (a^{1/2}\mathbf{r} \times \mathbf{N}^\alpha)_{,\alpha} d\eta^1 d\eta^2 + \int_{\dot{\eta}_1}\int_{\dot{\eta}_2} (a^{1/2}\mathbf{d} \times \mathbf{M}^\alpha)_{,\alpha} d\eta^1 d\eta^2 +$$

$$\int_{\dot{\eta}_1}\int_{\dot{\eta}_2} \hat{\rho} a^{1/2}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})d\eta^1$$

or

$$\int_{\hat{\mathcal{P}}} \{\overline{\hat{\rho} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})]} + \eta^{\alpha}_{\alpha}\hat{\rho}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})]\}d\hat{a} +$$

$$\int_{\hat{\mathcal{P}}} \{\overline{\hat{\rho}[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})]} + \eta^{\alpha}_{\alpha}\hat{\rho}[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})]\}d\hat{a} =$$

$$\int_{\partial\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N}^{\alpha})\nu_{\alpha}d\eta^1 d\eta^2 + \int_{\partial\hat{\mathcal{P}}} (\mathbf{d} \times \mathbf{M}^{\alpha})\nu_{\alpha}d\eta^1 d\eta^2 +$$

$$\int_{\hat{\mathcal{P}}} \rho(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})d\hat{a}$$

or

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\{\mathbf{r} \times (\mathbf{v} \times y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}d\hat{a} =$$

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M})d\hat{a} \qquad (16.20)$$

This is the conservation of moment of momentum of the shell-like micro-structure.

Finally we consider $(16.1)_b$, $(16.1)_c$ and form their scalar products with $\mathbf{v}$ and $\mathbf{w}$ respectively and add the resulting requations to the product of $(16.1)_e$ with $a^{1/2}$ to obtain

$$\hat{\rho}\, a^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \cdot \mathbf{v} + \hat{\rho} a^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \cdot \mathbf{w} + \hat{\rho} a^{1/2}(\dot{\varepsilon}) =$$

$$(\mathbf{N}^{\alpha} a^{1/2})_{,\alpha} \cdot \mathbf{v} + \hat{\rho}\, \hat{\mathbf{f}} \cdot \mathbf{v}\, a^{1/2} + (\mathbf{M}^{\alpha} a^{1/2})_{,\alpha} \cdot \mathbf{w}$$

$$- a^{1/2}\, \mathbf{m} \cdot \mathbf{w} + \hat{\rho}\, a^{1/2}\, \hat{\mathbf{l}} \cdot \mathbf{w}$$

$$+ a^{1/2}\, \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + a^{1/2}\, \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + a^{1/2}\, \mathbf{m} \cdot \mathbf{w} \qquad (16.21)$$

Rewriting (16.21), we obtain

$$\hat{\rho}\, a^{1/2}\, (\dot{\varepsilon}) + \hat{\rho}\, a^{1/2}(\mathbf{v} \cdot \dot{\mathbf{v}} + y^1\mathbf{v} \cdot \dot{\mathbf{w}} + y^1\dot{\mathbf{v}} \cdot \mathbf{w} + y^2\mathbf{w} \cdot \dot{\mathbf{w}}) =$$

$$\hat{\rho}\, a^{1/2}(\hat{\mathbf{f}} \cdot \mathbf{v} + \hat{\mathbf{l}} \cdot \mathbf{w}) +$$

$$(a^{1/2}\mathbf{N}^{\alpha} \cdot \mathbf{v})_{,\alpha} + (a^{1/2}\mathbf{M}^{\alpha} \cdot \mathbf{w})_{,\alpha} \qquad (16.22)$$

Integrating (16.22) with respect to η^1, η^2, making use of Stokes' theorem, and making use of

(16.7), (16.8), we obtain

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho} a^{1/2}(\dot{\hat{\varepsilon}}) d\eta^1 d\eta^2 + \int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho} a^{1/2}(\mathbf{v}\cdot\dot{\mathbf{v}} + y^1\mathbf{v}\cdot\dot{\mathbf{w}} + y^1\dot{\mathbf{v}}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\dot{\mathbf{w}}) d\eta^1 d\eta^2 =$$

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho} a^{1/2}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) d\eta^1 d\eta^2 +$$

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} (a^{1/2}\mathbf{N}^\alpha\cdot\mathbf{v})_{,\alpha} d\eta^1 d\eta^2 +$$

$$\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} (a^{1/2}\mathbf{M}^\alpha\cdot\mathbf{v})_{,\alpha} d\eta^1 d\eta^2$$

or

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\dot{\hat{\varepsilon}}) d\hat{a} + \int_{\hat{\mathcal{P}}} \frac{1}{2}\hat{\rho}(\overline{\mathbf{v}\cdot\mathbf{v} + 2y^1\mathbf{v}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\mathbf{w}})^{\cdot} d\hat{a} =$$

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) d\hat{a} +$$

$$\int_{\partial\hat{\mathcal{P}}} (\mathbf{N}^\alpha\cdot\mathbf{v} + \mathbf{M}^\alpha\cdot\mathbf{w}) ds$$

or

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\overline{\varepsilon + \mathcal{K}})^{\cdot} d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{f}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{N}^\alpha\cdot\mathbf{v} + \mathbf{M}^\alpha\cdot\mathbf{w}) ds$$

or

$$\int_{\hat{\mathcal{P}}} \{\hat{\rho}(\overline{\hat{\varepsilon} + \mathcal{K}})^{\cdot} + (\overset{\star}{\hat{\rho}} + \eta^\alpha_{\alpha}\hat{\rho})(\hat{\varepsilon} + \hat{\mathcal{K}})\} d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{N}^\alpha\cdot\mathbf{v} + \mathbf{M}^\alpha\cdot\mathbf{w}) ds$$

or

$$\int_{\hat{\mathcal{P}}} \{[\overline{\hat{\rho}(\hat{\varepsilon} + \mathcal{K})}]^{\cdot} + \eta^\alpha_{\alpha}[\hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}})]\} d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{N}^\alpha\cdot\mathbf{v} + \mathbf{M}^\alpha\cdot\mathbf{w}) ds$$

or

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}}) d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) d\hat{a} + \int_{\hat{\mathcal{P}}} (\mathbf{N}^\alpha\cdot\mathbf{v} + \mathbf{M}^\alpha\cdot\mathbf{w}) ds \qquad (16.23)$$

where in obtaining (16.23) we have used the fact that

$$\hat{\mathcal{K}} = \frac{1}{2}\left(\mathbf{v}\cdot\mathbf{v} + 2y^1\mathbf{v}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\mathbf{w}\right) \tag{16.24}$$

Equation (16.23) is the conservation of energy for the shell-like micro-structure.

17. Conservation laws for composite laminates

In this section we derive various conservation laws of a composite laminate (i.e. global forms of equations of motion) from the corresponding conservation laws of a shell-like micro-structure derived in section (16). We recall that the composite laminate is assumed to consist of infinitely many Cosserat surfaces. This assumption is justified by physical considerations since the thickness of each ply is small in comparison with the thickness of the laminate itself.

For convenience, we record below the conservation laws for an arbitrary part $\hat{\mathcal{P}}$ bounded by $\partial\mathcal{P}$ of the micro-structure

$$\begin{aligned}
&a: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} = 0 \\
&b: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{v} + y^1\mathbf{w})\, d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}\, \hat{\mathbf{f}}\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds \\
&c: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w})\, d\hat{a} = \int_{\hat{\mathcal{P}}} (\hat{\rho}\hat{\mathbf{l}} - \mathbf{m})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{M}\, ds \\
&d: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}\, d\hat{a} = \\
&\qquad \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})\, d\hat{a} + \int_{\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M})\, ds \\
&e: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}})\, d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{f} \cdot \mathbf{v} + \hat{\mathbf{l}} \cdot \mathbf{w})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{N}^{\alpha} \cdot \mathbf{v} + \mathbf{M}^{\alpha} \cdot \mathbf{w})\, ds
\end{aligned} \tag{17.1}$$

The first of (17.1) is a mathematical statement of the conservation of mass, the second that of the linear momentum, the third is the conservation of the director momentum, the fourth that of the moment of momentum, and the fifth is the conservation of energy. The various quantities appearing in (17.1) have been defined in the previous sections and in what follows we will make reference to these definitions when the need arises.

We observe that the basic structures of $(17.1)_{a,b,e}$ and their forms are analogous to the corresponding conservation laws of the classical 3-dimensional continuum mechanics. Equation $(17.1)_c$ does not exist in the classical continuum mechanics whereas equation $(17.1)_d$ although exist it has a simpler form. It should be noted that the conservation laws (17.1) are consistent with the invariance requirements under superposed rigid body motions, which have wide acceptance in continuum mechanics.

As described in section (10), we consider two sets of convected coordinate systems one of which is used to describe the behavior of the micro-structure and is designated by $\eta^i = \{\eta^1, \eta^2, \xi\}$. The second coordinate system is used to describe the behavior of the composite laminate (i.e. a continuum with micro-structure) and is designated by $\theta^i = \{\theta^1, \theta^2, \theta^3\}$. In general, the two sets are related by (10.1) subject to condition (10.2). As before, we also adopt (10.3), i.e.,

$$\theta^\alpha = \eta^\alpha \ ,$$
$$\theta^3 = \frac{1}{\varepsilon}\xi \ , \ \varepsilon \ll 1 \ . \tag{17.2}$$

Consider an arbitrary part $\mathcal{P}$ of the composite laminate in the present configuration and let it be bounded by a closed surface $\partial\mathcal{P}$. In view of the choice of the convected curvilinear coordinates θ^i we note that coordinate θ^3 is, roughly speaking, in the direction of the lamination stack up.

Considering the conservation of mass $(17.1)_1$, we write

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} = 0 \quad \Rightarrow \frac{d}{dt}\int_{\hat{\eta}_1}\int_{\hat{\eta}_2} \hat{\rho}\, a^{1/2} d\eta^1 d\eta^2 = 0$$

or in view of $(17.2)_1$:

$$\frac{d}{dt}\int_{\bar{\theta}_1}\int_{\bar{\theta}_2} \hat{\rho}\, a^{1/2} d\theta^1 d\theta^2 = 0 \tag{17.3}$$

where $\bar{\theta}_1, \bar{\theta}_2$ are appropriate ranges of integration within the region $\mathcal{P}$ of the composite laminate. We now integrate both sides of (17.3) with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3} \{ \frac{d}{dt} \int_{\bar{\theta}_1} \int_{\bar{\theta}_2} \hat{\rho}\, a^{1/2}\, d\theta^1 d\theta^2 \} d\theta^3 = \text{const.} \tag{17.4}$$

where $\bar{\theta}_3$ is the appropriate range of integration within the region $\mathcal{P}$. Since coordinates θ^i are convected, and since the quantity $\hat{\rho} a^{1/2}$ is independent of time, we obtain

$$\frac{d}{dt} \int_{\bar{\theta}_3} \int_{\bar{\theta}_2} \int_{\bar{\theta}_1} \hat{\rho}\, a^{1/2}\, d\theta^1 d\theta^2 d\theta^3 = 0 \tag{17.5}$$

The element of volume in terms of coordinates θ^i is

$$dv = g^{1/2}\, d\theta^1 d\theta^2 d\theta^3 \tag{17.6}$$

where g is the determinant of the metric of the space covered by the coordinates $\theta^1, \theta^2, \theta^3$. We now define *composite assigned mass density*, ρ, such that

$$\rho g^{1/2} = \hat{\rho} a^{1/2} = \int_0^{\xi_2} \rho^* g^{*1/2} d\xi = \int_0^{\xi_1} \rho_1^* g^{*1/2} d\xi + \int_{\xi_1}^{\xi_2} \rho_2^* g^{*1/2}$$

$$\rho = \rho(\theta^i, t) \tag{17.7}$$

where $a = \det(a_{\alpha\beta})$ and $g^* = \det(g^*_{ij})$. Substituting (17.7) into (17.5) and making use of (17.6), we obtain

$$\frac{d}{dt} \int_{\bar{\theta}_3} \int_{\bar{\theta}_2} \int_{\bar{\theta}_1} \rho g^{1/2} d\theta^1 d\theta^2 d\theta^3 = 0$$

or

$$\frac{d}{dt} \int_{\mathcal{P}} g^{1/2} \rho\, dv = 0 \tag{17.8}$$

This is the conservation of mass of the composite laminate. From (17.7) it is clear that since $\hat{\rho} a^{1/2}$ is independent of time, it follows that $\rho g^{1/2}$ is also independent of time, although both ρ

and $g^{1/2}$ may depend on t.

Remark

In the rest of this section we will frequently need to perform differentiation with respect to both coordinate systems η^i and θ^i in the same expression. For the sake of clarity in such occasions we will use lower case letters to designate differentiation with respect to η^i coordiantes while for the differentiation with respect to θ^i coordinates we will make use of capital letters. For example, $\mathbf{T}^{*i}{}_{,i}$ is equal to

$$\mathbf{T}^{*i}{}_{,i} = \frac{\partial \mathbf{T}^{*1}}{\partial \eta^1} + \frac{\partial \mathbf{T}^{*2}}{\partial \eta^2} + \frac{\partial \mathbf{T}^{*3}}{\partial \eta^3} = \frac{\partial \mathbf{T}^{*1}}{\partial \eta^1} + \frac{\partial \mathbf{T}^{*2}}{\partial \eta^2} + \frac{\partial \mathbf{T}^{*3}}{\partial \xi}$$

while $\mathbf{T}^A{}_{,A}$ is equivalent to

$$\mathbf{T}^A{}_{,A} = \frac{\partial \mathbf{T}^1}{\partial \theta^1} + \frac{\partial \mathbf{T}^2}{\partial \theta^2} + \frac{\partial \mathbf{T}^3}{\partial \theta^3}$$

This deviation from our usual notation is temporary and will be adopted when helps to clarify the derivation.

Next, we consider the conservation of the linear momentum of the micro-structure, i.e.,

$$\frac{d}{dt} \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{v} + y^1 \mathbf{w}) d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}\, \hat{\mathbf{f}}\, d\hat{a} + \int_{\partial \hat{\mathcal{P}}} \mathbf{N}\, ds \tag{17.9}$$

where

$$\hat{\rho}\, a^{1/2}\, \hat{\mathbf{f}} = \int_0^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} \tag{17.10}$$

and integrate with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3} \{ \frac{d}{dt} \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{v} + y^1 \mathbf{w}) d\hat{a} \} d\theta^3 = \int_{\bar{\theta}_3} \{ \int_{\hat{\mathcal{P}}} \hat{\rho}\, \hat{\mathbf{f}}\, d\hat{a} \} d\theta^3 + \int_{\bar{\theta}_3} \{ \int_{\partial \hat{\mathcal{P}}} \mathbf{N} ds \} d\theta^3 + \text{const}$$

We require that in the absence of the body and contact forces the total linear momentum of the composite laminate must remain constant at all times. In view of this and by making use of (17.10) and the fact that θ^3 is a convected coordinate we rewrite the above as

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}\hat{\rho}a^{1/2}(\mathbf{v}+y^1\mathbf{w})d\eta^1 d\eta^2 d\theta^3=\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}\hat{\rho}a^{1/2}\mathbf{f}\,d\eta^1 d\eta^2 d\eta^3+\int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}}\mathbf{N}^\alpha\nu_\alpha ds\,d\theta$$

or

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\rho g^{1/2}(\mathbf{v}+y^1\mathbf{w})d\theta^1 d\theta^2 d\theta^3=\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi+[\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2}\}d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}}\mathbf{N}^\alpha{}_{|\alpha}d\hat{a}d\theta^3$$

or

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi\}dv+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*3}{}_{,3}d\xi\}dv$$

$$+\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}(a^{1/2}\mathbf{N}^\alpha)_{,\alpha}d\eta^1 d\eta^2 d\theta^3$$

or

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi\}dv+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*3}{}_{,3}d\xi\}dv$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(\int_0^{\xi_2}\mathbf{T}^{*\alpha}{}_{,\alpha}d\xi)d\theta^1 d\theta^2 d\theta^3$$

or

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi\}dv+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*i}{}_{,i}d\xi\}dv \tag{17.11}$$

where in the above derivation, i.e., between (17.10) and (17.11) a comma denotes differentiation with respect to $\eta^i=\{\eta^\alpha,\xi\}$. Also, in obtaining (17.11) we have made use of (17.7), (17.10) and the Stokes' theorem. We now define the *composite assigned body for a density*, **b** and the *composite assigned stress vector* $\mathbf{T}^i$ such that

$$\rho g^{1/2}\mathbf{b} = \int_0^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi \tag{17.12}$$

and[1]

$$\mathbf{T}^A{}_{,A} = \mathbf{T}^i{}_{,i} = \int_0^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi \tag{17.13}$$

Substituting (17.12) and (17.13) in (17.11) we obtain

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho\mathbf{b}dv + \int_{\mathcal{P}} g^{-1/2}\mathbf{T}^A{}_{,A}dv \tag{17.14}$$

where a comma now denotes differentiation with respect to $\theta^i = \{\theta^1, \theta^2, \theta^3\}$. Making use of the divergence theorem from (17.14) we obtain

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho\mathbf{b}dv + \int_{\partial\mathcal{P}} g^{-1/2}\mathbf{T}^A n_A da \tag{17.15}$$

where $\mathbf{n} = n^i\mathbf{g}_i = n_i \cdot \mathbf{g}^i$ is the outward unit normal to the boundary surface $\partial\mathcal{P}$. Defininining[2]

$$\mathbf{t} = g^{-1/2}\mathbf{T}^A n_A \tag{17.16}$$

as the *composite assigned traction*, we obtain

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho\mathbf{b}dv + \int_{\partial\mathcal{P}} \mathbf{t}da \tag{17.17}$$

This is the conservation of the linear momentum for the composite laminate.

We now consider the conservation of the director momentum of the micro-structure, i.e.,

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} (y^1\mathbf{v} + y^2\mathbf{w})d\hat{a} = \int_{\hat{\mathcal{P}}} (\hat{\rho}\hat{\mathbf{l}} - \mathbf{m})d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{M}ds \tag{17.18}$$

[1,2] See remark after (17.8).

where

$$\hat{\rho}\,\hat{a}^{1/2}\,\hat{\mathbf{l}} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{1/2} \xi d\xi + [\mathbf{T}^{*3}\xi]_{\xi=0}^{\xi=\xi_2} \tag{17.19}$$

and integrate with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3} \{\frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w})d\hat{a}\}d\theta^3 = \int_{\bar{\theta}_3}\{\int_{\hat{\mathcal{P}}}\hat{\rho}\hat{\mathbf{l}}d\hat{a}\}d\theta^3 - \int_{\bar{\theta}_3}\{\int_{\hat{\mathcal{P}}}\mathbf{m}d\hat{a}\}d\theta^3$$

$$+ \int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}}\mathbf{M}ds\, d\theta^3 + \text{const.}$$

We require that in the absence of body and contact forces the total director momentum of the composite laminate must remain constant at all time. Hence, making use of (17.19) and the fact that θ^3 is a convected coordinate we reduce the above as follows:

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1} \hat{\rho}a^{1/2}(y^1\mathbf{v} + y^2\mathbf{w})d\eta^1 d\eta^2 d\theta^3 = \int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1} \rho a^{1/2}\hat{\mathbf{l}}d\eta^1 d\eta^2 d\theta^3$$

$$- \int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1} a^{1/2}\mathbf{m}d\eta^1 d\eta^2 d\theta^3$$

$$+ \int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}}\mathbf{M}^{\alpha}\nu_{\alpha}ds\, d\theta^3$$

or

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} \rho g^{1/2}(y^1\mathbf{v} + y^2\mathbf{w})d\theta^1 d\theta^2 d\theta^3 = \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\{\int_{o}^{\xi_2}\rho^*\mathbf{b}^* g^{*1/2}\xi d\xi + [\mathbf{T}^{*3}\xi]_{\xi=0}^{\xi=\xi_2}\}d\theta^1 d\theta^2 d\theta^3$$

$$- \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\{\int_{o}^{\xi_2}\mathbf{T}^{*3}d\xi\}d\theta^1 d\theta^2 d\theta^3$$

$$+ \int_{\bar{\theta}_3}\int_{\hat{\mathcal{P}}}\mathbf{M}^{\alpha}{}_{|\alpha}d\hat{a}\, d\theta^3$$

or

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi\}dv-\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*3}d\xi\}dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}(\mathbf{T}^{*3}\xi)_{,3}d\xi\}dv$$

$$+\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}(a^{1/2}\mathbf{M}^{\alpha})_{,\alpha}d\eta^1 d\eta^2 d\theta^3$$

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi\}dv-\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*3}d\xi\}dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}(\mathbf{T}^{*3}\xi)_{,3}d\xi\}dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}(\mathbf{T}^{*\alpha}\xi)_{,\alpha}d\xi\}dv$$

or

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi\}dv-\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*3}d\xi\}dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}(\mathbf{T}^{*i}\xi)_{,i}d\xi\}dv \qquad (17.20)$$

where a comma refers to differentiation with respect to $\eta^i=\{\eta^\alpha,\xi\}$. We now define the *composite assigned body couple*, $\mathbf{c}$, the *composite intrinsic director force*, $\mathbf{k}$, and the *composite assigned couple stress vector*, $\mathbf{S}^i$, respectively, by[3]

$$\rho g^{1/2}\mathbf{c}=\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi \qquad (17.21)$$

$$g^{1/2}\mathbf{k}=a^{1/2}\mathbf{m}=\int_0^{\xi_2}\mathbf{T}^{*3}d\xi \qquad (17.22)$$

$$\mathbf{S}^A{}_{,A}=\int_0^{\xi_2}(\mathbf{T}^{*i}\xi)_{,i}d\xi \qquad (17.23)$$

[3] See remark after (17.8).

Substituting (17.21) to (17.23) into (17.20), we obtain

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\mathcal{P}} g^{-1/2}\mathbf{S}^A{}_{,A}dv \tag{17.24}$$

Making use of the divergence theorem, we can reduce (17.24) as follows

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} g^{-1/2}\mathbf{S}^A n_A da \tag{17.25}$$

Defining

$$\mathbf{s} = g^{-1/2}\mathbf{S}^A n_A \tag{17.26}$$

as the *composite couple traction*, we obtain

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}da \tag{17.27}$$

This is the conservation of the director momentum for the composite laminate.

Next we consider the conservation of moment of momentum for the micro-structure, i.e.,

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \rho\{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}d\hat{a} =$$

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M})ds \tag{17.28}$$

and integrate with respect to θ^3

$$\int_{\bar{\theta}_3} \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}d\hat{a}\, d\theta^3 =$$

$$\int_{\bar{\theta}_3}\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})d\hat{a}d\theta^3 +$$

$$\int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M})ds\, d\theta^3 + \text{const.}$$

We require that in the absence of body and contact forces the angular momentum of the composite laminate must remain constant at all times. In view of this and since θ^3 is a convected coordinate, we may write

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}\hat{\rho}a^{1/2}\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}d\eta^1 d\eta^2 d\theta^3 =$$

$$\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}\{\mathbf{r}\times(\hat{\rho}a^{1/2}\hat{\mathbf{f}})+\mathbf{d}\times(\hat{\rho}a^{1/2}\hat{\mathbf{l}})\}d\eta^1 d\eta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}(\mathbf{r}\times\mathbf{N}^{\alpha})\nu_{\alpha}ds\, d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\hat{\eta}_2}\int_{\hat{\eta}_1}(\mathbf{d}\times\mathbf{M}^{\alpha})\nu_{\alpha}ds\, d\theta^3$$

or

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\rho g^{1/2}\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}d\theta^1 d\theta^2 d\theta^3 =$$

$$\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\{\mathbf{r}\times(\int_o^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi+[\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2})\}d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\{\mathbf{d}\times(\int_o^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi+[\mathbf{T}^{*3}\xi]_{\xi=0}^{\xi=\xi_2})\}d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(a^{1/2}\mathbf{r}\times\mathbf{N}^{\alpha})_{,\alpha}d\theta^1 d\theta^2 d\theta^3+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(a^{1/2}\mathbf{d}\times\mathbf{M}^{\alpha})_{,\alpha}d\theta^1 d\theta^2 d\theta^3$$

or

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(\mathbf{r}\times\int_o^{\xi_2}\mathbf{T}^{*3}{}_{,3}d\xi)d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(\mathbf{r}\times\int_o^{\xi_2}\mathbf{T}^{*\alpha}d\xi)_{,\alpha}d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(\mathbf{d}\times\int_o^{\xi_2}\mathbf{T}^{*3}\xi d\xi)_{,3}d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}(\mathbf{d}\times\int_o^{\xi_2}\mathbf{T}^{*\alpha}\xi d\xi)_{,\alpha}d\theta^1 d\theta^2 d\theta^3$$

$$=\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$+\int_{\mathcal{P}}g^{-1/2}(\mathbf{r}\times\int_o^{\xi_2}\mathbf{T}^{*i}d\xi)_{,i}dv$$

$$+\int_{\mathcal{P}}g^{-1/2}(\mathbf{d}\times\int_o^{\xi_2}\mathbf{T}^{*i}\xi d\xi)_{,i}dv \qquad (17.29)$$

where a comma denotes differentiation with respect to $\eta^i = \{\eta^\alpha,\xi\}$. Making use of (17.13) and (17.23) we can rewrite the above as:

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$\int_{\mathcal{P}}g^{-1/2}(\mathbf{r}\times\mathbf{T}^A)_{,A}dv$$

$$\int_{\mathcal{P}}g^{-1/2}(\mathbf{d}\times\mathbf{S}^A)_{,A}dv \qquad (17.30)$$

where a comma now refers to differentiation with respect to $\theta^i = \{\theta^1,\theta^2,\theta^3\}$ coordinates. Taking advantage of the divergence theorem, we proceed to reduce (17.30) as follows:

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$+\int_{\mathcal{P}}(g^{-1/2}\mathbf{r}\times\mathbf{T}^A)_{|A}dv$$

$$+\int_{\mathcal{P}}(g^{-1/2}\mathbf{d}\times\mathbf{S}^A)_{|A}dv$$

$$=\int\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$+\int_{\partial\mathcal{P}}g^{-1/2}\mathbf{r}\times\mathbf{T}^A n_A da$$

$$+\int_{\partial\mathcal{P}}g^{-1/2}\mathbf{d}\times\mathbf{S}^A n_A da$$

Then by (17.16) and (17.26) and the above we obtain

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=$$

$$\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv+\int_{\partial\mathcal{P}}(\mathbf{r}\times\mathbf{t}+\mathbf{d}\times\mathbf{s})da \tag{17.31}$$

This is the conservation of the moment of momentum for the composite laminate.

Finally, we consider the conservation of energy for the micro-structure, i.e.,

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}}\hat{\rho}(\hat{\varepsilon}+\hat{\mathcal{K}})d\hat{a}=\int_{\hat{\mathcal{P}}}\hat{\rho}(\hat{\mathbf{f}}\cdot\mathbf{v}+\hat{\mathbf{l}}\cdot\mathbf{w})d\hat{a}+\int_{\partial\hat{\mathcal{P}}}(\mathbf{N}\cdot\mathbf{v}+\mathbf{M}\cdot\mathbf{w})ds \tag{17.32}$$

We recall that in the context of purely mechanical theory $\hat{\varepsilon}=\hat{\varepsilon}(\eta^\alpha,t)$ is the specific internal energy while $\hat{\mathcal{K}}$ represents the kinetic energy of the Cosserat surface (micro-structure) in the present configuration and is given by

$$\hat{\mathcal{K}}=\frac{1}{2}(\mathbf{v}\cdot\mathbf{v}+2y^1\mathbf{v}\cdot\mathbf{w}+y^2\mathbf{w}\cdot\mathbf{w}) \tag{17.33}$$

We also define the momentum corresponding to the velocity v and the director momentum corresponding to $\mathbf{w}$ by

$$\hat{\rho}\,\frac{\partial\hat{\mathcal{K}}}{\partial \mathbf{v}} = \hat{\rho}(\mathbf{v} + y^1\mathbf{w})$$

$$\hat{\rho}\,\frac{\partial\hat{\mathcal{K}}}{\partial \mathbf{w}} = \hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w}) \tag{17.34}$$

We now integrate both sides of (17.32) with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3} \frac{d}{dt} \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}})d\hat{a}d\theta^3 = \int_{\bar{\theta}_3} \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\mathbf{f}} \cdot \mathbf{v} + \hat{\mathbf{l}} \cdot \mathbf{w})d\hat{a}d\theta^3 +$$

$$\int_{\bar{\theta}_3} \int_{\partial\hat{\mathcal{P}}} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w})d\hat{s}d\theta^3 + \text{const.} \tag{17.35}$$

We now require that within the context of purely mechanical theory and in the absence of body and contact forces the total energy of the composite laminate must remain constant at all times. In view of this and since θ^3 is a convected coordinate, we may write

$$\text{Left hand side of (17.35)} = \frac{d}{dt} \int_{\bar{\theta}_3} \int_{\hat{\eta}_2} \int_{\hat{\eta}_1} \hat{\rho} a^{1/2}(\hat{\varepsilon} + \hat{\mathcal{K}})d\eta^1 d\eta^2 d\theta^3$$

$$= \frac{d}{dt} \int_{\bar{\theta}_3} \int_{\bar{\theta}_2} \int_{\bar{\theta}_1} (\hat{\rho} a^{1/2}\hat{\varepsilon} + \hat{\rho} a^{1/2}\hat{\mathcal{K}})d\theta^1 d\theta^2 d\theta^3$$

or

$$\text{Left hand side of (17.35)} = \frac{d}{dt} \int_{\bar{\theta}_3} \int_{\bar{\theta}_2} \int_{\bar{\theta}_1} \left(\int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi + \rho g^{1/2} \hat{\mathcal{K}}\right) d\theta^1 d\theta^2 d\theta^3 \tag{17.36}$$

where in obtaining (17.36) we have made use of (15.27) and (17. 7). We now define the *composite assigned strain energy*, and the *composite assigned kinetic* energy, $\mathcal{K}$ both per unit mass of the composite such that

$$\rho g^{1/2} \varepsilon = \int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{17.37}$$

$$\mathcal{K} = \hat{\mathcal{K}} = \frac{1}{2}\,(\mathbf{v} \cdot \mathbf{v} + 2y^1 \mathbf{v} \cdot \mathbf{w} + y^2 \mathbf{w} \cdot \mathbf{w}) \tag{17.38}$$

We also record the momentum corresponding to the director velocity $\mathbf{w}$

$$\rho\,\frac{\partial \mathcal{K}}{\partial \mathbf{v}} = \rho(\mathbf{v} + y^1\mathbf{w})$$

$$\rho\,\frac{\partial \mathcal{K}}{\partial \mathbf{w}} = \rho(y^1\mathbf{v} + y^2\mathbf{w}) \tag{17.39}$$

Substituting (17.37) and (17.38) into (17.36), we obtain

$$\text{Left hand side of (17.35)} = \frac{d}{dt}\int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K})d\upsilon \tag{17.40}$$

Considering the right-hand side of (17.35), we write

$$\text{Right hand side of (17.35)} = \int_{\bar\theta_3}\int_{\hat\eta_2}\int_{\hat\eta_1} [(\hat\rho a^{1/2}\hat{\mathbf{f}})\cdot\mathbf{v} + (\hat\rho a^{1/2}\hat{\mathbf{l}})\cdot\mathbf{w}]d\eta^1 d\eta^2 d\theta^3$$

$$+\int_{\bar\theta_3}\int_{\hat\eta_2}\int_{\hat\eta_1} [(\mathbf{N}^\alpha\nu_\alpha)\cdot\mathbf{v} + (\mathbf{M}^\alpha\nu_\alpha)\cdot\mathbf{w}]ds\,d\theta^3$$

$$= \int_{\bar\theta_3}\int_{\bar\theta_2}\int_{\bar\theta_1} \{ \int_o^{\xi_2} \rho^* g^{*1/2}\mathbf{b}^* d\xi + [\mathbf{T}^*]_{\xi=0}^{\xi=\xi_2} \}\cdot\mathbf{v}\,d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar\theta_3}\int_{\bar\theta_2}\int_{\bar\theta_1} \{ \int_o^{\xi_2} \rho^* g^{*1/2}\mathbf{b}^*\xi d\xi + [\mathbf{T}^*\xi]^{\xi=\xi_2} \}\cdot\mathbf{w}\,d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\bar\theta_3}\int_{\hat{\mathcal{P}}} (\mathbf{N}^\alpha\cdot\mathbf{v})_{|\alpha} da\,d\theta^3 + \int_{\bar\theta_3}\int_{\hat{\mathcal{P}}} (\mathbf{M}^\alpha\cdot\mathbf{w})_{|\alpha} da\,d\theta^3$$

or

Right hand side of (17.35) $= \int_{\mathcal{P}} (g^{-1/2} \int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi) \cdot \mathbf{v}\, dv$

$$+ \int_{\mathcal{P}} (g^{-1/2} \int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi) \cdot \mathbf{w}\, dv$$

$$+ \int_{\bar{\theta}_3} \int_{\hat{\mathcal{P}}} a^{-1/2} \{ (a^{1/2} \mathbf{N}^\alpha \cdot \mathbf{v})_{,\alpha} + [\mathbf{T}^*]_{\xi=0}^{\xi=\xi_2} \cdot \mathbf{v} \} d\hat{a}\, d\theta^3$$

$$\int_{\bar{\theta}_3} \int_{\hat{\mathcal{P}}} a^{-1/2} \{ (a^{1/2} \mathbf{M}^\alpha \cdot \mathbf{w})_{,\alpha} + [\mathbf{T}^* \xi]_{\xi=0}^{\xi=\xi_2} \cdot \mathbf{w} \} d\hat{a}\, d\theta^3$$

$$= \int_{\mathcal{P}} \{ (g^{-1/2} \int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi) \cdot \mathbf{v} + (g^{-1/2} \int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi) \cdot \mathbf{w} \} dv$$

$$+ \int_{\mathcal{P}} g^{1/2} \{ [(\int_{o}^{\xi_2} \mathbf{T}^{*\alpha} d\xi) \cdot \mathbf{v}]_{,\alpha} + [(\int_{o}^{\xi_2} \mathbf{T}^{*3}{}_{,3} d\xi) \cdot \mathbf{v}] \} dv$$

$$\int_{\mathcal{P}} g^{-1/2} \{ [(\int_{o}^{\xi_2} \mathbf{T}^{*\alpha} \xi d\xi) \cdot \mathbf{w}]_{,\alpha} + [(\int_{o}^{\xi_2} (\mathbf{T}^{*3} \xi)_{,3} d\xi) \cdot \mathbf{w}] \} dv$$

or

Right hand side of (17.35) $= \int_{\mathcal{P}} g^{-1/2} \{ (\int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi) \cdot \mathbf{v} + (\int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi \cdot \mathbf{w} \} dv$

$$+ \int_{\mathcal{P}} g^{-1/2} (\int_{o}^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi) \cdot \mathbf{v} \} dv$$

$$+ \int_{\mathcal{P}} g^{-1/2} \{ (\int_{o}^{\xi_2} (\mathbf{T}^{*i} \xi)_{,i} d\xi) \cdot \mathbf{w} \} dv \qquad (17.41)$$

Making use of (17.12), (17.13), (17.21), (17.23) and the divergence theorem, we can rewrite (17.41) as follows

BASE

$$\text{Right hand side of (17.35)} = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv$$

$$+ \int_{\mathcal{P}} g^{-1/2}(\mathbf{T}^A \cdot \mathbf{v})_{,A} dv + \int_{\mathcal{P}} g^{-1/2}(\mathbf{S}^A \cdot \mathbf{w})_{,A} dv$$

$$= \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv$$

$$+ \int_{\mathcal{P}} (g^{-1/2}\mathbf{T}^A \cdot \mathbf{v})_{|A} dv + \int_{\mathcal{P}} (g^{-1/2}\mathbf{S}^A \cdot \mathbf{w})_{|A} dv$$

$$= \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv$$

$$+ \int_{\mathcal{P}} g^{-1/2}\mathbf{T}^A \cdot \mathbf{v} n_A da + \int_{\mathcal{P}} g^{-1/2}\mathbf{S}^A \cdot \mathbf{w} n_A da$$

$$= \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv + \int_{\mathcal{P}} \mathbf{t} \cdot \mathbf{v} da + \int_{\mathcal{P}} s \cdot \mathbf{w} \tag{17.42}$$

where in obtaining (17.41) we have also made use of (17.16) and (17.26). From (17.40) and (17.42) we obtain

$$\frac{d}{dt} \int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K}) dv = dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + s \cdot \mathbf{w}) da \tag{17.43}$$

This is the composite conservation of energy in purely mechanical theory.

18. Summary of basic principles for composite laminates

Considering the development in the previous section, we are now in a position to state the conservation laws (principles) for composite laminates. With reference to the present configuration, these conservation laws are summarized below:

$$a: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho \, dv = 0$$

$$b: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho \, \mathbf{b} \, dv + \int_{\partial\mathcal{P}} \mathbf{t} \, da$$

$$c: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s} \, da \qquad (18.1)$$

$$d: \quad \frac{d}{dt}\int_{\mathcal{P}} \{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}dv =$$

$$\int_{\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c})dv + \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s})da$$

$$e: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K})dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w})da$$

The first of (18.1) is the mathematical statement of conservation of mass, the second that of linear momentum principle, the third that of director momentum. the fourth is the principle of moment of momentum, and the fifth represents the balance of energy for composite laminates.

In (18.1) $\mathbf{r}, \mathbf{d}$ denote the *position vector* and the *director* associated with a *composite particle*, respectively, while the velocity and the director velocity of the composite particle are given by $\mathbf{v}$ and $\mathbf{w}$. The definition of the various field quantities in (18.1) and their relation to their counterparts in micro-structure and the similar three dimensional quantities are given below.

1) $\rho = \rho(\theta^i, t)$ is the *composite assigned mass density* in the present configuration given by

$$\rho g^{1/2} = \hat{\rho} \, a^{1/2} = \int_0^{\xi_2} \rho^* g^{1/2} d\xi \qquad (18.2)$$

where in (18.2) $\hat{\rho}$ is the mass density of the micro-structure, ρ^* is the classical 3-dimensional mass density, g is the determinant of the metric tensor g_{ij} associated with the composite coordinate system θ^i, g^* is the determinant of the metric tensor g^*_{ij} associated with the micro-structure coordinate system $\eta^i = \{\eta^\alpha,\xi\} = \{\theta^\alpha,\xi\}$, a is the determinant of the two-dimensional (surface) metric tensor $a_{\alpha\beta}$ associated with the Cosserat surface (micro-structure).

We notice that the dimensions of ρ^* and $\hat{\rho}$ are mass per unit volume and mass per unit area, respectively. However, the dimension of ρ is the dimension of integrated mass per unit volume of the composite.

2) $\mathbf{b} = \mathbf{b}(\theta^i,t)$ is the *composite assigned body force density* per unit of ρ, given by

$$\rho\, g^{1/2}\, \mathbf{b} = \int_0^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi \qquad (18.3)$$

where $\mathbf{b}^*$ is the classical 3-dimensional body force density. The dimension of $\mathbf{b}$ should be clear from (18.3).

3) $\mathbf{c} = \mathbf{c}(\theta^i,t)$ is the *composite assigned body couple density* per unit of ρ, given by[4]

$$\rho\, g^{1/2}\, \mathbf{c} = \int_0^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi \qquad (18.4)$$

The dimension of $\mathbf{c}$ should be clear from (18.4).

4) $\mathbf{t} = \mathbf{t}(\theta^i,t;\mathbf{n})$ is the *composite assigned traction* (per unit area of the composite) such that[5]

$$\mathbf{t} = g^{-1/2}\, \mathbf{T}^i n_i \qquad (18.5)$$

[4] **c** may also be called "composite assigned director force" emphasizing the "directed" nature of the present continuum theory. In the present context, however, we prefer the terminology in 3 above as it makes the physical nature of **c** more apparent.

[5] The nature of the definition (18.5) and (18.6) as well as (18.9) and (18.10) will be discussed and explained in section (19).

$$T^i{}_{,i} = \int_o^{\xi_2} T^{*i}{}_{,i} d\xi \tag{18.6}$$

$$T^\alpha = \int_o^{\xi_2} T^{*\alpha} d\xi = a^{1/2} N^\alpha \tag{18.7}$$

$$T^3{}_{,3} = T^{*3}{}_{|\xi=\xi_2} - T^{*3}{}_{|\xi=0} = \Delta T^{*3} \tag{18.8}$$

where T^{*i} is the classical stress vector and N^α is the resultant force of the micro-structure (i.e., Cosserat surface). We also recall that a comma on the left-hand side of (18.6) to (18.8) denotes partial differentiation with respect to θ^i. However, a comma on the right-hand side of (18.6) and in (18.8) denotes partial differentiation with respect to $\eta^i = \{\eta^\alpha, \xi\}$.

5) $\mathbf{s} = \mathbf{s}(\theta^i, t; \mathbf{n})$ is the *composite assigned couple traction* [6] per unit area of the composite such that

$$\mathbf{s} = g^{-1/2} \mathbf{S}^i n_i \tag{18.9}$$

$$S^i{}_{,i} = \int_o^{\xi_2} T^{*i}{}_{,i} \xi d\xi \tag{18.10}$$

$$S^\alpha = \int_o^{\xi} T^{*\alpha} \xi d\xi = a^{1/2} M^\alpha \tag{18.11}$$

$$S^3{}_{,3} = (T^{*3}\xi)_{|\xi=\xi_2} - (T^{*3}\xi)_{|\xi=0} = \Delta(T^{*3}\xi) \tag{18.12}$$

where M^α is the resultant couple of the micro-structure (i.e., Cosserat surface) and the same remark as in (4) above holds for commas and partial differentiation.

6) $\mathbf{k} = \mathbf{k}(\theta^i, t)$ is the *composite assigned intrinsic (director) force*, per unit volume of the composite, given by

$$g^{1/2} \mathbf{k} = a^{1/2} \mathbf{m} = \int_o^{\xi_2} T^{*3} d\xi \tag{18.13}$$

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[6] **s** may also be called "composite assigned contact director force" which reflects the "directed" nature of the present theory. However, the terminology given in 5 reflects the physical nature of s more clearly.

where **m** is the intrinsic director force of the micro-structure (i.e., Cosserat surface).

7) $y^\alpha = y^\alpha(\theta^i)$ are the inertia coefficients which are independent of time and are given by

$$y^\alpha = \int_0^{\xi_2} \rho^* g^{*1/2} \xi^\alpha d\xi \tag{18.14}$$

8) $\varepsilon = \varepsilon(\theta^i,t)$ is the *composite assigned specific internal energy* per unit of ρ given by

$$\rho\, g^{1/2}\, \varepsilon = \hat{\rho}\, a^{1/2}\, \hat{\varepsilon} = \int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{18.15}$$

where ε^* is the classical 3-dimensional specific internal energy and $\hat{\varepsilon}$ is the specific internal energy per unit $\hat{\rho}$ for the micro-structure (i.e., Cosserat surface).

9) $\mathcal{K} = \mathcal{K}(\theta^i,t)$ is the *composite assigned kinetic energy* per unit of ρ and is given by

$$\mathcal{K} = \hat{\mathcal{K}} = \frac{1}{2}\,(\mathbf{v}\cdot\mathbf{v} + 2y^1\mathbf{v}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\mathbf{w}) \tag{18.16}$$

where $\hat{\mathcal{K}}$ represents the kinetic energy per unit $\hat{\rho}$ of the micro-structure (i.e., Cosserat surface). The momentum corresponding to the velocity **v** and the director momentum corresponding to **w** are given by

$$\rho\,\frac{\partial \mathcal{K}}{\partial \mathbf{v}} = \rho(\mathbf{v} + y^1\mathbf{w}) \tag{18.17}$$

$$\rho\,\frac{\partial \mathcal{K}}{\partial \mathbf{w}} = \rho(y^1\mathbf{v} + y^2\mathbf{w}) \tag{18.18}$$

For simplicity in the rest of this development, when there is no possibility of confusion, we adopt the following simplified terminology:

ρ: "composite mass density"

b: "composite body force density"

c: "composite body couple density"

t: "composite traction"

s: "composite traction"

k: "composite intrinsic force"

ε: "composite specific internal energy"

$\mathcal{K}$: "composite kinetic energy"

19. Considerations on composite contact force and composite contact couple

In this section we discuss the physical nature of some of the field variables defined in section (16) and recorded in section (18).

We begin our discussion by considering the composite contact force **t**, and recall (18.5) and (18.6), i.e.,

$$\mathbf{t} = g^{-1/2}\mathbf{T}^i n_i \tag{19.1}$$

and

$$\mathbf{T}^A{}_{,A} = \mathbf{T}^i{}_{,i} = \int_0^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi \tag{19.2}$$

Expression (19.2) may be rewritten as

$$\mathbf{T}^A{}_{,A} - \int_0^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi = 0$$

or

$$(\mathbf{T}^\alpha - \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi)_{,\alpha} + (\mathbf{T}_{,3} - [\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2}) = 0 \tag{19.3}$$

We now identify $\mathbf{T}^\alpha$ as follows

$$\mathbf{T}^\alpha = \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi \tag{19.4}$$

From (19.3) and (19.4) we obtain

$$\mathbf{T}_{,3} = \mathbf{T}^{*3}{}_{|\xi=\xi_2} - \mathbf{T}^{*3}{}_{|\xi=0} = \Delta\mathbf{T}^{*3} \tag{19.5}$$

It is clear from (19.4) that $\mathbf{T}^\alpha = \{\mathbf{T}^1, \mathbf{T}^2\}$ are the stress resultants in the directions of $\mathbf{g}_\alpha = \{\mathbf{g}_1, \mathbf{g}_2\}$. In other words the composite stress vectors $\mathbf{T}^\alpha$ are in fact the stress resultants of the representative element (i.e., micro-structure). Differentiating both sides of (19.4) with respect to ξ, we obtain

$$\frac{\partial T^{\alpha}}{\partial \xi} = \frac{\partial T^{\alpha}}{\partial \theta^3} \frac{\partial \theta^3}{\partial \xi} = T^{*\alpha}$$

or

$$T^{*\alpha} = T^{\alpha}{}_{,\xi} = \frac{1}{\varepsilon} T^{\alpha}{}_{,\theta^3} = \frac{1}{\varepsilon} T^{\alpha}{}_{,3} \tag{19.6}$$

where $(,\xi)$ and $(,\theta^3)$ denote partial differentiation with respect to ξ and θ^3, respectively, and where we have made use of (17.2). Expression (19.6) shows that $T^{*\alpha}$ are the derivatives of T^{α} with respect to θ^3 divided by ε. We now consider (19.3) and write

$$\frac{\partial T^3}{\partial \theta^3} = \Delta T^{*3} = T^{*3}{}_{|\xi=\xi_2} - T^{*3}{}_{|\xi=0} \tag{19.7}$$

The right-hand side (RHS) of (19.7) can be written as

$$\Delta T^{*3} = \frac{T^{*3}{}_{\xi=\xi_2} - T^{*3}{}_{\xi=0}}{\xi_2}\, \xi_2 = \frac{(T^{*3}\xi_2)_{\xi=\xi_2} - (T^{*3}\xi_2)_{\xi=0}}{\Delta\theta^3} \tag{19.8}$$

where ξ_2 is associated with $\Delta\theta^3$. The continuity of displacement field at interface, as discussed in section 25, justifies this substitution. Left-hand side (LHS) of (19.7) can be written as

$$\frac{\partial T^3}{\partial \theta^3} = \lim_{\Delta\theta^3 \to 0} \frac{\Delta T^3}{\Delta \theta^3} = \lim_{\Delta\theta^3 \to 0} \frac{T^3(\theta^3 + \Delta\theta^3) - T^3(\theta^3)}{\Delta\theta^3} \tag{19.9}$$

Comparing RHS of (19.9) with RHS of (19.8) and considering the continuity constraints, we can associate the composite stress vector T^3 with $T^{*3}\xi_2$, i.e.,

$$T^3 = \xi_2 T^{*3} \tag{19.10}$$

This shows that T^3 is proportional to the interlaminar stress vector of the composite laminate. In other words, T^3 may be considered, in a mathematical sense, as the resultant of interlaminar stresses in a representative element across the planes parallel to the interface.

Next, we consider the last term on the right-hand side of $(18.1)_b$ which represents the total contact force on the part $\mathcal{P}$ of the composite body. Making use of (19.1) and (19.2) and the divergence theorem, we may write

$$\int_{\partial\mathcal{P}} \mathbf{t}\, da = \int_{\partial\mathcal{P}} g^{-1/2}\mathbf{T}^i n_i da = \int_{\mathcal{P}} (g^{-1/2}\mathbf{T}^i)_{|i} dv$$

$$= \int_{\mathcal{P}} g^{-1/2}\mathbf{T}^i_{,i} dv = \int_{\mathcal{P}} g^{-1/2}\{\int_o^{\xi_2} \mathbf{T}^{*i}_{,i} d\xi\} dv$$

$$= \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} \{\int_o^{\xi_2} \mathbf{T}^{*i}_{,i} d\xi\} d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\bar{\theta}_3} \{\int_{\bar{\eta}_2}\int_{\bar{\eta}_1}\int_o^{\xi_2} \mathbf{T}^{*i}_{,i} d\theta^1 d\theta^2 d\xi\} d\theta^3$$

$$= \int_{\bar{\theta}_3} \{\int_{\mathcal{P}^*} g^{*-1/2}\mathbf{T}^{*i}_{,i} dv^*\} d\theta^3$$

$$= \int_{\bar{\theta}_3} \{\int_{\mathcal{P}^*} (g^{*-1/2}\mathbf{T}^{*i})_{|i} dv^*\} d\theta^3$$

$$= \int_{\bar{\theta}_3} \{\int_{\partial\mathcal{P}^*} g^{*-1/2}\mathbf{T}^{*i} n_i^* da^*\} d\theta^3$$

Hence,

$$\int_{\partial\mathcal{P}} \mathbf{t}\, da = \int_{\bar{\theta}_3} (\int_{\partial\mathcal{P}^*} \mathbf{t}^* da^*) d\theta^3 \tag{19.11}$$

The above expression relates the composite stress vector to that of the representative element. In other words, (19.11) is a relationship between the stress vector of the micro-structure and the stress vector of the macro-structure showing that the total contact force on an arbitrary part $\mathcal{P}$ of the composite laminate is in fact the integral, in θ^3 direction of the total contact force on the corresponding part $\mathcal{P}^*$ of the micro-structure.

Next, we consider the composite contact couple s, and recall (18.9) and (18.10), i.e.,

$$s = g^{-1/2} S^i n_i \tag{19.12}$$

and

$$S^A{}_{,A} = S^i{}_{,i} = \int_o^{\xi_2} (T^{*i}\xi)_{,i} d\xi \tag{19.13}$$

The above expression may be rewritten as follows

$$S^i{}_{,i} - \int_o^{\xi_2} (T^{*i}\xi)_{,i} d\xi = 0$$

or

$$[S^\alpha{}_{,\alpha} - \int_o^{\xi_2} (T^{*\alpha}\xi)_{,\alpha} d\xi] + (S^3{}_{,3} - [T^{*3}\xi]_{\xi=0}^{\xi=\xi_2}) = 0 \tag{19.14}$$

We now identify S^α as follows:

$$S^\alpha = \int_o^{\xi_2} T^{*\alpha}\xi d\xi \tag{19.15}$$

From (19.14) and (19.15) we obtain

$$S^3{}_{,3} = (T^{*3}\xi)_{|\xi=\xi_2} - (T^{*3}\xi)_{|\xi=0} = \Delta(T^{*3}\xi) \tag{19.16}$$

It is clear from (19.15) that $S^\alpha = \{S^1, S^2\}$ are the stress couples in the directions of $g_\alpha = \{g_1, g_2\}$. In other words, the composite couple stress vectors S^α are in fact the stress couples of the representative elements (i.e., micro-structure). Differentiating both sides of (19.15) with respect to ξ, we obtain

$$\frac{\partial S^\alpha}{\partial \xi} = \frac{\partial S^\alpha}{\partial \theta^3} \frac{\partial \theta^3}{\partial \xi} = T^{*\alpha}\xi$$

or

$$T^{*\alpha}\xi = S^{\alpha}{}_{,\xi} = \frac{1}{\varepsilon} S^{\alpha}{}_{,\theta^3} = \frac{1}{\varepsilon} S^{\alpha}{}_{,3} \qquad (19.17)$$

where ($,\xi$) and ($,\theta^3$) denote partial differentiation with respect to ξ and θ^3, respectively, and where we have made use of (17.2). Expression (19.17) shows that ($T^{*\alpha}\xi$) are the derivatives of S^{α} with respect to θ^3 divided by ε. We now consider (19.16) and write

$$\frac{\partial S^3}{\partial \theta^3} = \Delta(T^{*3}\xi) = (T^{*3}\xi)_{|\xi=\xi_2} - (T^{*3}\xi)_{|\xi=0} = \int_o^{\xi_2} \frac{\partial (T^{*3}\xi)}{\partial \xi} d\xi \qquad (19.18)$$

By differentiating with respect to ξ we obtain

$$\frac{\partial^2 S^3}{\partial \xi \partial \theta^3} = \frac{\partial (T^{*3}\xi)}{\partial \xi} \qquad (19.19)$$

Making use of (16.2) we may write

$$\frac{\partial}{\partial \xi} \left(\frac{\partial S^3}{\partial \xi} \frac{\partial \xi}{\partial \theta^3}\right) = \frac{\partial (T^{*3}\xi)}{\partial \xi}$$

or

$$\frac{\partial}{\partial \xi} \left(\varepsilon \frac{\partial S^3}{\partial \xi}\right) = \frac{\partial (T^{*3}\xi)}{\partial \xi} \qquad (19.20)$$

We now integrate the above with respect to ξ to obtain

$$\varepsilon \frac{\partial S^3}{\partial \xi} = T^{*3}\xi + D_1 \qquad (19.21)$$

Assuming that for the unstressed state both T^{*i} and S^i and their space derivatives vanish, we can put $D_1 = 0$ and write (19.21) as follows:

$$\frac{\partial S^3}{\partial \xi} = \frac{T^{*3}\xi}{\varepsilon} \qquad (19.22)$$

where ε_1 is now a function of ξ and we have $\varepsilon_1 << 1$. We now integrate the above with respect to ξ and obtain

$$S^3 = \int_0^{\xi_2} \frac{1}{\varepsilon} (T^{*3}\xi) d\xi + D_2 \tag{19.23}$$

This is valid for all times, and in particular must be valid for the stress free state; hence, $D_2 = 0$. Then by the mean value theorem for integrals we may write

$$S^3 = \frac{1}{\varepsilon} (\xi_2 - 0)(T^{*3}\bar{\xi}) = \frac{1}{\varepsilon} \xi_2 T^{*3}\bar{\xi}$$
$$0 \le \bar{\xi} \le \xi^2 \tag{19.24}$$

Since the composite laminate is assumed to be composed of infinitely many Cosserat surfaces in the limit we have $\varepsilon \to 0$, $\xi_2 \to 0$, $\bar{\xi} \to 0$ and

$$S^3 = \lim_{\xi_2 \to 0, \bar{\xi} \to 0, \varepsilon \to 0} [\frac{1}{\varepsilon} \xi_2 T^{*3}\bar{\xi}] = 0 \tag{19.25}$$

Expression (19.25) shows that the component S^3 of the contact couple s is identically zero for the composite laminate. Next, we consider the last term on the right-hand side of $(18.1)_c$ which represents the total contact couple on the part $\mathcal{P}$ of the composite body. Making ue of (19.22), (19.13), (19.1) and the divergence theorem, we may write

BASE

$$\int_{\partial\mathcal{P}} s\,da = \int_{\partial\mathcal{P}} g^{-1/2} S^i n_i da = \int_{\mathcal{P}} (g^{-1/2} S^i)_{|i} dv$$

$$= \int_{\mathcal{P}} g^{-1/2} S^i_{,i} dv = \int_{\mathcal{P}} g^{-1/2} \{ \int_o^{\xi_2} (T^{*i}\xi)_{,i} d\xi \} dv$$

$$= \int_{\bar{\theta}_3} \int_{\bar{\theta}_2} \int_{\bar{\theta}_1} \{ \int_o^{\xi_2} T^{*i}\xi)_{,i} d\xi \} d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\bar{\theta}_3} \{ \int_{\bar{\eta}_2} \int_{\bar{\eta}_1} \int_o^{\xi_2} (T^{*i}\xi)_{,i} d\eta^1 d\eta^2 d\xi \} d\theta^3$$

$$= \int_{\bar{\theta}_3} \{ \int_{\mathcal{P}^*} g^{*-1/2} (T^{*i}\xi)_{,i} dv^* \} d\theta^3$$

$$= \int_{\bar{\theta}_3} \{ \int_{\mathcal{P}^*} (g^{*-1/2} T^{*i}\xi)_{|i} dv^* \} d\theta^3$$

$$= \int_{\bar{\theta}_3} \{ \int_{\partial\mathcal{P}^*} g^{*-1/2} (T^{*i}\xi) n_i^* da^* \} d\theta^3$$

Hence

$$\int_{\partial\mathcal{P}} s\,da = \int_{\bar{\theta}_3} (\int_{\partial\mathcal{P}^*} \mathbf{t}^* \,\xi\, da^*) d\theta^3 \qquad (19.26)$$

This expression relates the composite contact couple to the stress vector of the representative element. In other words, (19.26) is a relationship between the stress vector of the micro-structure and the stress couple vector of macro-structure showing that the total contact couple on an arbitrary part $\mathcal{P}$ of the composite laminate is in fact the integral, in θ^3 direction, of the total moment of the contact force, in ξ direction on the corresponding part $\mathcal{P}^*$ of the micro-structure.

Conditions (19.2) and (19.13) or equivalently (19.11) and (19.26) stipulate that the action of $\mathbf{T}^i$ and $\mathbf{S}^i$ over a portion of surface $\partial\mathcal{P}$, with outward unit normal $\mathbf{n}$, of the composite laminate is "equipolent" (i.e., equivalent in effectiveness) to the action of the stress vector $\mathbf{t}^*$ over the corresponding portion of the surface $\partial\mathcal{P}^*$, with outward unit normal $\mathbf{n}^*$, of the representative element (i.e., micro-structure).

20. Further consideration of the composite conservation laws.

In this section we proceed to obtain different froms of the basic principles for the composite laminate, namely expressions (18.1). The derivation of the alternative forms of these equations may be carried out using either Eulerian or convected coordinate systems. For our present development the use of a convected coordinate system simplifies the intermediate manipulations considerably. For an extensive discussion on convected coordinates, the reader is referred to [Oldroyd, 1950] and [Lodge, 1974] where formulae for differentiation with respect to time and various additional results can be found. It is, however, instructive to carry out the derivation by direct differentiation, with respect to time, of the integrals on the left hand sides of (18.1). Here we adopt this mode of derivation to derive the transport theorem in terms of convected coordinates.

Let $\mathcal{P}$ be an arbitrary part (or subset) of the laminated composite body $\mathcal{B}$ with a closed boundary surface $\partial\mathcal{P}$ in the present configuration at time t. The counterparts of $\mathcal{P}$ and $\partial\mathcal{P}$ in a fixed reference configuration will be denoted by $\mathcal{P}_0$ and $\partial\mathcal{P}_0$, respectively. Let ϕ be any scalar or tensor-valued field with the following representation in the present configuration at time t:

$$\phi = \phi(\theta^i, t) \tag{20.1}$$

and consider the volume integral

$$I = \int_{\mathcal{P}} \phi(\theta^i, t)\, dv \tag{20.2}$$

Often we encounter an expression of the type (20.2) and we need to calculate its time derivative $\frac{dI}{dt}$. Since θ^i are convected, we may write

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\mathcal{P}} \phi(\theta^i,t)dv = \frac{d}{dt}\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} \phi(\theta^i,t)g^{1/2}\, d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} \frac{d}{dt}(\phi g^{1/2}) d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} \{\dot{\phi} g^{1/2} + \phi(\dot{\overline{g^{1/2}}})\} d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} \{\dot{\phi} g^{-1/2} + \phi(\frac{1}{2}\, g^{1/2}\dot{g})\} d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1} g^{1/2}(\dot{\phi} + \frac{\dot{g}}{2g}\phi) d\theta^1 d\theta^2 d\theta^3$$

$$= \int_{\mathcal{P}} (\dot{\phi} + \frac{\dot{g}}{2g}\,\phi) dv \qquad (20.3)$$

where in (20.3), $\bar{\theta}_1$, $\bar{\theta}_2$, $\bar{\theta}_3$ denote appropriate ranges of integration for variables θ^i. Making use of the divergence theorem we can write (20.3) in the form

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\mathcal{P}} \phi(\theta^i,t)dv = \int_{\mathcal{P}} \frac{\partial\phi}{\partial t}\, dv + \int_{\partial\mathcal{P}} \phi\, \mathbf{v}\cdot\mathbf{n}\, da \qquad (20.4)$$

We now consider the conservation of mass $(18.1)_a$ and write

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\, dv = \int_{\mathcal{P}} (\dot{\rho} + \frac{\dot{g}}{2g}\,\rho) dv = 0 \qquad (20.5)$$

This must be valid for any part $\mathcal{P}$, hence, assuming that ρ is continuously differentiable we obtain

$$\dot{\rho} + \frac{\dot{g}}{2g}\,\rho = 0 \qquad (20.6)$$

where in obtaining (20.5) we have made use of (20.3). Equation (20.6) is the local form of the continuity equation for composite laminates.

Next we consider the conservation of linear momentum $(18.1)_b$, i.e.,

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})\,dv = \int_{\mathcal{P}} \rho\,\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}\,da \qquad (20.7)$$

Considering the left hand side of (20.7) and making use of (20.3) we can write

$$\text{Left hand side of (20.7)} = \int_{\mathcal{P}} \{\dot{\overline{[\rho(\mathbf{v} + y^1\mathbf{w})]}} + \frac{\dot{g}}{2g}\,[\rho(\mathbf{v} + y^1\mathbf{w})]\}\,dv$$

$$= \int_{\mathcal{P}} \{\dot{\rho}(\mathbf{v} + y^1\mathbf{w}) + \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) + \frac{\dot{g}}{2g}\,\rho(\mathbf{v} + y^1\mathbf{w})\}\,dv$$

$$= \int_{\mathcal{P}} \{(\dot{\rho} + \frac{\dot{g}}{2g})(\mathbf{v} + y^1\mathbf{w}) + \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})\}\,dv \qquad (20.8)$$

Substituting (20.8) into (20.7), we obtain

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})\,dv = \int_{\mathcal{P}} \rho\,\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}\,da \qquad (20.9)$$

where we have also made use of (20.6).

Adapting the same approach we consider the conservation of director momentum $(18.1)_c$ and we write

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})\,dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})\,dv + \int_{\partial\mathcal{P}} \mathbf{s}\,da \qquad (20.10)$$

or

$$\int_{\mathcal{P}} \{\dot{\overline{[\rho(y^1\mathbf{v} + y^2\mathbf{w})]}} + \frac{\dot{g}}{2g}\,[\rho(y^1\mathbf{v} + y^2\mathbf{w})]\}\,dv = \int_{\mathcal{P}} (\rho\,\mathbf{c} - \mathbf{k})\,dv + \int_{\partial\mathcal{P}} \mathbf{s}\,da$$

or

$$\int_{\mathcal{P}} \{\dot{\rho}(y^1\dot{\mathbf{v}} + y^2\mathbf{w}) + \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) + \frac{\dot{g}}{2g}\,\rho(y^1\mathbf{v} + y^2\mathbf{w})\}\,dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})\,dv + \int_{\partial\mathcal{P}} \mathbf{s}\,da$$

or

$$\int_{\mathcal{P}} \{(\dot{\rho} + \frac{\dot{g}}{2g}\,\rho)(y^1\dot{\mathbf{v}} + y^2\mathbf{w}) + \rho(y^1\mathbf{v} + y^2\dot{\mathbf{w}})\}\,dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})\,dv + \int_{\partial\mathcal{P}} \mathbf{s}\,da$$

or

$$\int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}\, da \qquad (20.11)$$

where in obtaining (20.11) we have made use of (20.6).

Next, we consider the conservation of linear momentum $(20.1)_d$, i.e.,

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}dv = \int_{\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c})dv$$

$$= \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s})da \qquad (20.12)$$

Following the same procedure, we can reduce the left hand side of (20.12) as follows:

$$\text{Left hand side of (20.12)} = \frac{d}{dt}\int_{\mathcal{P}} \rho\, \mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})dv + \frac{d}{dt}\int_{\mathcal{P}} \rho\, \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})dv$$

$$= \int_{\mathcal{P}} \{[\dot{\overline{\rho\, \mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})}}] + \frac{\dot{g}}{2g}[\rho\, \mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})]\}dv$$

$$+ \int_{\mathcal{P}} \{[\dot{\overline{\rho\, \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})}}] + \frac{\dot{g}}{2g}[\rho\, \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})]\}dv$$

$$= \int_{\mathcal{P}} \{\dot{\rho}\, \mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \rho[\dot{\overline{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})}}] + \frac{\dot{g}}{2g}\rho\, \mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})\}dv$$

$$+ \int_{\mathcal{P}} \{\dot{\rho}\, \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w}) + \rho[\dot{\overline{\mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})}}] + \frac{\dot{g}}{2g}\rho\, \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}dv$$

$$= \int_{\mathcal{P}} \{(\dot{\rho} + \frac{\dot{g}}{2g}\rho)[\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})] + \rho[\dot{\overline{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})}}]\}dv$$

$$+ \int_{\mathcal{P}} \{(\dot{\rho} + \frac{\dot{g}}{2g}\rho)[\mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})] + \rho[\mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})] + \rho[\dot{\overline{\mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})}}]\}dv$$

$$= \int_{\mathcal{P}} \rho[\dot{\overline{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w})}}]dv + \int_{\mathcal{P}} \rho[\dot{\overline{\mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})}}]dv$$

or

$$\text{Left hand side of(20.12)} = \int_{\mathcal{P}} \rho \{ \dot{\mathbf{r}} \times (\mathbf{v} + y^1 \mathbf{w}) + \mathbf{r} \times (\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) \} dv$$

$$+ \int_{\mathcal{P}} \rho \, \{ \dot{\mathbf{d}} \times (y^1 \mathbf{v} + y^2 \mathbf{w}) + \mathbf{d} \times (y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \} dv$$

$$= \int_{\mathcal{P}} \rho \{ \mathbf{r} \times (\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) + \mathbf{d} \times (y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \} dv$$

$$+ \int_{\mathcal{P}} \rho \, \{ \mathbf{v} \times (\mathbf{v} + y^1 \mathbf{w}) + \mathbf{w} \times (y^1 \mathbf{v} + y^2 \mathbf{w}) \} dv \qquad (20.13)$$

where in obtaining (20.13) we have again made use of (20.6) and the fact that

$$\dot{\mathbf{r}} = \mathbf{v} \quad , \quad \dot{\mathbf{d}} = \mathbf{w} \qquad (20.14)$$

Since

$$\mathbf{v} \times \mathbf{v} = 0 \quad , \quad \mathbf{w} \times \mathbf{w} = 0 \qquad (20.15)$$

and

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \qquad (20.16)$$

The second integral in (20.13) vanishes identically; hence, after substituting the result in (20.12), we obtain

$$\int_{\mathcal{P}} \rho \, \{ \mathbf{r} \times (\dot{\mathbf{v}} + y^1 \mathbf{w}) + \mathbf{d} \times (y^1 \dot{\mathbf{v}} + y^2 \mathbf{w}) \} dv = \int_{\mathcal{P}} \rho (\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c}) dv + \int_{\partial \mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s}) da \qquad (20.17)$$

Finally, we consider the conservation of energy $(18.1)_e$, i.e.,

$$\frac{d}{dt} \int_{\mathcal{P}} \rho [\mathcal{E} + \mathcal{K}] dv = \int_{\mathcal{P}} \rho (\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv + \int_{\partial \mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w}) da \qquad (20.18)$$

and write

$$\text{Left hand side of (20.18)} = \int_{\mathcal{P}} \{\overline{[\rho(\mathcal{E}+\mathcal{K})]}^{\boldsymbol{\cdot}} + \frac{\dot{g}}{2g}[\rho(\mathcal{E}+\mathcal{K})]\}dv$$

$$= \int_{\mathcal{P}} \{\dot{\rho}(\mathcal{E}+\mathcal{K}) + \rho\overline{(\mathcal{E}+\mathcal{K})}^{\boldsymbol{\cdot}} + \frac{\dot{g}}{2g}\rho(\mathcal{E}+\mathcal{K})\}dv$$

$$\text{Left hand side of (20.18)} = \int_{\mathcal{P}} \{(\dot{\rho} + \frac{\dot{g}}{2g}\rho)(\mathcal{E}+\mathcal{K}) + \rho(\dot{\mathcal{E}}+\dot{\mathcal{K}})\}dv$$

$$= \int_{\mathcal{P}} \rho(\dot{\mathcal{E}}+\dot{\mathcal{K}})dv \tag{20.19}$$

where in obtaining (20.19) we have again made use of (20.3) and (20.6). Substituting (20.19) into (20.18), we obtain

$$\int_{\mathcal{P}} \rho(\dot{\mathcal{E}}+\dot{\mathcal{K}})dv = \int_{\mathcal{P}} \rho(\mathbf{b}\cdot\mathbf{v} + \mathbf{c}\cdot\mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t}\cdot\mathbf{v} + \mathbf{s}\cdot\mathbf{w})da \tag{20.20}$$

For convenience and later use, below we summarize the results of this section:

$$\begin{aligned}
&a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \\
&b: \quad \int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\mathbf{b}dv + \int_{\partial\mathcal{P}} \mathbf{t}\,da \\
&c: \quad \int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}\,da \qquad (20.21) \\
&d: \quad \int_{\mathcal{P}} \rho\{\mathbf{r}\times(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) + \mathbf{d}\times(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})\}dv = \int_{\mathcal{P}} \rho(\mathbf{r}\times\mathbf{b} + \mathbf{d}\times\mathbf{c})dv + \int_{\partial\mathcal{P}} (\mathbf{r}\times\mathbf{t} + \mathbf{d}\times\mathbf{s})da \\
&e: \quad \int_{\mathcal{P}} \rho(\dot{\mathcal{E}}+\dot{\mathcal{K}})dv = \int_{\mathcal{P}} \rho(\mathbf{b}\cdot\mathbf{v} + \mathbf{c}\cdot\mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t}\cdot\mathbf{v} + \mathbf{s}\cdot\mathbf{w})da
\end{aligned}$$

21. Further consideration of composite contact force and composite contact couple

a) Existence of the composite stress tensor and its relationship to composite stress vector

Consider an arbitrary part of the composite laminate which occupies a region $\mathcal{P}$ in the present configuration at time t. Let $\mathcal{P}$ be divided into two regions $\mathcal{P}_1, \mathcal{P}_2$ separated by a surface, say σ (see figure 5). Further, let $\partial\mathcal{P}, \partial\mathcal{P}_1, \partial\mathcal{P}_2$ refer to the boundaries of $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$, respectively; and let

$$\partial\overline{\mathcal{P}_1} = \partial\mathcal{P}_1 \cap \partial\mathcal{P} \quad , \quad \partial\overline{\mathcal{P}_2} = \partial\mathcal{P}_2 \cap \partial\mathcal{P} \tag{21.1}$$

Thus, a summary of the above description is as follows:

$$\begin{aligned} &\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \quad , \quad \partial\mathcal{P} = \partial\overline{\mathcal{P}_1} \cup \partial\overline{\mathcal{P}_2} \\ &\partial\mathcal{P}_1 = \partial\overline{\mathcal{P}_1} \cup \sigma \quad , \quad \partial\mathcal{P}_2 = \partial\overline{\mathcal{P}_2} \cup \sigma \end{aligned} \tag{21.2}$$

We recall the principle of linear momentum in the form

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\mathbf{b}dv + \int_{\partial\mathcal{P}} \mathbf{t}(\mathbf{n})da \tag{21.3}$$

The above holds for any arbitrary part of the body including $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}$. Application of (21.3) to $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}$ yields

$$\int_{\mathcal{P}_1} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}_1} \rho\mathbf{b}dv + \int_{\partial\mathcal{P}_1} \mathbf{t}(\mathbf{n})da \tag{21.4}$$

$$\int_{\mathcal{P}_2} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}_2} \rho\,\mathbf{b}\,dv + \int_{\partial\mathcal{P}_2} \mathbf{t}(\mathbf{n})da \tag{21.5}$$

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\,\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}(\mathbf{n})da \tag{21.6}$$

We notice that if $\mathbf{n}$ is the outward unit normal at a point on σ when σ is a portion of $\partial\mathcal{P}_1$, the outward unit normal at the same point on σ when σ is a portion of $\partial\mathcal{P}_2$ is $-\mathbf{n}$. Subtracting (21.6) from the combination of (21.4) and (21.5), we obtain

$$\int_{\sigma} \{\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{n})\} da = 0 \tag{21.7}$$

over the arbitrary surface σ. Assuming that the composite stress vector is a continuous function of position and **n**, it follows that

$$\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{n}) = 0 \tag{21.8}$$

or

$$\mathbf{t}(\theta^i, t; \mathbf{n}) = -\mathbf{t}(\theta^i, t; -\mathbf{n}) \tag{21.9}$$

According to the result (21.9), **the composite stress vector acting on opposite sides of the same surface at a given point are equal in magnitude and opposite in direction**. This is the counterpart of Cauchy's lemma in the classical theory.

Consider an arbitrary part $\mathcal{P}$ of the composite laminate in the present configuration at time t which occupies the region $\mathcal{R}$ in the space covered by the coordinates θ^i. Consider some interior *macro-particle* P of $\mathcal{P}$ having the position vector **r** and the director **d** (note that the particles of the composite laminate are not like ordinary particles in the sense of classical continuum mechanics). We construct at P a curvilinear tetrahedron, lying entirely within $\mathcal{R}$, and in such a way that the side i (i = 1,2,3) is perpendicular to the coordinate direction θ^i and the inclined plane with outward unit normal **n** falls in the octant where $\theta^1,\theta^2,\theta^3$ are all positive. This means that the edges of the tetrahedron are formed by the coordinate curves PP_i of length ds_i (see figure 6). We refer to the side i of the tetrahedron by a_i and to the inclined plane by a, respectively. Now we recall that

$$\begin{aligned} ds_1 &= (\mathbf{g}_1 d\theta^1 \cdot \mathbf{g}_1 d\theta^1)^{1/2} = (\mathbf{g}_1 \cdot \mathbf{g}_1)^{1/2} d\theta^1 = (g_{11})^{1/2} d\theta^1 \\ ds_2 &= (\mathbf{g}_2 d\theta^2 \cdot \mathbf{g}_2 d\theta^2)^{1/2} = (\mathbf{g}_2 \cdot \mathbf{g}_2)^{1/2} d\theta^2 = (g_{22})^{1/2} d\theta^2 \\ ds_3 &= (\mathbf{g}_3 d\theta^3 \cdot \mathbf{g}_3 d\theta^3)^{1/2} = (\mathbf{g}_3 \cdot \mathbf{g}_3)^{1/2} d\theta^3 = (g_{33})^{1/2} d\theta^3 \end{aligned} \tag{21.10}$$

where

$$PP_i = g_i d\theta^i \quad \text{(no summation on i)} \tag{21.11}$$

Moreover,

$$da_1 = \frac{1}{2}\,|g_2 d\theta^2 \times g_3 d\theta^3| = \frac{1}{2}\,|g_2 \times g_3|\,d\theta^2 d\theta^3 = \frac{1}{2}\,|g^{1/2}\mathbf{g}^1|\,d\theta^2 d\theta^3$$

$$= \frac{1}{2}\,(g^{1/2})(\mathbf{g}^1 \cdot \mathbf{g}^1)^{1/2} d\theta^2 d\theta^3 = \frac{1}{2}\,(gg^{11})^{1/2} d\theta^2 d\theta^3 \tag{21.12}$$

In vector form we may write

$$\mathbf{da}_1 = \frac{1}{2}\,(g_2 d\theta^2 \times g_3 d\theta^3) = \frac{1}{2}\,g^{1/2} d\theta^2 d\theta^3 \mathbf{g}^1 = (g^{11})^{-1/2} ds_1 \mathbf{g}^1 \tag{21.13}$$

Similarly we obtain

$$da_2 = \frac{1}{2}\,(gg^{22})^{1/2} d\theta^1 d\theta^3$$

$$\mathbf{da}_2 = (g^{22})^{-1/2} ds_2 \mathbf{g}^2 \tag{21.14}$$

and

$$da_3 = \frac{1}{2}\,(gg^{33})^{1/2} d\theta^1 d\theta^2$$

$$\mathbf{da}_3 = (g^{33})^{-1/2} ds_3 \mathbf{g}^3 \tag{21.15}$$

For the inclined surface we have

$$da = \frac{1}{2}\,|P_3P_1 \times P_3P_2| = \frac{1}{2}\,|(g_1 d\theta^1 - g_3 d\theta^3) \times (g_2 d\theta^2 - g_3 d\theta^3)|$$

$$= \frac{1}{2}\,|(g_1 \times g_2) d\theta^1 d\theta^2 - (g_1 \times g_3) d\theta^1 d\theta^3 - (g_3 \times g_2) d\theta^2 d\theta^3|$$

$$= \frac{1}{2}\,|(g)^{1/2}\mathbf{g}^3 d\theta^1 d\theta^2 + (g)^{1/2}\mathbf{g}^2 d\theta^1 d\theta^3 + (g)^{1/2}\mathbf{g}^1 d\theta^2 d\theta^3|$$

$$= \frac{1}{2}\,|(g^{11})^{-1/2} ds_1 \mathbf{g}^1 + (g^{22})^{-1/2} ds_2 \mathbf{g}^2 + (g^{33})^{-1/2} ds_3 \mathbf{g}^3|$$

$$= |\mathbf{da}_1 + \mathbf{da}_2 + \mathbf{da}_3| \tag{21.16}$$

In vector form we may write

$$\mathbf{da} = (da)\mathbf{n} = (da)n_i\mathbf{g}^i = (da)(n_1\mathbf{g}^1) + (da)(n_2\mathbf{g}^2) + (da)(n_3\mathbf{g}^2)$$

$$= (g^{11})^{-1/2}da_1\mathbf{g}^1 + (g^{22})^{-1/2}da_2\mathbf{g}^2 + (g^{33})^{-1/2}da_3\mathbf{g}^3 \qquad (21.17)$$

where $\mathbf{n} = n_i\mathbf{g}^i$ is outward unit normal to da. From (21.17) we may write

$$da_i = n_i(g^{ii})^{1/2}\, da \quad \text{(no summation on i)} \qquad (21.18)$$

We also recall that the volume of the tetrahedron is given by

$$dv = (dh)\mathbf{n} \cdot \mathbf{da} = dh(n^i\mathbf{g}_i) \cdot ((da)n_j\mathbf{g}^j) = (dh)(da)n^i n_j \cdot \delta^i{}_j = (dh)(da)n^i n_i \qquad (21.19)$$

where dh is the height of the tetrahedron.

We now recall the principle of linear momentum in the form of $(20.21)_b$ and apply it to the tetrahedron under consideration; hence, we write

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\, \mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}\, da$$

or

$$\int_h \int_a \{\rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) - \rho\mathbf{b}\}dh\, da = \int_a \mathbf{t}\, da - \int_{a_1} \mathbf{t}_1 da_1 - \int_{a_2} \mathbf{t}_2 da_2 - \int_{a_3} \mathbf{t}_3 da_3$$

or

$$\int_h dh \int_a \{\rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) - \rho\mathbf{b}\}da = \int_a \mathbf{t}\, da - \int_a \mathbf{t}_1 n_1 (g^{11})^{1/2} da$$

$$- \int_a \mathbf{t}_2 n_2 (g^{22})^{1/2} da - \int_a \mathbf{t}_3 n_3 (g^{33})^{1/2} da$$

or

$$\int_h dh \int_a \{\rho(\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) - \rho\mathbf{b}\}da = \int_a \{\mathbf{t} - \sum_{i=1}^{i=3} \mathbf{t}_i n_i (g^{ii})^{1/2}\}da \qquad (21.20)$$

where in obtaining (21.20) we have made use of (21.18).

If we let $h \to 0$, we obtain

$$\int_a \{\mathbf{t} - \sum_{i=1}^{i=3} \mathbf{t}_i n_i (g^{ii})^{1/2}\} da = 0 \tag{21.21}$$

Since this must hold for any arbitrary surface a, we conclude

$$\mathbf{t} = \sum_{i=1}^{i=3} \mathbf{t}_i n_i (g^{ii})^{1/2} \tag{21.22}$$

Since under general transformation of coordinates, $\mathbf{t}$ is an invariant and n_i is a covariant vector, it then follows that $\mathbf{t}_i(g^{ii})^{1/2}$ transforms according to a contravariant type of transformation. We may therefore write

$$\mathbf{t}_i(g^{ii})^{1/2} = \tau^{ij}\mathbf{g}_j = \tau^i{}_j\mathbf{g}^j \tag{21.23}$$

where τ^{ij} and $\tau^i{}_j$ are contravariant and mixed components of the second order tensor which we call the *composite assigned stress tensor* or simply the *composite stress tensor.* Combining (21.22) and (21.23), we can now write

$$\mathbf{t} = \tau^{ij} n_i \mathbf{g}_j = g^{-1/2}\, \mathbf{T}^i\, n_i \tag{21.24}$$

where we have made use of (19.1). We also notice that by (21.22) and (21.24) we have

$$\mathbf{T}^i = (gg^{ii})^{1/2}\mathbf{t}_i \quad \text{(no summation on i)} \tag{21.25}$$

and

$$\mathbf{T}^i = g^{1/2}\, \tau^{ij}\, \mathbf{g}_j \tag{21.26}$$

b) Existence of the composite couple stress tensor and its relation to composite couple stress vector.

We now recall the conservation of director momentum in the form

$$\int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}(\mathbf{n})da \tag{21.27}$$

This holds for any arbitrary part of the body. With reference to figure (5) we apply (21.27) to $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}$ to obtain

$$\int_{\mathcal{P}_1} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}_1} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}_1} \mathbf{s}(\mathbf{n})da \tag{21.28}$$

$$\int_{\mathcal{P}_2} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}_2} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}_2} \mathbf{s}(\mathbf{n})da \tag{21.29}$$

$$\int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}(\mathbf{n})da \tag{21.30}$$

Recalling the remark made after (21.6), we subtract (21.30) from the combination of (21.28) and (21.29) to obtain

$$\int_{\sigma} \{\mathbf{s}(\mathbf{n}) + \mathbf{s}(-\mathbf{n})\}da = 0 \tag{21.31}$$

over the arbitrary surface σ. Assuming that the composite contact couple $\mathbf{s}$ is a continuous function of position and $\mathbf{n}$, it follows that

$$\mathbf{s}(\mathbf{n}) = -\mathbf{s}(-\mathbf{n}) \tag{21.32}$$

or

$$\mathbf{s}(\theta^i, t\, ; \mathbf{n}) = -\mathbf{s}(\theta^i, t\, ; -\mathbf{n}) \tag{21.33}$$

According to the result (21.33), **the composite couple stress (contact couple) vector acting on opposite sides of the same surface at a given point are equal in magnitude and opposite in direction.**

We recall the principle of director momentum in the form $(20.21)_c$ and apply to the tetrahedron in figure (6); hence, we write

$$\int_{\mathcal{P}} \rho(y^1\dot{v} + y^2\dot{w})dv = \int_{\mathcal{P}} (\rho c - k)dv + \int_{\partial\mathcal{P}} s da$$

or

$$\int_h \int_a \{\rho(\dot{v} + y^1\dot{w}) - (\rho c - k)\}dh\, da = \int_a s\, da - \int_{a_1} s_1 da_1 - \int_{a_2} s_2 ds_2 - \int_{a_3} s_3 ds_3$$

or

$$\int_h dh \int_a \{\rho(\dot{v} + y^1\dot{w}) - (\rho c - k)\}da = \int_a s da - \int_a s_1 n_1 (g^{11})^{1/2} da$$

$$- \int_a s_2 n_2 (g^{22})^{1/2} da - \int_a s_3 n_3 (g^{33})^{1/2} da$$

or

$$\int_h dh \int_a \{\rho(\dot{v} + y^1\dot{w}) - (\rho c - k)\}da = \int_a \{s - \sum_{i=1}^{i=3} s_i n_i (g^{ii})^{1/2}\}da \tag{21.34}$$

where in obtaining (21.34) we have made use of (21.18). If we let $h \to 0$, we obtain

$$\int_a \{s - \sum_{i=1}^{i=3} s_i n_i (g^{ii})^{1/2}\}da = 0 \tag{21.35}$$

Since this must hold for any arbitrary surface we conclude

$$s = \sum_{i=1}^{i=3} s_i n_i (g^{ii})^{1/2} \tag{21.36}$$

Since under general transformation of coordinates, s is an invariant and n_i is a covariant vector, it then follows that $s_i(g^{ii})^{1/2}$ transforms according to a contravariant type of transformation. We may therefore write

$$s_i(g^{ii})^{1/2} = s^{ij}g_j = s^i{}_j g^j \tag{21.37}$$

where s^{ij} and $s^i{}_j$ are contravariant and mixed components of the second order tensor which we call the *composite assigned couple stress tensor* or simply the *composite couple stress tensor.* Combining (21.36) and (21.37), we can now write

$$s = s^{ij} n_i g_j = g^{-1/2} S^i n_i \tag{21.38}$$

where we have made use of (19.12). We also notice that by (21.35) and (21.38), we can now write

$$S^i = (g g^{ii})^{1/2} s_i \quad \text{(no summation on i)} \tag{21.39}$$

and

$$S^i = g^{1/2} s^{ij} g_j \tag{21.40}$$

Recalling (19.25), we have

$$g^{1/2} s^{3j} g_j = 0$$

Since $g \neq 0$ and since g_j are linearly independent vectors, we arrive at

$$s^{3j} = 0 \tag{21.41}$$

22. Basic field equations for composite laminates

In this section we derive the basic field equations for a composite lamiante from the conservation laws of section (20). The local form of the conservation mass has already been obtained and we just rerecord it here for completeness

$$\dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \tag{22.1}$$

In the rest of this section we shall be concerned with the remaining four conservation laws $(20.21)_b$ to $(20.21)_c$. First we deduce the basic field equations in vector form using an invariant vector notation, and subsequently we reduce these equations in terms of tensor components.

a) <u>General field equations in vector form:</u>

To start we consider the conservation of linear momentum $(20.21)_b$ and make use of (21.24) to write

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\, \mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}\, da$$

or

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\, \mathbf{b}\, dv + \int_{\partial\mathcal{P}} g^{-1/2}\mathbf{T}^i n_i da \tag{22.2}$$

This with the help of the divergence theorem, i.e.,

$$\int_{\mathcal{P}} \mathbf{T}^i{}_{|i}\, dv = \int_{\partial\mathcal{P}} \mathbf{T}^i n_i da \tag{22.3}$$

and the identity

$$g^{1/2}\mathbf{T}^i{}_{|i} = (g^{1/2}\mathbf{T}^i)_{,i} \tag{22.3}$$

may be reduced as follows:

$$\int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})dv = \int_{\mathcal{P}} \rho\, \mathbf{b}\, dv + \int_{\mathcal{P}} (g^{-1/2}\mathbf{T}^i)_{|i} dv = \int_{\mathcal{P}} \rho\, \mathbf{b}\, dv + \int_{\mathcal{P}} g^{-1/2}\, \mathbf{T}^i{}_{,i}\, dv$$

or

$$\int_{\mathcal{P}} \{ g^{-1/2}\, \mathbf{T}^i{}_{,i} + \rho\, \mathbf{b} - \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \} dv = 0 \tag{22.4}$$

Since this must hold for all arbitrary parts $\mathcal{P}$ of the composite laminate, under the usual continuity assumption, we obtain

$$g^{-1/2}\, \mathbf{T}^i{}_{,i} + \rho\, \mathbf{b} - \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) = 0$$

or

$$\mathbf{T}^i{}_{,i} + \rho\, g^{1/2}\, \mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \tag{22.5}$$

which is the local form of the conservation of linear momentum.

Next we consider the conservation of director momentum $(20.21)_c$ and make use of (21.38) to write

$$\int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}\, da = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} g^{-1/2}\, \mathbf{S}^i\, n_i da \tag{22.6}$$

Recalling the divergence theorem in the form

$$\int_{\mathcal{P}} \mathbf{S}^i{}_{|i}\, dv = \int_{\partial\mathcal{P}} \mathbf{S}^i n_i da \tag{22.7}$$

and the identity

$$g^{1/2}\, \mathbf{S}^i{}_{|i} = (g^{1/2}\mathbf{S}^i)_{,i} \tag{22.8}$$

we can reduce (22.6) as follows:

$$\int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})dv - \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv = \int_{\mathcal{P}} (g^{-1/2}\mathbf{S}^i)_{|i}\, dv = \int_{\mathcal{P}} g^{-1/2}\, \mathbf{S}^i{}_{,i}\, dv$$

or

$$\int_{\mathcal{P}} \{ g^{-1/2} \mathbf{S}^i{}_{,i} + (\rho \mathbf{c} - \mathbf{k}) - \rho(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \} dv = 0 \tag{22.9}$$

This is valid for all arbitrary parts $\mathcal{P}$ of the composite laminate; thus, under the usual continuity assumption we obtain

$$g^{-1/2} \mathbf{S}^i{}_{,i} + (\rho \mathbf{c} - \mathbf{k}) - \rho(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) = 0$$

or

$$\mathbf{S}^i{}_{,i} + g^{1/2}(\rho \mathbf{c} - \mathbf{k}) = \rho g^{1/2}(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \tag{22.10}$$

This is the local form of the conservation of director momentum.

Considering the conservation of moment of momentum $(20.21)_d$, making use of the divergence theorem and the tensor identity of the form (22.3), we may write

$$\int_{\mathcal{P}} \rho \{ \mathbf{r} \times (\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) + \mathbf{d} \times (y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \} dv - \int_{\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c}) dv = \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s}) da$$

$$= \int_{\partial\mathcal{P}} \mathbf{r} \times g^{-1/2} \mathbf{T}^i n_i da + \int_{\partial\mathcal{P}} \mathbf{d} \times g^{-1/2} \mathbf{S}^i n_i da$$

$$= \int_{\mathcal{P}} (g^{-1/2} \mathbf{r} \times \mathbf{T}^i)_{|i} dv + \int_{\mathcal{P}} (g^{-1/2} \mathbf{d} \times \mathbf{S}^i)_{|i} dv$$

$$= \int_{\mathcal{P}} g^{-1/2} (\mathbf{r} \times \mathbf{T}^i)_{,i} dv + \int_{\mathcal{P}} g^{-1/2} (\mathbf{d} \times \mathbf{S}^i)_{,i} dv$$

or

$$\int_{\mathcal{P}} \{ g^{-1/2} [(\mathbf{r} \times \mathbf{T}^i)_{,i} + (\mathbf{d} \times \mathbf{S}^i)_{,i}] + \rho[(\mathbf{r} \times \mathbf{b}) + (\mathbf{d} \times \mathbf{c})] - \rho[\mathbf{r} \times (\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) +$$

$$\mathbf{d} \times (y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}})] \} dv = 0 \tag{22.11}$$

This must hold for all arbitrary parts $\mathcal{P}$ of the composite laminate; hence, assuming the usual continuity assumption we may write

$$g^{-1/2}[(\mathbf{r} \times \mathbf{T}^i)_{,i} + (\mathbf{d} \times \mathbf{S}^i)_{,i}] + \rho[(\mathbf{r} \times \mathbf{b}) + (\mathbf{d} \times \mathbf{c})] - \rho[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) +$$

$$\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] = 0 \qquad (22.12)$$

This may be further reduced as follows:

$$\{(\mathbf{r}_{,i} \times \mathbf{T}^i + \mathbf{r} \times \mathbf{T}^i_{,i}) + \rho g^{1/2}(\mathbf{r} \times \mathbf{b}) - \rho g^{1/2}\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})\} +$$

$$\{(\mathbf{d}_{,i} \times \mathbf{S}^i + \mathbf{d} \times \mathbf{S}^i_{,i}) + \rho g^{1/2}(\mathbf{d} \times \mathbf{c}) - \rho g^{1/2}\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})\} = 0$$

or

$$(\mathbf{r}_{,i} \times \mathbf{T}^i) + (\mathbf{d}_{,i} \times \mathbf{S}^i) + \mathbf{r} \times \{\mathbf{T}^i_{,i} + \rho g^{1/2}\mathbf{b} - \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})\} + \mathbf{d}_{,i} \times \mathbf{S}^i +$$

$$\mathbf{d} \times \{\mathbf{S}^i_{,i} + g^{1/2}(\rho\mathbf{c} - \mathbf{k}) - \rho g^{1/2}(y^2\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})\} + g^{1/2}\mathbf{d} \times \mathbf{k} = 0 \qquad (22.13)$$

Making use of (22.5) and (22.10), we obtain

$$\mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,i} \times \mathbf{S}^i + g^{1/2}\,\mathbf{d} \times \mathbf{k} = 0 \qquad (22.14)$$

where we have also used the fact that $\mathbf{r}_{,i} = \mathbf{g}_i$. Expression (22.14) is the local form of the conservation of moment of momentum and is the counterpart of the equation for symmetry of stress in the classical continuum mechanics.

Finally, we consider the conservation of energy $(20.21)_e$ and following the same procedure we obtain the local form of the principle of balance of energy as follows

$$\int_{\mathcal{P}} \rho(\dot{\mathcal{E}} + \dot{\mathcal{K}})\mathrm{d}v = \int_{\mathcal{P}} \rho(\mathbf{b}\cdot\mathbf{v} + \mathbf{c}\cdot\mathbf{w})\mathrm{d}v + \int_{\partial\mathcal{P}} (\mathbf{t}\cdot\mathbf{v} + \mathbf{s}\cdot\mathbf{w})\mathrm{d}a \qquad (22.15)$$

We consider each term of (22.15) individually

$$\int_{\partial\mathcal{P}}(\mathbf{t}\cdot\mathbf{v}+\mathbf{s}\cdot\mathbf{w})da=\int_{\partial\mathcal{P}}g^{-1/2}\,\mathbf{T}^i\cdot\mathbf{v}\,n_i\,da+\int_{\partial\mathcal{P}}g^{-1/2}\,\mathbf{S}^i\cdot\mathbf{w}\,n_i\,da$$

$$=\int_{\mathcal{P}}(g^{-1/2}\,\mathbf{T}^i\cdot\mathbf{v})_{|i}dv+\int_{\mathcal{P}}(g^{-1/2}\,\mathbf{S}^i\cdot\mathbf{w})_{|i}dv$$

$$=\int_{\mathcal{P}}g^{-1/2}(\mathbf{T}^i\cdot\mathbf{v})_{,i}dv+\int_{\mathcal{P}}g^{-1/2}(\mathbf{S}^i\cdot\mathbf{w})_{,i}dv$$

$$=\int_{\mathcal{P}}g(\mathbf{T}^i_{,i}\mathbf{v}+\mathbf{T}^i\cdot\mathbf{v}_{,i})dv+\int_{\mathcal{P}}g^{-1/2}(\mathbf{S}^i_{,i}\cdot\mathbf{w}+\mathbf{S}^i\cdot\mathbf{w}_{,i})dv \qquad (22.16)$$

where we have made use of the divergence theorem. Moreover,

$$\int_{\mathcal{P}}\rho(\dot{\mathcal{E}}+\dot{\mathcal{K}})dv=\int_{\mathcal{P}}\rho\,\dot{\mathcal{E}}dv+\int_{\mathcal{P}}\frac{1}{2}\,\overline{\rho(\mathbf{v}\cdot\mathbf{v}+2y^1\mathbf{v}\cdot\mathbf{w}+y^2\mathbf{w}\cdot\mathbf{w})}^{\cdot}dv$$

$$=\int_{\mathcal{P}}\rho\,\dot{\mathcal{E}}dv+\int_{\mathcal{P}}\rho(\mathbf{v}\cdot\dot{\mathbf{v}}+y^1\dot{\mathbf{v}}\cdot\mathbf{w}+y^1\mathbf{v}\cdot\dot{\mathbf{w}}+y^2\mathbf{w}\cdot\dot{\mathbf{w}})dv$$

$$=\int_{\mathcal{P}}\rho\,\dot{\mathcal{E}}dv+\int_{\mathcal{P}}\rho[(\dot{\mathbf{v}}+y^1\dot{\mathbf{w}})\cdot\mathbf{v}+(y^1\dot{\mathbf{v}}+y^2\dot{\mathbf{w}})\cdot\mathbf{w}]dv \qquad (22.17)$$

Substituting (22.16) and (22.17) into (22.15), we obtain

$$\int_{\mathcal{P}}\rho\,\dot{\mathcal{E}}dv+\int_{\mathcal{P}}\rho(\dot{\mathbf{v}}+y^1\dot{\mathbf{w}})\cdot\mathbf{v}dv+\int_{\mathcal{P}}\rho(y^1\dot{\mathbf{v}}+y^2\dot{\mathbf{w}})\cdot\mathbf{w}\,dvs=\int_{\mathcal{P}}\rho(\mathbf{b}\cdot\mathbf{v}+\mathbf{c}\cdot\mathbf{w})dv$$

$$+\int_{\mathcal{P}}g^{-1/2}(\mathbf{T}^i_{,i}\cdot\mathbf{v}+\mathbf{T}^i\cdot\mathbf{v}_{,i})dv$$

$$+\int_{\mathcal{P}}g^{-1/2}(\mathbf{S}^i_{,i}\cdot\mathbf{w}+\mathbf{S}^i\cdot\mathbf{w}_{,i})dv$$

or

$$\int_{\mathcal{P}}\rho\,\dot{\mathcal{E}}dv=\int_{\mathcal{P}}g^{-1/2}\{\mathbf{T}^i_{,i}+\rho g^{1/2}\mathbf{b}-\rho g^{1/2}(\dot{\mathbf{v}}+y^1\dot{\mathbf{w}})\}\cdot\mathbf{v}\,dv+\int_{\mathcal{P}}g^{-1/2}\,\mathbf{T}^i\cdot\mathbf{v}_{,i}\,dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\{\mathbf{S}^i_{,i}+g^{1/2}(\rho\mathbf{c}-\mathbf{k})-\rho g^{1/2}(y^1\dot{\mathbf{v}}+y^2\dot{\mathbf{w}})\}\cdot\mathbf{w}\,dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\mathbf{S}^i\cdot\mathbf{w}_n\cdot dv+\int_{\mathcal{P}}\mathbf{k}\cdot\mathbf{w}\,dv \qquad (22.18)$$

We now make use of (22.5) and (22.10) to obtain

$$\int_{\mathcal{P}} \{\rho\dot{\mathcal{E}} - g^{-1/2}\,\mathbf{T}^i \cdot \mathbf{v}_{,i} - g^{-1/2}\,\mathbf{S}^i \cdot \mathbf{w}_{,i} - \mathbf{k} \cdot \mathbf{w}\}dv = 0 \tag{22.19}$$

From (22.19) by the usual line of reasoning we obtain

$$\rho g^{1/2}\dot{\mathcal{E}} = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\,\mathbf{k} \cdot \mathbf{w} = g^{1/2}P \tag{22.20}$$

where P represents the mechanical power per element of volume of the composite laminate and is given by

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} \tag{22.21}$$

For future convenience, we summarize below the invariant vector form of the basic field equations:

$$\begin{aligned}
&a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\,\rho = 0 \\
&b: \quad \mathbf{T}^i{}_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \\
&c: \quad \mathbf{S}^i{}_{,i} + g^{1/2}(\rho\mathbf{c} - \mathbf{k}) = \rho g^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \\
&d: \quad \mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,i} \times \mathbf{S}^i + g^{1/2}\mathbf{d} \times \mathbf{k} = 0 \\
&e: \quad \rho g^{1/2}\dot{\mathcal{E}} = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} = g^{1/2}P
\end{aligned} \tag{22.22}$$

where P is given by (22.21).

b) Alternative form of the field equations

The basic field equations (22.22) are both simple and elegant in form. In practice, we usually work with the components of the various fields. Hence, we now proceed to deduce the basic field equations in tensor components. We introduce the contravariant and covariant components of acceleration (α^i,α_i), director acceleration (β^i,β_i), body force (b^i,b_i), body couple (c^i,c_i), and those of intrinsic force (k^i,k_i) as follows:

$$\dot{\mathbf{v}} = \alpha^i \mathbf{g}^i = \alpha_i \mathbf{g}^i \quad , \quad \dot{\mathbf{w}} = \beta^i \mathbf{g}_i = \beta_i \mathbf{g}^i$$

$$\mathbf{b} = b^i \mathbf{g}_i = b_i \mathbf{g}^i \quad , \quad \mathbf{c} = c^i \mathbf{g}_i = c_i \mathbf{g}^i \quad , \quad \mathbf{k} = k^i \mathbf{g}_i = k_i \mathbf{g}^i \tag{22.23}$$

Sustituting $(22.23)_{1,2,3}$ into $(22.22)_b$ and making use of (21.26), i.e.,

$$\mathbf{T}^i = g^{1/2}\, \tau^{ij} \mathbf{g}_j$$

we may write

$$(g^{1/2}\tau^{ij}\mathbf{g}_j)_{,i} + \rho g^{1/2} b^j \mathbf{g}_j = \rho g^{1/2}(\alpha^j \mathbf{g}_j + y^1 \beta^j \mathbf{g}_j)$$

or

$$g^{1/2}\tau^{ij}{}_{,i}\mathbf{g}_j + g^{1/2}\tau^{ij}\mathbf{g}_{j,i} + (g^{1/2})_{,i}\tau^{ij}\mathbf{g}_j + \rho g^{1/2} b^j \mathbf{g}_j = \rho g^{1/2}(\alpha^j \mathbf{g}_j + y^1 \beta^j \mathbf{g}_j) \tag{22.24}$$

Recall the following relations from tensor analysis

$$\mathbf{g}_{j,i} = \{{}_i{}^k{}_j\}\mathbf{g}_k \tag{22.25}$$

and

$$(g^{1/2})_{,i} = \frac{1}{2}\, g^{-1/2} g_{,i} = \{{}_m{}^m{}_i\} g^{1/2} \tag{22.26}$$

where $\{{}_j{}^i{}_k\}$ denotes the Christoffel symbol of the second kind. Then substitute (22.25) and (22.26) in (22.24) to obtain

$$g^{1/2}\tau^{ij}{}_{,i}\mathbf{g}_j + g^{1/2}\tau^{ij}\,\{{}_i{}^k{}_j\}\,\mathbf{g}_k + g^{1/2}\tau^{ij}\,\{{}_m{}^m{}_i\}\,\mathbf{g}_j + \rho g^{1/2} b^j \mathbf{g}_j = \rho g^{1/2}(\alpha^j + y^1\beta^j)\mathbf{g}_j$$

or

$$(\tau^{ij}{}_{,i} + \tau^{ik}\,\{{}_i{}^j{}_k\} + \tau^{ij}\,\{{}_m{}^m{}_i\}\,)\mathbf{g}_j + \rho b^j \mathbf{g}_j = \rho(\alpha^j + y^1\beta^j)\mathbf{g}_j \tag{22.27}$$

Since the base vectors $\mathbf{g}_j$ are linearly independent and since

$$\tau^{ij}{}_{|i} = \tau^{ij}{}_{,i} + \{{}_i{}^m{}_m\}\, \tau^{ij} + \{{}_k{}^j{}_i\}\, \tau^{ik} \tag{22.28}$$

we obtain

$$\tau^{ij}{}_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \tag{22.29}$$

This is the local form of the principle of linear momentum in component form. In terms of the various covariant and mixed components (22.29) takes the form

$$\tau^i{}_{j|i} + \rho b_j = \rho(\alpha_j + y^1\beta_j) \tag{22.30}$$

where $\tau^i{}_j$ is the mixed component of the composite stress tensor.

Next we consider $(22.22)_c$ and make use of (22.23) and (21.38) to write

$$(g^{1/2}s^{ij}\mathbf{g}_j)_{,i} + \rho g^{1/2}c^j\mathbf{g}_j - g^{1/2}k^j\mathbf{g}_j = \rho g^{1/2}(y^1\alpha^j\mathbf{g}_j + y^2\beta^j\mathbf{g}_j)$$

or

$$g^{1/2}s^{ij}{}_{,i}\mathbf{g}_j + g^{1/2}s^{ij}\mathbf{g}_{j,i} + (g^{1/2})_{,i}s^{ij}\mathbf{g}_j + \rho g^{1/2}c^j\mathbf{g}_j - k^j\mathbf{g}_j g^{1/2} = \rho g^{1/2}(y^1\alpha^j + y^2\beta^j)\mathbf{g}_j \tag{22.31}$$

Making use of (22.25) and (22.26) in (22.31) we obtain after simplification

$$(s^{ij}{}_{,i} + s^{ik}\,\{{}_i{}^j{}_k\} + s^{ij}\{{}_m{}^m{}_i\}\,)\mathbf{g}_j + (\rho c^j - k^j)\mathbf{g}_j = \rho(y^1\alpha^j + y^2\beta^j)\mathbf{g}_j \tag{22.32}$$

Since the base vectors $\mathbf{g}_j$ are linearly independent and since

$$s^{ij}{}_{|i} = s^{ij}{}_{,i} + s^{ik}\,\{{}_i{}^j{}_k\} + s^{ij}\,\{{}_m{}^m{}_i\} \tag{22.33}$$

we obtain

$$s^{ij}{}_{|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \tag{22.34}$$

This is the local form of the principle of director momentum in component form. In terms of the various covariant and mixed components (22.34) takes the following form

$$s_j^i|_i + (\rho c_j - k_j) = \rho(y^1\alpha^j + y^2\beta_j) \tag{22.35}$$

Let us now consider $(22.22)_d$ and make use of (21.26), (21.38), (22.23) and

$$\mathbf{d} = d^i\mathbf{g}_i = d_i\mathbf{g}^i$$

to write

$$\mathbf{g}_i \times (g^{1/2}\tau^{ij}\mathbf{g}_j) + (d^m\mathbf{g}_m)_{,i} \times (g^{1/2}s^{ij}\mathbf{g}_j) + g^{1/2}(d^i\mathbf{g}_i) \times (k^j\mathbf{g}_j) = 0$$

or

$$\varepsilon_{ijn}\tau^{ij}\mathbf{g}^n + (d^m{}_{,i}\mathbf{g}_m + d^m\mathbf{g}_{m,i}) \times (s^{ij}\mathbf{g}_j) + \varepsilon_{ijn}d^ik^j\mathbf{g}^n = 0$$

or

$$\varepsilon_{ijn}\tau^{ij}\mathbf{g}^n + d^i{}_{,m}s^{mj}\mathbf{g}_i \times \mathbf{g}_j + d^is^{mj}\mathbf{g}_{i,m} \times \mathbf{g}_j + \varepsilon_{ijn}d^ik^j\mathbf{g}^n = 0 \tag{22.36}$$

where ε_{ijn} denote the permutation symbol in 3 dimensions. Taking advantage of (22.25), we proceed to simplify (22.36) as follows:

$$\varepsilon_{ijn}\tau^{ij}\mathbf{g}^n + \{\varepsilon_{ijn}d^i{}_{,m}s^{mj}\mathbf{g}^n + s^{mj}d^i \left\{{}_m{}^n{}_i\right\} \mathbf{g}_n \times \mathbf{g}_j\} + \varepsilon_{ijn}d^ik^n\mathbf{g}^n = 0$$

or

$$\varepsilon_{ijn}\tau^{ij}\mathbf{g}^n + \{\varepsilon_{ijn}d^i{}_{,m}s^{mj}\mathbf{g}^n + s^{mj}d^n \left\{{}_m{}^i{}_n\right\} \mathbf{g}_i \times \mathbf{g}_j\} + \varepsilon_{ijn}d^ik^j\mathbf{g}^n = 0$$

or

$$\varepsilon_{ijn}\{\tau^{ij} + d^i{}_{|m}s^{mj} + d^ik^j\}\mathbf{g}^n = 0 \tag{22.37}$$

Since the base vectors $\mathbf{g}^n$ are independent, we obtain

$$\varepsilon_{ijn}(\tau^{ij} + d^i{}_{|m}s^{mj} + d^ik^j) = 0 \tag{22.38}$$

This condition is the consequence of the moment of momentum principle. Since ε_{ijn} is skew-symmetric with respect to i and j, it follows that the quantity in the parentheses in (22.37) must be symmetric with respect to i and j. Hence the quantity

$$\mathcal{T}^{ij} = \tau^{ij} + d^i{}_{|m}s^{mj} + d^ik^j = \tau^{ji} + d^j{}_{|m}s^{mi} + d^jk^i \qquad (22.39)$$

is symmetric. We call $\mathcal{T}^{ij}$ the *composite assigned symmetric tensor* or simply the *composite symmetric tensor*. We notice that in the absence of the director, i.e.,

$$\mathbf{d} = 0 \quad \text{or} \quad d^i = 0$$

the composite symmetric tensor $\mathcal{T}^{ij}$ reduces to the classical symmetric tensor.

Finally, we consider $(22.22)_e$ and by making use of (22.26), (22.38) and (22.23) reduce it as follows:

$$\rho g^{1/2}\dot{\mathcal{E}} = g^{1/2}\tau^{ij}\mathbf{g}_j \cdot \mathbf{v}_{,i} + g^{1/2}s^{ij}\mathbf{g}_j \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w}$$

or

$$\rho\dot{\mathcal{E}} = \tau^{ij}\mathbf{g}_j \cdot \mathbf{v}_{,i} + s^{ij}\mathbf{g}_j \cdot \mathbf{w}_{,i} + \mathbf{k} \cdot \mathbf{w} = P \qquad (22.40)$$

where P is now given by

$$P = \tau^{ij}\mathbf{g}_j \cdot \mathbf{v}_{,i} + s^{ij}\mathbf{g}_j \cdot \mathbf{w}_{,i} + \mathbf{k} \cdot \mathbf{w} \qquad (22.41)$$

Since

$$\mathbf{v}_{,i} = v^m{}_{|i}\mathbf{g}_m \qquad (22.42)$$

and

$$\mathbf{w}_{,i} = w^m{}_{|i}\mathbf{g}_m \qquad (22.43)$$

we can further reduce (22.40):

$$\rho\dot{\mathcal{E}} = \tau^{ij}\mathbf{g}_j \cdot (v^m{}_{|i}\mathbf{g}_m) + s^{ij}\mathbf{g}_j \cdot (w^m{}_{|i}\mathbf{g}_m) + (k^j\mathbf{g}_j) \cdot (w^m\mathbf{g}_m)$$

or

$$\rho\dot{\mathcal{E}} = \tau^{ij}v^m{}_{|i}g_{jm} + s^{ij}w^m{}_{|i}g_{jm} + k^jw^mg_{jm}$$

or

$$\rho\dot{\mathcal{E}} = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^iw_i = P \tag{22.44}$$

This is the local form of the principle of balance of energy in component form. The expression for mechanical power is now reduced to

$$P = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^iw_i \tag{22.45}$$

For later convenience we collect the component form of the field equations as follows:

$$\begin{aligned}
&a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \\
&b: \quad \tau^{ij}{}_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \\
&c: \quad s^{ij}{}_{|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \\
&d: \quad \varepsilon_{ijn}(\tau^{ij} + d^i{}_{|m}s^{mj} + d^ik^j) = 0 \\
&e: \quad \rho\dot{\mathcal{E}} = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^iw_i = P
\end{aligned} \tag{22.46}$$

where P is given by (22.45).

D. Elastic composite laminates (nonlinear theory)

This part is concerned mainly with the development of constitutive relations for elastic composite laminates, both by direct approach and from three-dimensional equations. Here we discuss the nonlinear constitutive relation and recapitulate the complete theory. While we confine our attention here to elastic composite laminates, it should be noted that the previous developments in part C are not limited to elastic laminated composites.

23. Constitutive equations for nonlinear elastic composite laminates. Direct approach.

Within the scope of the theory developed in part C, we discuss the constitutive relations for elastic composite laminates in the presence of finite deformation and in the context of purely mechanical theory.

We recall that a material is defined by a *constitutive* assumption which characterizes the mechanical behavior of the medium. The constitutive assumption places a restriction on the processes which are admissible in a body — here the composite laminate.

We define an elastic composite laminate by a set of response functions which, in the context of purely mechanical theory, depend on appropriate kinematic variables. In our present discussion the set of response functions consists of the following functions

$$\mathbf{T}^i \;,\; \mathbf{S}^i \;,\; \mathbf{k} \tag{23.1}$$

or an equivalent set

$$\tau^{ij}(\text{or } \tau^{ij}) \;,\; s^{ij} \;,\; k^i \tag{23.2}$$

We introduce constitutive relations which must hold at each composite material point (macro particle) and for all time (t) in terms of the response functions (23.1). In this connection, we recall that the displacement function $\mathbf{r}$ in (12.1) is the place occupied by the *composite particle* P (with coordinates θ^i) in the present configuration, and the function **d** in (12.1) is the director, at the same composite material point, representing the effect of micro-structure. Thus the local state of an elastic composite laminate can be defined by functions **r** and **d** and their gradients at each composite material point in the present configuration, namely

$$\mathbf{r} \ , \ \mathbf{r}_{,i} \ , \ \mathbf{d} \ , \ \mathbf{d}_{,i}$$

At this point, for convenience we recall the expression for mechanical power P, i.e.,

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} \tag{23.3}$$

or equivalently

$$P = \tau^{ij} v_{j|i} + s^{ij} w_{j|i} + k^i w_i \tag{23.4}$$

We continue our discussion by assuming the existence of a **strain energy** or stored energy $\psi = \psi(\theta^i,t)$ per unit mass ρ of the composite laminate such that $\rho\dot{\psi}$ is equal to the mechanical power defined by (23.4), i.e.,

$$P = \rho\dot{\psi} \tag{23.5}$$

In the development of nonlinear constitutive equations for elastic composite lamiantes, we assume that the strain energy density ψ at each material point of the composite laminate (macro-particle) and for all t is specified by a response function which depends on **r**, **d** and their partial derivatives with respect to θ^i. Hence, the constitutive relation for the composite strain energy density may be stated as

$$\psi = \psi(\mathbf{r},\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,i} \ ; \ X) \tag{23.6}$$

Since the response function must remain unaltered under superposed rigid body translational displacement, the dependence on **r** must be excluded. In addition, we have already shown that $\mathbf{S}^3$ vanishes identically. Therefore, the constitutive assumption for the strain energy density of the composite laminate can now be written as

$$\psi = \bar{\psi}(\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,\alpha}\,;\,X) \tag{23.7}$$

We also make similar constitutive assumptions for $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$. We make a note that in these constitutive equations, which represent the mechanical response of the medium, the dependence of the response function on the local geometrical properties of a reference state and material inhomogeneity is indicated through the argument X [1]. Here we limit the discussion to an elastic composite laminate which is homogeneous in its reference configuration and suppose that the dependence of the response functions on the properties of the reference state occurs through the values of the kinematical variables in the reference configuration. Therefore, in place of (23.7) we may write

$$\psi = \bar{\psi}(\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,\alpha}\,;\,\mathrm{R}_{,i},\mathbf{D},\mathbf{D}_{,\alpha}) \tag{23.8}$$

or

$$\psi = \bar{\psi}(\mathrm{g}_i,\mathbf{d},\mathbf{d}_{,i}\,;\,\mathbf{G}_i,\mathbf{D},\mathbf{D}_{,i}) \tag{23.9}$$

Since

$$\mathrm{g}_i = \mathbf{r}_{,i} \quad , \quad \mathrm{G}_i = \mathbf{R}_{,i} \tag{23.10}$$

Following the same argument, we can arrive at similar constitutive assumptions for $\mathbf{T}^i$,$\mathbf{S}^i$ and $\mathbf{k}$. From (23.9) we obtain [2]

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[1] See [Carroll and Naghdi, 1972].

[2] Operators of the form $\frac{\partial f}{\partial \mathbf{x}}$, where f is a scalar function of a vector variable occurring in (23.11) and elsewhere are defined as partial derivatives with respect to $\mathbf{x}$ satisfying

$$\lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{v}) - f(\mathbf{x})}{\varepsilon} = \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v}$$

for all values of the arbitrary vector v.

$$\dot{\psi} = \dot{\bar{\psi}} = \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}\cdot\dot{\mathbf{g}_i} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}}\cdot\dot{\mathbf{d}} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}}\cdot\dot{\mathbf{d}}_{,i} = \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}\cdot\mathbf{v}_{,i} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}}\cdot\mathbf{w} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}}\cdot\mathbf{w}_{,i} \tag{23.11}$$

Since

$$\dot{\mathbf{g}_i} = (\frac{\partial\mathbf{p}}{\partial\theta^i})^{\dot{}} = \frac{\partial}{\partial\theta^i}(\dot{\mathbf{p}}) = \frac{\partial\mathbf{v}}{\partial\theta^i} = \mathbf{v}_{,i} \tag{23.12}$$

and

$$\dot{\mathbf{d}} = \mathbf{w} \quad , \quad \dot{\mathbf{d}}_{,i} = \mathbf{w}_{,i} \tag{23.13}$$

Introducing (23.11) into $(22.22)_e$, we obtain

$$\rho g^{1/2}\{\frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}\cdot\mathbf{v}_{,i} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}}\cdot\mathbf{w} + \frac{\partial\bar{\psi}}{\mathbf{d}_{,i}}\cdot\mathbf{w}_{,i}\} = \mathbf{T}^i\cdot\mathbf{v}_{,i} + \mathbf{S}^i\cdot\mathbf{w}_{,i} + g^{1/2}\mathbf{k}\cdot\mathbf{w}$$

or

$$(\mathbf{T}^i - \rho g^{1/2}\frac{\partial\bar{\psi}}{\partial\mathbf{g}_i})\cdot\mathbf{v}_{,i} + (\mathbf{S}^i - \rho g^{1/2}\frac{\partial\bar{\psi}}{\partial\mathbf{d}_i})\cdot\mathbf{w}_{,i} + (g^{1/2}\mathbf{k} - \rho g^{1/2}\frac{\partial\bar{\psi}}{\partial\mathbf{d}})\cdot\mathbf{w} = 0 \tag{23.14}$$

This must hold for all arbitrary values of vectors $\mathbf{v}_{,i}$,$\mathbf{w}_{,i}$ and $\mathbf{w}$. Since the quantities in the parentheses are independent of $\mathbf{v}_{,i}$,,$\mathbf{w}_{,i}$ and $\mathbf{w}$, we conclude that

$$\mathbf{T}^i = \rho g^{1/2}\frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}$$

$$\mathbf{S}^i = \rho g^{1/2}\frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}} \tag{23.15}$$

$$g^{1/2}\mathbf{k} = \rho g^{1/2}\frac{\partial\bar{\psi}}{\partial\mathbf{d}}$$

These are the nonlinear constitutive equations for $\mathbf{T}^i$,$\mathbf{S}^i$ and $\mathbf{k}$ along with the condition

$$\mathbf{g}_i\times\mathbf{T}^i + \mathbf{d}_{,i}\times\mathbf{S}^i + g^{1/2}\,\mathbf{d}\times\mathbf{k} = 0 \tag{23.16}$$

which is imposed by the principle of the moment of momentum of the composite laminate and must be satisfied by the response function $\bar{\psi}$.

As with the equations of motion, it is convenient in applications to specific problems to obtain alternative forms of the above constitutive equations expressed in component forms. To obtain the appropriate constitutive equations for τ^{ij}, s^{ij} and k^i we proceed as follows.

Recall the formulas

$$\mathbf{r} = r^i \mathbf{g}_i = r_i \mathbf{g}^i \quad , \quad \mathbf{d} = d^i \mathbf{g}_i = d_i \mathbf{g}^i \tag{23.17}$$

$$\mathbf{g}_i = \mathbf{r}_{,i} = (r^m \mathbf{g}_m)_{,i} = r^m{}_{,i} \mathbf{g}_m + r^m \mathbf{g}_{m,i} = r^m{}_{,i} \mathbf{g}_m + r^m \{_i{}^n{}_m\} \mathbf{g}_n$$

$$= (r^m{}_{,i} + r^n \{_i{}^m{}_n\}) \mathbf{g}_m = r^m{}_{|i} \mathbf{g}_m = r_{m|i} \mathbf{g}^m \tag{23.18}$$

and

$$\mathbf{d}_{,i} = (d^m \mathbf{g}_m)_{,i} = d^m{}_{,i} \mathbf{g}_m + d^m \mathbf{g}_{m,i} = d^m{}_{,i} \mathbf{g}_m + r^n \{_i{}^m{}_n\} \mathbf{g}_m = d^m{}_{|i} \mathbf{g}_m = d_{m|i} \mathbf{g}^m \tag{23.19}$$

Substituting (23.17) to (23.19) into (23.9) and keeping in mind that ψ is a scalar valued function we may rewrite (23.9) as

$$\psi = \tilde{\psi}(r_{m|i}, d_m, d_{m|\alpha} ; R_{m|i}, D_m, D_{m|\alpha}) \tag{23.20}$$

where $\tilde{\psi}$ is now a different function than $\bar{\psi}$. From (23.20) by differentiation we obtain

$$\dot{\psi} = \dot{\tilde{\psi}} = \frac{\partial \tilde{\psi}}{\partial r_{m|i}} \overline{(r_{m|i})}^{\cdot} + \frac{\partial \tilde{\psi}}{\partial d_m} \dot{d}_m + \frac{\partial \tilde{\psi}}{\partial d_{m|i}} \overline{(d_{m|i})}^{\cdot} = \frac{\partial \tilde{\psi}}{\partial r_{m|i}} v_{m|i} + \frac{\partial \tilde{\psi}}{\partial d_m} v_m + \frac{\partial \tilde{\psi}}{\partial d_{m|i}} w_{m|i} \tag{23.21}$$

Substituting (23.21) into $(22.46)_e$, we obtain

$$\rho\{ \frac{\partial \tilde{\psi}}{\partial r_{m|i}} v_{m|i} + \frac{\partial \tilde{\psi}}{\partial d_m} v_m + \frac{\partial \tilde{\psi}}{\partial d_{m|i}} w_{m|i}\} = \tau^{ij} v_{d|i} + s^{ij} w_{j|i} + k^i w_i$$

or

$$(\tau^{ij} - \rho \frac{\partial \tilde{\psi}}{\partial r_{j|i}} \cdot v_{j|i} + (s^{ij} - \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}}) w_{j|i} + (k^i - \rho \frac{\partial \tilde{\psi}}{\partial d_i}) w_i = 0 \tag{23.22}$$

This must hold for all arbitrary values of $v_{j|i}$, $w_{j|i}$ and d_i. Since the quantities in the parentheses are independent of $v_{j|i}$, $w_{j|i}$ and d_i, we conclude that

$$\tau^{ij} = \rho \frac{\partial\tilde{\psi}}{\partial r_{j|i}}$$

$$s^{ij} = \rho \frac{\partial\tilde{\psi}}{\partial d_{j|i}} \tag{23.23}$$

$$k^i = \rho \frac{\partial\tilde{\psi}}{\partial d_i}$$

These are the component forms of the constitutive equations for τ^{ij}, s^{ij} and k^i along with the condition

$$\varepsilon_{ijn}\{\tau^{ij} + d^i{}_{|m}s^{mj} + d^ik^j\} = 0 \tag{23.24}$$

which is imposed by the principal of the moment of momentum of the composite laminate and must be satisfied by the response function $\tilde{\psi}$.

Before proceeding further, we obtain an alternative form of constitutive equations. To this end we consider the expression for mechanical power (23.3), i.e.,

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w}$$

and by taking advantage of the expressions (12.23), (21.26), we write

$$g^{1/2}P = (g^{1/2}\tau^{ij}g_j) \cdot [(\eta_{ki} + \omega_{ki})g^k] +$$

$$(g^{1/2}s^{ij}g_j) \cdot [\dot{\lambda}_{ki}g^k + \lambda l_i(\omega_{kl} - \eta_{kl})g^k] +$$

$$(g^{1/2}k^m g_m) \cdot [\dot{d}_k g^k + d^i(\omega_{ki} - \eta_{ki})g^k$$

$$= g^{1/2}\{\tau^{ij}\eta_{ki}(g_j \cdot g^k) + \tau^{ik}\omega_{ki}(g_j \cdot g^k) +$$

$$s^{ij}\dot{\lambda}_{ki}(g_j \cdot g^k) + s^{ij}\lambda l_i(\omega_{kl} - \eta_{kl})(g_j \cdot g^k) +$$

$$k^j\dot{d}_k(g_j \cdot g^k) + k^j d^i(\omega_{ki} - \eta_{ki})(g_j \cdot g^k)\}$$

or

$$g^{1/2}P = g^{1/2}\delta^k{}_j\{\tau^{ij}\eta_{ki} + \tau^{ij}\omega_{ki} + s^{ij}lbd_{ki} + s^{ij}\lambda l_i(\omega_{kl} - \eta_{kl})\ k^j\dot{d}_k + k^j d^i(\omega_{ki} - \eta_{ki})\}$$

or

$$P = (\tau^{ij} - s^{mj}\lambda^i{}_m - k^j d^i)\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i\dot{d}_i + (\tau^{ij} + s^{mj}\lambda^i{}_m + k^j\dot{d}^i)\omega_{ji} \qquad (23.25)$$

The last term on the right hand side of (23.25) is a produce to a symmetric and a skew-symmetric tensor component; hence it vanishes identically and we obtain

$$P = (\tau^{ij} - s^{mj}\lambda^i{}_m - k^j d^i)\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i\dot{d}_i \qquad (23.26)$$

We now define an alternative form for the symmetric commposite stress tensor $\bar{\tau}^{ij}$ by the relation

$$\bar{\tau}^{ij} = \frac{1}{2}\ \{(\tau^{ij} - \lambda^i{}_m s^{mj} - d^i k^j) + (\tau^{ji} - \lambda^j{}_m s^{mi} - d^j k^i)\} \qquad (23.27)$$

To show that $\bar{\tau}^{ij}$ is equivalent to $\mathcal{T}^{ij}$ defined by (22.39) we proceed as follows. Substituting (23.27) into (23.26), we obtain

$$P = \bar{\tau}^{ij}\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i\dot{d}_i \qquad (23.28)$$

In view of the symmetry of $\bar{\tau}^{ij}$ we may write

$$\varepsilon_{ijk}\bar{\tau}_{ij} = \frac{1}{2}\, \varepsilon_{ijk}\{(\tau^{ij} - \lambda^{i}{}_{m}s^{mj} - d^{i}k^{j}) + (\tau^{ji} - \lambda^{j}{}_{m}s^{mi} - d^{j}k^{i})\} = 0$$

$$= \frac{1}{2}\, \varepsilon_{ijk}\{(\tau^{ij} + \tau^{ji}) - (\lambda^{i}{}_{m}s^{mj} + \lambda^{j}{}_{m}s^{mi}) - (d^{i}k^{j} + d^{j}k^{i})\} = 0 \tag{23.29}$$

We now rewrite (23.29) as follows:

$$\varepsilon_{ijk}\tau^{ij} = \frac{1}{2}\{\varepsilon_{jik}(\tau^{ji} + \tau^{ij}) - \varepsilon_{ijk}(\lambda^{i}{}_{m}s^{mj} + \lambda^{j}{}_{m}s^{mi}) - \varepsilon_{ijk}(d^{i}k^{j} + d^{j}k^{i})\}$$

$$= \frac{1}{2}\ \{-\varepsilon_{ijk}(\tau^{ij} + \tau^{ji}) - \varepsilon_{ijk}(\lambda^{i}{}_{m}s^{mj} + \lambda^{j}{}_{m}s^{mi}) \cdot \varepsilon_{ijk}(d^{i}k^{j} + d^{j}k^{i})$$

$$= -\frac{1}{2}\ \{\varepsilon_{ijk}(\tau^{ij} + \lambda^{i}{}_{m}s^{mj} + d^{i}k^{j}) + \varepsilon_{ijk}(\tau^{ji} + \lambda^{j}{}_{m}s^{mi} + d^{j}k^{i})\}$$

$$= -\frac{1}{2}\, \varepsilon_{ijk}\ \{(\tau^{ij} + \lambda^{i}{}_{m}s^{mj} + d^{i}k^{j}) + (\tau^{ji} + \lambda^{j}{}_{m}s^{mi} + d^{j}k^{i})\} = 0 \tag{23.30}$$

Since

$$\tau^{ij} + \lambda^{i}{}_{m}s^{mj} + d^{i}k^{j} = \tau^{ji} + \lambda^{j}{}_{m}s^{mi} + d^{j}k^{i}$$

we obtain from (23.30)

$$\varepsilon_{ijk}(\tau^{ij} + \lambda^{i}{}_{m}s^{mj} + d^{i}k^{j}) = 0 \tag{23.31}$$

This shows that the relation (23.27) is equivalent to (22.38). Recall the kinematical variables

$$\gamma_{ij} = \frac{1}{2}\,(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \frac{1}{2}\,(g_{ij} - G_{ij})$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{23.32}$$

$$\gamma_i = d_i - D_i$$

From (23.32) we obtain

$$\dot{\gamma}_{ij} = \frac{1}{2}\,\dot{g}_{ij} = \frac{1}{2}\,(2\eta_{ij}) = \eta_{ij}$$

$$\dot{\mathcal{K}}_{ij} = \dot{\lambda}_{ij} \tag{23.33}$$

$$\dot{\gamma}_i = \dot{d}_i$$

The expression of power (23.26) in terms of the kinematical variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i is

$$\rho\dot{\varepsilon} = \mathcal{T}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i = P \tag{23.34}$$

where

$$P = \mathcal{T}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i \tag{23.35}$$

Rewriting the ψ as a function of the variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i, i.e.,

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \tag{23.36}$$

we obtain

$$\dot{\varepsilon} = \frac{\partial\psi}{\partial\gamma_{ij}}\,\dot{\gamma}_{ij} + \frac{\partial\psi}{\partial\mathcal{K}_{ij}}\,\dot{\mathcal{K}}_{ij} + \frac{\partial\psi}{\partial\gamma_i}\,\dot{\gamma}_i \tag{23.37}$$

From (23.34) and (23.36) we obtain

$$(\mathcal{T}^{ij} - \rho\,\frac{\partial\psi}{\partial\gamma_{ij}})\dot{\gamma}_{ij} + (s^{ij} - \rho\,\frac{\partial\psi}{\partial\mathcal{K}_{ij}})\dot{\mathcal{K}}_{ij} + (k^i - \rho\,\frac{\partial\psi}{\partial\gamma_i})\dot{\gamma}_i = 0 \tag{23.38}$$

Then by the usual procedure we obtain

$$\mathcal{T}^{ij} = \rho\,\frac{\partial\psi}{\partial\gamma_{ij}}$$

$$s^{ij} = \rho\,\frac{\partial\psi}{\partial\mathcal{K}_{ij}} \tag{23.39}$$

$$k^i = \rho\,\frac{\partial\psi}{\partial\gamma_i}$$

24. The complete theory

We recapitulate in this section the complete theory of elastic composite laminate in the context of purely mechanical theory.

The initial boundary value problem in the general theory.

The basic field equations of the nonlinear theory consist of the equations of motion and the energy equation given by (22.46) and repeated below for convenience:

$$\tau^{ij}{}_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \qquad (24.1)$$

$$s^{ij}{}_{|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \qquad (24.2)$$

$$\varepsilon_{ijn}(\tau^{ij} + \lambda^i{}_n s^{mj} + d^i k^j) = 0 \qquad (24.3)$$

$$\rho\dot{\varepsilon} = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^i w_i = P \qquad (24.4)$$

where P is given by

$$P = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^i w_i \qquad (24.5)$$

or equivalently by

$$P = \mathcal{T}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i \qquad (24.6)$$

The constitutive equations for an elastic composite laminate are specified by

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \qquad (24.7)$$

and

$$\mathcal{T}^{ij} = \rho\,\frac{\partial\psi}{\partial\gamma_{ij}} \qquad (24.8)$$

$$s^{ij} = \rho \frac{\partial \psi}{\partial \mathcal{K}_{ij}} \tag{24.9}$$

$$k^{i} = \rho \frac{\partial \psi}{\partial \gamma_i} \tag{24.10}$$

We recall that (24.9) is subjected to the condition

$$s^{i3} = 0 \tag{24.11}$$

We note that instead of (24.7) to (24.10), any other alternative equivalent expressions derived in section 23 may be used.

The above field equations and constitutive relations characterize the initial boundary-value problem in the nonlinear theory of an elastic composite laminate.

The problem of establishing boundary conditions is not always clear in the literature on continuum theory of composites. Even in the case of mathematically coherent continuum theories with micro-structure the physical interpretations are not given or are ambiguous. Indeed, most (if not all) of the problems that are treated using various continuum theories for composites deal with periodic wave propagation or those problems for which the boundary conditions are not of primary importance.

The nature of the boundary conditions in the present theory may be seen from the rate of work expression for the composite contact force and the composite contact couple, i.e.,

$$\mathcal{R}_c = \int_{\partial \mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w}) da \tag{24.12}$$

The conditions at the boundary surface of the composite laminate at which the surface forces $\bar{\mathbf{t}}$ and the surface couples are prescribed require that

$$\mathbf{t} = \bar{\mathbf{t}} \quad , \quad \mathbf{s} = \bar{\mathbf{s}} \tag{24.13}$$

If we express the surface forces $\bar{\mathbf{t}}$ and the surface couples $\bar{\mathbf{s}}$ in terms of their components, i.e.,

$$\bar{t} = \bar{t}^i g_i = \bar{t}_i g^i \tag{24.14}$$

$$\bar{s} = \bar{s}^i g_i = \bar{s}^i_{igb} \tag{24.15}$$

and then using (21.24) and (21.38) the boundary conditions take the following forms:

$$\tau^{ij} n_i = \bar{t}^j \quad , \quad \tau^i{}_j n_i = \bar{t}_j \tag{24.16}$$

$$s^{ij} n_i = \bar{s}^j \quad , \quad s^i{}_j n_i = \bar{s}_j \tag{24.17}$$

To elaborate, we recall that our choice of convected coordinates is such that at a point P with coordinates θ^i (i = 1,2,3) of the composite laminate, the coordinates θ^1, θ^2 are in fact the coordinate curves of the ply passing through the point P. Moreover, the coordinate θ^3 is in the direction of lay-up. This implies that for an arbitrary part $\mathcal{P}$, the boundary surface $\partial\mathcal{P}$ consists of two material surfaces of the form

$$\partial\mathcal{P}_1 \;:\; \theta^3 = \theta^3(\theta^\alpha) = C_1$$

and (24.18)

$$\partial\mathcal{P}_1 \;:\; \theta^3 = \theta^3(\theta^\alpha) = C_2$$

and a lateral material surface of the form

$$\partial\mathcal{P}_l \;:\; f(\theta^\alpha) = 0 \tag{24.19}$$

such that θ^3 = const. are closed smooth curves on the surface (24.19). With this background, it should now be clear that $\bar{t}^1, \bar{t}^2$ in $(24.16)_1$ are the stress resultants in $\mathbf{g}_1$, $\mathbf{g}_2$ directions, respectively, while $\bar{t}^3$ is the stress in $\mathbf{g}_3$ direction. Similarly, $\bar{s}^1, \bar{s}^2$ in $(24.17)_1$ are the stress couple resultants in $\mathbf{g}_1$, $\mathbf{g}_2$ directions, respectively, but $\bar{s}^3$ is the stress couple resultant in $\mathbf{g}_3$ direction, which is identically zero.

25. A constrained theory of composite laminates

So far our development of the continuum theory has been general and without any restriction/condition placed on the kinematical variables. Therefore the field equations and the constitutive relations are applicable to any elastic composite laminate. We did not introduce any kinematical constraints previously to keep the theory general enough so that it could be utilized for various physical situations. We now turn to the development of a constrained theory of our continuum model which may appropriately be called *Cosserat composite.* First we derive a set of constraint equations for the composite laminate. We then proceed to obtain the relevant response functions induced by the constraint. Finally we obtain a set of field equations in terms of the displacement and effected by the presence of the constraints.

We impose the condition that plies of the composite laminate do not separate from or slide over each other at all time during the motion of the composite laminate. This means the displacement vector of the material points throughout the body including at the interface must be single valued. Hence we require

$$\mathbf{r}(\theta^\alpha, \theta^3 + \Delta\theta^3) = \mathbf{r}(\theta^\alpha, \theta^3) + \xi_2\, \mathbf{d}(\theta^\alpha, \theta^3)$$

or

$$\mathbf{r}(\theta^\alpha, \theta^3 + \Delta\theta^3) - \mathbf{r}(\theta^\alpha,\theta^3) = \xi_2\, \mathbf{d}(\theta^\alpha,\theta^3)$$

or

$$\frac{\mathbf{r}(\theta^\alpha, \theta^3 + \Delta\theta^3) - \mathbf{r}(\theta^\alpha,\theta^3)}{\xi_2} = \mathbf{d}(\theta^\alpha,\theta^3) \qquad (25.1)$$

In the limit when $\xi_2 \to 0$ and $\mathbf{r}(\theta^\alpha,\theta^3 + \Delta\theta^3) \to \mathbf{r}(\theta^\alpha,\theta^3)$ we obtain

$$\lim_{\xi_2 \to 0} \frac{\mathbf{r}(\theta^\alpha,\theta^3 + \Delta\theta^3) - \mathbf{r}(\theta^\alpha,\theta^3)}{\Delta\theta^3} = \mathbf{d}(\theta^\alpha,\theta^3)$$

or

$$\mathbf{g}_3 = \mathbf{r}_{,3} = \mathbf{d} \tag{25.2}$$

where we have made use of the fact that

$$\Delta\theta^3 = \xi_2 \tag{25.2.a}$$

Expression (25.2) implies the following constraint condition

$$\mathbf{g}^\alpha \cdot \mathbf{d} = 0 \qquad (\alpha = 1,2) \tag{25.3}$$

Differentiating (25.3) with respect to time, we obtain

$$\dot{\mathbf{g}}^\alpha \cdot \mathbf{d} + \mathbf{g}^\alpha \cdot \mathbf{w} = 0 \qquad (\alpha = 1,2) \tag{25.4}$$

We recall

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i{}_j \tag{25.5}$$

Differentiating (25.5) with respect to time, we obtain

$$\dot{\mathbf{g}}^i \cdot \mathbf{g}_j + \mathbf{g}^i \cdot \dot{\mathbf{g}}_j = \dot{\mathbf{g}}^i \cdot \mathbf{g}_j + \mathbf{g}^i \cdot \mathbf{v}_{,j} = \dot{\mathbf{g}}^i \cdot \mathbf{g}_j + \mathbf{g}^i \cdot (v^m{}_{|j}\mathbf{g}_m)$$

$$= \dot{\mathbf{g}}^i \cdot \mathbf{g}_j + v^i{}_{|j} = \dot{\mathbf{g}}^i \cdot \mathbf{g}_j + v^i{}_{|m}\delta^m{}_j$$

$$= \dot{\mathbf{g}}^i \cdot \mathbf{g}_j + v^i{}_{|m}\mathbf{g}^m \cdot \mathbf{g}_j = (\dot{\mathbf{g}}^i + v^i{}_{|m}\mathbf{g}^m) \cdot \mathbf{g}_j = 0 \tag{25.6}$$

From this we obtain

$$\dot{\mathbf{g}}^i \cdot \mathbf{g}_j = -\,\mathbf{g}^i \cdot \mathbf{v}_{,j} \tag{25.7}$$

and

$$\dot{\mathbf{g}}^i = -\,v^i{}_{|m}\mathbf{g}^m \tag{25.8}$$

Substituting (25.7), (25.8) into (25.4), we arrive at

$$\dot{\mathbf{g}}^{\alpha} \cdot d^j \mathbf{g}_j + \mathbf{g}^{\alpha} \cdot \mathbf{w} = 0$$

or

$$d^i \mathbf{g}^{\alpha} \cdot \mathbf{v}_{,i} - \mathbf{g}^{\alpha} \cdot \mathbf{w} = 0 \qquad (25.9)$$

and

$$-v^{\alpha}{}_{|m} \mathbf{g}^m \cdot \mathbf{d} + \mathbf{g}^{\alpha} \cdot \mathbf{w} = 0$$

or

$$-v^{\alpha}{}_{|i} \mathbf{g}^i \cdot (d_m \mathbf{g}^m) + \mathbf{g}^{\alpha} \cdot (w_i \mathbf{g}^i)$$

$$d_m g^{im} g^{j\alpha} v_{j|i} - g^{i\alpha} w_i = 0 \qquad (\alpha = 1,2) \qquad (25.10)$$

This is another form of the constraints (25.3) which is more appropriate for our present development.

For a composite laminate with constraints we assume that each of the functions $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}^i$ are determined to within an additive constraint response so that

$$\mathbf{T}^i = \bar{\mathbf{T}}^i + \hat{\mathbf{T}}^i$$

$$\mathbf{S}^i = \bar{\mathbf{S}}^i + \hat{\mathbf{S}}^i \qquad (25.11)$$

$$\mathbf{k} = \bar{\mathbf{k}} + \hat{\mathbf{k}}$$

where

$$\hat{\mathbf{T}}^i \ , \ \hat{\mathbf{S}}^i \ , \ \hat{\mathbf{k}} \qquad (25.12)$$

are specified by constitutive equations and

$$\bar{\mathbf{T}}^i \ , \ \bar{\mathbf{S}}^i \ , \ \bar{\mathbf{k}} \qquad (25.13)$$

which represent the response due to constraints (25.9) are arbitrary functions of θ^i,t, are workless and independent of the kinematical variables $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$. Thus, recalling the expression (23.3) for mechanical power, we set

$$\tilde{\mathbf{T}}^i \cdot \mathbf{v}_{,i} + \tilde{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + g^{1/2}\tilde{\mathbf{k}} \cdot \mathbf{w} = 0 \tag{25.14}$$

This must hold for all values of the variables $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ subject to the constraint condition (25.9). Multiplying (25.9) by the Lagrange multipliers δ_α ($\alpha = 1,2$) and subtracting the results from (25.14), we obtain

$$\tilde{\mathbf{T}}^i \cdot \mathbf{v}_{,i} + \tilde{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + g^{1/2}\tilde{\mathbf{k}} \cdot \mathbf{w} - (\delta_\alpha d^i \mathbf{g}^\alpha \cdot \mathbf{v}_{,i} - \delta_\alpha \mathbf{g}^\alpha \cdot \mathbf{w}) = 0$$

or

$$(\tilde{\mathbf{T}}^i - \delta_\alpha d^i \mathbf{g}^\alpha) \cdot \mathbf{v}_{,i} + \tilde{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + (g^{1/2}\tilde{\mathbf{k}} + \delta_\alpha \mathbf{g}^\alpha) \cdot \mathbf{w} = 0 \tag{25.15}$$

From (25.15) and the fact that $\tilde{\mathbf{T}}^i$, $\tilde{\mathbf{S}}^i$ and $\tilde{\mathbf{k}}$ are independent of $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ it follows that

$$\tilde{\mathbf{T}}^i = \delta_\alpha d^i \mathbf{g}^\alpha \tag{25.16}$$

$$\tilde{\mathbf{S}}^i = 0 \tag{25.17}$$

$$g^{1/2}\tilde{\mathbf{k}} = -\,\delta_\alpha \mathbf{g}^\alpha \tag{25.18}$$

Expressions (25.16) to (25.18) represent the constraint response induced by the constraint equations (25.3). Substituting (25.16), (25.17) and (25.18) into linear momentum equation $(22.22)_b$ and the director momentum equation $(22.22)_c$, we obtain

$$[\hat{\mathbf{T}}^i + \delta_\alpha d^i \mathbf{g}^\alpha]_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) \tag{25.19}$$

and

$$\hat{\mathbf{S}}^i{}_{,i} + \rho g^{1/2}\mathbf{c} - [g^{1/2}\hat{\mathbf{k}} - \delta_\alpha \mathbf{g}^\alpha] = \rho g^{1/2}(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \tag{25.20}$$

Introducing the following temporary variables $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ by

$$\hat{\mathbf{b}} = \mathbf{b} - (\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}})$$

and (25.21)

$$\hat{\mathbf{c}} = \mathbf{c} - (y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}})$$

we can rewrite (25.19) and (25.20) as follows

$$\rho g^{1/2} \hat{\mathbf{b}} + \hat{\mathbf{T}}^i{}_{,i} + (\delta_\alpha d^i \mathbf{g}^\alpha)_{,i} = 0 \tag{25.22}$$

$$\rho g^{1/2} \hat{\mathbf{c}} + \hat{\mathbf{S}}^i{}_{,i} - g^{1/2} \hat{\mathbf{k}} + \delta_\alpha \mathbf{g}^\alpha = 0 \tag{25.23}$$

From (25.23) we obtain

$$(\delta_\alpha d^i \mathbf{g}^\alpha)_{,i} = -\,(\rho g^{1/2} d^i \hat{\mathbf{c}} + d^i \hat{\mathbf{S}}^j{}_{,j} - g^{1/2} d^i \hat{\mathbf{k}})_{,i} \tag{25.24}$$

Substitute (25.24) into (25.22) to obtain

$$\rho g^{1/2} \hat{\mathbf{b}} + \hat{\mathbf{T}}^i{}_{,i} - (\rho g^{1/2} d^i \hat{\mathbf{c}} + d^i \hat{\mathbf{S}}^j{}_{,j} - g^{1/2} d^i \mathbf{k})_{,i} = 0 \tag{25.25}$$

Moreover, from (25.23) and (25.3) we obtain

$$\rho g^{1/2} \mathbf{d} \cdot \hat{\mathbf{c}} + \mathbf{d} \cdot \hat{\mathbf{S}}^j{}_{,j} - g^{1/2} \mathbf{d} \cdot \hat{\mathbf{k}} = 0 \tag{25.26}$$

Recalling that $\hat{\mathbf{T}}^i$, $\hat{\mathbf{S}}^i$ and $\hat{\mathbf{k}}$ are specified as functions of various kinematical variables, it is clear that the system of equations (25.25) and (25.26) represent two equations for the determination of the primary unknowns $\mathbf{v}$ (or $\mathbf{r}$) and $\mathbf{d}$.

We now proceed to obtain the counterparts of (25.25) and (25.26) in component form. To this end, we assume, for an elastic composite laminate with constraint, the functions τ^{ij}, s^{ij} and k^i are determined to within an additive constraint response so that

$$\tau^{ij} = \tilde{\tau}^{ij} + \hat{\tau}^{ij}$$

$$s^{ij} = \tilde{s}^{ij} + \hat{s}^{ij} \tag{25.27}$$

$$k^i = \tilde{k}^i + \hat{k}^i$$

where

$$\hat{\tau}^{ij} \ , \ \hat{s}^{ij} \ , \ \hat{k}^{i} \tag{25.28}$$

are specified by constitutive equations and

$$\tilde{\tau}^{ij} \ , \ \tilde{s}^{ij} \ , \ \tilde{k}^{i} \tag{25.29}$$

which represent the response due to constraints (25.10), are arbitrary functions of θ^i,t, workless and independent of kinematical variables $v_{i|j}$, $w_{i|j}$ and w_i. Thus, recalling the expression (23.4) for mechanical power, we set

$$\tilde{\tau}^{ij}v_{j|i} + \tilde{s}^{ij}w_{j|i} + \tilde{k}^{i}w_i = 0 \tag{25.30}$$

which must hold for all values of the variables $v_{j|i}$, $w_{j|i}$ and w_i subject to the constraint conditions (25.10). Multiplying (25.10) by the Lagrange multipliers λ_α ($\alpha = 1,2$) and subtracting the results from (25.30), we obtain [3]

$$\tilde{\tau}^{ij}v_{j|i} + \tilde{s}^{ij}w_{j|i} + \tilde{k}^{i}w_i - (\lambda_\alpha d_m g^{im} g^{j\alpha} v_{j|i} - \lambda_\alpha g^{i\alpha} w_i) = 0$$

or

$$(\tilde{\tau}^{ij} - \lambda_\alpha d^i g^{j\alpha})v_{j|i} + \tilde{s}^{ij}w_{j|i} + (\tilde{k}^{i} + \lambda_\alpha g^{i\alpha})w_i = 0 \tag{25.31}$$

From (25.31) and the fact that $\tilde{\tau}^{ij}$, $\tilde{s}^{ij}$ and $\tilde{k}$ are independent of $v_{j|i}$, $w_{j|i}$ and w_i it follows that

$$\tilde{\tau}^{ij} = \lambda_\alpha d^i g^{j\alpha} \tag{25.32}$$

$$\tilde{s}^{ij} = 0 \tag{25.33}$$

$$\tilde{k}^{i} = -\lambda_\alpha g^{i\alpha} \tag{25.34}$$

Substituting (25.32), (25.33) and (25.34) into $(22.46)_b$ and $(22.46)_c$, we obtain

$$[\hat{\tau}^{ij} + \lambda_\alpha d^i g^{j\alpha}]_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \tag{25.35}$$

---

[3] Note that λ_α is now different from δ_α.

and

$$\hat{s}^{ij}{}_{|i} - [\hat{k}^j - \lambda_\gamma g^{j\gamma}] + \rho c^j = \rho(y^1\alpha^j + y^1\beta^j) \tag{25.36}$$

From (25.21) we have

$$\hat{b}^j = b^j - (\alpha^j + y^1\beta^j)$$

and

$$\hat{c}^j = c^j - (y^1\alpha^j + y^2\beta^j) \tag{25.37}$$

Making use of (25.37) we rewrite (25.35) and (25.36) as follows

$$\rho\hat{b}^j + \hat{\tau}^{ij}{}_{|i} + (\lambda_\gamma d^i g^{j\gamma})_{|i} = 0 \tag{25.38}$$

and

$$\rho\hat{c}^j + \hat{s}^{ij}{}_{|i} - \hat{k}^j + \lambda_\gamma g^{\gamma j} = 0 \tag{25.39}$$

From (25.39) we obtain

$$(\lambda_\gamma d^i g^{\gamma j})_{|i} = -(\rho d^i\hat{c}^j + d^i\hat{s}^{mj}{}_{|m} - d^i\hat{k}^j)_{|i} \tag{25.40}$$

and substitute into (25.38) to obtain

$$\rho\hat{b}^j + \hat{\tau}^{ij}{}_{|i} - (\rho d^i\hat{c}^j + \rho d^i\hat{s}^{mj}{}_{|m} - d^i\hat{k}^j)_{|i} = 0 \tag{25.41}$$

Moreover, from (25.39) we obtain

$$\rho d_j\hat{c}^j + d_j\hat{s}^{ij}{}_{|i} - d_j\hat{k}^j + \lambda_\alpha d_j g^{\beta j} = 0 \tag{25.42}$$

However, from (25.3) we have

$$d_j g^{\beta j} = 0 \tag{25.43}$$

Hence, by (25.42) and (25.43) we have

$$\rho d_j\hat{c}^j + d_j\hat{s}^{ij}{}_{|i} - d_j\hat{k}^j = 0 \tag{25.44}$$

Again recalling that $\hat{\tau}^{ij}$, $\hat{s}^{ij}$ and $\hat{k}^i$ are specified, by constitutive equations, as functions of relevant kinematical variables, it is clear that the system of equations (25.41) and (25.44) represent four equations for the determination of four primary unknowns v_i and d.

Before closing this section, we obtain a relation between the Lagrange multipliers δ_α and λ_α. Recalling (17.16), we may write (25.16) as follows

$$\tilde{\mathbf{T}}^i = g^{1/2}\tilde{\tau}^{ij}\mathbf{g}_j = \delta_\alpha d^i g^{j\alpha}\mathbf{g}_j$$

or

$$g^{1/2}(\tilde{\tau}^{ij} - g^{-1/2}\delta_\alpha d^i g^{j\alpha})\mathbf{g}_j = 0 \qquad (25.45)$$

Since $g^{1/2} \neq 0$ and $\mathbf{g}_j$ are linearly independent base vectors, we obtain

$$\tilde{\tau}^{ij} - g^{-1/2}\delta_\alpha d^i g^{j\alpha} = 0$$

or

$$\tilde{\tau}^{ij} = g^{-1/2}\delta_\alpha d^i g^{j\alpha} \qquad (25.46)$$

A comparison between (25.46) and (25.32) yields

$$\lambda_\alpha = g^{-1/2}\delta_\alpha \qquad (25.47)$$

Similarly, from (25.18) we obtain

$$g^{1/2}\tilde{k}^i\mathbf{g}_i = -\,\delta_\alpha g^{i\alpha}\mathbf{g}_i$$

or

$$g^{1/2}(\tilde{k}^i + g^{-1/2}\delta_\alpha g^{i\alpha}\mathbf{g}_i) = 0 \qquad (25.48)$$

Again, since $\mathbf{g}_i$ are independent base vectors and since $g^{1/2} \neq 0$, we obtain

$$\bar{k}^i = -g^{-1/2}\delta_\alpha g^{i\alpha} \tag{25.49}$$

Comparing (25.49) and (25.34), we obtain

$$\lambda_\alpha = g^{-1/2}\delta_\alpha \tag{25.50}$$

which confirms the results (25.47).

26. Constitutive equations of an elastic composite laminate: Derivation from three-dimensional classical continuum theory

The theory developed in the course of this investigation is exact in the context of nonlinear theory and is based on postulates (conservation laws) that are independent of those in classical continuum mechanics although their motivation inspired by the classical theory of continuum mechanics. Due to the material and geometric complexities inherent in composites and due to our rather limited (direct) knowledge of composite materials, the study of composite materials has always been conducted via three-dimensional classical continuum mechanics. In particular, constitutive relations for composites have always been derived from those of the constituents which are assumed to be elastic in the sense of classical theory of elasticity. It is therefore desirable to relate the various field quantities of the present theory to those of classical three-dimensional theory. This has already been accomplished, in part, through the relevant definitions. To complete the correspondence between the present theory and the classical continuum theory we need to establish appropriate relationships between the composite field quantities $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ with the classical stress vector $\mathbf{T}^{*i}$ of the constituents. This section is concerned with this task.

We recall that in the three-dimensional theory of classical (non-polar) continuum mechanics and within the context of purely mechanical theory the constitutive relation for the specific internal energy and the stress tensor of an elastic body can be expressed as follows [4]

$$\psi^* = \psi^*(\gamma_{ij}) \tag{26.1}$$

$$\tau^{*ij} = \rho^* \frac{\partial \psi^*}{\partial \gamma^*_{ij}} \tag{26.2}$$

[4] Whenever there is no danger of confusion we designate a function and its value with the same symbol. Moreover, the function ψ^* in (26.1) depends also on the reference values G^*_{ij}, but we have not exhibited this here. The partial derivative of a function with respect to a symmetric tensor such as that in (26.2) is understood to have the symmetric form $\frac{1}{2}\left(\frac{\partial \psi^*}{\partial \gamma^*_{ij}} + \frac{\partial \psi^*}{\partial \gamma^*_{ji}}\right)$.

We now proceed to deduce the counterparts of the above results for an elastic composite laminate. To this end, we first recall the expression for γ_{ij}^*, i.e.,

$$\gamma_{ij}^* = \gamma_{ji}^* = \frac{1}{2} (\mathbf{g}_i^* \cdot \mathbf{g}_j^* - \mathbf{G}_i^* \cdot \mathbf{G}_j^*) \tag{26.3}$$

and then observe the following relations:

$$\frac{\partial\psi^*}{\partial\mathbf{g}_k^*} = \frac{\partial\psi^*}{\partial\gamma_{ij}^*} \frac{\partial\gamma_{ij}^*}{\partial\mathbf{g}_k^*} = \frac{\partial\psi^*}{\partial\gamma_{ij}^*} \{\frac{\partial}{\partial\mathbf{g}_k^*} [\frac{1}{2} (\mathbf{g}_i^* \cdot \mathbf{g}_j^* - \mathbf{G}_i \cdot \mathbf{G}_j)]\}$$

$$= \frac{\partial\psi^*}{\partial\gamma_{ij}^*} \{\frac{1}{2} (\delta^k{}_i \mathbf{g}_j^* + \delta^k{}_j \mathbf{g}_i^*)\}$$

$$= \frac{1}{2} (\frac{\partial\psi^*}{\partial\gamma_{ij}^*} \delta^k{}_i \mathbf{g}_j^* + \frac{\partial\psi^*}{\partial\gamma_{ij}^*} \delta^k{}_j \mathbf{g}_i^*)$$

$$= \frac{1}{2} (\frac{\partial\psi^*}{\partial\gamma_{kj}^*} \mathbf{g}_j^* + \frac{\partial\psi^*}{\partial\gamma_{ik}^*} \mathbf{g}_i^*) = \frac{\partial\psi^*}{\partial\gamma_{ki}^*} \mathbf{g}_i^* \tag{26.4}$$

and

$$\frac{\partial\psi^*}{\partial\mathbf{a}_\alpha} = \frac{\partial\psi^*}{\partial\mathbf{g}_k^*} \frac{\partial\mathbf{g}_k^*}{\partial\mathbf{a}^\alpha} = \frac{\partial\psi^*}{\partial\gamma_{ki}^*} \mathbf{g}_i^* \delta^\alpha{}_k = \frac{\partial\psi^*}{\partial\gamma_{\alpha i}^*} \mathbf{g}_i^* \tag{26.5}$$

since

$$\mathbf{g}_\beta^* = \mathbf{a}_\beta + \xi\, \mathbf{d}_{,\beta} \quad , \quad \mathbf{g}_3^* = \mathbf{d} \tag{26.6}$$

$$\frac{\partial\mathbf{g}_\beta^*}{\partial\mathbf{a}_\alpha} = \delta^\alpha{}_\beta \quad , \quad \frac{\partial\mathbf{g}_3^*}{\partial\mathbf{a}_\alpha} = 0 \tag{26.7}$$

$$\frac{\partial\mathbf{g}_k^*}{\partial\mathbf{a}_\alpha} = \delta^\alpha{}_k \tag{26.8}$$

where in obtaining (26.4) we have also used (26.3). In addition we observe that

$$\frac{\partial\psi^*}{\partial\mathbf{d}} = \frac{\partial\psi^*}{\partial\mathbf{g}_k^*} \frac{\partial\mathbf{g}_k^*}{\partial\mathbf{d}} = \frac{\partial\psi^*}{\partial\gamma_{ki}^*} \mathbf{g}_i^* \frac{\partial\mathbf{g}_k^*}{\partial\mathbf{d}} = \frac{\partial\psi^*}{\partial\gamma_{ki}^*} \mathbf{g}_i^* \delta^3{}_k = \frac{\partial\psi^*}{\partial\gamma_{3i}^*} \mathbf{g}_i^* \tag{26.9}$$

where we have made use of (26.4) and the fact that from (26.6) we obtain

$$\frac{\partial g^*_\beta}{\partial d} = 0 \quad , \quad \frac{\partial g^*_3}{\partial d} = 1 \tag{26.10}$$

$$\frac{\partial g^*_k}{\partial d} = \delta^3{}_k \tag{26.11}$$

Further we observe

$$\frac{\partial \psi^*}{\partial d_{,\alpha}} = \frac{\partial \psi^*}{\partial g^*_k} \frac{\partial g^*_k}{\partial d_{,\alpha}} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}} g_i{}^* \frac{\partial g^*_k}{\partial d_{,\alpha}} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}} g_i{}^* \delta^\alpha{}_k \xi = \frac{\partial \psi^*}{\partial \gamma^*_{\alpha i}} g_i{}^* \xi \tag{26.12}$$

Since from (26.6) we have

$$\frac{\partial g^*_\beta}{\partial d_{,\alpha}} = \xi \, \delta^\alpha{}_\beta \quad , \quad \frac{\partial g^*_3}{\partial d_{,\alpha}} = 0 \tag{26.13}$$

$$\frac{\partial g^*_k}{\partial d_{,\alpha}} = \xi \, \delta^\alpha{}_k \tag{26.14}$$

and we have also made use of (26.4).

We now consider the constitutive equations for the components $\tau^{*\alpha i}$ in (26.2), i.e.,

$$\tau^{*\alpha j} = \rho^* \frac{\partial \psi^*}{\partial \gamma^*_{\alpha j}} \tag{26.15}$$

Recalling the formula

$$\mathbf{T}^{*\alpha} = g^{*1/2} \tau^{*\alpha j *} \mathbf{g}^*_j \tag{26.16}$$

and

$$\mathbf{T}^\alpha = \int_0^{\xi_2} \mathbf{T}^{*\alpha} \, d\xi = a^{1/2} \mathbf{N}^\alpha \tag{26.17}$$

we write

$$T^{\alpha} = \int_{o}^{\xi_2} T^{*\alpha} d\xi = \int_{o}^{\xi_2} g^{*1/2} \tau^{*\alpha j} g_j^* d\xi = \int_{o}^{\xi_2} \rho^* g^{*1/2} \frac{\partial \psi^*}{\partial \gamma_{\alpha j}^*} g_j^* d\xi$$

$$= \int_{o}^{\xi_2} \rho^* g^{*1/2} \frac{\partial \psi^*}{\partial g_{\alpha}^*} d\xi = \int_{o}^{\xi_2} \rho^* g^{*1/2} \frac{\partial \psi^*}{\partial \mathbf{a}_{\alpha}} d\xi$$

$$= \frac{\partial}{\partial \mathbf{a}_{\alpha}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^* d\xi \tag{26.18}$$

We recall that in terms of θ^i coordinates and in relation to $\mathbf{a}_{\alpha}$ and $\mathbf{g}_{\alpha}$ we have

$$\frac{\partial(\ )}{\partial \mathbf{a}_{\alpha}} = \frac{\partial(\ )}{\partial \mathbf{g}_{\beta}} \frac{\partial \mathbf{g}_{\beta}}{\partial \mathbf{a}_{\alpha}} = \frac{\partial(\ )}{\partial \mathbf{g}_{\beta}} \delta^{\beta}{}_{\alpha} = \frac{\partial(\ )}{\partial \mathbf{g}_{\alpha}} \tag{26.19}$$

Hence, we can write (26.18) as

$$T^{\alpha} = \frac{\partial}{\partial \mathbf{g}_{\alpha}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^* d\xi \tag{26.20}$$

Next, recall the formula

$$S^{\alpha} = \int_{o}^{\xi_2} T^{*\alpha} \xi \, d\xi = a^{1/2} \mathbf{M}^{\alpha} \tag{26.21}$$

and write

$$S^{\alpha} = \int_{o}^{\xi_2} T^{*\alpha} \xi \, d\xi = \int_{o}^{\xi_2} g^{*1/2} \tau^{*\alpha j} g_j^* \xi \, d\xi = \int_{o}^{\xi_2} \rho^* g^{*1/2} \frac{\partial \psi^*}{\partial \gamma_{\alpha j}^*} g_j^* \xi \, d\xi$$

$$= \int_{o}^{\xi_2} \rho^* g^{*1/2} \frac{\partial \psi^*}{\partial \mathbf{d}_{,\alpha}} d\xi = \frac{\partial}{\partial \mathbf{d}_{,\alpha}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^* d\xi \tag{26.22}$$

where we have made use of (26.16) and (26.12). Now recall the expression

$$g^{1/2} \mathbf{k} = a^{1/2} \hat{\mathbf{k}} = \int_{o}^{\xi} T^{*3} d\xi \tag{26.23}$$

and write

$$g^{1/2}\mathbf{k} = \int_o^{\xi_2} g^{*1/2}\tau^{*3j}\mathbf{g}_j^* d\xi = \int_o^{\xi_2} \rho^* g^{*1/2} \frac{\partial\psi^*}{\partial\gamma_{3j}^*} \mathbf{g}_j^* \, d\xi$$

$$= \int_o^{\xi_2} \rho^* g^{*1/2} \frac{\partial\psi^*}{\partial\mathbf{d}} d\xi = \frac{\partial}{\partial\mathbf{d}} \int_o^{\xi_2} \rho^* g^{*1/2}\psi^* d\xi \tag{26.24}$$

We have shown that the composite stress vector $\mathbf{T}^3$ corresponds to interlaminar stresses. We notice that interlaminar stress vector $\mathbf{T}^3$ acts as an applied contract force for the micro-structure. Hence the constitutive relation for $\mathbf{T}^3$ should be specified directly. This means $\mathbf{T}^3$ unlike $\mathbf{T}^\alpha$ may not be obtained from the strain energy of the constituents. However, like any other component of stress vector (or stress tensor), $\mathbf{T}^3$ may either be obtained by solving equations or motion or obtained through constitutive relations after determination of displacement vector from equations of motion.

Consider now the expression

$$\bar{\psi} = \frac{1}{\rho g^{1/2}} \int_o^{\xi_2} \rho^* g^{*1/2}\psi^*(\mathbf{g}_\alpha^*, \mathbf{d}, \mathbf{d}_{,\alpha}) d\xi \tag{26.25}$$

where the arguments of ψ^* have been defined before. Clearly, in view of kinematical relations (12.). the function $\bar{\psi}$ can be regarded as a function of the variables $\mathbf{g}_i$, $\mathbf{d}$ and $\mathbf{d}_{,i}$. Therefore, the constitutive equations for composite laminate will be given by

$$\psi = \bar{\psi}(\mathbf{g}_i, \mathbf{d}, \mathbf{d}_{,\alpha}) \tag{26.26}$$

$$\mathbf{T}^i = \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}$$

$$\mathbf{S}^i = \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}} \tag{26.27}$$

$$g^{1/2}\mathbf{k} = \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{d}}$$

where $\mathbf{S}^3$ vanishes identically since $\bar{\psi}$ is not a function of $\mathbf{d}_{,3}$. This is, of course, in agreement with (26.39). A comparison between (26.27) and (23.15) reveals that the two sets of constitutive relations will be equivalent if $\bar{\psi}$ is given by (26.25). This establishes the correspondence between the composite laminate's constitutive relations and those of the constituents.

For completeness, in the rest of this section we obtain the component forms of (26.27) using a different procedure than that used in section 22. To this end we recall the formulas

$$\mathbf{r} = r^i \mathbf{g}_i = r_i \mathbf{g}^i \quad , \quad \mathbf{d} = d^i \mathbf{g}_i = d_i \mathbf{g}^i \tag{26.28}$$

and

$$\mathbf{r}_{,i} = r^j{}_{|i} \mathbf{g}_j = r_{j|i} \mathbf{g}^j \quad , \quad \mathbf{d}_{,i} = d^j{}_{|i} \mathbf{g}_j = d_{j|i} \mathbf{g}^j \tag{26.29}$$

It is clear from (26.28) and (26.29) that the function $\bar{\psi}$ may be rewritten as

$$\psi = \bar{\psi}(\mathbf{g}_i, \mathbf{d}, \mathbf{d}_{,\alpha}) = \bar{\psi}(\mathbf{r}_{,i}, \mathbf{d}, \mathbf{d}_{,\alpha}) = \tilde{\psi}(r_{m|i}, d_m, d_{m|\alpha}) \tag{26.30}$$

With the help of the expression for $\mathbf{T}^i$, (26.28), (26.29) and the gradient of a scalar valued function of a vector, we write [5]

$$\mathbf{T}^i = g^{1/2} \tau^{ij} \mathbf{g}_j = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{g}_i} = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{r}_{,i}} = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial r_{j|i}} \mathbf{g}_j \tag{26.31}$$

Rewriting the above, we obtain

$$g^{1/2}\left(\tau^{ij} - \rho \frac{\partial \tilde{\psi}}{\partial r_{j|i}}\right) \mathbf{g}_j = 0 \tag{26.32}$$

Since the quantity in the parentheses is independent of $\mathbf{g}_j$, we conclude that

$$\tau^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial r_{j|i}} \tag{26.33}$$

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[5] Operators of the form $\frac{\partial f}{\partial \mathbf{x}}$ where f is a scalar valued function of a vector $\mathbf{x} = x^i \mathbf{g}_i = x_i \mathbf{g}^i$ were defined earlier. The component form of this operator which is in fact the gradient operator (derivative operator) is given by

$$\frac{df}{d\mathbf{x}} = \frac{\partial f}{\partial x_i} \mathbf{g}_i = \frac{\partial f}{\partial x^i} \mathbf{g}^i$$

In a similar manner, with the help of expression for S^i, (26.28) and (26.29) we obtain

$$S^i = g^{1/2} s^{ij} \mathbf{g}_j = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}_{,i}} = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial d_{j|i}} \mathbf{g}_j \tag{26.34}$$

From this we obtain

$$g^{1/2}(s^{ij} - \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}}) \mathbf{g}_j = 0 \tag{26.35}$$

Since the quantity in the parentheses is independent of $\mathbf{g}_j$, we obtain

$$s^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}} \tag{26.36}$$

Next, we consider $\mathbf{k}$ and making use of the same procedure we write

$$g^{1/2}\mathbf{k} = g^{1/2} k^i \mathbf{g}_i = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}} = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial d_i} \mathbf{g}_i \tag{26.37}$$

and

$$g^{1/2}(k^i - \rho \frac{\partial \tilde{\psi}}{\partial d_i}) \mathbf{g}_i = 0 \tag{26.38}$$

By the usual argument we obtain

$$k^i = \rho \frac{\partial \tilde{\psi}}{\partial d_i} \tag{26.39}$$

Collecting the results (26.33), (26.36) and (26.39), we have

$$\tau^{ij} = \rho \frac{\partial \bar{\psi}}{\partial r_{j|i}}$$

$$s^{ij} = \rho \frac{\partial \bar{\psi}}{\partial d_{j|i}} \tag{26.40}$$

$$k^{i} = \rho \frac{\partial \bar{\psi}}{\partial d_{i}}$$

which are the same as (23.23). It should be mentioned that the development after (23.23) of section (23) as well as the entire development of section (25) remains applicable and unchanged.

E. Linearized Theory

We now proceed to obtain the linearized version of the theory developed in previous sections. Linearization will be carried out for the kinematics, the field equations and the constitutive equations. We note that a vertical bar, in the linearized expressions, will denote covariant differentiation with respect to G_{ij} corresponding to the reference configuration.

27. Linearized kinematics

This section is devoted to the linearized form of the kinematical results of section (12). In particular, we deduce the linearized kinematic measures of a composite laminate with infinitesimal displacements and infinitesimal director displacements as a special case of the general results in section (12).

We recall the expressions

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \xi\mathbf{d}(\eta^\alpha,\theta^3,t) \tag{27.1}$$

and

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) \tag{27.2}$$

Within the context of linear theory of composite continuum we let [1]

$$\mathbf{p}^* = \mathbf{P}^* + \varepsilon\mathbf{u}^* \tag{27.3}$$

where ε is a non-dimensional parameter and $\mathbf{u}^*$ is a three-dimensional vector such that

$$\mathbf{u}^* = u^{*i}\mathbf{g}_i = u_i^*\mathbf{g}^i \tag{27.4}$$

$$\mathbf{u}^* = \mathbf{u}^*(\eta^\alpha,\xi,\theta^3,t) = \mathbf{u}(\eta^\alpha,\theta^3,t) + \xi\boldsymbol{\delta}(\eta^\alpha,\theta^3,t) \tag{27.5}$$

[1] The use of ε in this section is temporary, clear from the context and not to be confused with the use of the same notation in the previous section.

From (27.3) we obtain

$$\mathbf{v}^* = \varepsilon \dot{\mathbf{u}}^* \tag{27.6}$$

Introducing (27.5) into (27.3) and making use of (27.2), we obtain

$$\mathbf{p}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) + \varepsilon[\mathbf{u}(\eta^\alpha,\theta^3,t) + \xi\delta(\eta^\alpha,\theta^3,t)]$$

$$= [\mathbf{R}(\eta^\alpha,\theta^3) + \varepsilon\mathbf{u}(\eta^\alpha,\theta^3,t)] + \xi[\mathbf{D}(\eta^\alpha,\theta^3) + \varepsilon\delta(\eta^\alpha,t)] \tag{27.7}$$

By a comparison between (27.1) and (27.7) we conclude that

$$\begin{aligned} \mathbf{r}(\theta^i,t) &= \mathbf{R}(\theta^i) + \varepsilon\mathbf{u}(\theta^i,t) \\ \mathbf{d}(\theta^i,t) &= \mathbf{D}(\theta^i) + \varepsilon\delta(\theta^i,t) \end{aligned} \tag{27.8}$$

where we have identified η^α with θ^α. The velocity and the director velocity are readily obtained as

$$\begin{aligned} \mathbf{v} &= \varepsilon\dot{\mathbf{u}} \\ \mathbf{w} &= \varepsilon\dot{\delta} \end{aligned} \tag{27.9}$$

We say that the motion of a laminated composite continuum characterized by (27.8) describes infinitesimal deformation if the magnitudes of $\mathbf{u}$, δ and all their derivatives are bounded and are of the same order as $\mathbf{R}$ and $\mathbf{D}$ and if

$$\varepsilon \ll 1 \tag{27.10}$$

In what follows we shall be concerned with (scalar, vector or tensor) functions of position and time, determined by $\varepsilon\mathbf{u}$ and $\varepsilon\delta$ and their space and time derivatives. We denote these functions by the customary order symbol $O(\varepsilon^n)$ if there exists a real number C, independent of ε,$\mathbf{u}$,δ and their derivatives such that

$$| O(\varepsilon^n) | < C\,\varepsilon^n \qquad \varepsilon \to 0 \tag{27.11}$$

BASE

We would like to emphasize that the infinitesimal theory which we wish to obtain as a special case of the results in section (12) and in the sense of (27.10) is such that all kinematical quantities (including the displacement $\mathbf{u}$, the director displacement δ and other kinematical measures, as well as their space and time derivatives are all of $O(\varepsilon)$.

The base vectors $\mathbf{g}_i^*$ can be obtained from (27.7):

$$\mathbf{g}_\alpha^* = \mathbf{p}_{,\alpha}^* = (\mathbf{R}_{,\alpha} + \varepsilon\mathbf{u}_{,\alpha}) + \xi(\mathbf{D}_{,\alpha} + \varepsilon\delta_{,\alpha})$$

$$\mathbf{g}_3^* = \mathbf{p}_3^* = \mathbf{D} + \varepsilon\delta \tag{27.12}$$

Similarly the base vectors $\mathbf{g}_i$ are obtained from $(27.8)_1$

$$\mathbf{g}_i = \mathbf{r}_{,i} = (\mathbf{R} + \varepsilon\mathbf{u})_{,i} = \mathbf{G}_{,i} + \varepsilon\mathbf{u}_{,i} \tag{27.13}$$

We now proceed to obtain the relative kinematic measures γ_{ij}, $\mathcal{K}_{ij}$ and γ_i. To this end we first obtain

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = (\mathbf{G}_i + \varepsilon\mathbf{u}_i) \cdot (\mathbf{G}_j + \varepsilon\mathbf{u}_j) = G_{ij} + \varepsilon(\mathbf{G}_i \cdot \mathbf{u}_j + \mathbf{G}_{,j} \cdot \mathbf{u}_{,i}) + O(\varepsilon^2) \tag{27.14}$$

$$d_i = \mathbf{g}_i \cdot \mathbf{d} = (\mathbf{G}_i + \varepsilon\mathbf{u}_{,i}) \cdot (\mathbf{D} + \varepsilon\delta) = D_i + \varepsilon(\mathbf{G} \cdot \delta + \mathbf{u}_{,i} \cdot \mathbf{D}) + O(\varepsilon^2) \tag{27.15}$$

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = (\mathbf{G}_i + \varepsilon\mathbf{u}_{,i}) \cdot (\mathbf{D}_{,j} + \varepsilon\delta_{,j}) = \Lambda_{ij} + \varepsilon(\mathbf{G}_i \cdot \delta_{,j} + \mathbf{u}_{,i} \cdot \mathbf{D}_{,j}) + O(\varepsilon^2) \tag{27.16}$$

From (27.14) to (27.16) we obtain

$$\gamma_{ij} = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \frac{1}{2}\varepsilon(\mathbf{G}_i \cdot \mathbf{u}_{,j} + \mathbf{G}_j \cdot \mathbf{u}_{,i}) + O(\varepsilon^2) \tag{27.17}$$

$$\gamma_i = d_i - D_i = \varepsilon(\mathbf{G}_i \cdot \delta + \mathbf{u}_{,i} \cdot \mathbf{D}) + O(\varepsilon^2) \tag{27.18}$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} = \varepsilon(\mathbf{G}_i \cdot \delta_{,j} + \mathbf{u}_{,i} \cdot \mathbf{D}_{,j}) + O(\varepsilon^2) \tag{27.19}$$

At this stage it is desirable to elaborate on the manner in which the process of linearization may be carried out. To this end we take $\mathbf{u}'$ and δ' to be vector functions defined by

$$\mathbf{u}' = \varepsilon \mathbf{u} = O(\varepsilon) \qquad u'^i = \mathbf{A}^i \cdot \mathbf{u}' = O(\varepsilon) \tag{27.20}$$

$$\delta' = \varepsilon \delta = O(\varepsilon) \qquad \delta'^i = \mathbf{A}^i \cdot \delta' = O(\varepsilon) \tag{27.21}$$

Making use of (27.20) and (27.21) we may rewrite (27.17) to (27.19) as

$$\gamma'_{ij} = \frac{1}{2} (\mathbf{G}_i \cdot \mathbf{u}'_{,j} + \mathbf{G}_j \cdot \mathbf{u}'_{,i}) \tag{27.22}$$

$$\gamma'_i = (\mathbf{G}_i \cdot \delta' + \mathbf{u}'_{,i} \cdot \mathbf{D}) \tag{27.23}$$

$$\mathcal{K}'_{ij} = (\mathbf{G}_i \cdot \delta'_{,j} + \mathbf{u}'_{,i} \cdot \mathbf{D}_{,j}) \tag{27.24}$$

where we have introduced γ'_{ij}, γ'_i $\mathcal{K}'_{ij}$ which are of $O(\varepsilon)$ we have

$$\gamma_{ij} = \gamma'_{ij} + O(\varepsilon^2) = O(\varepsilon) \tag{27.25}$$

$$\gamma_i = \gamma'_i + O(\varepsilon^2) = O(\varepsilon) \tag{27.26}$$

$$\mathcal{K}_{ij} = \mathcal{K}'_{ij} + O(\varepsilon^2) = O(\varepsilon) \tag{27.27}$$

We also have

$$g^{*1/2} = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3$$

$$= [(\mathbf{G}_1 + \varepsilon \mathbf{u}_{,1}) \times (\mathbf{G}_2 + \varepsilon \mathbf{u}_{,2})] \cdot (\mathbf{G}_3 + \varepsilon \mathbf{u}_{,3})$$

$$= \{\mathbf{G}_1 \times \mathbf{G}_2) + \varepsilon(\mathbf{u}_{,1} \times \mathbf{G}_2 + \mathbf{G}_1 \times \mathbf{u}_{,2}) + \varepsilon^2(\mathbf{u}_{,1} \times \mathbf{u}_{,2})\} \cdot (\mathbf{G}_3 + \varepsilon \mathbf{u}_{,3})$$

$$= (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 + \varepsilon(\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{u}_{,3} +$$

$$\varepsilon(\mathbf{u}_{,1} \times \mathbf{G}_2) \cdot \mathbf{G}_3 + \varepsilon(\mathbf{G}_1 \times \mathbf{u}_{,2}) \cdot \mathbf{G}_3 +$$

$$\varepsilon^2(\mathbf{u}_{,1} \times \mathbf{G}_2) \cdot \mathbf{u}_{,3} + \varepsilon^2(\mathbf{G}_1 \times \mathbf{u}_{,2}) \cdot \mathbf{u}_{,3} +$$

$$\varepsilon^2(\mathbf{u}_{,1} \times \mathbf{u}_{,2}) \cdot \mathbf{G}_3 + \varepsilon^3(\mathbf{u}_{,1} \times \mathbf{u}_{,2}) \cdot \mathbf{u}_{,3}$$

$$= G^{1/2} + \varepsilon\{(\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{u}_{,3} + \mathbf{u}_{,1} \cdot (\mathbf{G}_2 \times \mathbf{G}_3) + \mathbf{u}_{,2} \cdot (\mathbf{G}_3 \times \mathbf{G}_1) + O(\varepsilon^2)$$

$$= G^{1/2} + \varepsilon(G^{1/2}\mathbf{G}^3 \cdot \mathbf{u}_{,3} + G^{1/2}\mathbf{G}^1 \cdot \mathbf{u}_{,1} + G^{1/2}\mathbf{G}^2 \cdot \mathbf{u}_{,2}) + O(\varepsilon^2)$$

or

$$g^{1/2} = G^{1/2} + \varepsilon G^{1/2}\mathbf{G}^i \cdot \mathbf{u}_{,i} + O(\varepsilon^2) = G^{1/2} + G^{1/2}(G^{ij} \cdot \gamma'_{ij}) + O(\varepsilon^2)$$

or

$$\left(\frac{g}{G}\right)^{1/2} = 1 + \gamma'^{i}_{i} + O(\varepsilon^2) \tag{27.28}$$

and

$$\frac{g}{G} = 1 + 2g^{ij}\gamma'_{ij} + O(\varepsilon^2) \tag{27.29}$$

We now retain only terms of $O(\varepsilon)$ in expressions such as (27.25) and hence approximate γ_{ij}, γ_i and $\mathcal{K}_{ij}$ by γ'_{ij}, γ'_i and $\mathcal{K}_{ij}$, etc. In order to avoid the introduction of unnecessary additional notations we proceed with linearization by retaining only terms of $O(\varepsilon)$ and after the approximation without loss of generality, we set $\varepsilon = 1$. In this manner the relative kinematic measures γ_{ij}, γ_i and $\mathcal{K}_{ij}$ reduce to

$$\gamma_{ij} = \frac{1}{2}\,(\mathbf{G}_i \cdot \mathbf{u}_{,j} + \mathbf{G}_j \cdot \mathbf{u}_{,i}) \tag{27.30}$$

$$\gamma_i = \mathbf{G}_i \cdot \delta + \mathbf{u}_{,i} \cdot \mathbf{D} \tag{27.31}$$

$$\mathcal{K}_{ij} = \mathbf{G}_i \cdot \delta_{,j} + \mathbf{u}_{,i} \cdot \mathbf{D}_{,j} \tag{27.32}$$

We also obtain

$$\rho g^{1/2} = \rho_o G^{1/2}$$

or

$$\rho G^{1/2}(1 + \gamma^i{}_i) = \rho_o G^{1/2}$$

or

$$\rho = \rho_o \frac{1}{1 + \gamma^i{}_i} = \rho_o(1 - \gamma^i{}_i) \tag{27.33}$$

28. Linearized field equations

Previously, with reference to the linearization of the kinematical results for a composite, it was assumed that all kinematic measures such as γ_{ij}, γ_i and $\mathcal{K}_{ij}$ as well as their space and time derivatives are of $O(\varepsilon)$. These must now be supplemented by additional assumptions in a complete infinitesimal theory. We now assume that the vector fields $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ are all zero in the reference configuration. We further assume that $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ (or their components) when expressed in suitable non-dimensional forms, as well as their space and time derivatives are all of $O(\varepsilon)$.

Recalling the linearization procedure of the previous section and avoiding the introduction of additional notations, we now regard $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ as infinitesimal quantities of $O(\varepsilon)$. As a result of linearization, all tensor quantities are now referred to the initial undeformed surface and covariant differentiation is with respect to G_{ij} in the reference configuration. It then follows that in the equations (22.30), (22.34), (22.38) each term is of $O(\varepsilon)$ and that d^i, and $d^i{}_{|m} = \lambda^i{}_m$ or λ_{im} must be replaced to the order of ε by D^i and $\Lambda^i{}_m$ or Λ_{im}, respectively. We omit the details since it is a straightforward calculation and merely record the linearized version of the equations of motion as follows:

$$\tau^i{}_{j\,|\,i} + \rho_o b_j = \rho_o(\ddot{u}_j + y^1\ddot{\delta}_j) \qquad (28.1)$$

$$s^i{}_{j\,|\,i} + (\rho_o c_j - k_j) = \rho_o(y^1\ddot{u}_j + y^2\ddot{\delta}_j) \qquad (28.2)$$

$$\varepsilon_{ijn}\{\tau^{ij} + s^{mj}\Lambda^i{}_m + D^i k^j\} = 0 \qquad (28.3)$$

where the vertical bar in (28.1) to (28.3) and the rest of this section denotes covariant differentiation with respect to G_{ij}. We also note that all quantities are now referred to the base vectors $\mathbf{G}_i$ of the reference configuration.

Moreover, upon linearization we obtain

$$\bar{\tau}^{ij} = \tau^{ij} - \Lambda^{i}{}_{m}s^{mj} - D^{i}k^{j} \tag{28.4}$$

In the light of the assumptions stated above and expression (28.4), the energy equation takes the form

$$\rho_o\dot{\varepsilon} = \bar{\tau}^{ij}\dot{\gamma}_{ji} + s^{ij}\dot{\mathcal{K}}_{ji} + k^{i}\dot{\gamma}_{i} = P \tag{28.5}$$

29. Linear constitutive relations for elastic composite laminates

This section is concerned with the derivation of the constitutive relations for a composite laminate in terms of those of its constituents. In what follows we assume that each of the constituents of the laminated composites is a homogeneous isotropic elastic material. We recall that within the scope of the linear theory all kinematical variables are referred to the reference configuration. Previously we showed that the strain energy function, ψ may be written as

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \qquad (29.1)$$

We assume that in the case of the linear theory ψ is given by a quadratic function of the infinitesimal kinematical variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i. We also recall that after systematic linearization of the expression for power, we obtained for the linear theory

$$\rho_0 \dot{\varepsilon} = \bar{\tau}^{ij} \dot{\gamma}_{ji} + s^{ij} \dot{\mathcal{K}}_{ji} + k^i \dot{\gamma}_i = P \qquad (29.2)$$

Since the rates $\dot{\gamma}_{ij}$, $\dot{\mathcal{K}}_{ij}$ and $\dot{\gamma}_i$ are all independent and since the coefficients are rate independent, after substituting (29.2) into (29.3) we obtain

$$\rho_0 \{ \frac{\partial \psi}{\partial \gamma_{ji}} \dot{\gamma}_{ji} + \frac{\partial \psi}{\partial \mathcal{K}_{ji}} \dot{\mathcal{K}}_{ji} + \frac{\partial \psi}{\partial \gamma_i} \dot{\gamma}_i \} = \bar{\tau}^{ij} \dot{\gamma}_{ji} + s^{ij} \dot{\mathcal{K}}_{ji} + k^i \dot{\gamma}_i \qquad (29.3)$$

or

$$(\bar{\tau}^{ij} - \rho_0 \frac{\partial \psi}{\partial \gamma_{ji}}) \dot{\gamma}_{ji} + (s^{ij} - \rho_0 \frac{\partial \psi}{\partial \mathcal{K}_{ji}}) \dot{\mathcal{K}}_{ji} + (k^i - \rho_0 \frac{\partial \psi}{\partial \gamma_i}) \dot{\gamma}_i = 0 \qquad (29.4)$$

Hence

$$\bar{\tau}^{ij} = \rho_o \frac{\partial \psi}{\partial \gamma_{ij}}$$

$$s^{ij} = \rho_o \frac{\partial \psi}{\partial \mathcal{K}_{ji}} \tag{29.5}$$

$$k^i = \rho_o \frac{\partial \psi}{\partial \gamma_i}$$

The relationship between the strain energy function ψ, per unit mass of the composite, and those of the constituents is given by

$$\psi = \frac{1}{\rho_o G^{1/2}} \int_o^{\xi_2} \rho_o^* G^{*1/2} \psi^* d\xi \tag{29.6}$$

or

$$\rho_o \psi = \int_o^{\xi_2} v(\rho_o^* \psi^*) d\xi = \int_o^{h_2} \mu(\rho_o^* \psi^*) d\xi = \int_o^{h_1} \mu(\rho_{o1}^* \psi_1^*) d\xi + \int_{h_1}^{h_2} \mu(\rho_{o2}^* \psi_2^*) d\xi$$

where ρ_{o1}. and ρ_{o2}^* denote mass densities of the constituents $\hat{\mathcal{B}}_1^*$ and $\hat{\mathcal{B}}_2^*$. We recall that in three-dimensional linear theory we have

$$\rho_o^* \psi^* = \frac{1}{2} E_{mn}^{*ij} \gamma_{ij}^* \gamma^{*mn} \tag{29.8}$$

and

$$\tau^{*ij} = E_{mn}^{*ij} \gamma^{*mn} \tag{29.9}$$

We also recall that for isotropic elastic materials we have

$$E_{mn}^{*ij} = \lambda^* G^{*ij} G_{mn}^* + \mu^* (\delta^i{}_m \delta^j{}_n + \delta^i{}_n \delta^j{}_m) \tag{29.10}$$

$$\tau^{*ij} = \mu^* (G^{*im} G^{*jn} + G^{*in} G^{*jm} + \frac{2v^*}{1-2v^*} G^{*ij} G^{*mn}) \gamma_{mn}^* \tag{29.11}$$

$$\lambda^* = \frac{2v^*}{1-2v^*} \mu^* \tag{29.12}$$

For an explicit set of constitutive relations the integration on the right hand side of (29.7) must be carried out using (29.8) for $\rho_0^* \psi^*$. Here we remark that as in the case of two dimensional theories of continuum mechanics (such as plates and shells), except possibly in very special cases, it appears to be extremely difficult to calculate the function ψ in (29.2) from the strain energy function ψ^* of the classical three dimensional theory. In the case of composite materials this becomes more complicated due to the existence of two (or more) materials.

Alternatively, in order to provide constitutive relations in which the coefficients are related to elastic constants of the constituents we can make use of the so-called specific Gibbs energy function. This method proves to be more convenient for the derivation of the linear constitutive equations for a composite laminate and will be described in the next section.

F. Application and comparison with the available theories

In this part we proceed to accomplish two goals. Frist we will apply the theory developed earlier (specifically in part E) to the case of initially flat composite laminates in which each ply is modeled as an initially flat Cosserat surface. We will also apply the theory to the case of initially clindrical composite laminates where each ply is modeled as a cylindrical shell (Cosserat surface). In addition, we develop an alternative method for the determination of the linear constitutive relations. This method which makes use of the Gibbs free energy function is more suitable for the application of the theory to various cases. Secondly, we will compare the present theory with the available continuum theories and point out the features that are unique in the present theory.

30. Preliminaries: Part I

Before proceeding further, we dispose of some results which are independent of linearization; however, they will be particular useful in the applications of the linear theory. First, we recall that the position vector $\mathbf{P}^*$, of the micro-body $\mathcal{B}^*$, in the reference configuration is given by

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \xi\, \mathbf{D}(\eta^\alpha,\theta^3) \tag{30.1}$$

In general, $\mathbf{D}$ in (30.1) is a three-dimensional vector having components D^1,D^2,D^3 in the directions of $\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3$. However, in the reference configuration without loss of generality we may specify $\mathbf{D}$ by

$$\mathbf{D} = D\mathbf{A}_3 \quad , \quad D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha,\theta^3) \tag{30.2}$$

where $\mathbf{A}_3 = \mathbf{A}_3\,(\eta^\alpha)$ is the unit normal to the Cosserat surface, i.e., the shell-like representative element at composite particle P. From (30.1) and (30.2) it follows that the base vectors $\mathbf{G}_i^*$ and the metric tensor G_{ij}^*, of the micro-structure, in the reference (initial) configuration are

$$\mathbf{G}_\alpha^* = \mathbf{R}_{,\alpha} + \xi\mathbf{D}_{,\alpha} = \mathbf{G}_\alpha + \xi(D\mathbf{A}_3)_{,\alpha} = \mathbf{G}_\alpha + \xi(D_{,\alpha}\mathbf{A}_3 + D\mathbf{A}_{3,\alpha}) = \mathbf{G}_\alpha + \xi D\mathbf{A}_{3,\alpha} + \xi D_{,\alpha}\mathbf{A}_3 \tag{30.3}$$

and

$$\mathbf{G}_3^* = \mathbf{D} = D\mathbf{A}_3 \tag{30.4}$$

We recall the results

$$\mathbf{A}_3 \cdot \mathbf{A}_\beta = 0 \;\Rightarrow\; \mathbf{A}_{3,\alpha} \cdot \mathbf{A}_\beta + \mathbf{A}_3 \cdot \mathbf{A}_{\beta,\alpha} = 0$$

Hence,

$$\mathbf{A}_\beta \cdot \mathbf{A}_{3,\alpha} = -\,\mathbf{A}_3 \cdot \mathbf{A}_{\beta,\alpha} = -\,B_{\beta\alpha} = -\,B_{\alpha\beta} \tag{30.5}$$

and

$$A_{3,\alpha} = -B_\alpha^\beta A_\beta \tag{30.6}$$

where $B_{\alpha\beta}$ are the components of the second fundamental form of the surface. By (30.3), (30.6) and the fact that $\mathbf{A}_\alpha = \mathbf{G}_\alpha$, we obtain

$$\mathbf{G}_\alpha^* = \mathbf{G}_\alpha - \xi D B^\beta{}_\alpha \mathbf{A}_\beta + \xi D_{,\alpha} \mathbf{A}_3 = \mathbf{G}_\beta \delta^\beta{}_\alpha - \xi D B^\beta{}_\alpha \mathbf{G}_\beta + \xi D_{,\alpha} \mathbf{A}_3$$

$$= (\delta^\beta{}_\alpha - \xi D B^\beta{}_\alpha)\mathbf{G}_\beta + \xi D_{,\alpha} \mathbf{A}_3 \tag{30.7}$$

Let

$$\nu^\beta{}_\alpha = \delta^\beta{}_\alpha - \xi D B^\beta{}_\alpha \tag{30.8}$$

Then by (30.4), (30.7) and (30.8) we have

$$\mathbf{G}_\alpha^* = \nu^\beta{}_\alpha \mathbf{G}_\beta + \xi D_{,\alpha} \mathbf{A}_3$$

$$\mathbf{G}_3^* = D\mathbf{A}_3 \tag{30.9}$$

and hence,

$$G_{\alpha\beta}^* = \nu^\gamma{}_\alpha \nu^\delta{}_\beta G_{\gamma\delta} + \xi^2 D_{,\alpha} D_{,\beta}$$

$$G_{\alpha 3}^* = \xi D D_{,\alpha} = \frac{1}{2}\xi (D^2)_{,\alpha} \tag{30.10}$$

$$G_{33}^* = D^2$$

Let us now introduce a set of curvilinear coordinates ζ^i such that $\zeta^\alpha = \eta^\alpha$ and where ζ^3 is measured to the scale of the rectangular Cartesian coordinates (say $x^i = x_i$) along the positive direction of the uniquely defined normal $\mathbf{A}_3$ of the Cosserat surface (i.e., micro-structure). Now in the reference configuration, which we take to be the initial configuration, the convected general curvilinear coordinates θ^i can always be related to ζ^i with ζ^3 as a specified function of η^α and ξ. For the purpose of this investigation and to avoid unnecessary complications, we denote ζ^3 simply by ζ and specify it by

$$\zeta = \zeta(\eta^\alpha)\xi \tag{30.11}$$

where ζ is a function of η^α only. In the special case that $\zeta(\eta^\alpha) = 1$ we obtain $\zeta = \xi$ in the reference configuration. The coordinate system $\{\eta^\alpha,\xi\}$ where ξ is measured along the normal to the Cosserat surface is called normal coordinate system. Thus with ζ specified by (30.11), the position vector $\mathbf{P}^*$ of the micro-body $\mathcal{B}^*$ in the reference configuration referred to the normal coordinates is given by

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \zeta\mathbf{A}_3(\eta^\alpha,\theta^3) \tag{30.12}$$

Let $\overline{\mathbf{G}}_i^*$ and $\overline{\mathbf{G}}_{ij}^*$ denote the base vectors and the metric tensor associated with the normal coordinates. From (30.12) we obtain

$$\overline{\mathbf{G}}_\alpha^* = \mathbf{R}_{,\alpha} + \zeta\mathbf{A}_{3,\alpha} = \mathbf{G}_\beta\delta^\beta{}_\alpha - \zeta B^\beta{}_\alpha\mathbf{A}_\beta = (\delta^\beta{}_\alpha - \zeta B^\beta{}_\alpha)\mathbf{G}_\beta$$

Hence, we have

$$\begin{aligned}\overline{\mathbf{G}}_\alpha^* &= \mu^\beta{}_\alpha\mathbf{G}_\beta \\ \overline{\mathbf{G}}_3^* &= \mathbf{A}_3\end{aligned} \tag{30.13}$$

where

$$\mu^\beta{}_\alpha = \delta^\beta{}_\alpha - \zeta B^\beta{}_\alpha \tag{30.14}$$

From (30.13) we have:

$$\begin{aligned}\overline{G}_{\alpha\beta}^* &= \mu^\gamma{}_\alpha\mu^\delta{}_\beta G_{\gamma\delta} \\ \overline{G}_{\alpha 3}^* &= 0 \\ \overline{G}_{33}^* &= 1\end{aligned} \tag{30.15}$$

A comparison between (30.1) and (30.2) with $\mathbf{D}$ specified by (30.2) reveals that

$$\zeta = D\xi \tag{30.16}$$

which is the transformation relation between ζ and ξ. Moreover, under this transformation, we obtain from (30.8), (30.14) and (30.15)

$$\nu^{\beta}{}_{\alpha} = \delta^{\beta}{}_{\alpha} - \xi D B^{\beta}{}_{\alpha} = \delta^{\beta}{}_{\alpha} - \zeta B^{\beta}{}_{\alpha} = \mu^{\beta}{}_{\alpha} \tag{30.17}$$

If we let det $(\nu^{\beta}{}_{\alpha}) = \frac{\nu}{D}$ and det $(\mu^{\beta}{}_{\alpha}) = \mu$ we obtain

$$\mu = \frac{\nu}{D} \tag{30.18}$$

It is worth noting that the metric tensors G^{*}_{ij} and $\overline{G}^{*}_{ij}$ become identical when evaluated on the surface $\xi = 0$ or $\zeta = 0$ in the reference configuration and are both given by

$$\begin{aligned} G^{*}_{\alpha\beta} &= \overline{G}^{*}_{\alpha\beta} = G_{\alpha\beta} \\ G^{*}_{\alpha 3} &= \overline{G}^{*}_{\alpha 3} = 0 \\ G^{*}_{33} &= \overline{G}^{*}_{33} = 1 \end{aligned} \tag{30.19}$$

We now proceed to obtain expressions for $\mathbf{G}_i$, G_{ij} and $G^{1/2}$ corresponding to coordinates θ^i. Consistent with the kinematic assumption (30.2) we take the function $\mathbf{R}(\theta^{\alpha},\theta^3)$ to be

$$\mathbf{R}(\theta^{\alpha},\theta^3) = \overline{\mathbf{R}}(\theta^{\alpha}) + \theta^3 \mathbf{A}_3(\theta^{\alpha}) \tag{30.20}$$

From this we have

$$\begin{aligned} \mathbf{G}_{\alpha} &= \mathbf{R}_{,\alpha} = \overline{\mathbf{R}}_{,\alpha} + \theta^3 \mathbf{A}_{3,\alpha} = \overline{\mathbf{R}}_{,\alpha} - \theta^3 B^{\gamma}{}_{\alpha} \mathbf{A}_{\gamma} \\ \mathbf{G}_3 &= \mathbf{R}_{,3} = (\theta^3 \mathbf{A}_3)_{,3} = \mathbf{A}_3 \end{aligned} \tag{30.21}$$

and

$$\begin{aligned} G_{\alpha\beta} &= (\overline{\mathbf{R}}_{,\alpha} - \theta^3 B^{\gamma}{}_{\alpha} \mathbf{A}_{\gamma}) \cdot (\overline{\mathbf{R}}_{,\beta} - \theta^3 B^{\delta}{}_{\beta} \mathbf{A}_{\delta}) \\ &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 B^{\gamma}{}_{\alpha} \mathbf{A}_{\gamma} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 B^{\delta}{}_{\beta} \mathbf{A}_{\delta} \cdot \overline{\mathbf{R}}_{,\alpha} + (\theta^3)^2 B^{\gamma}{}_{\alpha} B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \mathbf{A}_{\delta} \\ &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 (B^{\gamma}{}_{\alpha} \mathbf{A}_{\gamma} \cdot \overline{\mathbf{R}}_{,\beta} - B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \overline{\mathbf{R}}_{,\alpha}) + (\theta^3)^2 B^{\gamma}{}_{\alpha} B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \mathbf{A}_{\delta} \\ &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 (B^{\gamma}{}_{\alpha} \overline{\mathbf{R}}_{,\beta} + B^{\gamma}{}_{\beta} \overline{\mathbf{R}}_{,\alpha}) \cdot \mathbf{A}_{\gamma} + (\theta^3)^2 B^{\gamma}{}_{\alpha} B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \mathbf{A}_{\delta} \end{aligned}$$

Hence we have

$$G_{\alpha\beta} = \bar{\mathbf{R}}_{,\alpha} \cdot \bar{\mathbf{R}}_{,\beta} - \theta^3(B^\gamma{}_\alpha \bar{\mathbf{R}}_{,\beta} + B^\gamma{}_\beta \bar{\mathbf{R}}_{,\alpha}) \quad \mathbf{A}_\gamma + (\theta^3)^2 B^\gamma{}_\alpha B^\delta{}_\beta A_{\gamma\delta}$$

$$G_{\alpha 3} = (\bar{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{A}_\gamma) \cdot \mathbf{A}_3 = 0 \qquad (30.22)$$

$$G_{33} = \mathbf{A}_3 \cdot \mathbf{A}_3 = 1$$

Also,

$$G^{1/2} = [\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3] = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{A}_3$$

$$= [(\bar{\mathbf{R}}_{,1} - \theta^3 B^\gamma{}_1 \mathbf{A}_\gamma) \times (\bar{\mathbf{R}}_{,2} - \theta^3 B^\delta{}_2 \mathbf{A}_\delta)] \cdot \mathbf{A}_3$$

$$= (\bar{\mathbf{R}}_{,1} \times \bar{\mathbf{R}}_{,2}) \cdot \mathbf{A}_3 - \theta^3 [B^\gamma{}_1 (\mathbf{A}_\gamma \times \bar{\mathbf{R}}_{,2}) + B^\gamma{}_2 (\bar{\mathbf{R}}_{,1} \times \mathbf{A}_\gamma)] \cdot \mathbf{A}_3$$

$$+ (\theta^3)^2 B^\gamma{}_1 B^\delta{}_2 (\mathbf{A}_\gamma \times \mathbf{A}_\delta) \cdot \mathbf{A}_3 \qquad (30.23)$$

We now combine the assumptions (30.2) and (30.20) to obtain from (30.1)

$$\mathbf{P}^*(\theta^\alpha, \theta^3, \xi) = \mathbf{R} + \xi D \mathbf{A}_3 = \bar{\mathbf{R}} + \theta^3 \mathbf{A}_3 + \xi D \mathbf{A}_3 \qquad (30.24)$$

From this we obtain

$$\mathbf{G}^*_\alpha = \mathbf{R}_{,\alpha} + \xi (D \mathbf{A}_3)_{,\alpha} = \mathbf{R}_{,\alpha} + \xi D_{,\alpha} \mathbf{A}_3 - \xi D B^\gamma{}_\alpha \mathbf{A}_\gamma$$

Hence, we have

$$\mathbf{G}^*_\alpha = \mathbf{G}_\alpha - \xi D B^\gamma{}_\alpha \mathbf{G}_\gamma + \xi D_{,\alpha} \mathbf{A}_3 = \nu^\gamma{}_\alpha \mathbf{G}_\gamma + \xi D_{,\alpha} \mathbf{A}_3$$

$$\qquad (30.25)$$

$$\mathbf{G}^*_3 = D \mathbf{A}_3$$

where

$$\nu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \xi D B^\gamma{}_\alpha \qquad (30.26)$$

Moreover, from (30.25) we obtain

$$G^*_{\alpha\beta} = (v^\gamma{}_\alpha G_\gamma + \xi D_{,\alpha} A_3) \cdot (v^\delta{}_\beta G_\delta + \xi D_{,\beta} A_3) = v^\gamma{}_\alpha v^\delta{}_\gamma G_{\gamma\delta} + \xi^2 D_{,\alpha} D_{,\beta}$$

$$G^*_{\alpha 3} = (v^\gamma{}_\alpha G_\gamma + \xi D_{,\alpha} A_3) \cdot (DA_3) = \xi D D_{,\alpha} = \frac{1}{2}\xi (D^2)_{,\alpha} \tag{30.27}$$

$$G^*_{33} = (DA_3) \cdot (DA_3) = D^2$$

and

$$G^{*1/2} = [G^*_1, G^*_2, G^*_3] = (G^*_1 \times G^*_1) \cdot G^*_3$$

$$= \{(v^\gamma{}_1 G_\gamma + \xi D_{,1} A_3) \times (v^\gamma{}_2 G_\delta + \xi D_{,2} A_3)\} \cdot (DA_3)$$

$$= \{v^\gamma{}_1 v^\gamma{}_2 (G_\gamma \times G_\delta) + \xi D_{,2} v^\gamma{}_1 (G_\gamma \times A_3) + \xi D_{,1} v^\delta{}_2 (A_3 \times G_\delta)\} \cdot (DA_3)$$

$$= D v^\gamma{}_1 v^\delta{}_2 (G_\gamma \times G_\delta) \cdot A_3$$

$$= D\{v^1{}_1 v^2{}_2 (G_1 \times G_2) + v^2{}_1 v^1{}_2 (G_2 \times G_1)\} \cdot A_3$$

$$= D\{v^1{}_1 v^2{}_2 (G_1 \times G_2) + v^2{}_1 v^1{}_2 (G_2 \times G_1)\} \cdot A_3$$

$$= D(v^1{}_1 v^2{}_2 - v^1{}_2 v^2{}_1)(G_1 \times G_2) \cdot G_3 = DG^{1/2} \det(v^\alpha{}_\beta) \tag{30.28}$$

where in obtaining (30.28) we have made use of (30.23). Since

$$v = D \det(v^\beta{}_\alpha)$$

we obtain from (30.28)

$$v = D \det(v^\beta{}_\alpha) = \left(\frac{G^*}{G}\right)^{1/2} \tag{30.29}$$

31. Preliminaries: Part II

In the rest of this development we assume each ply of the composite is sufficiently thin and confine our attention to the field equations of the linearized theory (28.1) and (28.2). Moreover, for the position vector **R** and the director **D**, in the reference (initial) configuration, we adopt the assumptions (30.20) and (30.2). Hence, in the reference configuration we have

$$\mathbf{R}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}}(\theta^\alpha) + \theta^3\mathbf{A}_3 \tag{31.1}$$

$$\mathbf{D} = D\mathbf{A}_3 \quad , \quad D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha) = D(\theta^\alpha) \tag{31.2}$$

and

$$\mathbf{P}^* = \mathbf{R} + \xi D\mathbf{A}_3 = \bar{\mathbf{R}} + (\theta^3 + \xi D)\mathbf{A}_3 \tag{31.3}$$

As mentioned before, within the scope of the linear theory $\mathbf{g}_i, \mathbf{g}_i^*, \mathbf{a}_i, g^{1/2}, g^{*1/2}$ and $a^{1/2}$ may be replaced by their reference (initial) values in the definitions of the various resultants. We now proceed to obtain the resultants which occur in the linearized equations of motion. Consider $\mathbf{T}^\alpha$ and within the context of the linear theory make use of (18.7), (21.26) to write

$$\mathbf{T}^\alpha = G^{1/2}\tau^{\alpha j}\mathbf{G}_j = \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi = \int_0^{\xi_2} G^{*1/2}\tau^{*\alpha j}\mathbf{G}_j^* d\xi$$

or

$$G^{1/2}(\tau^{\alpha\beta}G_\beta + \tau^{\alpha3}G_3) = \int_o^{\xi_2} G^{*1/2}(\tau^{*\alpha\beta}G^*_\beta + \tau^{*\alpha3}G^*_3)d\xi$$

$$= \int_o^{\xi_2} G^{*1/2}\{\tau^{*\alpha\beta}(v^\gamma{}_\beta G_\gamma + \xi D_{,\beta}A_3) + \tau^{*\alpha3}DA_3\}d\xi$$

$$= \int_o^{\xi_2} vG^{1/2}\tau^{*\alpha\beta}v^\gamma{}_\beta G_\gamma d\xi$$

$$+ \int_o^{\xi_2} vG^{1/2}(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha3})A_3 d\xi \tag{31.4}$$

where in obtaining (31.4) we have made use of (30.25) and (30.29). Since G_β and A_3 are linearly independent vectors and since G_β, A_3 and $G^{1/2}$ are independent of ξ, it follows from (29.4) that

$$\tau^{\alpha\beta} = \int_o^{\xi_2} v\tau^{*\alpha\gamma}v^\beta{}_\gamma d\xi \quad , \quad \tau^{\alpha3} = \int_o^{\xi_2} v(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha3})d\xi \tag{31.5}$$

We note that the composite stress vector $\mathbf{T}^3$ is not related to $\mathbf{T}^{*3}$ (within each constituent of the composite) and must be specified by a constitutive relation separately. In a similar manner, we consider S^α and within the context of linear theory we use (18.11), (21.38) to write

$$S^\alpha = G^{1/2}s^{\alpha j}G_j = \int_o^{\xi_2} T^{*\alpha}\xi \, d\xi = \int_o^{\xi_2} G^{*1/2}\tau^{*\alpha j}G^*{}_j\xi \, d\xi$$

or

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$$G^{1/2}(s^{\alpha\beta}\mathbf{G}_\beta + s^{\alpha 3}\mathbf{G}_3) = \int_0^{\xi_2} G^{*1/2}(\tau^{*\alpha\beta}\mathbf{G}^*_\beta + \tau^{*\alpha 3}\mathbf{G}^*_3)\xi\, d\xi$$

$$= \int_0^{\xi_2} G^{*1/2}\{\tau^{*\alpha\beta}(v^\gamma{}_\beta\mathbf{G}_\gamma + \xi D_{,\beta}\mathbf{A}_3) + \tau^{*\alpha 3}D\mathbf{A}_3\}\xi\, d\xi$$

$$= \int_0^{\xi_2} vG^{1/2}\tau^{*\alpha\beta}v^\gamma{}_\beta\mathbf{G}_\gamma\xi\, d\xi$$

$$+ \int_0^{\xi_2} vG^{1/2}(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3})\mathbf{A}_3\xi\, d\xi \qquad (31.6)$$

where in obtaining (31.6) we have made use of (30.25) and (30.29). Since $\mathbf{G}_\beta$ and $\mathbf{A}_3$ are linearly independent and since $\mathbf{G}_\beta$, $\mathbf{A}_3$ and $G^{1/2}$ are not functions of ξ, we obtain from (30.6)

$$s^{\alpha\beta} = \int_0^{\xi_2} v\tau^{*\alpha\gamma}v^\beta{}_\gamma\xi\, d\xi \quad , \quad s^{\alpha 3} = \int_0^{\xi_2} v(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3})\xi\, d\xi \qquad (31.7)$$

We recall that

$$s^{3i} = 0 \qquad (31.8)$$

Next, we consider $\mathbf{k}$ and in the same manner we write

$$G^{1/2}\mathbf{k} = G^{1/2}(k^\alpha\mathbf{G}_\alpha + k^3\mathbf{G}_3) = \int_0^{\xi_2} \mathbf{T}^{*3}d\xi = \int_0^{\xi_2} G^{*1/2}\tau^{*3j}\mathbf{G}_j d\xi$$

or

$$G^{1/2}(k^\alpha G_\alpha + k^3 A_3) = \int_o^{\xi_2} G^{*1/2}(\tau^{*3\alpha} G_\alpha^* + \tau^{*33} G_3^*) d\xi$$

$$= \int_o^{\xi_2} G^{*1/2}\{\tau^{*3\alpha}(\nu\gamma_\alpha G_\gamma + \xi D_{,\alpha} A_3) + \tau^{*33} D A_3\} d\xi$$

$$= \int_o^{\xi_2} \nu G^{1/2} \tau^{*3\alpha} \nu\gamma_\alpha G_\gamma d\xi$$

$$+ \int_o^{\xi_2} \nu G^{1/2}(\xi D_{,\alpha}\tau^{*3\alpha} + D\tau^{*33}) A_3 d\xi \qquad (31.9)$$

where again in obtaining (31.9) we have made use of (30.25) and (30.29). By the usual argument it follows from (31.10)

$$k^\alpha = \int_o^{\xi_2} \nu\tau^{*3\gamma}\nu^\alpha{}_\gamma d\xi \quad , \quad k^3 = \int_o^{\xi_2} \nu(\xi D_{,\alpha}\tau^{*3\alpha} + D\tau^{*33}) d\xi \qquad (31.10)$$

Collecting the results of this section, we have

$$\tau^{\alpha\beta} = \int_o^{\xi_2} \nu\tau^{*\alpha\gamma}\nu^\beta{}_\gamma d\xi \quad , \quad \tau^{\alpha 3} = \int_o^{\xi_2} \nu(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha\beta}) d\xi$$

τ^{3i} or $\mathbf{T}^3$ must be specified directly by a constitutive equation.

$$s^{\alpha\beta} = \int_o^{\xi_2} \nu\tau^{*\alpha\gamma}\nu^\beta{}_\gamma \xi\, d\xi \quad , \quad s^{\alpha 3} = \int_o^{\xi_2} \nu(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha\beta})\xi\, d\xi \qquad (31.11)$$

$$s^{3i} = 0 \quad \text{or} \quad S^3 = 0$$

$$k^\alpha = \int_o^{\xi_2} \nu\tau^{*3\gamma}\nu^\alpha{}_\gamma d\xi \quad , \quad k^3 = \int_o^{\xi_2} \nu(\xi D_{,\alpha}\tau^{*3\alpha} + D\tau^{*33}) d\xi$$

The resultants in (31.11) are defined in terms of the stress tensor τ^{*ij} referred to the convected coordinates $\eta^i = \{\theta^\alpha, \xi\}$.

Next, we proceed to obtain the counterparts of (31.11) in terms of normal coordiantes $\zeta^i = \{\zeta^\alpha, \zeta^3\} = \{\eta^\alpha, \zeta\} = \{\theta^\alpha, \zeta\}$ where we have

$$\zeta = D\xi \tag{31.12}$$

Let the contravariant stress tensor in the context of classical continuum mechanics, referred to the normal coordinates ζ^i be denoted by $\bar{\tau}^{*ij}$. The relationship between $\bar{\tau}^{*ij}$ and τ^{*ij} is obtained by making use of the transformation law between two second order tensors as follows:

$$\partial\eta^l \, \tau^{*kl} \tag{31.13}$$

Hence,

$$\bar{\tau}^{*\alpha\beta} = \frac{\partial\zeta^\alpha}{\partial\eta^k} \frac{\partial\zeta^\beta}{\partial\eta^l} \tau^{*kl} = \frac{\partial\zeta^\alpha}{\partial\eta^k} \left(\frac{\partial\zeta^\beta}{\partial\eta^\lambda} \tau^{*k\lambda} + \frac{\partial\zeta^\beta}{\partial\eta^3} \tau^{*k3}\right)$$

$$= \frac{\partial\zeta^\alpha}{\partial\eta^\gamma} \frac{\partial\zeta^\beta}{\partial\eta^\lambda} \tau^{*\gamma\lambda} + \frac{\partial\zeta^\alpha}{\partial\eta^3} \frac{\partial\zeta^\beta}{\partial\eta^\lambda} \tau^{*3\lambda}$$

$$= \delta^\alpha{}_\gamma \delta^\beta{}_\lambda \tau^{*\gamma\lambda} = \tau^{*\alpha\beta} \tag{31.14}$$

and

$$\bar{\tau}^{*\alpha 3} = \frac{\partial\zeta^\alpha}{\partial\eta^k} \frac{\partial\zeta^3}{\partial\eta^l} \tau^{*kl} = \frac{\partial\zeta^\alpha}{\partial\eta^k} \left(\frac{\partial\zeta^3}{\partial\eta^\lambda} \tau^{*k\lambda} + \frac{\partial\zeta^3}{\partial\xi} \tau^{*k3}\right)$$

$$= \frac{\partial\zeta^\alpha}{\partial\eta^\gamma} \frac{\partial\zeta^3}{\partial\eta^\lambda} \tau^{*\gamma\lambda} + \frac{\partial\zeta^\alpha}{\partial\eta^3} \frac{\partial\zeta^3}{\partial\eta^\lambda} \tau^{*3\lambda} + \frac{\partial\zeta^\alpha}{\partial\eta^\gamma} \frac{\partial\zeta^3}{\partial\xi} \tau^{*\gamma 3} + \frac{\partial\zeta^\alpha}{\partial\xi} \frac{\partial\zeta^3}{\partial\xi} \tau^{*33}$$

$$= \delta^\alpha{}_\gamma (\xi D_{,\lambda}) \tau^{*\gamma\lambda} + D\delta^\alpha{}_\gamma \tau^{*\gamma 3} = \xi \, D^{,\beta} \tau^{*\alpha\beta} + D\tau^{*\alpha 3} \tag{31.15}$$

$$\bar{\tau}^{*33} = \frac{\partial\zeta^3}{\partial\eta^k} \frac{\partial\zeta^3}{\partial\eta^l} \tau^{*kl} = \frac{\partial\zeta^3}{\partial\eta^\gamma} \frac{\partial\zeta^3}{\partial\eta^l} \tau^{*\gamma l} + \frac{\partial\zeta^3}{\partial\xi} \frac{\partial\zeta^3}{\partial\eta^l} \tau^{*3l}$$

$$= \xi \, D_{,\gamma} \left(\frac{\partial\zeta^3}{\partial\eta^\lambda} \tau^{*\gamma\lambda} + \frac{\partial\zeta^3}{\partial\xi} \tau^{*\gamma 3}\right) + D\left(\frac{\partial\zeta^3}{\partial\eta^\lambda} \tau^{*3\lambda} + \frac{\partial\zeta^3}{\partial\xi} \tau^{*33}\right)$$

$$= \xi \, D_{,\gamma} (\xi \, D_{,\lambda} \tau^{*\gamma\lambda} + D\tau^{*\gamma 3}) + D(\xi \, D_{,\lambda} \tau^{*3\lambda} + D\tau^{*33})$$

$$= \xi^2 D_{,\alpha} D_{,\beta} \tau^{*\alpha\beta} + 2\xi D D_{,\alpha} \tau^{*\alpha 3} + D^2 \tau^{*33} \tag{31.16}$$

We note that if the thickness of the representative element in the direction of normal is h_2, we have

$$\zeta = 0 \quad \text{at} \quad \xi = 0$$
$$\zeta = h_2 \quad \text{at} \quad \xi = \xi_2 \tag{31.17}$$

Hence

$$D\xi_2 = h_2 \tag{31.18}$$

which relates ξ_2 to h_2 and D. In particular, if D = 1 we obtain $\mathbf{D} = \mathbf{A}_3$ and

$$\xi_2 = h_2 \tag{31.19}$$

We now define a new set of composite field quantities in terms of $\bar{\tau}^{*ij}$ as follows

$$\bar{\tau}^{\alpha\beta} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu_\gamma^\beta d\xi \quad , \quad V^\alpha = \bar{\tau}^{\alpha 3} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha 3} d\xi$$

$$\bar{\tau}^{3\alpha} = \theta^3 D_{,\beta}\tau^{\beta\alpha} + D\tau^{3\alpha}$$

$$\bar{\tau}^{33} = (\theta^3)^2 D_{,\alpha}D_{,\beta}\tau^{\alpha\beta} + \theta^3 DD_{,\alpha}(\tau^{\alpha 3} + \tau^{3\alpha}) + D^2\tau^{33} \tag{31.20}$$

$$\bar{s}^{\alpha\beta} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu_\gamma^\beta\xi\, d\xi \quad , \quad \bar{s}^{\alpha 3} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha 3}\xi\, d\xi$$

$$\bar{s}^{3i} = 0$$

$$V^3 = \int_0^{h_2} \mu(\bar{\tau}^{*33} - B_{\alpha\beta}\bar{\tau}^{*\alpha\gamma}\mu^\beta_\gamma\xi)d\xi$$

where μ^α_γ and μ are given previously by (30.17) and (30.18). Making use of (31.14) to (31.16) in (31.20) we obtain

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} \quad , \quad V^\alpha = \bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = Dk^\alpha + DB^\alpha_\gamma s^{\gamma 3} + s^{\beta\alpha}D_{,\beta}$$
$$\bar{s}^{ij} = Ds^{ij} \quad , \quad V^3 = Dk^3 - DB_{\alpha\beta}s^{\alpha\beta} + D_{,\alpha}s^{\alpha 3} \tag{31.21}$$

which relates the two sets of definitions (31.11) and (31.20).

32. Linear constitutive relations for composite laminates: An alternative procedure

In this section we introduce an alternative procedure for the derivation of the linear constitutive equations for a composite laminate. The method takes advantage of the specific Gibbs energy function[2].

We recall that the central idea in the derivation of the constitutive relation for an elastic composite laminate was that the specific internal energy is given by a function of the form (29.1) where in the case of the linear theory it reduces to a quadratic function of its arguments. As mentioned previously, although expression (29.6) is elegant, the explicit integration of (29.6) in most cases becomes exceedingly difficult. Here we provide and alternative approach for explicit derivation of the constitutive relations (for the linear theory of a composite laminate) in which the coefficients are related to the elastic constants of the constituents.

We recall that the constitutive equations of the classical linear theory of elasticity in the context of purely mechanical theory may be expressed in terms of the three-dimensional specific Gibbs free energy function, say ϕ^*, in the form[3]

$$\gamma^*_{ij} = -\rho^*_o \frac{\partial \phi^*}{\partial \tau^{*ij}} \tag{32.1}$$

where γ^*_{ij} is the infinitesimal strain and where ϕ^* and ψ^* are related through

$$\phi^* = \phi^*(\tau^{*ij}) = \psi^*(\gamma^*_{ij}) - \frac{1}{\rho^*_o} \tau^{*ij}\gamma^*_{ij} \tag{32.2}$$

and ϕ^* and ψ^* are quadratic functions of their arguments and both also depend on the reference

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[2] This idea was first introduced by Green, Naghdi and Wenner [1971], in the context of Cosserat surface theory.

[3] The partial derivative $\frac{\partial \phi^*}{\partial \tau^{*ij}}$ is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \phi^*}{\partial \tau^{*ij}} + \frac{\partial \phi^*}{\partial \tau^{*ij}}\right)$$

values of G^*_{ij}. It may be noted that the function ϕ^* defined by (32.2) is the negative of the expression for the complimentary energy density. We now recall that the Gibbs function ϕ^* for an initially homogeneous and isotropic material can be expressed as

$$\rho^*_o\phi^* = \{-\frac{1+\nu^*}{2E^*}G^*_{im}G^*_{jn} + \frac{\nu^*}{2E^*}G^*_{ij}G^*_{mn}\}\,\tau^{*ij}\tau^{*mn} \tag{32.3}$$

where G^*_{ij} is the initial metric tensor, E^* is Young's modulus of elasticity and ν^* is Poisson's ratio.

Within the scope of the linear theory and corresponding to (29.6) we define a *composite Gibbs free energy* (or a "composite complementary energy") ϕ as follows:

$$\rho_o G^{1/2}\phi = \int_o^{\xi_2} \rho^*_o G^{*1/2}\phi^* d\xi \tag{32.4}$$

From (32.2), by integration with respect to ξ between zero and ξ^2 we obtain

$$\int_o^{\xi_2} \rho^*_o G^{*1/2}\phi^* d\xi = \int_o^{\xi_2} \rho^*_o G^{*1/2}\psi^* d\xi - \int_o^{\xi_2} G^{*1/2}\tau^{*ij}\gamma^*_{ij} d\xi \tag{32.5}$$

Considering (23.6), (32.2) and (32.4), we may rewrite (32.5) as

$$\rho_o G^{1/2}\phi = \rho_o G^{1/2}\psi - \int_o^{\xi_2} \nu G^{1/2}\tau^{*ij}\gamma^*_{ij} d\xi$$

or

$$\phi = \psi - \frac{1}{\rho_o}\int_o^{\xi_2} \nu\tau^{*ij}\gamma^*_{ij} d\xi \tag{32.6}$$

where in obtaining (3.26) we have made use of (30.29). By making use of the expressions for τ^{*ij}, γ^*_{ij}, the expressions for various resultants and the kinematic assumptions for **R** and **D**, we can express the integral in (32.6) in terms of the various resultants and their corresponding relative kinematic measures. However, as before the constitutive relations for the interlaminar stress vectors T^i should be specified directly. Keeping this and expressions (32.2) and (32.6) in mind, we assume the existence of a Gibbs free energy function φ, such that

$$\rho_o\phi = \rho_o\bar{\phi}(\bar{\tau}^{ij}, s^{ij}, k^i) = \rho_o\psi - \{\bar{\tau}^{ij}\gamma_{ij} + s^{ij}\mathcal{K}_{ij} + k^i\gamma_i\} \tag{32.7}$$

Differentiating both sides of (32.7) with respect to t, we obtain

$$\rho_o\dot{\mathcal{E}} = \rho_o\dot{\psi} = \rho_o\dot{\bar{\phi}} + \overline{(\bar{\tau}^{ij}\gamma_{ij})}^{\cdot} + \overline{(s^{ij}\mathcal{K}_{ij})}^{\cdot} + \overline{(k^i\gamma_i)}^{\cdot}$$

$$= \rho_o\dot{\bar{\phi}} + \dot{\bar{\tau}}^{ij}\gamma_{ij} + \bar{\tau}^{ij}\dot{\gamma}_{ij} + \dot{s}^{ij}\mathcal{K}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + \dot{k}^i\gamma_i + k^i\dot{\gamma}_i \tag{32.8}$$

Next, we substitute (32.8) in the expression for power (31.3)

$$\rho_o\dot{\bar{\phi}} + \dot{\bar{\tau}}^{ij}\gamma_{ij} + \dot{s}^{ij}\mathcal{K}_{ij} + \dot{k}^i\gamma_i + \bar{\tau}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i = \bar{\tau}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i$$

or

$$\rho_o\left(\frac{\partial\bar{\phi}}{\partial\bar{\tau}^{ij}}\dot{\bar{\tau}}^{ij} + \frac{\partial\bar{\phi}}{\partial s^{ij}}\dot{s}^{ij} + \frac{\partial\bar{\phi}}{\partial k^i}\dot{k}^i\right) + \gamma_{ij}\dot{\bar{\tau}}^{ij} + \mathcal{K}_{ij}\dot{s}^{ij} + \gamma_i\dot{k}^i = 0$$

or

$$\left(\gamma_{ij} + \rho_o\frac{\partial\bar{\phi}}{\partial\bar{\tau}^{ij}}\right)\dot{\bar{\tau}}^{ij} + \left(\mathcal{K}_{ij} + \rho_o\frac{\partial\bar{\phi}}{\partial s^{ij}}\right)\dot{s}^{ij} + \left(\gamma_i + \rho_o\frac{\partial\bar{\phi}}{\partial k^i}\right)\dot{k}^i = 0 \tag{32.9}$$

where we have assumed the rates $\dot{\bar{\tau}}^{ij}$, $\dot{s}^{ij}$ and $\dot{k}^i$ are all independent and their coefficients are rate independent. From (32.9) it follows

$$\gamma_{ij} = -\rho_o\frac{\partial\bar{\phi}}{\partial\bar{\tau}^{ij}}$$

$$\mathcal{K}_{ij} = -\rho_o\frac{\partial\bar{\phi}}{\partial s^{ij}} \tag{32.10}$$

$$\gamma_i = -\rho_o\frac{\partial\bar{\phi}}{\partial k^i}$$

We note that the relationship between the Gibbs energy function ϕ, per unit mass of the composite, and those of the constituents is given by

$$\rho_o G^{1/2}\phi = \int_o^{\xi_2} \rho_o^* G^{*1/2}\phi^* d\xi \tag{32.11}$$

or

$$\rho_o\phi = \int_o^{\xi_2} \nu(\rho_o^*\phi^*)d\xi = \int_o^{h_2}\mu(\rho_o^*\phi^*)d\xi = \int_o^{h_1}\mu(\rho_{o1}^*\phi_1^*)d\xi + \int_{h_1}^{h_2}\mu(\rho_{o2}^*\phi_2^*)d\xi \tag{32.12}$$

where o^* for an isotropic elastic material is given by (32.3). The explicit determination of the various coefficients in constitutive relations is beyond the scope of this project and is left for a follow-on project.

33. Some results for the case of a normal director

We recall that in the context of the present Cosserat composite theory, director **d** is a three dimensional vector associated with each composite particle and in general the only restriction placed on **d** is that it cannot be tangent to any ply. The case in which the director **D**, at each composite particle in the reference configuration, is taken to be the unit normal to the ply is of special interest. In such case, in order to allude to the direction of **D** in the present configuration, we may refer to director as "normal director." This section contains some results for the case of a normal director. The results of this section will be helpful when we apply the theory to the cases of initially flat and initially cylindrical composite laminates. Therefore, in this section as well as in the rest of this development and within the context of the linearized theory we confine our attention to the case for which D is unit vector. Hence, we make the following kinematical assumptions in the reference (initial) configuration:

$$\mathbf{R}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}}(\theta^\alpha) + \theta^3\mathbf{A}_3 \tag{33.1}$$

$$\mathbf{D} = \mathbf{A}_3 \tag{33.2}$$

and

$$\mathbf{P}^*(\theta^\alpha,\theta^3,\xi) = \mathbf{R}(\theta^\alpha,\theta^3) + \xi\mathbf{D}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}} + (\theta^3 + \xi)\mathbf{A}_3 \tag{33.3}$$

where $\mathbf{P}^*$ is the position vector of an arbitrary point P^* of the micro-body, **R** is the position vector of the point P, corresponding to P^*, in the macro-body, and **D** is the director at point P. It is worth observing that in (33.1) and (33.3) the term involvin ξ accounts for the effect of micro-structure while the term involve θ^3 represents the continuum behavior of the macro-structure, namely the composite laminate. In this connection it is important to realize that if, at the outset, in (33.3), we discard ξ with respect to θ^3 we will lose the effect of the micro-structure in the continuum formulation of composite laminates.

From (33.2) it follows

$$D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha) = 1 \tag{33.4}$$

and

$$\zeta = \xi \tag{33.5}$$

From (33.1) we obtain

$$G_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} = \bar{\mathbf{R}}_{,\alpha} + \theta^3 \mathbf{A}_{3,\alpha} = \bar{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{A}_\gamma = \bar{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{G}_\gamma$$

$$G_3 = \frac{\partial \mathbf{R}}{\partial \theta^3} = \mathbf{R}_{,3} = \mathbf{A}_3 \tag{33.6}$$

Making use of (33.6) we write

$$G_{\alpha\beta} = \bar{\mathbf{R}}_{,\alpha} \cdot \bar{\mathbf{R}}_{,\beta} - \theta^3 (B^\gamma{}_\alpha \bar{\mathbf{R}}_{,\beta} + B^\gamma{}_\beta \bar{\mathbf{R}}_{,\alpha}) \cdot \mathbf{A}_\gamma + (\theta^3)^2 B^\gamma{}_\alpha B^\delta{}_\beta A_{\gamma\delta}$$

$$G_{\alpha 3} = (\bar{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{A}_\gamma) \cdot \mathbf{A}_3 = 0 \tag{33.7}$$

$$G_{33} = \mathbf{A}_3 \cdot \mathbf{A}_3 = 1$$

and

$$G^{1/2} = [\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3] = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{A}_3$$

$$= (\bar{\mathbf{R}}_{,1} \times \bar{\mathbf{R}}_{,2}) \cdot \mathbf{A}_3 - \theta^3 \beta^\gamma{}_1 (\mathbf{A}_\gamma \times \bar{\mathbf{R}}_{,2}) \cdot \mathbf{A}_3 - \theta^3 B^\gamma{}_2 (\bar{\mathbf{R}}_{,1} \times \mathbf{A}_\gamma) \cdot \mathbf{A}_3$$

$$+ (\theta^3)^2 B^\gamma{}_1 B^\delta{}_2 (\mathbf{A}_\gamma \times \mathbf{A}_\delta) \cdot \mathbf{A}_3 \tag{33.8}$$

Moreover, from (33.2) and (33.3) it follows

$$\mathbf{G}^*_\alpha = \frac{\partial \mathbf{P}^*}{\partial \eta^\alpha} = \frac{\partial \mathbf{P}^*}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} + \xi \mathbf{A}_{3,\alpha} = \mathbf{R}_{,\alpha} - \xi B^\gamma{}_\alpha \mathbf{A}_\gamma = \nu^\gamma{}_\alpha \mathbf{G}_\gamma$$

$$\mathbf{G}^*_3 = \frac{\partial \mathbf{P}^*}{\partial \xi} = \mathbf{A}_3 \tag{33.9}$$

where

$$\nu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \xi B^\gamma{}_\alpha = \mu^\gamma{}_\alpha \tag{33.10}$$

and

$$\nu = D\,\det(\nu^{\gamma}{}_{\alpha}) = \det(\nu^{\gamma}{}_{\alpha}) = \det(\mu^{\gamma}{}_{\alpha}) = \mu \tag{33.11}$$

Making use of (33.9), we obtain

$$G^{*}_{\alpha\beta} = \nu^{\gamma}{}_{\alpha}\nu^{\delta}{}_{\beta}G_{\gamma\delta} = \mu^{\gamma}{}_{\alpha}\mu^{\delta}{}_{\beta}G_{\gamma\delta}$$

$$G^{*}_{\alpha 3} = 0 \tag{33.12}$$

$$G^{*}_{33} = 1$$

and

$$G^{*1/2} = G^{1/2}\det(\nu^{\alpha}{}_{\beta}) = \nu G^{1/2} = \mu G^{1/2}$$

or

$$\nu = \left(\frac{G^{*}}{G}\right)^{1/2} = \mu \tag{33.13}$$

In view of (33.1) to (33.3) and (33.10) to (33.13) expressions (31.11) are reduced to

$$\tau^{\alpha\beta} = \int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha\gamma}\nu^{\beta}{}_{\gamma}d\xi \quad , \quad \tau^{\alpha 3} = \int_{0}^{\xi_2=h_2} \nu t a^{*\alpha 3}d\xi$$

τ^{3i} or T^3 are specified by a constitutive equation directly.

$$s^{\alpha\beta} = \int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha\gamma}\nu^{\beta}{}_{\gamma}\xi d\xi \quad , \quad s^{\alpha 3} = \int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha 3}\xi d\xi$$

$$s^{31} = 0 \qquad \text{or} \qquad S^3 = 0 \tag{33.14}$$

$$k^{\alpha} = \int_{0}^{\xi_2=h_2} \nu\tau^{*3\gamma}\nu^{\alpha}{}_{\gamma}d\xi \quad , \quad k^3 = \int_{0}^{\xi_2=h_2} \nu\tau^{*33}d\xi$$

while definitions (31.20) become

$$\bar{\tau}^{\alpha\beta} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu^{\beta}{}_{\gamma}d\xi \quad , \quad v^{\alpha} = \bar{\tau}^{\alpha 3} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha 3}d\xi$$

$$\bar{\tau}^{3\alpha} = \tau^{3\alpha} \quad , \quad \bar{\tau}^{33} = \tau^{33}$$

$$\bar{s}^{\alpha\beta} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu^{\beta}{}_{\gamma}\xi d\xi \quad , \quad \bar{s}^{\alpha 3} = \int_0^{h_2} \mu\bar{\tau}^{*\alpha 3}\xi d\xi \qquad (33.15)$$

$$v^3 = \int_0^{h_2} \mu(\bar{\tau}^{*33} - B_{\alpha\beta}\bar{\tau}^{*\alpha\gamma}\mu^{\beta}{}_{\gamma}\xi)d\xi$$

Also, the transformation between $\bar{\tau}^{*ij}$ and τ^{*ij}, namely expressions (33.14) to (33.16), are now given by

$$\bar{\tau}^{*ij} = \tau^{*ij} \qquad (33.16)$$

Finally the relations between the two sets of definitions (33.14) and (33.15) are reduced to

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} \qquad v^{\alpha} = \bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = k^{\alpha} + B^{\alpha}{}_{\gamma}s^{\gamma 3}$$

$$\bar{s}^{ij} = s^{ij} \qquad v^3 = k^3 - B_{\alpha\beta}s^{\alpha\beta} \qquad (33.17)$$

34. Theory of initially flat composite laminates

We are now in a position to apply the theory of Cosserat composite to special initial geometric configurations. In this section we apply the theory to the case of an initially flat composite laminate. The case of an initially cylindrical composite laminate will be considered in the next section.

Consider a composite laminate and let its plies be flat (i.e., having no curvature) in the reference (initial) configuration. Let $\mathbf{e}_i$ (i = 1,2,3) be the base vectors associated with a system of Cartesian coordinates x_i (i= 1,2,3). The position vector of a plane surface perpendicular to $\mathbf{e}_3$ and passing through the point (0,0,c) may be specified by

$$\mathbf{p}(x^i) = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + c\mathbf{e}_3 \tag{34.1}$$

where c is a constant. In view of (34.1) and recalling formulae (33.1) to (33.3) we adopt the following kinematical assumptions for an initially flat composite laminate:

$$\mathbf{R}(x^\alpha,x^3) = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \tag{34.2}$$

$$\mathbf{D} = \mathbf{A}_3 = \mathbf{e}_3 \tag{34.3}$$

and

$$\mathbf{P}^*(x^\alpha,x^3,\xi) = \mathbf{R} + \zeta\mathbf{e}_e = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + (x^3 + \zeta)\mathbf{e}_3 \tag{34.4}$$

We recall that (34.2) specifies the position of an arbitrary macro-particle of the composite laminate while the position vector of the micro-particle corresponding to the macro-particle is given by (34.4).

First we proceed to obtain various quantities associated with the surface (34.2). The base vectors of the surface are obtained from (34.2) as follows:

$$A_\alpha = \frac{\partial \mathbf{R}}{\partial x^\alpha}$$

$$A_1 = \frac{\partial \mathbf{R}}{\partial x^1} = \mathbf{e}_1 \quad , \quad A_2 = \frac{\partial \mathbf{R}}{\partial x^2} = \mathbf{e}_2 \tag{34.5}$$

The components of the surface metric tensor are

$$A_{\alpha\beta} = A_\alpha \cdot A_\beta$$

$$A_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 \quad , \quad A_{12} = A_{21} = A_1 \cdot A_2 = 0 \quad , \quad A_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1 \tag{34.6.a}$$

or

$$(A_{\alpha\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{34.6.b}$$

Moreover, we have

$$A^{\alpha\beta} A_{\beta\gamma} = \delta^\alpha{}_\gamma \Rightarrow A^{\alpha\beta} = (A_{\alpha\beta})^{-1} \tag{34.7}$$

Hence,

$$A^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{34.8}$$

The conjugate base vectors of the surface are given by

$$A^\alpha = A^{\alpha\beta} A_\beta$$

Therefore

$$A^1 = A^{11} A_1 + A^{12} A_2 = A_1 = \mathbf{e}_1 \quad , \quad A^2 = A^{21} A_1 + A^{22} A_2 = A_2 = \mathbf{e}_2 \tag{34.9}$$

The unit normal to the surface follows from (34.9):

$$A_3 = \frac{A_1 \times A_2}{|A_1 \times A_2|} = \mathbf{e}_3 \tag{34.10}$$

which confirms (34.3). In view of (34.5.b) and the expressions for Christoffel symbols of the first and second kind, i.e.,

$$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial A_{\gamma\beta}}{\partial x^{\alpha}} + \frac{\partial A_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial A_{\alpha\beta}}{\partial x^{\gamma}}\right)$$

and (34.11)

$$\{_{\alpha}{}^{\gamma}{}_{\beta}\} = a^{\gamma\delta}[\alpha\beta,\delta]$$

In view of (34.6.b), it is clear that all Christoffel symbols vanish, i.e.,

$$[\alpha\beta,\gamma] = \{_{\alpha}{}^{\gamma}{}_{\beta}\} = 0 \qquad (34.12)$$

Coefficients of the second fundamental form of the surface are given by

$$B_{\alpha\beta} = A_{\alpha,\beta} \cdot A_3 = - A_{\beta} \cdot A_{3,\alpha} \qquad (34.13)$$

It then follows from (34.4) and (34.13) that

$$B_{\alpha\beta} = B^{\alpha}{}_{\beta} = 0 \qquad (34.14)$$

This shows that for an initially flat ply (plate) the components of the second fundamental form of the surface vanish identically.

Next, we obtain the various kinematical quantities associated with micro and macro continuua for the case of initially flat composite laminate. From (33.6) it follows

$$G_{\alpha} = \frac{\partial R}{\partial \theta^{\alpha}} = \frac{\partial R}{\partial x^{\alpha}} = e_{\alpha}$$

(34.15)

$$G_3 = \frac{\partial R}{\partial \theta^3} = \frac{\partial R}{\partial x^3} = e_3$$

From (34.15) we obtain

$$G_{\alpha\beta} = \mathbf{R}_{,\alpha} \cdot \mathrm{RT}_{,\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$$

$$G_{\alpha 3} = \mathbf{R}_{,\alpha} \cdot \mathbf{G}_3 = \mathbf{e}_\alpha \cdot \mathbf{e}_3 = 0 \qquad (34.16.a)$$

$$G_{33} = \mathbf{G}_3 \cdot \mathbf{G}_3 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 0$$

or

$$(G_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (34.16.b)$$

Also,

$$G^{1/2} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1 \qquad (34.17)$$

Moreover, from (33.9), (34.4) and (34.14) we have

$$\mathbf{G}^*_\alpha = \frac{\partial \mathbf{P}^*}{\partial \eta^\alpha} = \frac{\partial \mathbf{P}^*}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} - \xi B^\gamma{}_\alpha \mathbf{A}_\gamma = \mathbf{e}_\alpha$$

$$\qquad (34.18)$$

$$\mathbf{G}^*_3 = \frac{\partial \mathbf{P}^*}{\partial \zeta} = \frac{\partial \mathbf{P}^*}{\partial \xi} = \mathbf{A}_3 = \mathbf{e}_3$$

Also, from (33.10) and (33.11) we obtain

$$\nu^\gamma{}_\alpha = \mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha \qquad (34.19)$$

and

$$\nu = D\,\det(\nu^\gamma{}_\alpha) = \mu = 1 \qquad (34.20)$$

Making use of (34.18), we obtain

$$G^*_{\alpha\beta} = \mathbf{G}^*_\alpha \cdot \mathbf{G}^*_\beta = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$$

$$G^*_{\alpha 3} = \mathbf{G}^*_\alpha \cdot \mathbf{G}^*_3 = \mathbf{e}_\alpha \cdot \mathbf{e}_3 = 0 \qquad (34.21.a)$$

$$G^*_{33} = \mathbf{G}^*_3 \cdot \mathbf{G}^*_3 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

or

$$(G^{*}_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{34.21.b}$$

It then follows that

$$G^{*1/2} = (\mathbf{G}^{*}_{1} \times \mathbf{G}^{*}_{2}) \cdot \mathbf{G}^{*}_{3} = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1 \tag{34.22}$$

By (34.17) and (34.22) we have

$$\left(\frac{G^{*}}{G}\right)^{1/2} = 1 = \nu = \mu \tag{34.23}$$

which confirms (34.20). In view of (34.2) to (34.4) and (34.23) formulae (33.14) and (33.15) simplify as follows:

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} = \int_{o}^{h_2} \tau^{*\alpha\beta} d\zeta \quad , \quad \bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = v^{\alpha} = k^{\alpha} = \int_{o}^{h_2} \tau^{*\alpha 3} d\zeta$$

$\bar{\tau}^{3i} = \tau^{3i}$ be specified by a constitutive relation directly

$$\bar{s}^{\alpha\beta} = s^{\alpha\beta} = \int_{o}^{h_2} \tau^{*\alpha\beta} \zeta d\zeta \quad , \quad \bar{s}^{\alpha 3} = s^{\alpha 3} = \int_{o}^{h_2} \tau^{*\alpha 3} \zeta d\zeta \tag{34.24}$$

$$s^{3i} = 0$$

$$k^3 = v^3 = \int_{0}^{h_2} \tau^{*33} d\zeta$$

where in obtaining formulae (34.24) we have noticed that

$$\bar{\tau}^{*ij} = \tau^{*ij} \tag{34.25}$$

where τ^{*ij} are now Cartesian components of the classical stress tensor.

We recall at this point that because all quantities are now referred to rectangular Cartesian axes, covariant differentiation with respect to metric tensory G_{ij} is reduced to partial differentiation with respect to x^i (or x_i)and no distinction needs to be made between superscripts and subscripts. In view of this, expressions (26.30) to (26.32) are reduced to [1]

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{34.26}$$

$$\gamma_i = \delta_i + u_{3,i} \tag{34.27}$$

$$\mathcal{K}_{ij} = \delta_{i,j} \tag{34.28}$$

Finally, with the help of (34.3) equations of motion for the case of an initially composite laminate are reduced to [1]

$$\tau_{ij,i} + \rho_o b_j = \rho_o(\ddot{u}_j + y^1\ddot{\delta}_j) \tag{34.29}$$

$$s_{ij,i} + (\rho_o c_j - k_j) = \rho_o(y^1\ddot{\delta}_j + y^2\ddot{\delta}_j) \tag{34.30}$$

$$\varepsilon_{ijn}\tau_{ij} = 0 \tag{34.31}$$

We observe that in (34.31), ε_{ijn} is skew-symmetric with respect to i and j; hence it follows that

$$\tau_{ij} = \tau_{ji} \tag{34.32}$$

This indicates that in the case of an initially flat composite laminate the components of the composite stress tensor are symmetric. The same conclusion can be reached from expressions $(34.24)_{1,2}$, i.e.,

$$\tau^{ij} = \tau_{ij} = \int_0^{h_2} \tau^*_{ij} d\xi$$

in view of the symmetry of the classical stress tensor.

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[1] In expressions like (34.27) the Greek letters such as δ_i denote components of the director displacement vectors, etc. This should not be confused with the use of Greek letters as indices in various expressions.

35. Theory of initially cylindrical composite laminates

In this section we continue to apply the theory of Cosserat composite to initially cylindrical composite laminates.

Consider a composite laminate and let its plies form a set of concentric right circular cylindrical surfaces. Let x^i (i = 1,2,3) and $\{r,\theta,z\}$ denote Cartesian and cylindrical coordinates with a common origin in a Euclidean three-dimensional space. Let $\mathbf{e}_i$ (i = 1,2,3) and $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ denote the unit base vectors in the foregoing coordinate systems, respectively. We recall that a right circular cylinder of radius r may be defined by a position vector of the form

$$\mathbf{P} = r\mathbf{e}_r + z\mathbf{e}_z \tag{35.1}$$

Recalling the relations between the unit base vectors in Cartesian and cylindrical coordinate systems, i.e.,

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \\ \mathbf{e}_\theta &= -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2 \\ \mathbf{e}_z &= \mathbf{e}_3 \end{aligned} \tag{35.2}$$

we obtain

$$\mathbf{P} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + z\mathbf{e}_3 \tag{35.3}$$

It is worth mentioning that sometimes it is more convenient to consider an alternative representation of the cylindriccal surface (35.3) as follows:

$$\mathbf{P} = (r\cos\frac{s}{r})\mathbf{e}_1 + (r\sin\frac{s}{r})\mathbf{e}_2 + z\mathbf{e}_3 \tag{35.4}$$

where $s = r\theta$ is the arclength measured from a fixed point ($\theta = 0$) along the section curve. Let us now introduce a set of coordinates θ^i (i = 1,2,3) such that

$$\theta^1 = r\vartheta \quad , \quad \theta^2 = z \quad , \quad \theta^3 = r \tag{35.5}$$

Hence, in terms of θ^i coordinates we have

$$\mathbf{P} = (\theta^3 \cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\theta^3 \sin \frac{\theta^1}{\theta^3})\mathbf{e}_2 + \theta^2 \mathbf{e}_3 \tag{35.6}$$

This representation will facilitate much of the intermediate steps especially in connection to calculation of the various quantities of the surface.

In view of the foregoing explanation, we now adopt the following kinematical assumptions for an initially cylindrical composite laminate

$$\bar{\mathbf{R}}(r,\theta,z) = r\mathbf{e}_r + z\mathbf{e}_z$$

$$\mathbf{D} = \mathbf{A}_3 = \mathbf{e}_r \tag{35.7}$$

$$\bar{\mathbf{P}}^*(r,\theta,z,\zeta) = (r+\zeta)\mathbf{e}_r + z\mathbf{e}_z$$

Making use of (35.5), we can rewrite this

$$\mathbf{R}(\theta^\alpha,\theta^3) = (\theta^3 \cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\theta^3 \sin \frac{\theta^1}{\theta^3})\mathbf{e}_2 + \theta^2 \mathbf{e}_3$$

$$\mathbf{D} = \mathbf{A}_3 = (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2 \tag{35.8}$$

$$\mathbf{P}^*(\theta^\alpha,\theta^3,\zeta) = [(\theta^3+\zeta)\cos \frac{\theta^1}{\theta^3}]\mathbf{e}_1 + [(\theta^3+\zeta)\sin \frac{\theta^1}{\theta^3}]\mathbf{e}_2 + \theta^2 \mathbf{e}_3$$

The base vectors of the surface are obtained from $(35.8)_1$ as follows

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha}$$

Hence we have

$$\mathbf{A}_1 = \mathbf{R}_{,\theta^1} = \frac{1}{\theta^3}\left(-\theta^3 \sin \frac{\theta^1}{\theta^3}\right)\mathbf{e}_1 + \frac{1}{\theta^3}\left(\theta^3 \cos \frac{\theta^1}{\theta^3}\right)\mathbf{e}_2$$

$$= -\left(\sin \frac{\theta^1}{\theta^3}\right)\mathbf{e}_1 + \left(\cos \frac{\theta^1}{\theta^3}\right)\mathbf{e}_2 = \mathbf{e}_\theta \qquad (35.9)$$

and

$$\mathbf{A}_2 = \mathbf{R}_{,\theta^2} = \mathbf{e}_3 = \mathbf{e}_z \qquad (35.10)$$

From (35.9) and (35.10) we obtain the components of the surface metric tensor $A_{\alpha\beta}$

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta$$

Therefore

$$A_{11} = \mathbf{A}_1 \cdot \mathbf{A}_1 = [-(\sin\theta^1)e_1 + \cos(\theta^1)e_2] \cdot [-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] = 1$$

$$A_{12} = A_{21} = \mathbf{A}_1 \cdot \mathbf{A}_2 = [-(\sin\theta^1)\mathbf{e}_1 + (\cos\theta^1)\mathbf{e}_2] \cdot \mathbf{e}_3 = 0 \qquad (35.11.a)$$

$$A_{22} = \mathbf{A}_2 \cdot \mathbf{A}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

or

$$(A_{\alpha\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (35.11.b)$$

Moreover, we have

$$A^{\alpha\beta}A_{\beta\gamma} = \delta^\alpha{}_\gamma \;\Rightarrow\; A^{\alpha\beta} = (A_{\alpha\beta})^{-1} \qquad (35.12)$$

Hence,

$$A^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (35.13)$$

The conjugate base vectors of the surface are given by

$$\mathbf{A}^\alpha = A^{\alpha\beta}\mathbf{A}_\beta$$

Thus,

$$A^1 = A^{11}A_1 + A^{12}A_2 = A_1 = e_\theta$$
$$A^2 = A^{21}A_1 + A^{22}A_2 = A_2 = e_3 = e_z \tag{35.14}$$

The unit normal to the surface follows from (35.14)

$$A_3 = \frac{A_1 \times A_2}{|A_1 \times A_2|} = \frac{1}{|A_1 \times A_2|}\{[-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] \times e_3\}$$

or

$$A_3 = \frac{1}{|A_1 \times A_2|}\{-(\sin\frac{\theta^1}{\theta^3})(e_1 \times e_3) + (\cos\frac{\theta^1}{\theta^3})(e_2 \times e_3)\}$$

$$= \frac{1}{|A_1 \times A_2|}\{-(\cos\frac{\theta^1}{\theta^3})e_1 + (\sin\frac{\theta^1}{\theta^3})e_2\}$$

$$= (\cos\frac{\theta^1}{\theta^3})e_1 + (\sin\frac{\theta^1}{\theta^3})e_2 = e_r$$

We note that A_3 could have been obtained from vector product of e_θ and e_z. However, to illustrate the general procedure we did not make use of e_θ and e_z. The Christoffel symbols of the first and second kind follow from (35.11)

$$[\alpha\beta,\gamma] = \{{}_\alpha{}^\gamma{}_\beta\} = 0 \tag{35.16}$$

and coefficients of the second fundamental form of the surface are given by

$$B_{\alpha\beta} = A_{\alpha,\beta} \cdot A_3 = - A_\alpha \cdot A_{3,\alpha}$$

hence,

$$B_{11} = \mathbf{A}_{1,1} \cdot \mathbf{A}_3 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2]_{,1} \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2]$$

$$= \frac{1}{\theta^3}[-(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 - (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = -\frac{1}{\theta^3} = -\frac{1}{r}$$

$$B_{12} = \mathbf{A}_{2,1} \cdot \mathbf{A}_3 = (\mathbf{e}_3)_{,1} \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$B_{21} = \mathbf{A}_{1,2} \cdot \mathbf{A}_3 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3}\mathbf{e}]_{,2} \cdot [\cos(\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$B_{22} = \mathbf{A}_{2,2} \cdot \mathbf{A}_3 = (\mathbf{e}_3)_{,2} \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

Therefore

$$(B_{\alpha\beta}) = \begin{bmatrix} -1/\theta^3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/r & 0 \\ 0 & 0 \end{bmatrix} \tag{35.17}$$

We also have

$$B^\alpha{}_\beta = A^{\alpha\gamma}B_{\gamma\beta}$$

Hence,

$$B^1{}_1 = A^{11}B_{11} + A^{12}B_{21} = -\frac{1}{\theta^3} = -\frac{1}{r}$$

$$B^1{}_2 = A^{11}B_{21} + A^{12}B_{22} = 0$$

$$B^2{}_1 = A^{21}B_{11} + A^{22}B_{21} = 0$$

$$B^2{}_2 = A^{21}B_{12} + A^{22}B_{22} = 0$$

or

$$(B^\alpha{}_\beta) = \begin{bmatrix} -1/\theta^3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/r & 0 \\ 0 & 0 \end{bmatrix} \tag{35.18}$$

Next, we obtain the various kinematical quantities associated with micro and macro continua for the case of initially cylindrical composite laminates. From (33.6) and (35.8) it follows that

$$\mathbf{G}_1 = \frac{\partial \mathbf{R}}{\partial \theta^1} = -(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2 = \mathbf{e}_\theta$$

$$\mathbf{G}_2 = \frac{\partial \mathbf{R}}{\partial \theta^2} = \mathbf{e}_3 = \mathbf{e}_z \tag{35.19}$$

$$\mathbf{G}_3 = \frac{\partial \mathbf{R}}{\partial \theta^3} = (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2 = \mathbf{e}_r$$

From (35.19) we obtain

$$G_{11} = \mathbf{G}_1 \cdot \mathbf{G}_1 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 1$$

$$G_{12} = G_{21} = \mathbf{G}_1 \cdot \mathbf{G}_2 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot \mathbf{e}_3 = 0$$

$$G_{13} = G_{31} = \mathbf{G}_1 \cdot \mathbf{G}_3 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$G_{22} = \mathbf{G}_2 \cdot \mathbf{G}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

$$G_{23} = G_{32} = \mathbf{G}_2 \cdot \mathbf{G}_3 = \mathbf{e}_3 \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$G_{33} = \mathbf{G}_3 \cdot \mathbf{G}_3 = [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 1$$

Hence

$$(G_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35.20}$$

We also have

$$G^{1/2} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3$$

$$= \{[-(\sin \frac{\theta^1}{\theta^3})\, \mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3})\, \mathbf{e}_2] \times \mathbf{e}_3\} \cdot [(\cos \frac{\theta^1}{\theta^3})\, \mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\, \mathbf{e}_2]$$

$$= \{[-(\sin \frac{\theta^1}{\theta^3})(-\mathbf{e}_2) + (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1] \cdot [(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] = 1 \qquad (35.21)$$

Moreover, from (33.9), (35.7) and (35.18) we have

$$\mathbf{G}_1^* = \frac{\partial \mathbf{P}^*}{\partial \theta^*} = \frac{\theta^3 + \zeta}{\theta^3}\, [-(\sin \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_2] = \frac{r+\zeta}{r}\, \mathbf{e}_\theta$$

$$\mathbf{G}_2^* = \frac{\partial \mathbf{P}^*}{\partial \theta^2} = \mathbf{e}_3 = \mathbf{e}_z \qquad (35.22)$$

$$\mathbf{G}_3^* = \frac{\partial \mathbf{P}^*}{\partial \zeta} = \frac{\partial \mathbf{P}^*}{\partial \zeta} = [(\cos \theta^1)\mathbf{e}_1 + (\sin\theta^1)\mathbf{e}_2] = \mathbf{e}_r$$

Also, from (33.10), (33.11) and (35.18) we obtain

$$\nu^\gamma{}_\alpha = \mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \zeta B^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \zeta B^\gamma{}_\alpha$$

$$\nu^1{}_1 = 1 - \zeta(-\frac{1}{\theta^3}) = 1 + \frac{\zeta}{r} = \frac{r+\zeta}{r}$$

$$\nu^1{}_2 = \nu^2{}_1 = 0$$

$$\nu^2{}_2 = 1$$

or

$$(\nu^\gamma{}_\alpha) = (\mu^\gamma{}_\alpha) = \begin{bmatrix} (r+\zeta)/r & 0 \\ 0 & 1 \end{bmatrix} \qquad (35.23)$$

and

$$\nu = D \det(\nu^\gamma{}_\alpha) = \det(\nu^\gamma{}_\alpha) = \mu = \frac{r+\zeta}{r} \qquad (35.24)$$

Making use of (35.22) we obtain

$$G^{*}_{11} = G^{*}_{1} \cdot G^{*}_{1} = (\frac{\theta^3+\zeta}{\theta^3})^2[-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] \cdot [-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] = (\frac{\theta^3+\zeta}{\theta^3})^2 = (\frac{r+\zeta}{r})^2$$

$$G^{*}_{12} = G^{*}_{21} = G^{*}_{1} \cdot G^{*}_{2} = (\frac{\theta^3+\zeta}{\theta^3})[-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] \cdot e_3 = 0$$

$$G^{*}_{13} = G^{*}_{31} = G^{*}_{1} \cdot G^{*}_{3} = (\frac{\theta^3+\zeta}{\theta^3})[-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] \cdot [-(\cos\theta^1)e_1 + (\sin\theta^1)e_2] = 0$$

$$G^{*}_{22} = G^{*}_{2} \cdot G^{*}_{2} = e_3 \cdot e_3 = 1$$

$$G^{*}_{23} = G^{*}_{32} = G^{*}_{2} \cdot G^{*}_{3} = e_3 \cdot [(\cos\theta^1)e_1 + (\sin\theta^1)e_2] = 0$$

$$G^{*}_{33} = G^{*}_{3} \cdot G^{*}_{3} = [(\cos\theta^1)e_1 + (\sin\theta^1)e_2] \cdot [(\cos\theta^1)e_1 + (\sin\theta^1)e_2] = 1$$

Hence,

$$(G^{*}_{ij}) = \begin{bmatrix} (1+\zeta/\theta^3)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1+\zeta/r)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35.25}$$

It then follows that

$$G^{*1/2} = \{\det(G^{*}_{ij})\}^{1/2} = (\frac{\theta^3+\zeta}{\theta^3}) = \frac{r+\zeta}{r} \tag{35.26}$$

By (35.21) and (35.26) we have

$$(\frac{G^{*}}{G})^{1/2} = \frac{r+\zeta}{r} = \nu = \mu \tag{35.27}$$

which confirms (35.24). In view of (35.8) and (35.27) formulae (33.14) and (33.15) reduce to

$$\bar{\tau}^{\alpha 1} = \tau^{\alpha 1} = \int_0^{h_2} (1 + \frac{\zeta}{r})\tau^{*(\alpha)}\nu^{1}_{1} d\zeta = \int_0^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha 1} d\zeta \tag{35.28.a}$$

$$\bar{\tau}^{\alpha 2} = \tau^{\alpha 2} = \int_0^{h_2} (1 + \frac{\zeta}{r})\tau^{*(\alpha)}\nu^{2}_{2} d\zeta = \int_0^{h_2} (1 + \frac{\zeta}{r})\tau^{*\alpha 2} d\zeta \tag{35.28.b}$$

$$\bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*\alpha 3} d\zeta = v^{\alpha} \tag{35.28.c}$$

$\bar{\tau}^{3i}$ be specified by a constitutive relation directly (35.28.d)

$$\bar{s}^{\alpha 1} = s^{\alpha 1} = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*\alpha\gamma} v^1{}_{\gamma} \zeta d\zeta = \int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha 1} \zeta d\zeta \tag{35.28.e}$$

$$\bar{s}^{\alpha 2} = s^{\alpha 2} = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*\alpha\gamma} v^2{}_{\gamma} \zeta d\zeta = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*\alpha 2} \zeta d\zeta \tag{35.28.f}$$

$$\bar{s}^{\alpha 3} = s^{\alpha 3} = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*\alpha 3} \zeta d\zeta \tag{35.28.g}$$

$$s^{3i} = 0 \quad \text{or} \quad S^3 = 0 \tag{35.28.h}$$

$$k^1 = \int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*31} d\zeta \ , \ k^2 = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*32} d\zeta \tag{35.28.i}$$

$$k^3 = \int_o^{h_2} (1 + \frac{\zeta}{r}) \tau^{*33} d\zeta \tag{35.28.j}$$

$$v^3 = \int_o^{h_2} (1 + \frac{\zeta}{r}) [\tau^{*33} - (1 + \frac{\zeta}{r}) \tau^{*11} \frac{\zeta}{r}] d\zeta \tag{35.28.k}$$

It is interesting to observe that when the radius of the cylindrical laminate becomes large (i.e., when the cylindrical surface approaches a flat surface) the value of $\frac{\zeta}{r}$ becomes small and may be neglected in comparison to unity (ideally $\frac{\zeta}{r}$ approaches zero) and the various expression obtained in this section will reduce to those obtained for an initially flat composite laminate.

The relative kinematical measures γ_{ij}, γ_i and $\mathcal{K}_{ij}$ are now given by

$$\gamma_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}) \tag{35.29}$$

$$\gamma_i = \delta_i + u_{3|i} \tag{35.30}$$

$$\mathcal{K}_{ij} = \delta_{i|j} \tag{35.31}$$

where a vertical bar (|) denotes covariant differentiation with respect to coordinates θ^i (i = 1,2,3) as specified by (35.5). Moreover, equations of motion are given by

$$\tau^{i}_{j|i} + \rho_o b_j = \rho_o(\alpha_j + y^1\beta_j) \tag{35.32}$$

$$s^{i}_{j|i} + (\rho_o c_i - k_j) = \rho_o(y^1\alpha_j + y^2\beta_j) \tag{35.33}$$

where all components in the above are referred to coordinates θ^i (i = 1,2,3).

For convenience and systematic reduction of various results of this section we adopted the coordinate system (35.5). However, most of the available results in continuum mechanics regarding cylindrical bodies are in terms of the cylindrical coordinates r,θ,z. In order to write the relevant results of this section in terms of r,θ,z we consider the representation (35.7) and adopt a system of cylindrical coordinates r,θ,z such that

$$\theta^1 = \theta \quad , \quad \theta^2 = z \quad , \quad \theta^3 = r \tag{35.34}$$

From $(35.7)_1$ and (35.34) it follows

$$G_1 = r e_\theta \quad , \quad G_2 = e_z \quad , \quad G_3 = e_r \tag{35.35}$$

and

$$(G_{ij}) = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{ij}) = \begin{bmatrix} 1/r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35.36}$$

$$G^{1/2} = r \tag{35.37}$$

Moreover, from $(35.7)_3$ we obtain

$$G_1^* = (r+\zeta)e_\theta \quad , \quad G_2 = e_z \quad , \quad G_3 = e_r \tag{35.38}$$

and

$$(G^{*}_{ij}) = \begin{bmatrix} (r+\zeta)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{*ij}) = \begin{bmatrix} 1/(r+\zeta)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (35.39)$$

$$G^{*1/2} = (r+\zeta) \qquad (35.40)$$

From (33.13), (35.37) and (35.40) it follows

$$v = \mu = (\frac{G^*}{G})^{1/2} = \frac{r+\zeta}{r} = (1 + \frac{\zeta}{r}) \qquad (35.41)$$

as before. In order to calculate expresisons involving covariant differentiation we need to calculate the Christoffel symbols of the first and second kind. Christoffel symbols of the first kind are given by

$$[ijk] = \frac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k}) \qquad (35.42)$$

The only non-vanishing Christoffel symbols of the first kind are

$$[311] = r \quad , \quad [131] = r \quad , \quad [113] = -r \qquad (35.43)$$

Christoffel symbols of the second kind are given by

$$\{_i{}^k{}_j\} = g^{km}[ijm] \qquad (35.44)$$

From (35.43) and (35.44) the only non-vanishing Christoffel symbols of the second kind are

$$\{_1{}^3{}_1\} = -r \quad , \quad \{_1{}^1{}_3\} = \frac{1}{r} \quad , \quad \{_3{}^1{}_1\} = \frac{1}{r} \qquad (35.45)$$

The physical components of the displacement vector **u** and the director displacement δ are given by

$$\bar{u}_1 = (g_{11})^{-1/2} u_1 = (g_{11})^{1/2} u^1 = u_\theta$$

$$\bar{u}_2 = (g_{22})^{-1/2} u_2 = (g_{22})^{1/2} u^2 = u_z \qquad (35.46)$$

$$\bar{u}_3 = (g_{33})^{-1/2} u_3 = (g_{33})^{1/2} u^3 = u_r$$

$$\bar{\delta}_1 = (g_{11})^{-1/2} \delta_1 = (g_{11})^{1/2} \delta^1 = \delta_\theta$$

$$\bar{\delta}_2 = (g_{22})^{-1/2} \delta_2 = (g_{22})^{1/2} \delta^2 = \delta_z \qquad (35.47)$$

$$\bar{\delta}_3 = (g_{33})^{-1/2} \delta_3 = (g_{33})^{1/2} \delta^3 = \delta_r$$

The physical components of γ_{ij} are

$$\gamma_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}$$

$$(35.48)$$

$$\gamma_{\theta z} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)$$

$$\gamma_{\theta r} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) - \frac{u_r}{r}$$

$$\gamma_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)$$

and the physical components of γ_i are

$$\gamma_\theta = \delta_\theta + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$

$$\gamma_z = \delta_z + \frac{\partial u_r}{\partial z} \qquad (35.49)$$

$$\gamma_r = \delta_r + \frac{\partial u_r}{\partial r}$$

Also, the physical components of K_{ij} are given by

$$\mathcal{K}_{\theta\theta} = \frac{1}{r}\,\frac{\partial \delta_\theta}{\partial \theta} + \frac{\delta_r}{r}$$

$$\mathcal{K}_{zz} = \frac{\partial \delta_z}{\partial z}$$

$$\mathcal{K}_{rr} = \frac{\partial \delta_r}{\partial r}$$

$$\mathcal{K}_{\theta z} = \frac{\partial \delta_\theta}{\partial z} \quad , \quad \mathcal{K}_{z\theta} = \frac{1}{r}\,\frac{\partial \delta_z}{\partial \theta}$$

$$\mathcal{K}_{\theta r} = \frac{\partial \delta_\theta}{\partial r} - \frac{\delta_\theta}{r} \quad , \quad \mathcal{K}_{r\theta} = \frac{1}{r}\,\frac{\partial \delta_r}{\partial \theta} - \frac{\delta_\theta}{r}$$

$$\mathcal{K}_{zr} = \frac{\partial \delta_z}{\partial r} \quad , \quad \mathcal{K}_{rz} = \frac{\partial \delta_r}{\partial z}$$

Next, we note that the physical components of the stress tensor and stress couple tensor may be written as

$$\tau_{\theta\theta} = r^2\tau^{11} = \frac{1}{r^2}\,\tau_{11}$$

$$\tau_{zz} = \tau^{22} = \tau_{22}$$

$$\tau_{rr} = \tau^{33} = \tau_{33}$$

$$\tau_{\theta z} = r\tau^{12} = \frac{1}{r}\,\tau_{12} \quad , \quad \tau_{z\theta} = r\tau^{21} = \frac{1}{r}\,\tau_{21} \tag{35.51}$$

$$\tau_{\theta r} = r\tau^{13} = \frac{1}{r}\,\tau_{13} \quad , \quad \tau_{r\theta} = r\tau^{31} = \frac{1}{r}\,\tau_{31}$$

$$\tau_{zr} = \tau^{23} = \tau_{23} \quad , \quad \tau_{rz} = \tau^{32} = \tau_{32}$$

$$s_{\theta\theta} = r^2 s^{11} = \frac{1}{r^2} s_{11}$$

$$s_{22} = s^{22} = s_{22}$$

$$s_{rr} = s^{33} = s_{33}$$

$$s_{\theta z} = r s^{12} = \frac{1}{r} s_{12} \quad , \quad s_{z\theta} = r s^{21} = \frac{1}{r} s_{21} \tag{35.52}$$

$$s_{\theta r} = r s^{13} = \frac{1}{r} s_{13} \quad , \quad s_{r\theta} = r s^{31} = \frac{1}{r} s_{31}$$

$$s_{zr} = s^{23} = s_{23} \quad , \quad s_{rz} = s^{32} = s_{32}$$

Moreover, the physical components of **b**, **c** and **k** are given by

$$b_\theta = r b^1 = \frac{1}{r} b_1 \quad , \quad b_z = b^2 = b_2 \quad , \quad b_r = b^3 = b_3 \tag{35.53}$$

$$c_\theta = r c^1 = \frac{1}{r} c_1 \quad , \quad c_z = c^2 = c_2 \quad , \quad c_r = c^3 = c_3 \tag{35.54}$$

$$k_\theta = r k^1 = \frac{1}{r} k_1 \quad , \quad k_z = k^2 = k_2 \quad , \quad k_r = k^3 = k_3 \tag{35.55}$$

Also, from (25.46) and (35.47) we have

$$\ddot{u}_\theta = r\alpha^1 = \frac{1}{r} \alpha_1 \quad , \quad \ddot{u}_z = \alpha^2 = \alpha_2 \quad , \quad \ddot{u}_r = \alpha^3 = \alpha_3 \tag{35.56}$$

and

$$\ddot{\delta}_\theta = r\beta^1 = \frac{1}{r} \beta_1 \quad , \quad \ddot{\delta}_z = \beta^2 = \beta_2 \quad , \quad \ddot{\delta}_r = \beta^3 = \beta_3 \tag{35.57}$$

where a superposed dot denotes partial differentiation with respect to time. With the help of (35.51) to (35.57) we are able to reduce the equations of motion (35.32) and (35.33) to

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \rho_o b_r = \rho_o(\ddot{u}_r + y^1\ddot{\delta}_r)$$

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial z} + \frac{2}{r}\tau_{\theta r} + \rho_o b_\theta = \rho_o(\ddot{u}_\theta + y^1\ddot{\delta}_\theta) \tag{35.58}$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r} + \rho_o b_z = \rho_o(\ddot{u}_z + y^1\ddot{\delta}_z)$$

and

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{s_{rr} - s_{\theta\theta}}{r} + (\rho_o c_r - k_r) = \rho_o(y^1\ddot{u}_r + y^2\ddot{\delta}_r)$$

$$\frac{\partial s_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial z_{rz}}{\partial z} + \frac{2}{r}s_{\theta r} + (\rho_o c_\theta - k_\theta) = \rho_o(y^1\ddot{u}_\theta + y^2\ddot{\delta}_\theta) \tag{35.59}$$

$$\frac{\partial s_{zr}}{\partial r} + \frac{1}{r}\frac{\partial s_{z\theta}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{zr}}{r} + (\rho_o c_z - k_z) = \rho_o(y^1\ddot{u}_z + y^2\ddot{\delta}_z)$$

where in obtaining (35.58) and (35.59) we have also made use of the expression for covariant differentiation of a second order tensor.

36. Comparison with the available theories

The use of advanced composite materials in aerospace and other related industries is rapidly increasing. This is due to significant advantages offered by composite materials in terms of efficiency and cost. A widespread application of composite materials requires a detailed and reliable knowledge of their physical properties and their behavior under the applied loads. One of the important subjects in this field is the development of a theoretically sound generalized continuum model for composite materials in general and for composite laminates in particular. There are a number of different theories that attempt to model the behavior of composite lamintes. These theories make use of a variety of approaches from analytical to numerical and from discrete modeling to continuum modeling. It is the purpose of this section to present a comparison between the theory developed in the course of this research project with the available continuum theories. To this end we first recall the main features of the present theory and then we proceed with a rather detailed comparison.

We record below the main features of the present continuum theory which will be referred to as "Cosserat composite theory."

a) It accounts for the effect of micro-structure.

In the present theory the motion of each material point P of the composite laminate is determined by two vector functions of position and time

$$\mathbf{r} = \mathbf{r}(\theta^i,t) \quad , \quad \mathbf{d} = \mathbf{d}(\theta^i,t) \quad , \quad i = 1,2,3 \tag{36.1}$$

where $\mathbf{r}$ is the position vector of the material point P and $\mathbf{d}$, called a director, is a deformable vector function assigned at each material point P of the composite representing the effect of micro-structure in the continuum. In other words, in the present theory a material point (particle), in addition to its mass, is endowed with a director (structuring). The kinematics and field quantities associated with the micro-structure are determined by $\mathbf{d}$ and its space and time derivatives and through relevant field quantities associated with the director **d**.

b) It accounts for the effect of geometric nonlinearity.

The vector functions **r** and **d** in (36.1) which determine the motion of the composite laminate represent finite quantities and are not restricted by any explicit or implied smallness assumptions. The complete theory presented in the course of this investigation has been developed totally in the context of the nonlinear theory.

c) It accounts for the effect of material nonlinearity.

The development of the constitutive relations in the present theory has been carried out in the context of nonlinear theory. The same is true for the development of the constraint theory of composite laminates. It should also be mentioned that although we have confined our attention to elastic composite laminates, theory is not restricted to only elastic materials and other types of materials (viscoelastic, plastic, etc.) can be treated as well.

d) It accounts for the effect of curvature.

The present theory has been developed with no restriction placed on the geometry of composite laminates. Hence, any type of initially curved composite laminate may be treated by the Cosserat composite theory. The specific cases of composite laminates such as laminated composite plates, laminated composite cylindrical shells, etc., are obtained as special cases of the present theory without any prior assumptions.

e) It accounts for the effect of interlaminar stresses.

In the present theory the intelaminar stresses are incorporated into the formulation of the theory in a natural and consistent manner and without any ad hoc assumptions. The three components of the interlaminar stress vector (i.e., one normal component and two tangential or shear components) can be extracted from the theory. This makes the theory applicable to problems involving delamination and edge effects in composite laminates both of which are considered extremely important.

f) It has a continuum character

The Cosserat composite theory possesses a character similar to that of the classical three-dimensional continuum mechanics. In particular the theory is represented by a set of conservation laws which are expressed in a coordinate-free notation. Consequently the form of the conservation laws are not changed under a transformation of the coordinate system. None of the available theories have been shown to exhibit this characteristic. From the conservation laws, in a systematic manner similar to classical continuum mechanics, we can obtain a set of basic field equations (local forms of equations of motion). The stress vector and stress couple vector in this theory exhibit similar characteristics to the stress vector in the classical continuum mechanics. The symmetry of the composite stress tensor does not hold as expected due to the presence of other field quantities. In the absence of the micro-structure when the composite laminate is reduced to a classical continuum body (i.e., $d = 0$) the conservation laws and their local forms are reduced systematically to those of the classical continuum mechanics and the symmetry property of the stress tensor is recovered automatically. The boundary conditions in the present theory are concise mathematically and are also clear from a physical point of view.

g) It is applicable to both static and dynamic problems.

Practically all continuum theories developed for composite laminates are eventually formulated in terms of displacements and aimed towards dynamic problems. Hence these theories are not normally capable of treating static problems or those with stress or mixed boundary conditions. The Cosserat composite theory does not suffer from such deficiency and without any further reformulation is capable of handling both static and dynamic problems.

h) It can be generalized to treat probiems with more than two constituents.

The Cosserat composite theory developed in the course of this study is, like all other continuum composite theories, applicable to a composite laminate with two constituents (bilaminates). However, due to choice of the configuration chosen for the representative element (micro-structure) and due to the coordinate systems adapted, this theory can be further generalized to include composite laminates with any number of constituents. The extension from bilaminate constituents to multi-laminate constituents in the available theories is not present in most available theories and for those that this generalization is possible, the resulting theories become extremely complicated.

Considering items (a) through (h) above, as a general assessment, it should be clear that there exists no single theory possessing the above characteristics collectively and at the same time having the relative simplicity of the Cosserat composite theory. Even in the cases where the available theories share some (but not all) of the above properties the Cosserat composite theory offers more generality and perhaps less complexity.

We now proceed to compare the Cosserat composite theory with some of the available theories. The conservation laws for composite laminates were summarized in section (17) and various field quantities were defined in that section. Also the complete theory of elastic composite laminates was recapitulated in section (23). In what follows we will frequently refer to these sections.

I) One of the earliest and rather elegant efforts in the field of composite materials is due to Sun, Achenbach and Herrmann in which they developed a linear continuum theory for a composite laminate. In their work, instead of introducing a representative homogeneous medium by means of "effective moduli," representative elastic moduli were used for the matrix, and the elastic and geometric properties of the reinforcing elements were combined into effective stiffnesses. With the aid of certain assumptions regarding the deformation of the reinforcing elements and by employing a smooth operation, approximate kinetic and strain energy densities for the composite laminate were obtained. By a subsequent application of Hamilton's principle to the expression for total energy of the composite, the displacement equations of motion were then obtained. In this effective stiffness theory the displacements of the reinforcing layers and the matrix layers were defined as a two term expansion about the mid-planes of the layers. The basic premise of effective stiffness theory is that a smoothing operation may be employed to replace the discrete layers of the laminate by a continuous medium. In other words the smoothing operation is a special assumption introduced so that a sum of discrete points can be replaced (mathematically) by an integral. This theory later was used to study the harmonic wave propagations in a laminated composite.

We now make a comparison between the effective stiffness theory of Sun, Achenbach and Herrmann with the Cosserat composite theory (CCT).

1) The Cosserat composite theory is a nonlinear theory whereas the effective stiffness theory is linear.

2) The Cosserat composite theory is a general theory applicable to any type of curvature while the effective stiffness theory is a special theory which is applicable to a flat composite laminate only.

3) The Cosserat composite theory is characterized by a set of well defined and coherent conservation laws (global-field equations) which are coordinate free and hence invariant under the transformation of the coordinate system(s). In contrast, the effective stiffness theory does not offer any conservation laws at all.

4) The Cosserat composite theory offers a formulation which is analogous to those of the classical continuum mechanics. In particular the local form of the basic field equations (equations of motion) are derived systematically from the conservation laws. The resulting equations

are in terms of various field quantities (i.e., stress, stress couple, etc.) which can easily be reduced to a system of displacement equations of motion. On the other hand, the effective stiffness theory offers a formulation which is based on ad hoc assumptions and is entirely in terms of the displacement variables and hence is only capable of treating a special class of problems.

5) Because of generality of its formulation the Cosserat composite theory is capable of treating problems with stress, displacement or mixed boundary conditions. However, the applicability of the effective stiffness theory seems to be limited to problems involving displacement boundary conditions only.

6) The Cosserat composite theory is capable of determining stresses, stress couples and the interlaminar stresses while in the effective stiffness theory the interlaminar stresses are not present.

7) The process of extension from the representative element to a continuum model in the two theories may be considered to have some conceptual or physical similarities, but are not the same. However, from a mathematical point of view in the effective stiffness theory the smoothing process is a special assumption whereas in the Cosserat composite theory the same type of results is obtained through a limiting process.

II) The basic concepts involved in the derivation of the linear effective stiffness theory were used by Grot and Achenbach to derive an approximate nonlinear theory to describe the mechanical behavior of a laminated composite consisting of alternating layers of homogeneous materials. The theory is based on two-term expansions of the motion across the thicknesses of the undeformed layers. The system of governing equations for the homogeneous continuum model of the laminated medium are derived in two stages. The first stage of the derivation involved certain assumptions and operations within the discrete system of layers. In particular, it was assumed that the motions of the individual layers can be described by two-term expansions in the local coordinate normal to the layering of the undeformed body. The kinematic variables that were introduced in the expansions were defined at the midplanes of the layers only. The local equations of linear momentum and moment of momentum for the individual layers were obtained by integration of the classical three-dimensional continuum mechanics across the thickness of the undeformed layers. Next, the definitions of average stress and couple-stress were introduced which were defined in discrete planes only. The stresses and couple stresses were then related to the relevant kinematical quantities through stress potentials which were obtained by integrating the local stress potentials across the undeformed thicknesses. In the second stage of the derivation a transition was made from the system of discrete layers to the homogeneous continuum model. The transition accomplished by essentially following the same line of argument used in the linear effective stiffness theory.

We now make a comparison between Grot and Achenbach theory (GA theory) with the Cosserat composite theory.

1) Both theories are nonlinear and hence are applicable to large deformations.

2) The Cosserat composite theory is a general theory applicable to any type of curvature while GA theory is applicable to flat composite laminates only.

3) The Cosserat composite theory is based on a set of well defined and coherent global principles (conservation laws) which are coordinate free and hence are invariant under coordinate transformations. In contrast, GA theory does not offer a set of conservation laws at all.

4) The Cosserat composite theory offers a systematic formulation similar to those of the classical continuum mechanics. The GA theory is also systematic (to some degree); but it is based on ad hoc assumptions.

5) The Cosserat composite theory is formulated in terms of general convected curvilinear coordinates in the present configuration resulting in simpler expressions. The transformation between the present (deformed) and the reference (undeformed) configurations is clear and well

defined. On the other hand, the GA theory is formulated in terms of field quantities defined in the reference (undeformed) configuration. The counterparts of the equations in the present configuration are not given. It should be mentioned that in the case of large deformations this latter formulation becomes important.

6) The Cosserat composite theory is applicable to laminates with variable mass densities (i.e., the mass densities of the constituents may be variable). On the other hand, in the GA theory the mass densities of the constituents is assumed to be constant.

7) The Cosserat composite theory rigorously establishes the existence of interlaminar stresses and accounts for their effect in the global and local field equations. In the GA theory, although these stresses appear in the equation for linear momentum but it is the consequence of a special assumption. The GA theory did not refer to these stresses as interlaminar stresses and did not elaborate on the nature of these stresses. Nevertheless, the assumptions that led to the existence of these stresses in the linear momentum equation, resulted in two equations, one for each of the constituents. These equations involve both the average stress couples across the thickness of the layers and stresses (ordinary three-dimensional) within each layer. This does not seem to be consistent.

8) A remark similar to that in (I-7) above also holds in this case.

The effective stiffness theory was later generalized, in the context of linear theory, where the displacement components were expressed in terms of Legendre polynomials. In this work Aboudi imposed the condition of continuity of displacement and stress components between the adjacent layers where all the continuity conditions were satisfied pointwise throughout the common boundary of the adjacent layers. Aboudi's formulation is more complicated than that of Achenbach et al. and hence less manageable. Aboudi has expanded on his theory and has been able to apply the theory to nonelastic laminated composites. Because of similarities of the main features of Aboudi's work and effective stiffness theory of Achenbach, the same comments (I through 7) are applicable to Aboudi's work too.

III) Very recently, Blinowski has recognized the need to formulate a continuum theory for composite laminates with curved layers. He has also commented on the lack of clarity about boundary conditions in the literature. Indeed, except for Blinowski's paper, the authors were not able to name any references dealing with curved geometry.

In his theory, Blinowski considered a nonlinear continuous mathematical model of a discrete laminated composite composed of a family of initially curved and parallel surfaces. Making use of a set of curvilinear coordinates he defined the kinematical variables and developed the various kinematical relations. Next, from the assumption that the elastic energy is a function of macro-deformation and the curvature variation of a family of initially parallel surfaces he obtained a quasi-static equation of equilibrium. These equations which involve resultant forces and resultant moments do not contain any dynamic terms. Blinowski stated that the set of equations derived describe a particular case of the Cosserat medium.

We now proceed to compare the Cosserat composite theory with the theory developed by Blinowski.

a) In the Cosserat composite theory the kinematical measures are clear and concise; in particular the director is clearly defined to be a deformable vector field which represent the effect of micro-structure. In contrast, the kinematic variables in Blinowski's theory are more complicated and the director is not clearly defined and implicitly assumed to be the normal to the surface.

b) In the Cosserat composite theory varius field quantities corresponding to the composite laminate are concisely defined and the relations betwen micro-structure field quantites and those of macro-structure as well as their physical nature are logical and clear. In contrast, this clarity does not appear in Blinowski's theory and the correspondence between micro- and macro-structural quantities is not explained. In particular, stress vector and couple stress vector are introduced by special assumptions which are motivated by Cauchy's postulate on stress in classical continuum mechanics.

c) In Blinowski's theory body couple density is not defined and does not appear in the equations of motion. Also the inertia terms are absent in the equations of motion. In addition, the principle of director momentum does not exist, and the role of the director is totally obscure. Although it is stated that the theory is a special case of the Cosserat surface theory of Green and Naghdi, the correspondence has not been shown. In contrast, these problems in the Cosserat composite theory do not occur and the correspondence to the Cosserat surface theory is absolutely clear.

d) It should be emphasized that Blinowski's theory, which is appropriate for bending, is the only theory that considers the effect of the curved geometry.

IV) Recently, a mixture theory for linear elastodynamics and periodically laminated media has been developed by Murakami et al. in which they introduced the concept of the director. In this theory the asymptotic method of multiple scales was used to construct a continuum theory with micro-structure for linear elastodynamics of a periodically laminated medium. The resulting theory is in the form of a homogeneous binary mixture theory of micromorphic materials with a common director oriented normal to the interfaces. The construction of the model was based upon the observation that, along a direction normal to the laminae, there are two length scales over which significant variations in displacement and stress profiles occur and that these scales differ by at least one order of magnitude in most problems of practical interest. The very important assumption of the theory is the periodicity of the medium under consideration. As pointed out by the authors, "the analysis may be valid only at points sufficiently far removed from the boundaries of the domain in which the solution to be obtained." This entails the fact that the theory is applicable to the problems in which the boundary conditions are not of primary importance. The basic field equations as well as the constitutive relations were obtained by an averaging operation. The model contains nine basic field equations, six for the linear momenta of both constituents and three for the director momentum. The constitutive relations are obtained asymptotically and with the help of the periodicity assumption.

We now proceed to compare the mixture theory of Murakami with the Cosserat composite theory.

1) The Cosserat composite theory is a nonlinear theory whereas the mixture theory of Murakami is linear.

2) The Cosserat composite theory is a general theory applicable to any type of curvature while the mixture theory of Murakami is a special theory which is applicable to a flat composite laminate only.

3) The Cosserat composite theory offers a formulation which is analogous to those of the classical continuum mechanics. In particular, the Cosserat composite theory is characterized by a set of well defined global conservation laws from which the local basic field equations are obtained systematically. In contrast the mixture theory of Murakami does not offer any global conservation laws. Hence, no conclusion may be reached regarding the character of various field quantities of the theory.

4) The use of the asymptotic method and the assumption of periodicity places an important restriction on the theory and makes it inadequate in the vicinity of the boundaries of the domain (at least in the direction of layering). Therefore the theory cannot be applied to problems in which the boundary conditions are of primary importance. In addition, when the material is not periodic (i.e., variable thickness plies) the theory becomes invalid. In contrast, the Cosserat composite theory is not restricted to the periodicity of the medium and can treat problems with variable thickness plies. Moreover, any type of boundary condition may be treated by the Cosserat composite theory.

5) The conservation of moment of momentum does not exist in the mixture theory of Murakami. In addition, equations of motion do not contain terms involving body force and body couple densities. The term called "interaction (body) force vector" which appears in the linear momentum equations does not exhibit the character of a body force density in the sense of the three-dimensional continuum mechanics. On the other hand, the Cosserat composite theory offers the moment of momentum equation and it also accounts for the effect of body force and body couple in the continuum.

6) In the Cosserat composite theory the nature of the various field quantities are quite clear, both physically and mathematically, while in the mixture theory of Murakami the physical nature of some of the field quantities is obscure. In particular, the term called "interaction (body) force vector" in their theory does not have the character of the body force. It may be shown or justified that this term is related to interlaminar stresses. However, this fact does not seem to have been recognized and emphasized in the development of the theory.

V) Another continuum theory attempting to describe the behavior of composite laminates is the multi-continuum (or diffusing continuum) theory of Bedford and Stern which is one of the earliest efforts in this field. Bedford and Stern developed a thermomechanical theory for composite materials in which the composite constituents were modeled by individual superimposed continua which may interact thermally and mechanically. The main ingredient of the theory is that each constituent is admitted to undergo an individual motion. The mechanical interaction between the individual constituent motions then provide a means of including composite structural effects in the theory. The mechanical interactions between the continua depend on the constituents relative displacements. This theory, which was developed in the context of nonlinear theory, considers each constituent individually which interact with other constituents only through an interaction term in the form of a body force entering into equations of motion of each constituent. However, there is no field equation (local or global) offered for the composite as a whole. Similarly, field quantities such as resultant stresses, etc., are not defined and do not play any roles in the theory. In this respect the theory seems to lag behind the effective stiffness theory and the mixture theory of Murakami. In addition, the correspondence between the field quantities of the constituents and the composite as a whole is not defined and explained in this theory. In general terms, this theory seems to place the emphasis on the continuum character of the individual constituents, while the composite Cosserat theory and the theories discussed earlier not only account for the continuum character of the constituents but also attempt to consider the continuum character of the composite laminate as a whole.

37. Conclusion

In the course of this investigation we have successfully developed a coherent continuum theory which is represented by a set of well defined conservation laws (global field equations) predicated on physical observations. The theory is complete, physically sound, and mathematically accurate. At the same time the theory enjoys characteristics similar to those of classical continuum mechanics and most of the techniques available in the classical three dimensional continuum mechanics may directly or with some modification, be applied to the present theory. Due to the use of Cosserat surface theory in development of the present theory we have appropriately named it as "*Cosserat composite theory*", after E. & F. Cosserat. We have demonstrated that within the context of purely mechanical theory the Cosserat composte theory exhibits the following features.

a) It acconts for the effect of micro-strctures.

b) It accounts for the effect of geometric nonlinearity.

c) It accounts for the interlaminar stresses and therefore delamination can be considered.

d) It is capable of incorporating the effect of material nonlinearity.

e) It accounts for the effect of curvature.

f) It possesses a continuum character.

g) It is applicable to both static and dynamic problems.

In addition, it is evident that the theory may further be developed to account for the effect of temperature. The theory may also be generalized to treat multi-constituent composite laminates.

In view of the above discussion it is clear that at the present time no other single theory offers the collective characteristics and the relative simplicity of the Cosserat composite theory. This makes the theory the ideal tool for treating the various problems concerning composite laminates. A proposal for further development of the theory, as mentioned in section 1, discusses future developments in detail in phase II of the present research.

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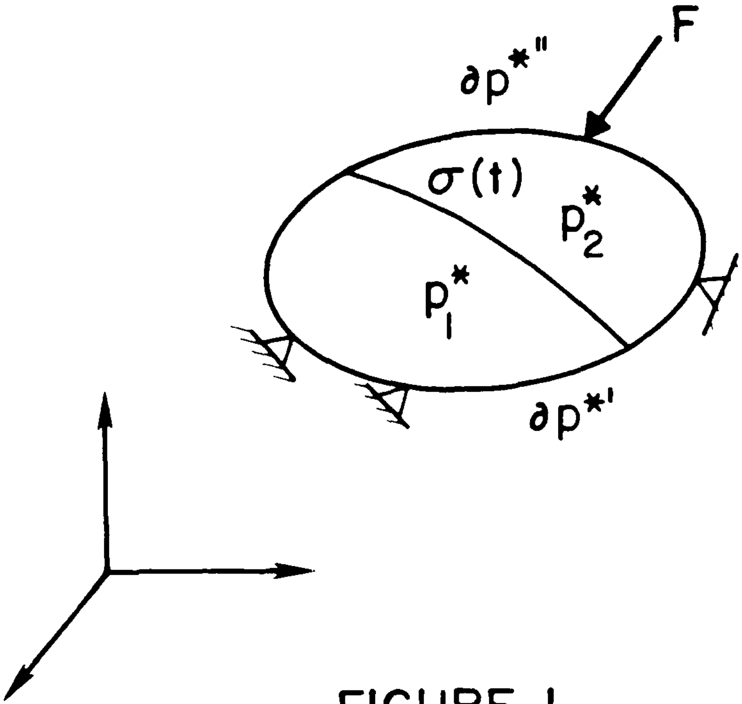

FIGURE 1

A CONTINUUM BODY WITH A SURFACE OF DISCONTINUITY

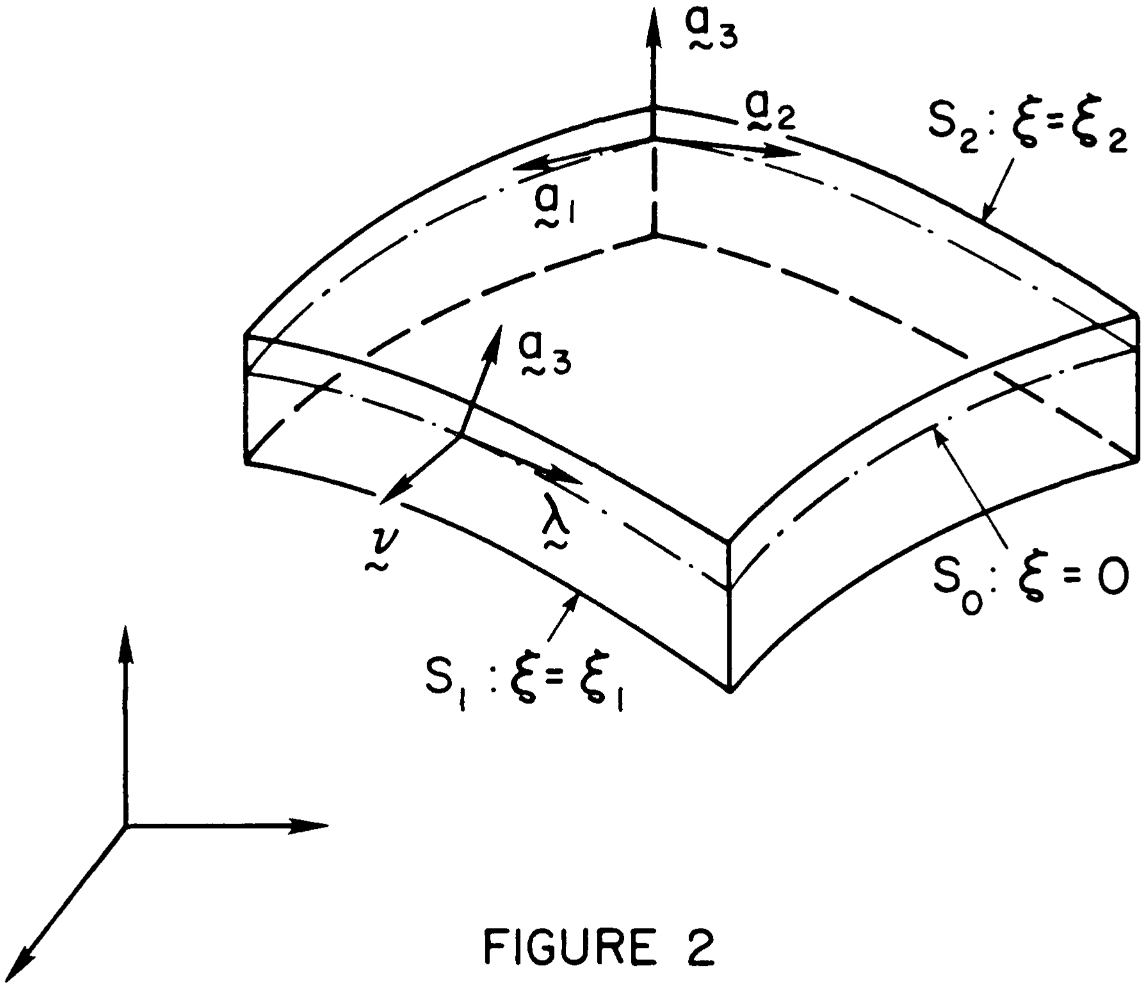

FIGURE 2

A TYPICAL SHELL-LIKE BODY

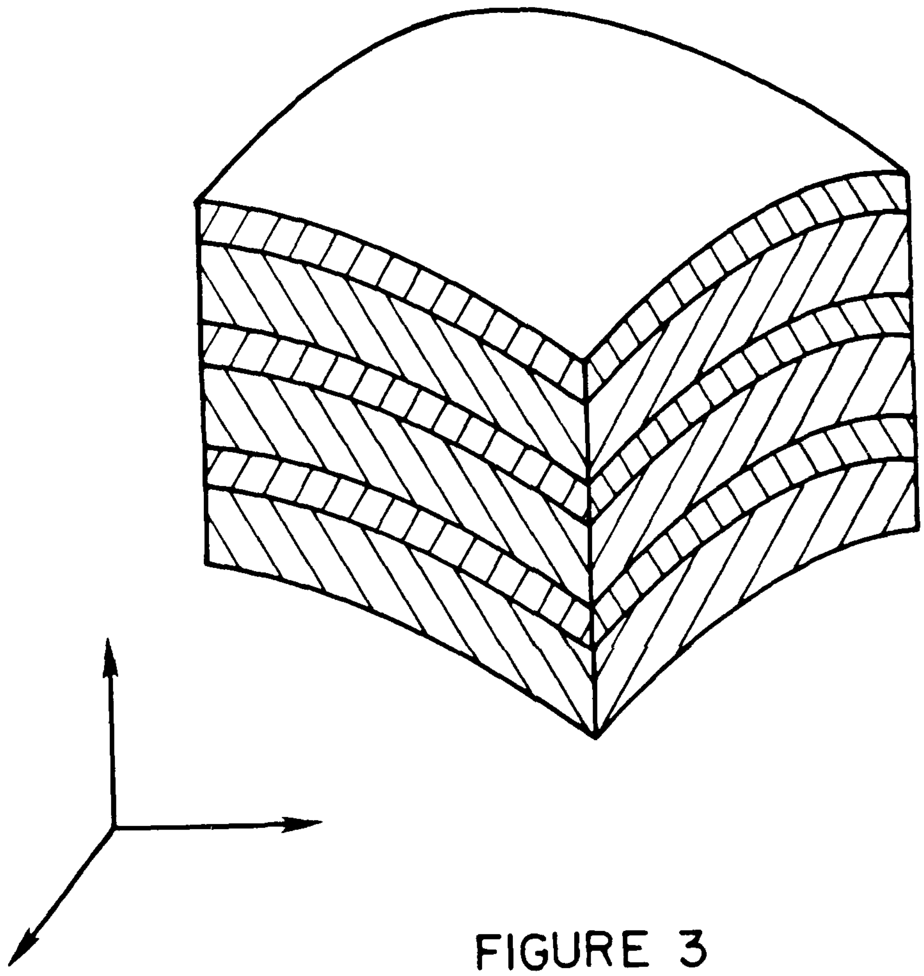

FIGURE 3

A COMPOSITE LAMINATE CONSISTING OF ALTERNATING LAYERS OF TWO MATERIALS

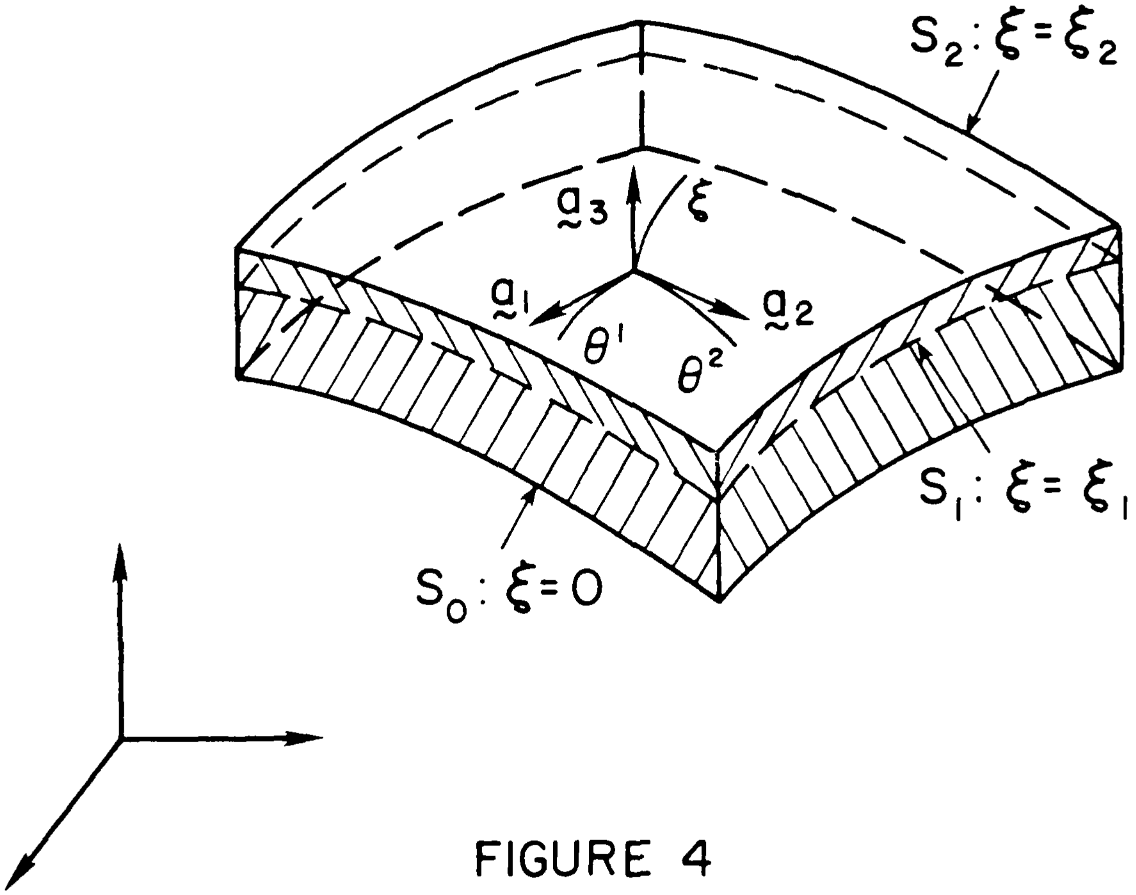

FIGURE 4

A SHELL-LIKE MICRO-STRUCTURE (REPRESENTATIVE ELEMENT)

BASE

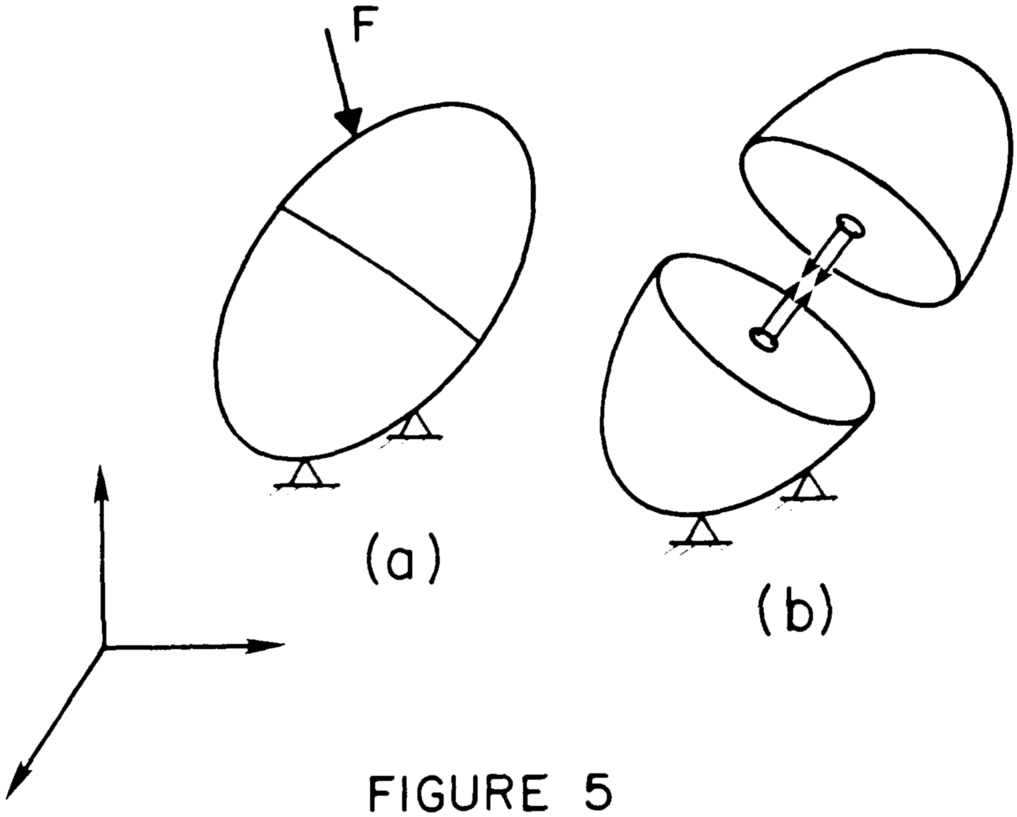

FIGURE 5

AN ARBITRARY PART OF A COMPOSITE LAMINATE

BASE

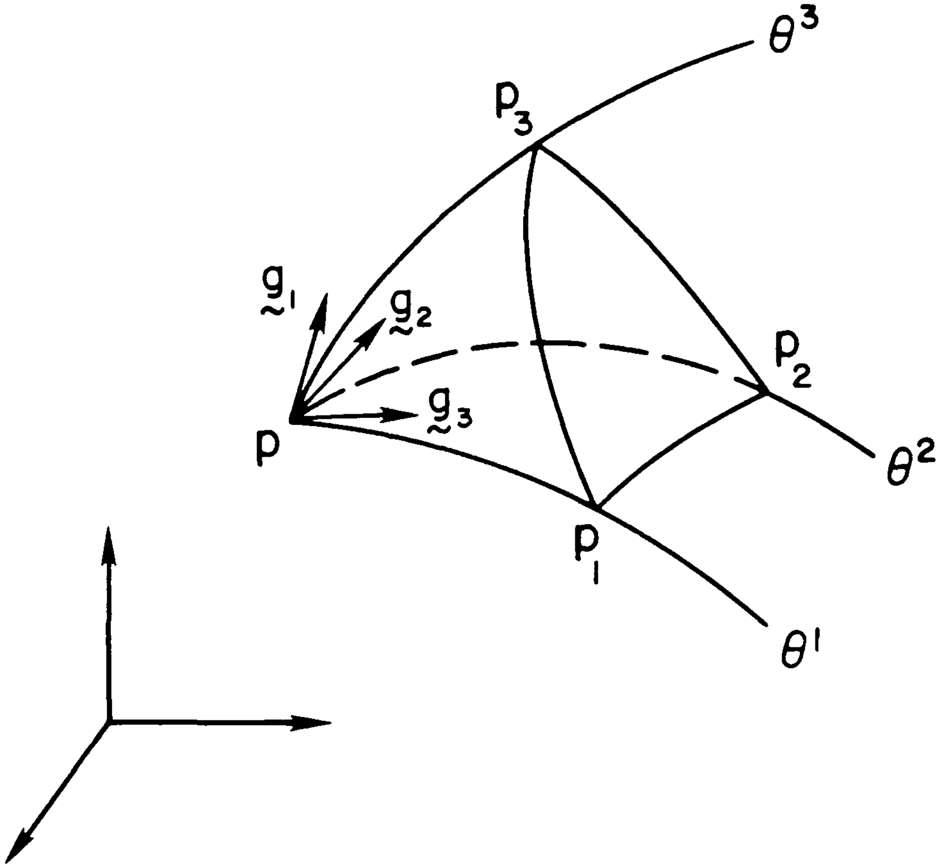

FIGURE 6

A CURVILINEAR TETRAHEDRON OF A COMPOSITE LAMINATE